Charles University in Prague
Faculty of Mathematics and Physics

MASTER THESIS

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Mathematical modelling of glass forming process

Mathematical Institute of Charles University

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Matematické modelování zpracování skla

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Abstrakt: Diplomová práce se zaměřuje na modelování výroby tabulového skla použitím aproximace proudění viskózního filmu. Po zprůměrování Navierových-Stokesových rovnic přes jednu prostorovou proměnnou transformujeme oblast, jejíž tvar není dopředu znám a je součástí řešení, do pevné výpočetní oblasti. Poté problém vyřešíme metodou konečných prvků za použití softwaru FEniCS. Na závěr diskutujeme vliv různých parametrů jako koleček, které umožňují měnit tloušťku skla, okrajových podmínek nebo síly, zamezující úplnému rozlivu skla, na výsledky numerických výpočtů.

Klíčová slova: matematické modelování, zpracovnání skla, aproximace proudění tenkého filmu, metoda konečných prvků

Title: Mathematical modelling of glass forming process

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Abstract: The thesis focus on modelling of float glass making process using viscose film type approximation. Navier-Stokes equations are averaged over one spatial variable. Then the domain with an a priory unknown shape, where the shape is a part of the solution, is transformed to a fixed computational domain. The problem is solved by finite element method using FEniCS software. In the end is discussed an influence of several parameters such as wheels, which regulates thickness of the glass and enforce an inner condition, boundary conditions or spreading coefficient on the numerical result.

Keywords: mathematical modelling, glass forming process, viscose film type approximation, finite element method

Contents

Introduction

The float process is the standard industrial scale process for making flat glass developed in the late 1950s. An excellent historical overview of earlier techniques for making flat glass, and the development of the float glass process is given in the survey by Pilkington (1969). Pilkington (1969) describes the process as follows:

> In the float process, a continuous ribbon of glass moves out of the melting furnace and floats along the surface of an enclosed bath of molten tin [...]. The ribbon is held in a chemically controlled atmosphere at a high enough temperature for a long enough time for the irregularities to melt out and for the surfaces to become flat and parallel. Because the surface of the molten tin is dead flat, the glass also becomes flat.
>
> The ribbon is then cooled down while still advancing across the molten tin until the surfaces are hard enough for it to be taken out of the bath without the rollers marking the bottom surface; so a ribbon is produced with uniform thickness and bright fire polished surfaces without any need for grinding and polishing.

Naturally, the importance of the process leads to the need to develop a mathematical model for the process.

The process starts by pouring glass on a melted tin, where it is bounded by restrictors. After a few meters glass leaves the restrictors and can spread freely on the tin bath. Approximately between 10 m and 20 m from the begin of the bath is located so called stretching region, where can be placed wheels, which helps to adjust thickness of the glass. About 60 m from the pouring area glass pulls out a device called lehr. We will focus on the part between the restrictors' tip and the lehr.

In what follows we model the process by viscous film type approximation and solve the model by finite element method. Then discuss the results and the influence of several parameters of the model to the result.

Chapter 1

Viscous film type approximation

In this part is presented some results concerning simplified viscous film type models for the description of float glass forming process. Models of this type were introduced by Narayanaswamy (1977, 1981), Popov (1982, 1983), M. and Mase (1991) and Kamihori T. (1994) to name a few.

1.1 Governing equations

The full system of governing equations describing motion of an incompressible fluid with non-constant density in a given domain reads

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \operatorname{div} (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbb{T} + \rho \boldsymbol{b}, \tag{1.1a}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \boldsymbol{v}) = 0, \tag{1.1b}$$

where $\boldsymbol{v}$ denotes the velocity, ρ the density, $\boldsymbol{b}$ external volume force and $\mathbb{T}$ the Cauchy stress tensor. In the following derivation of the simplified governing equations, we will consider the glass melt to be an incompressible fluid. The Cauchy stress is assumed to have the form

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S}, \tag{1.2}$$

where $\mathbb{S}$ is a symmetric traceless tensor.

1.1.1 Boundary conditions

The fluid is assumed to occupy the domain shown in Figure 1.1. The lighter fluid (glass) is assumed to form a thin film flowing in a bath formed by the heavier fluid (tin). The lighter fluid is dived in the heavier fluid by $H_1(x, y)$ and the height which floats above heavier fluid level is $H_2(x, y)$. Both heights are measured with respect to the free surface of the heavier fluid, and the coordinate system is chosen in such a way that the free surface of the heavier fluid is located at $z = 0$. Functions $H_1(x, y)$ and $H_2(x, y)$ are assumed to be slowly varying functions of x and y. This means that the partial derivatives $\frac{\partial H_1}{\partial x}$, $\frac{\partial H_1}{\partial y}$ and similarly for H_2 can be neglected in the appropriate formulae.

The boundary condition on the top surface (fluid–gas interface) reads

$$\mathbb{T}_g \boldsymbol{n} = \mathbb{T}_a \boldsymbol{n}, \tag{1.3a}$$

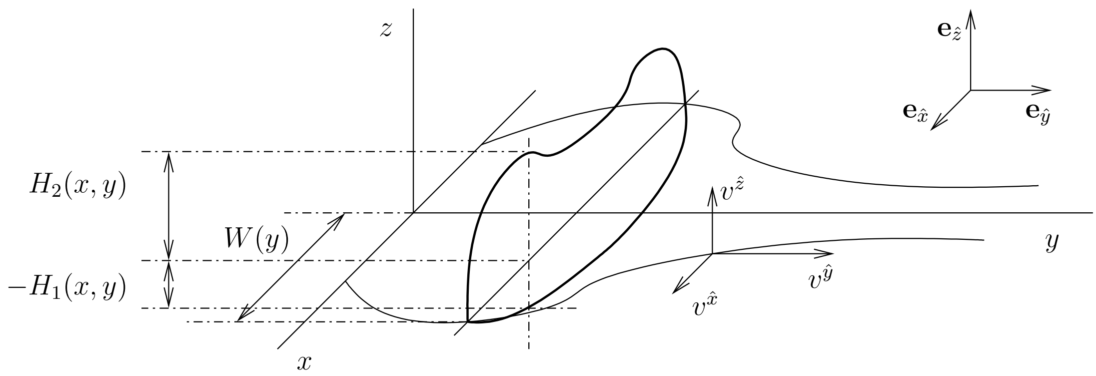

Figure 1.1: Problem geometry.

where $\boldsymbol{n}$ is the outward normal to the surface, $\mathbb{T}_a$ denotes the stress in the ambient gas and $\mathbb{T}_g$ denotes the Cauchy stress tensor for the lighter fluid. (It is assumed that no surface tension effects are acting on this surface.) Concerning the bottom surface, the boundary condition reads

$$\mathbb{T}_g \boldsymbol{n} = \mathbb{T}_t \boldsymbol{n}, \tag{1.3b}$$

where $\boldsymbol{n}$ is the outward normal to the surface and $\mathbb{T}_t$ denotes the Cauchy stress tensor for the heavier fluid. This means that we assume the continuity of the stress across the surface. (This boundary condition will not be, as we shall see later, enforced exactly.)

The (artificial) inflow and outflow boundary conditions will be specified later.

1.2 Auxiliary tools

Let us now recall some basic facts concerning the motion of surfaces, which will be crucial for capturing the motion of the fluid–fluid and fluid–gas interface. and some properties of the integral with respect to differentiation, because we will use these facts in following derivation.

Lemma 1. *A surface described by an implicit relation $f(\mathbf{x}, t) = 0$ is a material surface if and only if*

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = 0 \tag{1.4}$$

is satisfied for all points on the surface.

Lemma 2. *Outward normal to the surface $\to \Phi : [u, v] \in \mathbb{R}^2 \mapsto [x, y, z] \in \mathbb{R}^3$ described by the parametric equations*

$$x = u, \tag{1.5a}$$
$$y = v, \tag{1.5b}$$
$$z = \Phi(u, v), \tag{1.5c}$$

is given by the following formula

$$\boldsymbol{n} = \begin{bmatrix} -\frac{\partial \Phi}{\partial x} \\ -\frac{\partial \Phi}{\partial y} \\ 1 \end{bmatrix}. \tag{1.6}$$

The outward normal is understood in the sense that it is directed outward from the body $\{\boldsymbol{x} \in \mathbb{R}^3 : z \leq \Phi(x,y)\}$.

The simplified model will be essentially obtained by averaging the governing equations with respect to z variable, therefore we mention some properties of the integral with respect to differentiation. The following Lemma allows one to change the order of differentiation and integration which is the basic tool the averaging approach.

Lemma 3.

$$\frac{\partial}{\partial x}\int_{\xi=a(x)}^{b(x)} g(x,\xi)\,d\xi = \int_{\xi=a(x)}^{b(x)} \frac{\partial}{\partial x} g(x,\xi)\,d\xi + g(x,b(x))\frac{\partial b}{\partial x} - g(x,a(x))\frac{\partial a}{\partial x} \quad (1.7)$$

Proof. Let $G(x,\xi)$ be a primitive function to $g(x,\xi)$ with respect to ξ variable, that is $\frac{\partial G}{\partial \xi}(x,\xi) = g(x,\xi)$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial x}\int_{\xi=a(x)}^{b(x)} g(x,\xi)\,d\xi &= \frac{\partial}{\partial x}\left(G(x,b(x)) - G(x,a(x))\right) \\
&= \left(\frac{\partial}{\partial x}G(x,\xi)\bigg|_{\xi=b(x)} + \frac{\partial}{\partial \xi}G(x,\xi)\bigg|_{\xi=b(x)}\frac{\partial}{\partial x}b(x)\right) - \\
&\quad - \left(\frac{\partial}{\partial x}G(x,\xi)\bigg|_{\xi=a(x)} + \frac{\partial}{\partial \xi}G(x,\xi)\bigg|_{\xi=b(x)}\frac{\partial}{\partial x}a(x)\right) \\
&= \left(\int_{\xi=\xi_0}^{b(x)} \frac{\partial}{\partial x}g(x,\xi)\,d\xi + g(x,b(x))\frac{\partial}{\partial x}b(x)\right) - \\
&\quad - \left(-\int_{\xi=a(x)}^{\xi_0} \frac{\partial}{\partial x}g(x,\xi)\,d\xi + g(x,a(x))\frac{\partial}{\partial x}a(x)\right) \\
&= \int_{\xi=a(x)}^{b(x)} \frac{\partial}{\partial x}g(x,\xi)\,d\xi + g(x,b(x))\frac{\partial b}{\partial x} - g(x,a(x))\frac{\partial a}{\partial x}. \quad (1.8)
\end{aligned}$$

□

1.3 Derivation of a simplified two-dimensional model

If we rewrite the governing equations (1.1) in Cartesian coordinates, we get the following system of equations,

$$\frac{\partial\left(\rho_g v^x\right)}{\partial t}+\frac{\partial\left(\rho_g v^x v^x\right)}{\partial x}+\frac{\partial\left(\rho_g v^y v^x\right)}{\partial y}+\frac{\partial\left(\rho_g v^z v^x\right)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial S^{xx}}{\partial x}+\frac{\partial S^{xy}}{\partial y}+\frac{\partial S^{xz}}{\partial z}, \tag{1.9a}$$

$$\frac{\partial\left(\rho_g v^y\right)}{\partial t}+\frac{\partial\left(\rho_g v^x v^y\right)}{\partial x}+\frac{\partial\left(\rho_g v^y v^y\right)}{\partial y}+\frac{\partial\left(\rho_g v^z v^y\right)}{\partial z}=-\frac{\partial p}{\partial y}+\frac{\partial S^{yx}}{\partial x}+\frac{\partial S^{yy}}{\partial y}+\frac{\partial S^{yz}}{\partial z}, \tag{1.9b}$$

$$\frac{\partial\left(\rho_g v^z\right)}{\partial t}+\frac{\partial\left(\rho_g v^x v^z\right)}{\partial x}+\frac{\partial\left(\rho_g v^y v^z\right)}{\partial y}+\frac{\partial\left(\rho_g v^z v^z\right)}{\partial z}= -\frac{\partial p}{\partial z}+\frac{\partial S^{zx}}{\partial x}+\frac{\partial S^{zy}}{\partial y}+\frac{\partial S^{zz}}{\partial z}-\rho_g g, \tag{1.9c}$$

$$\frac{\partial \rho_g}{\partial t}+\frac{\partial\left(\rho_g v^x\right)}{\partial x}+\frac{\partial\left(\rho_g v^y\right)}{\partial y}+\frac{\partial\left(\rho_g v^z\right)}{\partial z}=0, \tag{1.9d}$$

where we have used the notation

$$\mathbb{T}_g=-p_g\mathbb{I}+\mathbb{S}_g=-p_g\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}S^{xx}&S^{xy}&S^{xz}\\S^{yx}&S^{yy}&S^{yz}\\S^{zx}&S^{zy}&S^{zz}\end{bmatrix} \tag{1.10}$$

and consider external gravitational force

$$\boldsymbol{b}=-\boldsymbol{e}_z g, \tag{1.11}$$

where g denotes gravitational acceleration.
In further derivation we assume the density ρ_g to be a known function of given temperature field θ, which is constant with respect to variable z.

$$\rho_g=\rho_g\left(\theta(x,y,t)\right). \tag{1.12}$$

1.3.1 Balance of mass

Let us now integrate the last equation (balance of mass) with respect to z variable, that is

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}\left(\frac{\partial \rho_g}{\partial t}+\frac{\partial\left(\rho_g v^x\right)}{\partial x}+\frac{\partial\left(\rho_g v^y\right)}{\partial y}+\frac{\partial\left(\rho_g v^z\right)}{\partial z}\right)dz=0. \tag{1.13}$$

First, we focus on a equation without time derivative, which we add later. Exploiting Lemma 3 yields

$$
\begin{aligned}
&\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\frac{\partial\left(\rho_g v^x\right)}{\partial x} + \frac{\partial\left(\rho_g v^y\right)}{\partial y} + \frac{\partial\left(\rho_g v^z\right)}{\partial z} \right) dz \\
&= \left(\frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^x \, dz - \rho_g v^x\big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial x} - \rho_g v^x\big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial x} \right) \\
&+ \left(\frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^y \, dz - \rho_g v^y\big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial y} - \rho_g v^y\big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial y} \right) \\
&\qquad + \left(\rho_g v^z\big|_{z=H_2(x,y,t)} - \rho_g v^z\big|_{z=-H_1(x,y,t)} \right),
\end{aligned}
\tag{1.14}
$$

which can be rewritten as

$$
\begin{aligned}
&\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\frac{\partial\left(\rho_g v^x\right)}{\partial x} + \frac{\partial\left(\rho_g v^y\right)}{\partial y} + \frac{\partial\left(\rho_g v^z\right)}{\partial z} \right) dz \\
&\qquad = \frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^x \, dz + \frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^y \, dz \\
&\quad + \left(\rho_g v^z\big|_{z=H_2(x,y,t)} - \rho_g v^x\big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial x} - \rho_g v^y\big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial y} \right) \\
&- \left(\rho_g v^z\big|_{z=-H_1(x,y,t)} + \rho_g v^x\big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial x} + \rho_g v^y\big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial y} \right),
\end{aligned}
\tag{1.15}
$$

where

$$
f\big|_{z=H(x,y,t)} =_{\text{def}} f(x, y, H(x, y, t), t) \tag{1.16}
$$

denotes the value of function f at point $[x, y, H(x, y, t)]$. The top and bottom surfaces, that is surfaces $z = H_2(x, y, t)$ and $z = -H_1(x, y, t)$, are *material surfaces*, therefore they must obey condition (1.4), which reads

$$
\left. \left(-\rho_g \frac{\partial H_2}{\partial t} - \rho_g v^x \frac{\partial H_2}{\partial x} - \rho_g v^y \frac{\partial H_2}{\partial y} + \rho_g v^z \right) \right|_{z=H_2(x,y,t)} = 0, \tag{1.17a}
$$

$$
\left. \left(\rho_g \frac{\partial H_1}{\partial t} + \rho_g v^x \frac{\partial H_1}{\partial x} + \rho_g v^y \frac{\partial H_1}{\partial y} + \rho_g v^z \right) \right|_{z=-H_1(x,y,t)} = 0. \tag{1.17b}
$$

These equalities imply that (1.15) can be rewritten as

$$
\begin{aligned}
&\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\frac{\partial\left(\rho_g v^x\right)}{\partial x} + \frac{\partial\left(\rho_g v^y\right)}{\partial y} + \frac{\partial\left(\rho_g v^z\right)}{\partial z} \right) dz \\
&= \frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^x \, dz + \frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^y \, dz + \rho_g \frac{\partial H_2}{\partial t} + \rho_g \frac{\partial H_1}{\partial t}.
\end{aligned}
\tag{1.18}
$$

If we introduce the notation

$$
H =_{\text{def}} H_1 + H_2, \tag{1.19a}
$$

$$
\overline{v^x}(x, y, t) =_{\text{def}} \frac{1}{H} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} v^x(x, y, z, t) \, dz, \tag{1.19b}
$$

$$
\overline{v^y}(x, y, t) =_{\text{def}} \frac{1}{H} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} v^y(x, y, z, t) \, dz \tag{1.19c}
$$

and apply independence ρ_g of z, we can finally rewrite (1.18) in the form

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\frac{\partial (\rho_g v^x)}{\partial x} + \frac{\partial (\rho_g v^y)}{\partial y} + \frac{\partial (\rho_g v^z)}{\partial z} \right) dz \\ = \rho_g \frac{\partial H}{\partial t} + \frac{\partial}{\partial x} \left(H \rho_g \overline{v^x} \right) + \frac{\partial}{\partial y} \left(H \rho_g \overline{v^y} \right). \quad (1.20)$$

Now we complete the balance of mass by the term with time derivative

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial \rho_g}{\partial t} dz = \frac{\partial}{\partial t} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g dz - \rho_g|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial t} - \rho_g|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial t} \\ = \frac{\partial (H \rho_g)}{\partial t} - \rho_g \frac{\partial H}{\partial t}, \quad (1.21)$$

where in the last equation we again used the fact that ρ_g is constant with respect to the z variable. Finally, together we get

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\frac{\partial \rho_g}{\partial t} + \frac{\partial (\rho_g v^x)}{\partial x} + \frac{\partial (\rho_g v^y)}{\partial y} + \frac{\partial (\rho_g v^z)}{\partial z} \right) dz \\ = \frac{\partial (H \rho_g)}{\partial t} + \frac{\partial}{\partial x} \left(H \rho_g \overline{v^x} \right) + \frac{\partial}{\partial y} \left(H \rho_g \overline{v^y} \right). \quad (1.22)$$

Apart from notation (1.19) it will be also convenient to introduce the "fluctuations" of the involved quantities with respect to its z averaged values

$$\widetilde{v^x} =_{\text{def}} v^x - \overline{v^x}, \quad (1.23a)$$

$$\widetilde{v^y} =_{\text{def}} v^y - \overline{v^y}, \quad (1.23b)$$

$$(1.23c)$$

By elementary observation we can see that the definition implies $\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \widetilde{v^x} \, dz = 0$ and $\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \widetilde{v^y}, dz = 0$.

