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Fractality of Stock Markets
A Comparative Study

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Prohlášení

Prohlašuji, že jsem rigorózní práci vypracoval samostatně a použil pouze uvedené prameny a literaturu.

Declaration

Hereby I declare that I compiled this rigorosis thesis independently, using only the listed literature and resources.

Prague, 14th May 2009

Ladislav Krištoufek

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Abstract

The main focus of the thesis is the introduction of new method for interpretation of fractality aspects of financial time series together with its application. We begin with description of various techniques of estimation of Hurst exponent – rescaled range, modified rescaled range and detrended fluctuation analysis. Further on, we present original theoretical results based on simulations of three mentioned procedures which have not been presented in literature yet. The results are then used in the new method of time-dependent Hurst exponent with confidence intervals developed in this thesis. Moreover, we show important advantage of using the mentioned techniques together to clearly distinguish between independent, trending, short-term dependent and long-term dependent properties of the time series. We eventually apply the proposed procedure on 13 different world stock indices and come to interesting results. To the author's best knowledge, the thesis presents the broadest application of time-dependent Hurst exponent on stock indices yet.

Keywords: fractality, time-dependent Hurst exponent, long-term memory, time series analysis, market efficiency

JEL Classification: G1, G10, G14, G15

Abstrakt

Tato práce se zaměřuje na prezentaci nové metody pro popis fraktality finančních časových řad. Popisujeme nejvíce používané techniky pro určení Hurstova exponentu – R/S, M-R/S a DFA. Dále prezentujeme vlastní simulace pro dané metody, které nebyly dříve uvedeny v literatuře. Výsledky jsou pak použity v nové metodě časově závislého Hurstova exponentu s konfidenčními intervaly. Navíc poukazujeme na výhody použití všech třech postupů najednou pro důsledné rozlišení mezi nezávislými procesy, procesy s trendy, procesy s krátkou pamětí a procesy s dlouhou pamětí. Nakonec aplikujeme navrženou metodu na 13 různých světových akciových indexů a přicházíme k zajímavým výsledkům. Podle autorova nejlepšího vědomí práce prezentuje zatím nejširší aplikaci časově závislého Hurstova exponentu na akciové indexy.

Klíčová slova: fraktalita, časově závislý Hurstův exponent, procesy s dlouhou pamětí, analýza časových řad, tržní efektivita

JEL klasifikace: G1, G10, G14, G15

Contents

List of abbreviations

AL76	Anis & Lloyd's (1976) rescaled range estimation method
ATX	Austrian Traded Index of Wiener Börse, Austria
BUX	Budapest Index of Budapest Stock Exchange, Hungary
CAC40	French Stock Market Index of Paris Bourse, France
CMH	Coherent markets hypothesis
DAX	Deutscher Aktien Index of Frankfurt Stock Exchange, Germany
DFA	Detrended fluctuation analysis
DFA-0	Detrended fluctuation analysis with constant trend filtering
DFA-1	Detrended fluctuation analysis with linear trend filtering
DFA-2	Detrended fluctuation analysis with quadratic trend filtering
DJI30	Dow Jones Industrial Average of NYSE, USA
DMA	Detrending moving average
EMH	Efficient markets hypothesis
F65	Market efficiency according to Fama (1965a)
FMH	Fractal markets hypothesis
FTSE	FTSE 100 Index of London Stock Exchange, UK
GBM	Geometric Brownian motion
GHE	Generalized Hurst exponent
IID	Independent and identically distributed process
LKZ	Larrain's KZ model
M-R/S	Modified rescaled range analysis
NASDAQ	NASDAQ Composite Index of NASDAQ, USA; and National Association of Securities Dealers Automated Quotations, New York City, New York, USA
NIKKEI	NIKKEI 225 of Tokyo Stock Exchange, Japan
NYSE	New York Stock Exchange, New York City, New York, USA
P94	Peters' (1994) rescaled range estimation
P94c	Corrected Peters' (1994) rescaled range estimation
PX	PX Index of Prague Stock Exchange (formerly PX50), Czech Republic
R/S	Rescaled range analysis
RW	Random walk
S65	Market efficiency according to Samuelson (1965)
SAX	Slovak Stock Index of Bratislava Stock Exchange, Slovak Republic
SD	Standard deviation
SP500	Standard & Poor's 500 of NASDAQ and NYSE, USA
SSEC	SSE Composite Index of Shanghai Stock Exchange, China
TSH	Time and scale dependent Hurst exponent
WIG20	Warsaw Index of Warsaw Stock Exchange, Poland

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❖ Introduction

Mainstream financial literature has been based on the assumption of normality of returns (Osborne, 1964). Even though this property has been questionable from the very beginning of the theory development (Mandelbrot, 1960; and Mandelbrot, 1963b), the efficient markets hypothesis won its first important battle in 1960s and defeated the theories based on different distributions mainly because of its easy application in the models (Markowitz, 1952; Sharpe, 1964; and Lintner, 1965). However, the years have passed and the hypothesis has been tackled on many fronts (Malkiel, 2003; Lo, 2008). Despite the fact that the efficient market hypothesis has survived, several competing theories have evolved and are ready to overtake the place of the mainstream paradigm.

One of the defeated theories from the 1960s was the theory of Benoît Mandelbrot which described the financial market as a system with fat tails, stable distributions and persistence (Mandelbrot, 1960; Mandelbrot, 1963a; Mandelbrot, 1963b; Mandelbrot, 1966; and Mandelbrot, 1967). The theory of Mandelbrot has evolved since 1960s but has retained the most important assumptions and was summarized by Edgar Peters as the fractal market hypothesis in early 1990s (Peters, 1994). Financial market is depicted as a complex dynamic system which can hardly be described by linear methods of efficient market theories. The complexity of the system consists of heterogeneous agents on the market who are not fully rational; moreover, they apply available information differently, invest at different investment horizons and react gradually to the information. The theory implicitly says that the dynamics of the system is fractal and therefore self-similar. Stock markets are stable as long as the returns of different investment horizons are self-similar in their distributions and thus there is no preferred investment horizon. When the market does not keep the self-similarity, it can easily break down (Peters, 1994).

For the fractality measurement, Hurst exponent is used in literature (Samorodnitsky, 2007) with respect to Edwin Hurst who developed basic tools for detecting long-term memory in the time series of water flows of the Nile River in 1950s (Hurst, 1951). As long-term memory is one of “symptoms” of fractality of the time series (Rose, 1996), the method was in turn applied in financial theory. Even though the long-term memory processes were out of the mainstream financial theory (Lo, 2008), they have gone through interesting development which was mostly recognized during 1990s and 2000s when lot of authors (e.g. Lo, 1991; Peters, 1994; Taqqu, Teverovky & Willinger, 1995; and Lillo & Farmer, 2004) applied the methods and found several interesting results of long-term memory processes in stock prices, FX returns and bond returns. In the last 15 years, there have been a lot of research papers which contradict the classic financial theory as a presence of long-term memory in the time series rejects a random walk hypothesis which is an important part of efficient markets theory (e.g. Peters, 1994; Matos *et al.*, 2008; Di Matteo, Aste & Dacorogna, 2005; and Los, 2008). However, as the theory is relatively new and the methods are not fully developed, majority of the papers solve the problems only partially.

The main purpose of this thesis is to compare the results of different authors and develop new method which avoids the shortcomings of the recent papers. The method uses time-dependent Hurst exponent, which was applied by several researchers recently (e.g. Grech & Mazur, 2004; Grech & Mazur, 2005; and Matos *et al.*, 2008), together with confidence intervals based on our original simulations for random time series. With this method, we are able to test the hypothesis that the market behaves independently (Weron, 2002). The alternative hypothesis states that market is dependent. For the purposes of detection of specific type of dependence, we use different methods of estimation of Hurst exponent. Rescaled range analysis (Hurst, 1951) is used as the first one as it provides the basic detection tool for dependence in the time series. Modified rescaled range (Lo, 1991) is used to distinguish between long-term and short-term dependence. The separation of two types of dependence is important as their implications are different. Short-term dependence implies that information about recent past of the time series is significant for the current state (Rose, 1996). On the other hand, long-term dependence indicates that even the information of far past is significant for the present state (Beran, 1994). Detrended fluctuation analysis (Peng *et al.*, 1993; and Peng *et al.*, 1994) is finally applied to check for significant trends in the time series which may bias both previously presented methods (Alvarez-Ramirez *et al.*, 2008). Therefore, we are able to estimate whether the time series is independent, short-term dependent, long-term dependent or only trending.

The first part of the thesis presents the definitions of self-similarity and fractality together with the implications for the examination of the time series. Efficient markets hypothesis (Fama, 1970) and fractal markets hypothesis (Peters, 1994) together with preceding theories of Larrain's KZ model (Larrain, 1991) and coherent markets hypothesis (Vaga, 1990) are presented. The description of the theories focuses on the most important aspects which are comparable with fractal markets hypothesis and with the application of Hurst exponent on the financial time series.

The second part describes the commonly used methods of estimation of Hurst exponent. Rescaled range analysis of Edwin Hurst (Hurst, 1951) and modified rescaled range analysis of Andrew Lo (Lo, 1991) are presented as these methods are the most applied and tested ones. Detrended fluctuation analysis of Peng (Peng *et al.,* 1994) is presented as it is the most used method from the range of methods which are resistant to non-stationarities in the time series (Alvarez-Ramirez *et al.*, 2008).

The third part of the thesis focuses on finite sample properties of all used methods as they are theoretically developed for infinite time series. The real world time series are finite and therefore, the implications of the methods must be tested. We simulate random time series for different lengths and test the properties of estimated Hurst exponents for each method. This way, we construct confidence intervals for each method which are further used for testing of hypothesis of random walk and martingale.

The fourth part applies all presented aspects of long-term memory processes on wide portfolio of world stock indices. The method we use is novel in the fact that we compare time-dependent Hurst exponent with estimated confidence intervals for specific time series length and thus test whether the time series follows martingale, random walk or is dependent. The important part of the method is the fact that not only we test the dependence of the time series but we also examine the changes of dynamics of the system as stock markets develop through time.

The last part concludes the most important results uncovered. We show that confidence intervals for time series with less than thousand observations are wide and thus the hypothesis of independence is rejected only for extreme values of Hurst exponent. Further on, R/S overestimates Hurst exponent compared to DFA methods. However, the confidence intervals for both methods have similar width. In the applied part, we show that indices of the Central Europe experienced very different evolution. WIG20 and BUX show no periods where long-term dependence is significantly present. On the other hand, ATX and PX show clear trend from a long-term dependent to an independent behavior. SAX shows very different behavior compared to all indices as it experienced rather reverse evolution when compared to ATX and PX. As for the Western Europe, DAX, CAC40 and FTSE show very similar behavior with almost two years on boundary between independent and anti-persistent behavior for FTSE and CAC40. The indices of the USA show no significant long-term dependence during whole examined period. NIKKEI is shown to be similar to the indices of Western Europe with all the attributes and SSEC shows stable path to an independent market.

Chapter 1 Fractals in finance

"When the number of fixes in recipes exceeds a certain number, that recipe collapses under its own weight and the need arises for a new start."

Benoît Mandelbrot[1]

In this chapter, we present the basic definitions of fractal and self-similar processes. As the definitions vary across the literature (compare Samorodnitsky & Taqqu, 1994; and Di Matteo, 2007), we provide those best suitable to the time series analysis. We follow the definitions with descriptions of efficient markets hypothesis and fractal markets hypothesis together with preceding models which mostly contributed to formation of fractal markets hypothesis. Let us start with definitions of self-similar and fractal processes.

1.1 Definitions of fractality and self-similarity

Fractality and self-similarity are two concepts which are confused in majority of the literature. However, they are not equal – self-similarity is a special case of fractality (Mandelbrot, Fisher & Calvet, 1997). We start with the definition of a self-similar process as it is widely used in literature (Samorodnitsky & Taqqu, 1994).

The simplest way of defining self-similar process is based on distributions (Samorodnitsky, 2006):

Definition 1-1 Self-similarity in distribution
Process $X=\left(X_t,-\infty<t<\infty\right)$ *is called self-similar if*

$$X(at)\rightarrow a^H X(t) \tag{1.1}$$

*for a positive factor **a** and non-negative self-similarity parameter **H**.*

To avoid confusion with self-affinity, we present the definitions of self-affinity of Mandelbrot & van Ness (1968) and Mandelbrot, Fisher & Calvet (1997), respectively:

Definition 1-2 Self-affinity of increments in distribution
The increments of process $X=\left(X_t,-\infty<t<\infty\right)$ *are said to be self-affine if*

$$X(t_0+\tau)-X(t_0)\rightarrow h^{-H}\left(X(t_0+h\tau)-X(t_o)\right) \tag{1.2}$$

for any $\boldsymbol{t_0}$*, a positive factor **h**, a positive time scale* $\boldsymbol{\tau}$ *and a non-negative parameter **H**.*

[1] Mandelbrot (2005), p. 195

Definition 1-3 Self-affinity in distribution

Process $X = (X_t, -\infty < t < \infty)$, $X(0) = 0$ *is called self-affine if*

$$\{X(ct_1), \dots, X(ct_k)\} \to \{c^H X(t_1), \dots, c^H X(t_k)\} \tag{1.3}$$

for non-negative factors ***c*** *and* ***k****, periods* $\boldsymbol{t_1, \dots, t_k}$ *and a positive parameter* ***H*** .

Self-affinity in distribution is thus a special case of self-similarity in distribution and should not be confused as these are often interchanged in literature (for comparison, see Mandelbrot, Fisher & Calvet, 1997; and Samorodnitsky & Taqqu, 1994). Let us return to self-similarity.

Self-similarity parameter *H* is called Hurst exponent after water engineer Harold Edwin Hurst who developed it for examining the behavior of the Nile River water flows to build appropriate reservoir that would never overflow and never become empty (Hurst, 1951). Notation *H* was given to the exponent by Benoît Mandelbrot (Mandelbrot & van Ness, 1968) who contributed mostly to self-similarity and fractals in not only physics and finance in pioneering years of the fractal theory (e.g. Mandelbrot & Wallis, 1969; Mandelbrot, 1970; and Mandelbrot, 1972).

However, more important implications of self-similarity for the time series are not based on distributions but on dynamic properties of the time series which are most basically defined by the autocorrelation function $\gamma(k)$ which we define as follows (Eichner *et al.*, 2007):

Definition 1-4 Autocorrelation function

Let $X = (X_t, t = 0,1,\dots,T)$ *be a covariance stationary stochastic process with mean* μ *and variance* σ^2*. Autocorrelation function* $\gamma(k), k \geq 0$ *dependent on number of lags* ***k*** *is then*

$$\gamma(k) = \frac{\sum_{t=0}^{T-k} (X_t - \mu)(X_{t+k} - \mu)}{\sigma^2 (T-k)}. \tag{1.4}$$

Self-similar processes according to Definition 1-1 have autocorrelation function which is defined exactly and asymptotically by following propositions based on Beran (1994) and Embrechts & Maejima (2002), respectively, who provide proofs as well:

Proposition 1-1 Autocorrelation function of self-similar process (1)

Let $X = (X_t, t = 0,1,\dots,T)$ *be self-similar process with* $0 < H < 1$ *and finite variance* $\sigma^2 < \infty$*. Then the correlations are given by autocorrelation function*

$$\gamma(k) = \frac{(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}}{2} \tag{1.5}$$

for $k \geq 0$*, where* ***H*** *is Hurst exponent.*

Proposition 1-2 Autocorrelation function of self-similar process (2)

Let $X = (X_t, t = 0,1,...,T)$ be self-similar process with $0 < H < 1$ and finite variance $\sigma^2 < \infty$. Then the correlations are asymptotically given by autocorrelation function

$$\gamma(k) \approx H(2H-1) * k^{2H-2} \tag{1.6}$$

*for $k \to \infty$, where **H** is Hurst exponent.*

The dividing value of Hurst exponent is 0.5 and indicates two possible processes. On the basis of Proposition 1-1, H being equal to 0.5 means an independent process (Beran, 1994). On the other hand, Proposition 1-2 suggests that if H of the process is 0.5, we have either an independent process, if Proposition 1-1 holds as well, or a short-term dependent process, if Proposition 1-1 is not valid (Lillo & Farmer, 2004). Thus, independent process is the one with zero correlations at all non-zero lags. On the other hand, short-term dependent process has significantly non-zero correlations at low lags but zero correlations at high lags[2]. Let us now follow with two cases which are more important for this thesis.

If $H > 0.5$, the process has significantly positive correlations at all lags and is said to be long-range dependent with positive correlations (Embrechts & Maejima, 2002). Note that there are several different notations for such process. The mostly used terms are used by Beran (1994), who employs a notion of "long-range dependence", Lillo & Farmer (2004), who calls the process as the one with "long-memory", Panas (2001) and Mandelbrot & van Ness (1968), who say that the process is "persistent", and Peters (1994), who marks the process as "black noise". The process has hyperbolically decaying correlations which are non-summable and $\sum_{k=0}^{\infty} \gamma(k) = \infty$ (Beran, 1994).

On the other hand, if $H < 0.5$, it has similar properties to the previous case as it has significantly negative correlations at all lags and the process is said to be long-range dependent with negative correlations (Embrechts & Maejima, 2002). Again, the process is called differently across literature. Beran (1994) uses term "short-range dependence", Panas (2001) and Mendelbrot & van Ness (1968) call the process "anti-persistent", Barkoulas, Baum & Travlos (2000) mark the process as the one with "intermediate memory" and Peters (1994) labels the process as "pink noise". Similarly to the previous case, the process has hyperbolically decaying correlations. However, the correlations are summable and thus $0 < \sum_{k=0}^{\infty} \gamma(k) < \infty$ (Embrechts & Maejima, 2002).

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[2] Kantelhardt (2008) specifies the autocorrelation function of short-term dependent process as $\gamma(k) \approx \exp(-k/t_x)$ where t_x is a characteristic time decay. The author further notes that e.g. for AR(1) process $x_t = \varphi * x_{t-1} + \varepsilon_t$, it holds that $t_x = -1/\log\varphi$. Note that the relationship holds for non-explosive processes with $\varphi < 1$ and only for positively autocorrelated processes with $\varphi > 0$.

As it has been mentioned, there are several notations used in the literature and some might cause confusion. To avoid this, we stick to terms “persistent” and “anti-persistent” throughout the following text. The persistent process implies that a positive movement is statistically more likely to be followed by another positive movement or vice versa. On the other hand, anti-persistent process implies that a positive movement is more statistically probable to be followed by a negative movement and vice versa (Vandewalle, Ausloos & Boveroux, 1997).

Let us turn to the definition of a fractal process. We present the most common definition[3] of the process of Mandelbrot, Fisher & Calvet (1997) with separation between multi-fractal and uni-fractal processes of Lux (2003):

Definition 1-5 Fractality in distribution

Process $X = (X_t, -\infty < t < \infty)$ *is called fractal if*

$$X(ct) \rightarrow c^{H(c)} X(t) \tag{1.7}$$

for a positive factor **c** *and an arbitrary non-negative function* ***H(c)***. *If* $H(c) = H$ *is a constant function, the process is said to be uni-fractal. Otherwise, the process is said to be multi-fractal.*

The fractal process is called persistent, anti-persistent and independent with respect to the same values of *H(c)* as for self-similar processes mentioned in the text above (von Seggern, 1993). The multi-fractal process is appealing mainly due to its ability to describe the process in more complex way and allow the distributions to follow more complicated functions of rescaling which is closer to the real world observations (Cont, 2001).

The processes then have following relations. A self-affine process is a special case of a self-similar process which is in turn a special case of a fractal process. All types of mentioned processes are connected by parameter *H* and its properties that show peristent, anti-persitent, short-term dependent and independent processes. The different values help us distinguish between specific definitions of efficient markets and fractal markets which are both discussed in the following subchapter

Before we turn to definitions of different markets hypotheses, we define the family of stable distributions which are also called “fractal” because of their self-similar properties (Mandelbrot, 1964b):

[3] For alternative definitions, see Fillol (2003) and Di Matteo (2007).

Definition 1-6 Stable distributions
Stable distributions are determined by characteristic function, natural logarithm of which is defined as

$$\phi(t) = i\delta t - |ct|^{\alpha}\left(1 - i\beta \frac{t}{|t|}\right)\tan\frac{\pi\alpha}{2} \tag{1.8}$$

for $\alpha \neq 1$ *and*

$$\phi(t) = i\delta t - |ct|^{\alpha}\left(1 + i\beta \frac{2}{\pi}\right)\ln|t| \tag{1.9}$$

for $\alpha = 1$*, where* $0 < \alpha \leq 2$ *is a characteristic exponent determining peakedness,* $\beta \leq |1|$ *is a skewness parameter,* $-\infty < \delta < \infty$ *is a location parameter and* $0 \leq c < \infty$ *is a scale parameter.*

There are several special cases of Stable distributions with important properties. If $\alpha = 2, \beta = 0$, we arrive at normal (Gaussian) distribution with $c = 2\sigma^2$. If moreover $\delta = 0$, we have standardized normal distribution with zero mean and unit variance. If $\alpha = 1, \beta = 0$, we have Cauchy distribution which has infinite variance and mean. The parameter α is crucial for an existence of variance. For $1 < \alpha < 2$, the distribution has infinite or undefined second moment and thus population variance. Moreover, for $0 < \alpha \leq 1$, the distribution has infinite mean as well (Peters, 1994).

The self-similar process is connected to stable distribution of the process by the relation between Hurst exponent and α (Panas, 2001):

$$\alpha = 1/H \tag{1.10}$$

This relation is very important for further testing of hypotheses which we further describe in Section 1.3. Let us now turn to the process of forming of fractal market hypothesis in the following subchapter.

1.2 From efficient to fractal markets

Fractal market hypothesis (FMH) by Peters (1994) is inspired by two hypotheses – Larrain's K-Z model (Larrain, 1991) and coherent market hypothesis (Vaga, 1990). Therefore, we present basic ideas of both theories and then turn to FMH itself. Moreover, we present the crucial points of efficient markets hypothesis (EMH) as it has been the theory of the mainstream for several decades and other hypotheses are the ones to oppose EMH or complement it. Let us start with EMH.

1.2.1 Efficient markets hypothesis

Efficient markets hypothesis was simultaneously developed by Eugene Fama (Fama, 1965a; Fama, 1965b; and Fama, 1970) and Paul Samuelson (Samuelson, 1965) during 1960s. Even though each author worked EMH out on different basis, both came to same implications (Lo, 2008). As EMH is well discussed in majority of finance textbooks (e.g. Reilly & Brown, 2002), we present only the most important implications of the theory.

The hypothesis of efficient markets was firstly summed by Fama (1970) who presented theory and empirical findings. The efficient market is described as *"...a market in which ... investors can choose among securities ... under the assumption that security prices at any time 'fully reflect' all available information."*[4] Fama (1970) presented three basic models – a fair game model, a martingale model and a random walk model. We will use martingale and random walk models as thresholds for efficiency as it is proposed by Samuelson (1965) and Fama (1965a), respectively. Let us move to the definitions.

Martingale process is defined on basis of semi-martingales. We provide the definition of Los (2008) where Φ_s is an information set at time *s* and *E* is an expected value operator:

Definition 1-7 Semi-martingale and martingale processes
A random process $(X(t),\phi_t : t = 1,2,...)$ *is called a submartingale if*

$$E\{|X(t)|\} < \infty \tag{1.11}$$

and

$$E\{X(t)|\phi_s\} \geq X(s), s < t \tag{1.12}$$

and a supermartingale if, instead,

$$E\{X(t)|\phi_s\} \leq X(s), s < t \tag{1.13}$$

and is a martingale if the process is both a submartinagel and a supermartingale.

Before we present the definition of random walk, we provide definitions of a Markov process (Kijima, 1997) and an independent process (Los, 2008).

Definition 1-8 Markov process
A random process $(X(t): t = 1,2,...)$ *is called a Markov process if, for each* ***n*** *and every* $i_o,...,i_n, j \in \aleph$,

$$P\{X_{n+1} = j|X_0 = i_o,...,X_n = i_n\} = P\{X_{n+1} = j|X_n = i_n\} \tag{1.14}$$

where P{.|.} denotes conditional probability.

[4] Fama (1970), pp. 383.

Definition 1-9 Independent process

Let $\{X(t): t = 1,2,...\}$ be a sequence of random variables on a given probability space (Ω, G, P) with $E\{X(t)\} = 0$ and $\{G_t : t = 1,2,...\}$ a current of σ-algebras on the measurable space (Ω, G), where Ω is the complete universe of all possible events. Then $\{X(t)\}$ is a sequence of independent random variables with respect to $\{G(t)\}$ if X(t) is measurable with respect to G_t and is independent of G_{t-1} for all $t = 1,2,...$

Now that we have defined Markov process and independence, we can define a random walk process. We define random walk (RW) and geometric Brownian motion (GBM) according to Los (2008) as follows:

Definition 1-10 Random Walk

A Random Walk is a Markov process with independent innovations

$$X(t) - X(t-1) = \varepsilon(t), \tag{2.15}$$

where $\varepsilon(t) \approx IID$, which stands for independent and identically distributed process.

Definition 1-11 Geometric Brownian Motion

A Geometric Brownian Motion is a random walk of natural logarithm of the original process X(t), where $x(t) = \ln X(t)$, so that

$$\Delta x(t) = x(t) - x(t-1) = \varepsilon(t), \tag{2.16}$$

where $\varepsilon(t) \approx IID$

Note that martingale is more general than random walk since semi-martingales allow for dependence in the process. Random walk thus implies martingale but martingale does not imply random walk in the process. With definitions of a martingale, independence and a random walk, we can follow with division of efficient markets hypothesis. Efficient markets are divided into three basic forms based on different information sets. Weak form states that only historical prices of stocks are available for current price formation. Semi-strong form broadens the information set by all publicly available information. Strong form includes insider information into the information set (Fama, 1970). Market is then said to be weakly efficient if investors cannot reach above-average risk-adjusted returns based on historical prices and similarly for the other forms (Malkiel, 2003).

The most problematic part of EMH is the fact that it can be hardly, if at all, tested as the "full" reflection of information in prices is hard to define[5]. The hypothesis was tackled on methodological basis quite early after its publication in LeRoy (1976). The author criticized all presented definitions of EMH by Fama (1970) and argued that all of them are rather tautologies and therefore impossible to test.

[5] The problem was mentioned by Fama himself in Fama (1970).

More importantly and critically, the biggest complication of EMH is joint-hypothesis problem which is touched by Fama himself in Fama (1991). The problem is that even when the potential inefficiency of the market is uncovered, it can be due to wrongly chosen asset-pricing model. Therefore, it is impossible to reject EMH in general and if one still wants to test EMH, he or she must state under which conditions (Lo, 2008).

Therefore, we present two approaches to the definition of efficiency which are testable. One is shown by Fama (1965a) and asserts that a market is weakly efficient if it follows a random walk process (Definition 1-9). Let us call this type of efficiency F65. The other one is presented by Samuelson (1965) and says that market is efficient if it follows a martingale process (Definition 1-7). Let us call this type of efficiency S65. Note that most recent researchers stick to the martingale process condition which is more general than random walk and allows for dependence in the process (Los, 2008)[6]. When we mention that market is efficient, we do so with regard to both S65 and F65. If the market is efficient only in the sense of one type, we stress the specific type of efficiency.

Implications of EMH are far-reaching and are basis for classical financial theory. The most important for our purposes are (Elton *et al.*, 2003):

- Homogeneity of investors based on their rationality[7];
- Normal distribution of returns[8];
- Standard deviation as a measure of volatility and thus risk[9];
- Tradeoff between risk and return[10];
- Unpredictability of future returns[11].

We will return to interconnection between EMH and fractal processes in the last part of the following chapter. Let us now turn to preceding models of FMH.

1.2.2 Larrain's K-Z model

Larrain's K-Z model (LKZ) is named after its creator Maurice Larrain who published the theory and empirical results in Larrain (1991). LKZ is a model of real interest rates but it

[6] E.g. AR(1) process does not follow random walk but is a semi-martingale.
[7] If all investors are rational and have the access to the same information, they necessarily arrive at the same expectations and are therefore homogeneous.
[8] Random walk can be represented by AR(1) process in the form of $P_t = P_{t-1} + \varepsilon_t$ which simply implies that $r_t = P_t - P_{t-1} = \varepsilon_t$ where $\varepsilon_t = N(0,\sigma)$ is independent normally distributed variable.
[9] As returns are normally distributed, it implies that standard deviation is stable and finite (Mandelbrot, 1963) and thus is a good measure of volatility.
[10] As standard deviation is stable and finite, there is a relationship between risk and return based on non-satiation and risk-aversion of the investors (Markowitz, 1952).
[11] As returns follow random walk, the known information is already incorporated in the prices and thus their prediction is impossible.

is presented here for the the idea that a system can be created by two separate mechanisms – one based on past behavior and the other based on interconnection with other fundamental variables.

The behavior of future real interest rates is based on two separate relationships:

$$r_{t+1} = f(r_{t-n}) \tag{1.17}$$

$$r_{t+1} = g(Z) \tag{1.18}$$

The first equation represents the assertion that future real interest rates r_{t+1} are dependent on present and past real interest rates r_t, r_{t-1}, .., r_{t-n}. The second equation states that future real interest rates r_{t+1} are dependent on fundamental variables represented by Z. The addition was made to (1.17) and it was reformulated to

$$r_{t+1} = a - c * r_t * (1 - r_t), \tag{1.19}$$

where a is an arbitrary constant and $c > 0$ is a constant.

Finally, after putting (1.18) and (1.19) together, we obtain

$$r_{t+1} = a - c * r_t * (1 - r_t) + g(Z). \tag{1.20}$$

Larrain (1991) also empirically tested the model and came to very strong results. All fundamental variables – real GNP, nominal money supply, consumer price index, real personal income and real personal consumption – had significant coefficients as well as the one of the past real interest rate. The result has an important implication that is used in FHM – both past prices and fundamental variables are important for the system and thus fundamentalists and technicians can be part of the market and both can influence the behavior of prices. Fundamentalists base their estimates on changes of expected cash-flows. On the other hand, technicians base their trading strategy on crowd behavior, short-term effects or past behavior of stock prices. This implication strongly contradicts two assumption of EMH – homogeneity of the investors and unusefulness of past prices for future prices prediction.

For more theory behind K-Z model, see already mentioned paper of Larrain (1991) or Peters (1991b).

1.2.3 Coherent market hypothesis

Coherent market hypothesis was developed by Tonis Vaga (Vaga, 1990) and is based on the theory of coherent systems in natural sciences (Peters, 1991b). The theory is based on a system with an order parameter which sums all external forces driving the system. The whole idea is based on Ising model of ferromagnetism where molecules behave randomly (their movement is normally distributed) up till order parameter (temperature of an iron bar in the original case) reaches certain level where molecules start to cluster and behave chaotically (Niss, 2004). The theory is in turn applied to behavior of social groups and finally to behavior of investors (Schöbel & Veith, 2006).

Vaga (1990) proposes following, quite complicated, probability formula for annualized return *f(q)* and additional equations

$$f(q) = \frac{Q(q) * e^{2\int_{-1/2}^{q} \frac{K(y)}{Q(y)} dy}}{c} \tag{1.21}$$

$$K(q) = \sinh(k * q + h) - 2q * \cosh(k * q + h) \tag{1.22}$$

$$Q(q) = \frac{\cosh(k * q + h)}{n} - 2q * \sinh(k * q + h) \tag{1.23}$$

$$c^{-1} = \int_{-1/2}^{1/2} Q^{-1}(q) * e^{2\int_{-1/2}^{q} \frac{K(y)}{Q(y)} dy} dq \,. \tag{1.24}$$

Variables *n*, *k* and *h* stand for a number of degrees of freedom (market participants), a degree of crowd behavior and a fundamental bias, respectively. There are five types of markets with respect to varying parameters *k* and *h* (Schöbel & Veith, 2006):

- Efficient market ($0 \le k << k_{critical}, h = 0$) where investors act independently of one another and a random walk is present;
- Coherent market ($k \approx k_{critical}, h << 0 \vee h >> 0$) where crowd behavior is in conjunction with strong bullish or bearish fundamentals and creates coherent market where traditional risk-return tradeoff is inverted and investors can earn above-average returns while facing below-average risk;
- Chaotic market ($k \approx k_{critical}, h \approx 0$) where crowd behavior is in conjunction with only weak bearish or bullish fundamentals and creates the situation of low returns with above-average risk;
- Repelling market ($k < 0, h = 0$) where opposite of crowd behavior is present, investors try to avoid having the same opinion as the majority;
- Unstable transitions consist of all market states that cannot be assigned to any of the former states.

For our purposes, the most important implication for FMH is very similar to those of LKZ – markets can exhibit various stages of behavior by combining fundamental and sentiment influences which again contradict assumptions of EMH (homogeneity of investors, independent identically distributed returns and risk-return tradeoff). Let us now follow with fractal markets hypothesis.

1.2.4 Fractal market hypothesis

Fractal markets hypothesis was originally presented by Peters (1994). The theory was based on criticism of efficient markets hypothesis and suggested that investors were heterogeneous with different investment horizons and reacted gradually to the information. Moreover, the normality of the returns was omited and the distribution was only suggested to be stable (Definition 1.6). However, it lacked formal definitions which were presented later by Rachev, Weron & Weron (1999) and which we present now.

Definition 1-12 FMH1
The market consists of many individuals with many different investment horizons.

FMH1 reacts to the fact that market consists of different types of investors with respect to their investment horizon. The market consists of the investors with investment horizon from several minutes (noise-traders) up to several years (pension funds).

Definition 1-13 FMH2
Information has a different impact on different investment horizons.

Investors with short investment horizon focus on technical information and crowd behavior of other market participants. On the other hand, investors with long investment horizon base their decisions on fundamental information and care little about crowd behavior. FMH1 and FM2 thus emphasize the heterogeneity of investors. Not only have investors different investment horizons but the information have different effects on investors in forming of expectations which is in contrary to EMH.

Definition 1-14 FMH3
The stability of the market is predominantly a matter of liquidity. Liquidity is available if FMH1 holds.

Liquidity is brought to market by many investors with many different investment horizons. If short-term investor experiences relatively high loss, it is a buying opportunity for a long-term investor and vice versa. If market switches to the place of many investors of several, or extremely only one, investment horizons, the trading becomes stuck and unstable. Single negative information can turn market into a downward spiral.

Definition 1-15 FMH4
Prices reflect a combination of short-term trading (technicians) and long-term valuation (fundamentalists).

Two basic cases of investors – technicians and fundamentalists – evaluate the market price absolutely differently. Of course, this division into two groups is only a simplification. There are literally thousands of possible trading rules for technicians with different trading horizons. On the other hand, fundamentalists have different investment horizons and put different weights to fundamental information.

Definition 1-16 FMH5
If a security has no bond with the economic cycle, there will be no long-term trend. Trading, liquidity and short-term information will be dominant.

The last assertion is an implication of FMH4 as if there is no bond of the stock to the fundamentals, there will be no fundamentalists and technicians will dominate. This kind of market is bound to be very volatile. However, if investors with many different investment horizons remain in the market, it will remain liquid and stable.

The main aim of FMH is to fit the real market. EMH works well if markets are stable and close to equilibrium. However, if market is close to or in turbulence, the models cease to work. One of important generalizations is that FMH does not restrict the data process to be of any specific distribution contrary to EMH which restricts the process to be normal (Osborne, 1964).

One important statement has not been made yet. FMH has "fractal" in its name because the distributions of different investment horizons are supposed to be self-similar and thus fractal (Section 1.1). If distributions of different time scales or investment horizons remain self-similar, FMH1-FMH3 are valid and market remains stable. FMH4 and FMH5 are rather empirical findings than theoretical assumptions. Let us now proceed with Hurst exponent estimation methods, which are followed with the implications between H, EMH and FMH.

Chapter 2 Hurst exponent estimation methods

"Statistics: The only science that enables different experts using the same figures to draw different conclusions"

Esar's Comic Dictionary

As we have shown, Hurst exponent is a crucial parameter of self-similar and fractal processes. There are several estimation methods which are used in the literature. We present the most used ones in the following text. The rescaled range analysis (Hurst, 1951) is introduced together with modified rescaled range analysis (Lo, 1991) as two interconnected methods which are together able to distinguish between short-term and long-term dependence in the process. Detrended fluctuation analysis (Peng *et al.*, 1994) is presented as method for non-stationary time series. In the last subchapter, we show the interconnections between Hurst exponent, efficient markets hypothesis and fractal markets hypothesis.

2.1 Rescaled range (R/S) and Modified Rescaled range (M-R/S)

Rescaled range method is the oldest one of the Hurst exponent estimation methods and was developed by Edwin Hurst while working as an engineer in Egypt (Hurst, 1951). As it was already mentioned in Section 1.1, the method was developed for the construction of ideal reservoir of water which would never become empty and never overflow. The method was later applied to financial time series by Mandelbrot (1970). We provide a detailed description of the method together with discussion of its weaknesses in the following subchapters.

2.1.1 Procedure

The guide to the R/S method is well reviewed by Edgar Peters (Peters, 1994) and we use the reference for definition of the method[12,13]:

- *Step 1:* Transform the original price series $(P_0, P_1, ..., P_{T-1}, P_T)$ to a series of logarithmic (continuous) returns $(r_1, r_2, ..., r_{T-1}, r_T)$, where

$$r_i = \log P_i - \log P_{i-1}, \text{ for } i = 1,2,...,T. \tag{2.1}$$

[12] Note that the procedure slightly differs across literature, for the discussion see Di Matteo (2007).
[13] For more thorough theory behind R/S analysis, we suggest Samorodnitsky (2007)

- *Step 2:* Divide time period T into N adjacent sub-periods of length υ while $N * \upsilon = T$. Each sub-period is to be labeled as I_n with $n = 1,2,...,N$. Moreover, each element in I_n is labeled $r_{k,n}$ with $k = 1,2,...,\upsilon$.
- *Step 3:* For each sub-period, calculate the average value as

$$\bar{r}_n = \frac{1}{\upsilon}\sum_{k=1}^{\upsilon} r_{k,n} \tag{2.2}$$

where $\bar{r}_n$ is the average value of r_i present in sub-period I_n of length υ.

- *Step 4:* Calculate new series of accumulated deviations from the arithmetic mean values (profile) for each sub-period as

$$X_{k,n} = \sum_{i=1}^{k}\left(r_{i,n} - \bar{r}_n\right). \tag{2.3}$$

- *Step 5:* Calculate the range defined as a difference between maximum and minimum value of $X_{k,n}$ for each sub-period as

$$R_{I_n} = \max\left(X_{k,n}\right) - \min\left(X_{k,n}\right). \tag{2.4}$$

- *Step 6:* Calculate the sample standard deviation of the profile as

$$S_{I_n} = \sqrt{\frac{1}{\upsilon}\sum_{k=1}^{\upsilon}\left(X_{k.n} - \bar{X}_{k,n}\right)^2} \tag{2.5}$$

where $\bar{X}_{k,n}$ is an arithmetic mean of the profile.

