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Logické základy forcingu
Logical background of forcing

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Abstract

This thesis examines the method of forcing in set theory and focuses on aspects that are set aside in the usual presentations or applications of forcing. It is shown that forcing can be formalized in Peano arithmetic (PA) and that consistency results obtained by forcing are provable in PA. Two ways are presented of overcoming the assumption of the existence of a countable transitive model. The thesis also studies forcing as a method giving rise to interpretations between theories. A notion of bi-interpretability is defined and a method of forcing over a non-standard model of ZFC is developed in order to argue that ZFC and ZF are not bi-interpretable.

Abstrakt

V předložené práci zkoumáme forcing jako metodu teorie množin a zaměřujeme se na okolnosti, které jsou při obvyklých výkladech a aplikacích forcingu ponechávány stranou. Ukážeme, že forcing lze formalizovat v Peanově aritmetice (PA) a že výsledky o relativních konzistencích teorií získané pomocí forcingu jsou dokazatelné v PA. Předvedeme dva způsoby, jak je možné překonat předpoklad existence spočetného tranzitivního modelu. Studujeme také forcing jako metodu, na jejímž základě je možné konstruovat interpretace teorií v teoriích jiných. Zavádíme pojem bi-interpretace a budujeme metodu forcingu přes nestandardní model ZFC, pomocí níž ukážeme, že teorie ZFC a ZF nejsou bi-interpretovatelné.

Klíčová slova: teorie množin, ZFC, forcing, interpretace, dokazatelnost, Peanova aritmetika, bi-interpretace, nestandardní model, spočetný tranzitivní model

Keywords: set theory, ZFC, forcing, interpretation, provability, Peano arithmetic, bi-interpretation, non-standard model, countable transitive model

Contents

Introduction

The method of forcing was first used by Paul Cohen in 1963 to prove independence of the axiom of choice and the continuum hypothesis from ZF. Since then, forcing developed into a general technique for obtaining independence and consistency results in set theory. Usually, forcing is seen as a method that allows us to construct a model $M[\mathbf{G}]$, called generic extension, from a countable transitive model M of ZFC – ground model – using a generic filter $\mathbf{G}$.

This thesis studies forcing as a general method and focuses on aspects that are set aside when forcing is used to obtain particular results.

First, the assumption of existence of a countable transitive model is useful for intuition; nevertheless, this assumption is problematic as existence of such a model is independent from ZFC. We introduce two ways how to overcome this problem – forcing in the theory ZFC^U and forcing over the universe. These two ways correspond to the two sections of Chapter 4.

Second, in various presentations of forcing, it is usually remarked that forcing can be formalized in Peano arithmetic (PA). In Section 4.1, we try to give details of the nature of such a formalization. In Section 4.1.2, we prove that independence results obtained by forcing are provable in PA.

Third, we show that forcing can be understood as a method giving rise to interpretations of extensions of ZFC in (extensions of) ZFC.

In the last chapter, we present an unusual application of forcing. We define the notion of bi-interpretation and show that ZFC and ZF are not bi-interpretable. In the proof, we use forcing over a non-standard model of ZFC to produce a certain automorphism. Unlike in the usual applications of forcing, this automorphism is not an element of the generic extension, it is an external object. Our method of using forcing over a non-standard model to construct such an object is, as far as we know, original.

Chapter 1

Preliminaries

1.1 Peano arithmetic

We assume the reader is familiar with Peano arithmetic (PA) and its ability to formalize, in certain cases, syntax. We do not go into detail in this respect but simply recall that PA is able to work with finite sequences and axiomatizations of recursive theories, can express notions of proof in a theory and consistency of a theory and allows us to use proofs by induction on the complexity of formulas. We often say that a metamathematical statement is formalizable in PA and mean by it that there is a natural, straightforward formalization, usually the one just translating the syntactical terms to their formal equivalents. Likewise, we may write PA $\vdash \phi$, where ϕ is some metamathematical statement, meaning that PA proves the formalization of ϕ.

1.2 Omitting types theorem

The omitting types theorem will be used in Chapter 3. Let us recall that an n-type (i.e. a type in n variables) Γ of a theory T is called *isolated* if there is a formula $\varphi(x_1, \ldots, x_n)$ such that $T \cup \{\exists \bar{x} \varphi(\bar{x})\}$ is consistent and $T \vdash \varphi(\bar{x}) \to \psi(\bar{x})$, for every $\psi \in \Gamma$. $\varphi(x_1, \ldots, x_n)$ is called an isolating formula for Γ. If Γ is not isolated, it is called non-isolated. The next theorem is one of the classical theorems of model theory; a proof of the following version can be found in Marker's book [7, p. 125].

Theorem 1 (Omitting Types Theorem)**.** *Let $\mathcal{L}$ be a countable language, T an $\mathcal{L}$-theory and Γ a non-isolated n-type. Then, there is a countable $\mathcal{M} \models T$ omitting Γ.*

Corollary 2. *Let $\mathcal{L}$ be a countable language, T an $\mathcal{L}$-theory and Γ an n-type. If Γ is realized in every model of T, then Γ is isolated.*

1.3 Some set theoretical definitions

Most of these definitions are well known and we present them mainly to introduce or recall the notation.

Definition 3 (The cumulative hierarchy and ranks). *By transfinite recursion, we define* V_α *for each* $\alpha \in \mathrm{Ord}$ *by:*

1. $\mathrm{V}_0 = \emptyset$,
2. $\mathrm{V}_{\alpha+1} = \mathcal{P}(\mathrm{V}_\alpha)$,
3. $\mathrm{V}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{V}_\alpha$ *if* λ *is a limit ordinal.*

$\mathrm{rk}(x)$ *denotes the rank of* x*, i.e. the least* α *such that* $x \in \mathrm{V}_{\alpha+1}$.

Definition 4 (Relativization). *Let* C *be a class,* φ *an* $\in$*-formula. The relativization* φ^C *of* φ *to* C *is defined by induction on the complexity of* φ*:*

- $(x = y)^C = (x = y)$,
- $(x \in y)^C = (x \in y)$,
- *relativization commutes with logical connectives,*
- $(\exists x \varphi)^C = (\exists x \in C)\varphi^C$,
- $(\forall x \varphi)^C = (\forall x \in C)\varphi^C$.

For any model $\mathcal{M}$ *of* ZFC *and any class* C *of* $\mathcal{M}$*, a formula* $\varphi(x_1, \ldots, x_n)$ *is called absolute for* C *if* $\mathcal{M}$ *satisfies*

$$(\forall x_1, \ldots, x_n \in C)(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^C(x_1, \ldots, x_n)).$$

Definition 5 (Gödel operations). *By Gödel operations we mean a collection of the following operations* G_1*–*G_{10}*:*
$G_1(x, y) = \{x, y\}$
$G_2(x, y) = x \times y$
$G_3(x, y) = \{(u, v); u \in x \wedge v \in y \wedge u \in v\}$
$G_4(x, y) = x \setminus y$
$G_5(x, y) = x \cap y$
$G_6(x) = \bigcup x$
$G_7(x) = \mathrm{dom}(x)$
$G_8(x) = \{(u, v); (v, u) \in x\}$

$G_9(x) = \{(u, v, w); (u, w, v) \in x\}$
$G_{10}(x) = \{(u, v, w); (v, w, u) \in x\}$

$\mathrm{cl}(x)$ *denotes the closure of* x *under Gödel operations.*

Definition 6. *The class* $\mathrm{OD}(A)$ *of all ordinal-definable sets over* A *is defined as follows:*

$$\mathrm{OD}(A) = \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{cl}(\{V_\beta; \beta < \alpha\} \cup \{A\} \cup A).$$

Definition 7. *The class* $\mathrm{HOD}(A)$ *of all hereditarily ordinal-definable sets over* A *is defined as follows:* $\mathrm{HOD}(A) = \{x; \mathrm{trcl}(\{x\}) \subseteq \mathrm{OD}(A)\}$*, where* trcl *denotes the transitive closure.*

1.4 Reflection principle

We state the well-known reflection principle for ZFC, sketch the proof and show that it can be formalized inside PA. There are many possible formulations of this principle, our formulation asserts the existence of a *countable transitive* set for which certain formulas are absolute. To ensure the countability, we use the axiom of choice.

Theorem 8. *Let* $\phi_1, \ldots, \phi_n$ *be sentences in the language of* ZFC.
$\mathrm{ZFC} \vdash \exists A(|A| = \omega \wedge \text{“}A \textit{ is transitive}\text{”} \wedge \phi_1 \leftrightarrow \phi_1^A \wedge \ldots \wedge \phi_n \leftrightarrow \phi_n^A)$

Proof. We first form a list of formulas $\phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_m$ that contains axiom of extensionality and is subformula-closed, i.e. if α is in the list, then every subformula of α is in the list. By induction on the complexity of formulas, it can be proved that, for any class C, the following are equivalent:

1. $\phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_m$ are absolute for C.
2. If ϕ_i is of the form $\exists x \phi_j(x, \bar{y})$, then
   $\forall \bar{y} \in C(\exists x \phi_j(x, \bar{y}) \rightarrow (\exists x \in C)\phi_j(x, \bar{y}))$ holds.

For each $1 \leq i \leq m$, we define a function $F_i : \mathrm{Ord} \rightarrow \mathrm{Ord}$. If ϕ_i is not of the form $\exists x \phi_j(x, \bar{y})$ for any $1 \leq j \leq m$, then $F_i(\alpha) = 0$. If there is $1 \leq j \leq m$ such that ϕ_i is $\exists x \phi_j(x, \bar{y})$, then $F_i(\alpha)$ is the least β such that for any $\bar{y} \in \mathrm{V}_\alpha$ if $\exists x \phi_j(x, \bar{y})$ holds, then $(\exists x \in \mathrm{V}_\beta)\phi_j(x, \bar{y})$. Let $\beta_0 = \omega$, $\beta_{i+1} = \max\{\beta_i + 1, F_1(\beta_i), \ldots, F_m(\beta_i)\}$, $\beta = \sup\{\beta_i; i \in \omega\}$. We claim that $\phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_m$ are absolute for V_β, as the construction of β guarantees that condition 2 is satisfied for $C = \mathrm{V}_\beta$. It holds that $\beta > \omega$, so $|\mathrm{V}_\beta| > \omega$.

To finish the proof, we fix a well-order $\lhd$ of V_β and make use of some "witness-choosing" functions once more. Let x_0 be the first element of V_β in the sense of $\lhd$. Let l_i be the number of free variables occuring in ϕ_i. $G_i : \mathrm{V}_\beta^{l_i} \to \mathrm{V}_\beta$ is defined as follows. If ϕ_i is of the form $\exists x \phi_j(x, \bar{y})$ for some $1 \leq j \leq m$ and $(\exists x \in \mathrm{V}_\beta)\phi_j(x, \bar{y})$, then $G_i(\bar{y})$ is the first such x in the sense of $\lhd$. (We include the case of $l_i = 0$, for which G_i is a nullary function picking one element of V_β.) If ϕ_i is not of the form $\exists x \phi_j(x, \bar{y})$ or it is $\neg(\exists x \in \mathrm{V}_\beta)\phi_j(x, \bar{y})$, then $G_i(\bar{y}) = x_0$. We denote by A' the closure of V_ω under $G_1, \ldots, G_m$. Then $\phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_m$ are absolute for A', as they satisfy the condition 2 from above. Also, $|A'| = \omega$, since A' is the closure of a countable set under finitely many functions.
The list $\phi_1, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_m$ contains axiom of extensionality, so it holds that $(\forall x, y \in A')((\forall z \in A')(z \in x \leftrightarrow z \in y) \to x = y)$. Thus we can apply the Mostowski collapse theorem to A' to obtain a transitive set A and an $\in$-isomorphism f from A' to A. So $|A| = |A'| = \omega$, A is transitive. For each $\phi_i(\bar{x})$ from the list and $x_1, \ldots, x_n \in A'$, it holds $\phi_i^{A'}(x_1, \ldots, x_n) \leftrightarrow \phi_i^{A}(f(x_1), \ldots, f(x_n))$. Therefore, if ϕ_i is a sentence, it holds that $\phi_i^{A} \leftrightarrow \phi_i^{A'} \leftrightarrow \phi_i$. $\square$

The reflection principle is not a theorem of ZFC. It is a metamathematical statement claiming that for any finite list of ZFC-sentences a certain formula χ (dependent of the list of sentences) is provable in ZFC, so it makes sense to say that the reflection principle is formalizable and its formalization is provable in PA. Clearly, the reflection principle can be formalized in PA, as PA allows us to talk about finite lists of ZFC-formulas and is able to construct the formula χ from the list. This formalization is provable in PA because all the metamathematical constructions in the proof (like forming the subformula closed list) can be done inside PA, and the rest of the proof just takes place in ZFC.

Corollary 9. *Let T be a theory extending* ZFC*, $\phi_1, \ldots, \phi_n$ axioms of T in the language of* ZFC*. Then*

$$T \vdash \exists A(|A| = \omega \wedge \text{“}A \textit{ is transitive”} \wedge \phi_1^A \wedge \ldots \wedge \phi_n^A).$$

For an arithmetically axiomatizable theory T, this corollary is formalizable and its formalization provable in PA from the same reasons as in the case of the reflection principle.

Chapter 2

Extensions of ZFC

We now define the extensions ZFC^U and $\mathrm{ZFC}^{U+\psi}$ of ZFC and prove some of their basic properties. These extensions will be used later in Section 4.1. The language of ZFC^U and $\mathrm{ZFC}^{U+\psi}$ contains one extra constant symbol U.

Definition 10. *The theory* ZFC^U *has the following axioms:*

- *all axioms of* ZFC,
- *"U is transitive and countable",*
- *φ^U, for every axiom φ of* ZFC.

Let ψ be a formula in the language of ZFC*. Then* $\mathrm{ZFC}^{U+\psi}$ *is* $\mathrm{ZFC}^U + \{\psi^U\}$.

Note that if ZFC is consistent, then ZFC^U is consistent as well. It is an easy consequence of the reflection principle.

The next lemma tells us that we can really view U as a model of ZFC. It also allows us to prove Corollary 12 without having to work syntactically with proofs and their transformations.

Lemma 11. *Let $\mathbb{V}$ be a model of* ZFC^U*, V the universe of $\mathbb{V}$ and $\in^{\mathbb{V}}$ the realization of $\in$ in $\mathbb{V}$. Let $\boldsymbol{U} = \{x \in V; \mathbb{V} \models x \in U\}$ and $\in^{\boldsymbol{U}}$ be the restiction of $\in^{\mathbb{V}}$ to $\boldsymbol{U}$. Then $\mathbb{W} = \left\langle \boldsymbol{U}, \in^{\boldsymbol{U}} \right\rangle$ is a model of* ZFC*. Moreover, for any $x_1, \ldots, x_n \in \boldsymbol{U}$ and any $\in$-formula φ it holds that $\mathbb{W} \models \varphi[x_1, \ldots, x_n]$ iff $\mathbb{V} \models \varphi^U[x_1, \ldots, x_n]$.*

Proof. Note that $\mathbb{W}$ is a substructure of $\mathbb{V}$ with respect to the language of ZFC. By induction on the complexity of the formula φ, we show that $\mathbb{W} \models \varphi[x_1, \ldots, x_n]$ iff $\mathbb{V} \models \varphi^U[x_1, \ldots, x_n]$. If φ is atomic, it follows from the definition of relativization and the fact that $\mathbb{W}$ is a substructure. The case for logical connectives is easy since relativization commutes with logical connectives. So let φ be $\exists x \psi(x)$, for brevity we suppress the mention of parameters $x_1, \dots, x_n$. Then $\mathbb{W} \models \exists x \psi(x)$ iff $\exists u \in \mathbf{U}(\mathbb{W} \models \psi(u))$ iff $\exists u(\mathbb{V} \models u \in U \wedge \psi^U(u))$ iff $\mathbb{V} \models (\exists x \psi(x))^U$. The second equivalence holds because of the induction hypothesis and the definition of $\mathbf{U}$. □

Corollary 12. *Let* $\mathrm{ZFC} \vdash \varphi$*, then* $\mathrm{ZFC}^U \vdash \varphi^U$*.*

Proof. Let φ be provable in ZFC. Using the notation of Lemma 11, we want to show that φ^U holds in $\mathbb{V}$. But, by Lemma 11, $\mathbb{W} \models \varphi$ and so $\mathbb{V} \models \varphi^U$. □

Theorem 13.

1. ZFC^U *is a conservative extension of* ZFC*.*
2. $\mathrm{ZFC}^{U+\psi}$ *is a conservative extension of* $\mathrm{ZFC} + \psi$*.*

Proof. For 1, let φ be a formula in the language of ZFC such that $\mathrm{ZFC}^U \vdash \varphi$. The proof of φ contains only finitely many axioms from $\mathrm{ZFC}^U \setminus \mathrm{ZFC}$. Let us denote these axioms by $\alpha_1, \dots, \alpha_n$ and suppose α_1 is the axiom "U is transitive and countable" (we may always add an axiom to the beginning of the proof). Then $\mathrm{ZFC} \vdash (\alpha_1 \wedge \dots \wedge \alpha_n) \to \varphi$.[1] Each α_i, for $2 \leq i \leq n$, is of the form β_i^U, where β_i is an axiom of ZFC. Therefore

$$\mathrm{ZFC} \vdash (|U| = \omega \wedge \text{“}U \text{ is transitive”} \wedge \beta_2^U \wedge \dots \wedge \beta_n^U) \to \varphi.$$

φ is in the language of ZFC, so it does not contain the symbol U. For a variable A not occurring in φ,

$$\mathrm{ZFC} \vdash \exists A(|A| = \omega \wedge \text{“}A \text{ is transitive”} \wedge \beta_2^A \wedge \dots \wedge \beta_n^A) \to \varphi.$$

By Corollary 9,

$$\mathrm{ZFC} \vdash \exists A(|A| = \omega \wedge \text{“}A \text{ is transitive”} \wedge \beta_2^A \wedge \dots \wedge \beta_n^A),$$

so

$$\mathrm{ZFC} \vdash \varphi.$$

The proof of 2 is similar. □

However, $\mathrm{ZFC}^{U+\psi}$ is not in general a conservative extension of ZFC. Let ψ be a Σ_1-sentence independent of ZFC. $\mathrm{ZFC} \vdash \forall x((x \text{ is transitive } \wedge \psi^x) \to \psi)$ since ψ is Σ_1. So $\mathrm{ZFC}^{U+\psi} \vdash (U \text{ is transitive } \wedge \psi^U) \to \psi$, but "$U$ is transitive" and ψ^U are axioms of $\mathrm{ZFC}^{U+\psi}$. Thus $\mathrm{ZFC}^{U+\psi}$ proves ψ, the sentence independent of ZFC.

