Positive Formulas for Some Substructural Logics

Pozitivní formule v některých substrukturálních logikách

Diplomová práce

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Pavel Truhlář

Abstract

We will examine which distributive substructural logics, as defined in the book of Restall [Res00], have the same positive fragment with and without the weak excluded middle axiom. The main result of this diploma thesis is that some substructural logics have this property.

We repeat the basic notions as described in [Res00], especially the consecution, natural deduction, frame semantics, Hilbert system. We will use the soundness and completeness theorems. We also will use the equivalence of natural deduction systems and Hilbert systems. All these important theorems are in [Res00].

We make the proof of our main result in the next part. We will use the semantics of frames, similarly as de Jongh and Zhao [dJZ13]. We will define the top model. After, we define the construction which converts a model to the top model. We define for each formula the positive part of it; this is the formula, which behaves the same way on the top models as the original formula. We use Hilbert type calculus to formulate our main theorem. We prove our main result using the deduction theorem for certain types of Hilbert type calculus. We list at the end the logics, for which we proved that have the same positive fragment with and without the weak excluded middle axiom.

Abstract

V této diplomové práci budeme zkoumat, které distributivní substrukturální logiky, tak jak jsou definovány v Resstalově knize [Res00] mají stejný positivní fragment s axiomem slabého vyloučeného třetího a bez něj. Hlavní výsledek této diplomové práce je, že některé substrukturální logiky tuto vlastnost mají.

Zopakujeme základní pojmy, tak jak jsou popsány v [Res00], zvláště pak konsekuce, přirozená dedukce, rámcová semantika, Hilbertův systém. Budeme používat věty o korektnosti a úplnosti. Také budeme potřebovat větu o ekvivalenci přirozené dedukce a Hilbertova důkazového systému. Všechny tyto důležité věty jsou v [Res00].

V další části dokážeme náš hlavní výsledek. Budeme používat sémantiku rámců podobně jako de Jongh and Zhao v článku [dJZ13]. Definujeme, co to je top model. Poté ukážeme, jak z daného modelu vytvoříme top model. Pro každou formuli definujeme její pozitivní část, to jest formuli, která se na top modelech chová stejně jako původní formule. Pro formulaci naší hlavní věty použijeme Hilbertův kalkulus. Dokážeme ji pomocí věty o dedukci, která platí pro některé typy Hilbertova kalkulu. Na závěr vypíšeme seznam logik, pro které jsme dokázali, že mají stejný pozitivní fragment se slabým zákonem vyloučeného třetího a bez něj.

Contents

1 Introduction

We consider in this paper certain distributive substructural logics and their extension with the axiom of weak excluded middle. We investigate whether these logics with and without the weak excluded middle axiom have the same positive fragment. Equivalently, we investigate whether adding the axiom of weak excluded middle is conservative over the positive fragment of the logics in consideration.

The weak excluded middle axiom is the following scheme: $\neg A \vee \neg\neg A$. We will denote this axiom WEM.

We extend the results of V.A.Jankov [V.A68] and de Jongh and Zhao [dJZ13]. Both results say that intuitionistic logic (which we will denote IL) and the IL with WEM (we will denote this logic JL as Jankov Logic) have the same positive fragment. The positive fragment of a language is the set of formulas which do not contain $\bot$, the positive fragment of a logic L are theorems of L which do not contain $\bot$. The positive fragment is the set of formulas without $\bot$ (and therefore without negation, since we will consider negation $\neg A$ as an abbreviation for $A \to \bot$. We will denote the positive fragment of the logic L as L^+. So the result in [V.A68] and [dJZ13] is $IL^+ = JL^+$. V.A. Jankov uses the algebraic approach to prove this result. Dick de Jongh and Zhiguang Zhao use the Kripke semantics approach.

Our goal will be to extend this result to some substructural logics, namely distributive substructural logics. We will use very similar approach as in [dJZ13] based on the frame semantics for distributive substructural logics as described in [Res00].

Our starting point is the distributive associative Full Lambek logic, with the twisted associativity axiom (which is also known as suffixing). We prove that adding the weak excluded middle axiom to this logic is conservative over its positive fragment, i.e. that these two logics have the same positive fragment.

In the proof, we will construct for each formula A its positive counterpart A^+ which is equivalent to A over so called top models (models with the greatest point, in which all atomic formulas are satisfied). This plus construction is an identity on positive formulas and it translates the axiom WEM into $\top$. Using Hilbert style axiomatization of the logic, which is complete and admits a deduction theorem, we show that the plus construction preserves provability, and the result follows.

The argument remains valid if we extend the basic system with (some of) structural axioms of commutativity, weakening and contraction. Therefore our work covers the extensions of distributive Full Lambek logic with commutativity (dFLe), with both commutativity and weakening (dFLew), intuitionistic logic (which is FLewc), and relevant logic R (dFLec).

An alternative approach is the one using the algebraic semantics of substructural logics given by residuated lattices, as has been done by Galatos in [Gal11] using an ordinal sum construction. This approach does not assume distributivity and is therefore more general. How exactly these two approaches relate to each other remains to be seen.

1.1 Structure of the Thesis

- We start with well known results. We describe at first the syntax of the substructural logics in the section 2.1. We describe the natural deduction in the section 2.2. Then we describe the frame semantics for these logics in the section 2.3. After, we state the soundness and completeness in the section 2.4. Then we show the Hilbert systems for these logics in the section 2.5. We state that natural deduction and Hilbert systems are equivalent in the section 2.6.

- We continue with our own results. We show in the section 2.7 how the frames with WEM axiom look. The next chapter 2.8 deals with positive formulas and top models. These notions are for substructural logics introduced in this paper. After, we will for each formula construct the positive part in the section 2.9. Then we will look at Hilbert systems, for which our main result holds in 2.10. The main result is that certain logics have the same positive fragment with WEM and without them. The proof of this follows. This main result with the proof is in the section 2.11 We list examples of the logics, for which it holds at the end of the work, in the section 3.1. We also recommend further work in this area in the section 3.2.

