Report on the doctoral thesis "Properties of weakly differentiable functions and mappings" by Luděk Kleprlík, submitted to Charles University in Prague, 2014.

Geometric function theory is largely concerned with the generalization of the theory of analytic functions to higher dimensions. The category of maps with the same geometric and function theoretic properties of analytic functions turned out to be mappings of bounded distortion, also called quasiregular mappings, or, if injective, quasiconformal mappings. Both kind of mappings solve uniformly elliptic partial differential equations in the plane. Moreover, these mappings preserve the natural Sobolev spaces which arise in consideration of the function theory and partial differential equations on subdomains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, or more generally n-manifolds. More precisely, let f be a homeomorphism from $\mathbb{X}$ onto $\mathbb{Y}$, then the composition operator $T_f(u) = u \circ f$ has the following property

$$T_f \colon W^{1,n}_{loc}(\mathbb{Y}, \mathbb{R}^n) \to W^{1,n}_{loc}(\mathbb{X}, \mathbb{R}^n), \qquad \text{provided } f \text{ has bounded distortion.}$$

The thesis of Luděk Kleprlík presents a detailed and comprehensive study of the composition operator between Sobolev spaces, Orlicz-Sobolev spaces and Lorentz-Sobolev spaces, together with striking information about the different behavior of T_f, between two seemingly similar spaces (both of these are close to the natural space $W^{1,n}$). The assumption of boundedness of the distortion is relaxed. Indeed, the thesis moves into the realm of degenerate elliptic systems where important applications lie. Usually, however, some control of the ellipticity bounds will be necessary to achieve concrete results. These often take the form of integral estimates in some Lebesgue or Sobolev space. Broadly speaking, this theory is about weakly differential mapping of finite distortion.

The thesis consists of four separate articles, together with an introductory review which presents the results of the articles in context, and demonstrates the internal unity of the thesis. Two of these articles are written in collaboration with a who's who of modern geometric analysis and two are singly authored.

The singly authored paper "Mappings of finite signed distortion: Sobolev spaces and composition of mappings" published in J. Math. Anal. Appl., raises the following question: for which f we have

$$T_f \colon W^{1,q}_{loc}(\mathbb{Y}, \mathbb{R}^n) \to W^{1,p}_{loc}(\mathbb{X}, \mathbb{R}^n)? \qquad 1 \le p \le q < \infty$$

This is proven to be the case for homeomorphims with finite distortion provided the distortion function $K_q(x) = |Df(x)|^q / |\det Df(x)|$ lies in $L^{\frac{p}{q-p}}(\mathbb{X})$. The dichotomy in the parameter q is an overarching theme in the article and appears, for example, in the validity of the Luzin condition (N^{-1}). For $1 \le q \le n$, a homeomorphism having the distortion $K_q \in L^{\frac{1}{q-1}}$ satisfies the fundamental Luzin condition (N^{-1}). For $n < q < \infty$ there exist homeomorphisms with bounded

K^q, known as q-quasiconformal, but not having the condition N^{-1}. It is known that q-quasiconformal characterize homeomorphisms f for which T_f is continuous between $W^{1,q}_{loc}$. The paper "Composition operators on W^1X are necessarily induced by quasiconformal mappings", to appear in Central European Journal of Mathematics, studies certain rearrangement invariant function spaces which are refinements of $W^{1,q}$. It is shown that the composition operator T_f is continuous between such spaces only when f is q-quasiconformal.

In the paper "Composition of q-quasiconformal mappings and functions in Orlicz-Sobolev Spaces", published in Illinois Journal of Mathematics, the authors show that T_f is bounded between the Orlicz-Sobolev spaces $WL^n \log^\alpha$, $\alpha \in \mathbb{R}$ (refinements of $W^{1,n}$) if f is a quasiconformal. The necessity of quasiconformality is also proven. The proof requires a Lebesgue type density theorem for Orlicz spaces which is new and interesting in its own right. Usually, the Orlicz-Sobolev spaces are an intermediate step towards to wider classes of mappings, called the Lorentz-Sobolev space. It is very surprising that the the composition operator behaves differently between the Lorentz-Sobolev space $WL^{n,s}$, $s \in \mathbb{R}$ than it does between the Orlicz-Sobolev spaces $WL^n \log^\alpha$, $\alpha \in \mathbb{R}$. This behavior is discovered in the paper "Composition operator and Sobolev-Lorentz spaces $WL^{n,q}$", which will appear in Studia Mathematica.

Luděk Kleprlík clearly showed that he masters in the techniques of Geometric Function Theory. His ability of finding examples shows his understanding of the structure of mappings of finite distortion. This thesis makes a definite contribution to an exciting and rapidly developing area of mathematics. It would exceed the standards of a Ph. D. thesis at University of Jyväskylä, Finland. I am very pleased to recommend that the thesis of Luděk Kleprlík be approved.

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