1.3.2 Balance of linear momentum

Let us now consider x component of the balance of linear momentum, that is (1.9a). We want to evaluate the z average of the equation (1.9a). Left hand side terms is

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\frac{\partial (\rho_g v^x)}{\partial t} + \frac{\partial (\rho_g v^x v^x)}{\partial x} + \frac{\partial (\rho_g v^y v^x)}{\partial y} + \frac{\partial (\rho_g v^z v^x)}{\partial z} \right) dz \\ = \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial (\rho_g v^x)}{\partial t} dz + \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial (\rho_g v^x v^x)}{\partial x} dz \\ + \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial (\rho_g v^y v^x)}{\partial y} dz + \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial (\rho_g v^z v^x)}{\partial z} dz. \quad (1.24)$$

Using Lemma 3 the time derivative can be rewritten as

$$\begin{aligned}
&\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial\left(\rho_g v^x\right)}{\partial t}\, dz \\
&= \frac{\partial}{\partial t}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^x\, dz - \rho_g v^x\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial t} - \rho_g v^x\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial t} \\
&= \frac{\partial}{\partial t}\left(H\rho_g\overline{v^x}\right) - \rho_g v^x\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial t} - \rho_g v^x\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial t}.
\end{aligned} \tag{1.25}$$

The spatial derivatives can be rewritten as follows

$$\begin{aligned}
&\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial\left(\rho_g v^y v^x\right)}{\partial y}\, dz \\
&= \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g v^x v^y\, dz - \rho_g v^x v^y\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial y} - \rho_g v^x v^y\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial y} \\
&= \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\left(\overline{v^x}+\widetilde{v^x}\right)\left(\overline{v^y}+\widetilde{v^y}\right) dz \\
&\quad - \rho_g v^x v^y\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial y} - \rho_g v^x v^y\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial y} \\
&= \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(\rho_g\overline{v^x v^y}\right) dz + \frac{\partial}{\partial y}\left(\rho_g\overline{v^y}\overbrace{\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}\widetilde{v^x}\, dz}^{=0}\right) \\
&\quad + \frac{\partial}{\partial y}\left(\rho_g\overline{v^x}\overbrace{\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}\widetilde{v^y}\, dz}^{=0}\right) + \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\widetilde{v^x}\widetilde{v^y}\, dz \\
&\quad - \rho_g v^x v^y\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial y} - \rho_g v^x v^y\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial y} \\
&= \frac{\partial}{\partial y}\left(H\rho_g\overline{v^x v^y}\right) - \rho_g v^x v^y\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial y} - \rho_g v^x v^y\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial y} \\
&\quad + \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\widetilde{v^x}\widetilde{v^y}\, dz
\end{aligned} \tag{1.26}$$

where we have used the fact that the z averages of the "fluctuations" $\widetilde{v^x}$ and $\widetilde{v^y}$ vanish. Later on we will neglect the last term as a part of the approximation, but for now let us keep the term in the computations in order to have a chance to explicitly track the "error" in the simplified model. (The term is quadratic in the "fluctuations" and therefore it is *presumably* small with respect to the other terms that contain either the averaged quantities or that depend linearly on the "fluctuations".) The remaining terms on the right hand side of (1.24) can be manipulated analogously, and we get

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial\left(\rho_g v^x\right)}{\partial t}\, dz = \frac{\partial}{\partial t}\left(H\rho_g\overline{v^x}\right) - \rho_g v^x\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial t} - \rho_g v^x\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial t}, \tag{1.27a}$$

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial}{\partial x}\left(\rho_g (v^x)^2\right) dz = \frac{\partial}{\partial x}\left(H\rho_g\left(\overline{v^x}\right)^2\right) - \rho_g\left(v^x\right)^2\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial x} - \rho_g\left(v^x\right)^2\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial x} + \frac{\partial}{\partial x}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\left(\widetilde{v^x}\right)^2 dz, \tag{1.27b}$$

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial}{\partial y}\left(\rho_g v^x v^y\right) dz = \frac{\partial}{\partial y}\left(H\rho_g\overline{v^x v^y}\right) - \rho_g v^x v^y\big|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial y} - \rho_g v^x v^y\big|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial y} + \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\widetilde{v^x}\widetilde{v^y}\, dz, \tag{1.27c}$$

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial}{\partial z}\left(\rho_g v^x v^z\right) dz = \rho_g v^x v^z\big|_{z=H_2(x,y,t)} - \rho_g v^x v^z\big|_{z=-H_1(x,y,t)}. \tag{1.27d}$$

Summing all the terms yields

$$\begin{aligned}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} &\left(\frac{\partial\left(\rho_g v^x\right)}{\partial t} + \frac{\partial}{\partial x}\rho_g\left(v^x\right)^2 + \frac{\partial}{\partial y}\left(\rho_g v^y v^x\right) + \frac{\partial}{\partial z}\left(\rho_g v^z v^x\right)\right) dz \\ &= \frac{\partial}{\partial t}\left(H\rho_g\overline{v^x}\right) + \frac{\partial}{\partial x}\left(H\rho_g\left(\overline{v^x}\right)^2\right) + \frac{\partial}{\partial y}\left(H\rho_g\overline{v^x v^y}\right) \\ &+ \frac{\partial}{\partial x}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\left(\widetilde{v^x}\right)^2 dz + \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\widetilde{v^x}\widetilde{v^y}\, dz \\ &+ \rho_g v^x\big|_{z=H_2(x,y,t)}\left(-\frac{\partial H_2}{\partial t} - \rho_g v^x\frac{\partial H_2}{\partial x} - \rho_g v^y\frac{\partial H_2}{\partial y} + \rho_g v^z\right)\bigg|_{z=H_2(x,y,t)} \\ &- \rho_g v^x\big|_{z=-H_1(x,y,t)}\left(\frac{\partial H_1}{\partial t} + \rho_g v^x\frac{\partial H_1}{\partial x} + \rho_g v^y\frac{\partial H_1}{\partial y} + \rho_g v^z\right)\bigg|_{z=-H_1(x,y,t)}.\end{aligned} \tag{1.28}$$

Now we can again exploit the fact that the top and bottom surface are material surfaces, see (1.17), and we see that the last two terms vanish. Finally, the formula for the averaged left hand side of the x component of the balance of linear momentum reads

$$\begin{aligned}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} &\left(\frac{\partial\left(\rho_g v^x\right)}{\partial t} + \frac{\partial}{\partial x}\rho_g\left(v^x\right)^2 + \frac{\partial}{\partial y}\left(\rho_g v^y v^x\right) + \frac{\partial}{\partial z}\left(\rho_g v^z v^x\right)\right) dz \\ &= \frac{\partial}{\partial t}\left(H\rho_g\overline{v^x}\right) + \frac{\partial}{\partial x}\left(H\rho_g\left(\overline{v^x}\right)^2\right) + \frac{\partial}{\partial y}\left(H\rho_g\overline{v^x v^y}\right) \\ &+ \frac{\partial}{\partial x}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\left(\widetilde{v^x}\right)^2 dz + \frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g\widetilde{v^x}\widetilde{v^y}\, dz.\end{aligned} \tag{1.29}$$

It remains to find the average of the right hand side of (1.9a), that is

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(-\frac{\partial p_g}{\partial x} + \frac{\partial S^{xx}}{\partial x} + \frac{\partial S^{xy}}{\partial y} + \frac{\partial S^{xz}}{\partial z} \right) dz. \tag{1.30}$$

We will again use Lemma 3. In particular

$$\begin{aligned}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial}{\partial x} \left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right) dz =& \frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right) dz \\ &- \left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\Big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial x} \\ &- \left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\Big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial x},\end{aligned} \tag{1.31a}$$

$$\begin{aligned}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial}{\partial y} S_{\mathrm{g}}^{\hat{x}\hat{y}}\, dz =& \frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} S_{\mathrm{g}}^{\hat{x}\hat{y}}\, dz \\ &- S_{\mathrm{g}}^{\hat{x}\hat{y}}\Big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial y} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\Big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial y},\end{aligned} \tag{1.31b}$$

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial}{\partial z} S_{\mathrm{g}}^{\hat{x}\hat{z}}\, dz = S_{\mathrm{g}}^{\hat{x}\hat{z}}\Big|_{z=H_2(x,y,t)} - S_{\mathrm{g}}^{\hat{x}\hat{z}}\Big|_{z=-H_1(x,y,t)}. \tag{1.31c}$$

Summing up all the terms yields

$$\begin{aligned}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} &\left(-\frac{\partial p_g}{\partial x} + \frac{\partial S_{\mathrm{g}}^{\hat{x}\hat{x}}}{\partial x} + \frac{\partial S_{\mathrm{g}}^{\hat{x}\hat{y}}}{\partial y} + \frac{\partial S_{\mathrm{g}}^{\hat{x}\hat{z}}}{\partial z} \right) dz \\ &= -\frac{\partial}{\partial x}\left(H\overline{p_g}\right) + \frac{\partial}{\partial x}\left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{x}}}\right) + \frac{\partial}{\partial y}\left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}}\right) \\ &+ \left(-\left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\Big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial x} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\Big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial y} + S_{\mathrm{g}}^{\hat{x}\hat{z}}\Big|_{z=H_2(x,y,t)} \right) \\ &+ \left(-\left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\Big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial x} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\Big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial y} - S_{\mathrm{g}}^{\hat{x}\hat{z}}\Big|_{z=-H_1(x,y,t)} \right).\end{aligned} \tag{1.32}$$

Here we extend the averaging notation as $\overline{S^{\bullet\bullet}}(x,y,t) =_{\mathrm{def}} \frac{1}{H}\int S^{\bullet\bullet}(x,y,z,t)dz$. The last two terms can be rewritten in the form

$$\begin{aligned}-\left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\Big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial x} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\Big|_{z=H_2(x,y,t)} \frac{\partial H_2}{\partial y} + S_{\mathrm{g}}^{\hat{x}\hat{z}}\Big|_{z=H_2(x,y,t)} \\ = \boldsymbol{e}_x \cdot \mathbb{T}_g\big|_{z=H_2(x,y,t)} \boldsymbol{n}_2,\end{aligned} \tag{1.33a}$$

$$\begin{aligned}-\left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\Big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial x} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\Big|_{z=-H_1(x,y,t)} \frac{\partial H_1}{\partial y} - S_{\mathrm{g}}^{\hat{x}\hat{z}}\Big|_{z=-H_1(x,y,t)} \\ = \boldsymbol{e}_x \cdot \mathbb{T}_g\big|_{z=-H_1(x,y,t)} \boldsymbol{n}_1,\end{aligned} \tag{1.33b}$$

where

$$\boldsymbol{n}_2 =_{\mathrm{def}} \begin{bmatrix} -\frac{\partial H_2}{\partial x} \\ -\frac{\partial H_2}{\partial y} \\ 1 \end{bmatrix}, \qquad \boldsymbol{n}_1 =_{\mathrm{def}} \begin{bmatrix} -\frac{\partial H_1}{\partial x} \\ -\frac{\partial H_1}{\partial y} \\ -1 \end{bmatrix}, \tag{1.34}$$

denote the outward normal to the surface $z = H_2(x, y, t)$ and $z = -H_1(x, y, t)$ respectively, see Lemma 2. This means that these terms can be specified using the boundary conditions (1.3a) and (1.3b). Finally,

$$\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \left(-\frac{\partial p_g}{\partial x} + \frac{\partial S_{\mathrm{g}}^{\hat{x}\hat{x}}}{\partial x} + \frac{\partial S_{\mathrm{g}}^{\hat{x}\hat{y}}}{\partial y} + \frac{\partial S_{\mathrm{g}}^{\hat{x}\hat{z}}}{\partial z} \right) dz$$
$$= -\frac{\partial}{\partial x} (H\overline{p_g}) + \frac{\partial}{\partial x} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{x}}} \right) + \frac{\partial}{\partial y} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} \right) + \boldsymbol{e}_x \cdot \left(\mathbb{T}_g|_{z=H_2(x,y,t)} \boldsymbol{n}_2 + \mathbb{T}_g|_{z=-H_1(x,y,t)} \boldsymbol{n}_1 \right). \tag{1.35}$$

Using (1.29) and (1.35) we can write down the averaged balance of linear momentum in $\boldsymbol{e}_x$ direction,

$$\begin{aligned} \left(\frac{\partial}{\partial t} (H\rho_g \overline{v^x}) + \frac{\partial}{\partial x} \left(H\rho_g \left(\overline{v^x} \right)^2 \right) + \frac{\partial}{\partial y} (H\rho_g \overline{v^x}\,\overline{v^y}) \right) \\ = -\frac{\partial}{\partial x} (H\overline{p_g}) + \frac{\partial}{\partial x} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{x}}} \right) + \frac{\partial}{\partial y} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} \right) \\ + \boldsymbol{e}_x \cdot \left(\mathbb{T}_g|_{z=H_2(x,y,t)} \boldsymbol{n}_2 + \mathbb{T}_g|_{z=-H_1(x,y,t)} \boldsymbol{n}_1 \right) \\ - \left(\frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g \left(\widetilde{v^x} \right)^2 dz + \frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g \widetilde{v^x} \widetilde{v^y} \, dz \right). \end{aligned} \tag{1.36}$$

A similar manipulation can be also done for the balance of linear momentum in $\boldsymbol{e}_y$ direction. In this case we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} (H\rho_g \overline{v^y}) + \frac{\partial}{\partial x} (H\rho_g \overline{v^x}\,\overline{v^y}) + \frac{\partial}{\partial y} \left(H\rho_g \left(\overline{v^y} \right)^2 \right) \right) \\ = -\frac{\partial}{\partial y} (H\overline{p_g}) + \frac{\partial}{\partial x} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} \right) + \frac{\partial}{\partial y} \left(H\overline{S_{\mathrm{g}}^{\hat{y}\hat{y}}} \right) \\ + \boldsymbol{e}_y \cdot \left(\mathbb{T}_g|_{z=H_2(x,y,t)} \boldsymbol{n}_2 + \mathbb{T}_g|_{z=-H_1(x,y,t)} \boldsymbol{n}_1 \right) \\ - \left(\frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g \widetilde{v^x} \widetilde{v^y} \, dz + \frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g \left(\widetilde{v^y} \right)^2 dz \right). \end{aligned} \tag{1.37}$$

1.3.3 Summary

The averaged counterparts of (1.9a), (1.9b) and (1.9d) read

$$\begin{aligned} \left(\frac{\partial}{\partial t} (H\rho_g \overline{v^x}) + \frac{\partial}{\partial x} \left(H\rho_g \left(\overline{v^x} \right)^2 \right) + \frac{\partial}{\partial y} (H\rho_g \overline{v^x}\,\overline{v^y}) \right) \\ = -\frac{\partial}{\partial x} (H\overline{p_g}) + \frac{\partial}{\partial x} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{x}}} \right) + \frac{\partial}{\partial y} \left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} \right) \\ + \boldsymbol{e}_x \cdot \left(\mathbb{T}_g|_{z=H_2(x,y,t)} \boldsymbol{n}_2 + \mathbb{T}_g|_{z=-H_1(x,y,t)} \boldsymbol{n}_1 \right) \\ - \left(\frac{\partial}{\partial x} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g \left(\widetilde{v^x} \right)^2 dz + \frac{\partial}{\partial y} \int_{z=-H_1(x,y,t)}^{H_2(x,y,t)} \rho_g \widetilde{v^x} \widetilde{v^y} \, dz \right). \end{aligned} \tag{1.38a}$$

$$\left(\frac{\partial}{\partial t}\left(H\rho_g\overline{v^y}\right)+\frac{\partial}{\partial x}\left(H\rho_g\overline{v^x v^y}\right)+\frac{\partial}{\partial y}\left(H\rho_g\left(\overline{v^y}\right)^2\right)\right)$$

$$=-\frac{\partial}{\partial y}\left(H\overline{p_g}\right)+\frac{\partial}{\partial x}\left(H\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}}\right)+\frac{\partial}{\partial y}\left(H\overline{S_{\mathrm{g}}^{\hat{y}\hat{y}}}\right)$$

$$+\boldsymbol{e}_y\cdot\left(\mathbb{T}_g\big|_{z=H_2(x,y,t)}\boldsymbol{n}_2+\mathbb{T}_g\big|_{z=-H_1(x,y,t)}\boldsymbol{n}_1\right)$$

$$-\left(\frac{\partial}{\partial x}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}\rho_g\widetilde{v^x}\widetilde{v^y}\,dz+\frac{\partial}{\partial y}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}\rho_g\left(\widetilde{v^y}\right)^2\,dz\right). \quad (1.38b)$$

and

$$\frac{\partial\left(H\rho_g\right)}{\partial t}+\frac{\partial}{\partial x}\left(H\rho_g\overline{v^x}\right)+\frac{\partial}{\partial y}\left(H\rho_g\overline{v^y}\right)=0. \quad (1.38c)$$

Or in vector notation

$$\frac{\partial\left(H\rho_g\right)}{\partial t}+\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\right)=0 \quad (1.39a)$$

$$\frac{\partial}{\partial t}\left(H\rho_g\overline{\boldsymbol{v}}\right)+\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}\right)=\operatorname{div}\left(-H\overline{p_g}\mathbb{I}+H\overline{\mathbb{S}}\right)$$

$$+\mathbb{I}\left(\mathbb{T}_g\big|_{z=H_2(x,y,t)}\boldsymbol{n}_2+\mathbb{T}_g\big|_{z=-H_1(x,y,t)}\boldsymbol{n}_1\right)-\operatorname{div}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}\rho_g\widetilde{\boldsymbol{v}}\otimes\widetilde{\boldsymbol{v}}dz \quad (1.39b)$$

Note that so far we have not made any simplification, the averaged governing equations are exact.