- *Step 7:* Each range R_{I_n} is standardized by corresponding standard deviation S_{I_n} and forms the rescaled range

$$\left(R/S\right)_{I_n} = \frac{R_{I_n}}{S_{I_n}}. \tag{2.6}$$

- *Step 8:* We repeat the process for each sub-period of length υ and get the average rescaled range as

$$\left(R/S\right)_{\upsilon} = \frac{1}{N}\sum_{n=1}^{N}\left(R/S\right)_{I_n}. \tag{2.7}$$

- *Step 9:* The length υ is increased and the whole process is repeated[14].

[14] The length υ is usually set as a divisor of T, which yields number of different lengths υ equal to the number of divisors (used in Peters, 1994). However, we use the procedure used in e.g. Weron (2002), so that we use the length υ equal to the power of a set integer value (the method is based on the theory of multiplicative cascades which are used for a construction of fractal time series, for more details see Lux, 2007, and Borland *et al.*, 2005). Thus, we set a basis b and a maximum power *pmax* so that we get sub-periods of length $\upsilon = b, b^2, ..., b^{p\max}$ and

- *Step 10:* We get average rescaled ranges $(R/S)_\upsilon$ for corresponding sub-interval lengths υ. Rescaled range then scales as[15]

$$(R/S)_\upsilon \approx c * \upsilon^H, \tag{2.8}$$

where c is a positive finite constant independent of υ (Taqqu, Teverovsky & Willinger, 1995).

The linear relationship in double-logarithmic scale indicates the power scaling (Weron, 2002). To uncover the scaling law, we use a simple ordinary least squares regression on logarithms of each side of the previous equation. We suggest using logarithm with basis equal to b. Thus, we get

$$\log_b (R/S)_\upsilon \approx \log_b c + H \log_b \upsilon, \tag{2.9}$$

where H is Hurst exponent.

2.1.2 Comments

Before we turn to the most problematic issues of R/S analysis, we present several recommendations of the use of optimal scales. Since the R/S analysis is based on range statistic (equation 2.4) and standard deviation (eqaution 2.5), the estimates of Hurst exponent can be biased. At low scales, sample standard deviation can strongly bias the final rescaled range as e.g. the standard deviation based on two observations can be equal to zero (for two same values) which implies infinite rescaled range. On the other hand, range statistic is very sensitive to outliers and its estimate can strongly bias the final rescaled range at high scales as the outliers are not averaged out (equation 2.8) as it is case at low scales (Di Matteo, 2007). Millen & Beard (2003) propose to use a minimum scale of at least 10 observations and a maximum scale of a half of the time series length. Weron (2002) suggests using a minimum scale of at least 50 trading days. However, Weron notes that if the time series is as short as 256 trading days, it is more suitable to use a minimum scale of 16 trading days as the estimates of Hurst exponent based on only three averaged rescaled ranges are quite volatile. Let us now turn to the issues of R/S analysis.

As the R/S analysis is known for a long time, it has been a subject to a lot of testing and criticism. The method is mostly criticized for its problematic use for heteroskedastic time series (Di Matteo, 2007) and for the series with short-term memory (Lo & MacKinlay, 1999; and Alfi *et al.*, 2008).

$b^{p\max} \leq T$. Moreover, we set a minimum power *pmin* because small sub-periods can strongly bias rescaled range (Peters, 1991a). After implementation of *pmin*, we get $\upsilon = b^{p\min}, b^{p\min+1}, \ldots, b^{p\max}$.

[15] See DiMatteo, 2007

The complicated use for heteroskedastic time series which is due to use of sample standard deviation (see equation 2.5) together with a filtration of a constant trend (see equation 2.3) makes R/S analysis sensitive to non-stationarities in the underlying process. Dealing with the non-stationarity problem means to move R/S analysis closer to detrended fluctuation analysis (DFA) methodology which we discuss later in the chapter. In this approach, one filters the profile not just from a constant trend but also from a trend of higher polynomials as linear and quadratic. However, the approach is not used in the literature and authors prefer methods which are developed for non-stationary time series such as already mentioned DFA (Peng *et al.*, 1994). To deal with short-term dependence in the time series, modified rescaled range (M-R/S) is the mostly used technique.

M-R/S presented by Lo (1991) differs only slightly from the original R/S and that is in the calculation of S_{I_n}. Nevertheless, it deals with both heteroskedasticity and short-term memory by modified definition of standard deviation. The new equation (compare with equation 2.5) is defined with a use of auto-covariance γ of the selected sub-interval I_n up to the lag ξ as follows

$$S_{I_n}^{M} = S_{I_n}^{2} + 2\sum_{j=1}^{\xi}\gamma_j\left(1-\frac{j}{\xi+1}\right). \tag{2.10}$$

Thus, R/S turns into a special case of M-R/S with $\xi = 0$ (Dülger & Ozdemir, 2005). The most problematic and also the crucial issue of the new standard deviation measure is the number of lags which are used for its estimation (Wang *et al.*, 2006). If the chosen lag is too low, it omits lags which may be significant and therefore still biases estimated Hurst exponent by the short-term memory in the time series. On the other hand, if the used lag is too high, the finite-sample distribution (which is the case of the samples we use) deviates significantly from its asymptotic limit (Teverovsky, Taqqu & Willinger, 1999).

There are two estimators of optimal lag suggested in the literature[16]. The first one proposed by Lo (1991) and Andrews (1991) is the more complicated and still the most used one. The optimal lag is based on the first-order autocorrelation coefficient $\hat{\rho}(1)$:

$$\xi^{*} = \left[\left(\frac{3\upsilon}{2}\right)^{\frac{1}{3}}\left(\frac{2\hat{\rho}(1)}{1-(\hat{\rho}(1))^{2}}\right)^{\frac{2}{3}}\right] \tag{2.11}$$

[16] Note that majority of authors does not deal with the optimal lag choice and set several different lags which they use and examine the differences of the results (e.g. Zhuang, Gree & Maggioni, 2000; and Alptekin, 2006).

The second one by Chin (2008) is based on the length of the sub-interval only and sets the optimal lag as

$$\xi^* = \left[4\left(\frac{\upsilon}{100} \right)^{\frac{2}{9}} \right]. \tag{2.12}$$

Note that optimal lag ξ^* is recalculated for each length of specific sub-period υ. Optimal lags for different sub-period lengths are shown in Chart 2-1.

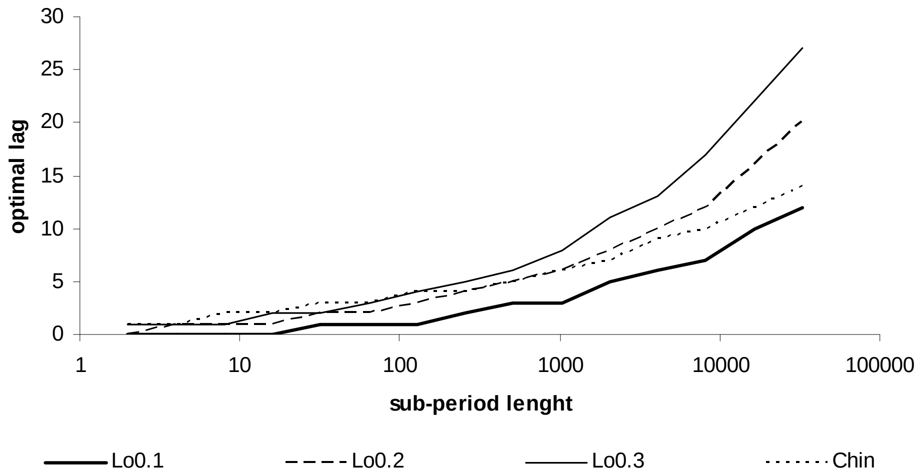

Chart 2-1 Comparison of different optimal lags for M-R/S
"Lo0.1", "Lo0.2" and "Lo0.3" stand for the first method with serial autocorrelations 0.1, 0.2 and 0.3, respectively, and "Chin" stands for the second method.

The method based on serial autocorrelations differs significantly with the changing correlations. In the case of low serial autocorrelations around 0.1, ξ^* is lower than the other method up to $\upsilon = 2^{16}$. However, if serial autocorrelation is doubled to 0.2 or even increased to 0.3, the differences between suggested lags ξ^* become significant. Couillard & Davison (2005) and Teverovsky, Taqqu & Willinger (1999) show that M-R/S is biased towards rejecting any long-term memory in the process when high number of lags is used[17]. It implies that in the case of significant short-term memory in the process, the method of Lo would lead to biased estimates of *H*. Moreover, if the short-term memory is not significant or low, the method of Lo does not significantly differ from the method of Chin. It is visible from Chart 2-1 that for sub-period lengths up to 500, which is the highest one used in the applied part of the thesis, there is no difference between both methods with the autoccorelation of 0.2 and only a difference of one lag for the autocorrelation of 0.3. Therefore, the use of rather complicated

[17] Method of Lo (1991) sets the optimal lag correctly only if the underlying process is AR(1) (Andreou & Zombanakis, 2006).

version with serial autocorrelations does not differ significantly for the most used time series lenghts. Furthermore, for the purposes of simulations which are performed in Section 3.2, the use of the method of Lo would not lead us to strong results as the first order auto-correlation of an independent process is equal to zero, the suggested optimal lag would be zero as well and M-R/S would turn to R/S. Hence, we stick to the method of Chin (2008).

The problem of choosing the correct lag can be partly overcome by short-term memory filtration (Peters, 1994). It is suggested to apply AR(1) on original (integrated) time series and then follow all steps of original procedure with the residuals of the estimated autoregressive process. However, two issues of this procedure can be questioned. The problem of setting the right lag appears again. AR(1) does not need to be satisfying for short-term memory filtration. Moreover, ARIMA(p,1,q) procedure can be applied as well. Moreover, application of any ARIMA(p,1,q) procedure on long time series ($T > 10000$) can be misleading or can be inefficient as a chance that the process retains its features (in sense of estimated ARIMA(p,1,q) coefficients) for such a long period is rather small (Mills, 1990).

Moreover, the use of any filtration on original data before any procedure is applied can bias the results to the point where there is no possibility to interpret them. The most problematic part is the fact that short-term and long-term memory processes cannot be perfectly separated on the basis of estimation as even a little bias in estimated coefficients can lead to significant break of long-term memory structure. Let us now turn to the method which is resistant to the non-stationarities in the time series – detrended fluctuation analysis.

2.2 Detrended fluctuation analysis (DFA)

Detrended fluctuation analysis was firstly proposed by Peng *et al.* (1994) while examining series of DNA nucleotides. Compared to the R/S analysis examined above, the DFA focuses on fluctuations around trend rather than a range of signal. Therefore, DFA is easily used for non-stationary time series, contrary to R/S and M-R/S. Let us describe the procedure of DFA and discussion of its properties in the following subchapters.

2.2.1 Procedure

Starting steps of the procedure are the same as the ones of R/S analysis (Step 1 to Step 4 of Section 2.1.1) as the whole series is divided into non-overlapping periods of length υ which is set on the same basis in the mentioned procedure (see Step 2 of Section 2.1.1). The following steps are based on Grech & Mazur (2005):

- *Step 5:* A polynomial fit $X_{\upsilon,l}$ of the profile is constructed for each sub-period I_n. The choice of order l of the polynomial is rather a rule of thumb. However, a linear or a quadratic trend is usually enough and higher degrees of polynomial do not add any significant information as of a behavior of Hurst exponent (Vandewalle, Ausloos & Boveroux, 1997). The procedure is then labeled as DFA-0, DFA-1 and DFA-2 for a constant, a linear and a quadratic trend filtering, respectively (Hu *et al.*, 2001).
- *Step 6:* A detrended signal $Y_{\upsilon,l}$ is then constructed as

$$Y_{\upsilon,l}(t) = X(t) - X_{\upsilon,\cdot}(t). \tag{2.13}$$

- *Step 7:* Fluctuation $F_{DFA}(\upsilon,l)$ is calculated as

$$F_{DFA}(\upsilon,l) = \sqrt{\frac{1}{T}\sum_{t=1}^{T} Y_{\upsilon,l}^2(t)}. \tag{2.14}$$

- *Step 8:* F_{DFA} then scales as follows[18]

$$F_{DFA}(\upsilon,l) \approx c * \upsilon^{H(l)}, \tag{2.15}$$

where again c is a constant independent of υ.

We then run the ordinary least squares regression on logarithms and estimate Hurst exponent $H(l)$ for set l-degree of polynomial trend in same way as for R/S and M-R/S (see equation 2.9) as

$$\log_b F_{DFA}(\upsilon,l) \approx \log_b c + H(l)\log_b \upsilon, \tag{2.16}$$

where $H(l)$ is Hurst exponent for particular degree of polynomial used for the filtration.

2.2.2 Comments

DFA, as mentioned above, can be based on different polynomial fits. Moreover, trend can be constructed on a basis of Fourier transforms (Chianca, Ticona & Penna, 2005), empirical mode decomposition (Jánosi & Müller, 2005), singular value decomposition (Nagarajan, 2006), different types of moving averages (Alessio *et al.*, 2002) and others. However, polynomial fit remains the most used method (Oh, Kim & Um, 2006; and Morariu *et al.*, 2007).

As DFA is still quite a new technique, there is not as much statistical testing available in recent literature. The most applied work on the use of DFA in financial time series is connected with Dariusz Grech and his colleagues (Grech & Mazur, 2004, 2005 and Czarnecki, Grech & Pamula, 2008). However, these papers do not provide much statistical background.

[18] See Weron (2002).

Grech & Mazur (2004) state one needs to be careful when choosing the optimal length of the time series investigated as well as the maximum scale υ_{max}. Authors propose to use the scale of $5 < \upsilon < T/5$. Moreover, a proposition of a method for choosing an optimal *T* is made. For a concrete time series (DJI30 in their paper), one should apply DFA procedure on several segments of the time series while using a range of scales. Authors chose 500 segments and $140 < T < 300$. Then, optimal *T* is chosen on the basis of minimum standard deviation of estimated *H*. Moreover, authors provide a measure of statistical uncertainty which is defined as $E_T(H)/\hat{\sigma}_T(H)$ and used for optimal scale choice as well. The optimal *T* is chosen as the one of a local minimum of standard deviation of *H* as well as of the statistical uncertainty. However, the choice of the optimal *T* still seems quite random as there are several local minima and there is no strong argument why the exact one was chosen. Nevertheless, authors propose to use $190 \leq T \leq 230$. Furthermore, the choice of DJI30 as a benchmark seems questionable as the analysis does not provide clear threshold for a time series still being treated as an independent process as we can't say whether DJI30 behaves independently or not.

Other authors deal with an optimal time scale as well. Matos *et al.* (2008) propose to use $\upsilon_{\max} \approx T/4$, while Alvarez-Ramirez, Rodriguez & Echeverria (2005) go further while proposing a minimum range as well $-\upsilon_{\min} \approx 5$ and $\upsilon_{\max} \approx T/4$. Very similarly, Einstein, Wu & Gil (2001) propose to use $\upsilon_{\min} \approx 4$ and $\upsilon_{\max} \approx T/4$. Quite thorough analysis of DFA is presented in Weron (2002)[19] who proposes to use $\upsilon_{\min} \approx 10$ for short time series of 256 or 512 observations and $\upsilon_{\min} \approx 50$ for longer ones. However, the author did not provide any suggestion of maximum scale and used all scales up from the minimum one for his estimates. Finally, Kantelhardt (2008) proposes to use $\upsilon_{\max} \approx T/4$ and suggests being very cautious for small ranges as they can significantly overestimate the resulting Hurst exponent. We proceed with interconnection between Hurst exponent, efficient and fractal markets hypothesis.

2.3 EMH, FMH and Hurst exponent

Hurst exponent has two properties which are important for the description of the market and its type – value and stability. Let us start with its stability which is not much discussed in the literature and then, we turn to implications of its value.

[19] Weron (2002) is more discussed in Section 3.3.

Stability of Hurst exponent is connected with the fact that a process is self-similar only when H is well defined. However, Hurst exponent can be only estimated and therefore, we cannot be sure whether the estimated value is actually the true one (Jagric, Podobnik & Kolanovic, 2005). Nonetheless, the stability of H can be examined on the basis of its characteristic values – rescaled ranges or fluctuations – as if the value is significantly higher for a specific scale, it represents the optimal investment horizon (Lo, 1991). However, the existence of optimal investment horizon contradicts both EMH and FMH. Efficient markets are challenged because optimal investment horizon implies potential predictability of the market. For fractal markets hypothesis, optimal investment horizon implies that self-similar structure of the market breaks down and therefore, the market can turn into a spiral. However, as we show in Section 3.4, we concentrate on time-dependent Hurst exponent (e.g. Grech & Mazur, 2005) for which the examination of different scales solely is not possible. Hence, we focus on significant changes in values of Hurst exponent as such a change implies significant shift of rescaled ranges or fluctuations at either low or high scales.

As for the values of Hurst exponent, there are several crucial implications. If H is equal to 0.5, the random walk (Definition 1-10) is implied (Karytinos, Andreou & Pavlides, 2000). Therefore, if we arrive at the value of H of 0.5, we have weakly efficient market of F65 type. Moreover, we know from (1.10) that such process has defined and finite second moment and thus finite variance which imlies martingale process as well which in turn indicates efficient market of S65 type. Let us now turn to more interesting cases – persistent and anti-persistent process.

Persistent process is characterized by Hurst exponent significantly higher than 0.5 and implies rejection of independence which in turn rejects random walk and consequently efficient market of F65 type (Embrechts & Maejima, 2002). However, the value of $1/2 < H < 1$ implies $1 < \alpha < 2$ which, as it was mentioned in Section 1.1, in turn indicates undefined or infinite variance. Such a result implies that also a square root of variance is infinite or undefined and thus martingale process is not present which leads to rejection of market efficiency of S65 type (Los, 2008).

On the other hand, anti-persitent processes do not lead to such strong implications. Even though the random walk and thus efficiecy of F65 type is rejected in the same way as for persistent process (Embrechts & Maejima, 2002), the situation is not so clear for S65 type. Hurst exponent which is in interval $0 < H < 1/2$ implies $2 < \alpha < \infty$ and thus the underlying distribution is not stable. However, the non-stable distributions are not well examined yet and there is only little literature focusing on them. Nonetheless, the crucial implication which is clear from the literature is that the process based on a non-stable distribution is not independent with identically distributed innovations (Der & Lee, 2006) and thus F65 efficiency is rejected. On the other hand, non-stable distributions were shown to have finite variance and thus S65 efficiency cannot be rejected (Da Silva *et al.*, 2005).

To be able to test the hypothesis of either F65 or S65, we need to estimate expected values and standard deviations for each method so that critical values can be calculated. The estimates are presented in the following chapter.

Chapter 3 Finite sample properties of R/S, M-R/S and DFA

"I didn't think; I experimented"

Anthony Burgess

The procedures presented in previous chapter work well only for very long (more than 10000 observations) or infinite time series (Weron, 2002). However, the financial time series are usually much shorter and we would need at least forty years of daily prices to reach the mentioned threshold. Moreover, we use time-dependent Hurst exponent and thus we would need much longer time series to be able to uncover the underlying dynamics. However, only several authors deal with the problem of finite samples and their properties (e.g. Grech & Mazur, 2005; and Weron, 2002).

In this chapter, we present the recent findings presented in the research papers for all described methods – R/S, M-R/S and DFA methods of different degree of detrending (DFA-0, DFA-1 and DFA-2). Moreover, we present our original results for different time series lengths. The crucial distinction of our procedure is the use of a minimum scale of 16 trading days and a maximum scale of one quarter of the time series length. The application of such scales is based on propositions presented in Section 2.1.2, Section 2.2.2 and results which are presented in Chapter 4.

3.1 R/S analysis

For the R/S analysis, we depict the results presented in recent research papers (Couillard & Davison, 2005; Weron, 2002; and Peters, 1994) and then, we turn to the results of our simulations. Note that we provide such division for R/S only as there are only several papers concerning M-R/S and DFA.

3.1.1 Recent results

R/S analysis has one significant advantage compared to the other methods – as it is known and tested for over 50 years, the methods for testing have been well developed and applied (Peters, 1991b).

The condition for a time series to reject long-term dependence is that $H = 0.5$[20]. However, it holds only for infinite samples and therefore is an asymptotic limit. The correction for finite samples is thoroughly tested in Couillard & Davison (2005). There are two methods used and both are based on estimating theoretical $(R/S)_\upsilon$.

The first method is the one of Anis & Lloyd (1976), which we note AL76, and states the expected value of rescaled range as[21]

$$E(R/S)_\upsilon = \frac{\Gamma\left(\frac{\upsilon-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\upsilon}{2}\right)} \sum_{i=1}^{\upsilon-1} \sqrt{\frac{\upsilon-i}{i}}\,. \tag{3.1}$$

Peters (1994) proposes "empirical correction", which we note P94 and defines expected rescaled range as

$$E(R/S)_\upsilon = \frac{\upsilon-\frac{1}{2}}{\upsilon} \sqrt{\frac{2}{\upsilon\pi}} \sum_{i=1}^{\upsilon-1} \sqrt{\frac{\upsilon-i}{i}}\,. \tag{3.2}$$

Peters (1994) argues that AL76 overestimates rescaled ranges for small υ. That is why he added $(2\upsilon-1)/2\upsilon$ into equation to make it fit better the real data for small υ[22]. Moreover, the gamma functions $\Gamma(\bullet)$ were substituted by $\sqrt{2/\upsilon}$ as when beta function $\mathrm{B}(\bullet)$[23] is used as a substitute of gamma function and Stirling's approximation is applied, Peters obtains $\Gamma((\upsilon-1)/2)/\Gamma(\upsilon/2) \approx \sqrt{2/\upsilon}$. It is needed to mention that Peters used an approximation of an approximation when stating the equality. The exact application of Stirling's approximation yields $\Gamma((\upsilon-1)/2)/\Gamma(\upsilon/2) \approx \sqrt{2/(\upsilon-1)}$ (Boisvert *et al.*, 2008)[24].

However, Couillard & Davison (2005) tested the assertion and came up with different results – AL76 estimates rescaled range for small samples ($\upsilon < 500$) much more accurately and underestimates rescaled range for large samples ($\upsilon > 500$) compared to P94. Note that the underestimation is insignificant.

Authors also tested the asymptotic standard deviation of H (we use the same notation $\hat{\sigma}(H)$) which is essential for hypothesis testing. They argue that the Peters' statement that $\hat{\sigma}(H) \approx 1/\sqrt{T}$ is again only an asymptotic limit and is significantly biased for finite

[20] See Section 1.1 and Section 1.2.1.

[21] For theory about gamma function $\Gamma(\bullet)$, see Appendix.

[22] Peters further proposes to use the minimum range of 10 trading days which in our case turns to 16 trading days as it is the first higher power of base 2.

[23] For theory about beta function $\mathrm{B}(\bullet)$, see Appendix.

[24] For detailed transformation and features of gamma and beta functions, see Appendix.

number of observations and come to new estimate based on simulations up to $T = 10000$. The estimate states that standard deviation of H behaves as $\hat{\sigma}(H) \approx 1/e\sqrt[3]{T}$.

Unfortunately, Couillard & Davison (2005) only tested the estimators up to $\upsilon = 1000$ and standard deviations up to $T = 10000$. However, the time series are often much longer[25]. Therefore, we present the results of our original simulations in following subchapter.

3.1.2 Original results

We performed original test[26] for time series lengths from $T = 512 = 2^9$ up to $T = 131072 = 2^{17}$. The lengths of the time series were chosen with respect to the fact that the time series of lower lengths were shown to be rather volatile (Weron, 2002). For purposes of the thesis, the need for an estimator of a standard deviation of Hurst exponent and the exponent itself is much more urgent than the estimators of rescaled ranges. Therefore, the simulations are performed for $E(H)$ and $\hat{\sigma}(H)$ only.

All steps of R/S analysis on 10000 time series drawn from standardized normal distribution $N(0,1)$ for $T = 2^9 = 512$ up to $T = 2^{17} = 131072$ were performed. $E_T(H)$ and $\hat{\sigma}_T(H)$ were estimated for each T[27,28].

Nonetheless, R/S estimators were tested against empirically obtained $E_T(H)$. We compared the simulated H with the ones estimated by AL76, P94 and corrected P94 procedure which is based on exact Stirling's approximation. We call the corrected procedure P94c further on. AL76 contained gamma functions up to $\upsilon = 2^8 = 256$ and approximation for higher ones[29]. $E_T(H)$ was obtained from rescaled ranges by log-log regression according to the power law mentioned in the Step 10 and below of R/S analysis in Section 2.1.

The results for estimated H based on AL76, P94 and P94c are summed in Table 3-1.

25 High-frequency time series can often contain over 100000 observations.
26 All simulations and estimations were run on TSP 5.0.
27 Hurst exponent was estimated by log-log regression according to the standard procedure. Approach used in majority of literature was applied here as well – averaged rescaled ranges applied in the regression were the ones for $2^4 \leq \upsilon \leq 2^{T-2}$. The logic behind this step is rather intuitive – very small scales can bias the estimate as standard deviations are based on very few observations; on the other hand, large scales can bias the estimate as outliers or simply extreme values are not averaged out (Peters, 1994). For details, see crossover detection section for all tested indices in Chapter 4 or only Section 4.6 for comments.
28 Weron (2002) simulated the time series for R/S and found out that the minimum scale of 16 trading days is not enough and proceeds with a minimum range of 64 trading days. However, we show further in the text that the omission of the highest scales is more important.
29 This method is used because the problem that was already tackled by Peters (1994) is still valid – gamma function for high values of υ can still cause problems to modern analytical software.

Table 3-1 Comparison of Anis & Llloyd's and Peters' formula for long series

T	*AL76*	*P96*	*P96c*
512	0,5686	0,5992	0,5858
1024	0,5611	0,5833	0,5729
2048	0,5513	0,5708	0,5624
4096	0,5455	0,5607	0,5540
8192	0,5411	0,5525	0,5470
16384	0,5361	0,5458	0,5412
32768	0,5318	0,5402	0,5363
65536	0,5282	0,5356	0,5322
131072	0,5254	0,5316	0,5287

We can see that all estimates are converging to 0.50 with increasing T which is as expected. Note that we don't get very close to asymptotic H even for very high T. However, we can't really say much about estimated Hurst exponents without the simulations.

The results for $E_T(H)$, $\hat{\sigma}_T(H)$ and corresponding descriptive statistics together with Jarque-Bera test (Jarque & Bera, 1981) for normality are summed in Table 3-2, probability functions are showed in Chart 0-1 in Appendix.

Table 3-2 Descriptive statistics of simulated of *H* for R/S

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*
mean	0,5763	0,5647	0,5570	0,5494	0,5430	0,5380	0,5338	0,5296	0,5267
SD	0,0551	0,0404	0,0310	0,0246	0,0199	0,0162	0,0138	0,0118	0,0102
skewness	0,0104	0,0003	-0,0231	-0,0316	-0,0223	-0,0331	-0,0329	0,0068	-0,0762
excess kurtosis	-0,1316	0,0730	-0,0595	-0,0567	0,0220	-0,0271	0,0136	-0,1108	0,0237
JB statistic	7,4569	2,1800	2,3895	3,0314	1,0196	2,1440	1,8737	5,2405	9,9080
p-value	0,0240	0,3362	0,3028	0,2197	0,6006	0,3423	0,3919	0,0728	0,0071

We can see that estimates of Hurst exponent are not equal to 0.5 as predicted by asymptotic theory. Therefore, one must be careful when accepting or rejecting hypotheses about long-term dependence present in time series solely on its divergence from 0.5. This statement is most valid for short time series. Chart 3-1 presents the idea together with estimations of H based on AL76, P94 and P94c. However, the Jarque-Bera test rejected normality of Hurst exponent estimates for time series lengths of 512, 65536 and 131072 and therefore, we should use percentiles rather than standard deviations for the estimation of confidence intervals (Weron, 2002). Nevertheless, the differences for mentioned estimates not normally distributed are only of the order of the tenths of the thousandth and therefore, we present confidence intervals based on standard deviations for R/S.

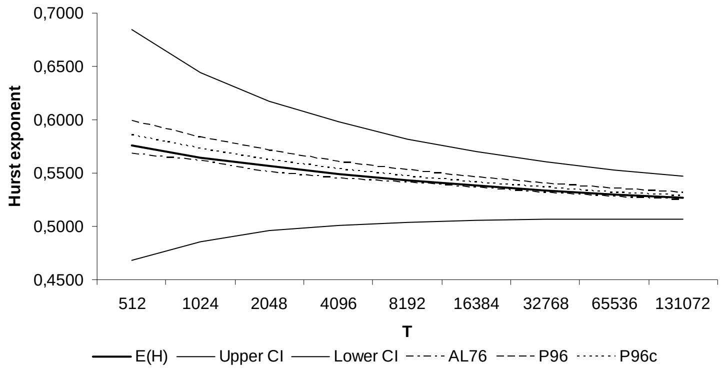

Chart 3-1 Simulated Hurst exponents with confidence intervals (R/S)
Hurst exponent estimation on simulated random series shows very broad 95% confidence intervals for short time series. AL76 outperforms P94 and P94c for all time series lengths.

From the chart, we can see that 95% confidence intervals are quite wide for short time series. Even if time series of 512 observations yields H equal to 0.65, we can't reject the hypothesis of a martingale process. Specific values are present in Table 3-3. The table shows that AL76 outperforms (measured by mean squared error - MSE[30]) both P94 and P94c. Interestingly, P94c strongly outperforms P94. Nonetheless, we suggest AL76 for expected value of H for different T than we have tested here.

Table 3-3 Simulated Hurst exponents compared with predicted ones for R/S

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*	*MSE*
E(H)	0,5763	0,5647	0,5570	0,5494	0,5430	0,5380	0,5338	0,5296	0,5267	
Upper CI	0,6843	0,6438	0,6178	0,5977	0,5820	0,5698	0,5608	0,5528	0,5466	
Lower CI	0,4684	0,4856	0,4962	0,5011	0,5040	0,5062	0,5068	0,5065	0,5069	
AL76	0,5686	0,5611	0,5513	0,5455	0,5411	0,5361	0,5318	0,5282	0,5254	0,000015
P96	0,5992	0,5833	0,5708	0,5607	0,5525	0,5458	0,5402	0,5356	0,5316	0,000160
P96c	0,5858	0,5729	0,5624	0,554	0,547	0,5412	0,5363	0,5322	0,5287	0,000028

[30] Mean squared error is defined as an average of squared deviations from estimated value for each time series length.

Standard deviations of Hurst exponent $\hat{\sigma}_T(H)$ were also tested and compared with the estimations of Peters (1994) and Couillard & Davison (2005). Just for reminder, authors propose that $\hat{\sigma}(H) \approx 1/\sqrt{T}$ and $\hat{\sigma}(H) \approx 1/e\sqrt[3]{T}$, respectively. Chart 3-2 shows the differences between predicted and simulated values[31].

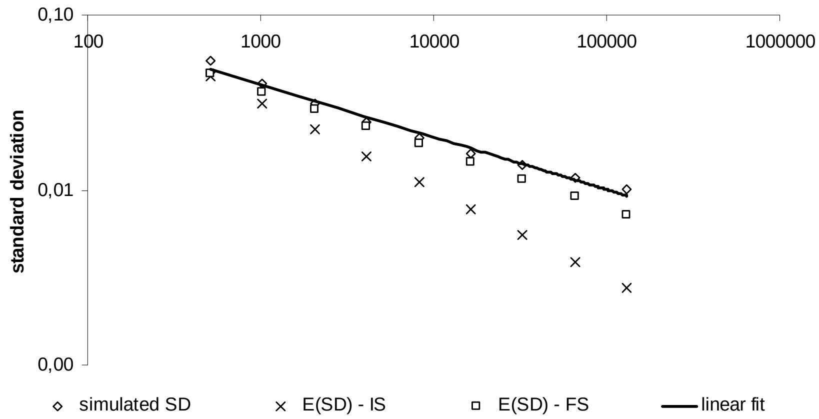

Chart 3-2 Standard deviation of Hurst exponent for R/S
"simulated SD" marks standard deviations based on our simulations, "E(SD) – IS" stands for an expected standard deviation of an infinite sampe from Peters (1994), "E(SD) – FS" stands for an expected standard deviation of a finite sample from Couillard & Davison (2005) and "linear fit" marks the ordinary least squares fit on double logarithmic scale.

Both estimators underestimate expected standard deviation $\hat{\sigma}_T(H)$. The estimator for infinite sample underestimates $\hat{\sigma}_T(H)$ more strongly. Therefore, we present new estimate of standard deviation, which is presented in Chart 3-2 as a solid line, as $\hat{\sigma}(H) \approx 1/\pi T^{0.3}$. Comparison of methods together with MSE is presented in Table 3-4[32].

Table 3-4 Comparison of standard deviations for R/S

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*	*MSE*
mean	0,5763	0,5647	0,5570	0,5494	0,5430	0,5380	0,5338	0,5296	0,5267	
SD	0,0551	0,0404	0,0310	0,0246	0,0199	0,0162	0,0138	0,0118	0,0102	
E(SD) – IS	0,0442	0,0313	0,0221	0,0156	0,0110	0,0078	0,0055	0,0039	0,0028	0,000077
E(SD) – FS	0,04598	0,0365	0,02897	0,02299	0,01825	0,01448	0,0115	0,00912	0,00724	0,000015
E(SD) - AFS	0,04899	0,03979	0,03232	0,02625	0,02132	0,01732	0,01407	0,01143	0,00928	0,000005

New method for estimation of expected standard deviation of Hurst exponent is three times more efficient than one of Couillard & Davison (2005) and fifteen times more efficient than one of Peters (1994) and therefore, we suggest it for estimation of $\hat{\sigma}_T(H)$ for any T from the tested interval and based on same procedure[33].

[31] Chart is presented as a log-log plot for more visible differences.
[32] E(SD) – AFS for an adjusted expected standard deviation of a finite sample.
[33] Different procedure can yield rather different results. For example, Weron (2002) estimates Hurst exponent using rescaled ranges for scales of at least 50 trading days but does not restrict scales from the top which results

Moreover, we have shown that a combination of a minimum scale of 16 trading days with a maximum scale of a fourth of the time series length yields Hurst exponent value which is very close to all AL76, P94 and P94c methods with standard deviations almost twice lower than those of Weron (2002). Therefore, it implies that omitting of high scales is more important and efficient than omitting of scales of 16 and 32 trading days for R/S analysis.

As an implication, we propose AL76 method for an estimation of expected value of H with our estimate of standard deviation for a construction of confidence intervals for the real world analysis in Chapter 4. Let us follow with M-R/S.

3.2 M-R/S analysis

M-R/S analysis is rather different from R/S analysis when the applications are compared. R/S analysis is usually based on estimation of Hurst exponent itself (Mandelbrot, 1970). On the other hand, only V statistics[34] is usually constructed for a specific investment horizon (scale in our case) and compared to critical values constructed by Lo (1991) in the case of M-R/S. The same procedure is then applied in several research papers – e.g. Eitelman & Vitanza (2008); Berg & Lyhagen (1998); Lillo & Farmer (2004); and Zhuang, Green & Maggioni (2000). However, this procedure can be hardly used for a sliding window as we would get multi-dimensional results which would be rather difficult to interpret. Therefore, we propose to simply use estimates of H based on M-R/S.

To make the results robust, we take the same path as for R/S and simulate the same random time series[35]. Unfortunately, there are no theoretical estimates of modified rescaled range itself and therefore, we must stick to simulated estimates only. Note that we use the method of Chin (2008), which was presented in Section 1.1.2, for estimation of optimal lag as it is the only method which bases the optimal lag on sub-period length only compared to the method of Lo (1991) which is based on autocorrelations which would imply zero optimal lag and would turn M-R/S into R/S and the simulations would be of no additional information.

The descriptive statistics for simulated random time series are summed in Table 3-5. There are several interesting results. The estimates of H based on M-R/S are obviously lower

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in standard deviations almost twice a value of estimates presented in this thesis. Furthermore, the author proposes the estimates for the time series length of 256 and shows 95% confidence intervals which are almost equal extreme values of 0.2 and 0.8 for lower and upper confidence interval for the null hypothesis of independence. This implies that if the same procedure is used for the real world time series of a length of 256 trading days, the interpretation is very close to imposible.

[34] See Section 3.4 for details about V statistics.

[35] We again simulated Hurst exponent for 10000 random time series drawn from standardized Gaussian distribution for minimum time series length of 2^9 and maximum one of 2^{17}. The minimum and maximum scales are set accordingly to R/S.

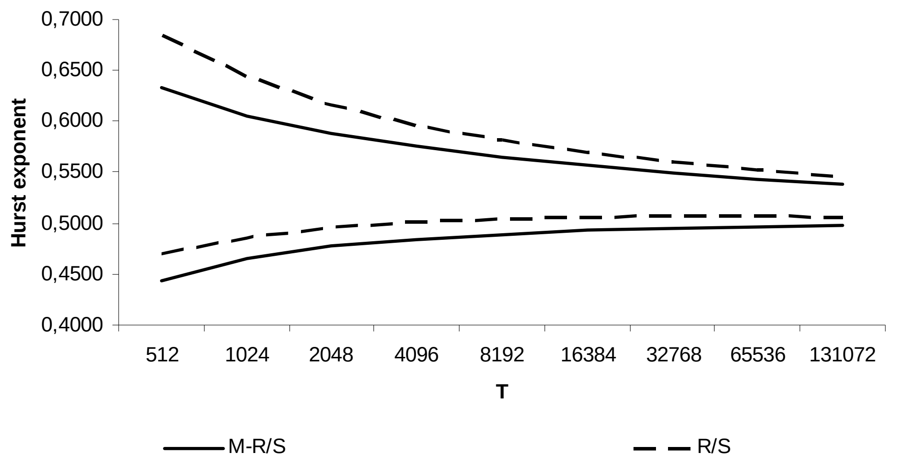
than those based on R/S. This finding suggests that one must be cautious when making conclusions based on comparison of Hurst exponents based on those two methods only. Moreover, standard deviations of estimates based on M-R/S are lower than the ones of R/S method and therefore, the estimates are more stable. On the other hand, distributions of Hurst exponent estimates are not normal for almost all lengths of the time series and therefore, we must stick to percentiles rather than standard deviation for the estimation of confidence intervals. The distributions are illustrated in Chart 0-1 in Appendix.

Table 3-5 Descriptive statistics of simulated of H for M-R/S

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*
mean	0,5393	0,5365	0,5337	0,5304	0,5278	0,5245	0,5223	0,5198	0,5182
SD	0,0485	0,0360	0,0284	0,0233	0,0192	0,0161	0,0139	0,0117	0,0101
skewness	-0,1088	-0,1048	-0,0393	-0,0693	-0,0824	0,0061	-0,0619	-0,0077	-0,0317
excess kurtosis	0,1919	0,0933	-0,0930	0,1823	0,0068	-0,0187	0,0282	0,1272	-0,0317
JB statistic	34,9582	21,8861	6,2216	21,7428	11,3207	0,2170	6,7039	6,7651	2,1094
p-value	0,0000	0,0000	0,0446	0,0000	0,0035	0,8972	0,0350	0,0340	0,3483

As the simulated estimates are not normally distributed, we do not present any fits for estimated standard deviation as their use would not be of any help. However, we present the confidence intervals based on percentiles[36] and show them together with confidence intervals for *H* based on R/S for comparison in Chart 3-3.