2 Positive Formulas in Substructural Logics

We will examine certain distributive substructural logics, using their syntax and frame semantics as defined in the book of Restall [Res00]. We will closely follow the book [Res00] in setting the necessary preliminaries and we refer the reader to consult it for further details.

2.1 Syntax of the Substructural Logics

Let us have the language L with binary connectives $\rightarrow$, $\leftarrow$, $\circ$, $\vee$ and $\wedge$. We will also consider unary symbols $\top$, t and $\bot$. We will also consider signs $\neg$ and $\sim$. These signs are just abbreviations: the $\neg A$ is an abbreviation for $A \rightarrow \bot$, and $\sim A$ is an abbreviation for $\bot \leftarrow A$. We also consider a countable set of atoms A_t.We will denote these atoms p, q and so on. We will consider as the language of substructural logics the set of all formulas created from these connectives and atoms in the usual way; we give a more precise definition below.

We will use a Hilbert style approach to capture axiomatization of the logics to prove the main result. However, to be able to refer to the book [Res00] as closely as possible, in particular, to use the completeness results proven there for a natural deduction calculi. We will also need to use the notion of structure and consecution.

The structure X is the structured collection of premises. We will provide the exact definition below. But before it, we have to give the definition of language (set of formulas) and punctuation mark.

Definition 1. Given a countable set $AForm$ of atomic formulas, and a finite set $Conn$ of connectives, disjoint from $AForm$, the language Lang$(AForm; Conn)$ is the smallest subset of $String(AForm \cup Conn)$ satisfying the following conditions:

- $AForm \subseteq Lang(AForm; Conn)$. This means that every atomic formula is a part of the language $Lang(AForm; Conn)$.
- If $c \in Conn$ and $a(c) = n$ (arity of the connective), then for every $A_1, ..., A_n \in Lang(AForm; Conn)$ the string $c \frown A_1 \frown ... \frown A_n$ is also in $Lang(AForm; Conn)$.

We will use in this paper the set of connectives $(\rightarrow, \leftarrow, \wedge, \vee, \circ)$.

Let us see, how the structures are built.

Definition 2. A punctuation mark p is an object together with a number $a(p) \in \omega$, its arity.

Definition 3. A collection $Struct$ of structures is made up of a language $Lang$ and a set $Punct$ of punctuation marks, disjoint from $Lang$. The set $Struct(Lang; Punct)$ on $Lang$ and $Punct$ is the smallest subset of $Sring(Lang \cup Punct)$ satisfying these conditions:

- *Lang* $\subseteq$ *Struct*(*Lang*; *Punct*). That is, every formula in *Lang* is a structure.
- If $p \in Punct$ and $a(p) = n$, then for any $X_1, ..., X_n \in Struct(Lang; Punct)$, the string $p \frown A_1 \frown ... \frown A_n \in Struct(Lang; Punct)$

We will use in our paper two binary punctuation marks semicolon **;** and comma **,** . We will use in some substructural logics the empty structure, we will denote it 0. We add just in this case to the definition of structure that 0 is a structure.

The consecution is the claim of the form, $X \vdash A$, where X is a body of premises and A is a consequence of X. We also provide the exact definition.

Definition 4. Given a set *Struct* of structures over a language *Lang*, a consecution on *Struct* is of the form $X \vdash A$, where $X \in Struct$ is the antecedent of the consecution and $A \in Lang$ is the consequent of the consecution.

2.2 Natural Deduction

We will describe now a system of the natural deduction $\mathcal{G}$. We have two rules for each connective. One rule tells us how we can derive the connective. It is called the introduction rule. The other rule to do with a connective tells how to eliminate the connective. It is called an elimination rule. These rules are in the detail described in the chapter 2 of [Res00]. Let us see these rules. We start with rules for implication:

$$\frac{X; A \vdash B}{X \vdash A \to B}\,(\to I)$$

$$\frac{X \vdash A \to B \qquad Y \vdash A}{X; Y \vdash B}\,(\to E)$$

The rules for converse implication $(\leftarrow I)$ and $(\leftarrow E)$ are:

$$\frac{A; X \vdash B}{X \vdash B \leftarrow A}\,(\leftarrow I)$$

$$\frac{X \vdash B \leftarrow A \qquad Y \vdash A}{Y; X \vdash B}\,(\leftarrow E)$$

The rules for fusion are:

$$\frac{X \vdash A \qquad Y \vdash B}{X; Y \vdash A \circ B}\ (\circ I)$$

$$\frac{X \vdash A \circ B \qquad Y(A; B) \vdash C}{Y(X) \vdash C}\ (\circ E)$$

The rules for the logical truth constant t are:

$$0 \vdash t\ (tI)$$

$$\frac{X \vdash t \qquad Y(0) \vdash A}{Y(X) \vdash A}\ (tE)$$

The exception is the trivial truth $\top$ and the trivial falsehood $\bot$. Each of them has only one rule:

$$X \vdash \top\ (\top I)$$

$$\frac{X \vdash \bot}{Y(X) \vdash A}\ (\bot E)$$

Similar rules are valid for the extensional connectives $\wedge$ and $\vee$.

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ (\wedge I)$$

$$\frac{X \vdash A \wedge B}{X \vdash A}\ (\wedge E_1)$$

$$\frac{X \vdash A \wedge B}{X \vdash B}\ (\wedge E_2)$$

$$\frac{X \vdash A}{X \vdash A \vee B}\ (\vee I_1)$$

$$\frac{X \vdash B}{X \vdash A \vee B}\ (\vee I_2)$$

$$\frac{Y(A) \vdash C \qquad Y(B) \vdash C \qquad X \vdash A \vee B}{Y(X) \vdash C} \; (\vee E)$$

The important notion in substructural logics is the structural rule.

Definition 5. Structural Rules

A rule

$$\frac{Y(X) \vdash A}{Y(X') \vdash A}$$

is a structural rule if it is closed under substitution of formulas. We will denote this structural rule $X \Leftarrow X'$.

We list several structural rules.