1.3.4 Simplification of the governing equations

Concerning the balance of linear momentum in $\boldsymbol{e}_z$ direction, see (1.9c), we will assume that all terms except the gravitational force and the pressure can be neglected. Equation (1.9c) is therefore considered in the form

$$0=-\frac{\partial p_g}{\partial z}-\rho_g g \quad (1.40)$$

and all other terms are ignored. This implies that

$$p_g=-\rho_g g z+C, \quad (1.41)$$

where C is a function of x, y and t. If we further assume that the stress field in the heavier fluid is given by the formula

$$\mathbb{T}_t=-p_t\mathbb{I}, \quad (1.42)$$

and that the stress field in the ambient gas is

$$\mathbb{T}_a=-p_a\mathbb{I}, \quad (1.43)$$

then the terms in the boundary conditions (1.3a) and (1.3b) read

$$\mathbb{T}_g \boldsymbol{n}_2 = \begin{bmatrix} -\left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\frac{\partial H_2}{\partial x} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\frac{\partial H_2}{\partial y} + S_{\mathrm{g}}^{\hat{x}\hat{z}} \\ -S_{\mathrm{g}}^{\hat{x}\hat{y}}\frac{\partial H_2}{\partial x} - \left(-p_g + S_{\mathrm{g}}^{\hat{y}\hat{y}}\right)\frac{\partial H_2}{\partial y} + S_{\mathrm{g}}^{\hat{y}\hat{z}} \\ -S_{\mathrm{g}}^{\hat{x}\hat{z}}\frac{\partial H_2}{\partial x} - S_{\mathrm{g}}^{\hat{y}\hat{z}}\frac{\partial H_2}{\partial y} + \left(-p_g + S_{\mathrm{g}}^{\hat{z}\hat{z}}\right) \end{bmatrix} \tag{1.44a}$$

$$\mathbb{T}_a \boldsymbol{n}_2 = -p_a \begin{bmatrix} -\frac{\partial H_2}{\partial x} \\ -\frac{\partial H_2}{\partial y} \\ 1 \end{bmatrix} \tag{1.44b}$$

$$\mathbb{T}_g \boldsymbol{n}_1 = \begin{bmatrix} -\left(-p_g + S_{\mathrm{g}}^{\hat{x}\hat{x}}\right)\frac{\partial H_1}{\partial x} - S_{\mathrm{g}}^{\hat{x}\hat{y}}\frac{\partial H_1}{\partial y} - S_{\mathrm{g}}^{\hat{x}\hat{z}} \\ -S_{\mathrm{g}}^{\hat{x}\hat{y}}\frac{\partial H_1}{\partial x} - \left(-p_g + S_{\mathrm{g}}^{\hat{y}\hat{y}}\right)\frac{\partial H_1}{\partial y} - S_{\mathrm{g}}^{\hat{y}\hat{z}} \\ -S_{\mathrm{g}}^{\hat{x}\hat{z}}\frac{\partial H_1}{\partial x} - S_{\mathrm{g}}^{\hat{y}\hat{z}}\frac{\partial H_1}{\partial y} - \left(-p_g + S_{\mathrm{g}}^{\hat{z}\hat{z}}\right) \end{bmatrix} \tag{1.44c}$$

$$\mathbb{T}_t \boldsymbol{n}_1 = -p_t \begin{bmatrix} -\frac{\partial H_1}{\partial x} \\ -\frac{\partial H_1}{\partial y} \\ -1 \end{bmatrix} \tag{1.44d}$$

The z component of the boundary condition (1.3a) reads

$$-\left.S_{\mathrm{g}}^{\hat{x}\hat{z}}\right|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial x} - \left.S_{\mathrm{g}}^{\hat{y}\hat{z}}\right|_{z=H_2(x,y,t)}\frac{\partial H_2}{\partial y} + \left(-\left.p_g\right|_{z=H_2(x,y,t)} + \left.S_{\mathrm{g}}^{\hat{z}\hat{z}}\right|_{z=H_2(x,y,t)}\right) = -p_{\mathrm{a}}. \tag{1.45}$$

We further assume that the first two terms can be neglected, therefore the boundary condition on the top surface can be rewritten as

$$-\left.p_g\right|_{z=H_2(x,y,t)} + \left.S_{\mathrm{g}}^{\hat{z}\hat{z}}\right|_{z=H_2(x,y,t)} = -p_{\mathrm{a}}. \tag{1.46}$$

If the fluid under consideration is the Navier–Stokes fluid (possibly with temperature dependent viscosity) then

$$S_{\mathrm{g}}^{\hat{z}\hat{z}} = 2\mu\frac{\partial v^z}{\partial z}. \tag{1.47}$$

The balance of mass (1.9d) implies that

$$\rho_g\frac{\partial v^z}{\partial z} = \frac{\partial\left(\rho_g v^z\right)}{\partial z} = -\left(\frac{\partial\left(\rho_g v^x\right)}{\partial x} + \frac{\partial\left(\rho_g v^y\right)}{\partial y} + \frac{\partial \rho_g}{\partial t}\right), \tag{1.48}$$

that can be rewritten as

$$\rho_g\frac{\partial v^z}{\partial z} = -\left(\frac{\partial\left(\rho_g \overline{v^x}\right)}{\partial x} + \frac{\partial\left(\rho_g \overline{v^y}\right)}{\partial y} + \left(\frac{\partial\left(\rho_g \widetilde{v^x}\right)}{\partial x} + \frac{\partial\left(\rho_g \widetilde{v^y}\right)}{\partial y}\right) + \frac{\partial \rho_g}{\partial t}\right). \tag{1.49}$$

Note that the first term does not depend on the z variable. If we neglect the "fluctuations", then we can substitute back into (1.46), which yields

$$-\left.p_g\right|_{z=H_2(x,y,t)} - \frac{2\mu}{\rho_g}\left(\frac{\partial\left(\rho_g \overline{v^x}\right)}{\partial x} + \frac{\partial\left(\rho_g \overline{v^y}\right)}{\partial y} + \frac{\partial \rho_g}{\partial t}\right) = -p_{\mathrm{a}}. \tag{1.50}$$

Let us now consider the z component of the boundary condition (1.3b), that is the equality

$$-\left.S_{\mathrm{g}}^{\hat{x}\hat{z}}\right|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial x} - \left.S_{\mathrm{g}}^{\hat{y}\hat{z}}\right|_{z=-H_1(x,y,t)}\frac{\partial H_1}{\partial y} - \left(-\left.p_g\right|_{z=-H_1(x,y,t)} + \left.S_{\mathrm{g}}^{\hat{z}\hat{z}}\right|_{z=-H_1(x,y,t)}\right) = p_{\mathrm{t}}. \tag{1.51}$$

Here we neglect the first two terms as well. The boundary condition then reads

$$-\left(-\left.p_g\right|_{z=-H_1(x,y,t)}+\left.S_{\mathrm{g}}^{\hat{z}\hat{z}}\right|_{z=-H_1(x,y,t)}\right)=p_{\mathrm{t}}. \tag{1.52}$$

Using the same arguments as above we can approximate (1.52) by

$$\left.p_g\right|_{z=-H_1(x,y,t)}+\frac{2\mu}{\rho_g}\left(\frac{\partial\left(\rho_g\overline{v^x}\right)}{\partial x}+\frac{\partial\left(\rho_g\overline{v^y}\right)}{\partial y}+\frac{\partial\rho_g}{\partial t}\right)=p_{\mathrm{t}} \tag{1.53}$$

If we assume that the pressure in the heavier fluid is the hydrostatic pressure, then pressure on the bottom surface is given by the formula

$$p_{\mathrm{t}}=p_{\mathrm{a}}+\rho_t g H_1. \tag{1.54}$$

Under this assumption the conditions (1.53) and (1.50) can be used to derive a relation between H_1 and H_2 and to determine the constant C in (1.41). Indeed, the sum of (1.53) and (1.50) reads

$$-\left.p_g\right|_{z=H_2(x,y,t)}+\left.p_g\right|_{z=-H_1(x,y,t)}=-p_{\mathrm{a}}+\left(p_{\mathrm{a}}+\rho_t g H_1\right), \tag{1.55}$$

that, upon substituting (1.41) for the pressure, yields

$$\rho_g g H_2+\rho_g g H_1=\rho_t g H_1, \tag{1.56}$$

that is

$$H_1=\frac{\rho_g}{\rho_t}H. \tag{1.57}$$

Further, substituting (1.41) into (1.50) reveals that

$$C=p_{\mathrm{a}}+\rho_g g H_2-\frac{2\mu}{\rho_g}\left(\frac{\partial\left(\rho_g\overline{v^x}\right)}{\partial x}+\frac{\partial\left(\rho_g\overline{v^y}\right)}{\partial y}+\frac{\partial\rho_g}{\partial t}\right). \tag{1.58}$$

The formula for the pressure p_g is therefore

$$p_g=p_{\mathrm{a}}+\rho_g g(H_2-z)-\frac{2\mu}{\rho_g}\left(\frac{\partial\left(\rho_g\overline{v^x}\right)}{\partial x}+\frac{\partial\left(\rho_g\overline{v^y}\right)}{\partial y}+\frac{\partial\rho_g}{\partial t}\right), \tag{1.59}$$

and the averaged pressure $\overline{p_g}=\frac{1}{H}\int_{z=-H_1(x,y,t)}^{H_2(x,y,t)}p_g\,dz$ is given by the formula

$$H\overline{p_g}=p_{\mathrm{a}}H+\rho_g g\frac{H^2}{2}-\frac{2\mu H}{\rho_g}\left(\frac{\partial\left(\rho_g\overline{v^x}\right)}{\partial x}+\frac{\partial\left(\rho_g\overline{v^y}\right)}{\partial y}+\frac{\partial\rho_g}{\partial t}\right). \tag{1.60}$$

In order to close the system of the equations, it remains to evaluate product $\boldsymbol{e}_x\cdot\left(\left.\mathbb{T}_g\right|_{z=H_2(x,y,t)}\boldsymbol{n}_2+\left.\mathbb{T}_g\right|_{z=-H_1(x,y,t)}\boldsymbol{n}_1\right)$ and $\boldsymbol{e}_y\cdot\left(\left.\mathbb{T}_g\right|_{z=H_2(x,y,t)}\boldsymbol{n}_2+\left.\mathbb{T}_g\right|_{z=-H_1(x,y,t)}\boldsymbol{n}_1\right)$ in equation (1.38a) and (1.38b) respectively. Let us consider the first product. The boundary conditions (1.3) imply that

$$\boldsymbol{e}_x\cdot\left(\left.\mathbb{T}_g\right|_{z=H_2(x,y,t)}\boldsymbol{n}_2+\left.\mathbb{T}_g\right|_{z=-H_1(x,y,t)}\boldsymbol{n}_1\right)=\boldsymbol{e}_x\cdot\left(\left.\mathbb{T}_a\right|_{z=H_2(x,y,t)}\boldsymbol{n}_2+\left.\mathbb{T}_t\right|_{z=-H_1(x,y,t)}\boldsymbol{n}_1\right) \tag{1.61}$$

which under our assumptions simplifies to

$$\boldsymbol{e}_x \cdot \left(\mathbb{T}_g\big|_{z=H_2(x,y,t)} \boldsymbol{n}_2 + \mathbb{T}_g\big|_{z=-H_1(x,y,t)} \boldsymbol{n}_1 \right) = -p_{\mathrm{a}} \boldsymbol{e}_x \cdot \boldsymbol{n}_2 - p_{\mathrm{t}} \boldsymbol{e}_x \cdot \boldsymbol{n}_1$$
$$= p_{\mathrm{a}} \frac{\partial H}{\partial x} + \rho_t g H_1 \frac{\partial H_1}{\partial x} = p_{\mathrm{a}} \frac{\partial H}{\partial x} + g H \rho_g \frac{\partial}{\partial x} \left(\frac{H\rho_g}{\rho_t} \right) = p_{\mathrm{a}} \frac{\partial H}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\left(H\rho_g\right)^2 g}{2\rho_t} \right), \tag{1.62}$$

where we have used (1.54) and (1.57). Similar manipulation yields

$$\boldsymbol{e}_y \cdot \left(\mathbb{T}_g\big|_{z=H_2(x,y,t)} \boldsymbol{n}_2 + \mathbb{T}_g\big|_{z=-H_1(x,y,t)} \boldsymbol{n}_1 \right) = p_{\mathrm{a}} \frac{\partial H}{\partial y} + \frac{\partial}{\partial y} \left(\frac{\left(H\rho_g\right)^2 g}{2\rho_t} \right). \tag{1.63}$$

1.3.5 Model

Now we are ready to go back to the system (1.38). Neglecting the terms quadratic in the "fluctuations" and using (1.59), (1.62) and (1.63), where we assume constant p_a, we arrive to the simplified system of governing equations

$$\left(\frac{\partial}{\partial t} \left(H\rho_g \overline{v^x}\right) + \frac{\partial}{\partial x} \left(H\rho_g \left(\overline{v^x}\right)^2 \right) + \frac{\partial}{\partial y} \left(H\rho_g \overline{v^x v^y}\right) \right) =$$
$$- \frac{\partial}{\partial x} \left(\rho_g \left(1 - \frac{\rho_g}{\rho_t}\right) g \frac{H^2}{2} - \frac{2\mu H}{\rho_g} \left(\frac{\partial \left(\rho_g \overline{v^x}\right)}{\partial x} + \frac{\partial \left(\rho_g \overline{v^y}\right)}{\partial y} + \frac{\partial \rho_g}{\partial t} \right) \right)$$
$$+ \frac{\partial}{\partial x} \left(H \overline{S_{\mathrm{g}}^{\hat{x}\hat{x}}} \right) + \frac{\partial}{\partial y} \left(H \overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} \right), \tag{1.64a}$$

$$\left(\frac{\partial}{\partial t} \left(H\rho_g \overline{v^y}\right) + \frac{\partial}{\partial x} \left(H\rho_g \overline{v^x v^y}\right) + \frac{\partial}{\partial y} \left(H\rho_g \left(\overline{v^y}\right)^2 \right) \right) =$$
$$- \frac{\partial}{\partial y} \left(\rho_g \left(1 - \frac{\rho_g}{\rho_t}\right) g \frac{H^2}{2} - \frac{2\mu H}{\rho_g} \left(\frac{\partial \left(\rho_g \overline{v^x}\right)}{\partial x} + \frac{\partial \left(\rho_g \overline{v^y}\right)}{\partial y} + \frac{\partial \rho_g}{\partial t} \right) \right)$$
$$+ \frac{\partial}{\partial x} \left(H \overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} \right) + \frac{\partial}{\partial y} \left(H \overline{S_{\mathrm{g}}^{\hat{y}\hat{y}}} \right), \tag{1.64b}$$

$$\frac{\partial \left(H\rho_g\right)}{\partial t} + \frac{\partial}{\partial x} \left(H\rho_g \overline{v^x}\right) + \frac{\partial}{\partial y} \left(H\rho_g \overline{v^y}\right) = 0 \tag{1.64c}$$

where

$$\overline{S_{\mathrm{g}}^{\hat{x}\hat{x}}} =_{\mathrm{def}} 2\mu \frac{\partial \overline{v^x}}{\partial x}, \tag{1.64d}$$
$$\overline{S_{\mathrm{g}}^{\hat{y}\hat{y}}} =_{\mathrm{def}} 2\mu \frac{\partial \overline{v^y}}{\partial y} \tag{1.64e}$$
$$\overline{S_{\mathrm{g}}^{\hat{x}\hat{y}}} =_{\mathrm{def}} \mu \left(\frac{\partial \overline{v^x}}{\partial y} + \frac{\partial \overline{v^y}}{\partial x} \right). \tag{1.64f}$$

This definition is motivated by the fact that for the Navier–Stokes fluid we have $S_{\mathrm{g}}^{\hat{x}\hat{x}} = 2\mu \frac{\partial v^x}{\partial x}$ and similarly for the other components of the extra stress tensor. Clearly, (1.64d) is a good approximation of the exact constitutive relation

$S_{\mathrm{g}}^{\hat{x}\hat{x}} = 2\mu\frac{\partial v^x}{\partial x}$. Note that the last equation (balance of mass) is still exact, no approximations have been made in this equation. System (1.64) has been derived, in the context of float glass processing, by Popov (1982), but it is the standard system of governing equations for thin fluid film flows.