Chart 3-3 Comparison of confidence intervals of simulated Hurst exponents based on R/S and M-R/S



The most obvious result is the fact that the estimates of M-R/S are lower than those of R/S. The difference is more profound for upper confidence interval and is very broad at lower scales which in turn shows that R/S overestimates *H* much more than M-R/S while the statement is more valid for lower scales.

[36] We present 95% confidence intervals and therefore we use 2,5% and 97,5% percentile.

As M-R/S has no theoretical models of expected rescaled ranges, we can only construct fits for confidence intervals. Nevertheless, we can provide the expected value of H as well. Therefore, we provide estimates for 95% two-tailed confidence intervals together with expected value of H[37]:

$$H_{UCI}(T) = \frac{0.6361}{T^{0.075}} \tag{3.3}$$

$$H_{LCI}(T) = \frac{0.4480}{T^{0.0519}} \tag{3.4}$$

$$E[H(T)] = \frac{0.5424}{T^{0.0189}} \tag{3.5}$$

Note that expected Hurst exponent decays rather slowly and does not reach a value of 0.50 up to very high time series lengths. Nevertheless, we propose the use of above mentioned estimates for the detection of significant long-term memory with short-term memory present as well and the usage of both R/S and M-R/S for comparison. We present the implications of the comparison between the methods in Section 3.4.

3.3 DFA

As DFA is still quite a new technique, there have been only several research papers concerning its finite sample properties. Let us introduce them.

Grech & Mazur (2005) tested time series of lengths from 100 to 100000 and for each one ran 65000 simulations. In their paper, they present results of DFA-1 for the time series of lengths of 1000, 10000 and 30000 with expected values of H close to 0.50 (0.500, 0.499 and 0.499, respectively) and low standard deviations (0.043, 0.024 and 0.016, respectively). Unfortunately, the authors do not provide any further statistics or tests for normality of Hurst exponent estimates and thus the usefulness of standard deviations for the construction of the confidence intervals might be problematic. More questionably, the authors take only the estimates of H which have yielded R^2 higher than 0.98 into consideration. Therefore, the estimates are only useful for "nicely" behaving estimates of fluctuations F as the "wrongly" behaving ones are not taken into consideration. Furthermore, there is no ratio of rejected estimates provided which might be useful. Therefore, the question what to do with the results which do not fall into "nicely" behaving category remains unanswered.

Weron (2002) ran 10000 simulations of DFA-1 on random time series of lengths from 256 to 65536 with minimum scales of 16 and 64. The results show the expected values of H

[37] Upper confidence interval, lower confidence interval and expected H, respectively

again very close to 0.5 with low standard deviations. As mentioned earlier, the author set no restriction on the highest scales. The standard deviation is lower for the lower minimum scale for the time length of 256 which shows that estimations of H based on only three points yield rather volatile results.

Unfortunately, there are no research papers dealing with finite sample properties of DFA-0 and DFA-2 or higher. Nevertheless, we ran new simulations for DFA-0, DFA-1 and DFA-2 based on similar procedure as for R/S such as the minimum range of 16 with the maximum scale of a quarter of the time series length (see Section 3.1). The standardized normal time series were simulated as they represent a constant trend and for linear and quadratic trend, the fact that they are not present in the time series makes no real difference as the methods are supposed to detect that. Probability functions for DFA-0, DFA-1 and DFA-2 are presented in Chart 0-1 in Appendix. We present the results in following subchapters.

3.3.1 DFA-0

Results of simulations for DFA-0 are presented in Table 3-6. The expected value of H is very close to 0.5, actually it equals to 0.50 if rounded to hundredths for all time series lengths. Further, Jarque-Bera statistics rejects normality only for two lowest lengths which enables us to use standard deviations for estimation of confidence intervals[38]. We follow the simulations results with DFA-1 in the next subchapter.

Table 3-6 Descriptive statistics of simulated of H for DFA-0

	512	*1024*	*2048*	*4096*	*8192*	*16384*	*32768*	*65536*	*131072*
mean	0,5027	0,5029	0,5017	0,5018	0,5013	0,5005	0,5008	0,5008	0,5003
SD	0,0727	0,0532	0,0405	0,0318	0,0255	0,0213	0,0179	0,0153	0,0132
skewness	0,0913	0,0416	0,0121	-0,0086	-0,0310	-0,0501	-0,0094	-0,0345	-0,0319
excess kurtosis	-0,0521	-0,0980	-0,0419	-0,0292	0,0419	-0,0078	-0,0402	0,0511	0,0233
JB statistic	15,0477	6,9304	0,9973	0,4922	2,3244	4,2051	0,8411	3,0479	1,9134
p-value	0,0005	0,0313	0,6074	0,7819	0,3144	0,1221	0,6567	0,2178	0,3841

3.3.2 DFA-1

The results of simulations for DFA-1 are summed in Table 3-7. We have very similar results to the ones of DFA-0 as expected values of H are between 0.50 and 0.51. The standard deviations are lower for DFA-1 than for DFA-0 which supports the fact that linear detrending does not significantly bias the estimates even though there are no trends in standardized normal series. The comparison of the standard deviations is presented in Section 3.3.4.

[38] Estimates of standard deviations are not well described by log-log fit for short time series as shown in the section for R/S (Section 3.1) and M-R/S (Section 3.2). Moreover, the difference between confidence intervals based on standard deviations and percentiles (Weron, 2002) is of order of thousandths and thus insignificant.

Similarly to DFA-0, the estimated Hurst exponents are normally distributed with exception of two lowest time series lengths which enables us to use the standard deviations for estimation of confidence intervals.

Table 3-7 Descriptive statistics of simulated of H for DFA-1

	512	1024	2048	4096	8192	16384	32768	65536	131072
Mean	0,5079	0,5062	0,5040	0,5031	0,5025	0,5022	0,5020	0,5015	0,5013
SD	0,0687	0,0500	0,0386	0,0304	0,0247	0,0202	0,0173	0,0149	0,0126
skewness	0,1189	0,0630	0,0430	-0,0069	0,0053	-0,0258	-0,0398	-0,0227	-0,0323
excess kurtosis	-0,0205	-0,0512	-0,0796	-0,0711	-0,0795	-0,0739	-0,0051	0,0109	-0,0919
JB statistic	23,7407	7,7276	5,7584	2,2171	2,7205	3,4246	2,6580	0,8990	5,3017
p-value	0,0000	0,0210	0,0562	0,3300	0,2566	0,1804	0,2647	0,6379	0,0706

If we compare the estimates of standard deviation with the ones of Weron (2002), our estimates are lower if compared to the ones with the same minimum scale as we use (16 trading days) which implies that the omitting of the two highest scales makes the estimates less volatile. For the ones of the author which use 64 trading days as a minimum range, the estimates presented in the paper are more efficient with exception of the ones of the time series length of 512 trading days. Nevertheless, the real world data[39] and the data with trend of a higher polynomial (Xu *et al.*, 2005) show that the two highest scales can cause problems and bias the estimates. Let us follow with results for DFA-2.

3.3.3 DFA-2

We present the results for DFA-2 in Table 3-8. The results are in hand with previous findings. Hurst exponent estimates are slightly higher which was expected as DFA is supposed to overestimate the exponent if the method of a higher degree than the actual degree of trend is used. Standard deviations, on the other hand, are lower when compared to the ones of DFA-1. Moreover, there are again only two time series lengths for which the normality of the estimates is rejected. We follow with comparison of all tested DFA methods.

Table 3-8 Descriptive statistics of simulated of H for DFA-2

	512	1024	2048	4096	8192	16384	32768	65536	131072
Mean	0,5141	0,5105	0,5080	0,5060	0,5048	0,5036	0,5031	0,5026	0,5022
SD	0,0601	0,0428	0,0325	0,0255	0,0206	0,0171	0,0144	0,0121	0,0105
Skewness	0,1184	0,0473	0,0318	0,0286	0,0334	0,0278	-0,0358	-0,0421	-0,0686
excess kurtosis	0,0025	-0,0651	-0,0579	0,0367	0,0215	0,0401	-0,0465	0,0189	0,0297
JB statistics	23,3699	5,5289	3,1048	1,9043	2,042	1,9362	3,0599	3,0943	8,1974
p-value	0,0000	0,0630	0,2117	0,3859	0,3602	0,3798	0,2166	0,2129	0,0166

39 See crossover detection parts in Chapter 4.

3.3.4 Comparison of DFA methods

The most important result is the fact that all methods yield estimates of Hurst exponent very close to 0.50 which is in hand with the results of other mentioned authors (Grech & Mazur, 2005; and Weron, 2002). However, it does not necessarily mean that DFA methods are superior to R/S analysis; it simply means that one must work with finite sample estimates of the methods and not with asymptotic limits.

Nevertheless, all DFA methods showed that almost all of the estimates for various time series lengths are normally distributed which enables us to use standard deviation for the estimation of confidence intervals and therefore, these standard deviations can be used for other scales and time series length. However, one must keep in mind the specifics of the procedure we use.

Chart 3-4 presents the estimated standard deviations of simulated time series[40]. We can see that the standard deviations of DFA-0 and DFA-1 are quite similar, whereas the ones of DFA-2 are clearly lower.

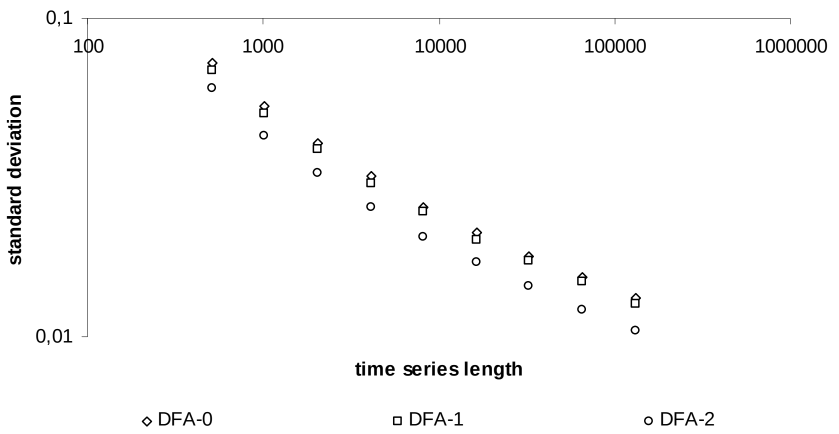


Chart 3-4Comparison of estimated standard deviations for DFA
Double logarithmic plot shows that standard deviations of DFA-2 are much lower than ones of DFA-0 and DFA-1. Best exponential fits show the same dacay of 0.3 with increasing time series length.

When we run the best linear fit to the presented log-log plot, we get the following results:

$$\hat{\sigma}_{DFA-0,H}(T) = \frac{0.4243}{T^{0.3}} \quad (3.6)$$

$$\hat{\sigma}_{DFA-1,H}(T) = \frac{0.3912}{T^{0.3}} \quad (3.7)$$

[40] We again use log-log scale for better clarity.

$$\hat{\sigma}_{DFA-2,H}(T) = \frac{0.3581}{T^{0.3}} \tag{3.8}$$

If we compare the estimates of standard deviations with the one of R/S in Section 3.1, we can see that all estimates of standard deviations decay at a rate of 0.3. However, such result does not imply any useful conclusions. Let us now move to the presentation of our original method.

3.4 Time series examination procedure

The theory of Hurst exponent estimation procedures in time series is based on asymptotic basis. However, infinite time series are not available for any real world phenomena. Moreover, most techniques work best with only weak trends or one trend at time. On the other hand, real world financial time series usually show strong trends or more trends at the same time (Peters, 1994). These effects can cause a change in scaling behavior at specific scale and estimates of Hurst exponent can be in turn strongly biased. The change of scaling behavior is called crossover and detection of the scale where it occurs is crucial for correct estimation of *H*.

McKenzie (2002) proposes a statistic that was firstly used by Hurst to test stability of the exponent (Hurst, 1951) and further for M-R/S analysis by Lo & MacKinlay (1991) – *V* statistic defined as

$$V_\upsilon = \frac{(R/S)_\upsilon}{\sqrt{\upsilon}}. \tag{3.9}$$

V statistic defines how rescaled range scales with increasing υ – *V* statistic is either constant for F65 and S65 efficient process or increasing for a persistent process or decreasing for an anti-persistent process. If the statistic behaves similarly for the whole tested period, we can see no crossovers and therefore the time series scales infinitely[41]. On the other hand, if the statistic changes its behavior (e.g. from increasing behavior to a decreasing or a constant one), a crossover is detected. Let us call the scale where the crossover is detected as a maximum scale υ_{max}. The maximum scale is the highest one taken into consideration when estimating *H* as the inclusion of higher scales would bias the results. This procedure can be used for R/S analysis, modified R/S analysis and DFA as the procedure for estimating *H* is very similar in all three cases.

[41] McKenzie (2002) and Peters (1994) uses *V* statistic to identify cycles in the time series – either periodic or non-periodic. However, detection of crossovers is in essence same to the identification of cycles.

Therefore, there are three cases of maximum scales which imply used time series length for R/S, M-R/S and DFA:

- Crossover is identified at $\upsilon_{\max} \leq T_{\max}/4$ and thus T is based on recommendations[42] so that $T = 4\upsilon_{\max}$;
- Crossover is identified at $T_{\max}/4 \leq \upsilon_{\max} \leq T_{\max}/2$ and thus T is chosen as the highest one possible so that $T = T_{\max}$ and $\upsilon_{\max} = T_{\max}/4$;
- Crossover is not identified and thus T is chosen as the highest one possible and therefore $T = T_{\max}$[43] and $\upsilon_{\max} = T_{\max}/4$.

Method of point to point derivatives of rescaled range or DFA fluctuation[44] can be used as an additional detection tool (Bashan *et al.*, 2008). However, this method is rather intuitive and there are no statistical tools developed. Therefore, we use it only as an additional tool if the results based on V statistics are not clear.

After we estimate maximum scales and time series lengths for the methods, we start the examination with R/S analysis and therefore use maximum scale and time length proposed for R/S. Time-dependent Hurst exponent is constructed for the time series of length T with a sliding window procedure applied (e.g. Grech & Mazur, 2004; and Carbone, Castelli & Stanley, 2004). However, we do not use the sliding window to construct histograms followed by an interpretation (e.g. Bartalozzi *et al.*, 2007) as this procedure is hard to defend when tackled on theoretical basis (McCauley *et al.*, 2007 and McCauley, *et al.*, 2008). We present a new method instead as we use the confidence intervals constructed in Section 3.1, Section 3.2 and Section 3.3 and compare them to the estimates of the particular method. If the estimate of H based on R/S analysis is out of its confidence interval, the null hypothesis of the random walk, and thus F65 efficiency, is rejected. However, as R/S analysis can be biased by short-term memory[45], we compare the Hurst exponent estimates of M-R/S with its confidence intervals as well. If the long-term memory is rejected, there is short-term memory present. DFA methods are then used if the time series still shows long-term memory. If these methods reject long-term dependence, the series is strongly influenced by trends. In case that all methods imply that the time series is long-term dependent, we can reject the hypothesis of

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[42] T_{max} is a number of observations and therefore maximal length of examined time series.
[43] The time series length is set as a highest power of set basis possible and thus the closest one to T_{max}. Same procedure is applied for the previous case.
[44] Point to point derivative is a slope of a line connecting two neighboring points (rescaled ranges or DFA fluctuations in our case).
[45] See Section 2.1.2.

F65 efficiency[46]. It is crucial to use the methods in proposed sequence as DFA is also vulnerable to short-term memory in the underlying process as it overestimates H similarly to R/S (Morariu *et al.*, 2007).

Note that possibility of R/S analysis to be biased by short-term memory process is actually an advantage of the method since we have shown in Section 1.1 that H equal to 0.5 can mean either independent or short-term dependent process. Therefore, H based on R/S which is out of confidence intervals only suggests that the process is dependent since short-term memory overestimates H. If R/S analysis could not be biased by short-term memory process, it would be impossible to say whether an estimate of $H=0.5$ means independence or short-term dependence, it would only reject long-term memory of the process. However, the use of both R/S and M-R/S enables us to distinguish between the two types of memory. If both methods show significant dependence, the process is long-term dependent. If R/S analysis shows significant dependence and M-R/S analysis does not, the process is short-term dependent. If R/S analysis shows no significant dependence the process is independent[47] and thus F65 and S65 efficient.

As we have already described in Chapter 1, two types of efficiency are connected to two types of long-term memory. Persistent behavior implies rejection of both F65 and S65 efficiency as it implies both dependence and infinite variance. On the other hand, anti-persistent behavior indicates rejection of F65 efficiency only as the process is dependent with finite variance. To sum the possibilities of rejection of null hypothesis, all of significant anti-persistent, persistent, short-term dependent and trending behavior implies rejection of F65 efficiency. Additionally, significant persistence leads to rejection of S65 efficiency as well.

We have also shown that sudden changes in Hurst exponent values indicate change of dynamics of the process and thus can be connected with significant changes in the behavior of market participants (Section 2.3). Therefore, we focus on such patterns in our analysis as well. We apply the proposed method in the following chapter.

[46] We use the same maximum scales and time series lengths for all methods where possible as it makes interpretation much more straightforward.

[47] If we use R/S and M-R/S separately, we cannot arrive at unambiguous results as R/S analysis can only tell us that the time series is either not long-term dependent or not independent and M-R/S analysis can only tell us that either the time series is not long-term dependent or it is. However, the rejection of long-term dependence on the basis of M-R/S would still leave us with two very different options – independence or short-term dependence.

Chapter 4 Fractality of world stock markets

"Experience without theory is blind, but theory without experience is mere intellectual play."

Immanuel Kant

In this chapter, we connect the findings from previous chapters and apply our new method of time-dependent Hurst exponent with confidence intervals. We present the results of recent applied literature which takes similar approach to ours and focus on the weak and strong points of the used methods. The rest of the chapter shows the results of the applied method together with connections between dynamics of Hurst exponent and specific stock indices. The interconnection of the markets is presented on the basis of correlations between H. As the data set covers the financial crisis of 2008, we introduce the similar reaction of the indices to the crisis. Let us start with overview of recent research papers.

4.1 Recent literature

There have been a lot of applied research papers examining long term memory in the financial time series during last several years (Peters, 1994; Di Matteo, Aste & Dacorogna, 2005; Di Matteo, 2007; Czarnecki, Grech & Pamula, 2008; Grech & Mazur, 2004; Carbone, Castelli & Stanley, 2004; Matos *et al.*, 2008; Vandewalle, Ausloos & Boveroux, 1997; and Alvarez-Ramirez *et al.*, 2008). Unfortunately, almost each paper separately takes quite different approach to the problem. Let us discuss these different approaches.

Peters (1994) examines DJI30 between years 1888 and 1990 and finds a long-term memory process in the time series using R/S analysis while comparing the estimates of rescaled ranges with those of P94. As an alternative approach in the case of short-term memory presence, Peters suggests to use AR(1) filtering of original integrated data and again interprets the results on a basis of comparison with P94. Importantly, persistent behavior remains even after the filtration. However, the author does not provide any testing of AR(1)-filtered time series[48]. Moreover, the interpretation is based on graphical methods only. Therefore, the results must be accepted with caution. Nonetheless, the author provides very

[48] The complications of AR(1) filtering were discussed in Section 2.1.2.

deep analysis[49] of trends in the time series and comparison for different frequencies of returns (daily, weekly and monthly) where all support the finding of stable persistent behavior over the whole period with a cycle of four trading years.

Di Matteo, Aste & Dacorogna (2005) examine the whole spectrum of indices (together with exchange rates and rates of T-bills) and compare them with respect to their assumed efficiency[50] while using a method of generalized Hurst exponent (GHE)[51]. The authors come to conclusion that NASDAQ, SP500, NIKKEI, CAC40 and FTSE exhibited a slightly anti-persistent behavior during 1997-2001 compared to an independent behavior during 1990-1996. On the other hand, DAX and DJI30 showed independent behavior during both periods. WIG20 and BUX show significantly persistent behavior during 1990-1996 in contrast to an independent behavior of WIG20 and only slightly persistent behavior of BUX during latter period[52]. Unfortunately, the authors have not used time-dependent method which could have uncovered dynamics of the whole system. GHE can be said to be brand new and there is therefore a lot of possibilities for further research not only of time-dependent generalized Hurst exponent, but also of multi-fractal estimates of the method together with general tests and simulations.

One step closer to time-dependent generalized Hurst exponent was taken by Di Matteo (2007) who presents more detailed findings of past work of Di Matteo, Aste & Dacorogna (2005) mentioned in the above paragraph. Di Matteo (2007) uses following methodology. The examined period of 1990-2001 is divided into ten sub-periods and generalized Hurst exponent for each one is estimated. The procedure is used on NASDAQ, NIKKEI and WIG20[53] and yields quite interesting results. WIG20 and NIKKEI exhibited a behavior much closer to independence when compared to NASDAQ. WIG20 also showed evident decreasing trend of Hurst exponent from a persistent to an independent or even an anti-persistent behavior. Quite interestingly, the US index showed significantly persistent behavior between 1990 and 1991 and then shifted to significantly anti-persistent behavior. The most problematic issue of the used method is the fact that there is no theory or even empirical estimates of generalized Hurst exponent for random time series and therefore, the results are based on comparison with asymptotic estimate of independent time series – H equal to 0.5. However, we have already

[49] Both publications – Peters (1991b) and Peters (1994) – are strongly suggested for deeper understanding of R/S analysis.

[50] By assumed efficiency, we mean that developed markets are expected to be more efficient than developing markets and therefore showing independent behavior or at least a behavior only slightly dependent.

[51] Method of generalized Hurst exponent is based on a scaling of moments of returns of the time series. For more details, see also other works of Di Matteo and his colleagues (for example Di Matteo, 2007).

[52] Note that we only present the results for indices which are important and also examined in this thesis.

[53] Indonesian index JSX was examined as well but is not relevant for the thesis.

shown that the estimates for finite samples can be very different for R/S and M-R/S analysis and neither Di Matteo (2007) nor Di Matteo, Aste & Dacorogna (2005) provide any proof or simulations that would show that generalized Hurst exponent for finite samples is equal or even close to 0.5.

Czarnecki, Grech & Pamula (2008) use DFA to analyze WIG20 and estimate its time-dependent Hurst exponent during 04/1991-01/2007. Authors use the maximum time series length based on findings of Grech & Mazur (2004) which was already tackled in Section 2.2.2[54]. Consequently, estimates of time-dependent Hurst exponent are very volatile reaching values from 0.3 up to almost 1.0. Authors thus chose to use a simple moving average of last 21 estimates and comment on a relationship between the moving average and potential huge swings in the index values. Even though the authors find several conditions which are fulfilled before the most important crashes of WIG20 during examined period, these conditions are not the sufficient ones as there are huge swings in returns of WIG20 which are not preceded by these conditions.

Grech & Mazur (2004) based their examination on the same basis as already mentioned in paragraph above. Authors used DFA on DJI30 for period 1995-2003 and constructed time-dependent Hurst exponents. In contrast to Czarnecki, Grech & Pamula (2008), the authors used a moving average of 5 last estimates of time-dependent Hurst exponent and again try to find patterns in relation between the moving average and huge swings in DJI30. Similarly to previous authors, the correlation between significant decreases of time-dependent Hurst exponent can be connected with upcoming market crash. However, the authors admit that *"…correlation range is too short with respect to* [time series length][55] *and* [time-dependent Hurst exponent] *loses its sensitivity to detect* [potential crashes]*…"*[56] and eventually summarize that *"…the prediction of market signal evolution with the use of local* [Hurst] *exponent becomes difficult in the period 1995-2003."*[57]

Carbone, Castelli & Stanley (2004) used the time-dependent Hurst exponent approach while using detrending moving average (DMA) technique[58]. The authors examine DAX and German government bonds in the period 1996-2002 and show that DAX exhibits interesting dynamics. Unfortunately, authors only conclude with the statement that *"…a more complex*

[54] The maximum suggested time series length is based on DJI30 which is rather questionable to be used for WIG20 as authors did not provide any comparison for other indices.
[55] Terms in brackets are provided by the author of the thesis as the authors of the paper use different marking and terminology.
[56] Grech & Mazur (2004), pp. 142
[57] Grech & Mazur (2004), pp. 143
[58] DMA is based on detrending of the signal by moving averages of different lengths and examining the scaling of deviations from the moving averages (for reference, see Xu *et al.* (2005) and Arianos & Carbone (2007)).

evolution dynamics characterizes the financial returns compared to artificial time series having the same average value of the Hurst exponent."[59] If we examine the statement more closely, we arrive at the conclusion that it contradicts itself in the way that was very aggressively criticized by McCauley *et al.* (2007) – interpretation made on average values and standard deviations of local Hurst exponents based on sliding window approach. The statement says on one hand that the time series is characterized by a complex dynamics and on the other hand sums all the local Hurst exponents into the average one. Therefore, the whole dynamics is interpreted on the basis of one number only. However, there already is one number that characterizes the dynamics of the time series – a global Hurst exponent. Global Hurst exponent is estimated for maximum time series length possible. The estimation of global Hurst exponent for the time series has stronger theoretical background and is not as controversial as average Hurst exponent based on an average of time-dependent Hurst exponents.

Matos *et al.* (2008) used DFA method in the form of, as the authors call it, time and scale dependent Hurst exponent (TSH). This method not only examines the evolution of Hurst exponent in time but also with a change of scale used. The method gets rid of one problem that was mentioned when discussing other papers – optimal maximum scale used. The authors examined NIKKEI, GSTPSE (Canadian index), Bovespa (Brazilian index) and PSI-20 (Portuguese index). The results are then interpreted on a basis of contour plot which shows dependence between time, scale and Hurst exponent with different shades of grey each corresponding to specific value of H. NIKKEI shows results around 0.5 while other indices show shift from highly persistent behavior to almost independent one. The interesting part of using contour plots is the examination of stripes in it as some of them go through all scales in consideration and can be interpreted with respect to significant events such as the "DotCom" crash which is visible in NIKKEI contour plot or disorder of high degree in PSI-20. This method will surely be further examined and used as it has high potential.

Vandewalle, Ausloos & Boveroux (1997) applied DFA method as well but used it for the most important exchange rates between 1980 and 1996. Even though the exchange rates are not part of our research, the authors touch one very important issue in the estimation of Hurst exponent – crossovers[60]. The estimates were calculated after the detection of crossovers and the anti-persistent behavior of majority of European exchange rates was based on the

[59] Carbone, Castelli & Stanley (2004), pp. 269
[60] Crossovers are discussed in Section 3.4.

scaling behavior up to the point where a crossover emerged. We apply similar procedure to estimate the maximum scale and corresponding time series.

Alvarez-Ramirez *et al.* (2008) examined long-range dependence of long time series of DJI30 (1928-2007) and SP500 (1950-2007) with DFA and focused on the break in Hurst exponent trending which collides with the end of Bretton-Woods system in 1972. The authors used the linear trends in Hurst exponent behavior and tried to connect the breaks in the trends of the exponent with important events which had happened in the market. However, the authors used confidence intervals based on standard errors of coefficients of regression which is used for the estimation of Hurst exponent. Yet, the confidence intervals of this type are not anyhow connected with the random data and thus are of no use for hypothesis testing.

In the next section, we contribute to the literature and use R/S, M-R/S and DFA with approach described in Section 3.4 on a set of 13 stock indices which cover the indices of the Central Europe, the indices of the Western Europe, the indices of the USA and the indices of Japan and China. We follow with detailed description of the data set.

4.2 Data set

We examine daily logarithmic returns of various indexes which are summed up in Table 4-1. Altogether we obtained 10 years of daily prices of shown indices with exception of WIG20 and BUX which were obtained for shorter time period.

We divided indices according to their economic, geographic and political properties. The countries of the Central Europe have similar recent history as they have reformed from centrally planned into market economy which was connected with a recreation of financial markets[61]. Even though the countries have chosen quite different approaches to privatization, the initial problems with liquidity of the markets ware widespread as the region was considered risky by foreign investors and domestic markets were not able to provide enough liquidity (Egert & Kocenda, 2005). For the comparison, we present the most liquid markets of the Western Europe together with the indices of the USA. The Western Europe has experienced different recent history as the evolution of the market economy was not interrupted after the World War II. As the consequence, the situation of Western and Central Europe was very different during 1990s when the countries of Central and Eastern Europe just started their transition (Bordo, 2000). As for the USA, we present three indices – DJI30, SP500 and NASDAQ – as each one is specific[62]. Out of the Asian region, we present Japan

[61] Except of ATX, which is included in the group on the geographical rather than political and economic basis.
[62] We discuss the specifics in Section 4.4.2 which focuses on the US indices.

and China as the economy strongly connected to the developed countries and the economy which has started to be a strong competitor on the global markets, respectively.

Hence, the indices were divided into four categories – Central Europe (Czech PX, Hungarian BUX, Polish WIG20 and Austrian ATX), Western Europe (British FTSE, German DAX and French CAC40), the USA (DJI30, NASDAQ and SP500) and Asia (Japanese NIKKEI and Chinese SSEC). We expect that Central European indices experienced persistent behavior with decreasing trend towards independent behavior as they started their transition to the market economy only 20 years ago and have not caught up to the most developed countries yet. Austrian index is included into the Central Europe mainly because of its history connected with the whole region. The indices of Germany, France, the UK and the USA are expected to be the most efficient ones as their markets are very liquid for long period of time and therefore independent behavior is expected for the whole time series. For different US indices, we expect similar behavior and differences are most likely to be connected with the diversification of each index. DJI30 comprises of only 30 stocks, SP500 includes 500 stocks and NASDAQ contains close to 3000 stocks. NASDAQ is thus expected to show the smoothest behavior off all US indices and DJI30 the other way around. As for Asian indices, we expect NIKKEI to be the most efficient of the region as it is very closely connected to the US economy, thus showing an independent behavior. SSEC is conversely expected to show decreasing trend similar to the one of the Central European countries as the market of China has come through similar economic development to the markets of the Czech Republic, Slovakia, Hungary and Poland.

Table 4-1 Summary of examined stock indices

Country	Index	Start Date	End Date	T_{max}
Czech Republic	PX	21.1.1999	20.1.2009	2485
Hungary	BUX	26.7.2001	20.1.2009	1839
Poland	WIG20	1.10.2003	20.1.2009	1363
Slovakia	SAX	21.1.1999	20.1.2009	2331
Austria	ATX	21.1.1999	20.1.2009	2474
Germany	DAX	21.1.1999	20.1.2009	2535
UK	FTSE	21.1.1999	20.1.2009	2524
France	CAC40	21.1.1999	20.1.2009	2545
	DJI30	21.1.1999	20.1.2009	2538
USA	NASDAQ	21.1.1999	20.1.2009	2533
	SP500	21.1.1999	20.1.2009	2532
Japan	NIKKEI	21.1.1999	20.1.2009	2421
China	SSEC	1.3.1999	20.1.2009	2367

We provide the evolution of index values and logarithmic returns of all indices in Chart 0-2 – Chart 0-9 in Appendix. Central European indices behave quite differently when compared to each other. Index values of ATX, BUX and PX are very alike; on the other hand, SAX behaves diversely. WIG20 behaves similarly to ATX, BUX and PX; however, the comparison is quite complicated as the time series for WIG20 is much shorter. Nevertheless, all indices, with exception of SAX, show rapid growth up till years 2007 and 2008 where the trend ends and slowly changes into very volatile and downward trending times of the financial crisis of 2008. SAX, on the other hand, experiences its peak much earlier (03/2005) but is hit by the crisis in similar way. On the contrary, indices of the Western Europe show almost the same behavior with two peaks in 2000 and 2007 which are connected with strong decreasing and strong increasing trend with bottom in 2003. The crisis hits the markets with the similar magnitude. The indices of the USA are alike with the ones of the Western Europe. However, the trends are not so profound for DJI30 and SP500. Moreover, NASDAQ shows more visible peak in 2000 which is followed by much stronger decreasing trend reminding of "DotCom" bubble on the US markets (Cooper, Dimitrov & Rau, 2002). As for the Asian markets, NIKKEI is again very close to the Western European ones, whereas SSEC shows unique behavior with rapid increasing trend peaking at the break of years 2007 and 2008 which is followed by similarly rapid fall of the index values.

As we have shown in Section 2.1 and 2.2, R/S and M-R/S are methods constructed for stationary time series, while DFA is immune to non-stationarities and thus can be used for both stationary and non-stationary data sets. To check whether the time series are stationary, we use Augmented Dickey-Fuller (ADF) test with non-zero mean (Dickey & Fuller, 1979) and KPSS test (Kwiatkowski *et al.*, 1992). The null hypothesis of ADF is a unit root and thus non-stationarity of the time series. On the other hand, the null hypothesis of KPSS is stationarity of the time series against the alternative hypothesis of non-stationarity. The results for both tests with critical values of 5% confidence level are presented in Table 4-2. The rejection of null hypothesis for each test is marked by bold italics of the value.

Table 4-2 Results of ADF and KPSS for stock indices

	ATX	PX	BUX	WIG20	SAX	DAX	CAC40
ADF	***-28,9039***	***-29,8903***	***-25,6288***	***-21,8626***	***-28,1588***	***-30,1393***	***-32,1223***
5% critical value				-2,86			
KPSS	***0,7275***	0,4346	***0,711***	***0,6156***	0,3089	0,1551	0,2211
5% critical value				0,463			

	FTSE	DJI30	NASDAQ	SP500	NIKKEI	SSEC
ADF	***-33,5078***	***-29,8886***	***-29,9127***	***-30,0279***	***-29,2647***	***-26,3987***
5% critical value			-2,86			
KPSS	0,1189	0,1854	0,1393	0,2128	0,2105	0,4276
5% critical value			0,463			

The results of stationarity tests indicate that ATX, BUX and WIG20 are non-stationary and all other indices are stationary. Thus we expect that R/S and M-R/S might be biased for three mentioned indices and DFA methods might yield different results. Nevertheless, the stationarity does not disqualify the use of DFA as it is useful for both stationary and non-stationary time series[63]. Moreover, non-stationarity of the time series implies rejection of F65 efficiency as the process is not identically distributed and thus is not IID. Thus, even before we start long-term memory examination, we reject F65 efficiency for ATX, BUX and WIG20. Nevertheless, the indices can still be S65 efficient which will be checked in following sections.

The rest of the chapter is divided into three subchapters, each examining one of the above defined groups of indices, while the indices of Western Europe and the USA are discussed in one subchapter, and summed up in the last subchapter where the comparison of the groups is made.

4.3 Central European Economies

We start our analysis with Austrian index and we present the procedure which is applied for the rest of the indices as well. For each tested market, we present the results of crossover detection on basis of which we set maximum scale and the time series length. We follow with results of time-dependent Hurst exponent based on R/S and M-R/S to distinguish between long-term and short-term dependence. Potential trends influence is then checked with DFA. After the separation between independent and dependent periods, we look for periods with interesting patterns of connection between time-dependent Hurst exponent and behavior of ATX. After examination of all Central European indices, we compare the results and show correlations of Hurst exponents. Last subchapter presents charts of time-dependent Hurst exponents for better comparison. Let us now turn to ATX.

4.3.1 Austria

We start the analysis with the crossovers and potential cycles detection in the behavior of ATX. Chart 0-10 in Appendix sums the V statistics and point to point derivatives of H for R/S, M-R/S, DFA-0, DFA-1 and DFA-2. The results are straightforward. Non-detrending methods (R/S, M-R/S and DFA-0) show a crossover at the scale of approximately one trading year. Detrending methods (DFA-1 and DFA-2) show no crossover. Thus, crossover detection showed that there is one year cycle which can be modeled by polynomial fits of the first and

[63] See Section 3.2.

second order. The results are supported by behavior of point by point derivatives which support the proposed method of using one fourth of the time series length as the maximum scale used for Hurst exponent estimation as the values of the derivatives for the highest scales deviate from others as well as the one with scale equal to 8 trading days[64]. The implications are as follows – we use R/S analysis as a starting point, check for short-term memory bias with M-R/S and eventually use DFA-1 to check for non-stationarity bias. We use only DFA-1 as DFA-0 shows no added value in the sense of a removal of the crossover. DFA-2 does not add more information since DFA-1 already filtered the crossover away.

We follow with the analysis of time-dependent Hurst exponent. We present all results for different markets at the end of this chapter so that the markets can be better compared with each other (Chart 4-2 in Section 4.3.7). Based on R/S, ATX index showed persistent behavior up to 27.6.2006 with couple periods where the hypothesis of martingale behavior cannot be rejected. After 27.6.2006, there is no single period where S65 efficiency can be rejected. ATX further experienced increasing trend of Hurst exponent up till 14.10.2005. This behavior was followed by clear decreasing trend between 14.10.2005 and 1.12.2006 which shows obvious transition from periods of alternation between persistent and S65 efficient to clear S65 efficient behavior. After the decreasing trend, the index stabilizes around constant trend up till the end of the examination period.

However, there are two periods which show visible relationship between Hurst exponent and ATX index values. Both patterns are summed in Chart 4-7 in Section 4.3.7. The first one is present between 17.10.2005 and 3.11.2005 and shows a strong decreasing trend which is followed by a relatively stable period up to 24.11.2005 where an increasing trend starts and peaks at 20.2.2006. The period from the bottom to the top of Hurst exponent values is connected to a cumulative return of 22.06%. The second one is connected with slow decreasing trend of Hurst exponent from 29.7.2003 to 27.11.2003 where the trend turns into rather slow increasing trend up till 10.3.2004 where the trend peaks. The trend is again accompanied by high accumulated returns of 22.86%. Both of the mentioned patterns are quite similar and can be connected with increasing uncertainty of investors which breaks into strong following of the trend. Let us now turn to other estimation methods of Hurst exponent.

The persistent behavior showed by R/S can be due to short-term memory present in the process. To check the robustness of the results presented above, we use M-R/S. The periods of persistent and independent behavior are very similar. Moreover, the trends of Hurst

[64] We, therefore, propose the minimum scale of 16 days for methods used. The results of point to point derivatives and their implications are similar for other indices (see sections below).

exponent showed by R/S are present as well. Therefore, based on R/S and M-R/S, there were periods where ATX exhibited significantly persistent behavior. Let us now turn to the detrending method.

DFA-1 shows some different aspects. Let us start with the common ones. ATX showed increasing trend of Hurst exponent for all methods used. The increasing trend was followed by a decreasing one. However, that is where common features end. Differences are more prevalent and more interesting. The decreasing trend is approximately 350 trading days longer for DFA-1 and the trend for the last part of examined period - from 26.6.2008 on - is rapidly increasing which indicates the increase of predictability. There were even periods of significant persistent behavior. Note that DFA-1 clears the linear trending and therefore the market was persistent even when cleared from the potential trends. Therefore, the increase can be interpreted as a strong belief in negative returns on the market which were already driving the market.