Name	Label	Rule
Associativity	B	$X; (Y; Z) \Leftarrow (X; Y); Z$
Twisted Associativity	B'	$X; (Y; Z) \Leftarrow (Y; X); Z$
Converse Associativity	B^c	$(X; Y); Z \Leftarrow X; (Y; Z)$
Strong Commutativity	C	$(X; Y); Z \Leftarrow (X; Z); Y$
Weak Commutativity	CI	$X; Y \Leftarrow Y; X$
Strong Contraction	W	$(X; Y); Y \Leftarrow X; Y$
Weak Contraction	WI	$X; X \Leftarrow X$
Mingle	M	$X \Leftarrow X; X$
Weakening	K	$X \Leftarrow X; Y$
Commuted Weakening	K'	$X \Leftarrow Y; X$

If all the associativity rules are present, the weak and strong versions of the rules become interderivable. If moreover commutativity is present, weakening and commuted weakening are interderivable.

Some of the structural rules are also known by different names: the commutativity is often called exchange and denoted by (e), contraction is often denoted by (c), weakening and commuted weakening are referred to as right weakening (o) and left weakening (o), and if both are present then simply (w).

2.3 Frame Semantics

In this section, we will describe the frame semantics for distributive substructural logics. We start with the important notion of frames. Frames are based on partially ordered sets of points and accessibility relations.

Definition 6. *(POINT SETS AND PROPOSITIONS)* A point set $\mathcal{P} =< P, \sqsubseteq>$ is a set P together with a partial order $\sqsubseteq$ on P. The set $Prop(\mathcal{P})$ of propositions on P is the set of all subsets X of P which are closed upwards: that is, if $x \in X$ and $x \sqsubseteq x'$ then $x' \in X$.

We will consider the plump accessibility relations R and C as ACCESSIBILITY RELATIONS defined in [Res00]. I will state here only the definition of the relation R, because the relation C is defined using R to fit our definition of negation, $\neg A$ is an abbreviation for $A \rightarrow \bot$. The compatibility relation for our two negation connectives is defined as xCy iff $\exists z Rxyz$.

Definition 7. A ternary relation R is a plump three-place accessibility relation on the point set $\mathcal{P}$ if and only if for any $x, y, x', y', z, z' \in \mathcal{P}$, where $Rxyz$, $x' \sqsubseteq x$, $y' \sqsubseteq y$ and $z \sqsubseteq z'$ then $Rx'y'z'$.

We state now, how the compatibility relation C looks, if negation is given by $A \rightarrow \bot$ (the dual negation $\bot \leftarrow A$ is given by the same compatibility relation). xCy if and only if $\exists z Rxyz$. The plump condition for C therefore translates to: for any $x, x', y, y' \in P$ where xCy and $x' \leqslant x$ and $y' \leqslant y$, it holds that $x'Cy'$.

We will call the point set $\mathcal{P}$ with any number of accessibility relations a frame $\mathcal{F}$. We will also consider the evaluation $\models$ on the frame $\mathcal{F}$ as defined in [Res00] and below in Definition 9.

Definition 8. TRUTH SETS FOR A TERNARY RELATION R
If R is a three-place accessibility relation on a point set P then for any subset $T \in Prop(P)$,

- T is a left truth set for R if and only if for each $x, y \in P, x \sqsubseteq y$ if and only if for some $z \in T, Rzxy$.
- T is a right truth set for R if and only if for each $x, y \in P, x \sqsubseteq y$ if and only if for some $z \in T, Rxzy$.

Definition 9. EVALUATIONS ON FRAMES A relation $\models$ between points of a frame $\mathcal{F}$ and formulas is said to be an evaluation if and only if for each the connectives in the language, the condition from the following list holds.

- $\{x \in \mathcal{F} : x \models p\} \in Prop(\mathcal{F})$
- $x \models \top$ for all $x \in \mathcal{F}$
- $x \models \bot$ for no $x \in \mathcal{F}$
- $x \models A \wedge B$ iff $x \models A$ and $x \models B$ for each $x \in \mathcal{F}$
- $x \models A \vee B$ iff $x \models A$ or $x \models B$ for each $x \in \mathcal{F}$
- $x \models A \supset B$ iff for each $y \in \mathcal{F}$, where $x \sqsubseteq y$, if $y \models A$ then $y \models B$
- $x \models t$ iff $x \in T$ for each $x \in \mathcal{F}$
- $x \models \sim A$ iff for each $y \in \mathcal{F}$, where xCy, $y \not\models A$
- $x \models \neg A$ iff for each $y \in \mathcal{F}$, where yCx, $y \not\models A$
- $x \models A \rightarrow B$ iff for each $y, z \in \mathcal{F}$, where $Rxyz$, if $y \models A$ then $z \models B$
- $x \models A \circ B$ iff for each $y, z \in \mathcal{F}$, where $Ryzx$, if $y \models A$ then $z \models B$
- $x \models A \leftarrow B$ iff for each $y, z \in \mathcal{F}$, where $Ryxz$, if $y \models A$ then $z \models B$

We also need to define the evaluation for structures:

- $x \models 0$ if and only if $x \in T$
- $x \models X, Y$ if and only $x \models X$ and $x \models Y$
- $x \models X \circ Y$ if and only if for some y,z, where $Ryzx$, $y \models X$ and $z \models Y$

2.4 Soundness and Completeness

We will use two important theorems from Restall's book [Res00]: Soundness and completeness of the natural deduction systems of substructural logics with respect to their corresponding classes of frames, and equivalence of natural deduction systems and Hilbert systems. The proofs of these two theorems are quite long, so we will state the theorems without proofs. The proofs can be found in [Res00].

Definition 10. Given a natural deduction system $\mathcal{G}$, a frame is fit for $\mathcal{G}$ if and only if every condition corresponding to each structural rule of the system G holds in the frame.

Definition 11.
- X entails A in the model $\mathcal{M}$, written $X \vDash_{\mathcal{M}} A$, iff for every point x, if $x \models X$ then $x \models A$
- X entails A in the frame $\mathcal{F}$, written $X \vDash_{\mathcal{F}} A$, iff for every model $\mathcal{M}$ based on $\mathcal{F}$ $X \vDash_{\mathcal{M}} A$

Theorem 1. *If $\mathcal{F}$ is fit for $\mathcal{G}$, and $X \vdash A$ is provable in $\mathcal{G}$, then $X \vDash_{\mathcal{F}} A$.*

See the THEOREM 11.20 on the page 251 of [Res00].