Let us further remark that if we denote

$$P =_{\mathrm{def}} \rho_g \left(1 - \frac{\rho_g}{\rho_t}\right) g\frac{H^2}{2}, \tag{1.65a}$$

$$\overline{\boldsymbol{v}} =_{\mathrm{def}} \begin{bmatrix} \overline{v^x} \\ \overline{v^y} \end{bmatrix}, \tag{1.65b}$$

$$\mathbb{T}_{gs} =_{\mathrm{def}} -P\mathbb{I} + \frac{2\mu H}{\rho_g}\left(\mathrm{div}\,(\rho_g\overline{\boldsymbol{v}}) + \frac{\partial \rho_g}{\partial t}\right)\mathbb{I} + 2\mu H\mathbb{D} \tag{1.65c}$$

where $\mathbb{D} =_{\mathrm{def}} \frac{1}{2}\left(\nabla\overline{\boldsymbol{v}} + (\nabla\overline{\boldsymbol{v}})^T\right)$ denotes the symmetric part of the velocity gradient, then we can rewrite the system as

$$\frac{\partial\,(H\rho_g)}{\partial t} + \mathrm{div}\,(H\rho_g\overline{\boldsymbol{v}}) = 0, \tag{1.66a}$$

$$\frac{\partial}{\partial t}\,(H\rho_g\overline{\boldsymbol{v}}) + \mathrm{div}\,\,(H\rho_g\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}) = \mathrm{div}\,\mathbb{T}_{gs}. \tag{1.66b}$$

If we further consider constant density ρ_g we will get in fact to equations for a “compressible” fluid with “density” dependent material coefficients

$$\frac{\partial H}{\partial t} + \mathrm{div}\,(H\overline{\boldsymbol{v}}) = 0, \tag{1.67a}$$

$$\rho_g\left[\frac{\partial}{\partial t}\,(H\overline{\boldsymbol{v}}) + \mathrm{div}\,\,(H\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}})\right] = \mathrm{div}\,\left[-P\mathbb{I} + 2\mu H\,(\mathrm{div}\,\overline{\boldsymbol{v}})\,\mathbb{I} + 2\mu H\mathbb{D}\right]. \tag{1.67b}$$

1.4 boundary conditions of simplified model

The governing equations must be solved in domain

$$\Omega_{x,y,t} =_{\mathrm{def}} \left\{\boldsymbol{x}\in\mathbb{R}^2 : 0 \le y \le L, -W(y,t) \le x \le W(y,t)\right\},$$

where L declares the length of the domain from restrictors’ tip to the lehr, where the glass is pulled out. The shape of the domain is specified by the function $W(y,t)$, which is a part of the problem. (Symmetry along y axis is assumed.)

1.4.1 Position of lateral free boundary

Since the position of lateral boundary is specified by a priori unknown function $W(y,t)$, we need a condition for its evaluation. The implicit equation of the boundary is $f^+(x,y,t) = x - W(y,t) = 0$ and $f^-(x,y,t) = x + W(y,t) = 0$, therefore we obtain the condition from Lemma 1

$$\frac{\partial f^+}{\partial t} + \overline{\boldsymbol{v}}\cdot\nabla f^+ = -\frac{\partial W}{\partial t} + \overline{v^x}|_{x=W(y,t)} - \overline{v^y}|_{x=W(y,t)}\frac{\partial W}{\partial y} = 0, \tag{1.68a}$$

and similarly for the other boundary

$$\frac{\partial f^-}{\partial t} + \overline{\boldsymbol{v}}\cdot\nabla f^- = \frac{\partial W}{\partial t} + \overline{v^x}|_{x=-W(y,t)} + \overline{v^y}|_{x=-W(y,t)}\frac{\partial W}{\partial y} = 0. \tag{1.68b}$$

Since we assume symmetry along y axis we do not need to consider the other boundary condition because it is automatically fulfilled. The symmetry of the problem further implies that $\left.\overline{v^x}\right|_{x=W(y,t)} = -\left.\overline{v^x}\right|_{x=-W(y,t)}$ and $\left.\overline{v^y}\right|_{x=W(y,t)} = \left.\overline{v^y}\right|_{x=-W(y,t)}$.

1.4.2 Lateral boundary condition

For better understanding how the fluid spreads in space, let us consider constant amount of glass dipped in tin with no movement in equilibrium state. It means that we have 1.67a zero velocity field $\overline{\boldsymbol{v}} = (0, 0)$ and linear momentum equation 1.66b reduces to

$$0 = \operatorname{div}(\mathbb{T}_{gs}) \tag{1.69}$$

and Cauchy stress tensor reads

$$\mathbb{T}_{gs} = -P\mathbb{I} + \frac{2\mu H}{\rho_g}\frac{\partial \rho_g}{\partial t}. \tag{1.70}$$

If we further assume constant glass density, then previous relations and 1.65a implies

$$\nabla\left(\rho_g\left(1 - \frac{\rho_g}{\rho_t}\right) g\frac{H^2}{2}\right) = 0, \tag{1.71}$$

which means

$$\rho_g\left(1 - \frac{\rho_g}{\rho_t}\right) g\frac{H^2}{2} = \gamma. \tag{1.72}$$

Physical meaning of constant γ can be seen from $[\gamma] = kg/s^2 = N/m$ and fact that equation 1.72 declares a relation between fluid's height H and γ, which can be also rewritten as

$$H^2 = \gamma\frac{2\rho_t}{g\rho_g(\rho_t - \rho_g)}. \tag{1.73}$$

It means, that γ is intensity of a force acting on unit length and reflects how the fluid spreads. So from now on we refer to γ as a spreading coefficient.

Relation 1.73, which we derived from linear momentum equation, can be also found in **??** "Pilkington", where is used for evaluating equilibrium thickness of the film.

The direction of the force acting on lateral boundary is equal to the spreading coefficient γ and its direction is normal to the boundary. Therefore we can generalize relation 1.72 and gain a dynamic boundary condition, which reads

$$\mathbb{T}_{gs}\boldsymbol{n} = -\gamma\boldsymbol{n}, \tag{1.74}$$

where $\boldsymbol{n}$ denotes the outward normal to the film boundary. As we can see from 1.73 the spreading coefficient should be positive if the height of the film is positive, therefore the force acting on the unit length of the boundary points inside the fluid film.

1.4.3 Inflow boundary condition

As an inflow boundary condition, we use Dirichlet condition for height $H(x, 0, t) = H_{\text{in}}(x, t)$ as well as for velocity $\overline{\boldsymbol{v}}(x, 0, t) = \boldsymbol{v}_{\text{in}}(x, t)$. Since function $H(x, y, t)$, which describes height of the glass, suppose to slowly vary, we use constant inflow height H_{in}. This constant should be slightly bigger then equilibrium thickness described in 1.73, because the glass enters the domain from restrictors and should fill all space between them. It also means we have inflow condition for width of the domain $W(0, t) = W_{\text{in}}$, which completes equation 1.68b.

The restrictors are two diverging walls meeting at given angle 2α. That is why it seems reasonable to use Jeffery-Hamel flow to capture the inflow velocity $\boldsymbol{v}_{\text{in}}$.

Jeffery Hamel flow

The flow between two planes that meet at an angle was first analyzed by Jeffery (1915) and Hamel (1916). Under suitable assumptions, the problem can be reduced to the solution of an ordinary differential equation. We shall use polar coordinates (r, θ), where $\theta \in [-\alpha, \alpha]$ and assume purely radial and steady flow, i.e., $\boldsymbol{v} = v^r(r, \theta)\boldsymbol{e}^r$. Under these assumptions the continuity equation reduces to

$$\frac{\partial}{\partial r}(r v^r) = 0, \tag{1.75}$$

which implies

$$v^r = \frac{\nu F(\theta)}{r}, \tag{1.76}$$

where F is a function of θ only and ν represents kinematic viscosity. Linear momentum equations simplifies to

$$v^r \frac{\partial v^r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v^r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v^r}{\partial \theta^2} + \frac{1}{r}\frac{\partial v^r}{\partial r} - \frac{v^r}{r^2}\right), \tag{1.77a}$$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\frac{2}{r^2}\frac{\partial v^r}{\partial \theta}. \tag{1.77b}$$

Substituting into the azimuthal momentum equation 1.77b, we obtain

$$\frac{\partial}{\partial \theta}\left(\frac{p}{\rho}\right) = \frac{\partial}{\partial \theta}\left(\frac{2\nu^2 F}{r^2}\right) \tag{1.78}$$

and after integration

$$\frac{p}{\rho} = \frac{2\nu^2 F}{r^2} + G(r), \tag{1.79}$$

where $G(r)$ is independent from θ. With combination of 1.76 and 1.79 we can rewrite radial momentum equation 1.77a as

$$\begin{aligned}\frac{\nu^2 F^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) &= -2\nu^2 F\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) - \frac{\partial G}{\partial r} \\ &\quad + \nu\left(\nu F\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\right) + \frac{\nu}{r^3}\frac{\partial^2 F}{\partial \theta^2} + \frac{\nu F}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\nu F}{r^3}\right).\end{aligned} \tag{1.80}$$

Because F and G are functions of only one variable, we shell use prime notation for differentiation $F' = \frac{\partial F}{\partial \theta}$ and $G' = \frac{\partial G}{\partial r}$ respectively. When we apply the derivations, we have

$$-\frac{\nu^2 F^2}{r^3} = \frac{4\nu^2 F}{r^3} - G' + \frac{2\nu^2 F}{r^3} + \frac{\nu^2}{r^3} F'' - \frac{\nu^2 F}{r^3} - \frac{\nu^2 F}{r^3}, \tag{1.81}$$

which is equivalent to

$$\frac{r^3 G}{\nu^2} = F'' + 4F + F^2. \tag{1.82}$$

Left hand side of this equation is a function of r only, while right hand side depends on θ. It means, that each side equals to the same constant C and for F we obtain ordinary differential equation

$$F'' + 4F + F^2 = C. \tag{1.83}$$

Constant C should be determined with respect to the fact, that the same mass of fluid Q passes per unit time through any cross-section $r = const.$, where Q could be evaluated as

$$Q = \rho \int_{-\alpha}^{\alpha} v^r r d\theta = \rho\nu \int_{-\alpha}^{\alpha} F d\theta. \tag{1.84}$$

Equation 1.83 should be completed with lateral boundary condition. Physical experiments suggests, that free slip boundary condition should be used.

As a simplification, we shall consider in the following the function $F(\phi)$ to be constant

$$v^r = v^r(r) = \frac{\nu F}{r}. \tag{1.85}$$

From the geometry of the problem, Fig. 1.2, we can easily see that $r = \sqrt{x^2 + r_c^2}$, $r_c = \frac{W_{\text{in}}}{\tan\alpha}$ and the following relations hold

$$v^x = v^r \sin\phi = \frac{\nu F}{r}\frac{x}{r} = \frac{\nu F x}{x^2 + r_c^2}, \tag{1.86a}$$

$$v^y = v^r \cos\phi = \frac{\nu F}{r}\frac{r_c}{r} = \frac{\nu F r_c}{x^2 + r_c^2}, \tag{1.86b}$$

Concerning the mass of fluid that passes per unit time through any cross-section $y = r_c =$ constant, we obtain

$$Q = \rho \int_{-W(y)}^{W(y)} v^y \,\mathrm{d}x = \rho \int_{-W(y)}^{W(y)} \frac{\nu F r_c(y)}{x^2 + r_c^2(y)} \,\mathrm{d}x = \rho\nu F \left[\arctan\left(\frac{x}{r_c(y)}\right)\right]_{-W(y)}^{W(y)}$$
$$= 2\rho\nu F \arctan \underbrace{\left(\frac{W(y)}{r_c(y)}\right)}_{\tan\alpha} = 2\alpha\rho\nu F. \tag{1.87}$$

Only v^y is responsible for the mass flux, since $\int_{-W(y)}^{W(y)} v^x \,\mathrm{d}x = 0$. Therefore we have a relation

$$F\nu = \frac{Q}{2\alpha\rho} \tag{1.88}$$

and inflow velocity $\boldsymbol{v}_{\text{in}}$ can determined only from given mass inflow Q and angle α under which the restrictiors meet. Indeed we have

$$\boldsymbol{v}_{\text{in}} = (v^x, v^y) = \left(\frac{\nu F x}{x^2 + r_c^2}, \frac{\nu F r_c}{x^2 + r_c^2}\right), \tag{1.89}$$

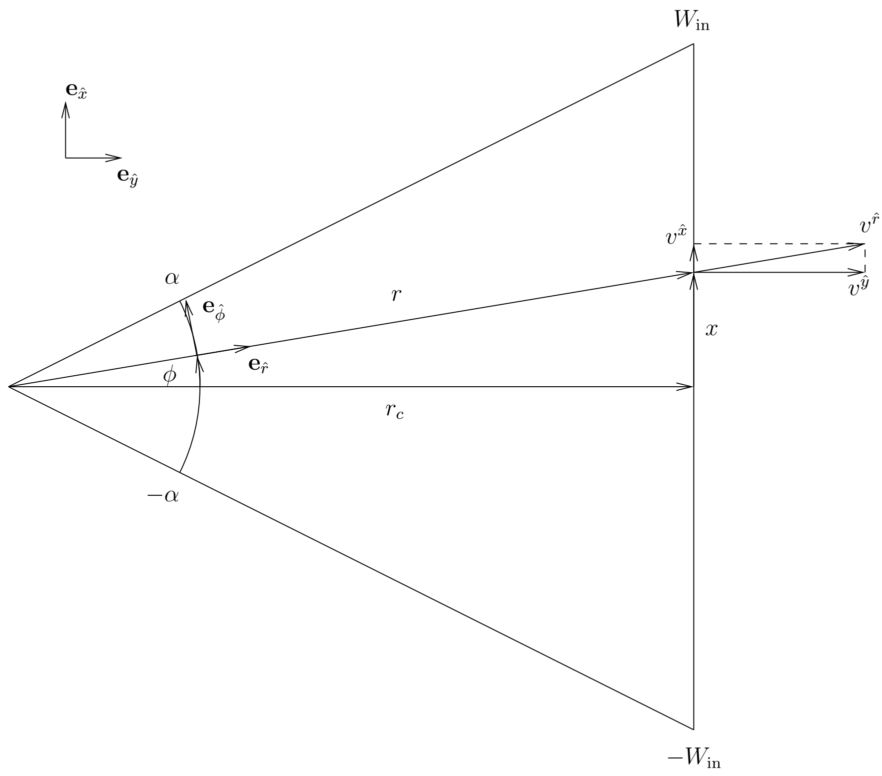

Figure 1.2: Jeffery Hamel flow geometry.

where $F\nu = \frac{Q}{2\alpha\rho}$ and $r_c = \frac{W_{\text{in}}}{\tan\alpha}$.

The problem of the Jeffery-Hamel flow can be also found in Rosenhead (1940) and Batchelor (2000).

1.4.4 Outflow boundary condition

At the end of the tin bath the glass is pulled out by a huge roller called lehr. By manipulating with speed of the lehr we can modify height and width of the ribbon, but for us it is important as a Dirichlet boundary condition at the outflow of the domain. Because the lehr moves with the same constant speed v_{out} over whole outflow, we can write the outflow boundary condition as

$$\overline{\boldsymbol{v}}(x, y, t)\big|_{y=L} = (0, v_{\text{out}}) = \boldsymbol{v}_{\text{out}}. \tag{1.90}$$

Chapter 2

Numerical solution

We have system

$$\frac{\partial\left(H\rho_g\right)}{\partial t}+\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\right)=0, \tag{2.1a}$$

$$\frac{\partial}{\partial t}\left(H\rho_g\overline{\boldsymbol{v}}\right)+\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}\right)=\operatorname{div}\mathbb{T}_{gs}, \tag{2.1b}$$

$$\frac{\partial W(y)}{\partial t}+\overline{v^x}(-W(y),y)+\overline{v^y}(-W(y),y)\frac{\partial W(y)}{\partial y}=0. \tag{2.1c}$$

where $\mathbb{T}_{gs}$ is declared in 1.65c. These equations will be solved in domain $\Omega=\{(x,y)\in\mathbb{R}^2;0\leq y\leq L,-W(y,t)\leq x\leq W(y,t)\}$ with following Dirichlet boundary conditions

$$W(0,t)=W_{\text{in}}, \tag{2.2}$$

$$H(x,0,t)=H_{\text{in}} \qquad -W(0,t)\leq x\leq W(0,t) \tag{2.3}$$

$$\overline{\boldsymbol{v}}(x,0,t)=\boldsymbol{v}_{\text{in}}(x) \qquad -W(0,t)\leq x\leq W(0,t), \tag{2.4}$$

$$\overline{\boldsymbol{v}}(x,L,t)=\boldsymbol{v}_{\text{out}} \qquad -W(L,t)\leq x\leq W(L,t), \tag{2.5}$$

and Neumann condition for sides of the ribbon

$$\mathbb{T}_{gs}\boldsymbol{n}^+=-\gamma\boldsymbol{n}^+ \tag{2.6}$$

$$\mathbb{T}_{gs}\boldsymbol{n}^-=-\gamma\boldsymbol{n}^- \tag{2.7}$$

where $\boldsymbol{n}^\pm=(\pm1,-\frac{\partial W}{\partial y})$ is outer normal of the ribbon. We further mark four parts of the boundary: inflow as Γ_{in}, outflow as Γ_{out} and lateral boundaries as $\Gamma^\pm_{\text{free}}$.