Moreover, note that Theorem 13 is formalizable and provable in PA. The reason is that the proof uses only Corollary 9 (that itself can be formalized and proved in PA) and simple manipulations with finite sequences of formulas.

[1] Formally, this is not correct; ZFC can not prove formulas containing the symbol U. But we may always extend the language of a theory with new symbols without changing the axioms.

Corollary 14.

1. $Con(\mathrm{ZFC}) \to Con(\mathrm{ZFC}^U)$,
2. $Con(\mathrm{ZFC} + \psi) \to Con(\mathrm{ZFC}^{U+\psi})$,
3. $\mathrm{PA} \vdash Con(\mathrm{ZFC}) \to Con(\mathrm{ZFC}^U)$,
4. $\mathrm{PA} \vdash Con(\mathrm{ZFC} + \psi) \to Con(\mathrm{ZFC}^{U+\psi})$.

Chapter 3

Interpretation

There are many possible definitions of interpretation, differing in allowing parameters or function symbols, translating equality, allowing more formulas with different number of free variables defining the domain of interpretation, etc. We first define the type of translation and interpretation we need for our purposes and comment later on other possible definitions.

In the next definitions, S, T are theories, $\mathcal{L}(S)$, $\mathcal{L}(T)$ their languages. For the sake of simplicity, we suppose that $\mathcal{L}(S)$ does not contain function and constant symbols, yet the definitions can be easily generalized to allow them as well. Also, to avoid mentioning free variables in the axioms of S and T, we suppose both theories are axiomatized by sentences.

Definition 15 (Translation)**.** *Let $\mathcal{L}(S)$, $\mathcal{L}(T)$ be as above. A translation τ from $\mathcal{L}(S)$ to $\mathcal{L}(T)$ specifies formulas $\delta(x)$, $\varepsilon(x, y)$ of $\mathcal{L}(T)$ and assigns to every n-ary relation symbol R of $\mathcal{L}(S)$ a formula $\varphi_R(x_1, \ldots, x_n)$ of $\mathcal{L}(T)$.*
For every formula ψ in $\mathcal{L}(S)$ the translation ψ^τ is defined inductively:

- *$R(x_1, \ldots, x_n)^\tau$ is $\varphi_R(x_1, \ldots, x_n)$,*
- *$(x = y)^\tau$ is $\varepsilon(x, y)$,*
- *translation commutes with logical connectives,*
- *$(\exists x \psi)^\tau$ is $\exists x(\delta(x) \wedge \psi^\tau)$,*
- *$(\forall x \psi)^\tau$ is $\forall x(\delta(x) \rightarrow \psi^\tau)$.*

Definition 16 (Interpretation)**.** *Let S, T, $\mathcal{L}(S)$, $\mathcal{L}(T)$ be as above. A translation τ from $\mathcal{L}(S)$ to $\mathcal{L}(T)$ is an interpretation of S in T if the following are provable in T:*

- *$\exists x \delta(x)$,*

- $\forall x(\delta(x) \to \varepsilon(x, x))$,
- $\forall x, y(\delta(x) \wedge \delta(y) \to (\varepsilon(x, y) \to \varepsilon(y, x)))$,
- $\forall x, y, z(\delta(x) \wedge \delta(y) \wedge \delta(z) \to (\varepsilon(x, y) \wedge \varepsilon(y, z) \to \varepsilon(x, z)))$,
- $\forall \bar{x}, \bar{y}(\delta(x_1) \wedge \delta(y_1) \wedge \ldots \wedge \delta(x_n) \wedge \delta(y_n) \wedge \varepsilon(x_1, y_1) \wedge \ldots \wedge \varepsilon(x_n, y_n) \to (\varphi_R(x_1, \ldots, x_n) \leftrightarrow \varphi_R(y_1, \ldots, y_n)))$, *for every R from $\mathcal{L}(S)$ with appropriate arity,*
- ψ^τ, *for every axiom ψ of S.*

We say that S is interpretable in T, writing $S \leq T$, if there exists some interpretation τ of S in T. We may write $S \leq_\tau T$ to indicate that S is interpretable in T via the translation τ.

From a model-theoretical point of view, the interpretation τ gives us a uniform way how to construct a model $\mathcal{N}$ of S from a model $\mathcal{M}$ of T. The relation defined by ε is a congruence with respect to φ_R on the nonempty set defined in $\mathcal{M}$ by δ. Therefore we may factorize the set defined by δ by the relation ε to get the universe of $\mathcal{N}$. The realization of a symbol R is given by φ_R. The fact that translations of all axioms of S hold in $\mathcal{M}$ implies that $\mathcal{N}$ is a model of S.

Definition 17 (Translation with finitely many parameters)**.** *The definition of translation with finitely many parametes is the same as the definition of translation without parameters, except that we allow finitely many extra free variables $p_1, \ldots, p_k$ to appear in any of the formulas $\delta, \varepsilon, \varphi_R$. We denote such a translation $\tau[p_1, \ldots, p_k]$.*

If φ is a formula, then its translation under $\tau[p_1, \ldots, p_k]$ is $\varphi^{\tau[p_1,\ldots,p_k]}$. Note that $p_1, \ldots, p_k$ appear in the translation as free variables. Thus we write $\varphi^\tau(p_1, \ldots, p_k)$ instead of $\varphi^{\tau[p_1,\ldots,p_k]}$.

For the concept of interpretation with finitely many parameters, different definitions can be found throughout the literature. We present two definitions – the first one follows the definition in Friedman's article [3], the second approach appears in the book by Hájek and Pudlák [4].

Definition 18. *Let S, T, $\mathcal{L}(S)$, $\mathcal{L}(T)$ be as above. A translation $\tau[p_1, \ldots, p_k]$ from $\mathcal{L}(S)$ to $\mathcal{L}(T)$ is a* ***model-theoretical interpretation*** *of S in T if for every model $\mathcal{M} \vDash T$ there exist $c_1, \ldots, c_k \in \mathcal{M}$ such that the following formulas are satisfied in $\mathcal{M}$:*

- $\exists x \delta(x, \bar{c})$,

- $\forall x(\delta(x, \bar{c}) \to \varepsilon(x, x, \bar{c}))$,
- $\forall x, y(\delta(x, \bar{c}) \wedge \delta(y, \bar{c}) \to (\varepsilon(x, y, \bar{c}) \to \varepsilon(y, x, \bar{c})))$,
- $\forall x, y, z(\delta(x, \bar{c}) \wedge \delta(y, \bar{c}) \wedge \delta(z, \bar{c}) \to (\varepsilon(x, y, \bar{c}) \wedge \varepsilon(y, z, \bar{c}) \to \varepsilon(x, z, \bar{c})))$,
- $\forall \bar{x}, \bar{y}(\delta(x_1, \bar{c}) \wedge \delta(y_1, \bar{c}) \wedge \ldots \wedge \delta(x_n, \bar{c}) \wedge \delta(y_n, \bar{c}) \wedge \varepsilon(x_1, y_1, \bar{c}) \wedge \ldots \wedge \varepsilon(x_n, y_n, \bar{c}) \to (\varphi_R(x_1, \ldots, x_n, \bar{c}) \leftrightarrow \varphi_R(y_1, \ldots, y_n, \bar{c})))$, *for every* R *from* $\mathcal{L}(S)$ *with appropriate arity,*
- $\psi^\tau(\bar{c})$, *for every axiom* ψ *of* S.

Definition 19. *Let* S, T, $\mathcal{L}(S)$, $\mathcal{L}(T)$ *be as above. A translation* $\tau[p_1, \ldots, p_k]$ *from* $\mathcal{L}(S)$ *to* $\mathcal{L}(T)$ *is a* ***syntactical interpretation*** *of* S *in* T *if there exists an* $\mathcal{L}(T)$*-formula* $\alpha(x_1, \ldots, x_k)$ *such that* T *proves the following:*

- $\exists \bar{s} \alpha(\bar{s})$,
- $\alpha(\bar{s}) \to (\exists x \delta(x, \bar{s}))$,
- $\alpha(\bar{s}) \to (\forall x(\delta(x, \bar{s}) \to \varepsilon(x, x, \bar{s})))$,
- $\alpha(\bar{s}) \to (\forall x, y(\delta(x, \bar{s}) \wedge \delta(y, \bar{s}) \to (\varepsilon(x, y, \bar{s}) \to \varepsilon(y, x, \bar{s}))))$,
- $\alpha(\bar{s}) \to (\forall x, y, z(\delta(x, \bar{s}) \wedge \delta(y, \bar{s}) \wedge \delta(z, \bar{s}) \to (\varepsilon(x, y, \bar{s}) \wedge \varepsilon(y, z, \bar{s}) \to \varepsilon(x, z, \bar{s}))))$,
- $\alpha(\bar{s}) \to (\forall \bar{x}, \bar{y}(\delta(x_1, \bar{s}) \wedge \delta(y_1, \bar{s}) \wedge \ldots \wedge \delta(x_n, \bar{s}) \wedge \delta(y_n, \bar{s}) \wedge \varepsilon(x_1, y_1, \bar{s}) \wedge \ldots \wedge \varepsilon(x_n, y_n, \bar{s}) \to (\varphi_R(x_1, \ldots, x_n, \bar{s}) \leftrightarrow \varphi_R(y_1, \ldots, y_n, \bar{s}))))$, *for every* R *from* $\mathcal{L}(S)$ *with appropriate arity,*
- $\alpha(\bar{s}) \to \psi^\tau(\bar{s})$, *for every axiom* ψ *of* S.

Theorem 20. *Let* S, T, $\tau[p_1, \ldots, p_k]$ *be as above. If* T *is complete and in a countable language, then the model-theoretical definition and the syntactical definition of interpretation with finitely many parameters are equivalent.*

Proof. Clearly, syntactical interpretation implies model-theoretical one, as for any model $\mathcal{M}$ of T we may choose as parameters $c_1, \ldots, c_k$ any k-tuple satisfying $\alpha(x_1, \ldots, x_n)$.

For the other direction, suppose that $\tau[p_1, \ldots, p_k]$ is a model-theoretical interpretation. To show that it is a syntactical interpretation as well, we have to find the formula $\alpha(x_1, \ldots, x_n)$. This will be done using the Omitting Types Theorem. The model-theoretical definition requires certain formulas to be satisfied by the chosen parameters. Let us denote the set of these formulas by Γ. Then Γ is a k-type realized in every model, and, by Corollary 2, Γ is isolated. Let $\alpha(x_1, \ldots, x_k)$

be the isolating formula. Then $T \cup \{\exists \bar{x} \alpha(\bar{x})\}$ is consistent. As T is complete, we have, in fact, $T \vdash \exists \bar{x} \alpha(\bar{x})$. Also, $\alpha(x_1, \ldots, x_k)$ implies in T all the formulas of Γ. So T proves all the formulas that Definition 19 requires it to. $\square$

We demanded T to be in a countable language because of the restriction of the Omitting Types Theorem. But all the theories we use are in a countable language anyway. The restriction to complete theories seems more serious. Note that we used the completeness of T in the proof just once, to prove that $T \vdash \exists \bar{x} \alpha(\bar{x})$. But even with an incomplete T it may often happen that $\exists \bar{x} \alpha(\bar{x})$ is provable. In fact, this will be the case with the interpretations in the next chapter – there, the isolating formula $\alpha(x)$ will be "x is generic".

Let us now comment on possible generalizations and restrictions of the definition of interpretation.

Instead of the formula $\delta(x)$, we could take finitely many formulas $\delta_1, \ldots, \delta_n$, where each of these formulas has different number of free variables. This generalizes the way in which the interpretation determines the new universe. The new universe (or what becomes universe after we perform the factorization) may consist of tuples of mixed lengths. Of course, we have to allow finitely many formulas instead of just one also in the case of $\varepsilon(x, y)$ and $\varphi_R(\bar{x})$. Instead of one formula $\varepsilon(x, y)$, we need a different formula for every pair of lenghts of tuples from the universe and in the case of $\varphi_R(\bar{x})$ we need formulas for all combinations of lenghts. Such an interpretation is called multidimensional.

As we have mentioned above, another possible generalization is to allow function symbols in $\mathcal{L}(S)$. This would require of the translation to assign to every n-ary function symbol its translation – a formula with $n+1$ free variables. This translation must provably act like a function on the set defined by $\delta(x)$ and ε has to be a congruence with respect to it.

Sometimes, the symbol $=$ is not allowed to be translated. In this case $\varepsilon(x, y)$ is always $x = y$. Such an interpretation is said to have absolute equality.

Chapter 4

Forcing

In this chapter, we present two main approaches to forcing – forcing over a countable transitive model and forcing over the universe. We show that forcing construction is formalizable in PA and can be looked at as an interpretation.

4.1 Forcing over a countable transitive model

4.1.1 Definitions and basic forcing theorems

In the naive version of this approach, one works within a countable transitive model M of ZFC. The naivety lies in the fact that one can not prove inside ZFC that such a model exists. We overcome this difficulty by working inside ZFC^U. However, by Theorem 13, ZFC^U can not prove existence of such a model either. In other words – ZFC^U does not know there exists a countable transitive model of ZFC, yet we know there always is one, namely the constant U.

Before we start with a presentation of the forcing technique, let us comment on two issues.

First, most of the work will take place "inside U" or "in the sense of U". By this expression we mean that we work in U viewed as a model of ZFC, or in other words, all of our work is relativized to U. So, if $\varphi(x)$ is a formula of ZFC describing some property, we say that y has this property in the sense of U iff $y \in U$ and $\varphi^U(y)$ holds. Similarly, if $\varphi(x)$ is a formula defining a new constant C in ZFC, we write C^U for the unique object satisfying the formula $\varphi^U(x)$, i.e. C^U is the constant C in the sense of U.

Second, we often use the fact that some formulas are absolute for U. It is a well-known fact that, for a transitive class, all Δ_0-formulas are absolute. Moreover, notions defined by recursion using only absolute notions are itself absolute for transitive models of ZFC.[1]

In the rest of this section, we work in ZFC^U. As we have remarked above, most of the work is done in the sense of U. The only crucial place where we step outside of U is, as we will see, when we choose a generic. The following definitions and theorems closely follow the presentation of forcing in Kunen's book [6]. The difference is the general setup – Kunen uses the naive approach, we work in ZFC^U.

Definition 21 (Forcing notion)**.** *A forcing notion is a triple $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in U$, where $\leq$ partially orders $\mathbb{P}$ and $\mathbb{1}$ is the largest element of this ordering. Elements of $\mathbb{P}$ are called forcing conditions. We say that $p, q \in \mathbb{P}$ are compatible, writing $p \parallel q$, if $(\exists r \in \mathbb{P})(r \leq p \wedge r \leq q)$. Otherwise, they are incompatible, $p \perp q$.*

We often write just $\mathbb{P}$ instead of $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$.

Definition 22 (Dense set)**.** *A set $D \subseteq \mathbb{P}$ is $\mathbb{P}$-dense if for every $p \in \mathbb{P}$ there exists $d \in D$ such that $d \leq p$.*
D is $\mathbb{P}$-dense below p_0 if $(\forall p \in \mathbb{P})(p \leq p_0 \rightarrow (\exists d \in D)(d \leq p))$.

If $\mathbb{P}$ is clear from the context, we write just dense instead of $\mathbb{P}$-dense. Note that we defined forcing notion to belong to U. Also, by absoluteness, $\leq$ partially orders $\mathbb{P}$ in the sense of U and $\mathbb{1}$ is the largest element in the sense of U. Compatibility and density are absolute as well, thus all of our definitions so far have been in the sense of U.

Definition 23 (Generic)**.** *Let $\mathbb{P}$ be a forcing notion. $\boldsymbol{G}$ is a $\mathbb{P}$-generic if $\boldsymbol{G}$ is a filter on $\mathbb{P}$ and for every $\mathbb{P}$-dense $D \in U$ it holds that $\boldsymbol{G} \cap D \neq \emptyset$.*

Again, we often write just generic instead of $\mathbb{P}$-generic. Let $\mathbf{G}$ be a generic, $p \in \mathbf{G}$. Note that if $D \in U$ is dense below p, then also $D \cap \mathbf{G} \neq \emptyset$.

Lemma 24. *For any forcing notion $\mathbb{P}$ and $p_0 \in \mathbb{P}$ there exists $\boldsymbol{G}$ which is a $\mathbb{P}$-generic and $p_0 \in \boldsymbol{G}$.*

Proof. U is countable, so we may consider an at most countable enumeration $D_1, D_2, D_3, \ldots$ of all dense sets from U. We start with p_0 and form a sequence $(p_n)_{n \in \omega}$, such that $p_0 \geq p_1 \geq p_2 \ldots$ and $p_n \in D_n$, for $n \geq 1$. This is enabled by the density of each D_n. We set $\mathbf{G}$ to be the filter generated by $\{p_n; n \in \omega\}$. □

Definition 25. *By induction on rank, we define the notion of $\mathbb{P}$-name. τ is a $\mathbb{P}$-name if τ is a relation, $\tau \in U$ and*

$$\forall \langle \pi, p \rangle \in \tau (\pi \textit{ is a } \mathbb{P}\textit{-name and } p \in \mathbb{P}).$$

We write $U^{\mathbb{P}}$ for the class of all $\mathbb{P}$-names.

[1]The exact formulation and its proof can be found in [6] as Theorem 5.6.

Note that once again, we defined names to belong to U, and, by absoluteness, names may be viewed as defined in the sense of U.

Definition 26. *Let $\mathbb{P}$ be a forcing notion, $x \in U$. By induction on rank, we define the canonical $\mathbb{P}$-name $\check{x}$ of x by $\check{x} = \{\langle \check{y}, \mathbb{1}\rangle; y \in x\}$.*

Definition 27. *Let $\mathbb{P}$ be a forcing notion and $\boldsymbol{G}$ a $\mathbb{P}$-generic. Then the $\boldsymbol{G}$-value $\tau_{\boldsymbol{G}}$ for a $\mathbb{P}$-name τ is defined as follows:*

$$\tau_{\boldsymbol{G}} = \{\pi_{\boldsymbol{G}}; \exists p \in \boldsymbol{G}(\langle \pi, p\rangle \in \tau)\}.$$

Moreover, we define $U[\boldsymbol{G}] = \{\tau_{\boldsymbol{G}}; \tau \in U^{\mathbb{P}}\}$. $U[\boldsymbol{G}]$ is called the $\boldsymbol{G}$-generic extension of U.