Theorem 2. *If $X \vDash_{\mathcal{F}} A$ for each frame $\mathcal{F}$ fit for $\mathcal{G}$, then $X \vdash A$ is provable in $\mathcal{G}$.*

See the THEOREM 11.37 on the page 259 of [Res00].

The previous two theorems in particular say that the distributive substructural logics containing some of the following structural rules are complete with respect to the corresponding classes of frames. We show several examples of this correspondence. We also add the corresponding Hilbert axiom.

- the associativity, the structural rule is $X;(Y;Z) \Leftarrow (X;Y);Z$, the frame must satisfy $R(xy)zw \to Rx(yz)w$, Hilbert axiom is $(A \to B) \to ((C \to A) \to (C \to B))$
- the twisted associativity, the structural rule is $X;(Y;Z) \Leftarrow (Y;X);Z$, the frame must satisfy $R(yx)zw \to Rx(yz)w$, Hilbert axiom is $(A \to B) \to ((B \to C) \to (A \to C))$
- the converse associativity, the structural rule is $(X;Y);Z \Leftarrow X;(Y;Z)$, the frame must satisfy $Rx(yz)w \to R(xy)zw$, Hilbert axiom is $((B \leftarrow C) \leftarrow (A \leftarrow C)) \leftarrow (B \leftarrow A)$
- the weak commutativity, $X;Y \Leftarrow Y;X$, $Rxyz \to Ryxz$, $A \to ((A \to B) \to B)$
- the strong commutativity, $(X;Y);Z \Leftarrow (X;Z);Y$, $R(xz)yw \to R(xy)zw$, $(A \to (B \to C)) \to (B \to (A \to C))$

- the weakening, $X \Leftarrow X;Y$, $Rxyz \to x \sqsubseteq z$, $A \to (B \to A)$
- the contraction, $(X;Y);Y \Leftarrow X;Y$, $Rxyz \to R(xy)yz$, $(A \to (A \to B)) \to (A \to B)$

Notice, that from associativity and commutativity (even from the weak commutativity) follows the twisted associativity.

2.5 Hilbert Systems

We define the Hilbert system $\mathcal{HG}$ as defined in [Res00]. Hilbert system contains rules and axioms. For better readability, we split them to the sets. We will use $\Longrightarrow$ for one-way rules and $\Longleftrightarrow$ for two-way rules.

Definition 12. The core rule set for a Hilbert system consists of the following axiom (the identity axiom) and rules

$$A \to A$$

$$A \circ B \to C \Longleftrightarrow A \to (B \to C)$$

$$A \to B, C \to D \Longrightarrow (B \to C) \to (A \to D)$$

$$t \to A \Longleftrightarrow A$$

$$A, A \to B \Longrightarrow B$$

Definition 13. The conjunction rule set is made up of the following axioms, together with one rule.

$$A \wedge B \to A$$

$$A \wedge B \to B$$

$$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$$

$$A, B \Longrightarrow A \wedge B$$

Definition 14. The disjunction rule set is made up of the following axioms.

$$A \to A \vee B$$

$$B \to A \vee B$$

$$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$$

Definition 15. The top and bottom axioms are the following axioms.

$$A \to \top$$

$$\bot \to A$$

The translation *transl* of a structure and a consecution into Hilbert type formula is defined:

Definition 16. THE TRANSLATION OF A STRUCTURE AND A CONSECUTION
If X is a structure then translation $transl(X)$ of X is a formula given recursively as follows.

- $transl(A) = A$
- $transl(0) = t$
- $transl(X; Y) = transl(X) \circ transl(Y)$
- $transl(X, Y) = transl(X) \wedge transl(Y)$

The translation of a consecution $transl(X \vdash A)$ is defined to be
$transl(X) \to A$.

We also must add axioms corresponding to the structural rules.

Definition 17. For any structural rules of the form $X \Leftarrow X'$, we add the axiom

$$transl(X') \to transl(X)$$

2.6 Equivalence of Natural Deduction Systems and Hilbert Systems

Hilbert style axiomatization of the substructural logics we will consider in what follows can be defined from the natural deduction systems as is done in the book of Restall [Res00]. We will use one of these Hilbert systems in the chapter 2.10. We will denote this as OH, which means our Hilbert to distinguish it from the Restall notation $\mathcal{HG}$ for the general Hilbert system used in the following two theorems.

Theorem 3. *If there is a proof of A in $\mathcal{HG}$, then in $\mathcal{G}$ $0 \vdash A$ can be proved.*

See the THEOREM 4.16 on the page 78 of [Res00].

Theorem 4. *If $X \vdash A$ is provable in the system $\mathcal{G}$, then $transl(X \vdash A)$ is provable in $\mathcal{HG}$.*

See the THEOREM 4.21 on the page 83 of [Res00].

2.7 Frames for the Weak Excluded Middle

We show in this subsection, how the class of the frames for the law weak excluded middle looks like, i.e. which frame condition corresponds to the WEM axiom. Let us consider the compatibility relation C as defined previously.

Theorem 5. *A frame $\mathcal{F}$ satisfies condition $\forall x \forall y \forall z (xCy \wedge xCz) \rightarrow yCz$ (I) if and only if the $\neg A \vee \neg\neg A$ is valid in all nodes of the frame $\mathcal{F}$ (II).*

Proof. Let us start with $(I) \rightarrow (II)$. The proof will be done by contradiction. Let us assume that (I) is valid and (II) is not valid. If (II) is not valid, there is a node x, where neither $\neg A$ nor $\neg\neg A$ is valid. If $\neg A$ is not valid, there must be at least one node y satisfying A, where xCy. Let us denote this node y_0. We show that in all z such that xCz, $\neg A$ is not valid. We know from our condition (I) $xCy_0 \wedge xCz \rightarrow y_0Cz$ and $xCz \wedge xCy_0 \rightarrow zCy_0$. The A is satisfied in y_0, so $\neg A$ cannot be valid in z. $\neg A$ is not valid in any z reachable from x and so $\neg\neg A$ is valid in x. And this is the contradiction with the assumption, that $\neg\neg A$ is not valid in x.