We will solve the system 2.1 by finite element method. For that purpose we need the equations in weak formulation.

2.1 Weak formulation

For further manipulation we neglect all terms with time derivative $\frac{\partial}{\partial t}$ from model 2.1, because want to study steady state of the ribbon. In this case, equations simplifies as follows

$$\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\right) = 0, \tag{2.8a}$$

$$\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}\right) = \operatorname{div}\mathbb{T}_{gs}, \tag{2.8b}$$

$$\frac{\partial W(y)}{\partial y} = -\frac{\overline{v^x}(-W(y),y)}{\overline{v^y}(-W(y),y)} \tag{2.8c}$$

and Cauchy stress tensor reads

$$\mathbb{T}_{gs} = -P\mathbb{I} + \frac{2\mu H}{\rho_g}\left(\operatorname{div}\left(\rho_g\overline{\boldsymbol{v}}\right)\right)\mathbb{I} + 2\mu H\mathbb{D}. \tag{2.9}$$

Moreover right hand side of the equation 2.8b can be rewritten with using continuum equation 2.8a in coordinates notation as follows

$$\frac{\partial H\rho_g v^i v^j}{\partial x_i} = \frac{\partial H\rho_g v^i}{\partial x_i}v^j + H\rho_g v^i\frac{\partial v^j}{\partial x_i} = \overbrace{\operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\right)}^{=0} v_j + H\rho_g v^i\frac{\partial v^j}{\partial x_i}. \tag{2.10}$$

Therefore equation 2.8b is equivalent to

$$H\rho_G\left(\nabla\overline{\boldsymbol{v}}\right)\overline{\boldsymbol{v}} = \operatorname{div}\mathbb{T}_{gs}. \tag{2.11}$$

If we want rewrite equations 2.1 in weak formulation, we multiply first equation 2.8a and third 2.8c by test functions $G \in \tilde{\Phi}$ and $\psi \in \tilde{\Psi}$ respectively and on second 2.8b we use scalar multiplication by $\boldsymbol{u} \in \tilde{\boldsymbol{V}}$, where $\tilde{\Phi}$ and $\tilde{\Psi}$ are function space and $\tilde{\boldsymbol{V}}$ vector function space, which will be specified later. Then, after integrating equations 2.8a and 2.8b over domain Ω and , we get

$$\int_\Omega \operatorname{div}\left(H\rho_g\overline{\boldsymbol{v}}\right)G d\mathrm{v} = 0, \tag{2.12a}$$

$$\int_\Omega \left(H\rho_G\left(\nabla\overline{\boldsymbol{v}}\right)\overline{\boldsymbol{v}}\right)\cdot\boldsymbol{u}d\mathrm{v} = \int_\Omega \operatorname{div}\left(\mathbb{T}_{gs}\right)\cdot\boldsymbol{u}d\mathrm{v}, \tag{2.12b}$$

$$\int_0^L \frac{\partial W(y)}{\partial y}\psi(y)dy = -\int_0^L \frac{\overline{v^x}(-W(y),y)}{\overline{v^y}(-W(y),y)}\psi(y)dy. \tag{2.12c}$$

On the right hand side of second equation 2.12b we use Green's theorem which yields

$$\int_\Omega \left(H\rho_G\left(\nabla\overline{\boldsymbol{v}}\right)\overline{\boldsymbol{v}}\right)\cdot\boldsymbol{u}d\mathrm{v} = -\int_\Omega \left(\mathbb{T}_{gs}\right):\nabla\boldsymbol{u}d\mathrm{v} + \int_{\partial\Omega}\mathbb{T}\boldsymbol{n}\cdot\boldsymbol{u}dS. \tag{2.13}$$

The boundary integral vanishes for the inflow and outflow, because test function $\boldsymbol{u}$ equals to zero, due to Dirichlet boundary condition on velocity. However on the rest of the boundary we have Neumann boundary condition $\mathbb{T}\boldsymbol{n}^\pm = -\gamma\boldsymbol{n}^\pm$, which reflects in equation as

$$\int_\Omega \left(H\rho_G\left(\nabla\overline{\boldsymbol{v}}\right)\overline{\boldsymbol{v}}\right)\cdot\boldsymbol{u}d\mathrm{v} = -\int_\Omega \left(\mathbb{T}_{gs}\right):\nabla\boldsymbol{u}d\mathrm{v} - \int_{\Gamma^\pm_{\mathrm{free}}}\gamma\boldsymbol{n}^\pm\cdot\boldsymbol{u}dS, \tag{2.14}$$

where $\Gamma^\pm_{\mathrm{free}}$ denotes lateral parts of the boundary and $\boldsymbol{n}^\pm$ outward normals.

Now we get to weak formulation of the problem. We define function spaces

$$\boldsymbol{V} = \left\{\overline{\boldsymbol{v}}(x,y) \in \mathbb{R}^2,\ \overline{\boldsymbol{v}}|_{\Gamma_{\text{in}}} = \boldsymbol{v}_{\text{in}},\ \overline{\boldsymbol{v}}|_{\Gamma_{\text{out}}} = \boldsymbol{v}_{\text{out}}\right\}, \tag{2.15a}$$

$$\Phi = \left\{H(x,y) \in \mathbb{R},\ H|_{\Gamma_{\text{in}}} = H_{\text{in}}\right\}, \tag{2.15b}$$

$$\Psi = \{W(y) \in \mathbb{R},\ W(0) = W_{\text{in}}\} \tag{2.15c}$$

such that all previous integrals are finite. Then we seek $H \in \Phi$, $\overline{\boldsymbol{v}} \in \boldsymbol{V}$ and $W \in \Psi$ that

$$\int_\Omega \text{div}\ (H\rho_g\overline{\boldsymbol{v}})\, G d\text{v} = 0, \tag{2.16a}$$

$$\int_\Omega (H\rho_g (\nabla\overline{\boldsymbol{v}})\,\overline{\boldsymbol{v}}) \cdot \boldsymbol{u} d\text{v} = -\int_\Omega (\mathbb{T}_{gs}) : \nabla \boldsymbol{u} d\text{v} - \int_{\Gamma^{\pm}_{\text{free}}} \gamma \boldsymbol{n}^{\pm} \cdot \boldsymbol{u} dS, \tag{2.16b}$$

$$\int_0^L \frac{\partial W(y)}{\partial y}\psi(y)dy = -\int_0^L \frac{\overline{v^x}(-W(y),y)}{\overline{v^y}(-W(y),y)}\psi(y)dy \tag{2.16c}$$

holds for all $G \in \tilde{\Phi}$, $\boldsymbol{u} \in \tilde{\boldsymbol{V}}$ and $\psi \in \tilde{\Psi}$, where

$$\tilde{\boldsymbol{V}} = \left\{\boldsymbol{u}(x,y) \in \mathbb{R}^2,\ \boldsymbol{u}|_{\Gamma_{\text{in}}} = \boldsymbol{0},\ \boldsymbol{u}|_{\Gamma_{\text{out}}} = \boldsymbol{0}\right\}, \tag{2.17a}$$

$$\tilde{\Phi} = \left\{G(x,y) \in \mathbb{R},\ G|_{\Gamma_{\text{in}}} = 0\right\}, \tag{2.17b}$$

$$\tilde{\Psi} = \{\psi(y) \in \mathbb{R},\ \psi(0) = 0\} \tag{2.17c}$$

are function spaces for test functions.

2.2 Domain transformation

Let us assume that we have a mapping χ that transforms a fixed domain $\Omega_{\mathbf{X}} \subset \mathbb{R}^2$ to the evolving domain $\Omega_{\mathbf{x},t} \subset \mathbb{R}^2$, see Figure 2.1. The relation between the position $\mathbf{X}$ in the fixed domain $\Omega_{\mathbf{X}}$ and the position $\mathbf{x}$ in the evolving domain $\Omega_{\mathbf{x},t}$ is

$$\mathbf{x} = \chi(\mathbf{X}, t). \tag{2.18}$$

In what follows will investigate transformation rules for certain volume and surface integrals that appear in the weak formulation of the problem. Using the derived transformation rules, we will be able to rewrite the weak formulation in a fixed computational domain, which will be convenient for the numerical solution of the problem.

Any function $\phi(\mathbf{x}, t)$ defined in the evolving domain $\Omega_{\mathbf{x},t}$ can be rewritten as a function on the fixed domain $\Omega_{\mathbf{X}}$ in terms of the following simple substitution,

$$\widehat{\phi}(\mathbf{X}, t) =_{\text{def}} \phi(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)}. \tag{2.19}$$

The derivatives of $\widehat{\phi}(\mathbf{X}, t)$ with respect to $\mathbf{X}$ can be easily found using the chain rule,

$$\frac{\partial\widehat{\phi}}{\partial X_i}(\mathbf{X}, t) = \frac{\partial}{\partial X_i}\left(\phi(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\right) = \frac{\partial \phi}{\partial x_j}(\mathbf{x}, t)\bigg|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{\partial \chi_j}{\partial X_i}(\mathbf{X}, t). \tag{2.20}$$

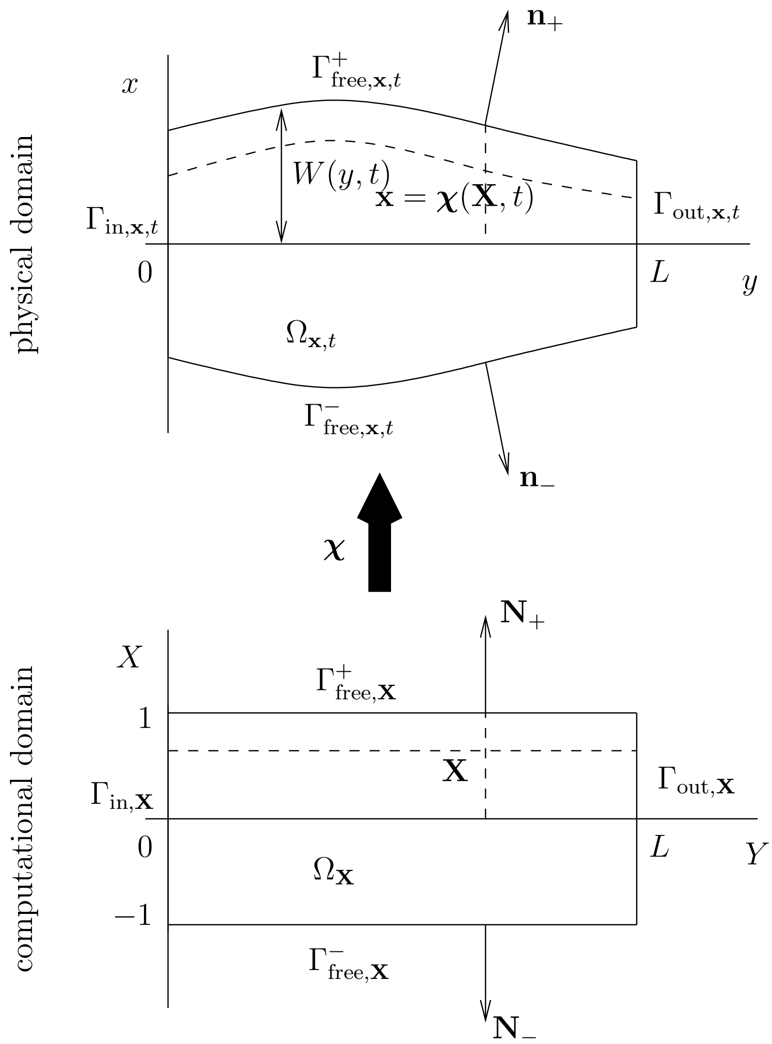

Figure 2.1: Change of variables.

If we define the transformation matrix $\mathbb{F}$ as follows,

$$\mathbb{F}(\mathbf{X},t) =_{\mathrm{def}} \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X},t), \tag{2.21}$$

which means that the matrix elements are given by the formula

$$\mathbb{F}(\mathbf{X},t) = \left[F_{ij}(\mathbf{X},t)\right]_{i,j=1,\ldots,3} = \left[\tfrac{\partial \chi_i}{\partial X_j}(\mathbf{X},t)\right]_{i,j=1,\ldots,3} \tag{2.22}$$

then the transformation rule (2.20) can be rewritten as

$$\nabla_{\mathbf{X}}\widehat{\phi}(\mathbf{X},t) = \mathbb{F}^T(\mathbf{X},t)\; \nabla_{\mathbf{x}}\phi(\mathbf{x},t)\big|_{\mathbf{x}=\chi(\mathbf{X},t)}\,, \tag{2.23}$$

or, in a compact notation, as $\nabla_{\mathbf{X}}\widehat{\phi} = \mathbb{F}^T\,(\nabla_{\mathbf{x}}\phi)$. (Note that (2.21) is frequently written as $\mathbb{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$.) It turns out that for transforming equations in weak formulation 2.16 we need to evaluate $(\nabla_{\mathbf{x}}\phi)$. For that purpose instead of $\mathbb{F}$ it is more convenient to work with the inverse of this matrix,

$$\mathbb{A}(\mathbf{X},t) =_{\mathrm{def}} \mathbb{F}^{-1}(\mathbf{X},t). \tag{2.24}$$

The standard manipulation yields—with a slight abuse of the notation—the formula

$$\mathbb{A}(\mathbf{X},t) = \left.\frac{\partial \mathbf{X}}{\partial \mathbf{x}}(\mathbf{x},t)\right|_{\mathbf{x}=\chi(\mathbf{X},t)} = \left.\frac{\partial \chi^{-1}}{\partial \mathbf{x}}(\mathbf{x},t)\right|_{\mathbf{x}=\chi(\mathbf{X},t)} \tag{2.25}$$

and by similar manipulation as in 2.20 we see that

$$\frac{\partial \phi}{\partial x_i}(x,t) = \frac{\partial}{\partial x_i}\left(\left.\widehat{\phi}(\mathbf{X},t)\right|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)} \right) = \left.\frac{\partial \widehat{\phi}}{\partial X_j}(\mathbf{X},t)\right|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)} \frac{\partial \chi_j^{-1}}{\partial x_i}(\mathbf{x},t), \tag{2.26}$$

which we can write as

$$\nabla_{\mathbf{x}}\phi(\mathbf{x},t) = \mathbb{A}^T(\mathbf{X},t)\, \nabla_{\mathbf{X}}\widehat{\phi}(\mathbf{X},t)\Big|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)}. \tag{2.27}$$

Concerning the vector valued functions, the counterpart of the transformation rule (2.19) reads

$$\widehat{\mathbf{f}}(\mathbf{X},t) =_{\text{def}} \mathbf{f}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}. \tag{2.28}$$

The application of the chain rule yields

$$\frac{\partial \widehat{f}_i}{\partial X_j}(\mathbf{X},t) = \frac{\partial}{\partial X_j}\left(f_i(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \right) = \left.\frac{\partial f_i}{\partial x_k}(\mathbf{x},t)\right|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{\partial \chi_k}{\partial X_j}(\mathbf{X},t), \tag{2.29}$$

which can be rewritten as

$$\nabla_{\mathbf{X}}\widehat{\mathbf{f}}(\mathbf{X},t) = \nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\, \mathbb{F}(\mathbf{X},t). \tag{2.30}$$

Similar relation holds for

$$\nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x},t) = \nabla_{\mathbf{X}}\widehat{\mathbf{f}}(\mathbf{x},t)\Big|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)} \mathbb{A}(\mathbf{X},t). \tag{2.31}$$

We can easily see, how to transform $\operatorname{div}_{\mathbf{X}}\widehat{\mathbf{f}}$ from previous relation by simple manipulation

$$\operatorname{div}_{\mathbf{X}}\widehat{\mathbf{f}}(\mathbf{X},t) = \operatorname{Tr}\left(\nabla_{\mathbf{X}}\widehat{\mathbf{f}}(\mathbf{X},t)\right) \overset{2.30}{\widehat{=}} \operatorname{Tr}\left(\nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\, \mathbb{F}(\mathbf{X},t)\right) \tag{2.32}$$

or we have with inverse transformation

$$\operatorname{div}_{\mathbf{x}}\mathbf{f}(\mathbf{x},t) = \operatorname{Tr}\left(\nabla_{\mathbf{X}}\widehat{\mathbf{f}}(\mathbf{x},t)\Big|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)} \mathbb{A}(\mathbf{X},t) \right). \tag{2.33}$$