The Austrian index was persistent up to 14.10.2005 which is connected with an end of exponential increase in a value of the index. After this period, market was gradually switching into random behavior which is shown by decreasing trend of Hurst exponent. This trend was broken and reversed by an arrival of the financial crisis on the global markets. Significant increase of Hurst exponent starting on 26.6.2008 shows that mood on the market[65] was becoming more positive about the widespreadness of the crisis and its negative effect on the stocks prices. Moreover, the break which preceded the significant losses can be interpreted by strong change of the mood. Further, the decreasing trend of Hurst exponent can be explained as increasing nervousness of the investors. Therefore, the increasing uneasiness on the market reached its highest point and was turned into ride on the negative prospects of the crisis. Dynamics of Hurst exponent during financial crisis is showed in Chart 4-7 in Section 4.3.7. The trend of the exponent is quite similar to the ones already mentioned – slow and long lasting decreasing trend is present from 8.2.2008 up to 29.5.2008 where strong decreasing trend begins. Hurst exponent shows 13 consecutive decreases and hits the bottom at 16.6.2008. Strong increasing trend of the exponent follows, then, and peaks at 26.11.2008. The trend is connected to huge losses of 87.36%. The dynamics can be interpreted as an increasing nervousness on the market even though the index was growing. The starting decrease of the index value is followed by a downward slide of investors' mood and hits the

[65] Hurst exponent can be also interpreted as a measure of mood on the market. The lower the exponent the more nervous the investors are and other way around (Lux, 2007).

bottom at already mentioned date of 16.6.2008. Investors were then convinced that the decrease is about to endure which was actually true.

To sum the findings up, ATX experienced significant persistent periods, which imply rejection of S65 efficiency. The persistent periods were gradually becoming less significant and turned into market of a martingale and thus S65 efficient. This kind of behavior is expected for the Central European indices. Note that ATX has been shown to be F65 inefficient as it is non-stationary. However, we need to examine the other indices of the region before we make strong conclusions about appropriateness of inclusion of ATX in the Central Europe. More important is the fact that the financial crisis of 2008 had significant impact on decreasing randomness of the market. Moreover, the break from nervous to “riding on a trend” mood is obvious before the full burst of the financial crisis.

4.3.2 Czech Republic

We follow our analysis with PX index. Let us first check the crossovers and potential cycles in the behavior of PX. Chart 0-11 in Appendix sums the results for R/S, M-R/S, DFA-0, DFA-1 and DFA-2. These are quite similar to the ones of ATX – non-detrending methods reveal a crossover at the scale of half a trading year and detrending methods do not show any significant crossovers and imply that detrended time series scale through the whole examined period. Similarly to ATX, we can state that there is a half trading year cycle which can be modeled by simple polynomial fits. Note that DFA-1 and DFA-2 again show big potential bias for high scales as the point to point derivatives of H jump significantly. Implications are the same as for ATX and thus R/S, M-R/S and DFA-1 are used as DFA-1 already cleared all potential crossovers and hence DFA-2 would overestimate Hurst exponent.

Let us now present the results of time-dependent Hurst exponent which are shown in Chart 4-3 in Section 4.3.7. We start with the estimates of R/S and M-R/S. Similarly to ATX, there are periods when PX exhibits persistent behavior. However, these periods are strongly present only between 2.2.2001 and 4.3.2002; other persistent periods, on the other hand, last for couple days only. Nonetheless, the comparison between R/S and M-R/S does not show any significant differences and therefore, we can again conclude that short-term memory processes are not biasing Hurst exponent estimations. Moreover, both methods show similar trends in Hurst exponent behavior. There is a decreasing trend from significantly persistent values starting at the beginning of the examination period and taking place up till 12.11.2003. This trend is followed by slightly increasing trend ending around 6.1.2006 which is in turn followed by either a slowly decreasing trend (R/S) or a constant one (M-R/S). Quite importantly, none of the trends is strong and therefore there can be no strong conclusions made. Nonetheless, PX index follows a trend to more efficient behavior in the long-term according to both R/S and M-R/S. We need to stress that comparison between R/S and M-R/S shows that long-term memory behavior is not caused by short-term process bias.

DFA-1 shows very similar behavior as R/S and M-R/S. Trends of time-dependent Hurst exponent are similar with stronger increasing trend in the second part of the examined period. Moreover, Hurst exponent is more volatile during the examined period and shows that even DFA-1 can be overestimating the exponent. Nonetheless, the periods of significant persistent behavior are again very alike. If the whole time period was divided into more sub-periods, there would be quite significant increasing trend starting at 24.6.2008 which is connected to the start of the financial crisis. Note that the date is very close to the turning point of ATX which was 26.6.2008 and therefore we can expect similar turning points for other indices.

However, not only the end of the time series shows possible patterns. The first breaking point in all three time series (12.11.2003) is connected with the start of very rapid increasing trend. The end of the decreasing trend of Hurst exponent (6.1.2006) is on the other hand very close to market turning point which was followed by significant negative trend. Therefore, if we put the results together, turning points of trends of time-dependent Hurst exponent are connected to significant events on the market. Thus, the decreasing trend suggests that something is about to happen and when the trend reverses to the increasing one, investors ride on the sentiment of the market.

Even though R/S analysis does not show any interesting patterns, DFA-1 does. One is present between 14.12.2001 and 24.6.2002. The dynamics starts with long lasting decreasing trend which hits bottom at 23.4.2002 and reverses to an increasing trend. Even though the trends are not connected to any significant gains or losses, the decreasing one shows an increasing negative mood in the market or it simply indicates that something significant is about to happen. Note that the increase of Hurst exponent is very significant as it starts at a value of 0.77 and bottoms at 0.49.

Other pattern covers behavior of Hurst exponent during financial crisis of 2008. There are two very strong trends of Hurst exponent which create V shape between 17.6.2008 and 22.8.2008 with a bottom at 14.7.2008. However, these significant trends are not connected to any strong losses or gains. Nonetheless, the immediately following and not such strong trends between 22.8.2008 and 24.11.2008 bottoming at 9.10.2008 are linked with a significant loss of 59.80% of PX index value. Therefore, the whole period shows that V shape of Hurst exponent needn't be joined by significant events on the market. Nevertheless, the erratic behavior of the exponent shows that investors were rather unsure about the situation on the market. Both patterns are shown in Chart 4-8 in Section 4.3.7. Let us proceed with BUX.

4.3.3 Hungary

We proceed with analysis of Hungarian index. Similarly to the previous cases, we begin with a crossover detection which is summed in Chart 0-12 in Appendix. The results are different in the way that all the methods show no crossover and therefore the statement that BUX scales indefinitely cannot be rejected for an examined period. The infinite scaling suggests no cycles in the time series. Therefore, suggested maximum time series length is the highest possible for the data set – 1024 – corresponding with maximum scale of 256 for all used methods. The result of crossover detection has impact on the used methods as well – R/S, M-R/S and DFA-1. DFA-1 is used rather than DFA-0 because the latter method is not detrending and we want to use at least one method which is robust against non-stationarities as BUX is not stationary and DFA-2 shows no added value again.

Let us turn to time-dependent Hurst exponent estimation which is presented in Chart 4-4 in Section 4.3.7 for all used methods. We start with R/S analysis which shows quite surprising results of an independent behavior for the whole examined period. As no long-term memory process is proposed by rescaled range method, there is no need to use M-R/S since the interpretation of results could be quite misleading. Let us now turn to DFA-1 as BUX has been shown to be non-stationary.

Even though the independence of BUX cannot be rejected at any period, there is still interesting dynamics of Hurst exponent present. The index shows, with respect to R/S, long-term decreasing trend of its Hurst exponent from the beginning of the examination period up till 1.10.2008 where the switch to an increasing trend occurs. Additionally, DFA-1 shows even stronger decreasing trend. However, the trend ends much earlier – at 25.10.2007 which equals almost a year difference. The dynamics become quite straightforward from this point on and linear trends are not presented as a consequence. Nonetheless, visible increasing trend starts on 25.10.2007 and ends on 8.4.2008 where another reversal occurs and changes to even stronger decreasing trend which lasts for 51 trading days up till 23.6.2008 where the evolution stabilizes. However, the calm period lasted for 70 trading days after which a rapid increase of Hurst exponent starts on 30.9.2008 and follows up till the end of the examination period.

Therefore, both used methods show a shift of behavior at a break of September and October 2008 which is most probably connected to the financial crisis of 2008. However, this turning point is present significantly later when compared to ATX and PX. Nevertheless, the end of significant decrease of Hurst exponent, which ends at 23.6.2008, is very close to the shift of two mentioned indices. These results imply that not only the shift from decreasing to increasing trend of Hurst exponent (or vice versa) but even shift from strong trend to stable behavior of the exponent which is later followed by reverse trend can indicate important change in investors' behavior. The situation can be interpreted as uncertainty about market behavior which occurred between June and September 2008 where the investors were not decisive whether the financial turbulences were about to last. On the break of September and October 2008, the financial crisis became widespread and the significant increase of Hurst exponent was connected with huge losses of the index.

Let us now show significant dynamics and patterns in more detail which are summed in Chart 4-9 in Section 4.3.7. The pattern between 10.5.2006 and 8.6.2006 starts with a strong increasing trend of Hurst exponent up to 17.6.2005 where it switches to strongly decreasing trend. The whole period is connected with a cumulative loss of 21.75%, the decreasing trend of the exponent with a loss of 15.78%. Even though this pattern is quite against the logics of previous patterns, it can be explained if we take a broader look at the dynamics of the downturn. Hurst exponent reaches its local maximum at 4.5.2006, an observation earlier than the peak of BUX index. Four consecutive decreases of the exponent follow and bottom at 10.5.2006 where the already mentioned dynamics starts. There is also strong increasing trend after 8.6.2006 which ends at 20.6.2006 and is connected to an additional cumulative loss of 10.22%. The peak of Hurst exponent is quite well synchronized with the peak of BUX index while being present two observations later. Nonetheless, the whole dynamics show that there was a clear trend behavior from 10.5.2006 to 17.5.2006 where the mood switched to rather uncertain one which was still connected with losses which were, however, not so significant and there was rather stable period between 19.5.2006 and 6.6.2006 connected with a small loss of 3.95%.

Next interesting behavior is connected with the financial crisis of 2008. There is long and stable period between 6.6.2008 and 6.10.2008 where the behavior of BUX is very close to an independent one. Very strong trend starts at 6.10.2008 and peaks at 28.10.2008 which is connected to a huge cumulative loss of 48.49% of BUX index. The trend stabilizes for several periods and starts again at 7.11.2008 with a peak at 19.11.2008. The second trend is connected with additional cumulative loss of 12.20% of the index value. The interpretation of the behavior is quite logical as the increasing trend is connected with investors' belief that current trend will continue. This belief slowed for short period and occurred again.

Even though BUX has been shown to be independent, the efficiency of F65 type is still rejected as the time series is non-stationary. Let us now turn to Polish index which was already shown to be non-stationary as well.

4.3.4 Poland

Analysis of Polish index follows. Standard crossover detection is presented in Chart 0-13 in Appendix. WIG20 shows two new situations which were not present in any of above examined indices. DFA-1 method is inconclusive as we can see changing behavior of *V* statistic for all scales between 8 and 128 days which implies that this method would not yield conclusive results and therefore, we omit it for the analysis of WIG20. Further, DFA-0 method shows a crossover at lower scale than R/S and M-R/S, which implies that the method should be used with a length of a time series only equal to 256. However, this method would be rather questionable while estimating *H* on the basis of three fluctuation statistics only. Therefore, DFA-0 is not suggested to be used at all for daily data of WIG20. Additionally, all remaining methods (R/S, M-R/S and DFA-2) show the same crossover at a scale of 128 days. Therefore, it is probable that the time series move in a cycle of a length of approximately half a trading year. However, this cycle is not covered by polynomial detrending methods we use and thus it suggests a cycle could be modeled by either higher polynomial fit or different detrending technique (Kantelhardt, 2008) or simply the series scales finitely up to the scale of 128 days. That fact can easily be a reason why *V* statistic of both DFA-0 and DFA-1 yields such inconclusive results.

The results for time-dependent Hurst exponent are illustrated in Chart 4-5 in Section 4.3.7. R/S analysis shows WIG20 to be behaving as a random process for the whole examined period. We use DFA-2 method as WIG20 has been shown to be non-stationary and thus the results of R/S might be biased[66]. The results as of independence of returns are quite similar. Even though there are two periods where *H* moves out of confidence intervals, these periods are not long and the movement outside is not significant. Nevertheless, WIG20 is non-stationary based on KPSS test (Table 4-2) which implies rejection of F65 efficiency.

There are again several breaks in the dynamics of WIG20. R/S shows four visible trends – the first starting at the beginning of the examination period and ending at 10.10.2006, the second from 11.10.2006 to 6.6.2007, the third starting at 7.6.2007 and switching to the last one at 11.3.2008. The most obvious common feature with other already discussed markets is the presence of strongly increasing trend of Hurst exponent during 2008. However, this trend

[66] M-R/S was not used as R/S showed no long-term memory in the time series (similarly as in the case of BUX).

starts more than three months earlier when compared to the markets examined above. DFA-2 shows even earlier start of the trend at 20.12.2007.

However, the dynamics of WIG20 is much more complex than the one of ATX, PX and BUX as it shows very strong trends of a short sequence together with many significant downward jumps of Hurst exponent which have usually preceded the mentioned trends (patterns are summed in Chart 4-10 in Section 4.3.7).

The first of such patterns was present between 29.3.2005, 19.4.2005 and 5.10.2005 where the dates represent start of significant decrease, the bottom of the trend and starting point of increasing trend, respectively. The starting and ending dates collide with the beginning and end of strong increasing trend of WIG20 values. This implies that the trend started as the shift of dynamics of the market (downward jump of Hurst exponent) and was followed up by increasing belief of the investors (increasing trend of Hurst exponent). Note that the strong increasing trend ceases several observations earlier than the trend of the index itself.

The second pattern starts right after the first one and the important dates are 6.10.2005, 28.10.2005 and 2.12.2005 with the same sequence of events as of the previous pattern. Again, starting and ending dates agree to the dates of beginning and ending trend. Yet, the trend is decreasing this time which again implies that the significant shifts (specifically downwards jumps) of Hurst exponent are connected to significant changes of behavior of the market. Furthermore, the subsequent increase of the exponent suggests that the change of dynamics of the market is about to last.

The third pattern is again connected to strong trend and crucial dates are 13.4.2006, 5.5.2006 and 2.6.2006 or 8.6.2006. The last one is not quite obvious as the most rapid part of the trend ends at 2.6.2006 but the growing trend of Hurst exponent is present up till 8.6.2006.

The last one of the similar patterns is connected to the current financial crisis. We can see that between 3.3.2008 and 11.3.2008, there is a downward jump of Hurst exponent which is followed by an increasing trend which becomes very strong after 31.3.2008. Note that the period between 3.3.2008 and 10.4.2008 is connected with a cumulative loss of 38% of the value of the index and is the most dramatic period of the financial crisis for WIG20 index. Let us now turn to the index of Slovakia.

4.3.5 Slovakia

We follow with the final index of the Central Europe – the one of Slovakia. Standard detection of crossovers is summed in Chart 0-14 in Appendix. SAX is the first index that shows strong and same crossover for all methods used – 512 trading days. However, this result does not necessarily mean that there is a cycle of the length of 2 years. The time series can either scale up to this finite period or can be cyclic in the way that cannot be filtered by simple polynomial trends or finally, the time series can scale infinitely as the estimates of V statistics for the last two scales can be biased[67]. This result of crossovers detection suggests three methods to be used – R/S, M-R/S and DFA-2[68]. As for the analysis of time-dependent Hurst exponent, we propose maximum scale of 256 trading days as if we use the maximum scale available – 512 – we would obtain estimates of time-dependent Hurst exponent of approximately 300 trading days only. Such a choice would not allow us to evaluate an evolution of Hurst exponent in longer period which is needed for assessment of the changes in efficiency of SAX.

Let us move to an analysis of time-dependent Hurst exponent (Chart 4-6 in Section 4.3.7). R/S analysis yields interesting results not seen yet – Hurst exponent follows an increasing trend up till exactly 14.3.2005, which is the peak value of SAX, and then stabilizes above confidence interval implying the persistence of the time series. We check the possibility of short-term memory bias with M-R/S. The result is very similar as for the general behavior of the exponent as an increasing trend of H up till 14.3.2005 and stabilization at persistent behavior with some periods where the independence cannot be rejected are present as well. However, very disturbing pattern starts at 8.1.2007 where Hurst exponent jumps down by approximately 0.08 compared to the previous one and this jump repeats every 16 trading days. When we take into consideration how a modified rescaled range is constructed, the problematic period can be detected. Firstly, 16 trading days is the lowest used scale and therefore, there is a bias at this range, most probably. Secondly, R/S analysis does not show this pattern and therefore, it is rather a problem of modified standard deviation estimator, more specifically, a problem of auto-covariance estimator. Thirdly, the pattern begins to occur at 8.1.2007. If we sum all the information together, it implies that a period between 13.12.2006 and 8.1.2007 is the one that causes problems. If we check this period, it contains seven consecutive observations with zero return and thus non-trading days. These returns not only significantly lower a variance estimator (which is the case for R/S as well) but also yield significantly negative first and second auto-covariance for the period which strongly biases the standard deviation estimator for M-R/S. This effect strongly overestimates rescaled range

[67] See Sections 2.2.2 and 3.3 for detailed discussion.

[68] As the proposed maximum scale is the maximum one available, we choose DFA2 rather than DFA1 as DFA2 does not show any bias based on crossover analysis. Moreover, there can be a crossover at the highest scales which can hardly be detected by crossover analysis and therefore, DFA2 is a safer procedure to use.

for the period which in turn biases average rescaled range for the scale of 16 trading days. The eventual consequence is a leverage effect on *H* which is strongly biased downwards.

To solve the problem, we check the results of M-R/S with higher minimum range of 32 trading days. Note that the estimation procedure changes and thus we cannot use the confidence intervals that were used up to this point. Therefore, we have simulated M-R/S procedure with only difference being a minimum scale of 32 trading days, other parameters remained unchanged. The resulting confidence intervals are much wider as a standard deviation of simulated *H* is approximately 1.4 times higher than the one based on minimum scale of 16 trading days. With the higher minimum scale, the repeating pattern of significant downward jumps disappeared. Nonetheless, the persistent behavior remains and therefore is not caused by short-term memory process. However, we still need to check whether the estimates based on R/S are not biased by the presence of trends in the time series.

The results are supported by the estimates of DFA-2. Therefore, the significant persistent behavior for a majority of the time series length is confirmed and efficiency of SAX is rejected for both F65 and S65 for the majority of the examined period. Nonetheless, common feature of the increasing trend of Hurst exponent during year of 2008 is present and shows that current turbulences in the financial markets affected even SAX, which shows very little common features with other Central European indices otherwise.

When we concentrate on patterns in the behavior of Hurst exponent, which are summed in Chart 4-11 in Section 4.3.7), we can identify several of them. The one that was already mentioned is connected with rapid increasing trend of SAX index. Note that DFA-2 does not show such an obvious pattern. Nonetheless, there is similar pattern of DFA-2 as observed before – significant decrease followed by significant increase of Hurst exponent connected with significant trends of levels of the index. The decrease starts at 4.11.2004[69] and hits bottom at 21.1.2005. The increasing trend then reaches its peak at 14.3.2005. The period between 21.1.2005 and 14.3.2005 is connected with cumulated return of 41.70%.

Quite similar pattern occurred between 18.3.2008 and 6.8.2008 with the bottom value of Hurst exponent at 13.5.2008. However, there are several differences. The increasing trend does not start right after the bottom is hit and the exponent remains rather constant around a value of 0.56. The trend starts to be more profound after 25.6.2008. Nonetheless, this trend is not connected with any significant cumulative returns. It is needed to be noted, though, that the examined period contains a lot of zero returns and is therefore a period of very shallow

[69] We can mark several different observations as the ones of the beginning of the trend. Nonetheless, it is the peak or bottom which is the most important (Alvarez-Ramirez *et al.*, 2008).

market where there is little liquidity present. Therefore, the estimates can be biased by this fact and hence this result does not necessarily contradict previous results. As a consequence, we can conclude that time-dependent Hurst exponent is connected with significant trends of the market as long as the market is liquid.

Examination of SAX has, thus, uncovered very important phenomenon which can strongly bias estimates of Hurst exponent and can yield very inconclusive results – liquidity. If the market is not liqiud, it can strongly bias both R/S and M-R/S. In the case of DFA, the estimation of trends in sub-periods can be easily biased as well as consecutive zero returns pull the estimation to a constant trend. Therefore, liquidity of the market is crucial for Hurst exponent analysis which is in hand with fractal markets hypothesis presented in Section 1.2.4. We follow with comparison of all examined Central European indices.

4.3.6 Comparison of Central European indices

We have examined the Central European indices and came to several interesting results. We have showed that Austrian index ATX behaves quite similarly to PX and less efficiently than WIG20 and BUX and therefore belongs in the category. Generally, all indices showed quite different behavior and thus it seems that there can be no strong conclusions made about Central European indices as a group. Nonetheless, we have estimated the correlations of time-dependent Hurst exponents, which were based on R/S analysis, for the indices (summed in Table 4-3). We estimated correlations of Hurst exponent constructed with the same maximum scale of 128 trading days as crossover analysis has not rejected this scale for any index. Moreover, days which are not trading days for all the indices were omitted to avoid bias. For the magnitude of correlation, we use a proposal of Cohen (1988) – absolute value of correlation from 0.1 to 0.3, from 0.3 to 0.5 and from 0.5 to 1.0 stand for weak, medium and strong correlation, respectively. We use such method for both negative and positive correlation.

Table 4-3 Correlations of time-dependent *H* (R/S) of Central European indices

	PX	*BUX*	*WIG*	*SAX*	*ATX*
PX	1				
BUX	0,16076	1			
WIG	0,358652	0,487661	1		
SAX	0,393539	0,211099	0,451508	1	
ATX	0,505671	0,173199	0,532348	0,431943	1

ATX is strongly positively correlated with PX and WIG20 and medium with SAX. Therefore, with its behavior, it obviously belongs to the Central European area. Additionally, WIG20 and BUX are almost strongly correlated. Most interesting result from the point of correlations is the fact that it is rather BUX than SAX which seems to be most atypical for the area of the Central Europe. Nevertheless, all the indices are positively correlated.

For comparison, we present the correlations for Hurst exponents based on both DFA-1 (Table 4-4) and DFA-2 (Table 4-5). The results are rather different for each method. However, there are several assertions which were further supported. BUX and WIG20 are strongly positively correlated (based on DFA-2). ATX still shows positive correlations with all indices (DFA-2 shows stronger correlations). On the other hand, other results must be taken with caution as they are method dependent.

Table 4-4 Correlations of time-dependent *H* (DFA-1) for Central European indices

	PX	*BUX*	*WIG*	*SAX*	*ATX*
PX	1				
BUX	0,082294	1			
WIG	0,200232	0,168306	1		
SAX	0,574923	0,098572	0,073722	1	
ATX	0,399175	0,260982	0,047427	0,267846	1

Table 4-5 Correlations of time-dependent *H* (DFA-2) for Central European indices

	PX	*BUX*	*WIG*	*SAX*	*ATX*
PX	1				
BUX	0,332123	1			
WIG	0,375363	0,559834	1		
SAX	0,234459	-0,07377	0,313052	1	
ATX	0,404997	0,274621	0,508811	0,216092	1

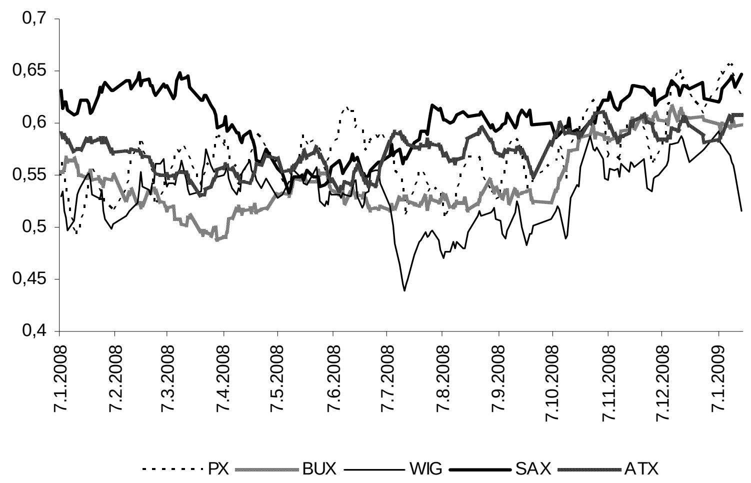
Moreover, all indices have shown similar patterns of Hurst exponent and index values. There have been two types of patterns uncovered. For the first one, a strong decrease of *H* followed by an increasing trend of *H* is connected with significant change in the evolution of the index (both negative and positive). The second one shows that long lasting decreasing trend of *H* again followed by rapid increase indicates that investors follow the commenced trend. Such findings are in hand with findings of Grech & Mazur (2004) and Czarnecki *et al.* (2008).

Further, as all indices have been hit by the financial crisis of 2008 and have shown patterns during the same period, we examine the period more thoroughly. The evolution of time-dependent Hurst exponent for all five indices based on DFA-2[70] is presented in Chart 4-1. The strong increasing trend of Hurst exponent starts at the beginning of October 2008

[70] DFA-2 is used because it was not rejected by crossover detection for any index and is resistant to non-stationarities which were shown to be present in majority of the Central European indices (ATX, BUX and WIG20).

which was connected to the start of the most significant losses of the indices in the sense of values. Moreover, the comparison again shows how SAX behaves differently compared to the other indices of the Central Europe.

Chart 4-1 Time-dependent *H* (DFA-2) for Central European indices during crisis of 2008

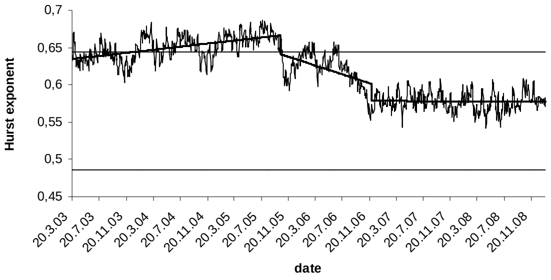


Last but not least is the fact that all indices have been inefficient during at least some periods. The most efficient market of the region is PX which moved from significantly inefficient to efficient in 2007 and is both S65 and F65 efficient for the rest of the examined period. However, PX still remains very close to upper confidence interval dividing persistent and efficient market. BUX and WIG20 have both shown S65 efficiency for the whole examination period. Nevertheless, they remain F65 inefficient due to non-stationarity. ATX is similar to BUX and WIG20 due to its non-stationarity and thus F65 inefficiency. Further, it is alike with PX as it experienced a shift from inefficient to efficient market in a sense of S65 in 2006. The last in the sense of efficiency is SAX which is persistent for the majority of the examined period and thus both F65 and S65 inefficient.

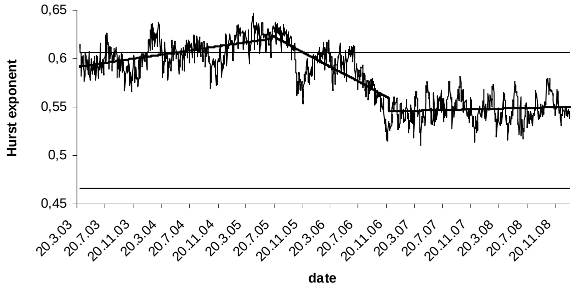
4.3.7 Charts for Central European indices

Chart 4-2 ATX time-dependent Hurst exponent

(a)



(b)



(c)

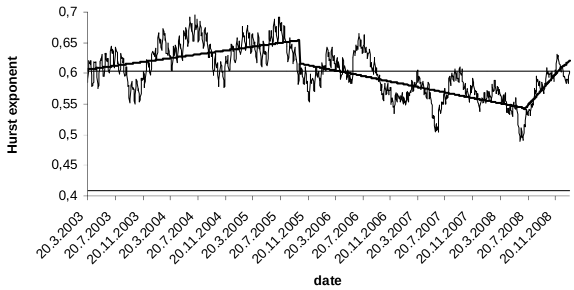
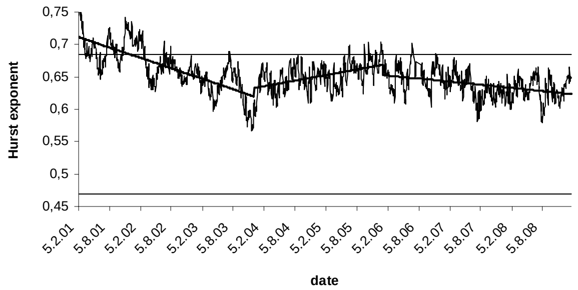


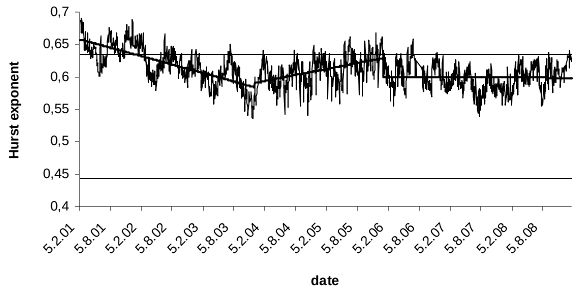
Chart 4-2: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on M-R/S, (c) Time-dependent *H* based on DFA-1: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-3 PX time-dependent Hurst exponent

(a)



(b)



(c)

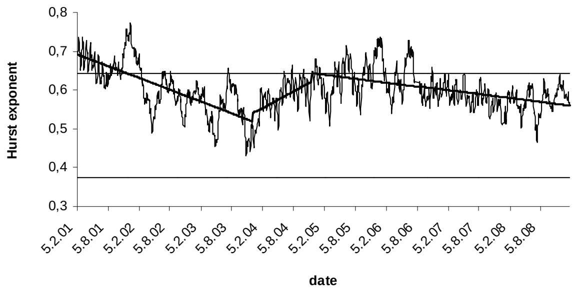


Chart 4-3: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on M-R/S, (c) Time-dependent *H* based on DFA-1: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-4 BUX time-dependent Hurst exponent

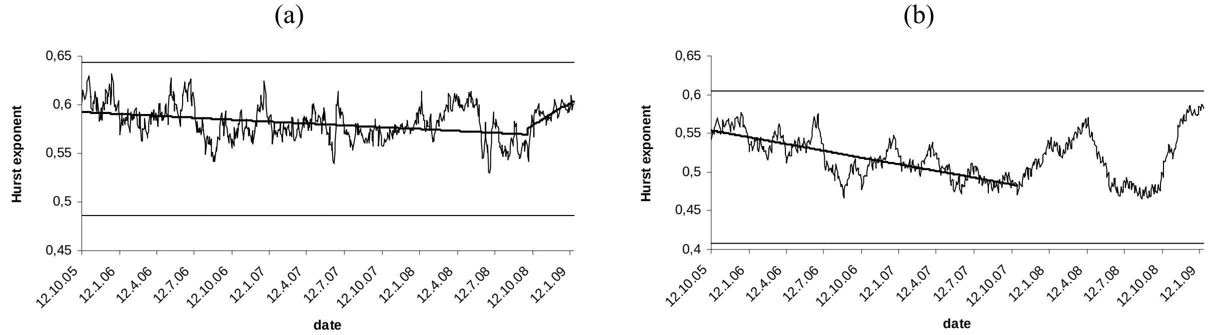


Chart 4-4: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on DFA-1: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-5 WIG20 time-dependent Hurst exponent

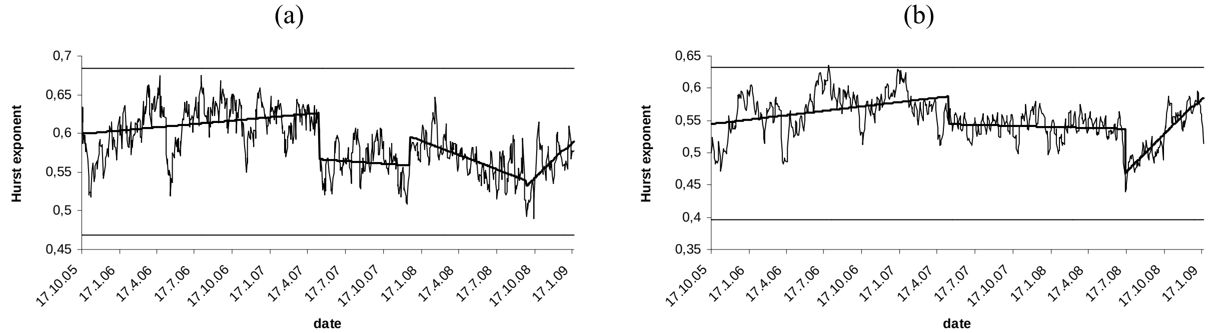


Chart 4-5: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on DFA-2: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-6 SAX time-dependent Hurst exponent

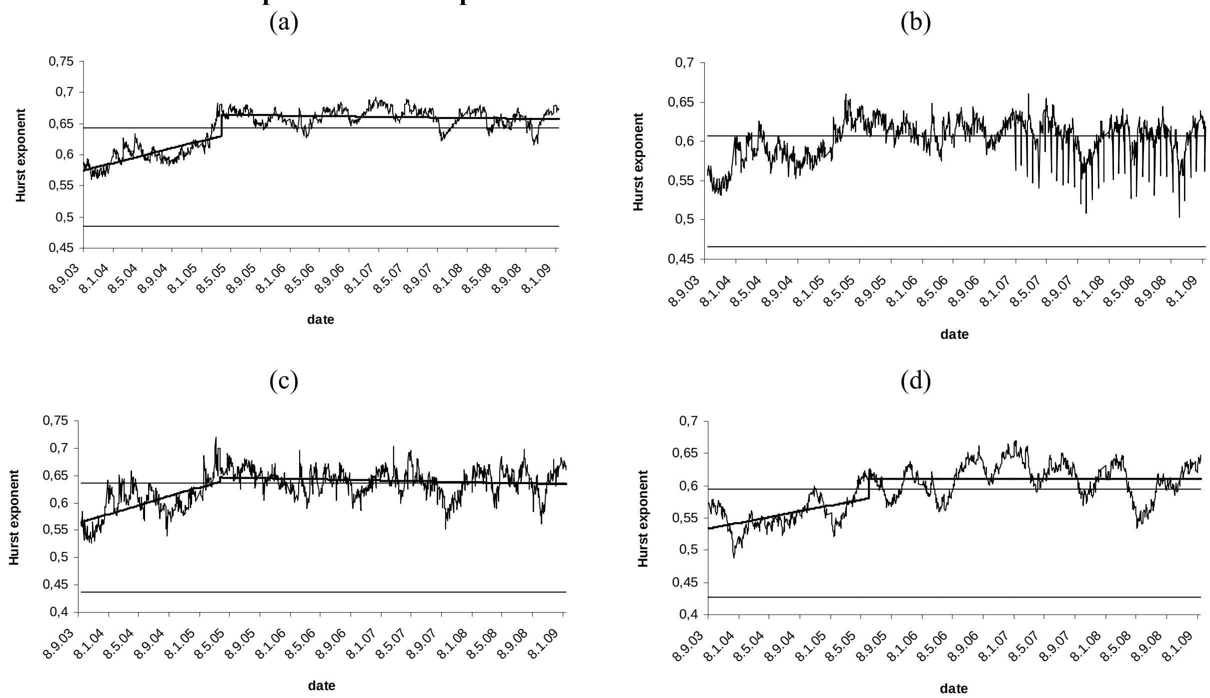
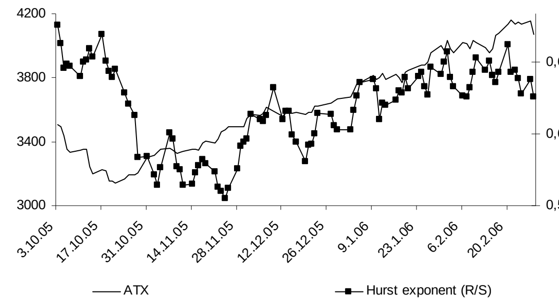


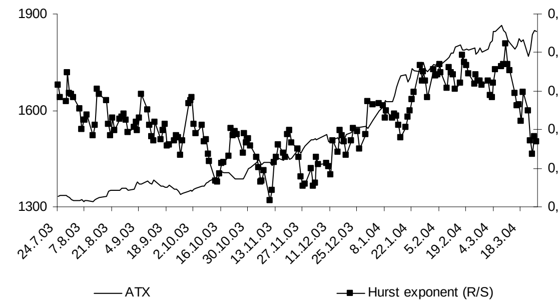
Chart 4-6: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on M-R/S, (c) Time-dependent *H* based on M-R/S with minimum scale of 32 trading days, (d) Time-dependent *H* based on DFA-2: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-7 ATX relationship between time-dependent Hurst exponent and index values

(a)



(b)



(c)

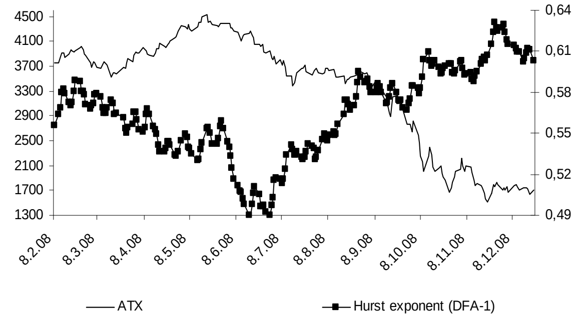
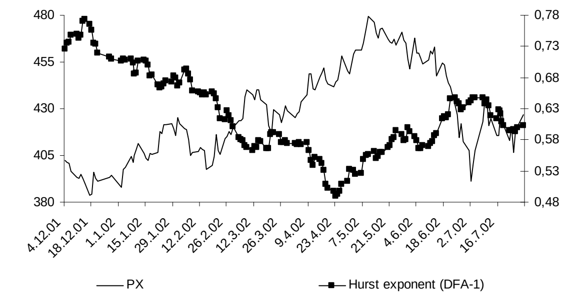


Chart 4-7: (a) *H* based on R/S, (b) *H* based on R/S, (c) *H* based on DFA-1: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in ATX values (left y-axis).

Chart 4-8 PX relationship between time-dependent Hurst exponent and index values

(a)



(b)

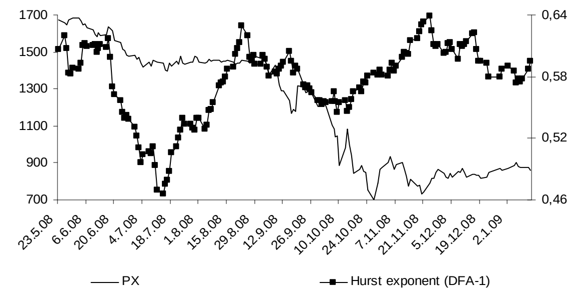
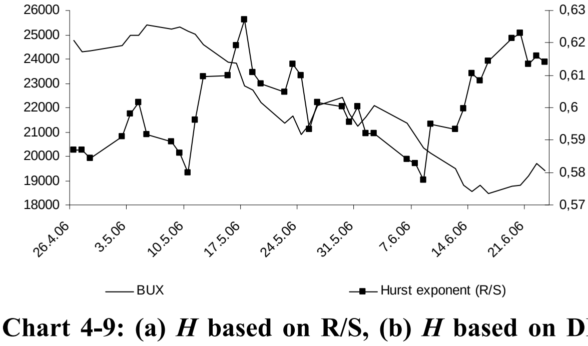


Chart 4-8: (a) *H* based on DFA-1, (b) *H* based on DFA-1 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in PX values (left y-axis).