We continue with $(II) \rightarrow (I)$. The proof will be done again by contradiction. Let us assume that (II) is valid and (I) is not valid. We negate (I) and we receive

$$\exists x \exists y \exists z (xCy \wedge xCz) \wedge \neg(yCz).$$

Let us denote the corresponding existing nodes x_0, y_0 and z_0. Let us evaluate y_0 with A and we let x_0 and z_0 without evaluation. We also let without evaluation all other nodes. Then $\neg A$ is not valid in the node x_0, because A is valid in y_0. Also $\neg\neg A$ is not valid in the node x_0, because $\neg A$ is valid in z_0 - in all nodes reachable from z_0 is not valid A . $\square$

We will reformulate $\forall x \forall y \forall z (xCy \wedge xCz) \rightarrow yCz$ with the help of our knowledge of the compatibility relation C. We know that this compatibility relation xCy is $\exists z Rxyz$. So we can write

$$\forall x \forall y \forall z (\exists u Rxyu \wedge \exists v Rxzv) \rightarrow \exists w Ryzw.$$

2.8 Positive Formulas and Top Models

Definition 18. Positive formula in the language of substructural logic L is a formula, which does not contain $\perp$. (It means it also does not contain the abbreviations $\neg$ and $\sim$.)

Definition 19. We say, that frame $\mathcal{F}$ with evaluation is a top model, iff there exists a node τ such that $\forall x\ x \sqsubseteq \tau$, $R\tau\tau\tau$ and all atoms are satisfied in the node τ. The frame must also satisfy that, if $R\tau yz$ or $Ry\tau z$ then $\tau = z$.

Definition 20. If we have frame $\mathcal{F}$ we define a plus frame $\mathcal{F}^+$ adding the element $\tau \in \mathcal{F}$ such that $\forall x\ x \sqsubseteq \tau$. We also expand the relation R. We add $R\tau\tau\tau$. We also take the accessibility closure, it means we add to the relation elements to assure that R remains plump accessibility relations. If we have $x \sqsubseteq \tau$ and $y \sqsubseteq \tau$, it means for arbitrary x and y, it must be $Rxy\tau$. It means that after the operation of adding τ, all points are compatible with each other. Observe, that this means that F^+ satisfies the frame condition for the weak excluded middle. For all x and y in $\mathcal{F}$ we have $Rxy\tau$. We also extend the truth set T, we add τ to T. We call the plus evaluation of $\mathcal{F}^+$ an evaluation where all atoms are satisfied in the node τ of the frame $\mathcal{F}^+$. If the original model was $\mathcal{M}$, we will name the corresponding model based on the frame $\mathcal{F}^+$ and plus evaluation $\mathcal{M}^+$.

Definition 21. A formula A has a top model property iff for all models $\mathcal{M}$ and for all nodes w $\mathcal{M}, w \models A$ iff $\mathcal{M}^+, w \models A$.

Definition 22. The consecution $X \vdash A$ has a top model property, iff the formula A and all formulas in X have the top model property.

Lemma 1. *In any top model all positive formulas are satisfied in the top node τ. In particular, this holds for any plus model $\mathcal{M}^+$.*

Proof. We will proceed by induction on the length of a formula. Atoms are satisfied in τ by the definition. We suppose A and B are satisfied in the node τ. Then the following formulas are satisfied in the node τ.

1. $A \vee B$

2. $A \wedge B$

3. $A \circ B$

4. $A \rightarrow B$

5. $A \leftarrow B$

See the or case 1. $A \vee B$ is valid from the definition of valuation. Similarly the case 2. Regarding 3 we have to use $R\tau\tau\tau$. Because A is satisfied in τ and B is satisfied in τ, $A \circ B$ is satisfied in τ. Let us have a look to 4. To show that $\tau \models A \rightarrow B$ we take y and z such that $R\tau yz$. From our definition of top models it must be that $z = \tau$. But τ satisfies B as desired. The case 5 is analogous to the case 4. □

Theorem 6. *The appropriate classes of frames (described in section 2.4) are closed under the plus construction.*

Proof. Let us denote the original frame F and the corresponding plus frame F^+.

1. Commutativity, we must show, that on the plus frame it is valid that if $Rxyz$ than $Ryxz$. If no x, y and z is our top element τ, we know this from the commutativity of the frame F. If one of the x, y or z is τ, we are ready, because the z is also τ, from our construction of R on the plus frame, so $Ryxz$.

2. Associativity, we show if $R(xy)zw$ then $Rx(yz)w$, if all x, y, z and w are from F, we know it from the associativity from the frame F. If one of the x, y, z and w is τ, then w is τ and the result is valid trivially.

3. Weakening, we have, that from $Rxyz$ follows $x \sqsubseteq z$. If no x, y and z is τ, we know this from this property of the frame F. If one of the x, y and z is τ, then also z is τ, and it is valid $x \sqsubseteq z$.

4. Contraction, if we have $Rxxx$ and x is from F, we know it from the assumption, that F is contractive. We know from our F^+ construction, that $R\tau\tau\tau$ is satisfied.

□

2.9 Constructing the Positive Part of the Formula

Theorem 7. *For any formula A, there exists the formula A^+ which is positive or it is identically equal to $\perp$, such that for any top model $\mathcal{M}$ and its node w we have $w \models A$ iff $w \models A^+$.*

Proof. We will now construct for every formula A the formula A^+, which does not contain $\perp$ or it is identically equal to $\perp$. This plus formula A^+ is valid in the plus evaluation of the framework if and only if the original formula A is valid in the plus evaluation of the framework, i.e., A and A^+ are equivalent over all top models.

We will create the plus formula in several steps. We remove $\perp$ according to the following table, where $\equiv$ denotes logical equivalence.

Remove $\perp$
$\perp \wedge\ A \equiv A \wedge\ \perp \equiv \perp$
$\perp \vee\ A \equiv A \vee\ \perp \equiv A$
$\perp \circ\ A \equiv A \circ\ \perp \equiv \perp$
$\perp \rightarrow\ A \equiv \top$
$A \rightarrow \perp \equiv \neg A$
$A \leftarrow \perp \equiv \top$
$\perp \leftarrow A \equiv \sim A$
$\neg\perp \equiv \top$
$\sim \perp \equiv \top$

We prove these equivalences, they are not obvious.