The weak formulation of the governing equations will in principle include volume integrals such $\int_{\Omega_{\mathbf{x},t}} f(\mathbf{x},t)\, d\mathrm{v}$ and $\int_{\Omega_{\mathbf{x},t}} \mathbf{f}(\mathbf{x},t)\, d\mathrm{v}$, where f, $\mathbf{f}$ are some scalar valued or vector valued functions on $\Omega_{\mathbf{x},t}$ respectively. Let us now investigate what needs to be done if one wants to integrate over the fixed domain $\Omega_{\mathbf{X}}$ instead of the evolving domain $\Omega_{\mathbf{x},t}$. Since $\Omega_{\mathbf{x},t} = \chi(\Omega_{\mathbf{X}},t)$, the straightforward application of the substitution theorem yields

$$\begin{aligned}\int_{\Omega_{\mathbf{x},t}} f(\mathbf{x},t)\, d\mathrm{v} = \int_{\chi(\Omega_{\mathbf{X}},t)} f(\mathbf{x},t)\, d\mathrm{v} = \int_{\Omega_{\mathbf{X}}} f(\chi(\mathbf{X},t),t)\det\mathbb{F}(\mathbf{X},t)\, d\mathrm{V} \\ = \int_{\Omega_{\mathbf{X}}} \widehat{f}(\mathbf{X},t)\det\mathbb{F}(\mathbf{X},t)\, d\mathrm{V}\end{aligned} \tag{2.34}$$

and

$$\int_{\Omega_{\mathbf{x},t}} \mathbf{f}(\mathbf{x},t)\, d\mathrm{v} = \left(\int_{\Omega_{\mathbf{x},t}} f_1(\mathbf{x},t)\, d\mathrm{v}, \int_{\Omega_{\mathbf{x},t}} f_2(\mathbf{x},t)\, d\mathrm{v} \right)$$
$$= \left(\int_{\Omega_{\mathbf{X}}} \widehat{f}_1(\mathbf{X},t) \det \mathbb{F}(\mathbf{X},t)\, d\mathrm{V}, \int_{\Omega_{\mathbf{X}}} \widehat{f}_2(\mathbf{X},t) \det \mathbb{F}(\mathbf{X},t)\, d\mathrm{V} \right)$$
$$= \int_{\Omega_{\mathbf{X}}} \widehat{\mathbf{f}}(\mathbf{X},t) \det \mathbb{F}(\mathbf{X},t)\, d\mathrm{V}. \quad (2.35)$$

The integration by parts that is used in the weak formulation of the problem will lead to the following integral over the boundary of the physical domain,

$$\int_{\partial \Omega_{\mathbf{x},t}} \mathbf{a} \cdot \mathbf{n}\, d\mathrm{l}, \quad (2.36)$$

where $\mathbf{a}$ is a vector valued function on $\Omega_{\mathbf{x},t}$, $\mathbf{n}$ is the unit outward normal to the boundary and $d\mathrm{l}$ is the length of the infinitesimal element of the boundary. The transformation rule for this integral can be best found by the following trick. All quantities considered so far are two dimensional objects. We will formally "extend", see Figure 2.2, all the quantities to three dimensional objects, in the following manner

$$\mathbf{x}_{3\mathrm{D}} =_{\mathrm{def}} \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix}, \quad (2.37a)$$
$$\mathbf{X}_{3\mathrm{D}} =_{\mathrm{def}} \begin{bmatrix} \mathbf{X} \\ Z \end{bmatrix}, \quad (2.37b)$$
$$\mathbf{a}_{3\mathrm{D}}(\mathbf{x}_{3\mathrm{D}}, t) =_{\mathrm{def}} \begin{bmatrix} \mathbf{a}(\mathbf{x},t) \\ 0 \end{bmatrix}, \quad (2.37c)$$
$$\chi_{3\mathrm{D}}(\mathbf{X}_{3\mathrm{D}}, t) =_{\mathrm{def}} \begin{bmatrix} \chi(\mathbf{X},t) \\ Z \end{bmatrix}, \quad (2.37d)$$
$$\mathbb{F}_{3\mathrm{D}}(\mathbf{X}_{3\mathrm{D}}, t) =_{\mathrm{def}} \frac{\partial \chi_{3\mathrm{D}}}{\partial \mathbf{X}_{3\mathrm{D}}} = \begin{bmatrix} \mathbb{F}(\mathbf{X},t) & 0 \\ 0 & 1 \end{bmatrix}, \quad (2.37e)$$
$$\Omega_{\mathbf{x}_{3\mathrm{D}},t} =_{\mathrm{def}} \Omega_{\mathbf{x},t} \times (-l, l), \quad (2.37f)$$
$$\Omega_{\mathbf{X}_{3\mathrm{D}}} =_{\mathrm{def}} \Omega_{\mathbf{X}} \times (-l, l), \quad (2.37g)$$

where l is a positive number. Using this notation, we see that

$$\int_{\partial \Omega_{\mathbf{x}_{3\mathrm{D}},t}} \mathbf{a}_{3\mathrm{D}}(\mathbf{x}_{3\mathrm{D}}, t) \cdot \mathbf{n}_{3\mathrm{D}}(\mathbf{x}_{3\mathrm{D}}, t)\, d\mathrm{s} = \int_{z=-l}^{l} \left(\int_{\partial \Omega_{\mathbf{x},t}} \mathbf{a}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x},t)\, d\mathrm{l} \right) dz. \quad (2.38)$$

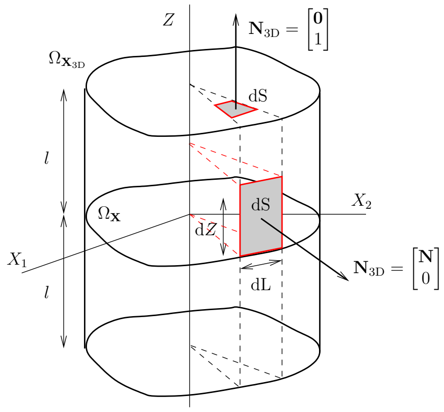

Figure 2.2: Formal extension to three dimensional space.

On the other hand, the substitution theorem for surface elements yields

$$\int_{\partial\Omega_{\mathbf{x}_{3D},t}} \mathbf{a}_{3D}(\mathbf{x}_{3D},t)\cdot\mathbf{n}_{3D}(\mathbf{x}_{3D},t)\,ds = \int_{\chi_{3D}(\partial\Omega_{\mathbf{X}_{3D}})} \mathbf{a}_{3D}(\mathbf{x}_{3D},t)\cdot\mathbf{n}_{3D}(\mathbf{x}_{3D},t)\,ds$$
$$= \int_{\partial\Omega_{\mathbf{X}_{3D}}} \mathbf{a}_{3D}(\chi_{3D}(\mathbf{X}_{3D}),t)\cdot\left(\det\mathbb{F}_{3D}(\mathbf{X}_{3D},t)\mathbb{F}_{3D}^{-T}(\mathbf{X}_{3D},t)\mathbf{N}_{3D}(\mathbf{X}_{3D})\right)\,d\mathrm{S}$$
$$= \int_{\partial\Omega_{\mathbf{X}_{3D}}} \widehat{\mathbf{a}}_{3D}(\mathbf{X}_{3D},t)\cdot\left(\det\mathbb{F}_{3D}(\mathbf{X}_{3D},t)\mathbb{F}_{3D}^{-T}(\mathbf{X}_{3D},t)\mathbf{N}_{3D}(\mathbf{X}_{3D})\right)\,d\mathrm{S}$$
$$= \int_{\partial\Omega_{\mathbf{X}_{3D}}} \det\mathbb{F}(\mathbf{X},t)\left(\mathbb{F}^{-1}(\mathbf{X},t)\widehat{\mathbf{a}}(\mathbf{X},t)\cdot\mathbf{N}(\mathbf{X})\right)\,d\mathrm{S}$$
$$= \int_{Z=-l}^{l}\left(\int_{\partial\Omega_{\mathbf{X}}} \det\mathbb{F}(\mathbf{X},t)\left(\mathbb{F}^{-1}(\mathbf{X},t)\widehat{\mathbf{a}}(\mathbf{X},t)\cdot\mathbf{N}(\mathbf{X})\right)\,d\mathrm{L}\right)\,dZ, \quad (2.39)$$

where we have used the fact that, informally, $d\mathrm{S} = d\mathrm{L}dZ$. The dot product in (2.39)—in the integration over the lateral surfaces of $\Omega_{\mathbf{X}_{3D}}$—has been simplified by in virtue of the identity

$$\widehat{\mathbf{a}}_{3D}(\mathbf{X}_{3D},t)\cdot\left(\det\mathbb{F}_{3D}(\mathbf{X}_{3D},t)\mathbb{F}_{3D}^{-T}(\mathbf{X}_{3D},t)\mathbf{N}_{3D}(\mathbf{X}_{3D})\right)$$
$$= \det\mathbb{F}_{3D}(\mathbf{X}_{3D},t)\left(\mathbb{F}_{3D}^{-1}(\mathbf{X}_{3D},t)\widehat{\mathbf{a}}_{3D}(\mathbf{X}_{3D},t)\cdot\mathbf{N}_{3D}(\mathbf{X}_{3D})\right)$$
$$= \det\begin{bmatrix}\mathbb{F}(\mathbf{X},t) & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}\mathbb{F}^{-1}(\mathbf{X},t) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{a}}(\mathbf{X},t)\\ 0\end{bmatrix}\cdot\begin{bmatrix}\mathbf{N}(\mathbf{X})\\ 0\end{bmatrix}\right)$$
$$= \det\mathbb{F}(\mathbf{X},t)\left(\mathbb{F}^{-1}(\mathbf{X},t)\widehat{\mathbf{a}}(\mathbf{X},t)\cdot\mathbf{N}(\mathbf{X})\right) \quad (2.40)$$

that holds on the lateral surfaces of $\Omega_{\mathbf{X}_{3D}}$. Concerning the top and bottom surface of $\Omega_{\mathbf{X}_{3D}}$, the integrand in (2.39) vanishes on these surfaces since

$$\widehat{\mathbf{a}}_{3D}(\mathbf{X}_{3D},t)\cdot\left(\mathbb{F}_{3D}^{-T}(\mathbf{X}_{3D},t)\mathbf{N}_{3D}(\mathbf{X}_{3D})\right) = \begin{bmatrix}\widehat{\mathbf{a}}(\mathbf{X},t)\\ 0\end{bmatrix}\cdot\begin{bmatrix}\mathbb{F}^{-T}(\mathbf{X},t) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix} = \begin{bmatrix}\widehat{\mathbf{a}}(\mathbf{X},t)\\ 0\end{bmatrix}\cdot\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix} = 0. \quad (2.41)$$

Finally, comparing the right hand sides of (2.38) and (2.39), we see that

$$\int_{\partial\Omega_{\mathbf{x},t}} \mathbf{a}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)\,d\mathrm{l} = \int_{\partial\Omega_{\mathbf{X}}} \det\mathbb{F}(\mathbf{X},t)\left(\mathbb{F}^{-1}(\mathbf{X},t)\widehat{\mathbf{a}}(\mathbf{X},t)\cdot\mathbf{N}(\mathbf{X})\right)\,d\mathrm{L}, \quad (2.42)$$

which can be rewritten in a compact form and with using relation $(\det\mathbb{F})\,\mathbb{F}^{-T} = \mathrm{cof}\mathbb{F}$ as

$$\int_{\partial\Omega_{\mathbf{x},t}} \mathbf{a}\cdot\mathbf{n}\, dl = \int_{\partial\Omega_{\mathbf{X}}} (\mathbb{A}\widehat{\mathbf{a}})\cdot\mathbf{N}\det\mathbb{F}\, d\mathrm{L} = \int_{\partial\Omega_{\mathbf{X}}} \widehat{\mathbf{a}}\cdot\mathrm{cof}\mathbb{F}\mathbf{N}\, d\mathrm{L}. \tag{2.43}$$

2.3 Summary

When we define transformation $\mathbf{x} = \chi(\mathbf{X}, t)$

$$x = W(y,t)X \tag{2.44a}$$
$$y = Y \tag{2.44b}$$

we we obtain a fixed computational domain

$$\Omega_{\mathbf{X}} =_{\mathrm{def}} \left\{[X,Y]\in\mathbb{R}^2,\ -1 < X < 1,\ 0 < Y < L\right\} \tag{2.45}$$

with lateral boundaries

$$\Gamma^{+}_{\mathrm{free},\mathbf{X}} =_{\mathrm{def}} \{[X,Y]\in\partial\Omega_{\mathbf{X}},\ X = 1\}, \qquad \Gamma_{\mathrm{in},\mathbf{X}} =_{\mathrm{def}} \left\{[X,Y]\in\mathbb{R}^2,\ Y = 0\right\},$$
$$\Gamma^{-}_{\mathrm{free},\mathbf{X}} =_{\mathrm{def}} \{[X,Y]\in\partial\Omega_{\mathbf{X}},\ X = -1\}, \qquad \Gamma_{\mathrm{out},\mathbf{X}} =_{\mathrm{def}} \left\{[X,Y]\in\mathbb{R}^2,\ Y = L\right\},$$

where each lateral boundary $\Gamma^{+}_{\mathrm{free},\mathbf{X}}$ and $\Gamma^{-}_{\mathrm{free},\mathbf{X}}$ has a unit outward normal $\mathbf{N}^{+} = (1,0)$ and $\mathbf{N}^{-} = (-1,0)$, respectively. From relation 2.44 we obtain transformation matrices

$$\mathbb{F} = \begin{bmatrix} \widehat{W} & X\frac{\partial\widehat{W}}{\partial Y} \\ 0 & 1 \end{bmatrix}, \qquad \mathbb{A} = \begin{bmatrix} \frac{1}{\widehat{W}} & -\frac{X}{\widehat{W}}\frac{\partial\widehat{W}}{\partial Y} \\ 0 & 1 \end{bmatrix}. \tag{2.46}$$

Before we define function spaces on computational domain, we need to transform Dirichlet boundary conditions. Since W_{in}, H_{in} and $\boldsymbol{v}_{\mathrm{out}}$ are constants, the only function that will be changed is $\boldsymbol{v}_{\mathrm{in}}$. Applying 2.44 on condition 1.89 implies

$$\begin{aligned}\left(\frac{\nu F x}{x^2 + r_c^2}, \frac{\nu F r_c}{x^2 + r_c^2}\right) &= \boldsymbol{v}_{\mathrm{in}}(\mathbf{x}) = \boldsymbol{v}_{\mathrm{in}}(\chi(\mathbf{X}, t)) \\ &= \left(\frac{\nu F X W(Y,t)}{(XW(Y,t))^2 + r_c^2}, \frac{\nu F r_c}{(XW(Y,t))^2 + r_c^2}\right) = \widehat{\boldsymbol{v}_{\mathrm{in}}}(\mathbf{X}).\end{aligned} \tag{2.47}$$

Now we define function spaces on transformed fixed domain

$$\widehat{\boldsymbol{V}} = \left\{\widehat{\boldsymbol{v}}(X,Y)\in\mathbb{R}^2,\ \widehat{\boldsymbol{v}}\Big|_{\Gamma_{\mathrm{in},\mathbf{X}}} = \widehat{\boldsymbol{v}_{\mathrm{in}}},\ \widehat{\boldsymbol{v}}\Big|_{\Gamma_{\mathrm{out},\mathbf{X}}} = \widehat{\boldsymbol{v}_{\mathrm{out}}}\right\}, \tag{2.48a}$$
$$\widehat{\Phi} = \left\{\widehat{H}(X,Y)\in\mathbb{R},\ \widehat{H}\Big|_{\Gamma_{\mathrm{in},\mathbf{X}}} = \widehat{H}_{\mathrm{in}}\right\}, \tag{2.48b}$$
$$\widehat{\Psi} = \left\{\widehat{W}(Y)\in\mathbb{R},\ \widehat{W}(0) = \widehat{W}_{\mathrm{in}}\right\} \tag{2.48c}$$

and test function spaces

$$\widehat{\widehat{\boldsymbol{V}}} = \left\{\widehat{\boldsymbol{u}}(X,Y)\in\mathbb{R}^2,\ \widehat{\boldsymbol{u}}|_{\Gamma_{\mathrm{in},\mathbf{X}}} = \mathbf{0},\ \widehat{\boldsymbol{u}}|_{\Gamma_{\mathrm{out},\mathbf{X}}} = \mathbf{0}\right\}, \tag{2.49a}$$
$$\widehat{\widehat{\Phi}} = \left\{\widehat{G}(X,Y)\in\mathbb{R},\ \widehat{G}\Big|_{\Gamma_{\mathrm{in},\mathbf{X}}} = 0\right\}, \tag{2.49b}$$
$$\widehat{\widehat{\Psi}} = \left\{\widehat{\psi}(Y)\in\mathbb{R},\ \widehat{\psi}(0) = 0\right\}. \tag{2.49c}$$