We may define, in the sense of U, the name $\gamma = \{\langle \check{p}, p\rangle; p \in \mathbb{P}\}$. Then for any generic $\mathbf{G}$, $\gamma_{\mathbf{G}} = \mathbf{G}$. This implies $\mathbf{G} \in U[\mathbf{G}]$. The goal is to show that $U[\mathbf{G}]$ is a model of ZFC, i.e. all axioms of ZFC hold relativized to $U[\mathbf{G}]$. To do so, we define a forcing relation and prove basic forcing theorems.

Definition 28. *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, $\tau_1, \ldots, \tau_n$ $\mathbb{P}$-names. The forcing relation $\Vdash^*$ is defined for all* ZFC*-formulas by induction on complexity as follows:*

(a) By induction on pairs of ranks under the canonical well-ordering,

$p \Vdash^* \tau_1 = \tau_2$ *if*

(i) for all $\langle \pi_1, s_1\rangle \in \tau_1$

$$\{q \leq p; q \leq s_1 \rightarrow \exists \langle \pi_2, s_2\rangle \in \tau_2(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is dense below p, and

(ii) for all $\langle \pi_2, s_2\rangle \in \tau_2$

$$\{q \leq p; q \leq s_2 \rightarrow \exists \langle \pi_1, s_1\rangle \in \tau_1(q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is dense below p.

(b) $p \Vdash^* \tau_1 \in \tau_2$ if

$$\{q; \exists \langle \pi, s\rangle \in \tau_2(q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p.

(c) $p \Vdash^* \phi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n)$ if

$$p \Vdash^* \phi(\tau_1, \ldots, \tau_n) \text{ and } p \Vdash^* \psi(\tau_1, \ldots, \tau_n).$$

(d) $p \Vdash^* \neg\phi(\tau_1, \ldots, \tau_n)$ *if there is no* $q \leq p$ *such that* $q \Vdash^* \phi(\tau_1, \ldots, \tau_n)$.

(e) $p \Vdash^* \exists x \phi(x, \tau_1, \ldots, \tau_n)$ *if*

$$\{r; \exists \sigma \in U^{\mathbb{P}}(r \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n))\}$$

is dense below p.

The nature of this definition deserves some comments. First of all, we did not, in fact, define a new relation $\Vdash^*$ in ZFC^U – it would have to be a relation between forcing conditions from $\mathbb{P}$ and formulas, which, formally, makes no sense. Nevertheless, we can look at the Definition 28 as defining for each formula $\phi(x_1, \ldots, x_n)$ a relation F_ϕ so that $F_\phi(\tau_1, \ldots, \tau_n, \mathbb{P}, p)$ is equivalent to $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$. Thus the induction on the complexity of formulas takes place in the metatheory. Therefore, formally, the proof of the next theorem uses metamathematical induction as well. We have to keep this fact in mind for the later comments on formalizing the forcing method in PA.

Also, note that in the clauses (a)–(d) all of the quantifiers are bound by an element of U. Only in the clause (e) we have $\exists \sigma \in U^{\mathbb{P}}$, but $U^{\mathbb{P}}$ is a subclass of U; hence, by absoluteness, Definition 28 is in the sense of U. For some $p \in \mathbb{P}$ and $\varphi(x)$, let us define the class $C = \{\tau; \tau \in U^{\mathbb{P}} \wedge p \Vdash^* \varphi(\tau)\}$. Then C is a class in the sense of U. Similarly, for any $\tau \in U^{\mathbb{P}}$ and $\varphi(x)$, the set $D = \{p; p \in \mathbb{P} \wedge p \Vdash^* \varphi(\tau)\}$ is a set in the sense of U. This is important; as we often define dense sets in a similar way, this fact allows us to conclude that they belong to U and therefore have nonempty intersections with the generic.

Lemma 29. *Let* $\mathbb{P}$ *be a forcing notion,* $\boldsymbol{G}$ *a generic,* $p \in \mathbb{P}$. *Then for any formula* φ *the following holds:*

$$((p \Vdash^* \varphi) \wedge (r \leq p)) \rightarrow r \Vdash^* \varphi.$$

Proof. By induction on the complexity of φ. For atomic formulas we use the fact that if a set is dense below p, then it is also dense below any $r \leq p$. The induction step for the logical connectives is obvious. The case for $\exists$ is similar to the case for atomic formulas. □

Theorem 30. *Let* $\mathbb{P}$ *be a forcing notion,* $\boldsymbol{G}$ *a* $\mathbb{P}$*-generic,* $\tau_1, \ldots, \tau_n$ $\mathbb{P}$*-names and* $\phi(x_1, \ldots, x_n)$ *a* ZFC*-formula. Then the following holds:*

(1) If $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$ *and* $p \in \boldsymbol{G}$, *then* $\phi(\tau_{1\boldsymbol{G}}, \ldots, \tau_{n\boldsymbol{G}})^{U[\boldsymbol{G}]}$.

(2) If $\phi(\tau_{1\boldsymbol{G}}, \ldots, \tau_{n\boldsymbol{G}})^{U[\boldsymbol{G}]}$, *then* $\exists p \in \boldsymbol{G}(p \Vdash^* \phi(\tau_1, \ldots, \tau_n))$.

Proof. By metamathematical induction on the complexity of ϕ.

(a) Let ϕ be of the form $x_1 = x_2$. The proof proceeds by induction on pairs of ranks under the cannonical well-ordering.
For (1), assume $p \Vdash^* \tau_1 = \tau_2$ and $p \in \mathbf{G}$. We show that $\tau_{1\mathbf{G}} \subset \tau_{2\mathbf{G}}$ and $\tau_{2\mathbf{G}} \subset \tau_{1\mathbf{G}}$, concluding $\tau_{1\mathbf{G}} = \tau_{2\mathbf{G}}$. Fix $a \in \tau_{1\mathbf{G}}$. There is some $\langle \pi_1, s_1 \rangle \in \tau_1$ such that $a = \pi_{1\mathbf{G}}$ and $s_1 \in \mathbf{G}$. Fix some $r \in \mathbf{G}$ such that $r \leq p$ and $r \leq s_1$. By Definition 28,

$$\{q \leq p; q \leq s_1 \rightarrow \exists \langle \pi_2, s_2 \rangle \in \tau_2 (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is dense below p and therefore also dense below r. Fix t belonging to this set such that $t \leq r$, $t \in \mathbf{G}$. As $t \leq r \leq s_1$, it holds that $\exists \langle \pi_2, s_2 \rangle \in \tau_2$ $(t \leq s_2 \wedge t \Vdash^* \pi_1 = \pi_2)$. Fix one such $\langle \pi_2, s_2 \rangle$. $t \in \mathbf{G}$ and $t \leq s_2$ imply $s_2 \in \mathbf{G}$ and $\pi_{2\mathbf{G}} \in \tau_{2\mathbf{G}}$. We have $t \Vdash^* \pi_1 = \pi_2$ and $t \in \mathbf{G}$; thus, by induction hypothesis, $a = \pi_{1\mathbf{G}} = \pi_{2\mathbf{G}}$. Therefore $a \in \tau_{2\mathbf{G}}$, showing $\tau_{1\mathbf{G}} \subset \tau_{2\mathbf{G}}$. The case of $\tau_{2\mathbf{G}} \subset \tau_{1\mathbf{G}}$ is similar.

To prove (2), we use a density argument. Assume $\tau_{1\mathbf{G}} = \tau_{2\mathbf{G}}$. We construct a set D and show it is a dense set. D consists of all $r \in \mathbb{P}$ such that one of the following holds:

(a) $r \Vdash^* \tau_1 = \tau_2$,

(b) $\exists \langle \pi_1, s_1 \rangle \in \tau_1$ such that

$$r \leq s_1 \wedge (\forall \langle \pi_2, s_2 \rangle \in \tau_2)(\forall q \in \mathbb{P})[(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2) \rightarrow q \perp r],$$

(c) $\exists \langle \pi_2, s_2 \rangle \in \tau_2$ such that

$$r \leq s_2 \wedge (\forall \langle \pi_1, s_1 \rangle \in \tau_1)(\forall q \in \mathbb{P})[(q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2) \rightarrow q \perp r].$$

The conditions (b) and (c) were chosen so that they prevent any $r \in \mathbf{G}$ from satisfying them. For contradiction, suppose $r \in \mathbf{G}$ and fix $\langle \pi_1, s_1 \rangle$ satisfying the condition (b). As $r \leq s_1$, it holds that $s_1 \in \mathbf{G}$, so $\pi_{1\mathbf{G}} \in \tau_{1\mathbf{G}} = \tau_{2\mathbf{G}}$. Let us then fix $\langle \pi_2, s_2 \rangle \in \tau_2$ with $s_2 \in \mathbf{G}$ and $\pi_{1\mathbf{G}} = \pi_{2\mathbf{G}}$. By induction hypothesis, there is $t \in \mathbf{G}$ with $t \Vdash^* \pi_1 = \pi_2$. We fix $q \in \mathbf{G}$ such that $q \leq t$ and $q \leq s_2$. By Lemma 29, it follows that $q \Vdash^* \pi_1 = \pi_2$. By (b) we have $q \perp r$. This is a contradiction as both $q, r \in \mathbf{G}$. The case for the condition (c) is similar.

Now, if D is dense, then there exists $r \in \mathbf{G} \cap D$ such that $r \Vdash^* \tau_1 = \tau_2$, so (2) holds.

To check the density of D, fix $p \in \mathbb{P}$. If $p \Vdash^* \tau_1 = \tau_2$, then $p \in D$ and we are done. If not, then one of the conditions (i), (ii) from Definition 28 is not satisfied. Let us analyze the case when (i) fails, the other case is similar. Fix $\langle \pi_1, s_1 \rangle \in \tau_1$ for which

$$\{q \le p; q \le s_1 \to \exists \langle \pi_2, s_2 \rangle \in \tau_2 (q \le s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is not dense below p. Further, fix $r \le p$ such that

$$\forall q \le r[q \le s_1 \wedge (\forall \langle \pi_2, s_2 \rangle \in \tau_2) \neg (q \le s_2 \wedge q \Vdash^* \pi_1 = \pi_2)].$$

Now, it can be easily checked that r satisfies the condition (b) for the fixed $\langle \pi_1, s_1 \rangle$, so $r \in D$.

(b) Let ϕ be of the form $x_1 \in x_2$.
We assume $p \Vdash^* \tau_1 \in \tau_2$ and $p \in \mathbf{G}$. By (b) of Definition 28, the set

$$\{q; \exists \langle \pi, s \rangle \in \tau_2 (q \le s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p. Therefore we may fix $q \in \mathbf{G}$ and $\langle \pi, s \rangle \in \tau_2$ so that $q \le s$ and $q \Vdash^* \pi = \tau_1$. Then $\pi_{\mathbf{G}} \in \tau_{2\mathbf{G}}$ and, by induction hypothesis, $\pi_{\mathbf{G}} = \tau_{1\mathbf{G}}$. Thus $\tau_{1\mathbf{G}} \in \tau_{2\mathbf{G}}$ and (1) is proved.

For (2), assume $\tau_{1\mathbf{G}} \in \tau_{2\mathbf{G}}$. We need to find some $p \in \mathbf{G}$ such that

$$\{q; \exists \langle \pi, s \rangle \in \tau_2 (q \le s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p. Fix $\langle \pi, s \rangle \in \tau_2$ such that $s \in \mathbf{G}$ and $\pi_{\mathbf{G}} = \tau_{1\mathbf{G}}$. By induction hypothesis there is $r \in \mathbf{G}$ such that $r \Vdash^* \pi = \tau_1$. Fix $p \in \mathbf{G}$ so that $p \le s$ and $p \le r$. It is easy to check that p is the forcing condition we have been looking for as for every $t \le p$ it holds that $t \le s$ and $t \Vdash^* \pi = \tau_1$ (because $t \le r$ and $r \Vdash^* \pi = \tau_1$).

(c) The case for $\wedge$ is just easy checking of definitions.

(d) The induction step for negation.
For (1), assume $p \in \mathbf{G}$ and $p \Vdash^* \neg\phi(\tau_1, \dots, \tau_n)$. We want to show that $\neg\phi(\tau_{1\mathbf{G}}, \dots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$ holds. Suppose for contradiction $\phi(\tau_{1\mathbf{G}}, \dots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$. By induction hypothesis, there is $q \in \mathbf{G}$ with $q \Vdash^* \phi(\tau_1, \dots, \tau_n)$. By choosing $r \in \mathbf{G}$ so that $r \le q$ and $r \le p$, we reach contradiction with $p \Vdash^* \neg\phi(\tau_1, \dots, \tau_n)$.

For (2), assume $\neg\phi(\tau_{1\mathbf{G}}, \dots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$. We construct the set of all conditions that already decide ϕ, i.e.

$$D = \{p \in \mathbb{P}; p \Vdash^* \phi(\tau_1, \dots, \tau_n) \vee p \Vdash^* \neg\phi(\tau_1, \dots, \tau_n)\}.$$

D is dense by Definition 28 and belongs to U. Fix $p \in \mathbf{G} \cup D$. $p \Vdash^* \phi(\tau_1, \dots, \tau_n)$ would, by induction hypothesis, imply $\phi(\tau_{1\mathbf{G}}, \dots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$, contradicting our assumption. Thus $p \Vdash^* \neg\phi(\tau_1, \dots, \tau_n)$ and we are done.

(e) The induction step for $\exists$.
For (1), assume $p \in \mathbf{G}$ and $p \Vdash^* \exists x \phi(x, \tau_1, \ldots, \tau_n)$. Then the set

$$\{r; \exists \sigma \in U^{\mathbb{P}}(r \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n))\}$$

is dense below p and belongs to U. Thus we may fix some $r \in \mathbf{G}$ and $\sigma \in U^{\mathbb{P}}$ such that $r \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n)$. Then, by induction hypothesis, it holds that $\phi(\sigma_{\mathbf{G}}, \tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$ and $\sigma_{\mathbf{G}} \in U[\mathbf{G}]$. By the definition of relativization, we have $(\exists x \phi(x, \tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}}))^{U[\mathbf{G}]}$.

For (2), assume $(\exists x \phi(x, \tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}}))^{U[\mathbf{G}]}$. Fix some $\sigma \in U^{\mathbb{P}}$ such that $\phi(\sigma_{\mathbf{G}}, \tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$. By induction hypothesis, there is $p \in \mathbf{G}$ such that $p \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n)$. Therefore $r \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n)$ for every $r \leq p$. So, by Definition 28, $p \Vdash^* \exists x \phi(x, \tau_1, \ldots, \tau_n)$, and we are done.

□

As we have noted above, it is not formally correct to use the symbol $\Vdash^*$ in ZFC^U-formulas. However, for any ZFC-formula ϕ it is possible to rewrite Theorem 30 in a formally correct way. Thus, there is a formula α, provable in ZFC^U, claiming that Theorem 30 holds for atomic formulas. As ZFC$^U \vdash \alpha$, it holds that PA $\vdash$ "ZFC^U proves α". Likewise, for any formula ϕ there is a formula α_ϕ claiming that Theorem 30 holds for ϕ. As before, ZFC$^U \vdash \alpha_\phi$ for any ϕ, so PA $\vdash$ "ZFC^U proves α_ϕ" for any ϕ. To say that Theorem 30 is formalizable in PA we need a bit more, namely that PA $\vdash$ "ZFC^U proves α_ϕ for any formula ϕ". To argue that this holds, observe that the function mapping ϕ to α_ϕ is recursive and that the metamathematical induction in the proof above can be performed inside PA.

Definition 31. *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, $\phi(x_1, \ldots, x_n)$ a* ZFC*-formula, $\tau_1, \ldots, \tau_n$ $\mathbb{P}$-names. The forcing relation $\Vdash$ is defined as follows: $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ if*

$$\forall \boldsymbol{G}[(\boldsymbol{G} \textit{ is a } \mathbb{P}\textit{-generic } \wedge p \in \boldsymbol{G}) \rightarrow \phi(\tau_{1\boldsymbol{G}}, \ldots, \tau_{n\boldsymbol{G}})^{U[\boldsymbol{G}]}].$$

This definition is not in the sense of U. It quantifies over all generics and it is often the case that a generic does not belong to U. As with $\Vdash^*$, the forcing relation $\Vdash$ is not formally a relation of ZFC^U. In this case we can view $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ just as a shortcut for $\forall \mathbf{G}[(\mathbf{G} \text{ is } \mathbb{P}\text{-generic } \wedge p \in \mathbf{G}) \rightarrow \phi(\tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}]$.

The next theorem shows that forcing relations $\Vdash^*$ and $\Vdash$ are in fact equivalent.

Theorem 32. *Let $\mathbb{P}$ be a forcing notion, $\phi(x_1, \ldots, x_n)$ a* ZFC*-formula, $\tau_1, \ldots, \tau_n$ $\mathbb{P}$-names, $\boldsymbol{G}$ a $\mathbb{P}$-generic. Then*

1. $\forall p \in \mathbb{P}(p \Vdash \phi(\tau_1, \ldots, \tau_n) \leftrightarrow p \Vdash^* \phi(\tau_1, \ldots, \tau_n))$.

2. $\phi(\tau_{1\boldsymbol{G}}, \ldots, \tau_{n\boldsymbol{G}})^{U[\boldsymbol{G}]} \leftrightarrow \exists p \in \boldsymbol{G}(p \Vdash \phi(\tau_1, \ldots, \tau_n))$.

Proof. The second statement follows immediately from the first one and Theorem 30. As for the first statement, the implication from right to left is immediate from Theorem 30. We prove the other implication.

First, we argue that to conclude $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$ it suffices to show that $D = \{r; r \Vdash^* \phi(\tau_1, \ldots, \tau_n)\}$ is dense below p. This follows, by induction on the complexity of ϕ, from Definition 28 and the fact that if D is dense below p and for each $d \in D$ there is H_d dense below d, then $\bigcup_{d \in D} H_d$ is dense below p.

