Charles University in Prague
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DOCTORAL THESIS

Luděk Kleprlík

Properties of weakly differentiable functions and mappings

Department of Mathematical Analysis

Supervisor of the doctoral thesis: doc. RNDr. Stanislav Hencl, Ph.D.
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In .... date .... 2014 Luděk Kleprlík

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Název práce: Vlastnosti slabě diferencovatelných funkcí a zobrazení
Autor: Luděk Kleprlík
Katedra (ústav): Katedra matematické analýzy
Vedoucí diplomové práce: doc. RNDr. Stanislav Hencl, Ph.D.
e-mail vedoucího: hencl@karlin.mff.cuni.cz

Abstrakt: V předložené práci studujeme optimální podmínky na homeomorfismus $f : \Omega \to \mathbb{R}^n$, která nám zaručí, že složení $u \circ f$ je slabě diferencovatelné a slabá derivace patří do nějakého vhodného prostoru funkcí. Ukážeme, má-li f konečnou distorzi a q-distorze $K_q = |Df|^q / J_f$ je dostatečně integrovatelná, potom operátor složení $T_f(u) = u \circ f$ zobrazuje funkce z $W^{1,q}_{\text{loc}}$ do prostoru $W^{1,p}_{\text{loc}}$ a navíc platí známé řetízkové pravidlo. Pro důkaz tohoto tvrzení budeme muset nejdříve zjistit, kdy inverzní zobrazení f^{-1} zobrazuje množiny nulové míry na množiny nulové míry (tj. splňuje Luzinovu (N^{-1}) podmínku). Ukážeme optimální podmínky pro Sobolev-Lorentzův prostor $WL^{n,q}$ a pro Sobolev Orliczův prostor $WL^q \log L$, kde $q \geq n$ a $\alpha > 0$ nebo $1 < q \leq n$ a $\alpha < 0$. Nalezneme také nutnou podmínku na homeomorfismus f pro funkce s derivací v prostoru funkcí invariantnímu vůči nerostoucímu přerovnání X blízko k L^q, t.j. X je q-škálující.

Klíčová slova: Homeomorfismus s konečnou distorzí, (N^{-1}) Luzinova podmínka, Operátor složení, Sobolevovy prostory, Orliczovy prostorz, Lorentzovy prostory, Prostory invariantní k nerostoucímu přerovnání, Lebesgueovy body hustoty

Title: Properties of weakly differentiable functions and mappings
Author: Luděk Kleprlík
Department: Department of Mathematical Analysis
Supervisor: doc. RNDr. Stanislav Hencl Ph.D.
Supervisor's e-mail address: hencl@karlin.mff.cuni.cz

Abstract: We study the optimal conditions on a homeomorphism $f : \Omega \to \mathbb{R}^n$ which guarantee that the composition $u \circ f$ is weakly differentiable and its weak derivative belongs to the some function space. We show that if f has finite distortion and q-distortion $K_q = |Df|^q / J_f$ is integrable enough, then the composition operator $T_f(u) = u \circ f$ maps functions from $W^{1,q}_{\text{loc}}$ into space $W^{1,p}_{\text{loc}}$ and the well-known chain rule holds. To prove it we characterize when the inverse mapping f^{-1} maps sets of measure zero onto sets of measure zero (satisfies the Luzin (N^{-1}) condition). We also fully characterize conditions for Sobolev-Lorentz space $WL^{n,q}$ for arbitrary q and for Sobolev Orlicz space $WL^q \log L$ for $q \geq n$ and $\alpha > 0$ or $1 < q \leq n$ and $\alpha < 0$. We find a necessary condition on f for Sobolev rearrangement invariant function space WX close to WL^q, i.e. X has q-scaling property.

Keywords: Homeomorphism of finite distortion, (N^{-1}) Luzin condition, Composition operator, Sobolev spaces, Orlicz spaces, Lorentz spaces, Rearrangement invariant spaces, Lebesgue's density theorem

CHAPTER 1

Introduction

This Ph.D. thesis consists of four papers:

- [T1] L. Kleprlík: *Mappings of finite signed distortion: Sobolev spaces and composition of mappings*, J. Math. Anal. Appl., **386** no.2 (2012), 870–881.
- [T2] S. Hencl, L. Kleprlík: *Composition of q-quasiconformal mappings and functions in OrliczSobolev Spaces*, Illinois Journal of Mathematics **56** no.3 (2012), 931–955.
- [T3] S. Hencl, L. Kleprlík, J. Malý: *Composition operator and Sobolev-Lorentz spaces $WL^{n,q}$*, to appear in Studia Mathematica.
- [T4] L. Kleprlík: *Composition operators on W^1X are necessarily induced by quasiconformal mappings*, to appear in Central European Journal of Mathematics.