Chart 4-9 BUX relationship between time-dependent Hurst exponent and index values

(a)



(b)

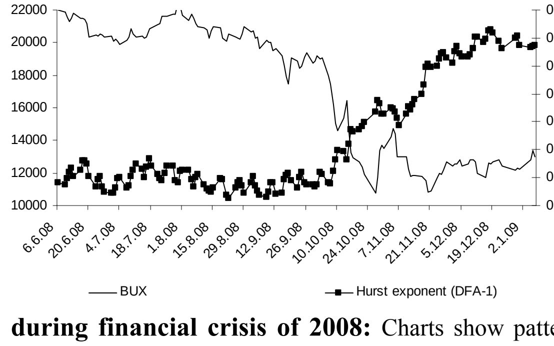


Chart 4-9: (a) *H* based on R/S, (b) *H* based on DFA-1 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in BUX values (left y-axis).

Chart 4-10 WIG20 relationship between time-dependent Hurst exponent and index values

(a) (b)

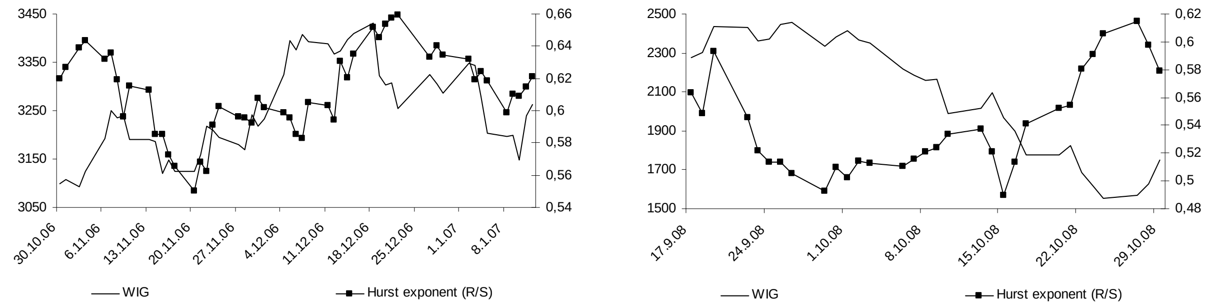


Chart 4-10: (a) *H* based on R/S, (b) *H* based on R/S during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in WIG20 values (left y-axis).

Chart 4-11 SAX relationship between time-dependent Hurst exponent and index values

(a) (b)

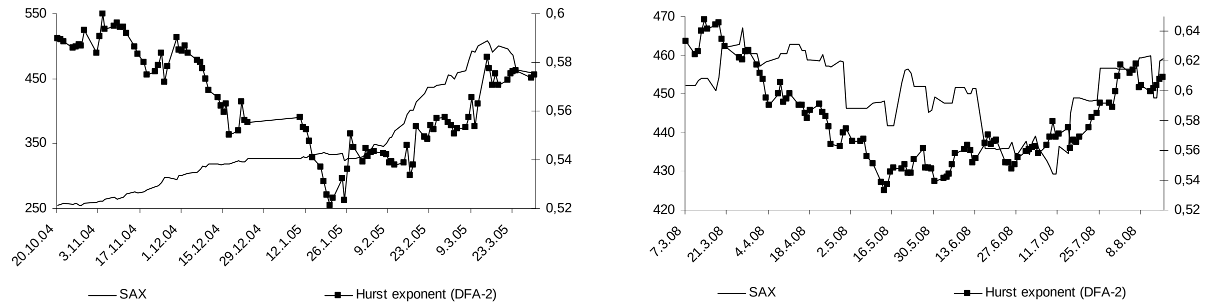


Chart 4-11: (a) *H* based on DFA-2, (b) *H* based on DFA-2 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in SAX values (left y-axis).

4.4 Western Economies

We follow our analysis with the indices of the Western Europe, which we analyse first, and the USA, which is presented further. The charts for each section are included after the analysis (Section 4.4.2 and Section 4.4.4, respectively). The last subchapter sums and compares the findings.

4.4.1 Western Europe

We present the procedure for the economies of the Western Europe in different way than the ones of the Central Europe. As the economies of France, Germany and the UK are much interconnected, they are expected to yield similar results and a direct comparison of the results will be more transparent. We start with a crossover analysis.

The results for crossover detection are very similar for all indices as it was expected. Evolution of *V* statistics and point to point derivatives of *H* for different methods and indices are shown in Chart 0-15, Chart 0-16 and Chart 0-17 in Appendix for CAC40, DAX and FTSE, respectively.

Detected crossovers are summed in Table 4-6. All indices show same crossover of 128 trading days for R/S and M-R/S. However, the results are quite different for the other methods. CAC40 shows the same crossover of 128 trading days for all DFA methods and thus, there are no simple trends (constant, linear or quadratic) which could strongly bias the results of R/S and M-R/S. On the other hand, crossovers are detected at a scale of 512 trading days for DFA-1 for DAX and FTSE which indicates that there is a significant linear trend for both indices which can influence the estimates of rescaled range methods. However, it is needed to note that crossovers for all methods and especially for DFA methods were rather insignificant and therefore, we expect the indices to exhibit independent behavior. The results suggest a use of R/S, M-R/S and DFA-1 with maximum scale of 128 trading days and 512 trading days long estimation period. DFA-1 is used for all indices as it clears all trends for DAX and FTSE and does not bias the estimates for CAC40 as it shows same crossover for all DFA methods.

Let us start the long-term dependence examination with the results of R/S analysis (charts are provided in Chart 4-12 in Section 4.4.2). Most clear results are obtained for DAX which does not show any dependence during whole examined period. On the other hand, CAC40 and FTSE show approximately two years (from September 2003 to September 2005) for which the behavior is anti-persistent or very close to confidence interval separating martingale and anti-persistent behavior. Note that anti-persistent behavior is very little discussed in the literature (e.g. Di Matteo, 2007) even though it yields strong implications about market efficiency as was shown in Section 2.3. Let us now turn to M-R/S to exclude the possibility of short-term memory bias.

Table 4-6 Crossover detection results for CAC40, DAX and FTSE

Method	υ_{max}			T		
	CAC40	DAX	FTSE	CAC40	DAX	FTSE
R/S	128	128	128	512	512	512
M-R/S	128	128	128	512	512	512
DFA-0	128	512	128	512	2048	512
DFA-1	128	512	512	512	2048	2048
DFA-2	128	256	128	512	1024	512

The results of M-R/S analysis are quite straightforward for FTSE and indicate that short-term memory does not bias the estimates of R/S for this index. On the other hand, CAC40 seems to exhibit short-term memory which influences the estimates of R/S. Nevertheless, there are still periods which exhibit significant anti-persistence. However, the results must be interpreted with caution as M-R/S analysis show several persistent periods for both CAC40 and FTSE even though there has been none shown for R/S. Nevertheless, the anti-persistence might still be caused by trends in the time series. Thus, we turn to analysis based on DFA-1.

However, even DFA-1 analysis shows anti-persistent behavior for both CAC40 and FTSE and thus we can conclude that these indices exhibited several significantly anti-persistent periods between September 2003 and September 2005. Let us now focus on patterns which show significant relation between time-dependent Hurst exponent and behavior of the indices (Chart 4-13, Chart 4-14 and Chart 4-15 in Section 4.4.2 for CAC40, DAX and FTSE, respectively).

We start our analysis with CAC40. There are again several periods which show patterns of Hurst exponent behavior. There is a strong decreasing trend of H which starts at 2.2.2001, bottoms at 14.2.2001 and is followed by an increasing trend which stabilizes at 1.3.2001. Behavior is then independent as H is very close to 0.5 and deviates again from 10.4.2001. The period between 2.2.2001 and 1.3.2001 is connected to 11% loss of CAC40 index.

Similar pattern starts at 7.8.2001 where a break from stable to strongly decreasing trend of Hurst exponent occurs. The decreasing trend stops at 14.8.2001 and is stable up till 29.8.2001 where an increasing trend begins and reaches its maximum at 21.9.2001. This behavior is again connected to very significant loss of the index as between 29.8.2001 and 21.9.2001, CAC40 index lost 30.20% of its value.

Both of the mentioned patterns were present when R/S analysis was used. DFA-1 shows different breaks in the behavior of Hurst exponent. Pattern similar to those already showed by R/S analysis starts at 16.3.2006 where significant decreasing trend begins and bottoms at 25.4.2006 which is in turn followed by increasing trend up 20.6.2006. Even though the significant increasing trend of Hurst exponent is connected with not such a significant cumulative loss of CAC40 (9.34%), the pattern again supports previous findings.

There are two more interesting parts of Hurst exponent behavior present. The first one is an obvious transition which happened around 3.9.2003 for all used methods which is connected to change of market structure as the break is followed by long lasting increasing trend of index values. The second one is a pattern already showed in preceding indices – strongly increasing Hurst exponent during the financial crisis of 2008, mainly in the second half of the year. Interestingly, the pattern for CAC40 is connected with other of its patterns as it shows strong decreasing trend of H from 15.9.2008 which inverses at 1.10.2008 into strong increasing trend up till 13.10.2008. The period between 1.10.2008 and 10.10.2008 is connected to a cumulative loss of 23.85%; on the other hand, a day of 13.10.2008 is connected to a gain of 10.59%. Nonetheless, the dynamics of Hurst exponent is again strongly connected to the behavior on the market.

We follow our analysis with DAX. The pattern, which starts at 4.7.2001 with slowly decreasing trend of Hurst exponent down to 30.8.2001 where the trend bottoms and reverses into strongly increasing trend up till 20.9.2001, is connected to a huge loss of 33.90% of DAX value. Other pattern shows again a strong decreasing (from 19.8.2002 to 4.9.2002) and increasing trend (from 26.9.2002 to 10.10.2002) of Hurst exponent. However, the reversal between the two is not immediate and there is a period of quite stable or only slightly increasing trend of the exponent. The period from the bottom of the decreasing trend to the end of the increasing trend is connected with a cumulative loss of 21.80%. The whole period is then connected with even more severe loss of 29.87%.

Another pattern is linked to the financial crisis of 2008. Behavior is again connected with the switch between strong decreasing and strong increasing trend of Hurst exponent. The critical dates are 24.9.2008, 1.10.2008 and 16.10.2008. The increasing trend is joined by a cumulative loss of 23.22% and whole period is connected with even deeper cumulative loss of 27.21%. Similarly to other indices, this period shows the most significant losses of the financial crisis.

The last pattern of DAX is different from the ones presented up till now. The main difference is the fact that significant increasing trend is not preceded by any significant decreasing trend and is rather preceded by stable Hurst exponent behavior. The strong increasing trend is present between 15.5.2006 and 15.6.2006 which is connected to a cumulative loss of 7.71% which is rather modest. Nonetheless, the loss in value of DAX starts quite earlier at 9.5.2006 and bottoms at 13.6.2006 and is equal to 14.66%, almost a double of a loss connected visibly to Hurst exponent trending. Therefore, the behavior of Hurst exponent suggests that it took investors several days to react to the trend or rather to believe that the trend is not only a short term episode.

The last index of the Western Europe, which we examine, is FTSE. If we check for patterns, we can't find any of the most typical type observed for above examined indices (strong decreasing trend of Hurst exponent followed immediately by strong increasing one). However, there are two visible by the naked eye and these are two waves between 26.4.2002 and 16.4.2003 of DFA-1.

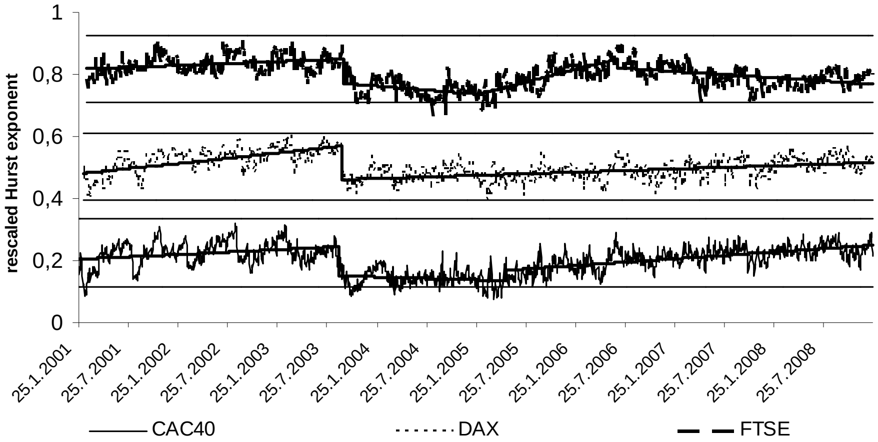
The wave shows a strong increasing trend starting at 9.5.2002 and peaking at 16.7.2002 and is connected with a cumulative loss of 25.87%. The other wave shows similar relationship. Increasing trend starting at 2.12.2002 and reaching its peak at 4.2.2003 is connected with a cumulative loss of 14.96%. Therefore, both waves of Hurst exponent show very similar connection to the returns of FTSE index.

The last pattern is again connected with the biggest losses during the financial crisis of 2008. We can see quite modest decreasing trend of Hurst exponent starting at 1.9.2008 which turns into several consecutive downward movements from 23.9.2008 to 1.10.2008 which are followed by an upward movement of Hurst exponent. An increasing trend follows and peaks at 16.10.2008. The whole mentioned period is connected with a huge cumulative loss of 37.83%. Nevertheless, the period between the shift of behavior from a decreasing to an increasing trend is connected with a cumulative loss of 25.03% which is even more severe if considering the length of the period (11 trading days compared to 34 of the whole period). The interpretation is similar to the ones above – decreasing Hurst exponent indicates increasing uncertainty and investors' nervousness which turns into panic while the trend breaks.

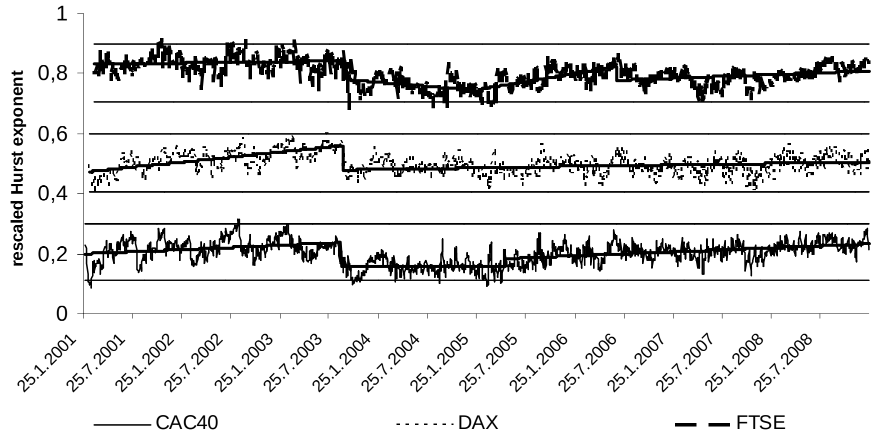
4.4.2 Charts for the Western Europe

Chart 4-12 Time-dependent Hurst exponent for CAC40, DAX and FTSE

(a)



(b)



(c)

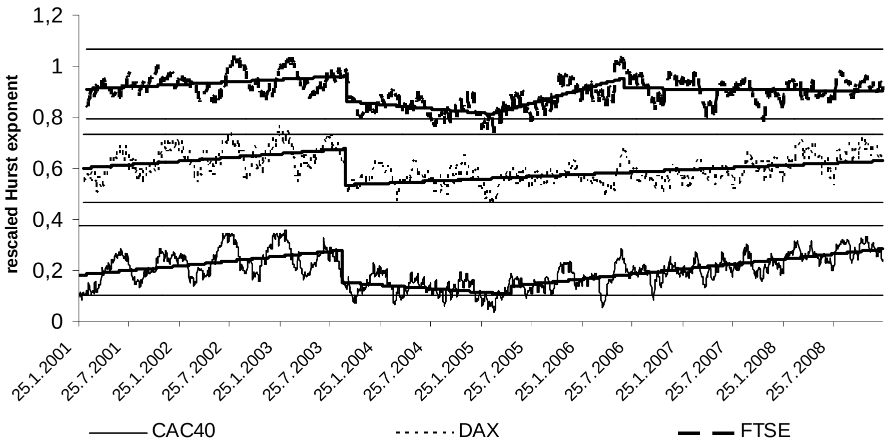


Chart 4-12: (a) based on R/S, (b) based on M-R/S, (c) based on DFA-1: Time-dependent Hurst exponent and related confidence intervals are rescaled so that 0.2, 0.5 and 0.8 are in the middle of the confidence intervals for CAC40, DAX and FTSE, respectively. All indices very similar behavior of time-dependent Hurst exponent with almost two years long period of anti-persistent and very close to anti-persistent behavior between 09/2003 and 09/2005 for CAC40 and FTSE.

Chart 4-13 CAC40 relationship between time-dependent Hurst exponent and index values

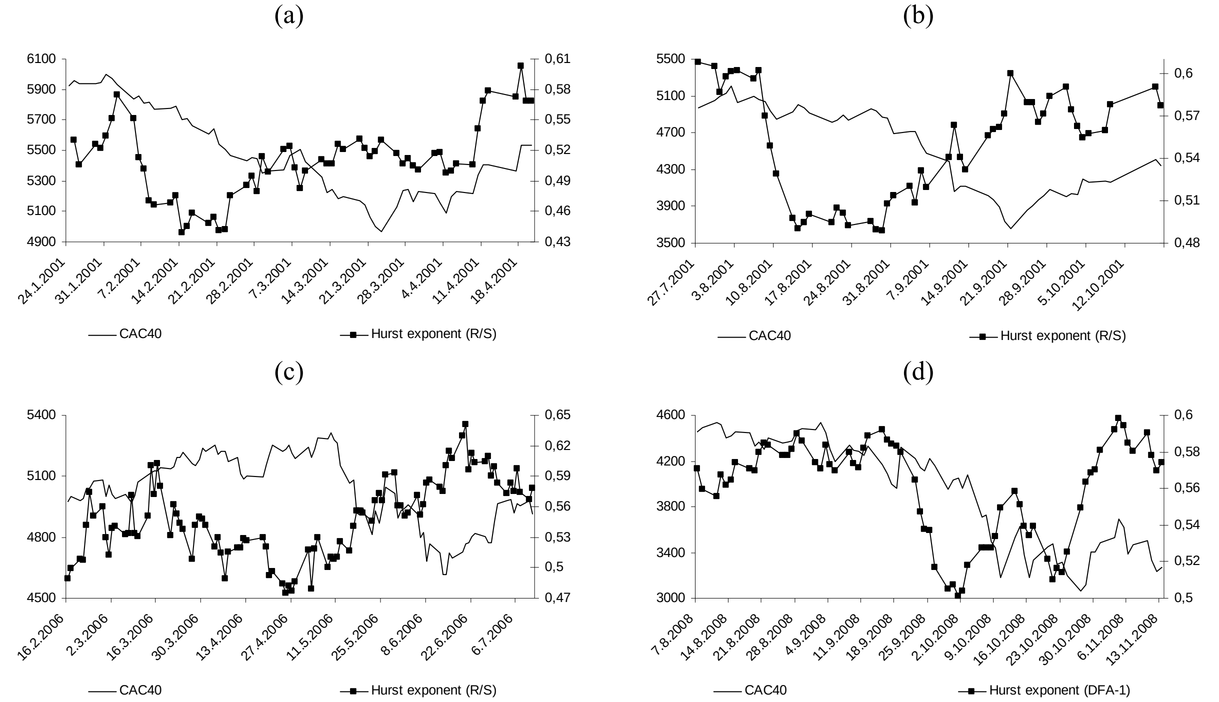


Chart 4-13: (a) *H* based on R/S, (b) *H* based on R/S, (c) *H* based on R/S, (d) *H* based on DFA-1 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in CAC40 values (left y-axis).

Chart 4-14 DAX relationship between time-dependent Hurst exponent and index values

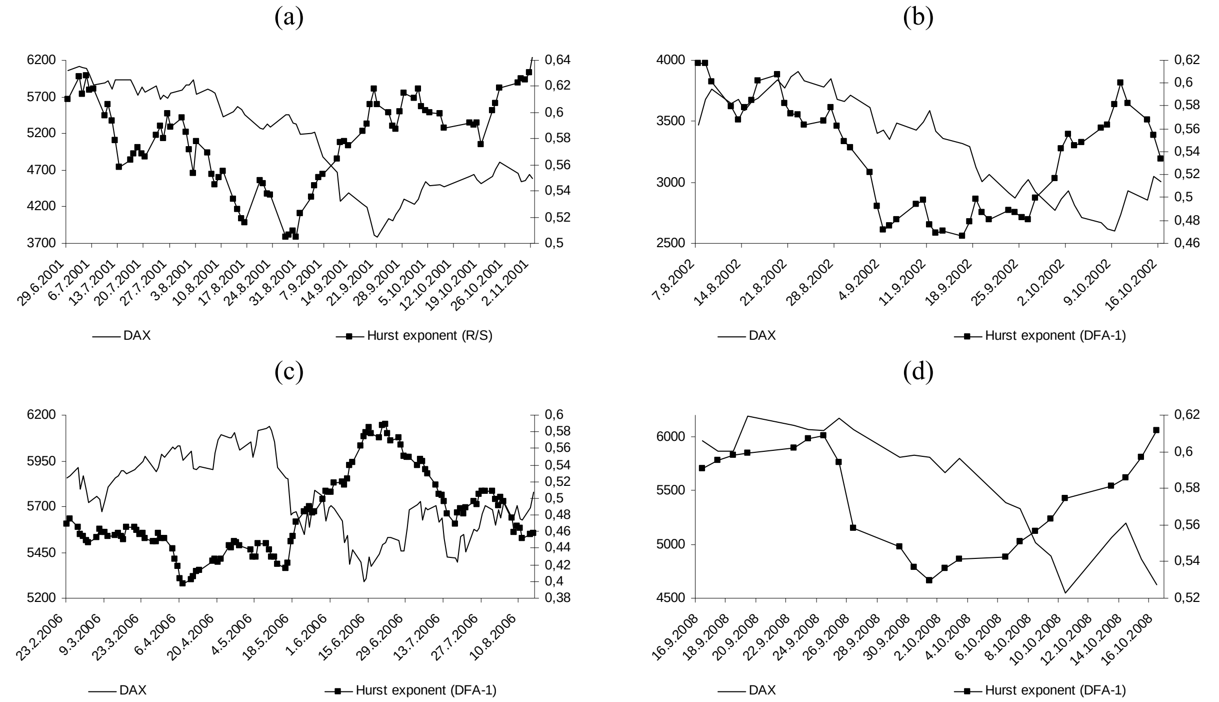
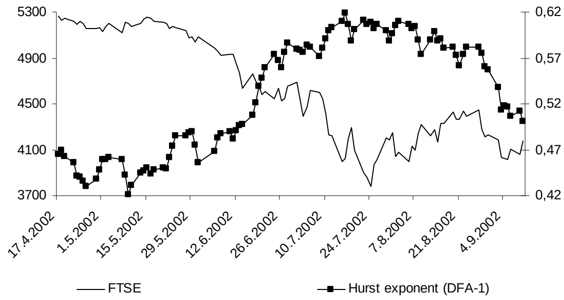


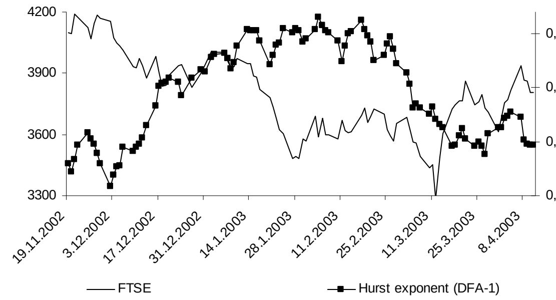
Chart 4-14: (a) *H* based on R/S, (b) *H* based on DFA-1, (c) *H* based on DFA-1, (d) *H* based on DFA-1 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in DAX values (left y-axis).

Chart 4-15 FTSE relationship between time-dependent Hurst exponent and index values

(a)



(b)



(c)

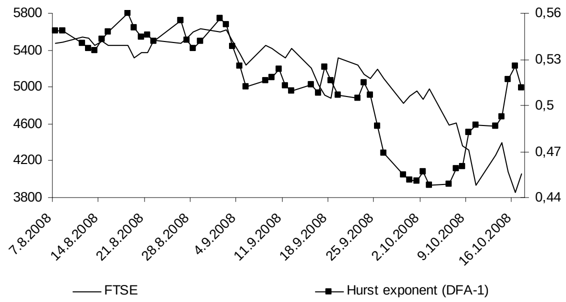


Chart 4-15: (a) *H* based on DFA-1, (b) *H* based on DFA-1, (c) *H* based on DFA-1 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in FTSE values (left y-axis).

4.4.3 USA

The methodology of the research of the US indices is similar to the one of the Western European indices. We have chosen this approach because we have different indices for one economy. There can be several separations done. Firstly, DJI30 is the index of NYSE; NASDAQ is the index of NASDAQ, obviously; and SP500 is the index of both NASDAQ and NYSE. Secondly, DJI30 is a sector index and the other two are broad market indices. Thirdly, the breadth of the indices is quite different. The number of stocks included in indices is quite obvious from their names in case of DJI30 and SP500. NASDAQ consists of almost 3000 stocks[71].

Therefore, there are several expectations based on above noted differences. DJI30 is expected to show the least independent behavior or even dependent during several periods. On the other hand, NASDAQ is expected to behave the most efficiently with some respect to potential biases caused by trends which might be present as NASDAQ generally lists stocks of technologic companies. The same is in turn expected for DJI30 – trending may bias the results. Let us follow with standard long-term dependence detection procedure.

We present standard crossover detections for DJI30, SP500 and NASDAQ in Chart 0-18, Chart 0-19 and Chart 0-20 in Appendix, respectively, and sum the proposed maximum

[71] See finance.yahoo.com for specific stocks.

scales in Table 4-7. The results lead to several conclusions. All indices stock a common feature of maximum scale proposed for DFA-2 which is lower than the one proposed for DFA-1. This behavior leads to strong conclusion that a maximum trending is linear while the quadratic one biases the estimates. Therefore, we omit DFA-2 from the examination of all the US indices. The same is true for DFA-1 in the case of DJI30 and therefore only DFA-0 is recommended for this index. Nevertheless, we use DFA-1 as a detrending method as well for the sake of comparison. Further, DJI30 scales very differently when compared with other indices and shows a scaling behavior up to only half a trading year. However, if M-R/S and DFA-0 methods are used, it shows the maximum available scale. Therefore, short-term memory is expected to be present in the process of DJI30. It must be noted that the most of the crossovers found were weak and the behavior of V statistics was close to constant which implies that all indices are expected to be independent or very close to independent.

Table 4-7 Crossover detection results for US indices

Method	υ_{max}			T		
	DJI30	SP500	NASDAQ	DJI30	SP500	NASDAQ
R/S	128	256	256	512	1024	1024
M-R/S	512	256	256	2048	1024	1024
DFA-0	512	512	256	2048	2048	1024
DFA-1	256	512	512	1024	2048	2048
DFA-2	128	256	256	512	1024	1024

Let us move to time-dependent Hurst exponent analysis (Chart 4-16 in Section 4.4.4). We start with the results of R/S analysis. NASDAQ shows an independent behavior during whole examined period which is in hand with expectations. The second index, which is closest to independent behavior, is DJI30 which is quite surprising. There are three periods which are persistent but short, though. Nonetheless, we will check for short-term memory or trends later. SP500 shows an independent behavior up to approximately 3.5.2007 where H gets very close to lower confidence interval and in many cases even slips to an anti-persistent behavior.

M-R/S analysis shows that short term memory process biased the estimates of R/S for SP500. DJI30 remains persistent in the mentioned periods as for R/S estimates. However, a problem of a pattern that was already tackled while examining SAX occurs. Unfortunately, we cannot use the same method as we did for SAX as using minimum range of 32 trading days together with maximum range of 128 trading days leaves us with only three rescaled range statistics[72] to estimate *H*. Therefore, we need to stick with minimum range of 16 trading days which implies that the presence of long-term memory was not caused by a short term memory bias.

As no long-term dependence was shown for SP500 and NASDAQ, we could examine only DJI30 with DFA. However, we present the results for DFA-1 for sake of classification of possible interesting patterns which were shown to be present even for independent time series (e.g. BUX and WIG). Moreover, DFA methods have shown interesting dynamics during the crisis of 2008. The results of DFA-1 are quite similar to the ones of R/S and M-R/S. The method cleared away almost all the persistence of DJI30. Nevertheless, DFA-1 is used mainly to uncover interesting patterns in the behavior of Hurst exponent of the indices. Let us now turn to these patterns (Chart 4-17, Chart 4-18 and Chart 4-19 in Section 4.4.4 for NASDAQ, SP500 and DJI30, respectively)

We start with the examination of NASDAQ. There is a pattern between 23.11.2007 and 10.3.2008. The strong increasing trend between mentioned dates is connected with a cumulative loss of 16.64% of the index. Even though the loss is not huge considering the length of the period, the behavior of Hurst exponent again shows that there is a connection between the increasing Hurst exponent and ongoing trends of the indices as was shown in almost all examined indices.

Other pattern is connected to the financial crisis of 2008. We compare NASDAQ values and estimates of Hurst exponent based on both R/S and DFA-1. R/S analysis shows a decreasing trend of Hurst exponent starting at 20.8.2008 up to 15.9.2008 where the trend reverses to an increasing trend which peaks at 10.10.2008. The whole period is connected with a cumulative loss of 36.85% while the increasing trend of the exponent is connected to a loss of 31.54%. DFA-1 shows quite different results. The decreasing trend of the exponent starts already at 18.8.2008 and stops at 22.9.2008 where the dynamics stabilize or at least slows and quite a strong increasing trend starts at 6.10.2008 and tops at 24.10.2008. The last trend is connected to a cumulative loss of 22.69%. The whole period is in hand with a loss of 45.76%. Therefore, we can see similar behavior of the investors which was present at the other indices. The positive mood of the investors vanished and turned into increasing nervousness and uncertainty which eventually turned into following of the decreasing trend of the index.

[72] See Weron (2002) for the estimates based on only three rescaled range statistics which show large standard deviations.

SP500 shows the most interesting dynamics during the financial crisis of 2008. The peak of the index is the same as the peak of Hurst exponent at 16.5.2008. The exponent then falls into six consecutive decreases and bottoms at 27.5.2008 where the behavior switches into slow increasing trend peaking at 2.7.2008. Even though the latter trend is connected with a cumulative loss of only 8.68%, it shows rather fast transition of the market mood. The position of the market then stays in rather slowly increasing trend of the exponent up till 20.8.2008 where the behavior again switches into strongly decreasing trend which bottoms at 22.9.2008 where the reversal into strong increasing trend again occurs. The period of the increasing trend is connected with significant cumulative loss of 32% of the index. Therefore, the whole period starting at 19.2.2008 to the end of the examined period at 20.1.2009 can be described as the end of growing trend which was driven by a positive mood of the investors and which was in turn switched into negative expectations which turned out to be true.

The evolution of Hurst exponent based on DFA-1, however, yields quite different results. The decreasing trend of the exponent starts already at 6.3.2008 and bottoms at 28.5.2008 where it suddenly inverses into an increasing trend which peaks at 10.11.2008. The period of the increasing Hurst exponent is connected with a cumulative loss of 41.02% of SP500 index. The interpretation is the same – increasing uncertainty followed by strong decreasing trend in the index.

DJI30 shows more interesting dynamics compared to the other indices as was expected. The pattern which, we can now say, is quite typical shows a decreasing Hurst exponent from 8.2.2002 to the bottom at 5.4.2002 where the trend turns into an increasing trend which peaks at 22.7.2002. The increasing trend is again connected to a significant cumulative loss of 26.95% of the value of DJI30.

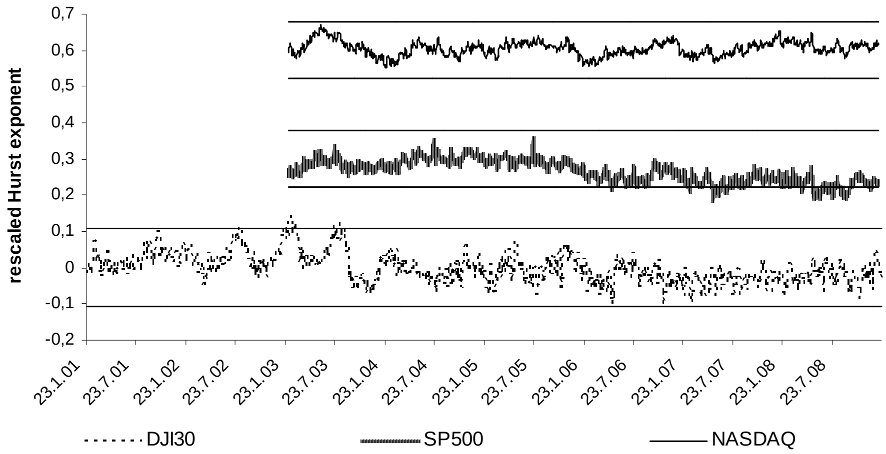
The other pattern is again rather typical and shows increasing uncertainty on the market despite the increasing trend of the index. The decreasing trend is visible for more than 100 trading days and hits its bottom at 12.5.2006 where it turns into strong increasing trend which peaks at 13.6.2006. Even though a cumulative loss connected to the latter trend equals only to 7.16% of the index, the pattern is similar to the ones already presented. Therefore, we can see that the patterns are not sufficient conditions for the market change but rather necessary.

The last pattern is quite expectedly connected to the financial crisis of 2008. The decreasing trend connected with an increasing uncertainty bottoms at 22.9.2008 and turns into the increasing one peaking at 9.10.2008 which is connected to a cumulative loss of 28.33%. After the last mentioned observation, the decreasing trend comes and ends at 5.11.2008 and reaches its maximum at 17.11.2008. The latter trend is again connected to significant cumulative losses of 15.13%.

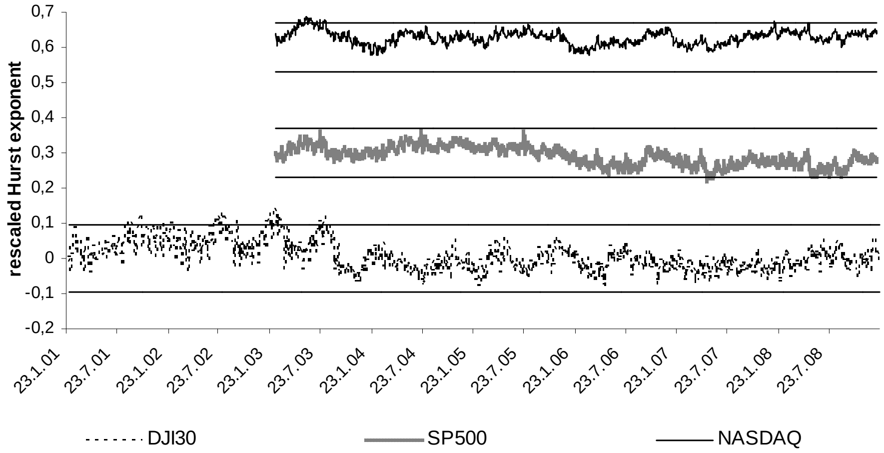
4.4.4 Charts for the USA

Chart 4-16 Time-dependent Hurst exponent for DJI30, SP500 and NASDAQ

(a)



(b)



(c)

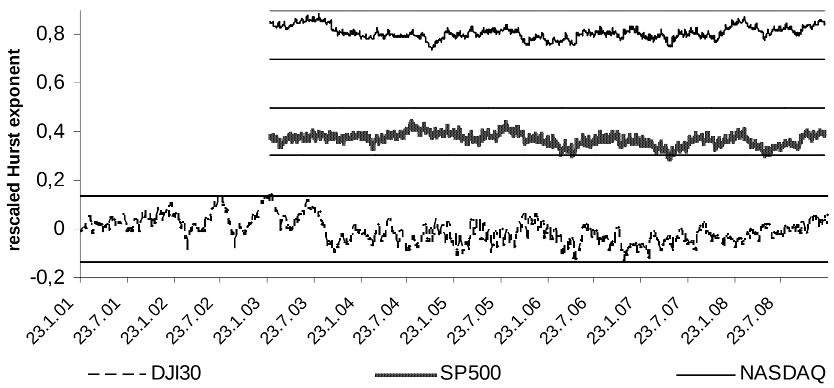
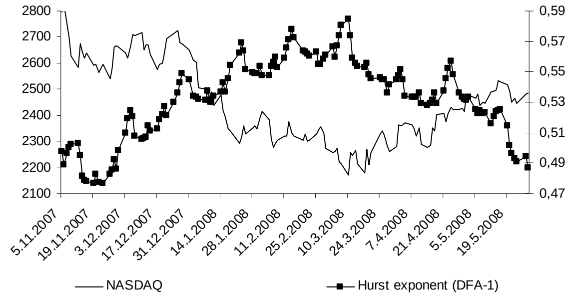


Chart 4-16: (a) based on R/S, (b) based on M-R/S, (c) based on DFA-1: Time-dependent Hurst exponent and related confidence intervals are rescaled so that 0, 0.4 and 0.8 are in the middle of the confidence intervals for DJI30, SP500 and NASDAQ, respectively. DJI30 shows that long-term dependence is caused by trends in the time series, NASDAQ is efficient for the whole examined period and SP500 shows several periods of anti-persistent behavior which is not caused by short-term memory or trends.

Chart 4-17 NASDAQ relationship between time-dependent Hurst exponent and index values

(a) (b)



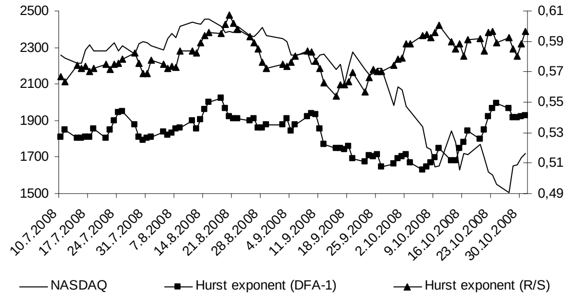
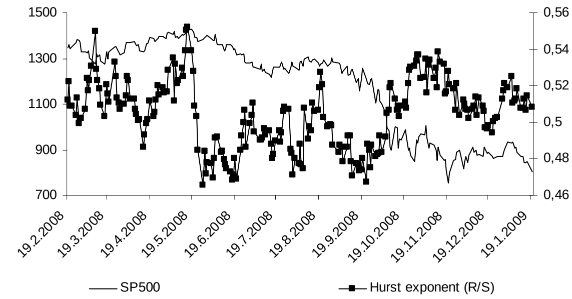


Chart 4-17: (a) *H* based on DFA-1, (b) *H* based on DFA-1, (c) *H* based on DFA-1 and R/S during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in NASDAQ values (left y-axis).