So suppose, for contradiction, that $D = \{r; r \Vdash^* \phi(\tau_1, \ldots, \tau_n)\}$ is not dense below p. Fix $q \leq p$ such that $\neg(\exists r \leq q)(r \in D)$. By Definition 28, we have $q \Vdash^* \neg\phi(\tau_1, \ldots, \tau_n)$, and so $q \Vdash \neg\phi(\tau_1, \ldots, \tau_n)$. Fix some generic $\mathbf{G}$ such that $q \in \mathbf{G}$. By definition of $\Vdash$, we have $\neg\phi(\tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$. But $q \leq p$ implies $p \in \mathbf{G}$, and as we assume $p \Vdash \phi(\tau_1, \ldots, \tau_n)$, we reach contradiction by deducing $\phi(\tau_{1\mathbf{G}}, \ldots, \tau_{n\mathbf{G}})^{U[\mathbf{G}]}$. □

This theorem is formalizable and provable in PA. We used metamathematical induction on the complexity of formulas in the second paragraph of the proof above, but we can perform this induction in PA. The rest of the proof takes place in ZFC^U using only Theorem 30, that is itself provable in PA, as we noted above.

Theorem 33. *Let $\mathbb{P}$ be a forcing notion and $\boldsymbol{G}$ be a $\mathbb{P}$-generic. Then $U[\boldsymbol{G}]$ satisfies* ZFC*, i.e. $\phi^{U[\boldsymbol{G}]}$ holds for every axiom ϕ of* ZFC.

Proof.

- Extensionality: It is immediate from the definition that $U[\mathbf{G}]$ is transitive, therefore extensionality holds.

- Foundation: Foundation holds relativized to any class or set.

- Pairing: Let $x, y \in U[\mathbf{G}]$ have names τ, σ respectively. Then $\{\langle \tau, \mathbb{1} \rangle, \langle \sigma, \mathbb{1} \rangle\}$ is the name of their pair in the sense of $U[\mathbf{G}]$.

- Separation: Let $\phi(a, b, \bar{x})$ be a formula, $\sigma, \bar{\tau}$ be names. We want to verify that
  $$\{x \in \sigma_{\mathbf{G}}; \phi(x, \sigma_{\mathbf{G}}, \bar{\tau}_{\mathbf{G}})^{U[\mathbf{G}]}\} \text{ belongs to } U[\mathbf{G}].$$
  We construct a name for this set:
  $$\{\langle \pi, p \rangle; \pi \in \mathrm{dom}(\sigma) \land p \in \mathbb{P} \land p \Vdash (\pi \in \sigma \land \phi(\pi, \sigma, \bar{\tau}))\}.$$

- Union: We show that for any $x \in U[\mathbf{G}]$ there exists $y \in U[\mathbf{G}]$ such that $(\bigcup x \subseteq y)^{U[\mathbf{G}]}$. This is sufficient, since we may use separation in the sense of $U[\mathbf{G}]$. Let $x = \tau_{\mathbf{G}}$, we set $\pi = \bigcup \mathrm{dom}(\tau)$. If $a \in \tau_{\mathbf{G}}$ and $b \in a$, then $b = \sigma_{\mathbf{G}}$ for some $\langle \sigma, p \rangle$, where $p \in \mathbf{G}$ and $\langle \sigma, p \rangle \in \bigcup \mathrm{dom}(\tau)$. Therefore $b \in \pi_{\mathbf{G}}$, and $\pi_{\mathbf{G}}$ is the y we look for.

- Power set: Let $x \in U[\mathbf{G}]$. We want to show that there exists $y \in U[\mathbf{G}]$ such that $(\forall a \in U[\mathbf{G}])(a \subseteq x \to a \in y)$. Let $x = \tau_{\mathbf{G}}$. We set $P = \{\sigma \in U^{\mathbb{P}}; \mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)\}$ and $\pi = \{\langle \sigma, \mathbb{1} \rangle; \sigma \in P\}$. Fix any $a \in U[\mathbf{G}]$ such that $a \subseteq \tau_{\mathbf{G}}$. We show that $a \in \pi_{\mathbf{G}}$, thus concluding the proof. Let $a = \mu_{\mathbf{G}}$. We set $\vartheta = \{\langle \sigma, p \rangle; \sigma \in \mathrm{dom}(\tau) \wedge p \Vdash \sigma \in \mu\}$. Note that $\vartheta \in P$, so $\vartheta_{\mathbf{G}} \in \pi_{\mathbf{G}}$. To finish the argument, we show that $\vartheta_{\mathbf{G}} = \mu_{\mathbf{G}}$. Let $b \in \vartheta_{\mathbf{G}}$. Then $b = \sigma_{\mathbf{G}}$ for some $\langle \sigma, p \rangle \in \vartheta$, $p \in \mathbf{G}$. Since $p \Vdash \sigma \in \mu$, by Theorem 32 we have $\sigma_{\mathbf{G}} \in \mu_{\mathbf{G}}$. Thus $\vartheta_{\mathbf{G}} \subseteq \mu_{\mathbf{G}}$. Let $b \in \mu_{\mathbf{G}}$. Since $\mu_{\mathbf{G}} \subseteq \tau_{\mathbf{G}}$, there exists $\sigma \in \mathrm{dom}(\tau)$ such that $b = \sigma_{\mathbf{G}}$. By Theorem 32, there is a $p \in \mathbf{G}$ such that $p \Vdash \sigma \in \mu$. So $\langle \sigma, p \rangle \in \vartheta$ and $\sigma_{\mathbf{G}} \in \vartheta_{\mathbf{G}}$. This implies $\mu_{\mathbf{G}} \subseteq \vartheta_{\mathbf{G}}$.

- Infinity: It holds, since $\check{\omega}_{\mathbf{G}} = \omega \in U[\mathbf{G}]$.

- Replacement: Let $\phi(x, y)$ be a formula. Again, we suppress mentioning of any possible parameters in ϕ for the sake of brevity. Suppose we have $(\forall x \exists ! y(\phi(x, y)))^{U[\mathbf{G}]}$. We want to show that for every $x \in U[\mathbf{G}]$ there is $y \in U[\mathbf{G}]$ such that

$$((\forall a \in x)(\exists b \in y)(\phi(a, b)))^{U[\mathbf{G}]}.$$

Fix x, and let $x = \tau_{\mathbf{G}}$. We define

$$P = \{\sigma \in U^{\mathbb{P}}; (\exists \pi \in \mathrm{dom}(\tau))(\exists p \in \mathbb{P})(p \Vdash \phi(\pi, \sigma)))\}.$$

P is a class in the sense of U. We change the definition a little to make sure we get a set. For given π, p, a name σ gets to P' iff $p \Vdash \phi(\pi, \sigma)$ and, moreover, there is no μ with a smaller rank such that $p \Vdash \phi(\pi, \mu)$. We define $\vartheta = \{\langle \sigma, \mathbb{1} \rangle; \sigma \in P'\}$ and show that $\vartheta_{\mathbf{G}}$ is the y we look for. Let $\pi_{\mathbf{G}} \in \tau_{\mathbf{G}}$. Then there is some $a \in U[\mathbf{G}]$ such that $\phi(\pi_{\mathbf{G}}, a)^{U[\mathbf{G}]}$ holds. Let $a = \sigma_{\mathbf{G}}$; by Theorem 32, there is $p \in \mathbf{G}$ with $p \Vdash \phi(\pi, \sigma)$. Therefore there exists $\sigma' \in P'$ such that $p \Vdash \phi(\pi, \sigma')$. So it holds $\sigma'_{\mathbf{G}} \in \vartheta_{\mathbf{G}}$ and $\phi(\sigma'_{\mathbf{G}}, \vartheta_{\mathbf{G}})^{U[\mathbf{G}]}$. This concludes the proof.

- Choice: Instead of the axiom of choice, we work with an equivalent statement – for every x we may find some $\alpha \in \mathrm{Ord}$ such that there exists a function f with $\mathrm{dom}(f) = \alpha$ and $\mathrm{rng}(f) \supseteq x$. Let $x = \tau_{\mathbf{G}} \in U[\mathbf{G}]$. By the axiom of choice in the sense of U, we may enumerate $\mathrm{dom}(\tau)$ by some ordinal α. So $\text{dom}(\tau) = \{\pi_\beta; \beta < \alpha\}$. By a construction similar to the one we used to prove the axiom of pairing, we may construct for any two names σ_1, σ_2 a name $\langle\langle\sigma_1, \sigma_2\rangle\rangle$ such that $\langle\langle\sigma_1, \sigma_2\rangle\rangle_{\mathbf{G}} = \langle\sigma_{1\mathbf{G}}, \sigma_{2\mathbf{G}}\rangle$. Let $T = \{\langle\langle\check{\beta}, \pi_\beta\rangle\rangle; \beta < \alpha\}$ and $\vartheta = \{\langle t, \mathbb{1}\rangle; t \in T\}$. It is easy to see that $\vartheta_{\mathbf{G}}$ is the function we look for.

$\square$

Like all the theorems above and from the same reasons, Theorem 33 can be formalized and proved in PA.

4.1.2 Formalization of forcing in Peano arithmetic

As we have noted above, the forcing costruction can be formalized in PA. The next theorem shows that PA also proves the usual consistency results.

Theorem 34. *Let φ be a* ZFC*-sentence and let* ZFC^U *prove that there exists a forcing notion $\mathbb{P}$ and a generic $\boldsymbol{G}$ such that $\varphi^{U[\boldsymbol{G}]}$ holds. Then* $\text{PA} \vdash Con(\text{ZFC}) \to Con(\text{ZFC} + \varphi)$.

Proof. Let us work in PA. Let $\alpha_1(\bar{v_1}), \ldots, \alpha_n(\bar{v_n})$ be a proof in the theory ZFC$+\varphi$. We want to show that

$$\text{ZFC}^U \vdash (\exists\, \mathbb{P}, \mathbf{G})(\forall \bar{x_1}, \ldots, \bar{x_n} \in U[\mathbf{G}])(\alpha_1(\bar{x_1})^{U[\mathbf{G}]} \wedge \ldots \wedge \alpha_n(\bar{v_x})^{U[\mathbf{G}]}).$$

By writing $\mathbb{P}$, $\mathbf{G}$ we indicate, as usual, that $\mathbb{P}$ is a forcing notion, $\mathbf{G}$ a generic. We did drop this supposition above for brevity.

Let us fix $\mathbb{P}$, $\mathbf{G}$ as in the assumption of this theorem. If α_i is an axiom of ZFC, its relativization to $U[\mathbf{G}]$ holds by Theorem 33. If α_i is φ, then its relativization holds by the assumption of this theorem.

Let α_i be a logical axiom[2]. If it is a propositional axiom, then $\alpha^{U[\mathbf{G}]}$ is an axiom as well, since the relativization commutes with logical connectives. The language of ZFC does not contain function symbols, so the logical axiom of specification has the form $\phi(y) \to \exists x \phi(x)$. We want to show that the formula $\phi(y)^{U[\mathbf{G}]} \to ((\exists x \in U[\mathbf{G}])\, \phi(x)^{U[\mathbf{G}]})$ is satisfied by any element y of $U[\mathbf{G}]$, which is obvious.

Finaly, let α_i be deduced by modus ponens or the rule of generalization from formulas earlier on the list. Generalization allows us to deduce from $\phi(x) \to \psi$ the formula $\exists x \phi(x) \to \psi$. So suppose $\phi(x)^{U[\mathbf{G}]} \to \psi^{U[\mathbf{G}]}$ is satisfied by all elements x of $U[\mathbf{G}]$, then $(\exists x \in U[\mathbf{G}])\, \phi(x)^{U[\mathbf{G}]} \to \psi^{U[\mathbf{G}]}$ holds. The case for modus ponens is obvious.

[2]The version of a predicate calculus we use for this proof can be found in [8].

Let us suppose $\neg Con(\mathrm{ZFC} + \varphi)$; we want to conclude $\neg Con(\mathrm{ZFC})$. Fix a proof $\alpha_1, \dots, \alpha_n$ in $\mathrm{ZFC} + \varphi$ of some contradiction; e.g. let α_n be $\exists x(x \neq x)$. Then $\mathrm{ZFC}^U \vdash (\exists \mathbb{P}, \mathbf{G})(\exists x \in U[\mathbf{G}])(x \neq x)$. So we have $\neg Con(\mathrm{ZFC}^U)$, and, by Corollary 14, we get $\neg Con(\mathrm{ZFC})$. □

4.1.3 Forcing as interpretation

Theorem 35. *Let φ be a sentence. Suppose that for every model of* ZFC^U *there is a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-generic $\boldsymbol{G}$, such that φ holds in $U[\boldsymbol{G}]$. Then there exists a model-theoretical intepretation of the theory* $\mathrm{ZFC} + \varphi$ *in* ZFC^U*.*

Proof. As we have seen, ZFC^U can define $U[\mathbf{G}]$ from given $\mathbb{P}$ and $\mathbf{G}$. Let us, for now, write explicitly $U[\mathbf{G}]_{\mathbb{P}}$ to indicate the dependency on $\mathbb{P}$. Let $\delta(x, \mathbb{P}, \mathbf{G})$ be the formula $x \in U[\mathbf{G}]_{\mathbb{P}}$. Let further $\varepsilon(x, y)$ be $x = y$ and $\varphi_{\in}(x, y)$ be $x \in y$. We want to show that the parametric translation $\tau[p_1, p_2]$ given by the formulas $\delta(x, p_1, p_2)$, $\varepsilon(x, y)$ and $\varphi_{\in}(x, y)$ is a model-theoretical interpretation of $\mathrm{ZFC} + \varphi$ in ZFC^U. Let $\mathcal{M}$ be a model of ZFC^U and let $\mathbb{P}$, $\mathbf{G}$ be as in the assumptions of this theorem. Realize the parameters p_1 and p_2 by $\mathbb{P}$ and $\mathbf{G}$ respectively. We want to show that certain formulas, given by the Definition 18, hold in $\mathcal{M}$. The formulas expressing that ε is a congruence hold because τ has absolute equality. $\exists x\, \delta(x, \mathbb{P}, \mathbf{G})$ holds as well because $U[\mathbf{G}]_{\mathbb{P}}$ is nonempty. For any formula ϕ, the translation $\phi^{\tau}(\mathbb{P}, \mathbf{G})$ is $\phi^{U[\mathbf{G}]_{\mathbb{P}}}$. Therefore, by the Theorem 33, $\phi^{\tau}(\mathbb{P}, \mathbf{G})$ holds in $\mathcal{M}$ for every axiom ϕ of ZFC, and $\varphi^{\tau}(\mathbb{P}, \mathbf{G})$ holds by the assumption of this theorem. □

Theorem 36. *Let φ be a sentence. Suppose that for every model of* ZFC^U *there is a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-generic $\boldsymbol{G}$, such that φ holds in $U[\boldsymbol{G}]$. Then there exists a syntactical intepretation of the theory* $\mathrm{ZFC} + \varphi$ *in* ZFC^U*.*

Proof. Let $\tau[p_1, p_2]$ be the model-theoretical interpretation from the proof above. In the notation of Definition 19, we want to find a suitable formula $\alpha(x_1, x_2)$. Let $\alpha(x_1, x_2)$ be the formula "x_1 is a forcing notion $\wedge$ x_2 is a x_1-generic $\wedge \varphi^{U[x_2]}$". Then $\mathrm{ZFC}^U \vdash (\exists x_1, x_2)\alpha(x_1, x_1)$. Definition 19 demands that ZFC^U proves some more formulas. But their provability is easy consequence of the proof above. □

Usually, one does not work with an arbitary $\mathbb{P}$, but a particular forcing notion is defined for an intended result. So we may get rid of the parameter p_1 in τ by replacing it by the definition of the particular $\mathbb{P}$.

Still, the parameter p_2 stays. In this case, the formula $\alpha(x_2)$ would be "x_2 is a suitable generic", where suitable means such that φ holds in the corresponding generic extension.

Sometimes it is possible to get rid of the parameters at all. Many forcing arguments work with the constructible universe; in our setup this means to work in the theory $\mathrm{ZFC}^U + \mathbf{V} = \mathbf{L}$. The axiom $\mathbf{V} = \mathbf{L}$ implies that there exists a definable well-ordering of the universe. [3] This allows us to define $\mathbf{G}$ as the first $\mathbb{P}$-generic in this ordering and replace the parameter p_2 by this definition.

Finally, note that the forcing construction may be performed in the theory $\mathrm{ZFC}^{U+\psi}$, for a suitable ψ. The forcing construction is not affected by an additional axiom. But the axiom ψ^U may give us new opportunities for defining a forcing notion $\mathbb{P}$. Consistency results obtained by forcing in $\mathrm{ZFC}^{U+\psi}$ are of the form $Con(\mathrm{ZFC} + \psi) \to Con(\mathrm{ZFC} + \varphi)$ and the corresponding interpretation interprets a theory $\mathrm{ZFC} + \varphi$ in $\mathrm{ZFC}^{U+\psi}$. Apart from that, all the results proved in this section for ZFC^U hold for $\mathrm{ZFC}^{U+\psi}$ as well.

4.1.4 Examples of forcing constructions

As an illustration, we define two well-known forcing constructions – Cohen forcing and Sacks forcing. The details, proofs and many other forcing constructions can be found in [6] or [5].

We define Cohen forcing for adding a new subset of ω. There are many variants of Cohen forcing; the original forcing notion used by Cohen adds ω_2 distinct subsets of ω. We define the forcing notion for Cohen forcing as follows: $\mathbb{P}$ is the collection of all functions f such that $\mathrm{dom}(f)$ is a finite subset of ω and $\mathrm{rng}(f) \subseteq \{0, 1\}$; the ordering $\leq$ is reverse inclusion; the maximal element $\mathbb{1}$ is $\emptyset$. If $\mathbf{G}$ is generic, then $\bigcup \mathbf{G}$ is the characteristic function of a new subset of ω. This subset is called a Cohen generic real.