Let us prove $\perp \circ\ A \rightarrow \perp$. This is clear because $\perp$ is valid nowhere. The opposite implication is also valid from the same reason.

We prove now $(\perp \rightarrow\ A) \rightarrow \top$. The $\perp$ is valid nowhere so $\perp \rightarrow\ A$ is valid everywhere. We will prove $\top \rightarrow (\perp \rightarrow\ A)$. It is also valid, because $\perp \rightarrow\ A$ is valid everywhere.

$A \leftarrow \perp \equiv \top$ is analogous to the $\perp \rightarrow\ A \equiv \top$

$A \rightarrow \perp \equiv \neg A$ is our definition of negation.

We prove now $\neg\perp \equiv \top$. $\neg\perp$ is by definition $\perp \rightarrow \perp$. This is satisfied everywhere.

We will make a modification of the similar construction of A^+ in [dJZ13]. The construction proceeds in even and odd steps as follows. We eliminate $\perp$ in the even stage, using the equivalences in the table. In contrast to [dJZ13], it is enough to take only the column Remove $\perp$ of our table. We will repeatedly apply the rules from this column and we receive the formula without $\perp$ or identically equals $\perp$. If we receive $\perp$ we finish. If not, we make an odd step. If we have formula without $\perp$ we search a subformula $\neg A$ where A is a positive formula. We substitute $\neg A$ with $\perp$. We make then even step again, if needed followed by the odd step. The described procedure will come to an end, since all steps reduce the number of symbols in the formula.

Each step preserves the validity of formula in the top models. We have already shown, that the even step preserves the validity, it was shown without the assumption of the top model. The odd steps preserve the validity as well, we will need to assume we have the top model for this case. Let us denote the negated positive formula A. This A is a positive formula, so it is valid in τ. We want to show that $A \to \perp$ is never valid, it means it is equivalent with $\perp$. If I had a node x where $A \to \perp$ we know, that $R\tau x\tau$. But in τ is valid A and not valid $\perp$. This is a contradiction. □

Theorem 8. *A formula A in substructural logic L has a top model property iff it is equivalent to a $\perp \neg$ free formula (in fact A^+) or $\perp$.*

Proof. The direction from the right to the left is just lemma 1. Let us prove the other direction. Assume, that A has the top model property, but it is not equivalent to A^+. Then there is a model $\mathcal{M}$ with a world w so that A and A^+ have different truth values in $\mathcal{M}$ node w. Then, because both have the top-model property, they have different truth values in $\mathcal{M}^+$ w as well. But that contradicts the fact given by the previous theorem that A and A^+ behave identically on top models. □

Theorem 9. 1. *If $A(B_1, B_2, ...B_k)$ arises from the simultaneous substitution of $B_1, B_2, ...B_k$ for $p_1, p_2, ...p_k$ in $A(p_1, p_2, ...p_k)$, then*

$$(A(B_1, B_2, ...B_k))^+ = (A(B_1^+, B_2^+, ...B_k^+))^+$$

2. $(A \to B)^+ = A^+ \to B^+$, *if B^+ is not $\perp$*

$(B \leftarrow A)^+ = B^+ \leftarrow A^+$, *if B^+ is not $\perp$*

$(A_1 \wedge A_2 ... \wedge A_k)^+ = A_1^+ \wedge A_2^+ ... \wedge A_k^+$

$(A_1 \vee A_2 ... \vee A_k)^+ = A_1^+ \vee A_2^+ ... \vee A_k^+$

$(A_1 \circ A_2 ... \circ A_k)^+ = A_1^+ \circ A_2^+ ... \circ A_k^+$

Proof. 1. By the fact that the construction of the +-formula in Theorem 7 is inside out. We can construct $(A(B_1, B_2, ...B_k))^+$ by first applying the +-operation to the formulas $B_1, B_2, ...B_k$ in $A(B_1, B_2, ...B_k)$ to obtain $A(B_1^+, B_2^+, ...B_k^+)$ and then continue to work on the remainder to obtain $(A(B_1^+, B_2^+, ...B_k^+))^+$.

2. We will use the previous point in our proof.

   Let us start with the first implication. There are only two possibilities, how the plus construction can finish. It can finish as positive formula or $\perp$. So we have 2 possibilities, because of the condition that B^+ is not $\perp$. If A^+ is $\perp$, we have $\top$ on the left and right of our equality. If A^+ is positive, we also have the same positive formula on both sides of our equality.

   We can use the same argument for the converse implication.

   Regarding conjunction, we also use the fact, that A^+ is a positive formula or $\perp$. If one of the A_i is $\perp$, then we have $\perp$ on both sides of our equality. If none of them is $\perp$ we have the same positive formula on both sides.

   Regarding disjunction if one of the ${A_i}^+$ is positive we have the same positive formula on both sides. If all ${A_i}^+$ are $\perp$ we have $\perp$ on both sides of our equality.

   Regarding fusion if one of the A_i is $\perp$, then we have $\perp$ on both sides of our equality. If none of them is $\perp$ we have the same positive formula on both sides.

□

Remark 1. *There is an error in the Theorem 7 in [dJZ13]. The error is in the point 5 which states that $(\phi \to \psi)^+ = \phi^+ \to \psi^+$. But it is not true. If we take ϕ as an atom p and as ψ we take $\perp$, then we have $p \to \perp$ on the left side, which translates to $\perp$. But there is $p^+ \to \perp^+$ on the right side. But $\perp$ translates to $\perp$. So we have $p \to \perp$. The left side does not equal to right side.*

Let us see several examples of the plus construction.

Let us calculate $(p \wedge (q \vee \neg p))^+$. The p translates to p in the first step as well as q. The $\neg p$ translates to $\bot$. So we have, after first step $p \wedge (q \vee \bot)$. We continue with the translation and we get $p \vee q$. This is a positive formula.

Let us calculate $p \to (\neg q \to p)$. So we get in the first step $p \to (\bot \to p)$. We receive in the second step $p \to \top$. This is the searched positive formula.