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be domains and let $f : \Omega_1 \to \Omega_2$ be a homeomorphism. The general question we are interested in is the following. Given a function space X we would like to characterize mappings f for which the composition operator $T_f : \ T_f(u) = u \circ f$ maps $X(\Omega_2)$ into $X(\Omega_1)$ continuously.

This problem has been studied for many function spaces and one of the most important is the following well-known result: The composition operator $T_f : \ T_f(u) = u \circ f$ maps $W^{1,n}_{\text{loc}}(\Omega_2)$ into $W^{1,n}_{\text{loc}}(\Omega_1)$ if $f : \Omega_1 \to \Omega_2$ is a quasiconformal mapping ([**8**], [**10**]). Moreover each homeomorphism f which maps $W^{1,n}_{\text{loc}}(\Omega_2)$ into $W^{1,n}_{\text{loc}}(\Omega_1)$ continuously is necessarily a quasiconformal mapping up to a reflection. Similarly it is possible to characterize homeomorphism for which the composition operator is continuous from $W^{1,q}_{\text{loc}}$ to $W^{1,q}_{\text{loc}}$ and we obtain a class of q-quasiconformal mappings [**2**]. Here the homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is called a q-quasiconformal mapping if there is a constant K such that the distortion inequality

$$|Df(x)|^q \leq K|J_f(x)| \text{ holds for a.e. } x \in \Omega \ . \tag{1.1}$$

Later the question of characterization of composition operator was studied also for mappings f such that $u \circ f \in W^{1,p}$ for every $u \in W^{1,q}$, $q \geq p$, by Ukhlov [**9**]. However the proof there seems to contain gaps and it was not clear if the statement is valid. In the first paper [T1] we have given a full and correct proof of the statement. We show that if $K_q = |Df|^q/J_f$ belongs to $L^{p/(q-p)}$ then the composition operator $T_f(u) = u \circ f$ is continuous from $W^{1,q} \cap C$ (respectively $W^{1,q}$ if $q \leq n$) to $W^{1,p}$. To prove it we characterize when the inverse mapping f^{-1} maps sets of measure zero onto sets of measure zero (satisfies Luzin (N^{-1}) condition). Further we have shown the chain rule and we explained the role of the correct representative of u. This important result was included as one chapter in the recent monograph [**4**].

In general one could expect that different function spaces have a different class of morphisms unless the answer is somehow trivial. Surprisingly this is not the case as many examples indicate. The class of n-quasiconformal mappings serves as the best class of morphisms not only for $W^{1,n}_{\mathrm{loc}}$ functions but also for other function spaces that are 'close' to $W^{1,n}_{\mathrm{loc}}$. Let us mention for example the stability under quasiconformal mappings for the BMO space [**7**], fractional Sobolev spaces $\dot{M}^s_{n/s,q}$, $s \in (0,1]$, [**6**, Theorem 1.3], absolutely continuous functions of several variables AC^n_λ [**3**] and exponential Orlicz space $\exp L(\Omega)$ in the plane [**1**].

The general aim of our research project was to study this phenomena more closely. Is it for example true that all function spaces close to $W^{1,n}$ (resp. $W^{1,p}$) are stable under n-quasiconformal (resp. p-quasiconformal) mappings? Is this condition necessary?

In [T2] we have shown that Sobolev Orlicz space $WL^n \log^\alpha L$, $\alpha \in \mathbb{R}$, is stable under n-quasiconformal mapping and this condition is also necessary. Similar conclusion holds for $WL^p \log^\alpha L$ and p-quasiconformal mappings if ($p > n$ and $\alpha > 0$) or ($p < n$ and $\alpha < 0$). As a new tool, we proved a Lebesgue density type theorem for Orlicz spaces. On the other hand, we have constructed a counterexample to stability in the remaining cases ($p > n$ and $\alpha < 0$) or ($p < n$ and $\alpha > 0$). Thus the answer to stability is nontrivial at least for $p \neq n$.

Somewhat surprisingly the class of n-quasiconformal mappings does not serve as a suitable class for all function spaces close to $W^{1,n}$, i.e the answer to stability is nontrivial also for $p = n$. In [T3] we have shown that if the composition operator T_f maps the Sobolev-Lorentz space $WL^{n,q}$ to $WL^{n,q}$ for some $q \neq n$ then f must be a locally bilipschitz mapping. In the same time this was shown also for homogeneous Besov spaces $\dot{B}_{n/s,q}$, $sq \neq n$ [**5**]. However the first step in the proofs of these two results is to show that each morphism is n-quasiconformal and using this additional regularity one then proves that it must be bilipschitz.

In the last paper of the thesis [T4] we have shown that if composition operator T_f maps W^1X to W^1X for an rearrangement invariant function space X, which is close to $L^q(\Omega)$ then f is necessarily a q-quasiconformal mapping. This shows that q-quasiconformality is indeed a necessary and crucial condition for boundedness of the composition operator. We also give some new results for the sufficiency of this condition for the composition operator.

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