Chart 4-18 SP500 relationship between time-dependent Hurst exponent and index values

(a) (b)



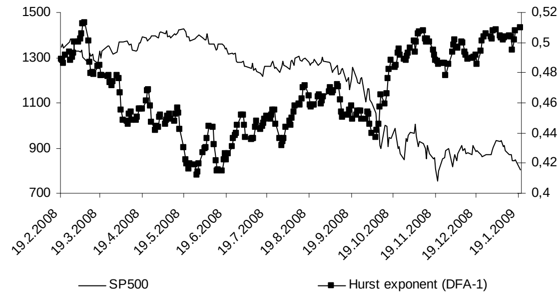
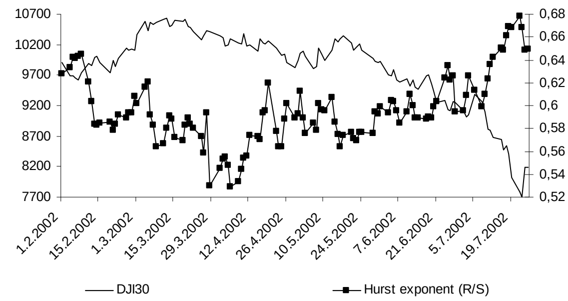
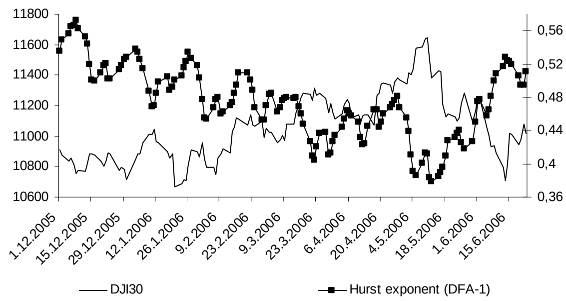


Chart 4-18: (a) *H* based on R/S during financial crisis of 2008, (b) *H* based on DFA-1 and R/S during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in SP500 values (left y-axis).

Chart 4-19 DJI30 relationship between time-dependent Hurst exponent and index values

(a) (b)





(c)

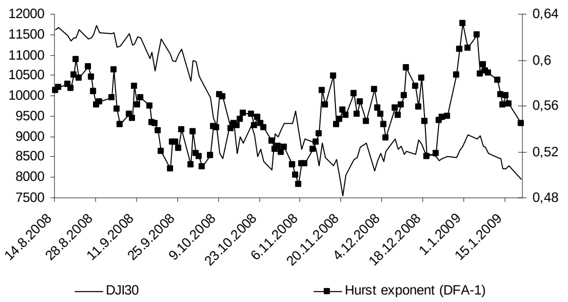


Chart 4-19: (a) *H* based on R/S, (b) *H* based on DFA-1, (c) *H* based on DFA-1 during financial crisis of 2008: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in DJI30 values (left y-axis).

4.4.5 Comparison of indices of Western Europe and the USA

The analysis of the indices of developed economics has shown several interesting results. We showed that CAC40, FTSE and SP500 were anti-persistent during several periods which points out to the mere existence of this phenomena in financial time series. There are only several researchers who found anti-persistence; however, the anti-persistence was mostly insignificant (Di Matteo, 2007). Moreover, some quite recent papers about long-term memory processes have not taken anti-persistence connected to the financial markets into consideration at all (e.g. Lillo & Farmer, 2004 and Taqqu, Teverovsky & Willinger, 1995). However, our results show significant anti-persistence and such behavior should be subject to further research. We have also shown that the indices of the Western Europe are rather different from the ones of the USA in a sense of time-dependent *H* behavior.

For more detailed relationship analysis of Western Europe and the USA, we compare correlations of time-dependent Hurst exponent, which were again based on a maximum scale of 128 trading days, and we present Table 4-8, Table 4-9 and Table 4-10.

Table 4-8 Correlations of time-dependent *H* (R/S) for indices of Western Europe and the USA

	DAX	FTSE	CAC	DJI	NASDAQ	SP
DAX	1					
FTSE	0,431708	1				
CAC	0,730905	0,558112	1			
DJI	0,306398	0,311075	0,12111	1		
NASDAQ	0,3264	0,648306	0,444827	0,349786	1	
SP	0,249142	0,299979	0,033824	0,793742	0,347699	1

The mostly correlated indices are quite expectedly pairs SP500 - DJI30 and DAX – CAC40. Further, all the indices of the Western Europe are strongly correlated (with exception of pair FTSE – DAX which shows medium correlation). Quite interestingly, CAC40 shows weak or no correlation at all with DJI30 and SP500. Also, NASDAQ shows only medium correlation with SP500 and DJI30; on the other hand, NASDAQ is strongly correlated with FTSE. For the comparison and confirmation, we present results for DFA-1 and DFA-2.

Table 4-9 Correlations of time-dependent *H* (DFA-1) for indices of Western Europe and the USA

	DAX	FTSE	CAC	DJI	NASDAQ	SP
DAX	1					
FTSE	0,61355	1				
CAC	0,851346	0,746557	1			
DJI	0,716294	0,464963	0,532585	1		
NASDAQ	0,489885	0,661039	0,509113	0,485312	1	
SP	0,627521	0,469759	0,423351	0,854654	0,588312	1

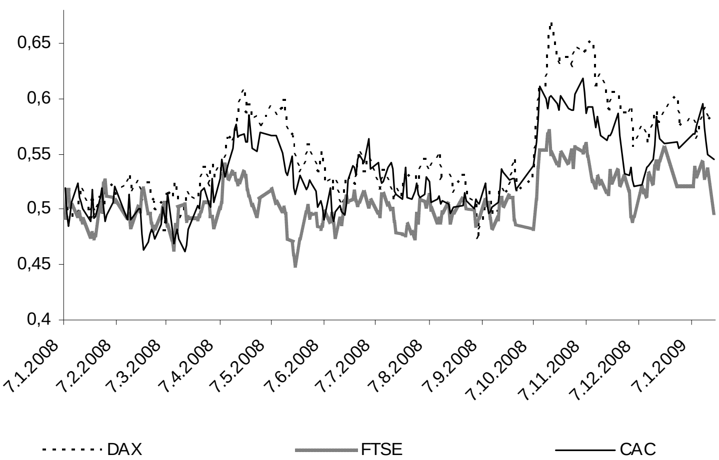
Both methods show quite similar results – all the correlations are even stronger for all pairs of the indices. Therefore, we can state an important conclusion – the indices of the Western Europe and the USA are positively correlated in the sense of changes of market dynamics and mood.

Table 4-10 Correlations of time-dependent *H* (DFA-2) for indices of Western Europe and the USA

	DAX	*FTSE*	*CAC*	*DJI*	*NASDAQ*	*SP*
DAX	1					
FTSE	0,523119	1				
CAC	0,844103	0,629008	1			
DJI	0,573599	0,393236	0,311605	1		
NASDAQ	0,478637	0,577908	0,310696	0,83379	1	
SP	0,592486	0,412485	0,339927	0,928694	0,811595	1

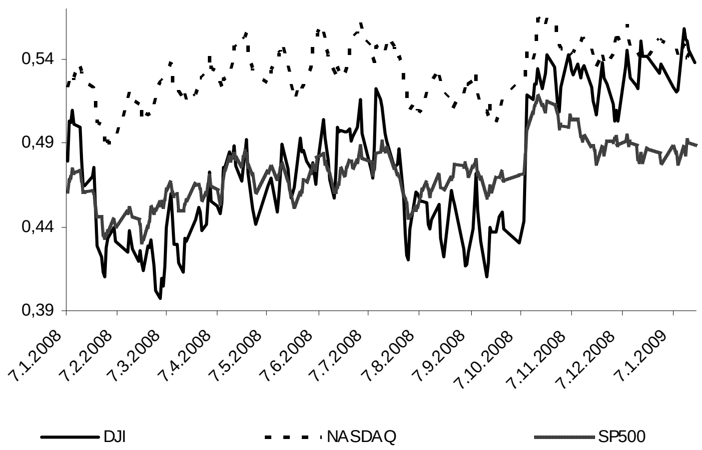
The indices of the Western Europe reacted rather similarly to the financial crisis of 2008 (Chart 4-20). However, the reaction is quite different from the one of the indices of the Central Europe. The examined indices behaved independently for the first four months of 2008. After that period, quite strong increasing trend of Hurst exponent occurs for all three markets. The trend is connected with the first wave of significant losses. However, the most severe negative returns occur after quite long decreasing period of Hurst exponent when the significant jump of the exponent occurs. Quite importantly, the jump is present for all three indices at the same time and is followed by slow decreasing trend of the exponent for all indices.

Chart 4-20 Comparison of CAC40, DAX and FTSE during financial crisis of 2008



The dynamics is rather different for the US indices (Chart 4-21). The year of 2008 starts with a decreasing trend of Hurst exponent for all DJI30, SP500 and NASDAQ and quickly (after a month) switches into an increasing trend of the exponent which lasted to the half of July 2008 for all indices. The exponent then fell significantly and performed rather stable for almost a quarter of the year. After the stable period, significant jump of similar magnitude appears and is again connected to the most significant losses of the whole financial crisis of 2008. Importantly, the jump occurs between 7th and 9th October 2008 which is the same as for the indices of the Western Europe. Thus, the Western Europe and the USA reacted rapidly to the biggest losses of the crisis while the Central Europe reacted rather slowly. This implies that the belief of negative prospects spilled over from Western countries to the ones of the Central Europe.

Chart 4-21 DJI, NASDAQ and SP500 and time-dependent *H* during financial crisis of 2008



When we compare the efficiency of the markets, we have clear winners in NASDAQ and DAX, which have not shown a single long-term dependent period, are stationary and thus both S65 and F65 efficient. All the other indices are S65 efficient as there has been no persistent behavior shown but none of them has been F65 efficient for the whole examined period. CAC40, FTSE and SP500 have experienced significantly anti-persistent behavior. Moreover, SP500 has also exhibited significant short-term memory which rejects the hypothesis of a random walk as well. On the other hand, DJI30 has followed significant trends during several periods and thus again rejects a random walk model of its behavior. Nevertheless, the efficiency of last two years can be summed by both F65 and S65 efficient markets for all indices but one. SP500 has been anti-persistent during several short periods and thus is the least efficient of the Western countries. Let us turn to the analysis of the indices of China and Japan.

4.5 Asian Economies

At last, we present the analysis of two indices – SSEC and NIKKEI. As China and Japan have undergone different evolution of the financial markets, we expect rather different results. We present standard procedures for long-term memory detection for both indices. We start with examination of SSEC and follow with NIKKEI. Two last parts of the chapter compare the results and present the charts.

4.5.1 China

We start our analysis with a crossover detection which is presented in Chart 0-21 in Appendix. R/S, M-R/S, DFA-0 and DFA-2 all show no crossovers in the process and therefore no cycles. On the other hand, DFA-1 shows a crossover at a scale of 128 trading days. Even though the crossover is not significant, it is supported by the evolution of point to point derivatives of *H*. However, if we check the *V* statistic evolution of DFA-0, we recognize that the statistic behaves almost constantly between scales of 128 and 256 trading days. The result is again supported by point to point derivatives. Therefore, the results suggest that there is a quadratic trend in the time series which causes crossovers in DFA-0 and DFA-1. However, neither R/S nor M-R/S show such a crossover. Therefore, we use R/S, M-R/S and DFA-2 as the other DFA methods can be biased. Despite the fact that all methods with exception of DFA-1 show no significant crossover, we choose a maximum scale of 256 trading days as it does not lead to biased results and the longer period can be examined[73]. Let us proceed with time-dependent Hurst exponent analysis which is summed in Chart 4-33 in Section 4.5.4.

R/S analysis shows a behavior which varies around the higher confidence interval, which separates an independent and a persistent behavior, for the most of the examined period. Interestingly, *H* decreases significantly from 22.7.2008 to the end of examined period which is in contrast with majority of already examined indices. M-R/S analysis shows very similar results to R/S analysis. However, there is no persistent behavior after 25.9.2007 according to M-R/S which implies that there was an increasing influence of short-term memory process in the last year of examination. Nonetheless, there may still be a bias caused by trends and non-stationarity of the time series.

DFA-2 shows stronger decreasing trend of Hurst exponent for the whole examined period. Thus, the decreasing trend is supported by all used methods and therefore SSEC shows clear movement to more efficient market. Moreover, the decrease below upper confidence interval, which continues to the end of examined period (20.1.2009), is again

[73] The use of maximum scale of 512 trading days would lead to only 333 trading days with estimated *H*. However, we want to compare the evolution of *H* for longer periods.

present for all used methods and suggests that the financial crisis of 2008 rather helped SSEC in the sense of efficiency.

Quite interestingly, there are no patterns in the relationship between Hurst exponent and SSEC index but one which is connected with the global peak of SSEC at 16.10.2007. The peak is preceded by significantly growing Hurst exponent that peaks at the same point as SSEC. The dynamics significantly changes, then, as there is very strong decreasing trend bottoming at 9.11.2007 which in turn switches to rather volatile behavior up till 27.12.2007 where very strong increasing trend begins. The exponent stabilizes around upper confidence interval, then. Therefore, Hurst exponent shows that turning point of the index is connected with belief of investors. The pattern is presented in Chart 4-35 in Section 4.4.4.

What is, however, even more interesting and for the sake of comparison more important is the fact that all used methods show strong decreasing trend of Hurst exponent. Therefore, we can make strong conclusion that SSEC switched to more efficient market during the financial crisis of 2008. We now follow with the analysis of the last index – NIKKEI.

4.5.2 Japan

We present the crossovers detection for NIKKEI in Chart 0-22. However, the analysis shows quite inconclusive results for all methods as there is a crossover at a scale of 64 trading days which would give us only three points for Hurst exponent estimation which was discarded in the text above, already. Therefore, we choose a maximum scale of 128 trading days as it is the closest one to the one detected and point to point derivatives of H do not show volatile behavior for all methods with an exception of DFA-1. Therefore, we omit the method based on linear detrending as its results would not be reliable. Nonetheless, all the methods imply that the behavior of NIKKEI is either independent or very close to it. Moreover, we can expect several periods of an anti-persistent behavior which is in hand with the results of CAC40, FTSE and SP500. We follow with the examination of behavior of Hurst exponent (Chart 4-34 in Section 4.5.4)

As expected, R/S analysis shows periods of anti-persistent behavior during similar periods as indices of developed economies did. Moreover, M-R/S analysis shows that a short-term memory process does not bias the estimates of H based on R/S. DFA-2 cleared most of the dependence in the time series. Nonetheless, there are still several anti-persistent periods remaining and even one persistent period emerges at 5.11.2008. Even though the periods marked as anti-persistent by R/S and M-R/S are independent if checked by DFA-2, they still remain rather close to the lower confidence interval. Despite the fact that the behavior of Hurst exponent for all methods is rather volatile in time, there are still two periods with interesting patterns.

The first one is similar to the most common pattern present in the time series, slowly decreasing trend of Hurst exponent starts at 19.11.2007 and bottoms at 28.12.2007 where it reverses into rather rapid increasing trend which vanishes at 22.1.2008. The strong increasing trend is connected with a cumulative loss of 21.35% of the index. The interpretation is again the same – increasing uncertainty which turns into strong belief of continuing downward trend.

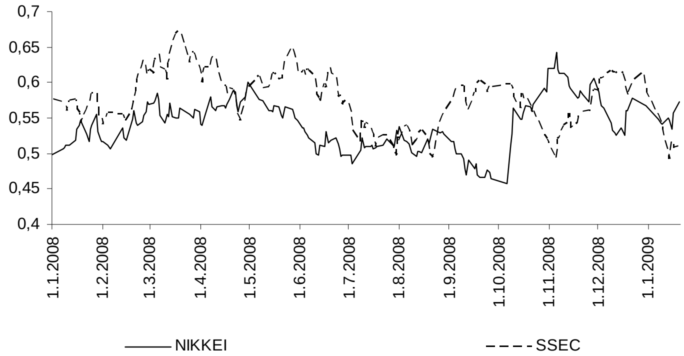
The second pattern starts with rather stable behavior of Hurst exponent followed by strong increasing trend of the exponent which is connected with significant negative returns. The strong increasing trend begins at 30.7.2007 and peaks at 17.8.2007. This trend is linked with a cumulative loss of 12.36% and can be explained as a change of behavior of the investors who came to believe that the slowly starting decreasing trend of NIKKEI is about to persist.

As it was already mentioned, DFA-2 shows significant upward jump in Hurst exponent during the year of 2008. However, the evolution is rather different from the ones of comparable indices (CAC40, DAX and FTSE). It is needed to point out that we use DFA-2 as a detrending method compared to DFA-1 used for indices of the Western Europe and therefore, the comparison is not straightforward. Nonetheless, the significant increase of Hurst exponent is present as well and is foregone by downward sloping trend of the exponent. Interestingly, Hurst exponent jumps to significantly persistent behavior and even though it goes below the confidence interval after three persistent observations between 1.9.2008 and 3.9.2008, the estimates of the exponent remain close to the critical value. In spite of the fact that persistent values of Hurst exponent are not directly connected to the start of the most significant losses of the index (60.89% between 11.8.2008 and 27.10.2008), these losses are again in hand with increasing trend of Hurst exponent. We can again see the decreasing trend of the exponent as the mood on the market was worsening and eventually turned into huge losses connected with belief of the investors that the turbulence is not to pass quickly. Let us now compare the results for Asian indices in the following section.

4.5.3 Comparison of Asian indices

Both indices have shown different behavior and therefore, there is little to compare[74]. Nonetheless, we present the dynamics of Hurst exponent during the financial crisis of 2008 (Chart 4-22). We can see that Hurst exponent of both indices behaved rather differently and the dynamics is almost perfectly negatively correlated during last quarter of 2008. Nevertheless, NIKKEI shows the same behavior as the indices of the Western Europe and the USA – slow decreasing trend of the exponent with rapid increase in the beginning of October 2008 followed by slow decreasing trend.

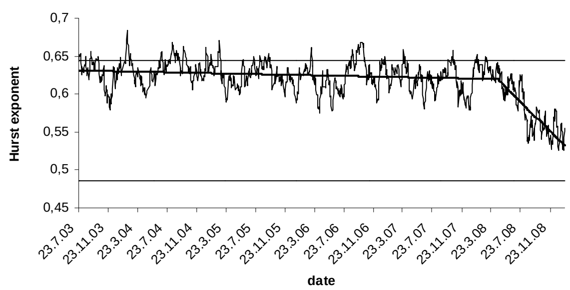
Chart 4-22 Comparison of SSEC and NIKKEI during financial crisis of 2008



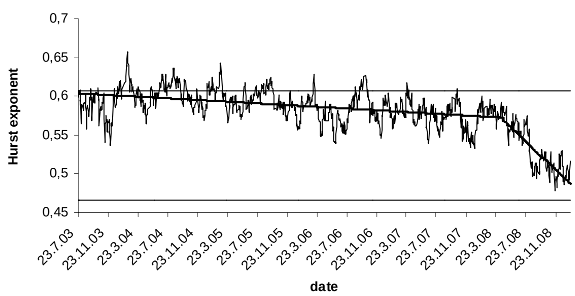
4.5.4 Charts for Asian indices

Chart 4-23 SSEC time-dependent Hurst exponent

(a)



(b)



(c)

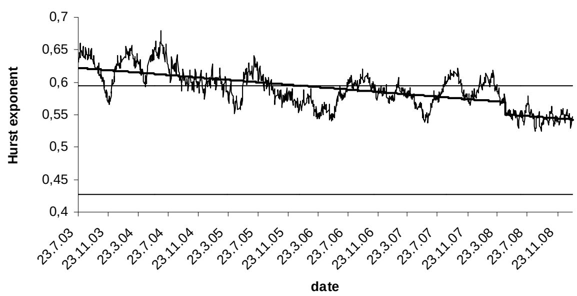


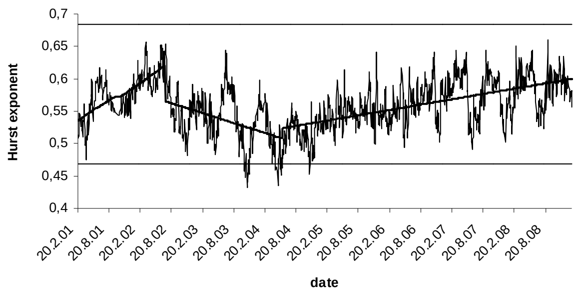
Chart 4-23: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on M-R/S, (c) Time-

[74] Comparison of both NIKKEI and SSEC with other indices is present in Section 4.6.

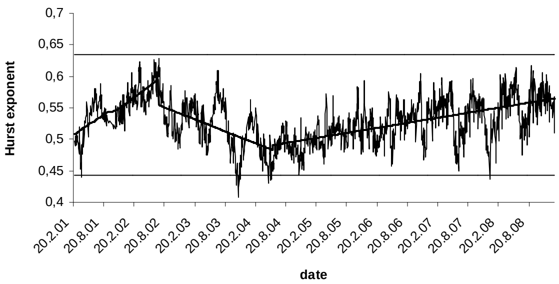
dependent *H* based on DFA-2: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-24 NIKKEI time-dependent Hurst exponent

(a)



(b)



(c)

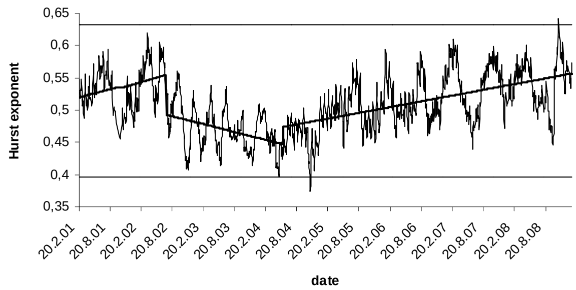
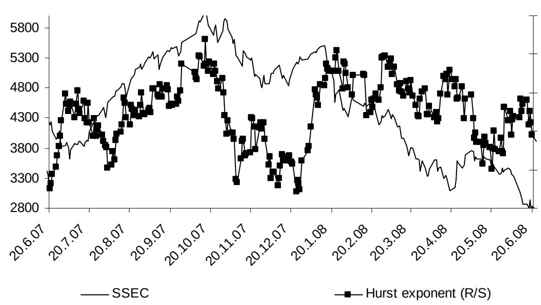


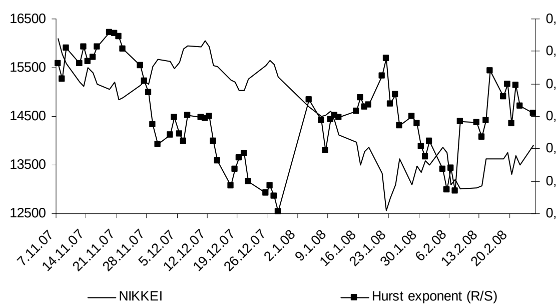
Chart 4-24: (a) Time-dependent *H* based on R/S, (b) Time-dependent *H* based on M-R/S, (c) Time-dependent *H* based on DFA-2: Constant solid lines present upper and lower confidence intervals (2.5% and 97.5%). Curved solid lines show linear trends of Hurst exponent.

Chart 4-25 SSEC and NIKKEI relationship between time-dependent Hurst exponent and index values

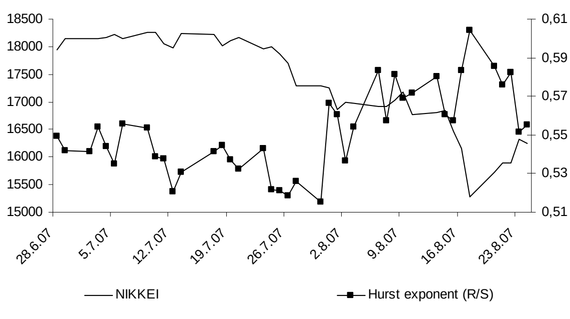
(a)



(b)



(c)



(d)

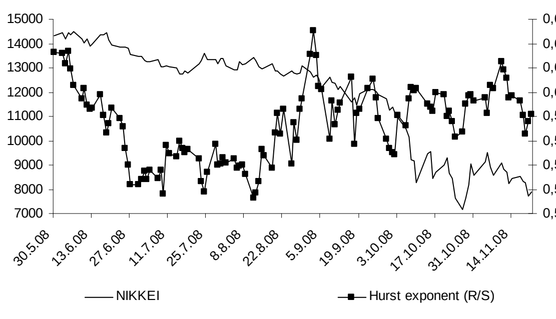


Chart 4-25: (a) *H* based on R/S for SSEC, (b) *H* based on R/S for NIKKEI, (c) *H* based on R/S for NIKKEI, (d) *H* based on R/S during financial crisis of 2008 for NIKKEI: Charts show patterns between time-dependent Hurst exponent (right y-axis) and significant movements in SSEC and NIKKEI values (left y-axis).

4.6 Comparison and conclusions

The examination of daily time series has shown several interesting results which were not, to our best knowledge, covered in any literature yet.

The investigation of Central European indices has revealed that all the indices have to be considered separately as the results for long term memory tests have yielded very different evolution for each market or couple of markets at most. Czech and Austrian indices (PX and ATX, respectively) have shown decreasing trend of Hurst exponent estimates (starting in a persistent phase and eventually reaching an independent one). Nonetheless, there still is quite a difference in the fact that PX was persistent for shorter period but remains very close to upper confidence interval which separates independent and persistent phase up to recent days. On the other hand, ATX started at higher levels of persistence but showed very quick transition to independent behavior far from both confidence intervals. Hungarian and Polish indices (BUX and WIG20, respectively), quite surprisingly, have shown an independent behavior through the whole examination period. Slovakian index (SAX) has shown significantly different results from other indices of this group. The index has experienced an increasing trend of Hurst exponent which has not been caused by either short term memory or trends in the time series. The trend has stabilized in the persistent region, then. Nevertheless, SAX examination has displayed a pitfall of M-R/S which has not been obvious during the investigation of other indices. The modified standard deviation estimator can be significantly biased when the time series includes several consequent zero returns. The estimator is thus underestimated and in turn overestimates the rescaled range leveraging downwards the estimate of Hurst exponent. Therefore, the special caution is suggested when a repeating pattern of significant jumps is present in time-dependent Hurst exponent estimates.

Indices of the Western Europe have shown very similar behavior. The indices of Germany, the UK and France (DAX, FTSE and CAC40, respectively) are independent in the vast majority of the estimation period. DAX has shown no dependent period at all. However, it has shown that the use of DFA techniques can yield dependent behavior in several periods which agreed to the fact that DFA methods yield biased estimates of Hurst exponent when there are no non-stationarities present in the time series. FTSE and CAC40 have shown an anti-persistent behavior during similar periods. However, the dependency is much lower for FTSE. Moreover, behavior of FTSE seems to be partly caused by trends in the time series. Nonetheless, the anti-persistency was not cleared even by DFA-2. CAC40, on the other hand, shows quite similar anti-persistency no matter the estimation method. Last but not least is the fact that all three indices show the same cycle of the length of half a trading year.

As for the US indices, they yield expected results in the sense of independence as all examined indices are either independent or dependent during only several periods. However, the results have showed some new features which were not covered by other indices. Estimates of SP500 were biased by short-term memory. On the other hand, estimates of DJI30 were biased by trends. Generally, DJI30 behaved differently when compared to the other US indices. Not only did it show persistent behavior in several periods, it also exhibited a cycle of half a trading year in comparison to the whole trading year of all other indices.

The Asian indices show strongly different performance compared to each other. NIKKEI index behaves in a very similar manner as the indices of the Western Europe – several periods of an anti-persistent behavior, which are caused by neither short-term memory nor trending of the time series. On the other hand, the crossover detection showed a cycle of only 64 days or no cycles at all as V statistics behaved in rather non-monotonous way and point to point derivatives of H were inconclusive as well. SSEC has shown behavior, which is on the edge of independence and persistence, for almost whole examination period. The persistent features of the index vanish during last trading year where Hurst exponent shows a decreasing trend to an independent region. This trend is in contrast with the results of other indices (with exception of SAX) as they show rather increasing trend of Hurst exponent during 2008.

The correlation analysis has revealed quite strong connection of ATX to the rest of Central European region. Other results supported previous findings such as a close connection between CAC40, DAX and FTSE as well as the expected connection between SP500 and DJI30. However, we still need to check the correlation for all pairs of indices. The results are summed in Table 0-1, Table 0-2 and Table 0-3 in Appendix for R/S, DFA-1 and DFA-2, respectively. The most interesting implications, which were not covered in previous sections, are as follows.

PX as the only one from the Central European indices has medium positive correlation with all three examined US indices based on all methods used for correlation analysis. Moreover, the index shows medium positive correlation with FTSE and weak positive one with DAX, CAC40 and NIKKEI. On the other hand, BUX and WIG20 show no significant correlations with any of the indices from other regions. Interestingly, SAX is weakly positively correlated with DAX and CAC40 while being medium positively correlated with NASDAQ. ATX shows medium positive correlation with FTSE. NIKKEI, quite expectedly, shows medium positive correlation with all of CAC40, FTSE and DAX. SSEC, on the other hand, shows no significant correlations with exception of rather random medium positive correlation with ATX. Therefore, the correlation analysis has shown that NIKKEI is strongly connected to the indices of the Western Europe as well as is PX which is additionally correlated with the US indices.

As for the efficiency of the indices, NASDAQ and DAX are the only indices which were efficient during whole examined period. Other indices showed at least several periods for which they were inefficient in either F65 or S65 sense. ATX, BUX and WIG20 have been shown to be non-stationary and thus inefficient in F65 even before long-term memory analysis. Nevertheless, BUX and WIG20 were independent for the whole examined period and thus S65 efficient. ATX, on the other hand, moved from inefficient to S65 efficient market in 2006. PX moved from inefficient to efficient market in 2007 and remained so until the end of the examined period. CAC40, FTSE, NIKKEI and SP500 were efficient during majority of the examined period while experiencing several anti-persistent periods which are connected with F65 inefficiency. SSEC showed stable movement towards efficiency of both types while SAX, reversely, was becoming less efficient in time and thus is the least efficient overall.

For the methodological part, the detection of crossovers based on point to point derivatives of Hurst exponent has supported the assertion made in Chapter 3 which is the use of minimum scale of 16 trading days. The reason is straightforward as point to point derivatives were in vast majority of cases higher than one and usually three or even four times higher than average value of the rest of the derivatives. In the same way, the use of one fourth of time series length as a maximum scale was supported in a similar way as again for a vast majority of cases, point to point derivatives significantly deviated from the other ones.

❖ Conclusion

We have shown that neither methodology nor applications of Hurst exponent estimations are uniform throughout the literature. The unification and improvement of current techniques have been the main aim of the thesis. We have arrived to several interesting theoretical as well as applied results.

On the methodological part, we have shown that asymptotic limits of *H* being equal to 0.5 are far from finite sample estimates for random time series for R/S and M-R/S analysis. Both methods have shown that the estimates are higher than the limit in infinity. On the other hand, we have confirmed the findings of several recent papers that DFA with a constant, a linear and a quadratic detrending show estimates of the exponent very close to its asymptotic limit. However, we argue that the result does not discredit R/S and M-R/S analysis. On the contrary, the simulations have showed that R/S and M-R/S analysis are not biased but need to be applied together with confidence intervals which have been constructed for various time series lengths for all presented methods. The use of confidence intervals also strongly rejects the method used by several authors who reject independence of the time series just on the basis of inequality of estimated *H* to its asymptotic limit of 0.5. The method has been shown to be absolutely incorrect as the confidence intervals are rather wide.

Moreover, we have uncovered a weak point of M-R/S which occurs when the time series contains several consecutive zero returns. The problem was uncovered while examining SAX index and pointed out that the method can be highly biased as the estimate of modified standard deviation for low scales is highly underestimated. As a consequence, the estimate of modified rescaled range is strongly overestimated and the estimated Hurst exponent is in turn leveraged down. Therefore, liquidity of the market has been shown to be quite important for estimates of the exponent.

As an interconnection between methodological and applied findings, we have shown that estimates which take into consideration all available scales can be unstable. On the basis of *V* statistic, we have shown that two highest scales are rather different from the others. Moreover, we have presented the results based on point to point derivatives of *H* which have supported the findings based on *V* statistic and additionally uncovered that the lowest scales act differently. The use of the lowest scales for the estimation of *H* yields overestimated results and the assertion has been shown to be stronger for DFA methods. Therefore, we propose to use the minimum scale of 16 trading days and the maximum scale of one fourth of the time series length for all of used methods.

The interesting results of applied part are summed in the previous section. Nonetheless, we present the most important ones. Indices are strongly interconnected at both their stage of independence and the dynamics. The most similar are the indices of the Western Europe (DAX, CAC40 and FTSE) which are also close to the index of Japan (NIKKEI). These indices also show the phenomena which has been discussed rarely in the literature – anti-persistence. When the long-term dependence was uncovered in the stock markets, it was mostly persistence which was driving the market. However, the indices of Western Europe and Japan show almost two years where the anti-persistence was present.

Further, there have been several patterns present in almost all the indices. The most frequent one has been the change of market dynamics from a decreasing trend of Hurst exponent, which is connected with growing uncertainty of the investors, to an increasing one, which shows that the investors came to believe that the starting trend is about to continue for several periods. Such pattern was mostly connected with significantly negative returns.

Furthermore, almost all indices reacted very similarly to the financial crisis of 2008. The crisis was preceded by slowly decreasing trend of Hurst exponent which reversed into slowly increasing trend of the exponent (Central Europe) or rapid increase of the exponent followed by relatively stable behavior close to the borderline between independent and persistent behavior (Western Europe, Japan and the USA). The index of Slovakia (SAX) has showed only mild reaction to the crisis. On the other hand, SSEC has been influenced by the crisis in different way. The start of the crisis has been connected with beginning of the decreasing trend of Hurst exponent which has led SSEC into independent behavior. Therefore, the financial crisis has led the index of China to higher efficiency.

To sum the conclusions up, we have presented several new ideas and applications of the fractal approach to the financial markets and have showed that the fractality and specifically long-term dependence has been present in almost all the examined markets for at least some periods. Moreover, if the independence could not be rejected, there was interesting dynamics in the market which could be further examined.

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❖ Appendix

Gamma function and beta function

Gamma function Γ is defined as

$$\Gamma(n) = (n-1)! \text{ for } n \in \aleph \tag{0.1}$$

and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \text{ for } z \in \Re . \tag{0.2}$$

Equation (0.2) holds for complex numbers as well. However, we work with real numbers only. Following part covers only definitions needed for purposes of this thesis. For more detailed description, see Boisvert *et al.* (2008).

Beta function B is defined as

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \text{ for } x, y \in \Re_+ . \tag{0.3}$$

The most important relationship between both functions for our purposes is

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} . \tag{0.4}$$

Stirling's asymptotic approximation of B for high x and high y states that

$$\mathrm{B}(x, y) \approx \sqrt{2\pi} \frac{x^{x-\frac{1}{2}} y^{y-\frac{1}{2}}}{(x+y)^{x+y-\frac{1}{2}}} . \tag{0.5}$$

Nevertheless, this approximation does not solve the computational problem for very high values of x and y. For high value of x and low value of y, Stirling's asymptotic approximation of B is stated as

$$\mathrm{B}(x, y) \approx \Gamma(y) x^{-y} . \tag{0.6}$$

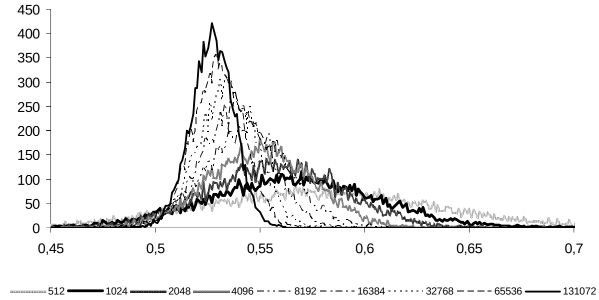
Therefore, $\Gamma\left(\frac{\upsilon-1}{2}\right) \Big/ \Gamma\left(\frac{\upsilon}{2}\right)$ from equation of Anis & Lloyd (1976) can be transformed for high values of υ. If we set $x = \frac{\upsilon-1}{2}$ and $y = \frac{1}{2}$ and use (0.4) and (0.6), we get the exact approximation as following:

$$\frac{\Gamma\left(\frac{\upsilon-1}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)} = \frac{\mathrm{B}\left(\frac{\upsilon-1}{2}, \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{\upsilon-1}{2}\right)^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} = \sqrt{\frac{2}{\upsilon-1}} . \tag{0.7}$$

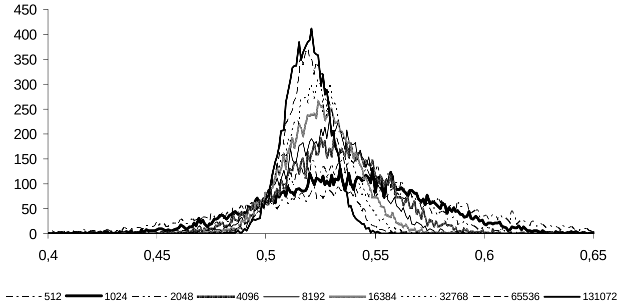
Distributions of simulated Hurst exponents

Chart 0-1 Distributions of simulated Hurst exponents

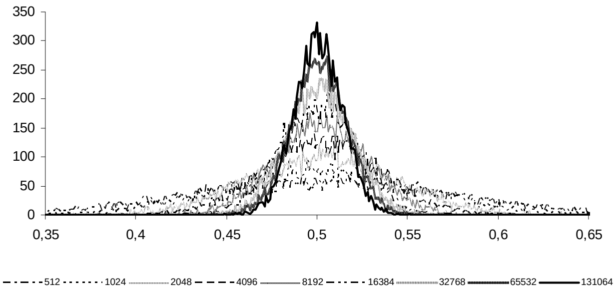
(a)



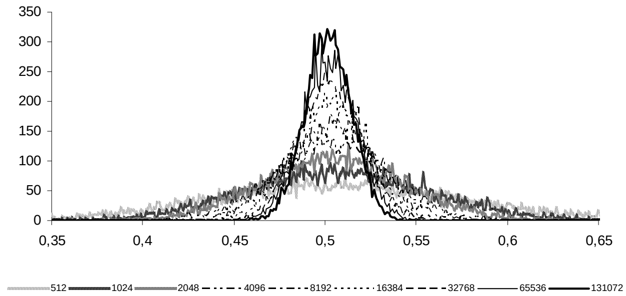
(b)



(c)



(d)



(e)

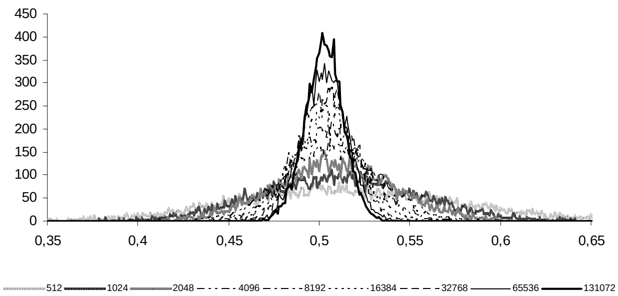


Chart 0-1: (a) R/S, (b) M-R/S, (c) DFA-0, (d) DFA-1, (e) DFA-2: Charts show distributions of simulated Hurst exponents. For each time series length, 10000 simulations have been run with minimum scale of 16 and maximum scale of one fourth of the time series length.