We say that $t \subseteq {}^{<\omega}2$ is a tree if t is closed under initial segments. A nonempty tree t is called perfect if for every $a \in t$ there exists $b \supseteq a$ such that both $b^\frown 0$ and $b^\frown 1$ are in t. The forcing notion for Sacks forcing is defined as follows: $\mathbb{P}$ is the set of all perfect trees; the ordering $\leq$ is inclusion; $\mathbb{1}$ is the full binary tree. Like the Cohen forcing, Sacks forcing adds a new subset of ω – a Sacks real.[4]

4.2 Forcing over the universe

In this section, we show another approach to forcing. We define a Boolean universe $V^{\mathcal{B}}$ and show how to assing Boolean values to formulas – we construct a Boolean-valued model. We then show how this Boolean-valued model can be used to define the usual two-valued model and a corresponding interpretation. The

[3] This result may be found in [6] as Lemma 4.4.

[4] Sacks real is a real with minimal degree of constructibility. Definition and details are in [5, p. 244].

presentation of this approach closely follows Jech's book [5] with one important difference. We do not use generic filters and do not define a generic extension. Instead, we show how to collapse a Boolean-valued model to a two-valued model by any ultrafilter.

In the whole section, we work in ZFC.

In the next, $\mathcal{B}$ denotes a complete Boolean algebra. The role of $\mathcal{B}$ is similar to the role of a partial ordering $\mathbb{P}$ in the previous section. The reason for starting with a complete Boolean algebra insted of an arbitrary partial ordering is that the existence of suprema and infima allows us to define Boolean values of formulas (Definition 38). Neverheless, this is not a crucial restriction as for every forcing notion $\mathbb{P}$ there exists a Boolean algebra $\mathcal{B}$ such that $\mathbb{P}$ and $\mathcal{B}$ produce the same generic extensions.[5]

Definition 37. *The Boolean universe $V^{\mathcal{B}}$ is defined by induction:*

1. $V_0^{\mathcal{B}} = \emptyset$,
2. $V_{\alpha+1}^{\mathcal{B}} = \{f; f \text{ is a function } \wedge \operatorname{dom}(f) \subseteq V_\alpha^{\mathcal{B}} \wedge \operatorname{rng}(f) \subseteq \mathcal{B}\}$,
3. $V_\beta^{\mathcal{B}} = \bigcup_{\alpha<\beta} V_\alpha^{\mathcal{B}}$, *if β is a limit ordinal,*
4. $V^{\mathcal{B}} = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha^{\mathcal{B}}$.

We assign to every $b \in V^{\mathcal{B}}$ its rank; $\operatorname{rank}(b)$ is the least α such that $b \in V_{\alpha+1}^{\mathcal{B}}$. If $x, y \in V^{\mathcal{B}}$, then x, y are, in fact, functions. Therefore the notation $x(t)$ makes sense for $t \in \operatorname{dom}(x)$.

Definition 38. *By induction on the complexity of formulas, we define, for each formula $\varphi(v_1, \ldots, v_n)$ and each $x_1, \ldots, x_n \in V^{\mathcal{B}}$, the Boolean value $\|\varphi(x_1, \ldots, x_n)\|$.*

The definition for atomic formulas proceeds by induction on pairs of ranks under the canonical well-ordering:

$$\|x \in y\| = \sum_{t\in\operatorname{dom}(y)} (\|x = t\| \cdot y(t)),$$

$$\|x \subseteq y\| = \prod_{t\in\operatorname{dom}(x)} (-x(t) + \|t \in y\|),$$

$$\|x = y\| = \|x \subseteq y\| \cdot \|y \subseteq x\|.$$

[5]The proof can be found in [5] as Lemma 14.13.

For non-atomic formulas we define:

$$\|\neg\varphi(x_1,\dots,x_n)\| = -\|\varphi(x_1,\dots,x_n)\|,$$
$$\|(\varphi\wedge\psi)(x_1,\dots,x_n)\| = \|\varphi(x_1,\dots,x_n)\|\cdot\|\psi(x_1,\dots,x_n)\|,$$
$$\|(\varphi\vee\psi)(x_1,\dots,x_n)\| = \|\varphi(x_1,\dots,x_n)\|+\|\psi(x_1,\dots,x_n)\|,$$
$$\|(\varphi\rightarrow\psi)(x_1,\dots,x_n)\| = \|(\neg\varphi\vee\psi)(x_1,\dots,x_n)\|,$$
$$\|\exists y\varphi(y,x_1,\dots,x_n)\| = \sum_{x\in V^{\mathcal{B}}}\|\varphi(x,x_1,\dots,x_n)\|,$$
$$\|\forall y\varphi(y,x_1,\dots,x_n)\| = \prod_{x\in V^{\mathcal{B}}}\|\varphi(x,x_1,\dots,x_n)\|.$$

Formally, we do not define a function mapping formulas to their Boolean values. We in fact define countably many functions, one for each formula, where the arity of the function corresponds to the number of free variables in the formula. E.g. for the formula $u\in v$, we define the function f_ϵ. The notation $\|x\in y\|$ stands for $f_\epsilon(x,y)$.

Let us define the Boolean operation $\Rightarrow$ as follows: let $a,b\in\mathcal{B}$, then $a\Rightarrow b = -a+b$. This allows us to write for the implication $\|\varphi\rightarrow\psi\| = \|\varphi\|\Rightarrow\|\psi\|$. Note that $\|\varphi\rightarrow\psi\|=1$ iff $\|\varphi\|\leq\|\psi\|$.

Lemma 39. *The following is provable in* ZFC*:*

a) $\|x=x\|=1$,

b) $\|x=y\|=\|y=x\|$,

c) $\|x=y\|\cdot\|y=z\|\leq\|x=z\|$,

d) $\|x\in y\|\cdot\|v=x\|\cdot\|w=y\|\leq\|v\in w\|$.

Proof.

a) By induction on $\operatorname{rank}(x)$. It suffices to show $\|x\subseteq x\|=1$. Fix $t\in\operatorname{dom}(x)$; by the induction hypothesis, $\|t=t\|=1$. Then, by definition of $\|t\in x\|$, $x(t)\leq\|t\in x\|$. So $x(t)\Rightarrow\|t\in x\|=1$ and $\|x\subseteq x\|=1$ follows by definition.

b) Obvious.

We prove c) and d) together. Note that for d) it suffices to prove that $\|x \in y\| \cdot \|v = x\| \leq \|v \in y\|$ and $\|v \in y\| \cdot \|w = y\| \leq \|v \in w\|$. If we rename the variables it means to show that $\|x \in y\| \cdot \|x = z\| \leq \|z \in y\|$ and $\|y \in x\| \cdot \|x = z\| \leq \|y \in z\|$. So we prove these two inequalities and c) simultaneously by induction on triples of ranks.

For c), it suffices to prove that $\|x \subseteq y\| \cdot \|y = z\| \leq \|x \subseteq z\|$ and $\|z \subseteq y\| \cdot \|x = y\| \leq \|z \subseteq x\|$. We prove the first, the second is a just renaming of the first. Let us fix $t \in \mathrm{dom}(x)$; we want to show that

$$(x(t) \Rightarrow \|t \in y\|) \cdot \|y = z\| \leq x(t) \Rightarrow \|t \in z\|.$$

The following holds by the induction hypothesis:

$$-x(t) + (\|t \in y\| \cdot \|y = z\|) \leq -x(t) + \|t \in z\|.$$

The fact that $\|y = z\| \cdot (-x(t) + \|t \in y\|) \leq -x(t) + (\|t \in y\| \cdot \|y = z\|)$ finishes the proof.

The proof of $\|x \in y\| \cdot \|x = z\| \leq \|z \in y\|$ and $\|y \in x\| \cdot \|x = z\| \leq \|y \in z\|$ is similar. Again, we rewrite the inequalities using Definition 38 and use the induction hypothesis.

□

Lemma 39 shows that all the axioms of equality have Boolean value 1. It is easy to see that all the other axioms of the predicate calculus have value 1 as well. If a formula $\varphi(\bar{x})$ has value 1 for all possible $\bar{x} \in V^{\mathcal{B}}$, then the same is true for $\psi(\bar{x})$ derived from it by some rule of the calculus. Thus if φ is provable from $\alpha_1, \ldots, \alpha_n$ and $\|\alpha_i\| = 1$ whenever $1 \leq i \leq n$, then $\|\varphi\| = 1$.

Lemma 40. *Let $\varphi(v_0, \ldots, v_n)$ be a formula, $x, a_1, \ldots, a_n \in V^{\mathcal{B}}$. Then*

$$\|(\exists y \in x)\varphi(y, a_1, \ldots, a_n)\| = \sum_{y \in \mathrm{dom}(x)} (x(y) \cdot \|\varphi(y, a_1, \ldots, a_n)\|),$$

$$\|(\forall y \in x)\varphi(y, a_1, \ldots, a_n)\| = \prod_{y \in \mathrm{dom}(x)} (x(y) \Rightarrow \|\varphi(y, a_1, \ldots, a_n)\|).$$

Proof. We prove the first equality, the proof of the second is similar. By definition

$$\|(\exists y \in x)\varphi(y, \bar{a})\| = \|\exists y(y \in x \wedge \varphi(y, \bar{a}))\| = \sum_{t \in V^{\mathcal{B}}} (\|t \in x\| \cdot \|\varphi(t, \bar{a})\|).$$

As

$$\|t \in x\| = \sum_{y \in \mathrm{dom}(x)} (\|t = y\| \cdot x(y));$$

it holds that

$$\sum_{t \in V^{\mathcal{B}}} (\|t \in x\| \cdot \|\varphi(t, \bar{a})\|) \geq \sum_{y \in \mathrm{dom}(x)} (x(y) \cdot \|\varphi(y, \bar{a})\|).$$

To conclude the proof, we show that the inequality $\leq$ holds as well. Let us fix $t \in V^{\mathcal{B}}$. We want to show that

$$\|t \in x\| \cdot \|\varphi(t, \bar{a})\| \leq \sum_{y \in \mathrm{dom}(x)} (x(y) \cdot \|\varphi(y, \bar{a})\|).$$

Since the formula $(t = y \wedge \varphi(t, \bar{a})) \rightarrow \varphi(y, \bar{a})$ is provable, we have

$$\|t = y\| \cdot \|\varphi(t, \bar{a})\| \leq \|\varphi(y, \bar{a})\|,$$

$$\|t = y\| \cdot x(y) \cdot \|\varphi(t, \bar{a})\| \leq \|\varphi(y, \bar{a})\| \cdot x(y),$$

$$\sum_{y \in \mathrm{dom}(x)} (\|t = y\| \cdot x(y)) \cdot \|\varphi(t, \bar{a})\| \leq \sum_{y \in \mathrm{dom}(x)} (\|\varphi(y, \bar{a})\| \cdot x(y)).$$

□

Definition 41. *If x is a set, then it has the canonical name $\check{x} \in V^{\mathcal{B}}$. Canonical names are defined by $\in$-induction.*

1. $\check{\emptyset} = \emptyset$,
2. $\mathrm{dom}(\check{x}) = \{\check{y}; y \in x\}$ *and* $\check{x}(t) = 1$ *for all* $t \in \mathrm{dom}(\check{x})$.

Let $\varphi(x_1, \ldots, x_n)$ be a Δ_0-formula; then, by induction on the complexity of φ, we can prove that $\varphi(x_1, \ldots, x_n)$ is equivalent to $\|\varphi(\check{x}_1, \ldots, \check{x}_n)\| = 1$. Thus if $\varphi(x_1, \ldots, x_n)$ is Σ_1, then $\varphi(x_1, \ldots, x_n)$ implies $\|\varphi(\check{x}_1, \ldots, \check{x}_n)\| = 1$.

Theorem 42. *$V^{\mathcal{B}}$ is a Boolean-valued model of* ZFC, *i.e. $\|\varphi\| = 1$ for every axiom φ of* ZFC.

Proof.

- Extensionality: We want to show that $\|\forall x(x \in y \leftrightarrow x \in z)\| \leq \|y = z\|$. We first prove that $\|\forall x(x \in y \rightarrow x \in z)\| \leq \|y \subseteq z\|$. By definition

$$\|\forall x(x \in y \rightarrow x \in z)\| = \prod_{t \in V^{\mathcal{B}}} (\|t \in y\| \Rightarrow \|t \in z\|)$$

and

$$\|y \subseteq z\| = \prod_{t \in \mathrm{dom}(y)} (y(t) \Rightarrow \|t \in z\|),$$

so it suffices to show

$$\prod_{t \in V^{\mathcal{B}}} (\|t \in y\| \Rightarrow \|t \in z\|) \leq \prod_{t \in \mathrm{dom}(y)} (y(t) \Rightarrow \|t \in z\|).$$

But this holds since for every $t \in \mathrm{dom}(y)$ we have $\|t \in y\| \geq y(t)$. The proof of $\|\forall x(x \in z \to x \in y)\| \leq \|z \subseteq y\|$ is similar. We may then conclude $\|\forall x(x \in y \leftrightarrow x \in z)\| \leq \|y = z\|$.

- Foundation: Note that for any $a, b, c \in \mathcal{B}$, $a \Rightarrow c \geq b$ implies $a \cdot b \leq c$. Let $x \in V^{\mathcal{B}}$. We prove that

$$\|\exists y(y \in x) \to (\exists y \in x)(\forall z \in y)(z \notin x)\| = 1.$$

We proceed by contradiction. So let $b \in \mathcal{B}$ be such that

$$\|\exists y(y \in x) \wedge (\forall y \in x)(\exists z \in y)(z \in x)\| = b \neq 0.$$

The fact that $\|\exists y(y \in x)\| \geq b$ implies that there exists $y \in V^{\mathcal{B}}$ such that $\|y \in x\| \cdot b \neq 0$. Let us fix one such y with the least possible rank. Since $\|y \in x\| \Rightarrow \|(\exists z \in y)(z \in x)\| \geq b$, we have $\|y \in x\| \cdot b \leq \|(\exists z \in y)(z \in x)\|$. Thus $\|(\exists z \in y)(z \in x)\| \cdot b \neq 0$, so there exists $z \in \mathrm{dom}(y)$ such that $\|z \in x\| \cdot b \neq 0$. Since $\mathrm{rank}(z) < \mathrm{rank}(y)$, we reached a contradiction.

- Pairing: Let $a, b \in V^{\mathcal{B}}$. Let c be such that $\mathrm{dom}(c) = \{a, b\}$ and $c(a) = c(b) = 1$. It is easy to verify that $\|a \in c \wedge b \in c \wedge (\forall x \in c)(x = a \vee x = b)\| = 1$.

- Separation: Let $x \in V^{\mathcal{B}}$, $\varphi(v)$ be a formula, possibly with parameters from $V^{\mathcal{B}}$. We want to find some $y \in V^{\mathcal{B}}$ such that $\|y \subseteq x\| = 1$ and $\|(\forall z \in x)(\varphi(z) \leftrightarrow z \in y)\| = 1$. It suffices to define y as follows: $\mathrm{dom}(y) = \mathrm{dom}(x)$ and $y(t) = x(t) \cdot \|\varphi(t)\|$ for every $t \in \mathrm{dom}(y)$.

- Union: We show that for every $x \in V^{\mathcal{B}}$ there is $y \in V^{\mathcal{B}}$ such that $\|(\forall a \in x)(\forall b \in a)(b \in y)\| = 1$. This, together with separation, implies that the axiom of union has value one. For a fixed x we define y as follows: $\mathrm{dom}(y) = \bigcup\{\mathrm{dom}(a); a \in \mathrm{dom}(x)\}$ and $y(t) = 1$ for every $t \in \mathrm{dom}(y)$. It is easy to check that y satisfies the equality above.

- Power set: As above, we prove a weak version of the power set axiom. The usual version is implied by the result for the axiom of separation. Hence, we want to show that for every $x \in V^{\mathcal{B}}$ there exists $y \in V^{\mathcal{B}}$ such that $\|\forall a(a \subseteq x \to a \in y)\| = 1$. Fix x and define

$$\mathrm{dom}(y) = \{a \in V^{\mathcal{B}}; \mathrm{dom}(a) = \mathrm{dom}(x) \wedge (\forall t \in \mathrm{dom}(a))(a(t) \leq x(t))\}$$

and $y(a) = 1$, for every $a \in \mathrm{dom}(y)$. We want to check that for every $b \in V^{\mathcal{B}}$ it holds that $\|b \subseteq x\| \leq \|b \in y\|$. So fix $b \in V^{\mathcal{B}}$ and define a so that $\mathrm{dom}(a) = \mathrm{dom}(x)$ and $a(t) = x(t) \cdot \|t \in b\|$. Obviously, $a \in \mathrm{dom}(y)$ and $\|a \in y\| = 1$. Therefore, by Lemma 39, $\|a = b\| \leq \|b \in y\|$. To conclude the proof, it suffices to show that $\|b \subseteq x\| \leq \|a = b\|$.

It is easy to check that $\|a \subseteq b\| = 1$. We show that $\|b \subseteq x\| \leq \|b \subseteq a\|$ by proving $\|(\forall v \in b)(v \in x \to v \in a)\| = 1$. Let us fix $t \in \mathrm{dom}(b)$; we show that $b(t) \Rightarrow (\|t \in x\| \Rightarrow \|t \in a\|) = 1$ or equivalently

$$b(t) \cdot \|t \in x\| \leq \|t \in a\|.$$

This can be written as

$$b(t) \cdot \sum_{s \in \mathrm{dom}(x)} x(s) \cdot \|s = t\| \leq \sum_{s \in \mathrm{dom}(a)} a(s) \cdot \|s = t\|,$$

$$\sum_{s \in \mathrm{dom}(x)} b(t) \cdot x(s) \cdot \|s = t\| \leq \sum_{s \in \mathrm{dom}(a)} x(s) \cdot \|s \in b\| \cdot \|s = t\|.$$

The last inequality holds since $\mathrm{dom}(x) = \mathrm{dom}(a)$ and, by definition of $\|s \in b\|$, $b(t) \cdot \|s = t\| \leq \|s \in b\|$.

- Infinity: It is easy to check that $\|\check{\omega}$ is an inductive set$\| = 1$.

- Replacement: We show a stronger statement, namely the collection principle – let $\varphi(u, v)$ be a formula, possibly with parameters form $V^{\mathcal{B}}$; then for every $x \in V^{\mathcal{B}}$ there exists $y \in V^{\mathcal{B}}$ such that

$$\|(\forall u \in x)(\exists v \varphi(u, v) \to (\exists v \in y)\varphi(u, v))\| = 1.$$

For a fixed u the value $\|\exists v \varphi(u, v)\|$ is defined as the supremum of values $\|\varphi(u, v)\|$ for all $v \in V^{\mathcal{B}}$. Note that we can find a set $S_u \subseteq V^{\mathcal{B}}$ such that the supremum over $V^{\mathcal{B}}$ equals the supremum over S_u. So the following holds:

$$\|\exists v \varphi(u, v)\| = \sum_{v \in V^{\mathcal{B}}} \|\varphi(u, v)\| = \sum_{v \in S_u} \|\varphi(u, v)\|.$$

So it suffices to define $\mathrm{dom}(y) = \bigcup\{S_u; u \in \mathrm{dom}(x)\}$ and $y(t) = 1$ for every $t \in \mathrm{dom}(y)$.