Weak formulation of the governing equations for the motion inside the fixed computational domain $\Omega_{\mathbf{X}}$ means to find $\widehat{\widehat{\boldsymbol{v}}} \in \widehat{\boldsymbol{V}}$ and $\widehat{H} \in \widehat{\Phi}$ such that the equations hold

$$\int_{\Omega_{\mathbf{X}}} \operatorname{Tr}\left[\left(\nabla_{\mathbf{X}}\left(\widehat{H}\widehat{\rho}_g\widehat{\widehat{\boldsymbol{v}}}\right)\right)\mathbb{A}\right]\widehat{G}\det\mathbb{F}\,d\mathrm{V} = 0 \tag{2.50a}$$

$$\int_{\Omega_{\mathbf{X}}}\left(\widehat{H}\widehat{\rho}_{\mathrm{g}}\left[\left(\nabla_{\mathbf{X}}\widehat{\widehat{\boldsymbol{v}}}\right)\mathbb{A}\right]\widehat{\widehat{\boldsymbol{v}}}\right)\cdot\widehat{\boldsymbol{u}}\det\mathbb{F}\,d\mathrm{V} = -\int_{\Omega_{\mathbf{X}}}\widehat{\mathbb{T}_{\mathrm{gs}}} : \left[\left(\nabla_{\mathbf{X}}\widehat{\boldsymbol{u}}\right)\mathbb{A}\right]\det\mathbb{F}\,d\mathrm{V}$$
$$-\int_{\Gamma^+_{\mathrm{free},\mathbf{X}}}\widehat{\gamma}\widehat{\boldsymbol{u}}\cdot\left(\operatorname{cof}\mathbb{F}\mathbf{N}^+\right)d\mathrm{L} - \int_{\Gamma^-_{\mathrm{free},\mathbf{X}}}\widehat{\gamma}\widehat{\boldsymbol{u}}\cdot\left(\operatorname{cof}\mathbb{F}\mathbf{N}^-\right)d\mathrm{L} \tag{2.50b}$$

for any $\widehat{\boldsymbol{u}} \in \widehat{\widetilde{\boldsymbol{V}}}$ and $\widehat{G} \in \widehat{\widetilde{\Phi}}$. Cauchy stress tensor $\widehat{\mathbb{T}_{\mathrm{gs}}}$ in the fixed computational domain $\Omega_{\mathbf{X}}$ reads

$$\widehat{\mathbb{T}}_{\mathrm{gs}} = -\widehat{P}\mathbb{I} + 2\widehat{\mu}\frac{\widehat{H}}{\widehat{\rho}_g}\operatorname{Tr}\left[\left(\nabla_{\mathbf{X}}(\widehat{\rho}_g\widehat{\widehat{\boldsymbol{v}}})\right)\mathbb{A}\right]\mathbb{I} + 2\widehat{\mu}\widehat{H}\widehat{\mathbb{D}}, \tag{2.51a}$$

$$\widehat{\mathbb{D}} = \frac{1}{2}\left(\left(\nabla_{\mathbf{X}}\widehat{\widehat{\boldsymbol{v}}}\right)\mathbb{A} + \mathbb{A}^T\left(\nabla_{\mathbf{X}}\widehat{\widehat{\boldsymbol{v}}}\right)^T\right), \tag{2.51b}$$

$$\widehat{P} = \widehat{\rho}_{\mathrm{g}}\left(1 - \frac{\widehat{\rho}_{\mathrm{g}}}{\widehat{\rho}_{\mathrm{t}}}\right)g\frac{\widehat{H}^2}{2}. \tag{2.51c}$$

Position of the free boundary, $f^{\pm}(\mathbf{x},t) = 0$, in physical domain and $\widehat{f}^{\pm}(\mathbf{X},t) = 0$, in computational domain reads

$$f^{\pm}(\mathbf{x},t) = x \mp W(y,t), \qquad \widehat{f}^{\pm}(\mathbf{X},t) = (X \mp 1)\,\widehat{W}(Y,t). \tag{2.52}$$

If we define areal Jacobian as

$$\mathrm{J}_- = \det\mathbb{F}\frac{\left|\mathbb{F}^{-T}\left(\nabla_{\mathbf{X}}\widehat{f}^-\right)\right|}{\left|\nabla_{\mathbf{X}}\widehat{f}^-\right|} \tag{2.53}$$

then we can write weak formulation of the governing equations for the motion of the free boundary $\Gamma^-_{\mathrm{free},\mathbf{X}}$. We need to find $\widehat{W} \in \widehat{\Psi}$ such that the equation

$$\int_{\Gamma^-_{\mathrm{free},\mathbf{X}}}\left(\mathbb{A}\widehat{\widehat{\boldsymbol{v}}}\right)\cdot\left(\nabla_{\mathbf{X}}\widehat{f}^-\right)\widehat{\psi}\mathrm{J}_-\,d\mathrm{L} = 0 \tag{2.54}$$

holds for any $\widehat{\psi} \in \widehat{\widetilde{\Psi}}$. Since we have

$$\nabla_{\mathbf{X}}\widehat{f}^-(-1,Y,t) = \left(\widehat{W}(Y,t), (-1+1)\frac{\partial\widehat{W}(Y,t)}{\partial Y}\right) = \left(\widehat{W}(Y,t), 0\right), \tag{2.55a}$$

$$\frac{\det\mathbb{F}}{\left|\nabla_{\mathbf{X}}\widehat{f}^-\right|}(-1,Y) = \frac{\widehat{W}(Y,t)}{\widehat{W}(Y,t)} = 1, \tag{2.55b}$$

$$\left|\mathbb{F}^{-T}\left(\nabla_{\mathbf{X}}\widehat{f}^-\right)\right|(-1,Y,t) = \left|\mathbb{A}^T\left(\nabla_{\mathbf{X}}\widehat{f}^-\right)\right|(-1,Y,t) = \begin{bmatrix}\frac{1}{\widehat{W}} & 0\\ \frac{1}{\widehat{W}}\frac{\partial\widehat{W}}{\partial Y} & 1\end{bmatrix}\begin{bmatrix}\widehat{W}\\ 0\end{bmatrix} = 1 \tag{2.55c}$$

on boundary $\Gamma^-_{\text{free},\mathbf{X}}$, we also have for areal Jacobian $\jmath_-(-1,Y) = 1$. We can similarly compute

$$\left(\mathbb{A}\widehat{\overline{\boldsymbol{v}}}\right)\cdot\left(\nabla_{\mathbf{X}}\widehat{f}^-\right)(-1,Y) = \begin{bmatrix} \frac{1}{\widehat{W}} & \frac{1}{\widehat{W}}\frac{\partial \widehat{W}}{\partial Y} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{v^x} \\ \widehat{v^y} \end{bmatrix} \cdot \begin{bmatrix} \widehat{W} \\ 0 \end{bmatrix}$$
$$= \frac{1}{\widehat{W}}\widehat{v^x}\widehat{W} + \frac{1}{\widehat{W}}\frac{\partial \widehat{W}}{\partial Y}\widehat{v^y}\widehat{W} = \widehat{v^x} + \frac{\partial \widehat{W}}{\partial Y}\widehat{v^y}. \quad (2.56)$$

Therefore equation 2.54 for the motion of the free boundary $\Gamma^-_{\text{free},\mathbf{X}}$ reduces to

$$\int_{\Gamma^-_{\text{free},\mathbf{X}}} \left(\widehat{v^x} + \frac{\partial \widehat{W}}{\partial Y}\widehat{v^y}\right)\widehat{\psi}\, d\mathrm{L} = 0. \quad (2.57)$$

2.4 Finite element method

The transformed problem is solved by finite element method. First step when using finite element methodis to declare a mesh on which we compute the solution. However while finding function W, which gives us the position of the free boundary, from equation 2.57 is one dimensional problem, finding velocity field $\overline{\boldsymbol{v}}$ and height of the ribbon H from system 2.50 is problem two dimensional. Therefore it seems reasonable to separate these two problems and iterate between them as described below.

Two dimensional problem requires a rectangle mesh, which represents a transformed fixed domain $\Omega_{\mathbf{X}}$ 2.45. The viscosity μ depends on temperature exponentially and inflow temperature is about $500°C$ higher then outflow temperature. It means that viscosity at inflow is significantly lower so it seems appropriate to make mesh denser towards inflow. On the other hand one dimensional problem 2.57 is much smaller, so we can afford denser mesh on whole 1D domain without significant lose of computational time. It means that 1D mesh is an equidistantly divided interval with length L.

In second step we define finite dimensional function spaces. We use quadric Lagrange elements on 2D mesh for approximating velocity $\overline{\boldsymbol{v}}$, linear Lagrange elements also on 2D mesh for height of the ribbon H and linear Lagrange elements on 1D mesh for width of the ribbon W.

Since we separated the problem in two parts we need to iterate between them. At first we define initial conditions, which we use as an initial approximation for unknown functions. Because it is desirable to preserve boundary conditions, we declare initial velocity $\boldsymbol{v}_{\text{init}}$ as linear interpolation between inflow velocity $\boldsymbol{v}_{\text{in}}$ 1.89 and outflow velocity $\boldsymbol{v}_{\text{out}}$ 1.90

$$\boldsymbol{v}_{\text{init}} = \boldsymbol{v}_{\text{in}} + (\boldsymbol{v}_{\text{out}} - \boldsymbol{v}_{\text{in}})\frac{Y}{L}. \quad (2.58)$$

Initial height could be declared as inflow height H_{in}, however when we use equilibrium height H_{eq} 1.73 we receive smaller initial residuum and save some computational time. Width W starts as constant W_{in}, which means that for first iteration we have a straight channel.

Now we find piecewise quadric velocity $\widehat{\overline{\boldsymbol{v}}}_{\text{ap}}$ and piecewise linear height $\widehat{H}_{\text{ap}}$ from above defined Lagrange spaces, which satisfy system 2.50 discretisated in virtue of finite element methodwith given width $\widehat{W}_{\text{ap}}$. Then we substitute computed velocity $\widehat{\widehat{\boldsymbol{v}}}_{\text{ap}}$ into equation for position of the free boundary 2.57 again dicretisated in virtue of finite element methodand evaluate new piecewise linear width $\widehat{W}_{\text{ap}}$. After that we use new width $\widehat{W}_{\text{ap}}$ in equations 2.50 and again solve them with respect to $\widehat{\widehat{\boldsymbol{v}}}_{\text{ap}}$ and $\widehat{H}_{\text{ap}}$. And then repeat whole process again. Each time, after computing new width, we assemble the system 2.50 with new height $\widehat{H}_{\text{ap}}$, velocity $\widehat{\widehat{\boldsymbol{v}}}_{\text{ap}}$ and new width $\widehat{W}_{\text{ap}}$ and find L^2 norm of the residuum. When the residuum is sufficiently small, we stop the process and proclaim $\widehat{\widehat{\boldsymbol{v}}}_{\text{ap}}$, $\widehat{H}_{\text{ap}}$ and $\widehat{W}_{\text{ap}}$ an approximated solution.

Algorithm is implemented using FEniCS. FEniCS is a software for solving differential equations by finite element method.

Somewhere in ribbon are placed wheels, which stretches the glass in order to obtain appropriate thickness. These wheels gives us an inner Dirichlet condition, which can be easily implemented in FEniCS.

However a slight problem is, that even we have a given placement of the wheels in the physical domain, we do not know their position in computational domain, since we change transformation function $\widehat{W}_{\text{ap}}$ during calculation. For example if we have a wheel placed in physical domain at $(x, y) = (2.4, 15.5)$ and start iteration with $\widehat{W}_{\text{ap}} = 1.6$, then the wheel should be placed in the computational domain at $(X, Y) = (x/\widehat{W}_{\text{ap}}(Y), y) = (1.5, 15.5)$, which in fact is not in the domain, since it has a property $X \in [-1, 1]$. But even in this case it seem reasonable to place the wheel at the edge of the domain, because this way glass stretches more to next iteration and more likely we end up with wheel inside the domain. Now each time we recalculate width of the physical domain $\widehat{W}_{\text{ap}}$ we need to adjust position of wheels. For example if we obtain width with property $\widehat{W}_{\text{ap}}(15.5) = 3.2$ then the wheel moves in computational domain to point $(X, Y) = (x/\widehat{W}_{\text{ap}}(Y), y) = (2.4/3.2, 15.5) = (0.75, 15.5)$.

In FEniCS we represent the wheel as a pointwise Dirichlet condition. It means that we prescribe a velocity in given node of the mesh. Since we use quadric Lagrangian elements for velocities, the nodes are vertices of the triangulated mesh or middle of the triangle's facet. So we find the closest node to the given point and place the condition there.

In the end the whole cycle looks like this:

```
> move wheel
> solve velocity
> apply velocity on boundary to width equation
> solve width
> apply width on system for velocity and height equations
> compute residuum of velocity and height equations
> store computed solution to file
> if residuum < tolerance then stop cycle
```

After the iteration stops we can transform the computational rectangle domain to the physical domain according to computed width.

Chapter 3

Results

There are plenty of parameters it he system, we wnat to solve, such as spreading coefficient γ, outflow velocity $\boldsymbol{v}_{\text{out}}$, inflow velocity, which is characterized by inflow mass Q and angle α at which the glass enters the domain. Some parameters are not easily measured and some could be changed, while different thickness of glass is required.

3.1 Spreading coefficient

First mentioned was the spreading coefficient γ. One way, how to determine this value, was suggested in 1.73 as it has a direct connection to equilibrium height of the glass. Another way, how to measure γ, can be obtained by considering the balance of forces acting on a drop of glass poured in tin bath. This way, γ depends on measured angles α, β and θ shown in figure 3.1. and the partial

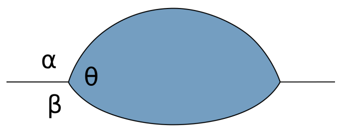

Figure 3.1: Drop of glass

surface tensions between glass-air γ_{ga}, glass-tin γ_{gt} and tin-air γ_{ta}. Then we have $\gamma = \gamma_{ga} + \gamma_{gt} - \gamma_{ta}$. See for example Popov (1981) and Langmuir (1933). However both methods proves to be inaccurate, because liquid glass is photographed in closed container, since it needs high temperatures to melt down. And from these photographs is hard to get precise data.

Therefore we calculated several results with slightly and even roughly different values fo γ. The most important results of the model is final width of the ribbon over the domain and height of the glass which we gain at the outflow. In figure 3.2 is shown dependence of the half-width of the ribbon according to various γ. As we can see, the smaller is the γ the wider is the ribbon. So the representation of γ as a force acting on unit length of the edge of the ribbon seems correct. In virtue of supporting this hypothesis, a table 3.1 is attached. There is a position of maximal width for case. And again the larger is the force, the closer is maximum from beginning of the domain.

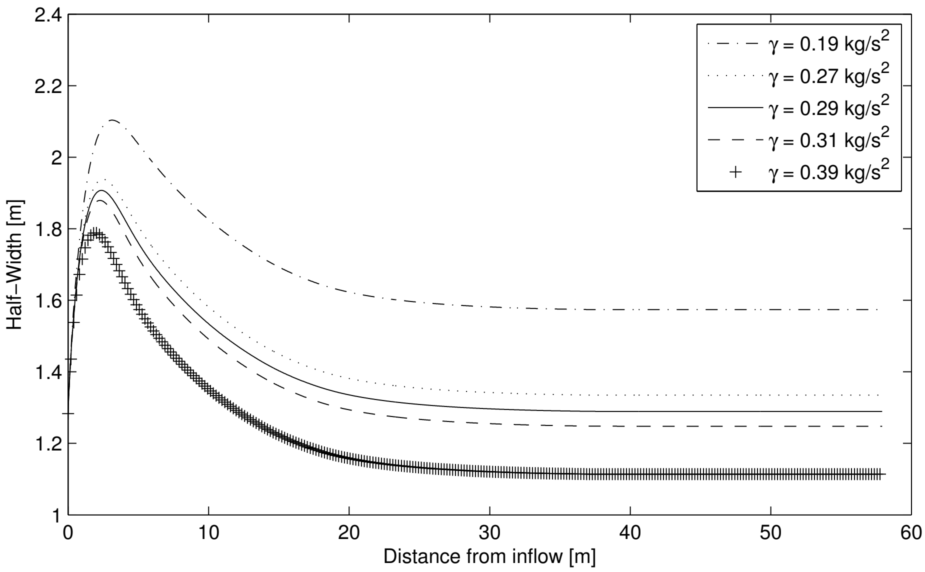

Figure 3.2: Ribbon width with different γ

Position [m]	**Half-width [m]**				
	$\gamma = 0.19$ kg/s^2	$\gamma = 0.27$ kg/s^2	$\gamma = 0.29$ kg/s^2	$\gamma = 0.31$ kg/s^2	$\gamma = 0.39$ kg/s^2
1.94 (max $\gamma = 0.39$kg/s^2)	2.0373	1.9215	1.8965	1.8728	1.7886
2.26 (max $\gamma = 0.31$kg/s^2)	2.0708	1.9356	1.9065	1.8791	1.7825
2.37 (max $\gamma = 0.29$kg/s^2)	2.0796	1.9376	1.9072	1.8785	1.7778
2.49 (max $\gamma = 0.27$kg/s^2)	2.0869	1.9383	1.9066	1.8767	1.7719
3.13 (max $\gamma = 0.19$kg/s^2)	2.1036	1.9226	1.8847	1.8492	1.7268
57.91 (exit)	1.5737	1.3346	1.2892	1.2478	1.1137

Table 3.1: Ribbon width with different γ

Height of the ribbon at the end of the domain is a crucial result, because it refers to the final profile of the glass. In figure 3.3 we observe the influence of spreading coefficient γ to the outflow height. Again appears significant dependence of height on γ. For example, when we increase γ by 6%, the height increase by 3%, what can be calculated from table 3.2 with hard data received from the program.