Index values and returns charts

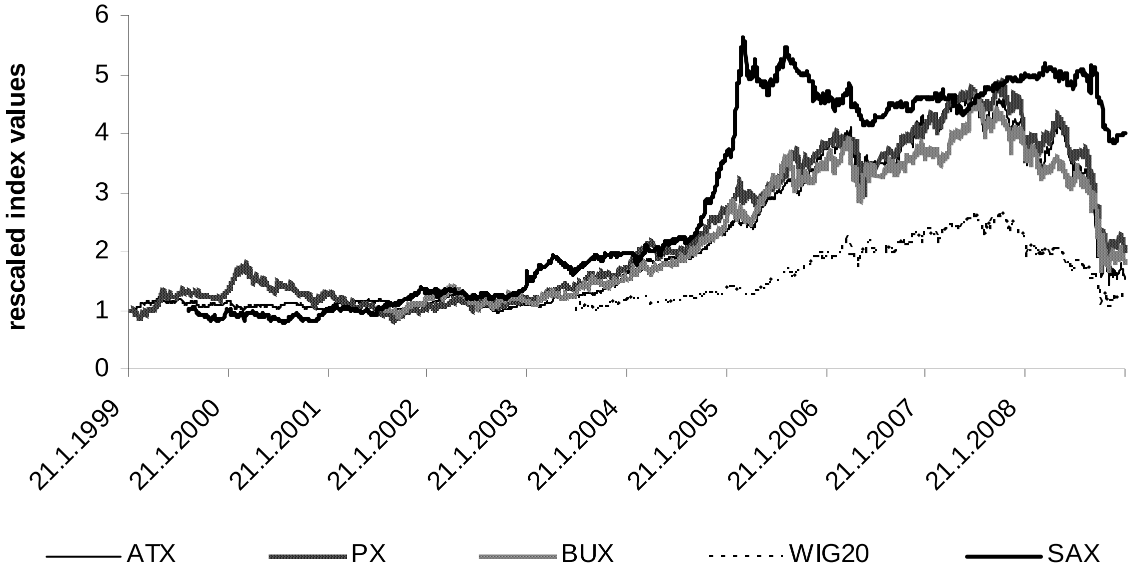


Chart 0-2 Index values of ATX, PX, BUX, WIG20 and SAX

Index values are divided by initial value. Chart shows that PX, BUX and ATX behave very similarly; WIG20 has shorter data set but still behaves similarly to mentioned indices; SAX shows different evolution with much earlier peak.

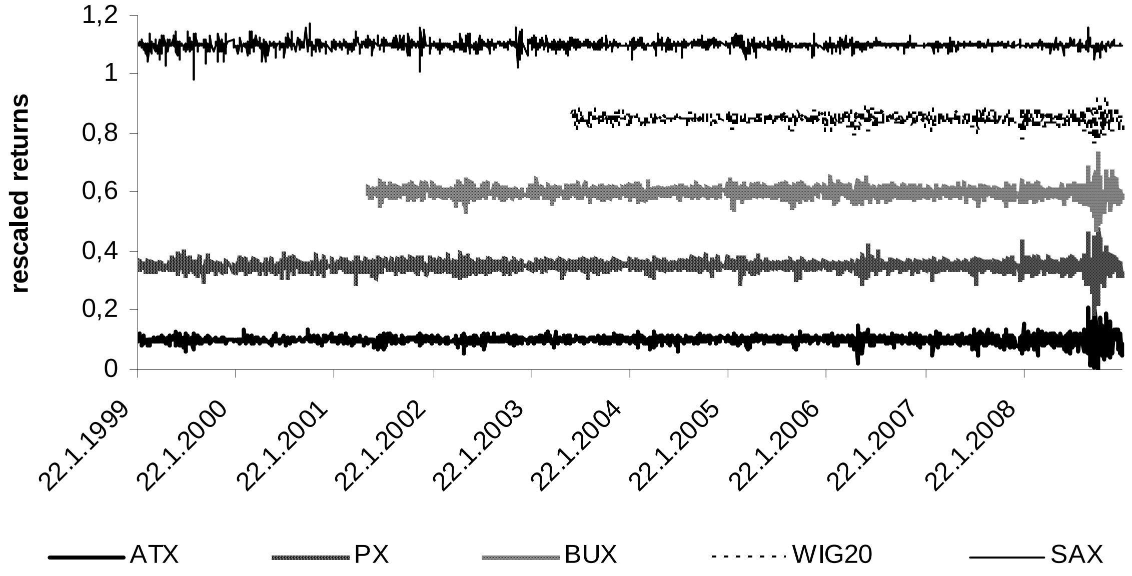


Chart 0-3 Logarithmic returns of ATX, PX, BUX, WIG20 and SAX

Logarithmic returns are adjusted by addition of 0.1, 0.35, 0.6, 0.85 and 1 to ATX, PX, BUX, WIG20 and SAX, respectively. Chart shows that SAX differs significantly from other indices as it is less liquid with many non-trading days. Other indices behave similarly to each other.

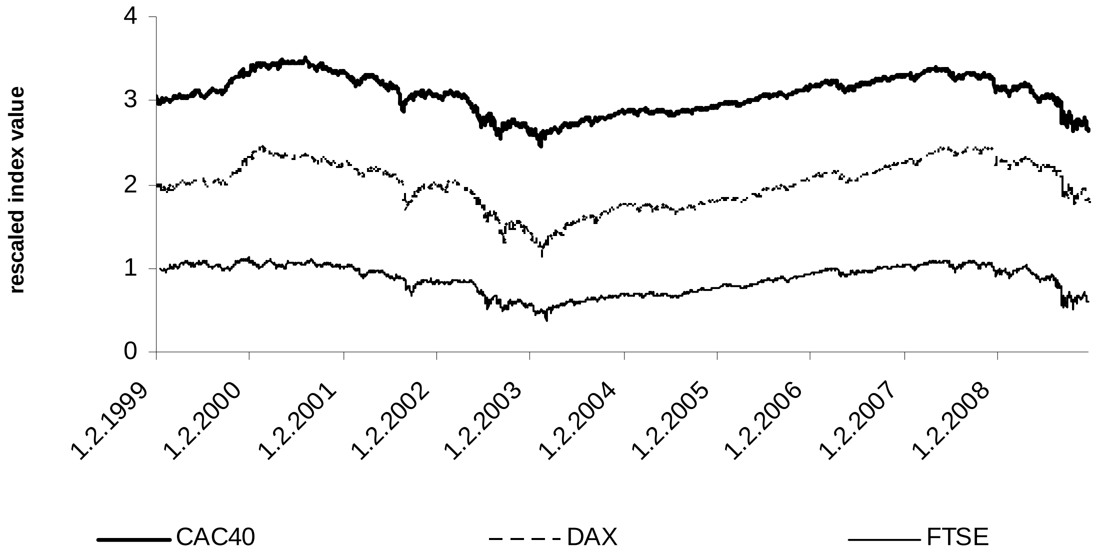


Chart 0-4 Prices of CAC40, DAX and FTSE

Index values are divided by initial value (at 21.1.1999) and 0, 1, and 2 is added for FTSE, DAX and CAC40, respectively. All indices show very similar behavior with two peaks and two strong trends.

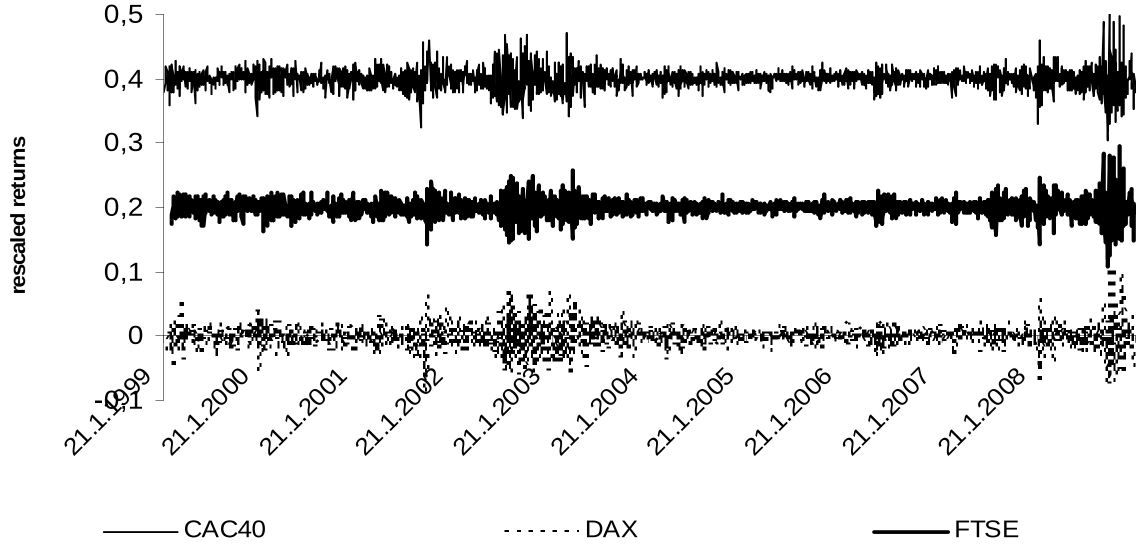


Chart 0-5 Returns for CAC40, DAX and FTSE

Returns are rescaled by adding 0, 0.2 and 0.4 for DAX, FTSE and CAC40, respectively. All indices show increased volatility during same periods.

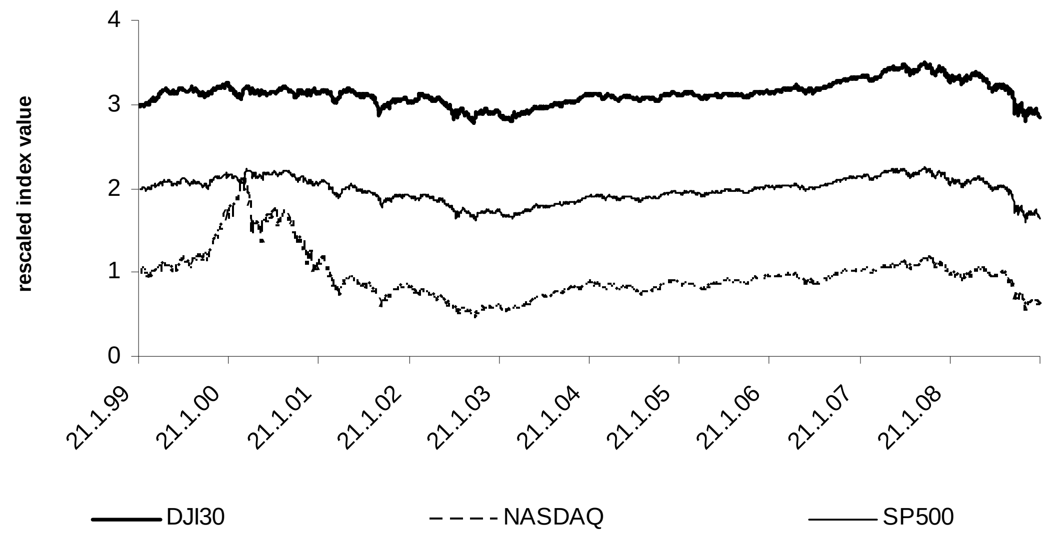


Chart 0-6 Prices of DJI30, NASDAQ and SP500

Index values are divided by initial value (at 21.1.1999) and 0, 1, and 2 is added for NASDAQ, SP500 and DJI30, respectively. NASDAQ behaves differently during "DotCom" bubble; otherwise, the indices show similar behavior.

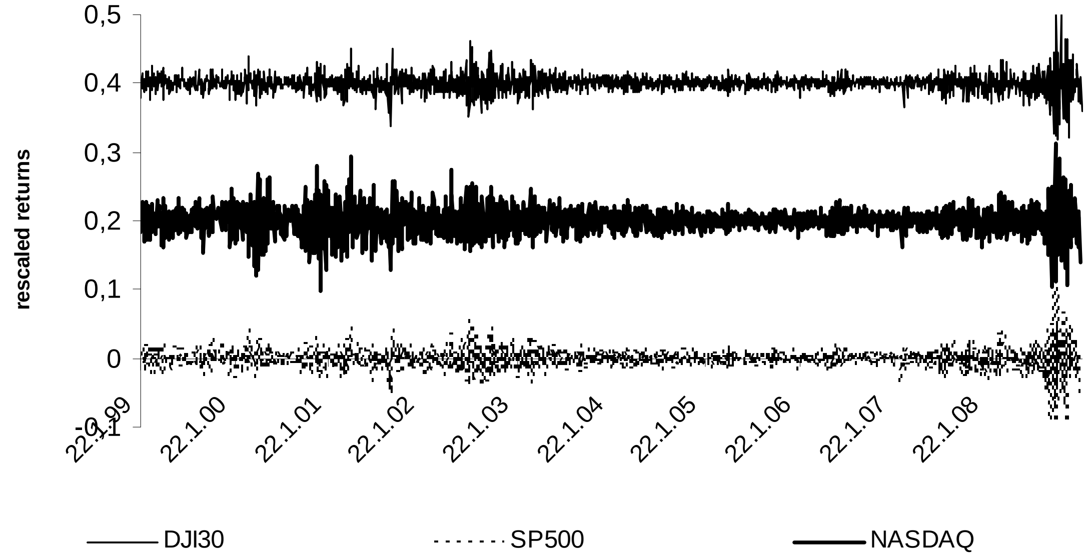


Chart 0-7 Returns of DJI30, NASDAQ and SP500

Returns are rescaled by adding 0, 0.2 and 0.4 for SP500, NASDAQ and DJI30, respectively. All indices show increased volatility during the financial crisis of 2008. NASDAQ differs during "DotCom" bubble and following years where its volatility increased significantly compared to SP500 and DJI30.

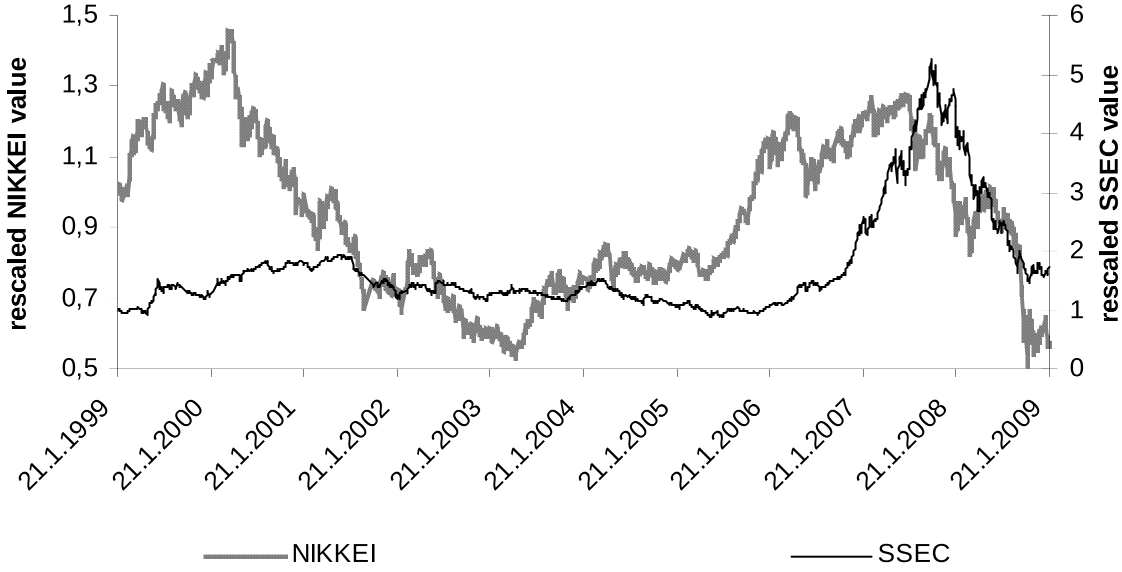


Chart 0-8 Prices of SSEC and NIKKEI

Index values are divided by initial value (at 21.1.1999). NIKKEI (left y-axis) shows behavior very similar to CAC40, DAX and FTSE (Chart 8-3). Behavior of SSEC (right y-axis) is unique when compared to other indices.

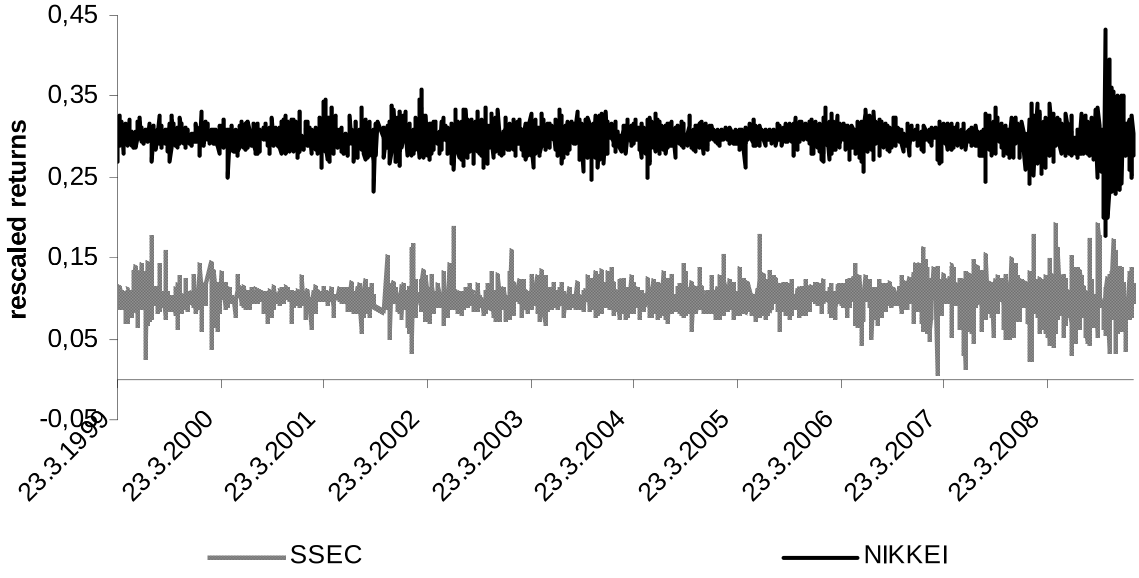


Chart 0-9 Returns of SSEC and NIKKEI

Returns are rescaled by adding 0.1 and 0.3 for SSEC and NIKKEI, respectively. Increased volatility during financial crisis of 2008 is visible for both indices. However, the increase of volatility started earlier for SSEC. Moreover, SSEC did not trade between 8.2.2002 and 25.2.2002

Crossover detection charts

Chart 0-10 Crossover detection of ATX

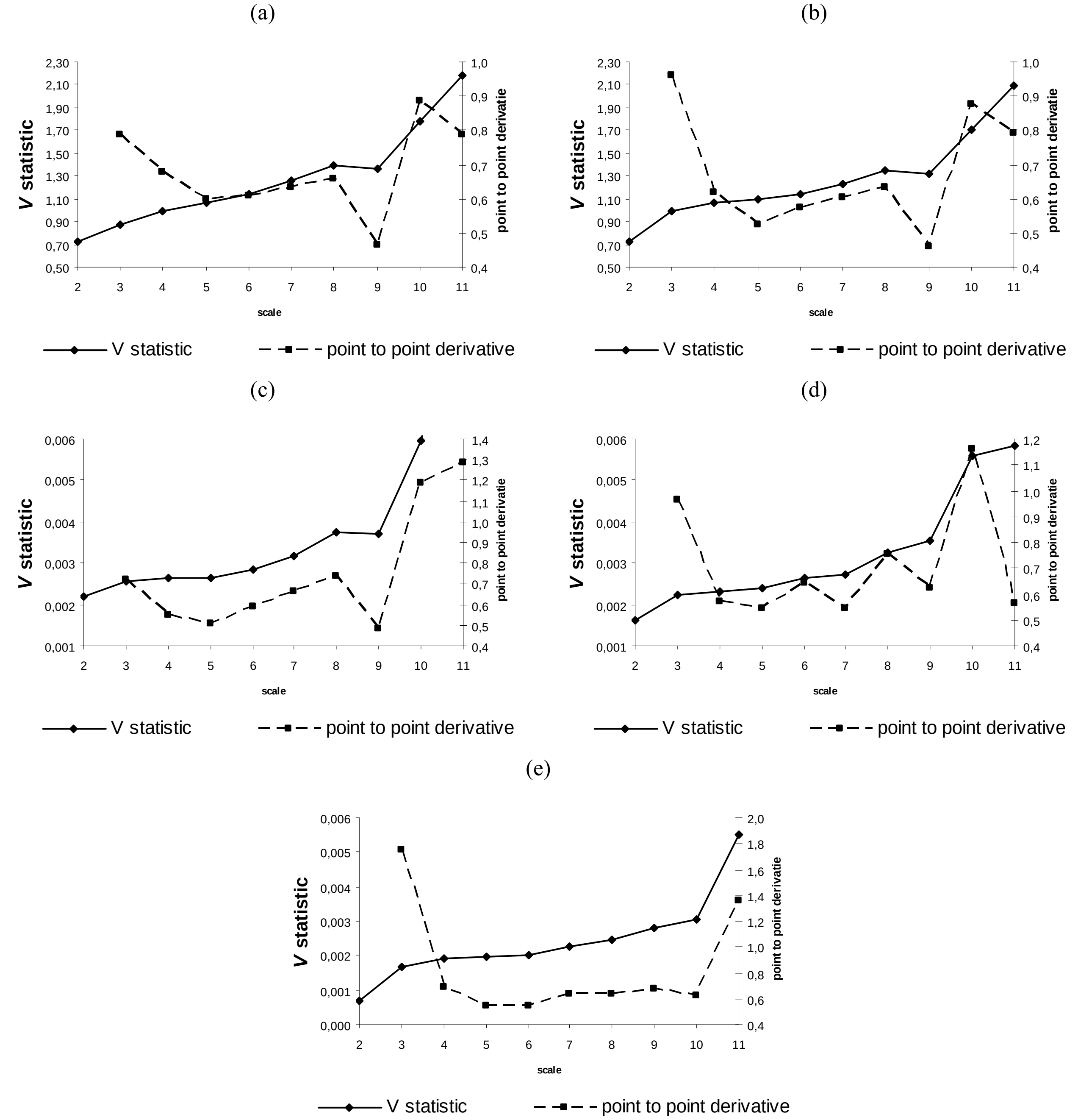
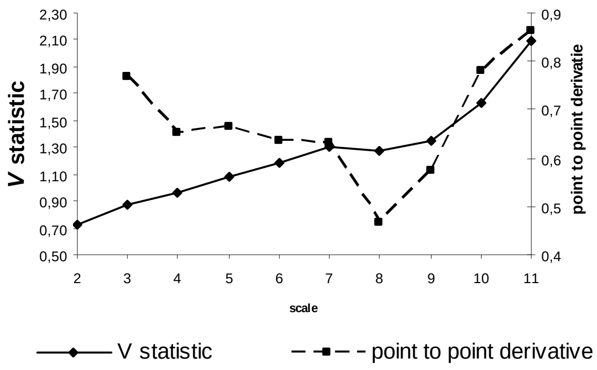


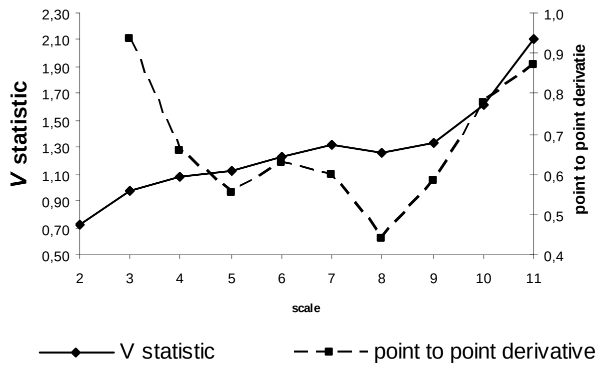
Chart 0-10: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S analysis (a), M-R/S (b) and DFA-0 (c) show a crossover at scale of 256 trading days while DFA-1 (d) and DFA-2 (e) show no crossover implying that a crossover is caused by linear trend in the time series. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. DFA methods show volatile behavior of point to point derivatives of *H* at two highest scales and thus support the use of one fourth of the time series length as a maximum scale. Moreover, DFA methods have much higher point to point derivative of *H* at the lowest scale of 8 trading days and thus support the use of 16 trading days as a minimum scale. R/S and M-R/S show similar results which imply the same minimum and maximum scale for *H* estimation.

Chart 0-11 Crossover detection of PX

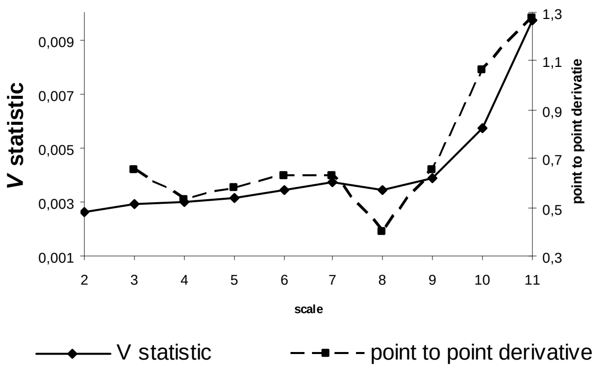
(a)



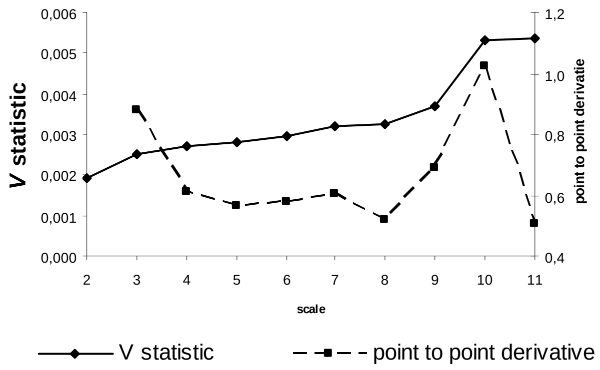
(b)



(c)



(d)



(e)

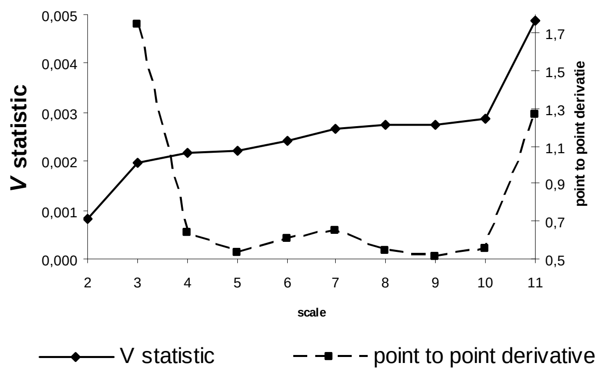
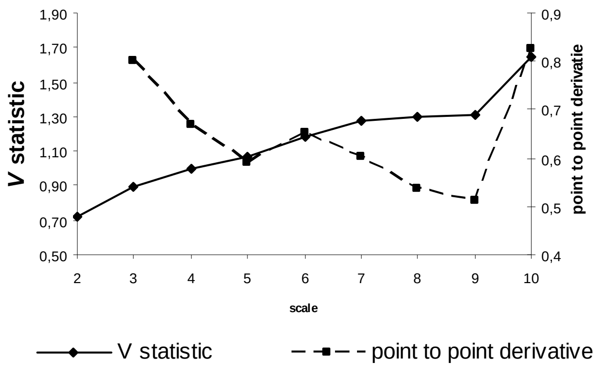


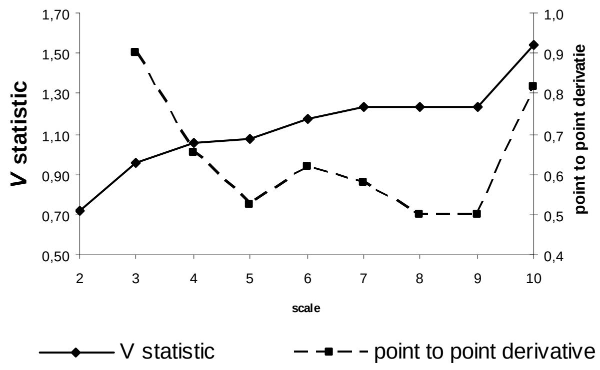
Chart 0-11: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S analysis (a), M-R/S (b) and DFA-0 (c) show a crossover at scale of 128 trading days while DFA-1 (d) and DFA-2 (e) show no crossover implying that a crossover is caused by linear trend in the time series. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. DFA methods show volatile behavior of point to point derivatives of *H* at two highest scales and thus support the use of one fourth of the time series length as a maximum scale. Moreover, DFA-1 and DFA-2 have much higher point to point derivative of *H* at the lowest scale of 8 trading days and thus support the use of 16 trading days as a minimum scale. R/S and M-R/S show unstable behavior at scales higher than proposed maximum scale and at the lowest scale which support the minimum and maximum scale for *H* estimation.

Chart 0-12 Crossover detection of BUX

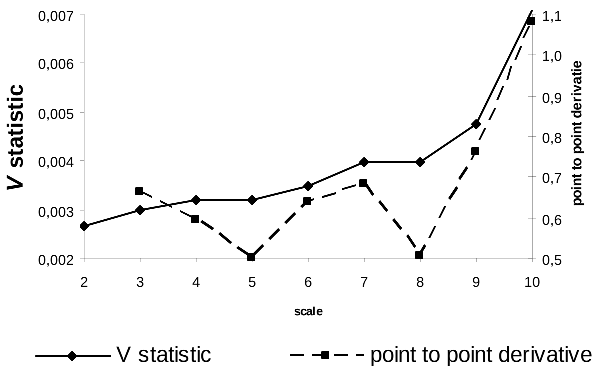
(a)



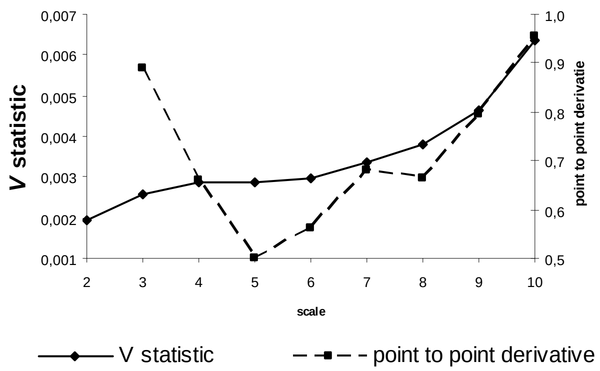
(b)



(c)



(d)



(e)

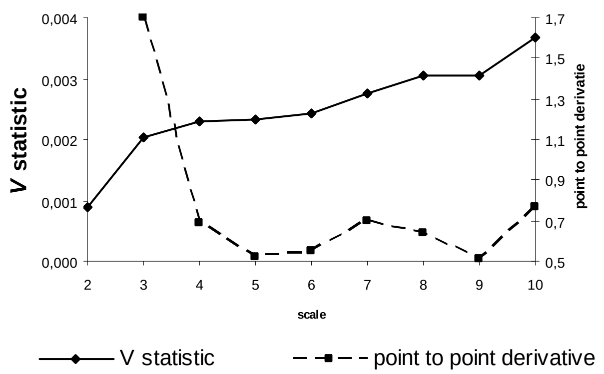


Chart 0-12: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. All methods show no crossover implying that there is no trend in the time series. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. DFA-0 and DFA-1 show point to point derivatives of *H* well above the average at two highest scales and thus support the use of one fourth of the time series length as a maximum scale. Moreover, DFA-1 and DFA-2 have much higher point to point derivative of *H* at the lowest scale of 8 trading days and thus support the use of 16 trading days as a minimum scale. R/S and M-R/S show above average point to point derivatives at the lowest and the highest scale.

Chart 0-13 Crossover detection of WIG20

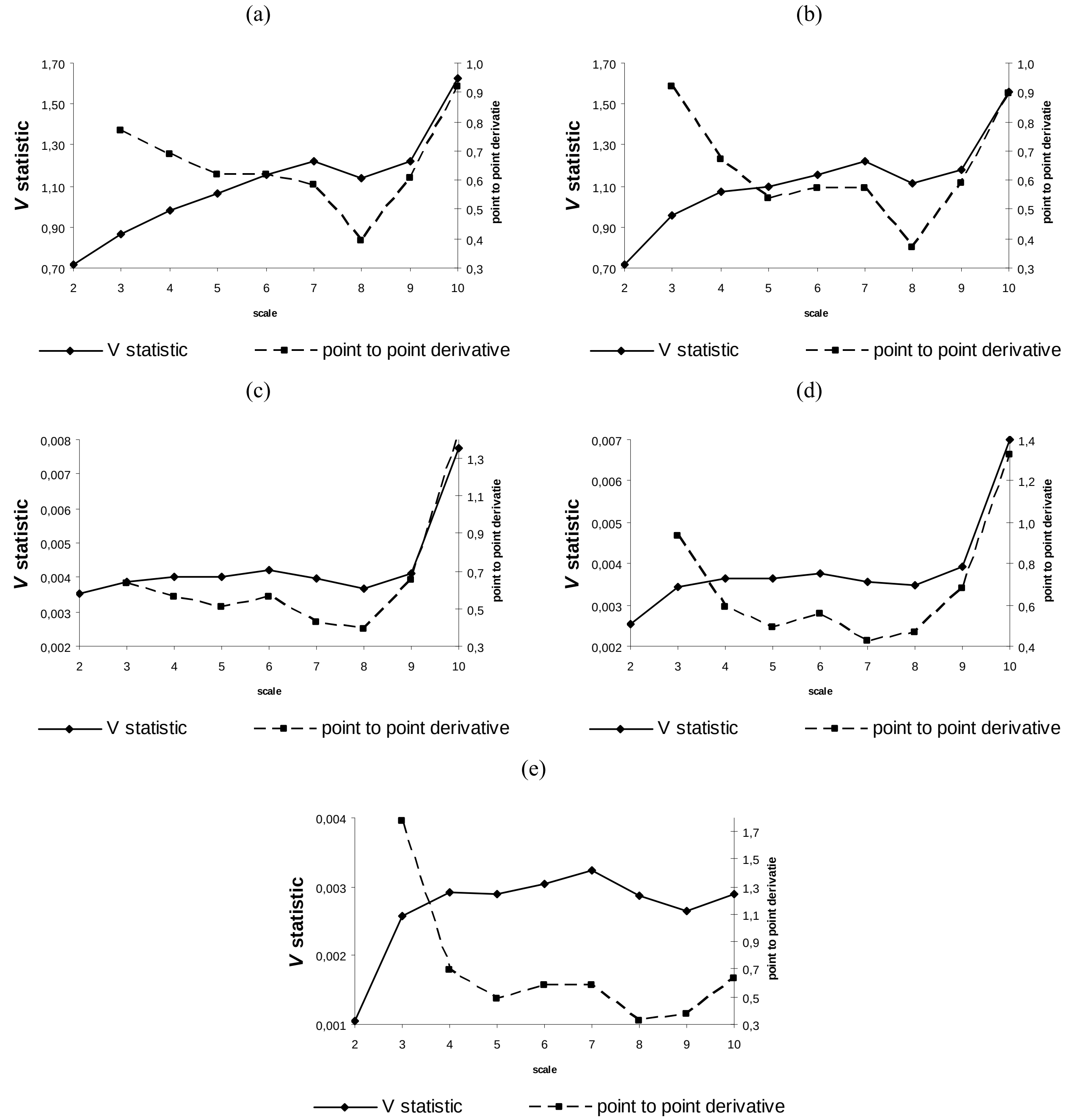


Chart 0-13: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a), M-R/S (b) and DFA-2 (e) show a crossover at scale of 128 trading days. DFA-0 (c) and DFA-1 (d) show a crossover at scale of 64 trading days and thus are inferior to DFA-2. Such results lead to use of R/S, M-R/S and DFA-2 in the analysis of time-dependent Hurst exponent. DFA-0 and DFA-1 show point to point derivatives of *H* well above the average at the highest scale. Moreover, DFA-1 and DFA-2 have much higher point to point derivative of *H* at the lowest scale of 8 trading days and thus support the use of 16 trading days as a minimum scale. R/S and M-R/S show above average point to point derivatives at the lowest and the highest scale.