The last example will be useful for our main theorem. We translate the WEM axiom $\neg A \vee \neg\neg A$. There are two possibilities A^+ is positive or A^+ is $\bot$. If A^+ is positive we have the translation $\bot \vee \neg\bot$ in the first step. After we have $\bot \vee \top$ which translates to $\top$. If A^+ is $\bot$, we have the translation $\top \vee \neg\top$, after we get $\top \vee \bot$, which gives $\top$.

Definition 23. We define X^+ for structure X as the structure, which is created by substituting all formulas A by A^+.

Definition 24. Positive consecution is a consecution with any formula in the consecution positive. We turn the consecution into the positive consecution turning each formula in the consecution into the positive formula.

Theorem 10. *1. If $X \vdash A$ then $X^+ \vdash A^+$. (This means provable in the natural deduction.)*

Proof. We are using the completeness here. Let X^+ is satisfied in some $\mathcal{F}$, then X^+ is satisfied in $\mathcal{F}^+$, because it has a top model property. But then also X is satisfied in $\mathcal{F}^+$. We know then A is satisfied in $\mathcal{F}^+$. So also A^+ is satisfied in $\mathcal{F}^+$. So A^+ is satisfied in $\mathcal{F}$, because it has a top model property. □

We formulate now our main theorem.

2.10 Hilbert System for dFL with Suffixing

In what follows we will consider the Hilbert type systems for substructural logics as described in the book of Restall [Res00]. In particular, we will define a Hilbert style axiomatization of the distributive Full Lambek with suffixing and some of its extension with structural axioms. The Hilbert type systems are fully equivalent with the systems of natural deduction as shown in [Res00] and recalled previously in Theorems 3 and 4. We will denote $\vdash_H$ the Hilbert deducibility. To proof the main theorem, we will also need to prove the deduction theorem as stated in the book of Restall [Res00].

We will specify the semantics for Hilbert deducibility.

We will denote the following Hilbert style system OH as the abbreviation of the words our Hilbert. We will use in the following text the H as the abbreviation for OH, when no confusion can arise.

Definition 25. Let us define the Hilbert system OH as the system, which satisfies the following rules and axioms:

The rules are the modus ponens and the conjunction rule:

(R1) $A, A \to B \Longrightarrow B$

(R2) $A, B \Longrightarrow A \wedge B$

The axioms are

(A1) $A \to A$

(A2) $(A \to B) \to ((C \to A) \to (C \to B))$

(A3) $((B \leftarrow C) \leftarrow (A \leftarrow C)) \leftarrow (B \leftarrow A)$

(A4) $(A \to B) \to ((B \to C) \to (A \to C))$

(A5) $(A \circ B \to C) \leftrightarrow (A \to (B \to C))$

(A6) $A \leftrightarrow (t \to A)$

(A7) $(A \wedge B) \to A$

(A8) $(A \wedge B) \to B$

(A9) $(A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$

(A10) $A \to A \vee B$

(A11) $B \to A \vee B$

(A12) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(A13) $A \to \top$

(A14) $\bot \to A$

(A15) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

This is an axiomatization of the distributive, associative Full Lambek calculus with suffixing (the axiom A4, which corresponds to twisted associativity). We consider this extension of the Full Lambek Logic because it can be axiomatized using only R1 and R2 as rules, and therefore it has a nice deduction theorem, which will be crucial to prove our main result.

Definition 26. CONFUSION

A *confusion* of the propositions in the set Σ is defined inductively as follows:

$\top$, t and any element of Σ are confusions of Σ

If C_1 and C_2 are confusions of Σ, so are $C_1 \circ C_2$ and $C_1 \wedge C_2$.

Theorem 11. *HILBERT DEDUCTION THEOREM*
In any Hilbert system in which the only rules are modus ponens and conjunction introduction, $\Sigma \cup A \vdash_H B$ if and only if $\Sigma \vdash_H C \to B$ where C is some confusion of $\{A\}$.

We prove at first a useful lemma about confusions.

Lemma 2. *In the Hilbert system defined above, $\Sigma \vdash_H B$, whenever B is a confusion of Σ.*

Proof. We will make the proof by induction on the complexity of B. For the base case we have $\Sigma \vdash_H \top$ and $\Sigma \vdash_H C$, for any $C \in \Sigma$.

For the induction step, if we take $\Sigma \vdash_H C_1$ and $\Sigma \vdash_H C_2$ then we have $\Sigma \vdash_H C_1 \wedge C_2$ by the conjunction rule. We also know $\vdash_H C_1 \to (C_2 \to C_1 \circ C_2)$. We apply modus ponens twice. And we get $\Sigma \vdash_H C_1 \circ C_2$. □

Proof. Let us prove now the Hilbert Deduction Theorem. Let us assume $\Sigma \vdash_H C \to B$. We know from the lemma $\Sigma \cup A \vdash_H C$, so $\Sigma \cup A \vdash_H B$ by modus ponens.

We prove now the converse implication. We assume that $\Sigma \cup A \vdash_H B$. Assume that the proof is of length m and the result holds for proofs of smaller length. Now B is either in $\Sigma \cup \{A\}$ or is axiom. In this case we have $\Sigma \vdash_H \tau \wedge A$.
We will examine the two rules. Let us start with the modus ponens. So we have confusions C_1 and C_2, where $\Sigma \vdash_H C_1 \to (B' \to B)$ and $\Sigma \vdash_H C_2 \to B'$. We know that $C_1 \to (B' \to B), C_2 \to B' \vdash_H C_1 \circ C_2 \to B$. It gives us $\Sigma \vdash_H C_1 \circ C_2 \to B$ and $C_1 \circ C_2$ is the desired confusion.
Let us take now the conjunction introduction rule. We have confusions C_1

and C_2, where $\Sigma \vdash_H C_1 \to B_1$ and $\Sigma \vdash_H C_2 \to B_2$, where $B = B_1 \wedge B_2$. So $C_1 \wedge C_2$ is the desired confusion, because $\Sigma \vdash_H C_1 \wedge C_2 \to B$. $\square$

Definition 27. We say, that A is valid iff for all models $\mathcal{M}$ and for all nodes w which are elements of the Truth set, it holds that $\mathcal{M}, w \models A$. We will write in this case $\models A$.