- Choice: First, note that if α is an ordinal, then $\|\check{\alpha}$ is an ordinal$\| = 1$. This is because the axiom of foundation allows us to express the property of being an ordinal by a Δ_0-formula.

  Second, let x be a set, $\alpha \in \mathrm{Ord}$ and f a bijection between x and α. Then

  $$\|\check{f} \text{ is a bijection between } \check{x} \text{ and } \check{\alpha}\| = 1.$$

  Together it gives us that $\|\check{x}$ can be well-ordered$\| = 1$, for all x. To conclude the proof it suffices to show that for any $y \in V^{\mathcal{B}}$ there is x and $f \in V^{\mathcal{B}}$ such that

  $$\|f \text{ is a function } \wedge \mathrm{dom}(f) = \check{x} \wedge \mathrm{rng}(f) \supseteq y\| = 1.$$

  Fix y. We define $x = \mathrm{dom}(y)$. Let $\langle\langle a, b\rangle\rangle$ denote an ordered pair in the sense of $V^{\mathcal{B}}$. We set $\mathrm{dom}(f) = \{\langle\langle \check{z}, z\rangle\rangle; z \in x\}$ and $f(t) = 1$ for all $t \in \mathrm{dom}(f)$. It is easy to check that x and f satisfy the equality above.

$\square$

Theorem 43. *Let $\varphi(x, x_1, \ldots, x_n)$ be a formula, $b_1, \ldots, b_n \in V^{\mathcal{B}}$. Then there is $b \in V^{\mathcal{B}}$ such that*

$$\|\varphi(b, b_1, \ldots, b_n)\| = \|\exists x \varphi(x, b_1, \ldots, b_n)\|.$$

We say that $V^{\mathcal{B}}$ is full.

Proof. For every $b \in V^{\mathcal{B}}$, it holds that $\|\varphi(b, b_1, \ldots, b_n)\| \leq \|\exists x \varphi(x, b_1, \ldots, b_n)\|$. We want to find $b \in V^{\mathcal{B}}$ such that $\|\varphi(b, b_1, \ldots, b_n)\| \geq \|\exists x \varphi(x, b_1, \ldots, b_n)\|$. In the next, we suppress mentioning of $b_1, \ldots, b_n$ for the sake of brevity. Let $\|\exists x \varphi(x)\| = v$. Note that there exists a set $S \subseteq V^{\mathcal{B}}$ such that

$$\sum_{s \in S} \|\varphi(s)\| = v.$$

Let $V = \{\|\varphi(s)\|; s \in S\}$; then $\sup(V) = v$. Moreover, we can find a set W such that its elements are pairwise disjoint, $\sup(W) = \sup(V) = v$ and $(\forall w \in W)(\exists s \in S)(\|\varphi(s)\| \geq w)$. We fix for every $w \in W$ one such s and denote it by s_w. We define b as follows: $\mathrm{dom}(b) = \bigcup\{\mathrm{dom}(s_w); w \in W\}$, and for every $t \in \mathrm{dom}(b)$ we set $b(t) = \sum\{s_w(t) \cdot w; w \in W \wedge t \in \mathrm{dom}(s_w)\}$. Let $t \in \mathrm{dom}(b)$, $w \in W$. If $t \in \mathrm{dom}(s_w)$, then $w \cdot b(t) = w \cdot s_w(t)$ since elements of W are pairwise disjoint. This implies $-b(t) + s_w(t) \geq w$. If $t \notin \mathrm{dom}(s_w)$, then $-b(t) \geq w$. Thus

$$\prod_{t \in \mathrm{dom}(b)} (b(t) \Rightarrow \|t \in s_w\|) \geq w,$$

or equivalently $\|b \subseteq s_w\| \geq w$. Similarly, we can show that $\|s_w \subseteq b\| \geq w$. So $\|b = s_w\| \geq w$ and, by the definition of s_w, $\|\varphi(s_w)\| \geq w$, which implies $\|\varphi(b)\| \geq w$. Since this can be proved for every $w \in W$, we have $\|\varphi(b)\| \geq v$. $\square$

Let us fix an ultrafilter F on $\mathcal{B}$. We use F to define a two-valued model – we collapse the Boolean values of formulas from $\mathcal{B}$ to $\{0,1\}$ according to F.

Note that the notation $\|x = y\|$ and $\|x \in y\|$ makes sense only for $x, y \in V^{\mathcal{B}}$. But we may always define $\|a = b\| = 0$ if $a \notin V^{\mathcal{B}}$ or $b \notin V^{\mathcal{B}}$.

We define a relation ε on $V^{\mathcal{B}}$ as follows: $\varepsilon(x, y)$ iff $\|x = y\| \in F$. By Lemma 39 and the fact that F is a filter, ε is an equivalence relation. Let $V^{\mathcal{B}}_{\varepsilon}$ be the factorization of $V^{\mathcal{B}}$ over ε and let $[x]$ denote the factor-class of x. Lemma 39 also ensures that the following definition of the relation E on $V^{\mathcal{B}}_{\varepsilon}$ is correct, i.e. it does not depend on the choice of representatives. For any $x, y \in V^{\mathcal{B}}$ we define that $[x]E[y]$ holds iff $\|x \in y\| \in F$. So $\mathcal{M}_F = (V^{\mathcal{B}}_{\varepsilon}, E)$ is a model of the language of ZFC.

Lemma 44. *Let $\mathcal{M}_F$ be defined as above. Then for any formula $\varphi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_n \in V^{\mathcal{B}}$ it holds that*

$$\mathcal{M}_F \models \varphi([a_1], \ldots, [a_n]) \Leftrightarrow \|\varphi(a_1, \ldots, a_n)\| \in F.$$

Proof. By induction on the complexity of φ.

For atomic formulas it holds by definition. Let $\varphi = \neg\psi$. Then $\mathcal{M}_F \models \varphi$ iff $\mathcal{M}_F \not\models \psi$ iff $\|\psi\| \notin F$ iff $-\|\psi\| \in F$ iff $\|\varphi\| \in F$. Similarly for other connectives.

If φ is $\exists x\psi(x)$, we use the fact that $V^{\mathcal{B}}$ is full. It holds that $\mathcal{M}_F \models \exists x\psi(x)$ iff $(\exists [a] \in V^{\mathcal{B}}_{\varepsilon})(\mathcal{M}_F \models \psi([a]))$ iff $(\exists a \in V^{\mathcal{B}})(\|\psi(a)\| \in F)$ iff $\|\exists x\psi(x)\| \in F$. The first equivalence is obvious, the second is the induction hypothesis and the third uses fullness of $V^{\mathcal{B}}$. □

By Theorem 42, all axioms of ZFC have the Boolean value 1. So

Corollary 45. *$\mathcal{M}_F$ is a model of* ZFC.

Note that the Boolean values of formulas are defined with respect to a given $\mathcal{B}$, thus we can write $\|\varphi\|_{\mathcal{B}}$ instead of $\|\varphi\|$ to make the dependence on $\mathcal{B}$ explicit.

Theorem 46. *Let φ be a sentence. If* ZFC *proves that there exists a complete Boolean algebra $\mathcal{B}$ such that $\|\varphi\|_{\mathcal{B}} \neq 0$, then there is a model-theoretical interpretation of* ZFC $+ \varphi$ *in* ZFC.

Proof. We define a translation $\tau[p_1, p_2]$ and show that $\tau[p_1, p_2]$ is in fact an interpretation. To define $\tau[p_1, p_2]$ means to specify formulas δ, ε and $\varphi_{\in}$. We want the formula δ to define the Boolean universe $V^{\mathcal{B}}$, but instead of $\mathcal{B}$ we have to use a parameter, so $\delta(x, p_1)$ is $x \in V^{p_1}$. The formula ε is defined as above, but, again, we replace F by the parameter, so $\varepsilon(x, y, p_1, p_2)$ is $\|x = y\|_{p_1} \in p_2$. Finally, $\varphi_{\in}$ mimics the relation E from above, so $\varphi_{\in}(x, y, p_1, p_2)$ is $\|x \in y\|_{p_1} \in p_2$.

Let $\mathcal{M}$ be a model of ZFC. Inside $\mathcal{M}$, fix $\mathcal{B}$ such that $\|\varphi\|_{\mathcal{B}} \neq 0$. There exists an ultrafilter F on $\mathcal{B}$ such that $\|\varphi\|_{\mathcal{B}} \in F$. We let $\mathcal{B}$, F be the values of parameters p_1, p_2. To show that $\tau[p_1, p_2]$ is an interpretation, we have to argue that the following formulas are satisfied in $\mathcal{M}$:

- $\exists x \delta(x, \mathcal{B})$,
- $\forall x(\delta(x, \mathcal{B}) \to \varepsilon(x, x, \mathcal{B}, F))$,
- $\forall x, y[\delta(x, \mathcal{B}) \wedge \delta(y, \mathcal{B}) \to (\varepsilon(x, y, \mathcal{B}, F) \to \varepsilon(y, x, \mathcal{B}, F))]$,
- $\forall x, y, z[\delta(x, \mathcal{B}) \wedge \delta(y, \mathcal{B}) \wedge \delta(z, \mathcal{B}) \to$
  $\to (\varepsilon(x, y, \mathcal{B}, F) \wedge \varepsilon(y, z, \mathcal{B}, F) \to \varepsilon(x, z, \mathcal{B}, F))]$,
- $\forall x_1, x_2, y_1, y_2[\delta(x_1, \mathcal{B}) \wedge \delta(y_1, \mathcal{B}) \wedge \delta(x_2, \mathcal{B}) \wedge \delta(y_2, \mathcal{B}) \wedge$
  $\wedge\, \varepsilon(x_1, y_1, \mathcal{B}, F) \wedge \varepsilon(x_2, y_2, \mathcal{B}, F) \to (\varphi_\epsilon(x_1, x_2, \mathcal{B}, F) \leftrightarrow \varphi_\epsilon(y_1, y_2, \mathcal{B}, F))]$,
- $\varphi^\tau(\mathcal{B}, F)$,
- $\psi^\tau(\mathcal{B}, F)$, for every axiom ψ of ZFC.

The first formula is $\exists x(x \in V^{\mathcal{B}})$ and so is obviously satisfied. The next four formulas are satisfied as a consequence of Lemma 39 and the fact that F is a filter.

Let $\phi(\bar{x})$ be a formula; we want to show that $\phi^\tau(\bar{a}, \mathcal{B}, F) \leftrightarrow \|\phi(\bar{a})\|_{\mathcal{B}} \in F$ for any $\bar{a} \in V^{\mathcal{B}}$. We proceed by induction, for brevity we suppress mentioning of paremeters $\bar{a}$. For atomic formulas, the equivalence follows immediately from the definition of τ. Let $\phi = \neg\psi$. Then, by the induction hypothesis, $\psi^\tau(\mathcal{B}, F) \leftrightarrow \|\psi\|_{\mathcal{B}} \in F$. So $(\neg\psi)^\tau(\mathcal{B}, F) \leftrightarrow \neg\psi^\tau(\mathcal{B}, F) \leftrightarrow \|\psi\|_{\mathcal{B}} \notin F \leftrightarrow \|\neg\psi\|_{\mathcal{B}} \in F$. In the case of the other logical connectives, the proof is similar. Finaly, let $\phi = \exists x\psi(x)$. Then $(\exists x\psi(x))^\tau(\mathcal{B}, F) = \exists x(x \in V^{\mathcal{B}} \wedge \psi^\tau(x, \mathcal{B}, F))$. By the induction hypothesis, $\exists x(x \in V^{\mathcal{B}} \wedge \psi^\tau(x, \mathcal{B}, F))$ implies $\exists x(x \in V^{\mathcal{B}} \wedge \|\psi(x)\|_{\mathcal{B}} \in F)$. As $\|\exists x\psi(x)\|_{\mathcal{B}} \geq \|\psi(x)\|_{\mathcal{B}}$, it follows that $\|\exists x\psi(x)\|_{\mathcal{B}} \in F$. For the other direction, suppose $\|\exists x\psi(x)\|_{\mathcal{B}} \in F$. By the fullness of $V^{\mathcal{B}}$, it holds that $\exists a(a \in V^{\mathcal{B}} \wedge \|\psi(a)\|_{\mathcal{B}} \in F)$. By the induction hypothesis, it implies $(\exists x\psi(x))^\tau(\mathcal{B}, F)$.

So $\varphi^\tau(\mathcal{B}, F)$ holds as F was chosen to contain the Boolean value of φ. If ψ is an axiom of ZFC, then $\psi^\tau(\mathcal{B}, F)$ holds since, by the Theorem 42, $\|\psi\|_{\mathcal{B}} = 1$. □

The next theorem shows that under the assumptions of the theorem above, there exists not only a model-theoretical interpretation but even a syntactical one.

Theorem 47. *Let φ be a sentence. If* ZFC *proves that there exists a complete Boolean algebra $\mathcal{B}$ such that $\|\varphi\|_{\mathcal{B}} \neq 0$, then there is a syntactical interpretation of* ZFC $+ \varphi$ *in* ZFC.

Proof. Let $\tau[p_1, p_2]$ be as in the proof of the previous theorem. By Definition 19, we have to find a suitable formula $\alpha(x_1, x_2)$. We take for $\alpha(x_1, x_2)$ the formula "x_1 is a complete Boolean algebra such that $\|\varphi\|_{x_1} \neq 0$ and x_2 is an ultrafilter on x_1 such that $\|\varphi\|_{x_1} \in x_2$".

Then ZFC $\vdash (\exists s_1, s_2)\alpha(s_1, s_2)$. The rest of the formulas demanded by Definition 19 to be provable in ZFC are provable as a consequence of the previous theorem. □

Note that, unlike in the previous section, the interpretation τ does not have absolute equality. But the remarks about the possibility of obtaining an interpretation without parameters can be repeated here as well. In applications of forcing we define a particular $\mathcal{B}$ to suit our needs, so we may use the definition of $\mathcal{B}$ instead of the parameter p_1. And if there exists a definable well-ordering of the universe, as in the case of the constructible universe, we may define F as the first suitable ultrafilter in this ordering and use the definition instead of p_2.

Chapter 5

Bi-interpretation

Let τ be an interpretation of a theory S in a theory T. Then, as we discussed in Chapter 3, this interpretation uniformly defines from any model $\mathcal{M}$ of T a model of S. Let us denote such a model by $\tau(\mathcal{M})$.

Definition 48 (Bi-interpretation)**.** *Let S, T be theories, τ an interpretation of S in T, σ an interpretation of T in S. The pair of interpretations $\langle \tau, \sigma \rangle$ is a bi-interpretation of S and T if*

- *There exists a formula $\varphi(x, y)$ in the language of theory T such that for any model $\mathcal{M} \models T$ the formula $\varphi(x, y)$ defines an isomorphism between $\mathcal{M}$ and $\sigma(\tau(\mathcal{M}))$.*
- *There exists a formula $\psi(x, y)$ in the language of theory S such that for any model $\mathcal{N} \models S$ the formula $\psi(x, y)$ defines an isomorphism between $\mathcal{N}$ and $\tau(\sigma(\mathcal{N}))$.*

Theories S, T are said to be bi-interpretable if there exists a bi-interpretation of S and T.

It seems that bi-interpretability is strictly stronger than mutual interpretability of two theories. We show that this is really the case by showing that ZFC and ZF are mutually interpretable, but not bi-interpretable.[1]

Theorem 49. *There exists an intepretation of* ZFC *in* ZF *and vice versa; in other words,* ZFC *and* ZF *are mutually interpretable.*

Proof. We describe the translations that give rise to the interpretations.
Clearly, ZF $\leq$ ZFC, as any model of ZFC is, by definition, a model of ZF. The translation here is trivial: $\delta(x)$ is $x = x$, $\varepsilon(x, y)$ is $x = y$ and $\varphi_\epsilon(x, y)$ is $x \in y$.

[1]The structure of this proof is from [2].

To show that ZFC $\leq$ ZF, we use the well known result that **L** is a model of ZFC. Thus the formula $\delta(x)$ in this case should restrict the universe to **L**. Therefore, the translation is as follows: $\delta(x)$ is $x \in \mathbf{L}$, $\varepsilon(x, y)$ is $x = y$ and $\varphi_\epsilon(x, y)$ is $x \in y$. □

To show that ZFC and ZF are not bi-interpretable, we first prove a general lemma about groups of automorphisms. Note that the next lemma talks about all automorphisms, not just definable ones.

Lemma 50. *Let S and T be bi-interpretable, τ, σ, φ, ψ being as in Definition 48. Let $\mathcal{M} \vDash T$, then the groups of automorphisms of $\mathcal{M}$ and $\tau(\mathcal{M})$ are isomorphic.*

Proof. We present a proof for the case when τ and σ have absolute equality. A proof for the general case is similar, only we have to involve factorizations at several places in the proof. This is not problematic as these factorizations are definable by formulas, it only makes the notation in the proof more complicated.

First, let us observe that if a is an automorphism of $\mathcal{M}$ and $\bar{a}$ is its restriction to the universe of $\tau(\mathcal{M})$, then $\bar{a}$ is an automorphism of $\tau(\mathcal{M})$. This is simply because $\tau(\mathcal{M})$ is defined by T-formulas. Still using the notation $\bar{a}$ for the restriction of a, we have a function i mapping automorphisms of $\mathcal{M}$ to automorphisms of $\tau(\mathcal{M})$ so that $i(a) = \bar{a}$. We want to show that i is the isomorphism we look for.

Clearly, if e is identity, then $\bar{e}$ is identity as well, so i preserves the identity element. Similarly, if $a \circ b = e$, then $\bar{a} \circ \bar{b} = \bar{e}$, so i preserves inverses as well. Finaly, $\overline{a \circ b} = \bar{a} \circ \bar{b}$.