Chart 0-14 Crossover detection of SAX

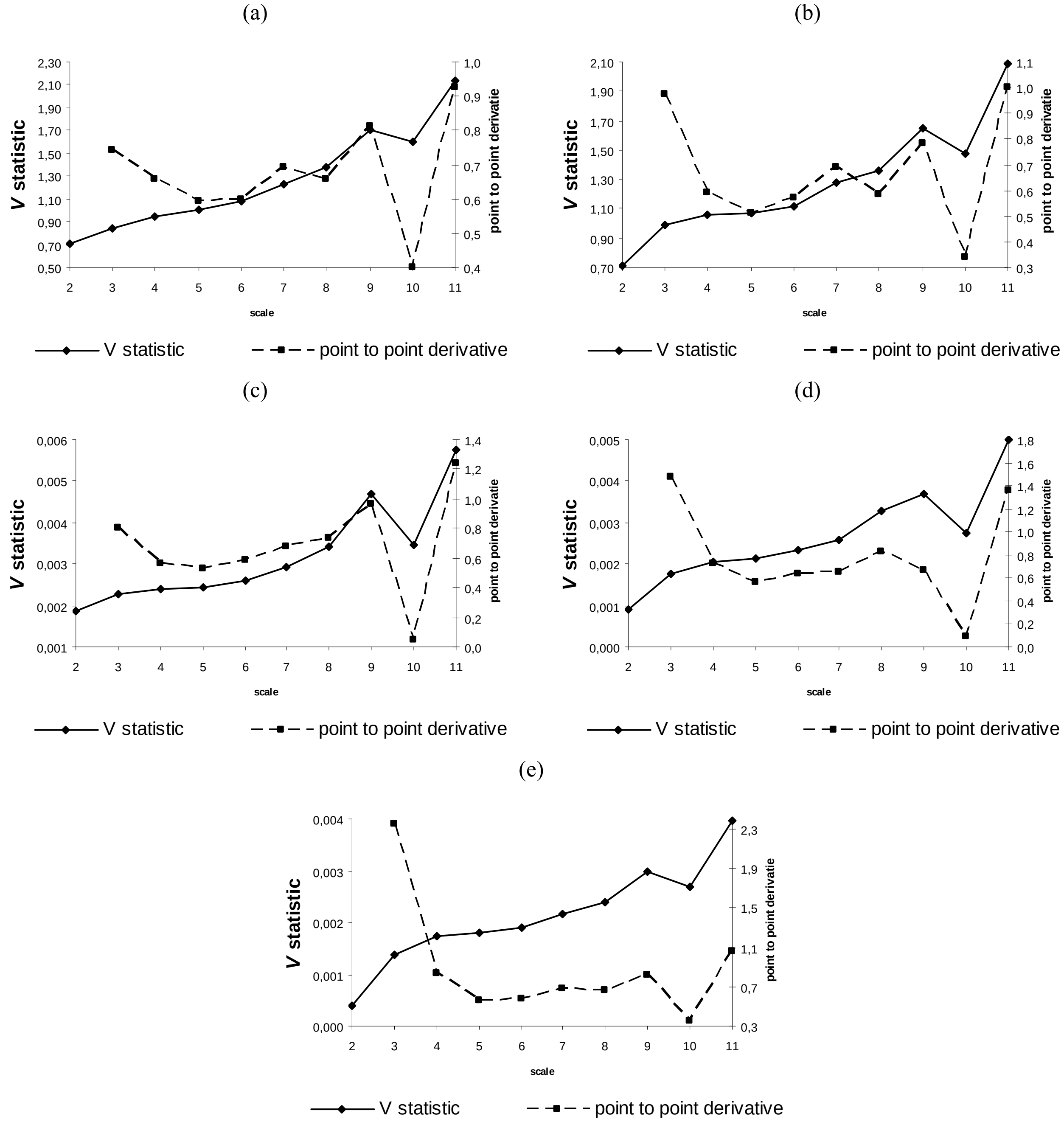
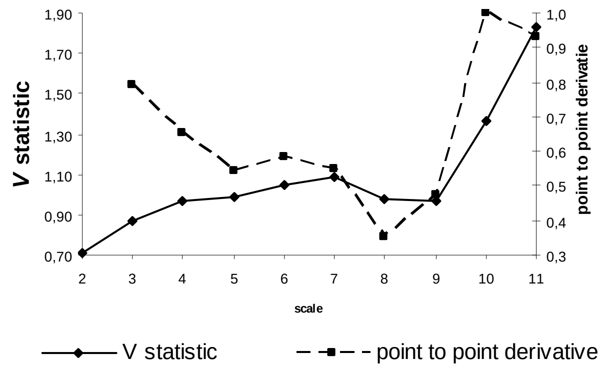


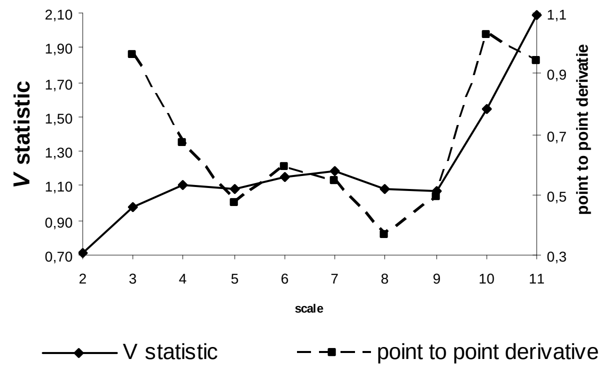
Chart 0-14: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. All methods show a crossover at scale of 512 trading days. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. All methods show volatile behavior of point to point derivatives of *H* at two highest scales and above-average value at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-15 Crossover detection of CAC40

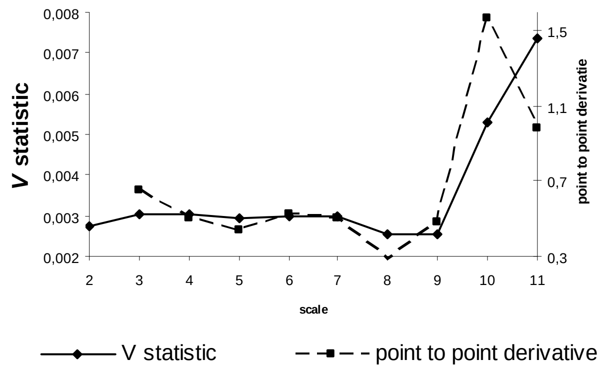
(a)



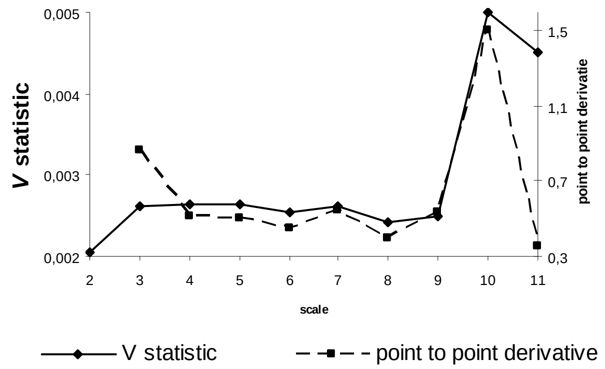
(b)



(c)



(d)



(e)

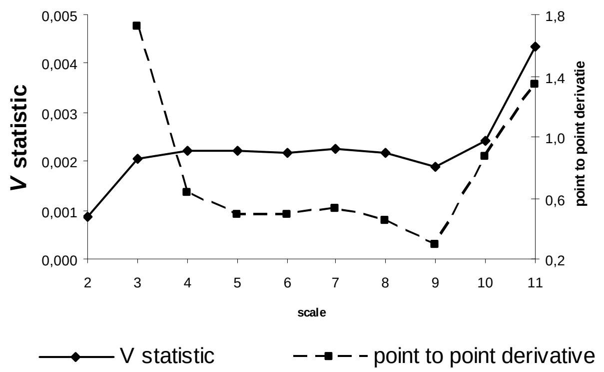


Chart 0-15: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. All methods show a crossover at scale of 128 trading days. However, the results are not clear for DFA methods. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-16 Crossover detection of DAX

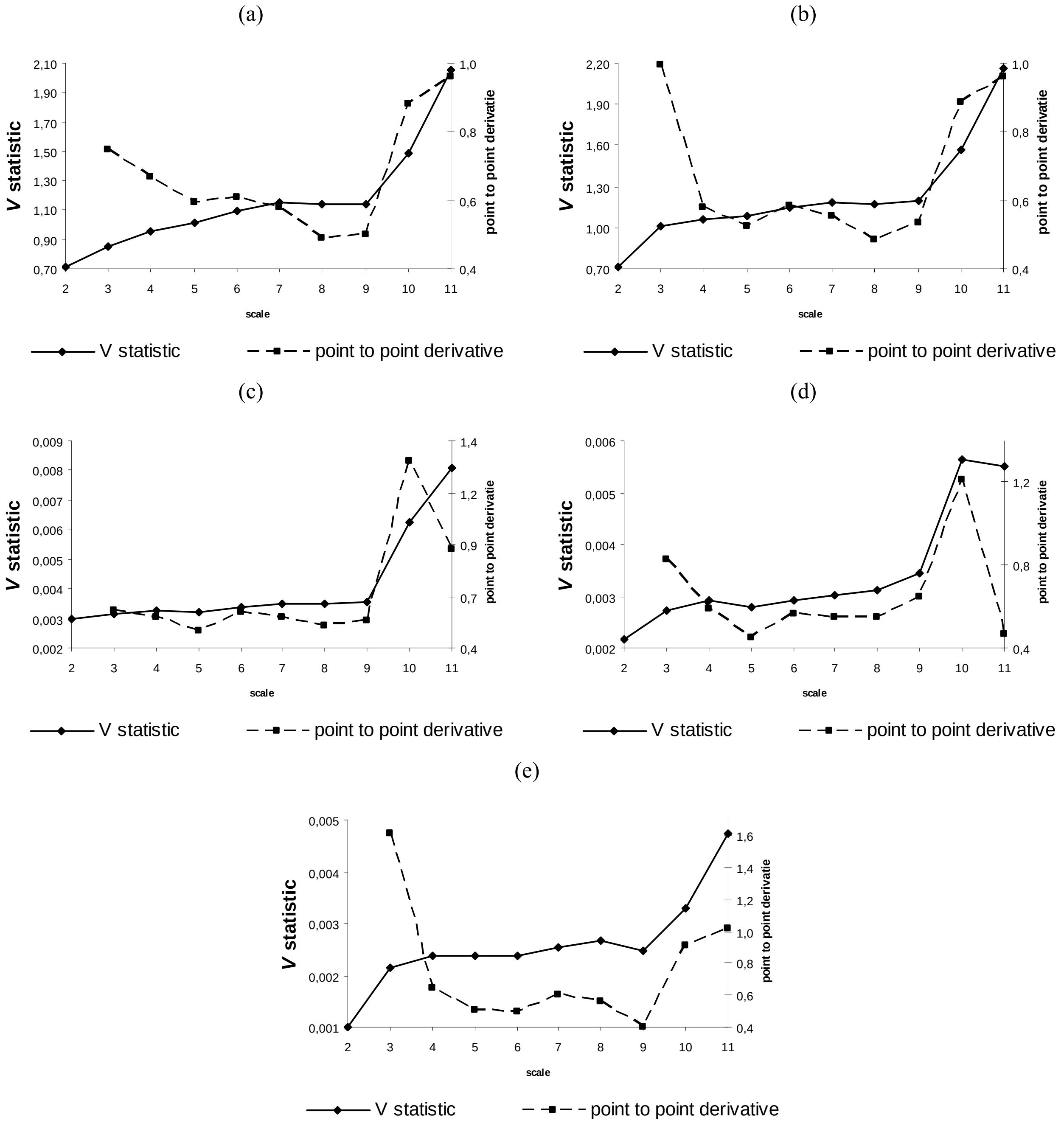


Chart 0-16: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a) and M-R/S (b) show a crossover at scale of 128 trading days. DFA-0 (c) and DFA-1 (d) show no crossover and DFA-2 (e) shows a crossover at scale of 256 trading days. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-17 Crossover detection of FTSE

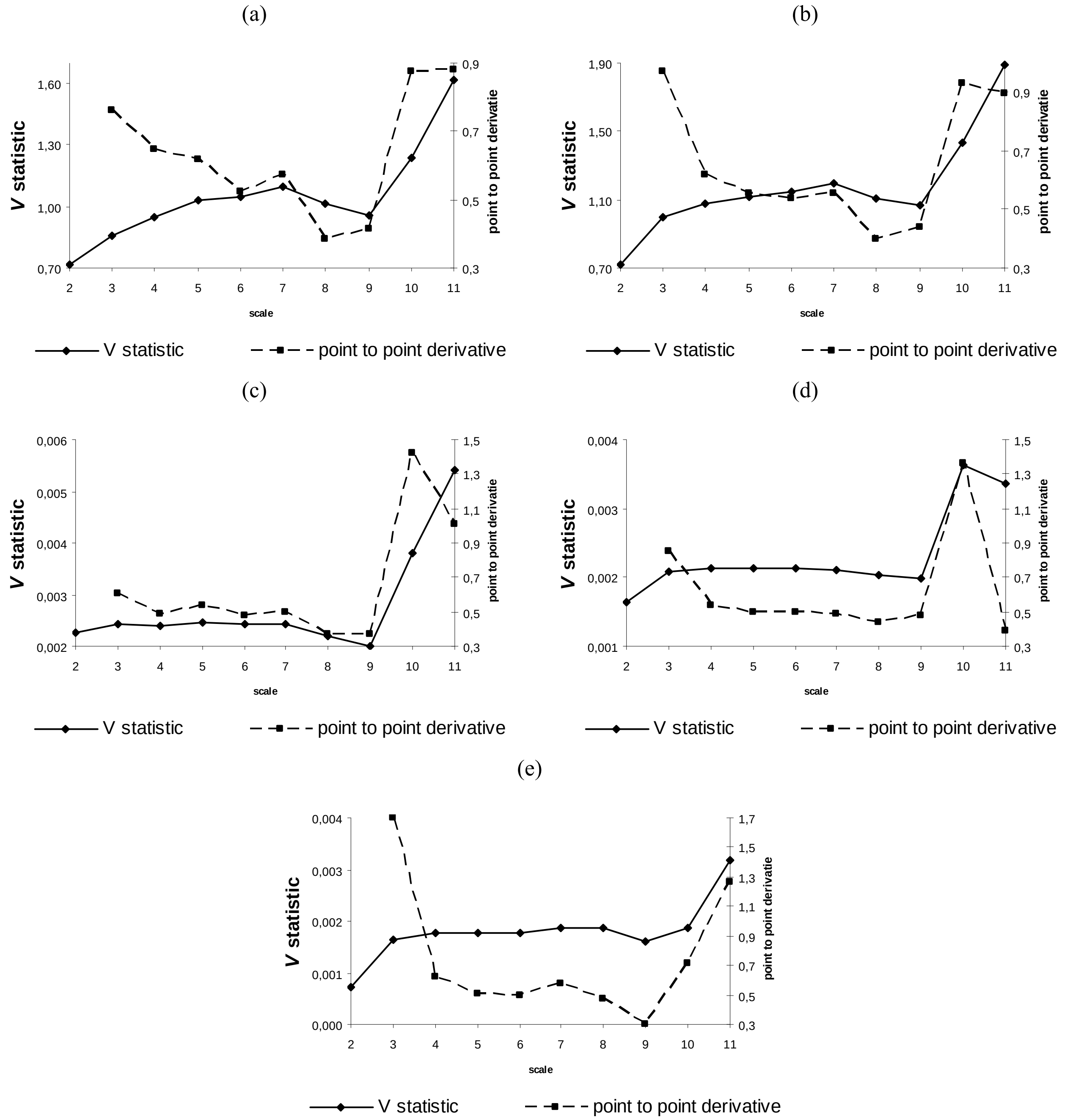
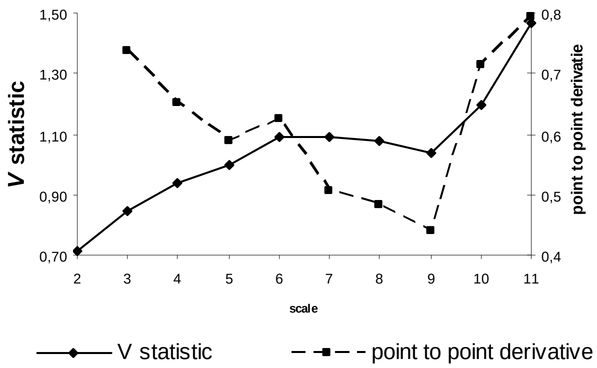


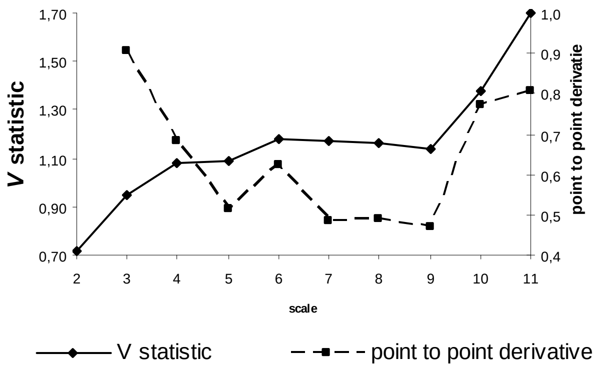
Chart 0-17: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a), M-R/S (b) and DFA-0 (c) show a crossover at scale of 128 trading days. DFA-1 (d) shows a crossover at scale of 512 trading days and DFA-2 (e) shows a crossover at scale of 256 trading days. Note that DFA-1 shows a crossover from anti-persistent to persistent behavior whereas the other methods show the reverse. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-18 Crossover detection of DJI30

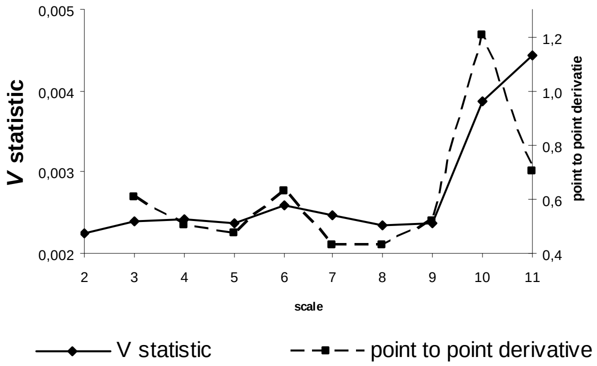
(a)



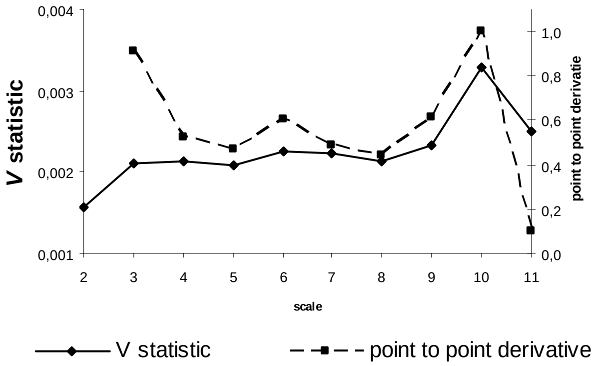
(b)



(c)



(d)



(e)

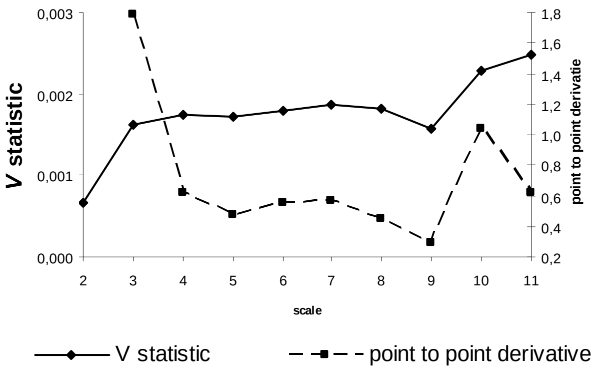
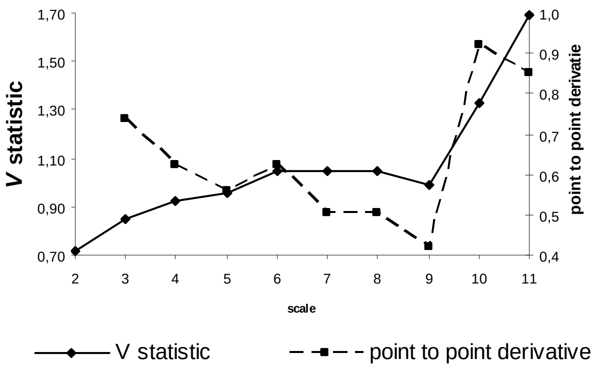


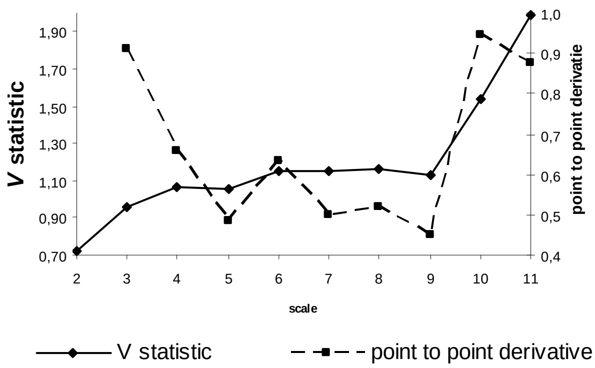
Chart 0-18: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a) shows a crossover at scale of 128 trading whereas M-R/S (b) shows a crossover at a scale of 512 trading days. Note that *V* statistics differ only slightly for both methods and different crossovers are not significant. DFA-0 (c) shows same behavior as M-R/S, DFA-1 (d) shows a crossover at a scale at 256 trading days and DFA-2 (e) at 512 trading days. Such results lead to very unclear conclusions and R/S, M-R/S, DFA-0 and DFA-1 are proposed in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-19 Crossover detection of SP500

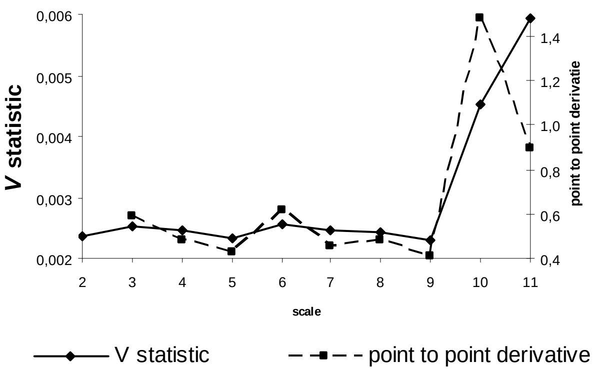
(a)



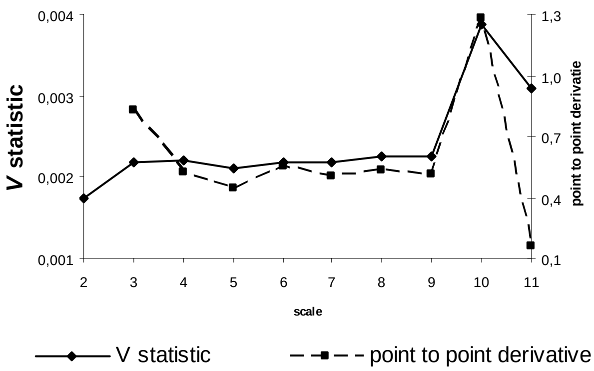
(b)



(c)



(d)



(e)

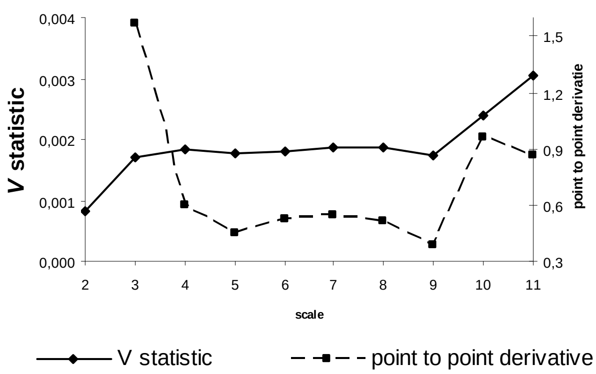
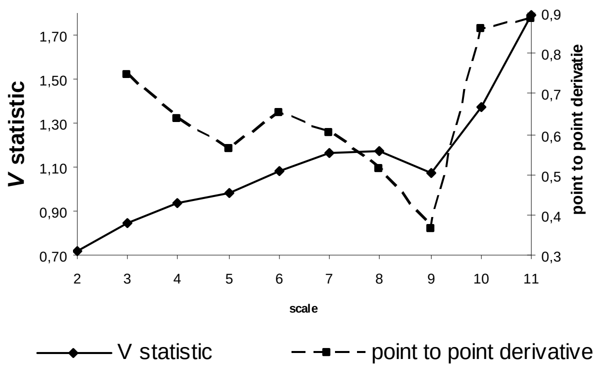


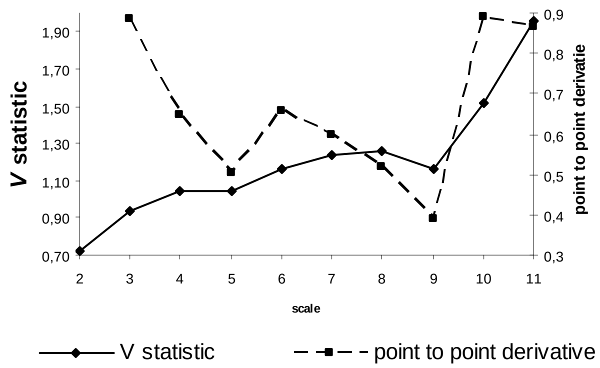
Chart 0-19: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a) and M-R/S (b) show a weak crossover at a scale of 256 trading days. DFA-0 (c) and DFA-1 (d) show a crossover at a scale of 512 trading days while DFA-2 (e) shows the same crossover as R/S and M-R/S. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-20 Crossover detection of NASDAQ

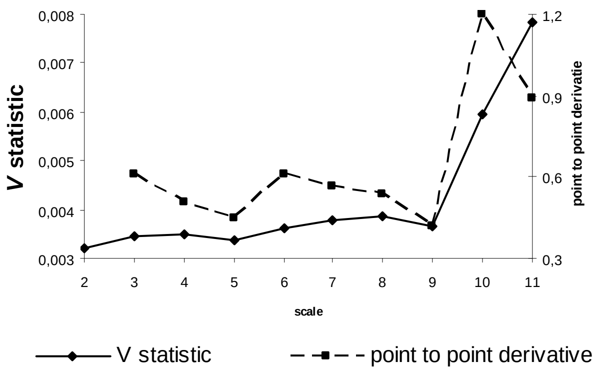
(a)



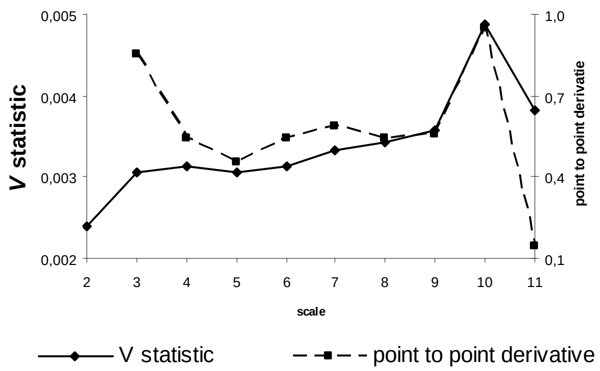
(b)



(c)



(d)



(e)

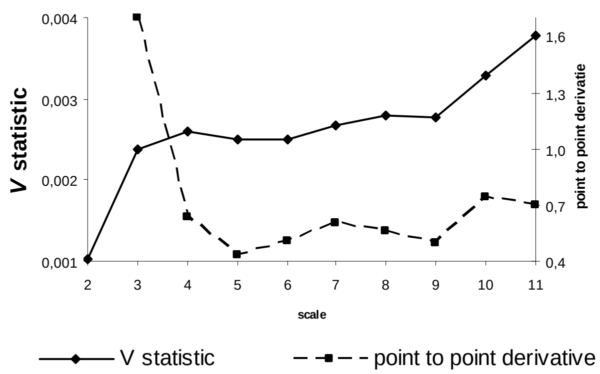
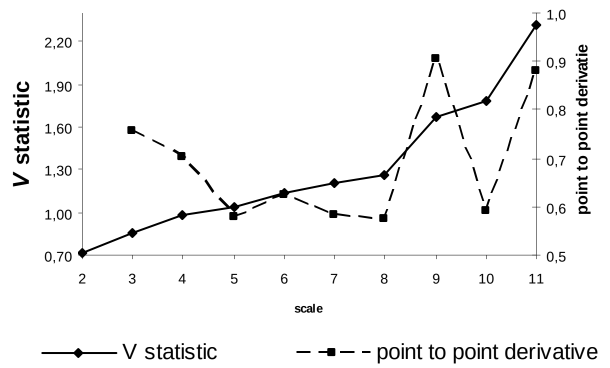


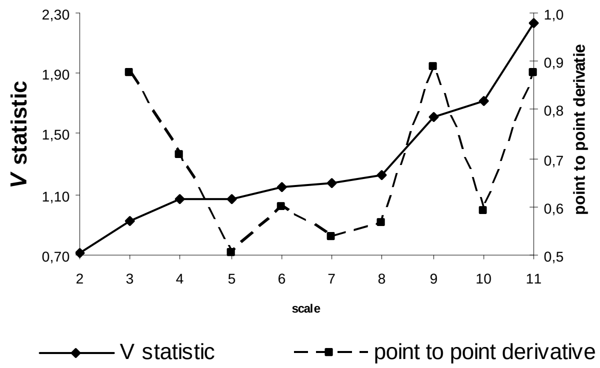
Chart 0-20: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a), M-R/S (b) and DFA-0 (c) show a weak crossover at a scale of 256 trading days. DFA-1 (d) shows no crossover and DFA-2 (e) shows the same crossover as R/S, M-R/S and DFA-0. Such results lead to use of R/S, M-R/S and DFA-1 in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Chart 0-21 Crossover detection of SSEC

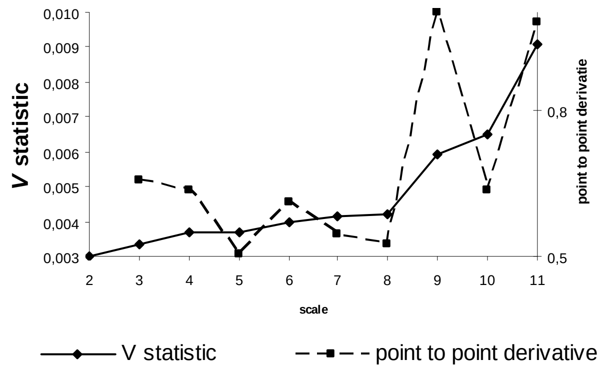
(a)



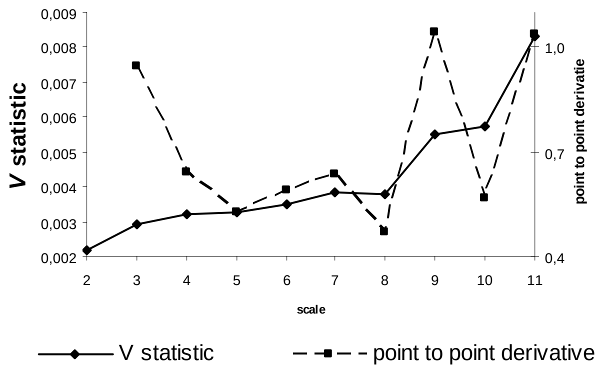
(b)



(c)



(d)



(e)

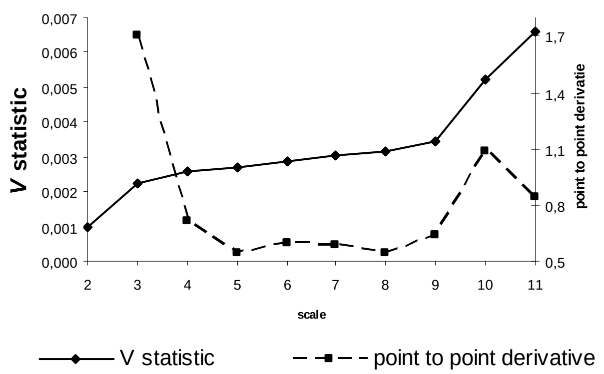
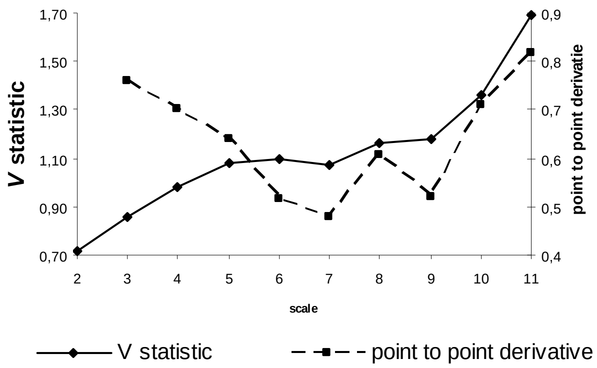
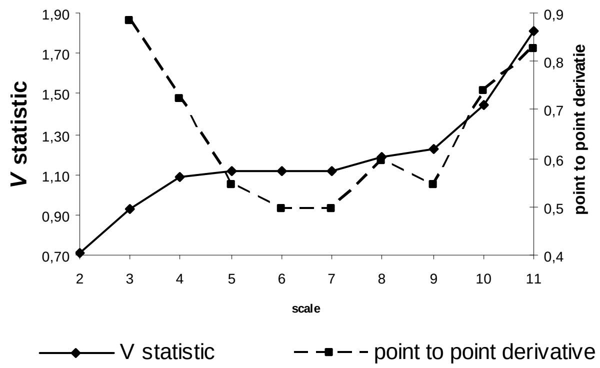


Chart 0-21: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. R/S (a), M-R/S (b), DFA-0 (c) and DFA-2 (e) show no crossover. DFA-1 (d) shows weak crossover at scale of 128 trading days. Such results lead to use of R/S, M-R/S and DFA-2 in the analysis of time-dependent Hurst exponent. All methods with exception of DFA-2 show unstable behavior of point to point derivatives of *H* at three highest scales. DFA-2 shows very volatile behavior of point to point derivatives at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length for DFA-2, respectively.

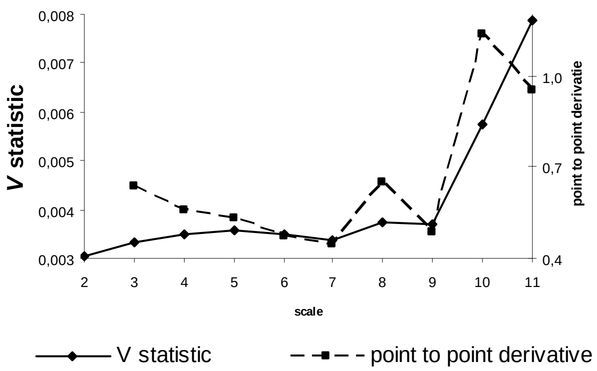
Chart 0-22 Crossover detection of NIKKEI

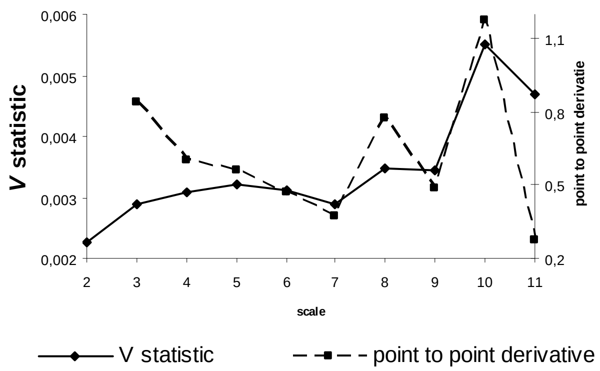
(a) (b)





(c) (d)





(e)

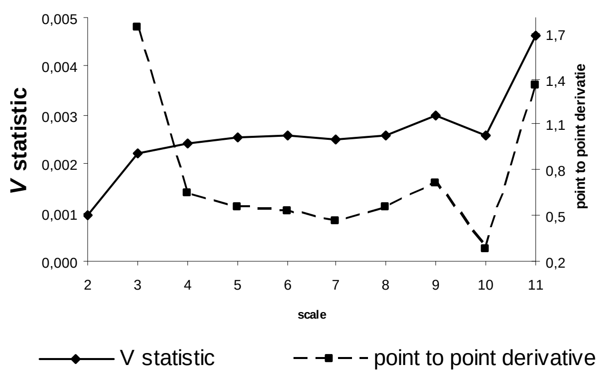


Chart 0-22: (a) R/S based, (b) M-R/S based, (c) DFA-0 based, (d) DFA-1 based, (e) DFA-2 based: *V* statistics (left y-axis) and point to point derivatives of *H* (right y-axis) are presented for all possible scales. Crossover detection based on *V* statistics is unclear as the behavior changes from to scale to scale for both R/S (a) and M-R/S (b). DFA-0 (c) and DFA-1 (d) show a crossover from anti-persistent to persistent behavior at scale of 128 trading days. DFA-2 (e) shows a crossover at scale of 512 trading days. Such results lead to use of R/S, M-R/S and DFA-2 in the analysis of time-dependent Hurst exponent. All methods show unstable behavior of point to point derivatives of *H* at two highest scales. Moreover, all methods except of DFA-0 show above average point to point derivatives of *H* at the lowest scale. Such results support the use of minimum and maximum scale of 16 trading days and a fourth of time series length, respectively.

Hurst exponent correlations

Table 0-1 Correlations of time-dependent Hurst exponents for all indices (R/S)

	PX	*BUX*	*WIG20*	*SAX*	*ATX*	*DAX*	*FTSE*	*CAC40*	*DJI30*	*NASDAQ*	*SP500*	*NIKKEI*	*SSEC*
PX	1,0000												
BUX	0,1608	1,0000											
WIG20	0,3587	0,4877	1,0000										
SAX	0,3935	0,2111	0,4515	1,0000									
ATX	0,5057	0,1732	0,5323	0,4319	1,0000								
DAX	0,1246	-0,2519	-0,3194	-0,2780	0,0561	1,0000							
FTSE	0,3275	0,0048	0,2613	0,2661	0,4391	0,4317	1,0000						
CAC40	0,1149	-0,3426	-0,3483	-0,1550	0,1103	0,7309	0,5581	1,0000					
DJI30	0,3547	0,2125	0,0135	0,0929	0,2518	0,3064	0,3111	0,1211	1,0000				
NASDAQ	0,4433	0,0362	0,4421	0,4234	0,3375	0,3264	0,6483	0,4448	0,3498	1,0000			
SP500	0,2940	0,3275	0,1737	-0,0396	0,2421	0,2491	0,3000	0,0338	0,7937	0,3477	1,0000		
NIKKEI	0,2131	-0,3166	0,0592	0,0002	0,2724	0,4241	0,3905	0,5616	0,0987	0,3807	0,0621	1,0000	
SSEC	0,1144	-0,0803	0,3086	0,1156	0,3164	0,0139	0,0960	0,1159	-0,0415	0,0958	-0,1597	0,1201	1,0000

Table 0-2 Correlations of time-dependent Hurst exponents for all indices (DFA-1)

	PX	*BUX*	*WIG*	*SAX*	*ATX*	*DAX*	*FTSE*	*CAC*	*DJI*	*NASDAQ*	*SP*	*NIKKEI*
PX	1,0000											
BUX	0,0823	1,0000										
WIG	0,2002	0,1683	1,0000									
SAX	0,5749	0,0986	0,0737	1,0000								
ATX	0,3992	0,2610	0,0474	0,2678	1,0000							
DAX	0,0705	0,1716	-0,2367	-0,2380	0,1128	1,0000						
FTSE	0,2553	0,0153	-0,3543	0,1379	0,2727	0,6136	1,0000					
CAC	0,0136	-0,0284	-0,4159	-0,2111	0,1188	0,8513	0,7466	1,0000				
DJI	0,1930	0,1660	-0,1175	-0,1454	0,1473	0,7163	0,4650	0,5326	1,0000			
NASDAQ	0,4383	-0,1599	-0,2812	0,1664	0,0985	0,4899	0,6610	0,5091	0,4853	1,0000		
SP	0,2647	0,2083	0,0119	-0,0839	0,1760	0,6275	0,4698	0,4234	0,8547	0,5883	1,0000	
NIKKEI	0,1454	-0,2215	-0,3406	0,0040	0,2866	0,2965	0,3290	0,4032	0,1110	0,4734	0,1550	1,0000

Table 0-3 Correlations of time-dependent Hurst exponents for all indices (DFA-2)

	PX	*BUX*	*WIG*	*SAX*	*ATX*	*DAX*	*FTSE*	*CAC*	*DJI*	*NASDAQ*	*SP*	*NIKKEI*	*SSEC*
PX	1,0000												
BUX	0,3321	1,0000											
WIG	0,3754	0,5598	1,0000										
SAX	0,2345	-0,0738	0,3131	1,0000									
ATX	0,4050	0,2746	0,5088	0,2161	1,0000								
DAX	0,1750	0,0854	0,0082	-0,2582	0,2627	1,0000							
FTSE	0,1584	0,0723	0,3865	0,2773	0,5212	0,5231	1,0000						
CAC	0,0500	0,0619	-0,0486	-0,2204	0,2211	0,8441	0,6290	1,0000					
DJI	0,4213	0,0236	0,1763	0,0884	0,3503	0,5736	0,3932	0,3116	1,0000				
NASDAQ	0,4091	-0,0479	0,2865	0,3057	0,4519	0,4786	0,5779	0,3107	0,8338	1,0000			
SP	0,4022	0,1223	0,2015	0,0291	0,3983	0,5925	0,4125	0,3399	0,9287	0,8116	1,0000		
NIKKEI	0,1908	0,1259	0,2327	0,0698	0,4002	0,4074	0,3466	0,4927	0,0274	0,1597	0,0397	1,0000	
SSEC	-0,0141	0,1615	0,2333	-0,2019	0,3673	0,3061	0,4292	0,4561	0,0268	0,1193	0,1265	0,4476	1,0000