Theorem 12. *Soundness and Completeness for Hilbert Semantics*
$\models A$ *iff* $\vdash_H A$

Proof. We will prove first soundness. If we have $\vdash_H A$ we have $0 \vdash A$, see the THEOREM 4.16 on the page 78 of [Res00]. So it means that A is valid everywhere in our models, everywhere, where 0 is valid, which is the truth set.

Let us go to the completeness. Let us assume it is not the case that $\vdash_H A$. Then we know from [Res00] that it is not the case that $0 \vdash A$. Let us take the theory $\{B, 0 \vdash B\}$. A is not in this theory. So we can extend $\{B, 0 \vdash B\}$ to the prime theory which does not include A. So we have the model of all prime theories $\mathcal{M}$ and the node $\{B, 0 \vdash B\}$, where A is not satisfied. $\square$

Theorem 13. *If* $\vdash_H A$ *then* $\vdash_H A^+$.

Proof. We use our Soundness and Completeness theorem. Assume, that $\vdash_H A^+$ is not valid. Then also $\models_H A^+$ is not satisfied. It means we have model $\mathcal{M}$ and the node w, where A^+ is not satisfied. Because A^+ is positive it is not satisfied in $\mathcal{M}^+$. So A is not satisfied in $\mathcal{M}^+$. It contradicts the fact, that $\models_H A$ and $\vdash_H A$. $\square$

Theorem 14. *If* $\Sigma \vdash_H D$ *then* $\Sigma^+ \vdash_H D^+$*, provided* D^+ *is not* $\perp$.

Proof. Let us consider that Σ is the set $\{A_1, A_2, ..., A_n\}$ We can take gradually $A_1, A_2, ..., A_n$ and we repeat the Hilbert deduction theorem. We get

$\vdash_H C_1 \to (C_2 \to (... \to (C_n \to D))...)$, where $C_1, C_2, ..., C_n$ are confusions of $A_1, A_2, ..., A_n$

We use the previous theorem, so we know:

$\vdash_H (C_1 \to (C_2 \to (... \to (C_n \to D))...))^+$

From this and from the theorem 9, we know $\vdash_H C_1^+ \to (C_2^+ \to (... \to (C_n^+ \to D^+))...)$. This is because when we have implication, which right side does not translate to false, then all the implication does not translate to false (easy to see from examining the cases). From the plus construction we know if C is the confusion of A, then C^+ is the confusion A^+. Now we use the Hilbert deduction theorem in the opposite direction repeatedly. So we get $\{A_1^+, A_2^+, ..., A_n^+\} \vdash_H D^+$, which is $\Sigma^+ \vdash_H D^+$. $\square$

2.11 Main Theorem

Theorem 15. *The Hilbert system OH for substructural logic dFL with suffixing with the additional axiom of weak excluded middle has the same positive fragment as without it. It means if B is a positive formula we have $\vdash_H B$ if and only if $\{A_1, A_2, ..., A_n\} \vdash_H B$, where $\{A_1, A_2, ..., A_n\}$ are instances of the axiom of the weak excluded middle.*

Proof. We will prove the nontrivial implication. We will use the previous theorem. As B we take any positive formula and we assume that $A_1, ..., A_n \vdash_H B$. Then $B = B^+$. From previous theorem we have $\{A_1^+, A_2^+, ..., A_n^+\} \vdash_H B^+$. We just take all instances A_1, A_2, ... , A_n of the axiom of weak excluded middle $\neg D_i \vee \neg\neg D_i$, used in a proof of B. If we take A_i^+ we receive $\bot \vee \top$ which is equivalent to $\top$ (in the case D_i^+ is positive) or $\top \vee \bot$ which is equivalent to $\top$ (in the case D_i^+ is $\bot$) . B is then provable only from $\top$ and therefore provable. $\square$

3 Conclusion

3.1 The logics, where our result is valid

In the proof of the main result we substantially used the deduction theorem. For it to remain valid, we are in general only allowed to add axioms (and not rules). The basic calculus contains the axioms corresponding to B(associativity), B'(twisted associativity) and B^c(conversed associativity). We can use only modus ponens rule and conjunction introduction rule. The other rules cannot be added in general if we want the deduction theorem to remain valid, it means we have to substitute them by axioms. The axioms we consider are those that correspond to some of the structural rules.

We know that the result remains valid for axiomatic extensions of the basic calculus from Definition 25 with axioms of commutativity, weakening or contraction. In particular, the result holds for the following logics:

1. commutative Full Lambek $dFLe$, axiom

   $(A \to (B \to C)) \to (B \to (A \to C))$,

2. commutative Full Lambek with weakening $dFLew$ axiom from previous item and

   $A \to (B \to A)$,

3. intuitionistic logic $dFLewc$ axioms from previous items and

   $(A \to (A \to B)) \to (A \to B)$,

4. relevant logic, commutative with contraction, axioms

   $(A \to (B \to C)) \to (B \to (A \to C))$,

   $(A \to (A \to B)) \to (A \to B)$.

Let us see that our proof remains valid if we add (some of) the axioms mentioned above:

The soundness and completeness theorem, with respect to the corresponding classes of frames, is valid, if we add the corresponding condition of the relevant structural rule on the frame. It is the chapter 11 of [Res00]. We have seen in Theorem 6 that the corresponding classes of frames are closed under the plus construction.

The deduction theorem remains valid, if we add additional axioms and we do not add any rules. In fact, the deduction theorem even simplifies: if weakening is present, the notion of confusion simplifies (it is enough to consider fusions of finitely many copies of the formula). If moreover contraction is present (and we are therefore in IL) we obtain the usual deduction theorem.

3.2 Further work

We conjecture that the method used in this thesis can be suitably generalized and applied to non-distributive substructural logics.

The proof could be given also by the algebraic semantics as was done by Galatos in [Gal11].

It remains to be seen how the approach based on the frame semantics relates to the algebraic approach using the duality theory. Namely, how the plus construction on frames relates to the ordinal sum construction used in [Gal11].

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