To conclude that i is really an isomorphism, we have to show that i is a bijection. For injectivity, we prove that if $a \neq b$, then $\bar{a} \neq \bar{b}$. So let us fix x, y_1, y_2 so that $a(x) = y_1$, $b(x) = y_2$ and $y_1 \neq y_2$. The formula $\varphi(x, y)$ defines an isomorphism between $\mathcal{M}$ and $\sigma(\tau(\mathcal{M}))$, so for every $u \in \mathcal{M}$ there is a unique $\tilde{u} \in \sigma(\tau(\mathcal{M}))$ such that $\varphi(u, \tilde{u})$ holds in $\mathcal{M}$; we use the tilde-notation for this unique element in the next. Since, by definition, $\sigma(\tau(\mathcal{M})) \subseteq \tau(\mathcal{M})$, it holds that $\tilde{x}, \tilde{y_1}, \tilde{y_2} \in \tau(\mathcal{M})$. Therefore, it suffices to show that $a(\tilde{x}) = \tilde{y_1}$, $b(\tilde{x}) = \tilde{y_2}$ and $\tilde{y_1} \neq \tilde{y_2}$. It holds in $\mathcal{M}$ that $\varphi(x, \tilde{x})$ and a is an automorphism, so $\varphi(a(x), a(\tilde{x}))$ holds as well. We have $a(x) = y_1$, so also $\varphi(y_1, a(\tilde{x}))$ holds, and $\tilde{y_1}$ is the unique element such that $\varphi(y_1, \tilde{y_1})$, so $a(\tilde{x}) = \tilde{y_1}$. The proof of $b(\tilde{x}) = \tilde{y_2}$ is similar. Finally, $y_1 \neq y_2$ and φ defines an isomorphism, so $\tilde{y_1} \neq \tilde{y_2}$.

For surjectivity, fix an automorphism b of $\tau(\mathcal{M})$; we want to find an automorphism a of $\mathcal{M}$ such that $\bar{a} = b$. We write $\bar{b}$ for a restriction of b to $\sigma(\tau(\mathcal{M}))$. There is a natural way how to define an automorphism of $\mathcal{M}$ from $\bar{b}$ using $\varphi(x, y)$ – namely take the "φ-isomorphic image" of b. We denote such an automorphism $\bar{b}_\varphi$; it holds that $\bar{b}_\varphi(x) = y$ iff $\bar{b}(\tilde{x}) = \tilde{y}$. We show that $\bar{b}_\varphi$ is the a we look for.

Fix $u, v \in \tau(\mathcal{M})$ so that $b(u) = v$. It suffices to show that $\bar{b}_\varphi(u) = v$. From $b(u) = v$ we get $b(\tilde{u}) = \tilde{v}$ as in the argument for injectivity. As $\tilde{u}, \tilde{v} \in \sigma(\tau(\mathcal{M}))$, we have $\bar{b}(\tilde{u}) = \tilde{v}$. From the definition of $\bar{b}_\varphi$, we get $\bar{b}_\varphi(u) = v$ concluding the argument. □

Let us recall that an automorphism a has order two iff a is not the identity, but $a \circ a$ is.

Theorem 51. *No model of* ZFC *admits an automorphism of order two.*

Proof. Let us fix $\mathcal{M} \models$ ZFC.

First, we show that any automorphism a of $\mathcal{M}$ of order two preserves ordinals; i.e. $a(\alpha) = \alpha$ for any $\alpha \in$ Ord. If $a(\alpha) = \beta$, then $a(\beta) = \alpha$, as a is of order two. Suppose $\alpha < \beta$, then also $a(\alpha) < a(\beta)$, so $\alpha < \beta < \alpha$, which is a contradiction. Similarly, it may not happen that $\alpha > \beta$.

Now, we show that any automorphism of $\mathcal{M}$ preserving ordinals is the identity. Let us fix $x \in \mathcal{M}$. We write $\mathrm{trcl}(x)$ for the transitive closure of x. By the axiom of choice, there exists a function f and an ordinal α such that f is a bijection between α and $\mathrm{trcl}(x)$. Let $X = \{\langle u, v\rangle; u, v \in \mathrm{trcl}(x) \wedge u \in v\}$ and $A = \{\langle \beta, \gamma\rangle; \beta, \gamma \in \alpha \wedge \langle f(\beta), f(\gamma)\rangle \in X\}$. The relation A is well-founded, set-like and extensional on α because X is. Let C be the Mostowski collapsing function of α, A; the relation A was defined straightforwadly to satisfy $C(f^{-1}(y)) = y$ for any $y \in \mathrm{trcl}(x)$. This fact is expressible by a ZFC-formula; so fix $\varphi(x, y, z, v)$ meaning "the Mostowski collapsing function of x, y maps z to v". In particular, $\varphi(\alpha, A, f^{-1}(x), x)$ holds in $\mathcal{M}$. Let a be an automorphism of $\mathcal{M}$ preserving ordinals, then also $\varphi(a(\alpha), a(A), a(f^{-1}(x)), a(x))$ holds. As $\alpha, f^{-1}(x) \in$ Ord, $A \subseteq \alpha \times \alpha$ and a preserves ordinals, we have $a(\alpha) = \alpha$, $a(A) = A$, $a(f^{-1}(x)) = f^{-1}(x)$. Combining $\varphi(\alpha, A, f^{-1}(x), x)$ and $\varphi(\alpha, A, f^{-1}(x), a(x))$, we get $a(x) = x$. □

Theorem 52. ZFC *and* ZF *are not bi-interpretable.*

Proof. In the next section, we show that there is a model of ZF that admits an automorphism of order two. Let $\mathcal{M}$ be such a model and a be an automorphism of $\mathcal{M}$ of order two. Suppose, for contradiction, that $\langle \tau, \sigma\rangle$ is a bi-interpretation of ZFC and ZF. By Lemma 50, the groups of automorphisms of $\mathcal{M}$ and $\tau(\mathcal{M})$ are isomorphic. Clearly, automorphisms of order two are mapped by any isomorphism on automorphisms of order two. Moreover, $\tau(\mathcal{M})$ is a model of ZFC, so the isomorphic image of a is its automorphism of order two. This contradicts Theorem 51. □

5.1 Model of ZF with an automorphism of order two

In this section, we construct a model of ZF that admits an automorphism of order two. A construction of such a model is sketched in Cohen's article [1], but the details of the construction are missing. Yet, it is the details that are insteresting as they involve an interplay between a non-standard model of ZFC and a metamathematical level of the construction. We develop a method that allows such an interplay during the construction. As far as we know, this method is original.

As we have already mentioned, the usual approach to forcing starts with a countable transitive model, the ground model. In the following construction, the ground model M will be non-standard; more precisely, it will be a model with non-standard ω. The problem here is to choose the point of view from which M is non-standard. In the forcing construction, we have to step outside of M to choose the generic filter, and this step outside deserves some preliminary comments. We could step outside from M to our real, metamathematical world, from which we know M is non-standard. Further on, we call this metamathematical world external. We say we work externally to express that we work in the external world. So we could choose a generic filter $\mathbf{G}$ externally. But problems would soon arise as how to construct the generic extension. M-names are defined by transfinite induction inside M and their $\mathbf{G}$-interpretations should take place in a model which admits such an induction; in particular, this model should have the same ω as M. The solution to this problem will be similar to our approach to forcing in Section 4.1. We will, in fact, start with V $\vDash$ ZFC such that there is M $\in$ V, model of ZFC, countable and transitive in the sense of V. Inside V, we use forcing over M as usual.

5.1.1 Models M, V and forcing conditions

Theory ZFC$^+$ is a theory in the language $\mathcal{L} = \{\in, M, n\}$, where $\in$ is a binary relation symbol and M, n are constant symbols. Axioms of ZFC$^+$ are:

- all axioms of ZFC,
- $n \in \omega$,
- $n > i$, for each $i \in \mathbb{N}$,
- ϕ^M, for all ϕ axioms of ZFC,
- "M is transitive and countable".

Note that, in this definition, $\mathbb{N}$ stands for the external, i.e. standard, natural numbers while ω is, as usual, defined inside the theory.

If ZFC is consistent, then by the reflection principle every finite subtheory of ZFC^+ is consistent, and by compactness theorem ZFC^+ is consistent. For the rest of the construction, let us fix an (externally) countable model of ZFC^+. We denote this fixed model by V and write M, n for the realizations of symbols M, n inside V.

By Lemma 11, axioms of ZFC^+ ensure that M is a countable transitive model of ZFC. We know that ω is absolute for transitive models, so $\omega^{\mathrm{V}} = \omega^{\mathrm{M}}$ and, from transitivity, $\mathrm{n} \in \mathrm{M}$. M has the same non-standard natural numbers as V. It is crucial to realize that from the point of view of V and M, n is not non-standard. Non-standardness is an external notion. Therefore, if we say that we use forcing over a non-standard model, it is an external statement. From the point of view of V and M, where the whole forcing construction takes place, there are no non-standard numbers.

In the rest of this section, we work mainly inside M. Our goal is to add ω-many new sets of each rank from ω to $\omega+\mathrm{n}$. To do so, we define a suitable forcing notion.

Definition 53 (Atomic condition, forcing notion $\mathbb{P}$)**.**

Atomic conditions are:

- $\{a, \langle i, m\rangle\}$, *where* $a \in \mathrm{M}_\omega$, $i \in \omega$, $m \in \{0,1\}$,
- $\{\langle \alpha, j\rangle, \langle i, m\rangle\}$, *where* $\alpha = \omega + l$ *for some* $1 \leq l \leq \mathrm{n}$, $j, i \in \omega$, $m \in \{0,1\}$.

We say that a set p *of atomic conditions is consistent if there are no* $c_1, c_2 \in p$ *such that* $c_1 = \{\langle \alpha, j\rangle, \langle i, 0\rangle\}$ *(resp.* $c_1 = \{a, \langle i, 0\rangle\}$*) and* $c_2 = \{\langle \alpha, j\rangle, \langle i, 1\rangle\}$ *(resp.* $c_2 = \{a, \langle i, 1\rangle\}$*).*

$\mathbb{P}$ *is the collection of all finite consistent sets of atomic conditions. The ordering of* $\mathbb{P}$ *is by reverse inclusion.*

Note that an atomic condition is not, in fact, a condition of $\mathbb{P}$, nevertheless if c is an atomic condition, then $\{c\}$ is a $\mathbb{P}$-condition. So we identify atomic conditions with their singletons where necessary.

Notation 54 ($x^j_\alpha \in x^i_{\alpha+1}$)**.**

We write $x^j_\alpha \in x^i_{\alpha+1}$ *(resp.* $x^j_\alpha \notin x^i_{\alpha+1}$*) instead of* $\{\langle \alpha, j\rangle, \langle i, 1\rangle\}$ *(resp.* $\{\langle \alpha, j\rangle, \langle i, 0\rangle\}$*) and* $a \in x^i_\omega$ *(resp.* $a \notin x^i_\omega$*) instead of* $\{a, \langle i, 1\rangle\}$ *(resp.* $\{a, \langle i, 0\rangle\}$*.*

The obvious abuse of language present in this notation should be justified by Lemma 56.

Definition 55 (Names $\hat{x}^i_\omega, \hat{x}^i_\alpha$)**.** *For $a \in \mathrm{M}$, let $\check{a}$ be the canonical name of a. We set $\hat{x}^i_\omega = \{\langle\check{a}, \{a \in x^i_\omega\}\rangle\,; a \in \mathrm{M}_\omega\}$. By induction, we define $\hat{x}^i_{\alpha+1} = \{\langle\hat{x}^j_\alpha, \{x^j_\alpha \in x^i_{\alpha+1}\}\rangle\,; j \in \omega\}$, for $\omega \leq \alpha < \omega + \mathrm{n}$.*

Lemma 56. *Let $\boldsymbol{G}$ be a generic filter for $\mathbb{P}$, $\mathrm{M}[\boldsymbol{G}]$ the generic extension of M. Then the following holds:*

1. *Let $i \in \omega$. Then $\hat{x}^i_\omega$ is a name and $\hat{x}^i_{\omega\boldsymbol{G}}$ has rank ω. For all $j \neq i$, it holds that $\hat{x}^i_{\omega\boldsymbol{G}} \neq \hat{x}^j_{\omega\boldsymbol{G}}$, and for all $a \in \mathrm{M}$, it holds that $\hat{x}^i_{\omega\boldsymbol{G}} \neq \check{a}_{\boldsymbol{G}}$. Moreover, $\hat{x}^i_{\omega\boldsymbol{G}} = \{a;$ condition $a \in x^i_\omega$ is in $\bigcup \boldsymbol{G}\}$.*

2. *Let $i \in \omega$, $\omega + 1 \leq \alpha \leq \omega + \mathrm{n}$. Then $\hat{x}^i_\alpha$ is a name and $\hat{x}^i_{\alpha\boldsymbol{G}}$ has rank α. For all $j \neq i$, it holds that $\hat{x}^i_{\alpha\boldsymbol{G}} \neq \hat{x}^j_{\alpha\boldsymbol{G}}$, and for all $a \in \mathrm{M}$, it holds that $\hat{x}^i_{\alpha\boldsymbol{G}} \neq a$. Moreover, $\hat{x}^i_{\alpha\boldsymbol{G}} = \{\hat{x}^j_{\alpha-1\boldsymbol{G}};$ condition $x^j_{\alpha-1} \in x^i_\alpha$ is in $\bigcup \boldsymbol{G}\}$.*

Proof.

1. $\hat{x}^i_\omega$ is a name by definition. Let us first show $\hat{x}^i_{\omega\mathbf{G}} = \{a;$ condition $a \in x^i_\omega$ is in $\bigcup\mathbf{G}\}$. It follows immidiately from the definition of $\hat{x}^i_\omega$ that $\hat{x}^i_{\omega\mathbf{G}} = \{a;$ $\{a \in x^i_\omega\}$ is in $\mathbf{G}\}$, so it suffices to show that for any atomic condition c: $c \in \bigcup\mathbf{G} \Leftrightarrow \{c\} \in \mathbf{G}$. The right-to-left direction is obvious. The opposite direction follows from the fact that if $c \in p \in \mathbf{G}$, then $\{c\} \geq_{\mathbb{P}} p$ and therefore $\{c\} \in \mathbf{G}$.

   $a \in x^i_\omega$ is an atomic condition only for $a \in \mathrm{M}_\omega$; from above, we can conclude $\hat{x}^i_{\omega\mathbf{G}} \subseteq \mathrm{M}_\omega = \mathrm{M}[\mathbf{G}]_\omega$. To prove $\mathrm{rk}(\hat{x}^i_{\omega\mathbf{G}}) = \omega$, we need to show that $\hat{x}^i_{\omega\mathbf{G}} \notin \mathrm{M}_\omega$. Let $a \in \mathrm{M}$, $a \subseteq \mathrm{M}_\omega$; we set $D_a = \{p \in \mathbb{P}; (\exists b \in \mathrm{M}_\omega)(b \in x^i_\omega$ is in $p \wedge b \notin a) \vee (b \in a \wedge b \notin x^i_\omega$ is in $p)\}$. D_a is dense and $\forall p \in D_a : p \Vdash (\check{a} \neq \hat{x}^i_\omega)$. It follows that $\hat{x}^i_{\omega\mathbf{G}} \neq \check{a}_{\mathbf{G}}$ and $\mathrm{rk}(\hat{x}^i_{\omega\mathbf{G}}) = \omega$.

   It is clear that for all $a \in \mathrm{M}$ it holds that $\hat{x}^i_{\omega\mathbf{G}} \neq \check{a}_{\mathbf{G}}$, as we have proved both $\hat{x}^i_{\omega\mathbf{G}} \subseteq \mathrm{M}_\omega$ and for all $a \in \mathrm{M}$: $(a \subseteq \mathrm{M}_\omega) \rightarrow (\hat{x}^i_{\omega\mathbf{G}} \neq \check{a}_{\mathbf{G}})$.

   We use a density argument to show that, for all $j \neq i$, it holds that $\hat{x}^i_{\omega\mathbf{G}} \neq \hat{x}^j_{\omega\mathbf{G}}$. For fixed $i \neq j$, let us set $D = \{p \in \mathbb{P}; (\exists a \in \mathrm{M}_\omega)(a \in x^i_\omega$ is in $p \wedge a \notin x^j_\omega$ is in $p)\}$. D is dense and all $p \in D$ force $\hat{x}^i_\omega \neq \hat{x}^j_\omega$.

2. Similar, by induction on l.

□

5.1.2 Permutation σ

Inside V, let us fix $\mathbf{G}$, a generic filter for $\mathbb{P}$. Our goal is to construct some automorphism of order two. Nevertheless, by Theorem 51, there is no such automorphism on $\mathrm{M}[\mathbf{G}]$, as $\mathrm{M}[\mathbf{G}] \models \mathrm{ZFC}$. The idea is to define a suitable permutation on some set X, construct a model M^* of ZF consisting of sets definable from X and then show how to extend the permutation to an automorphism of the model M^*.

In this section, we define σ, a permutation on

$$X = \{x^i_\alpha; \omega \leq \alpha \leq \omega + \mathrm{n}, i \in \omega\}.$$

X is the set of all "symbols" occuring in atomic conditions of $\mathbb{P}$. Formally, X would be the set of all $\langle \alpha, i \rangle$, for $\omega \leq \alpha \leq \omega + \mathrm{n}$, $i \in \omega$; but, as in Notation 54, we identify $\langle \alpha, i \rangle$ with x^i_α. Obviously, $X \in \mathrm{M}$.

We want σ to satisfy the following:

1. σ is a permutation on X,
2. σ preserves ranks; that is if $\sigma(x^i_\alpha) = x^j_\beta$, then $\alpha = \beta$,
3. $\sigma^2 = \mathrm{Id}$,
4. $x^i_\alpha \in x^j_{\alpha+1}$is in $\bigcup \mathbf{G} \Leftrightarrow \sigma(x^i_\alpha) \in \sigma(x^j_{\alpha+1})$ is in $\bigcup \mathbf{G}$,
5. $a \in x^i_\omega$ is in $\bigcup \mathbf{G} \Leftrightarrow a \in \sigma(x^i_\omega)$ is in $\bigcup \mathbf{G}$,
6. if $\sigma(x^i_\alpha) \neq x^i_\alpha$, then $\alpha = \omega + m$ where m is non-standard; we say that σ moves only objects of non-standard rank.