3.2 Inflow velocity

The condition for velocity at the inflow is quiet delicate problem, since we don't exactly know, what happens, when glass leaves the restrictors. It is not easy to determine accurate rule for any situation, anyway we chose the Jeffery-Hamel flow approximation. With this approximation we can easily change inflow angle α. It means that the fluid enters the domain from channel in the shape of 'V' and at different angles. First attempt was to copy the angle of restrictors. But physical measuring of the shape of the ribbon shows, that around the restrictors'

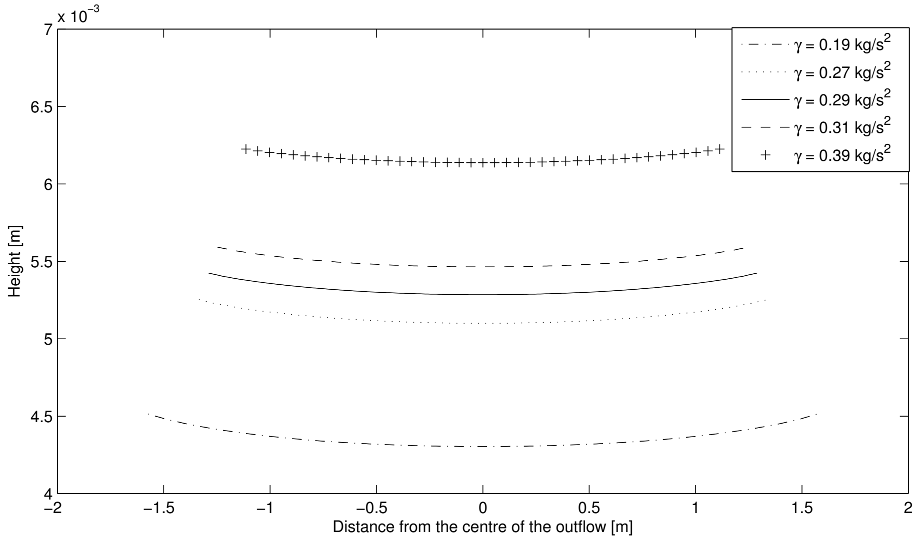

Figure 3.3: Ribbon height at the end of the domain with different γ

	Height [m]				
	$\gamma = 0.19$ kg/s^2	$\gamma = 0.27$ kg/s^2	$\gamma = 0.29$ kg/s^2	$\gamma = 0.31$ kg/s^2	$\gamma = 0.39$ kg/s^2
Centre, position [m]	0.0	0.0	0.0	0.0	0.0
Centre, value [m]	0.00430	0.00510	0.00528	0.00546	0.00614
Edge, position [m]	1.5737	1.3346	1.2892	1.2478	1.1137
Edge, value [m]	0.00451	0.00525	0.00542	0.00559	0.00623

Table 3.2: Ribbon height at the end of the domain with different γ

tip the angle opens a little bit.

First graph 3.4 represents ribbon's half-width while we change the angle at which the glass enters the domain. As we can see in the graph 3.4 or also in table 3.3, width at the end of the domain is not so sensitive to inflow angle α as the maximal width. However different shape of the ribbon reflects in outflow height. Besides the fact that height is slightly different through whole outflow, which reflects different outflow width, important are differences between height at the edge and height at the centre of the ribbon. Or in other words how much is the glass bent. When we increase angle α we also increase a little the rate of bending. This effect is illustrated in graph 3.5 and in table 3.4.

3.3 Lehr speed

Lehr pulls glass out of the tin bath at the end of the domain. It's speed generates for us outflow velocity. The speed of the lehr can be easily adjust during the process in order to influence the outflow height. Therefore it is one of the parameters, which we want to study and learn how it influence our calculations.

Again first we present figure 3.6 and table 3.5 how outflow speed changes half-width of the ribbon, where we can see how different velocities spreads back

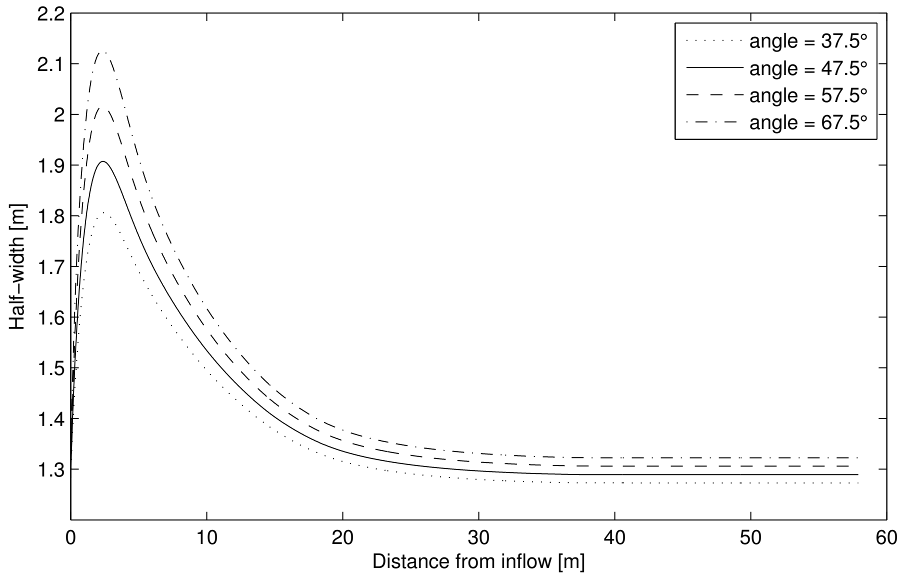

Figure 3.4: Ribbon width with different inflow angle

Position [m]	**Half-width [m]**			
	37.5°	47.5°	57.5°	67.5°
1.01	1.6817	1.7698	1.8637	1.9587
2.35 (max for 67.5°)	1.8057	1.9072	2.0163	2.1254
2.43 (max for 37.5°)	1.8061	1.9071	2.0158	2.1247
5.01	1.6903	1.7598	1.8356	1.9132
57.91 (exit)	1.2730	1.2892	1.3060	1.3224

Table 3.3: Ribbon width with different inflow angle

through domain and has an effect on ribbon's width.

Main reason, why to change outflow velocity is to influence height of the glass exiting the domain. In figure 3.7 we can see, that this way height moves by 1mm on any side. This way we can also observe in table 3.6 that with increasing speed the glass becomes more bent.

3.4 Wheels

The industry demands thinner and even thicker glass then was presented above. For that purpose wheels are dug in the glass on sides of the ribbon approximately in the middle of the domain, in the stretching region, where glass gain larger viscosity. The wheels stretches the glass and make it thinner and wider or thicker and more narrow.

Figures 3.8 with half width of glass and 3.9 with height at the outflow shows that wheels are very effective, when we want to obtain different thickness of the glass. But sometimes appears little waves on the glass, so the glass is not so plane and it is really undesirable.

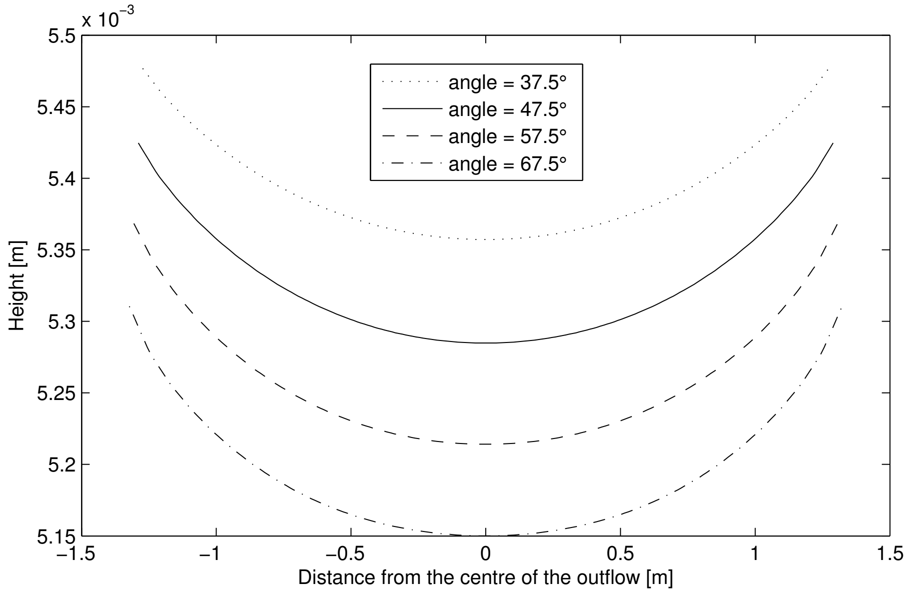

Figure 3.5: Ribbon height at the end of the domain with different inflow angle

Position [m]	**Height [m]**			
	37.5°	47.5°	57.5°	67.5°
Centre, position [m]	0.0	0.0	0.0	0.0
Centre, value [m]	0.00536	0.00528	0.00521	0.00515
Edge, position [m]	1.2730	1.2892	1.3060	1.3224
Edge, value [m]	0.00548	0.00542	0.00537	0.00531

Table 3.4: Ribbon height at the end of the domain with different inflow angle

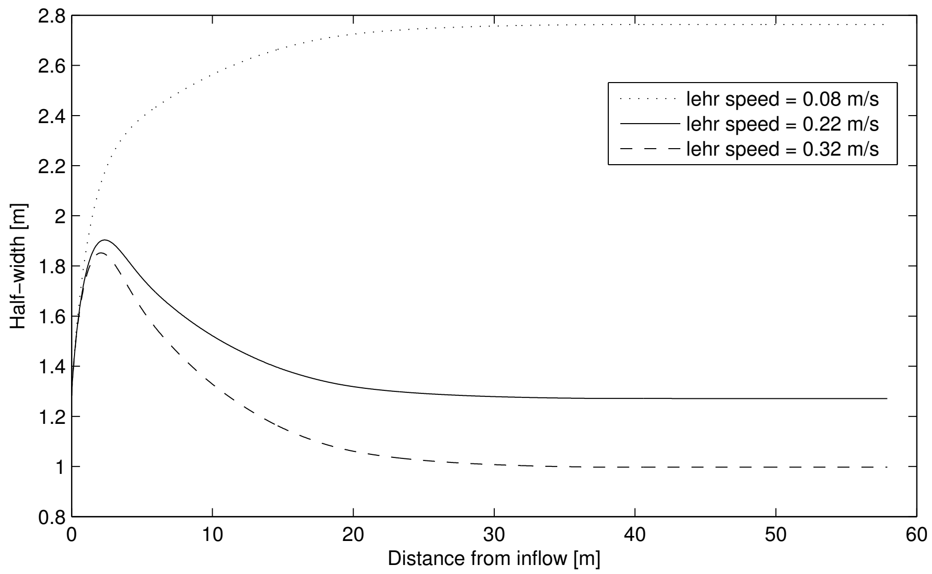

Figure 3.6: Half-width of the ribbon with different lehr speed

Position [m]	Half-width [m]		
	0.08m/s	0.22m/s	0.32m/s
1.01	1.8559	1.7686	1.7502
2.20	2.1485	1.9023	1.8512
9.99	2.5632	1.5218	1.3289
20.01	2.7247	1.3184	1.0609
57.91 (exit)	2.7630	1.2712	0.9971

Table 3.5: Ribbon width with different outflow speed

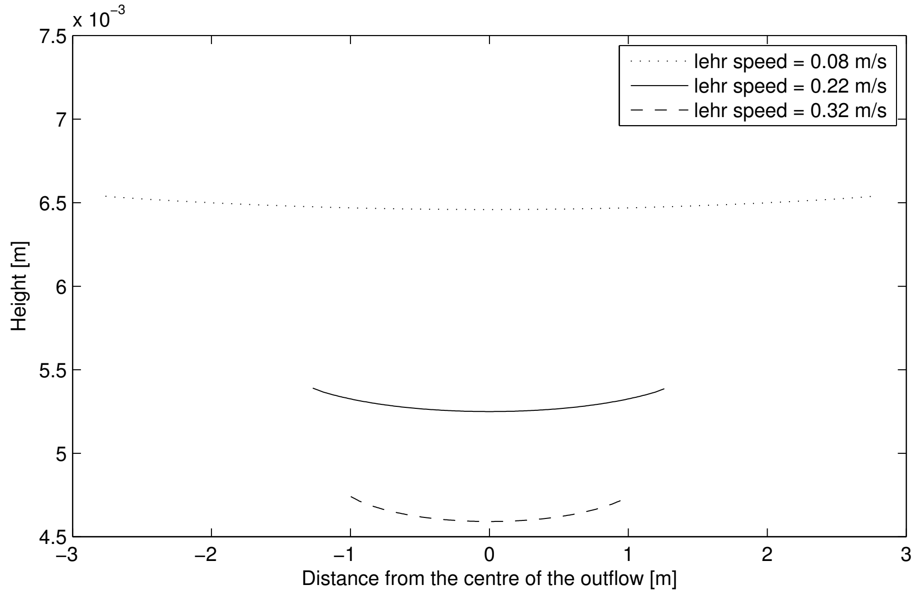

Figure 3.7: Ribbon height at the end of the domain with different lehr speed

Position [m]	Height [m]		
	0.08m/s	0.22m/s	0.32m/s
Centre, position [m]	0.0	0.0	0.0
Centre, value [m]	0.00646	0.00525	0.00459
Edge, position [m]	2.7630	1.2712	0.9971
Edge, value [m]	0.00654	0.00539	0.00474

Table 3.6: Ribbon height at the end of the domain with different outflow speed

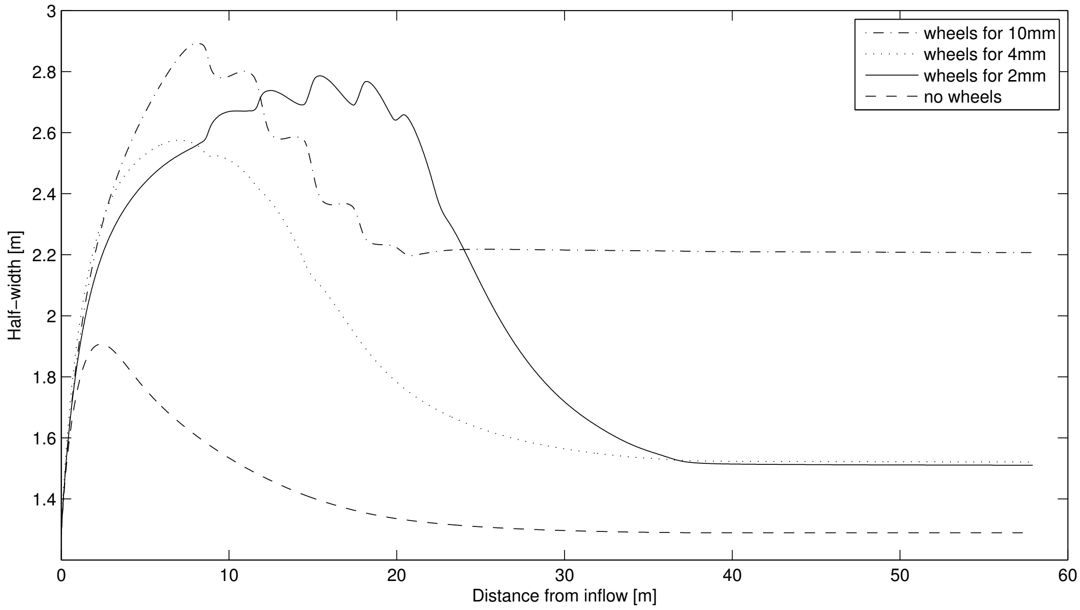

Figure 3.8: Half-width of the ribbon with wheels

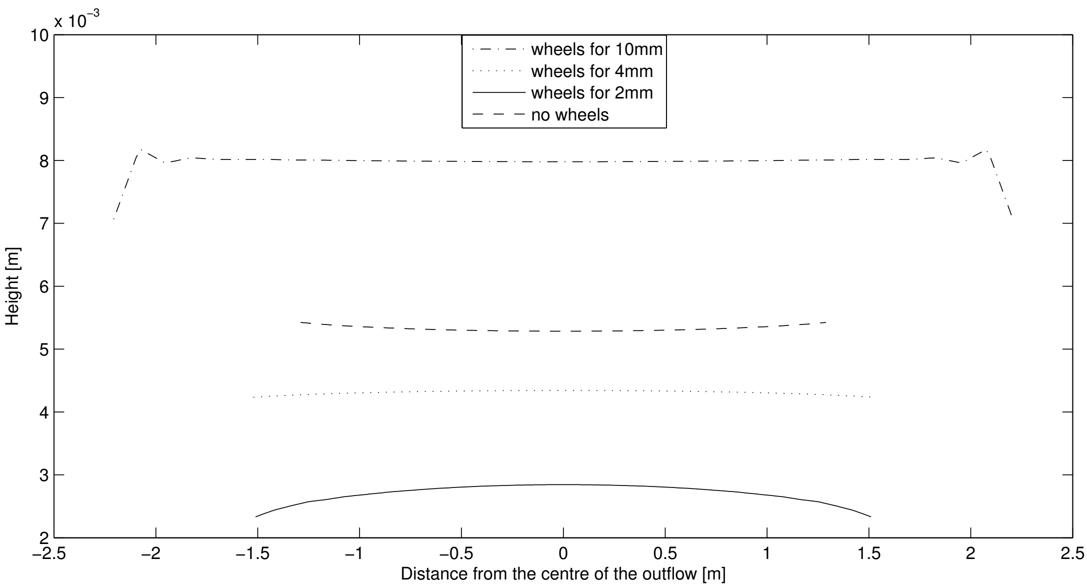

Figure 3.9: Ribbon height at the end of the domain with wheels

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