The last condition talks about non-standard numbers. We can not use the notion of non-standardness inside $\mathrm{M}[\mathbf{G}]$ or V, so the condition 6 is formulated externally. We construct σ externally, but during the construction we pay some respect to $\mathrm{M}[\mathbf{G}]$. The model M^* will be defined inside $\mathrm{M}[\mathbf{G}]$ and we would encounter problems when trying to extend the purely external σ to M^*. Thus we choose a halfway approach. We define σ as the union of $\{\sigma_i\}_{i \in \mathbb{N}}$, where $\mathbb{N}$ denotes the external, standard natural numbers, but we make sure that for every $i \in \mathbb{N}$ it holds that $\sigma_i \in \mathrm{M}[\mathbf{G}]$, in fact $\sigma_i \in \mathrm{M}$.

$\mathrm{S}(p)$ denotes all "symbols" occuring in p, i.e. for an atomic condition we set $\mathrm{S}(x^j_\alpha \in x^k_{\alpha+1}) = \{x^j_\alpha, x^k_{\alpha+1}\}$ (resp. $\mathrm{S}(a \in x^i_\omega) = \{x^i_\omega\}$). For p a condition, $\mathrm{S}(p) = \bigcup\{\mathrm{S}(a); a \in p\}$. As with X, a formally correct definition of $\mathrm{S}(p)$ would use $\langle \alpha, i \rangle$ instead of x^i_α. We call a condition p full if for all $a \in \mathrm{M}_\omega$ occuring in p and all $x^i_\omega, x^j_\alpha, x^k_{\alpha+1} \in \mathrm{S}(p)$ either $a \in x^i_\omega$ (resp. $x^j_\alpha \in x^k_{\alpha+1}$) or $a \notin x^i_\omega$ (resp. $x^j_\alpha \notin x^k_{\alpha+1}$) is in p. Inside $\mathrm{M}[\mathbf{G}]$, we define

$$\mathbf{G}_f = \{p \in \mathbf{G}; p \text{ is full}\}.$$

Let us choose some non-standard natural number $m \in \mathrm{M}[\mathbf{G}], m < \mathrm{n}$. By a density argument, there is some $g \in \mathbf{G}_f$ such that for some $i, j, k, l \in \omega$

$$g = \{x^i_{\omega+m} \in x^j_{\omega+m+1}, x^k_{\omega+m} \in x^l_{\omega+m+1}, x^i_{\omega+m} \notin x^l_{\omega+m+1}, x^k_{\omega+m} \notin x^j_{\omega+m+1}\}.$$

Let us recall that we supposed V to be countable. Therefore we can, externally, enumerate $\mathbf{G}_f$ starting with g. We denote this enumeration $\{g_i\}_{i\in\mathbb{N}}$, so we have $g_0 = g$.

The next lemma uses notation $\overline{g_n}$ and σ_n that has not been explained yet. For the purpose of this lemma, we may understand $\overline{g_n}$ and σ_n as arbitrary sets. The use of this notation will be justified later.

Lemma 57. *Inside* $\mathrm{M}[\boldsymbol{G}]$. *Let* $\overline{g_n}, g_{n+1} \in \boldsymbol{G}_f$. *Let* σ_n *satisfy conditions:*

1' σ_n *is a permutation on* $\mathrm{S}(\overline{g}_n)$,

2' σ_n *preserves rank,*

3' $\sigma_n^2 = Id$,

4' $x^i_\alpha \in x^j_{\alpha+1}$ *is in* $\overline{g}_n \Leftrightarrow \sigma_n(x^i_\alpha) \in \sigma_n(x^j_{\alpha+1})$ *is in* $\overline{g}_n$,

5' $a \in x^i_\omega$ *is in* $\overline{g}_n \Leftrightarrow a \in \sigma_n(x^i_\omega)$ *is in* $\overline{g}_n$,

6' σ_n *moves only objects of non-standard rank.*

Then there are $\sigma_{n+1}, \overline{g_{n+1}}$ *such that:*

- σ_{n+1} *is a permutation extending* σ_n,
- σ_{n+1} *satisfies conditions 1' – 6' with* n *replaced by* $n+1$,
- $\overline{g_{n+1}} \in \boldsymbol{G}_f$,
- $\overline{g_{n+1}} \leq_{\mathbb{P}} \overline{g_n} \cup g_{n+1}$.

Proof. Let us denote $g' = \overline{g_n} \cup g_{n+1}$. It holds that $g' \in \mathbf{G}$. g' may not be full, but there is some $g \in \mathbf{G}_f$ such that $g \leq_{\mathbb{P}} g'$. Let $N = \{x \in \mathrm{S}(g) - \mathrm{S}(\overline{g_n}); \exists y \in \mathrm{S}(\overline{g_n})\colon \sigma_n(y) \neq y$ and $\mathrm{rk}(x)$ is one greater or one less than $\mathrm{rk}(y)\}$.

N contains all the symbols that σ_{n+1} may have to move to satisfy conditions 4' and 5'. We want to find some N', an image-to-be of N under σ_{n+1}. Let us set

$D = \{p \in \mathbb{P}; (\exists N' \subseteq \mathrm{S}(p))(\exists f : N' \to N)$ such that a)–c) are fulfilled$\}$

a) f is a rank-preserving bijection,

b) $N' \cap \mathrm{S}(g) = \emptyset$; we say that N' contains fresh symbols,

c) $(\forall n \in N')$: if $x \in \mathrm{S}(g) \cup N'$ is of an appropriate rank, i.e. one less (resp. one greater) than rank of n, then one of $x \in n, x \notin n$ (resp. $n \in x, n \notin x$) is in p and it holds that

- for $x \in N$: conditions $x \notin n, n \notin x$ are in p,
- for $x \in N'$: $x \in n$ (resp. $n \in x$) is in $p \Leftrightarrow f(x) \in f(n)$ (resp. $f(n) \in f(x)$) is in g,
- for $x \in \mathrm{S}(\overline{g_n})$: $x \in n$ (resp. $n \in x$) is in $p \Leftrightarrow \sigma_n(x) \in f(n)$ (resp. $f(n) \in \sigma_n(x)$) is in g,
- for $x \in \mathrm{S}(g) - (N \cup \mathrm{S}(\overline{g_n}))$: $x \in n$ (resp. $n \in x$) is in $p \Leftrightarrow x \in f(n)$ (resp. $f(n) \in x$) is in g.

It is easy to check that D is dense; for any $\mathbb{P}$-condition p, we may choose a set of fresh symbols of respective ranks (with respect to the ranks of symbols in N) to satisfy a) and b) and add to p conditions in which these fresh symbols occur to satisfy c). Let us fix some $p' \in D \cap \mathbf{G}$; let us fix corresponding N' and f witnessing that p' belongs to D. We denote by p the restriction of p' to conditions in which only symbols from $N' \cup \mathrm{S}(g)$ occur.

We put $\overline{g_{n+1}} = g \cup p$. It is $\overline{g_{n+1}} \in \mathbf{G}$ as $g \in \mathbf{G}$ and $p \in \mathbf{G}$ (since p is a restriction of $p' \in \mathbf{G}$). g is full and p contains, by c), all conditions needed to conclude that $\overline{g_{n+1}}$ is full as well. So, in fact, $\overline{g_{n+1}} \in \mathbf{G}_f$. Clearly, $\overline{g_{n+1}} \leq_{\mathbb{P}} g \leq_{\mathbb{P}} g' = \overline{g_n} \cup g_{n+1}$, thus we have $\overline{g_{n+1}} \leq_{\mathbb{P}} \overline{g_n} \cup g_{n+1}$.

We finish the proof by defining σ_{n+1}:

- for $x \in N'$: $\sigma_{n+1}(x) = f(x)$,
- for $x \in N$: $\sigma_{n+1}(x) = f^{-1}(x)$,
- for $x \in \mathrm{S}(\overline{g}_n)$: $\sigma_{n+1}(x) = \sigma_n(x)$,
- for other $x \in \mathrm{S}(\overline{g_{n+1}})$: $\sigma_{n+1}(x) = x$.

σ_{n+1}, clearly, extends σ_n. To finish the proof, we have to show that it satisfies 1'–6'. 1'–3' are easy. By the assumption, σ_n moves only objects of non-standard rank. σ_{n+1} moves only objects that σ_n already moves plus objects from $N \cup N'$. Every element of $N \cup N'$ has rank one greater or one less than rank of some object moved by σ_n and thus is of non-standard rank. Therefore σ_{n+1} moves only objects of non-standard rank.

We show that σ_{n+1} satisfies conditions 4' and 5'. Suppose $a, b \in \mathrm{S}(\overline{g_{n+1}})$ have appropriate ranks. We want to show that $a \in b$ is in $\overline{g_{n+1}} \Leftrightarrow \sigma_{n+1}(a) \in \sigma_{n+1}(b)$ is in $\overline{g_{n+1}}$. $\mathrm{S}(\overline{g_{n+1}}) = \mathrm{S}(g) \cup N'$. We distinguish possible cases.

Suppose $a \in \mathrm{S}(\overline{g}_n)$.

- Let $b \in \mathrm{S}(\overline{g}_n)$. This case is immediate from the assumptions about σ_n.
- Let $b \in N$. We want to show that $a \in b$ is in $\overline{g_{n+1}} \Leftrightarrow \sigma_n(a) \in f^{-1}(b)$ is in $\overline{g_{n+1}}$. It holds that $\sigma_n(a) \in \mathrm{S}(\overline{g}_n)$, $f^{-1}(b) \in N'$. The equivalence holds as a consequence of the third bullet in c).
- Let $b \in N'$. This case is, again, a consequence of the third bullet in c).
- Let $b \in \mathrm{S}(g) - (N \cup \mathrm{S}(\overline{g_n}))$. The fact that $b \notin N$ implies that a is not moved by σ_n. Therefore, σ_{n+1} does not move either of a, b and the equivalence is trivial.

  The other cases are similar.

□

We now construct, externally, sequences $\{\overline{g}_i\}_{i\in\mathbb{N}}$ and $\{\sigma_i\}_{i\in\mathbb{N}}$ from the sequence $\{g_i\}_{i\in\mathbb{N}}$. We set $\overline{g}_0 = g_0$ and define σ_0, a permutation on $\mathrm{S}(\overline{g}_0)$:

$$\sigma_0(x^i_{\omega+m}) = x^k_{\omega+m},$$

$$\sigma_0(x^k_{\omega+m}) = x^i_{\omega+m},$$

$$\sigma_0(x^j_{\omega+m+1}) = x^l_{\omega+m+1},$$

$$\sigma_0(x^l_{\omega+m+1}) = x^j_{\omega+m+1}.$$

It is easy to check that σ_0 and $\overline{g}_0$ satisfy the conditions 1'–6' for $n = 0$. Thus we may apply Lemma 57 and define $\{\overline{g}_i\}_{i\in\mathbb{N}}$, $\{\sigma_i\}_{i\in\mathbb{N}}$ by external induction using the elements of the sequence $\{g_i\}_{i\in\mathbb{N}}$. Note that σ_0, $\overline{g}_0 \in \mathrm{M}$. The construction of Lemma 57 uses $\mathbf{G}$, which might seem to imply that σ_n, $\overline{g}_n$ are not necessarily elements of M. But $\overline{g}_n \in \mathbb{P}$, so in fact $\overline{g}_n \in \mathrm{M}$, although inside M we may not be able to decide $\overline{g}_{n+1}$ from $\overline{g}_n$, g_{n+1} and σ_n. Similarly, $\sigma_n \in \mathrm{M}$.

The elements of the sequence extend each other, so we may set $\sigma = \bigcup\{\sigma_i\}_{i\in\mathbb{N}}$. We now check that conditions 1–6 hold for σ. σ is a permutation on X, as every x^i_α is in $\mathrm{S}(\overline{g}_n)$ for some $n \in \mathbb{N}$, and σ_n is a permutation on $\mathrm{S}(\overline{g}_n)$. It is rank preserving as every σ_n is and similarly for conditions 3–6.

5.1.3 Model M^* and automorphism σ^*

In the next, π denotes a rank-preserving permutation on X, such that $\pi^2 = \mathrm{Id}$. Such a π can be naturally extended to act on $\mathbb{P}$-conditions and names. For an atomic condition $x \in y$ (resp. $x \notin y$), we define $\pi(x \in y) = \pi(x) \in \pi(y)$ (resp.

$\pi(x \notin y) = \pi(x) \notin \pi(y))$. If, in the previous, $x \in \mathrm{M}_\omega$, then we write x instead of $\pi(x)$. For p a $\mathbb{P}$-condition, $\pi(p) = \{\pi(c); c \in p\}$. For $\dot{a}$ a $\mathbb{P}$-name, we define by induction $\pi(\dot{a}) = \{\langle \pi(\dot{b}), \pi(p)\rangle; \langle \dot{b}, p\rangle \in \dot{a}\}$.

It is obvious that π preserves finiteness and ranks and easy to check that p is consistent iff $\pi(p)$ is consistent. Thus p is a $\mathbb{P}$-condition iff $\pi(p)$ is a $\mathbb{P}$-condition. Also, $p \leq_{\mathbb{P}} q$ iff $\pi(p) \leq_{\mathbb{P}} \pi(q)$.

Lemma 58. *Inside* M. *Let π be a rank-preserving permutation on X, $\pi^2 = \mathrm{Id}$, $a_1, a_2, \ldots, a_n$ names, p a $\mathbb{P}$-condition, ϕ a formula. It holds that:*

a) $p \Vdash^* a_1 = a_2 \Leftrightarrow \pi(p) \Vdash^* \pi(a_1) = \pi(a_2)$
$p \Vdash^* a_1 \in a_2 \Leftrightarrow \pi(p) \Vdash^* \pi(a_1) \in \pi(a_2)$

b) $p \Vdash^* \phi(a_1, \ldots, a_n) \Leftrightarrow \pi(p) \Vdash^* \phi(\pi(a_1), \ldots, \pi(a_n))$

Proof.

a) By induction on the ranks of the names. For the left-to-right direction in the first statement we have to show, by Definition 28, that if for all $\langle b_1, s_1\rangle \in a_1$

$$D_{\langle b_1, s_1\rangle} = \{q \leq p; q \leq s_1 \rightarrow \exists \langle b_2, s_2\rangle \in a_2 (q \leq s_2 \wedge q \Vdash^* b_1 = b_2)\}$$

is dense below p, then for all $\langle b_1', s_1'\rangle \in \pi(a_1)$

$$D_{\langle b_1', s_1'\rangle} = \{q \leq \pi(p); q \leq s_1' \rightarrow \exists \langle b_2', s_2'\rangle \in \pi(a_2)(q \leq s_2' \wedge q \Vdash^* b_1' = b_2')\}$$

is dense below $\pi(p)$. Let us fix some $\langle b_1', s_1'\rangle \in \pi(a_1)$, we want to show that $D_{\langle b_1', s_1'\rangle}$ is dense below $\pi(p)$. $\langle b_1', s_1'\rangle$ is in fact $\langle \pi(b_1), \pi(s_1)\rangle$ for some $\langle b_1, s_1\rangle \in a_1$, and it holds that $D_{\langle \pi(b_1), \pi(s_1)\rangle} = \{\pi(q); q \in D_{\langle b_1, s_1\rangle}\}$. Thus if $D_{\langle b_1, s_1\rangle}$ is dense below p, then $D_{\langle \pi(b_1), \pi(s_1)\rangle}$ is dense below $\pi(p)$. To complete the left-to-right direction we have to show that if for all $\langle b_2, s_2\rangle \in a_2$ the set $D_{\langle b_2, s_2\rangle}$ is dense below p, then for all $\langle b_2', s_2'\rangle \in \pi(a_2)$ the set $D_{\langle b_2', s_2'\rangle}$ is dense below $\pi(p)$. The proof is similar, as are the proofs of the opposite direction and of the second statement.

b) By induction on the complexity of formula ϕ.

□

In the previous subsection, we have constructed sequences $\{\overline{g}_i\}_{i\in\mathbb{N}}$ and $\{\sigma_i\}_{i\in\mathbb{N}}$. Every σ_i is a permutation on $\mathrm{S}(\overline{g}_i)$. Using the notation from above, conditions 4' and 5' of Lemma 57 can be written as $\sigma_i(\overline{g}_i) = \overline{g}_i$. σ_i is not defined on the whole X, but we can suppose that σ_i acts on X and put $\sigma_i(x) = x$ for $x \notin \mathrm{S}(\overline{g}_i)$.

Lemma 59. *Let π be a rank-preserving permutation on X. If $\pi(x^i_\alpha) = x^j_\alpha$, then $\pi(\hat{x}^i_\alpha) = \hat{x}^j_\alpha$.*

Proof. By induction on α. Let $\alpha = \omega$ and $\pi(x^i_\omega) = x^j_\omega$. Then it holds that $\hat{x}^i_\omega = \{\langle \check{a}, \{a \in x^i_\omega\}\rangle; a \in \mathrm{M}_\omega\}$, so $\pi(\hat{x}^i_\omega) = \{\langle \pi(\check{a}), \{a \in \pi(x^i_\omega)\}\rangle; a \in \mathrm{M}_\omega\} = \{\langle \pi(\check{a}), \{a \in x^j_\omega)\}\rangle; a \in \mathrm{M}_\omega\}$. To conclude that $\pi(\hat{x}^i_\omega) = \hat{x}^j_\omega$ it suffices to show that $\pi(\check{a}) = \check{a}$. But $\check{a}$ is a canonical name, and the only condition used to define a canonical name is $\mathbb{1}_\mathbb{P} = \emptyset$. Since $\pi(\emptyset) = \emptyset$, we have $\pi(\mathbb{1}_\mathbb{P}) = \mathbb{1}_\mathbb{P}$ and thus $\pi(\check{a}) = \check{a}$.

The induction step: Let $\pi(x^i_{\alpha+1}) = x^j_{\alpha+1}$. Then the following holds: $\hat{x}^i_{\alpha+1} = \{\langle \hat{x}^k_\alpha, \{x^k_\alpha \in x^i_{\alpha+1}\}\rangle; k \in \omega\}$, $\pi(\hat{x}^i_{\alpha+1}) = \{\langle \pi(\hat{x}^k_\alpha), \{\pi(x^k_\alpha) \in x^j_{\alpha+1}\}\rangle; k \in \omega\}$. π is a rank-preserving permutation on X, so $(\forall l \in \omega)(\exists k \in \omega)$ such that $\pi(x^k_\alpha) = x^l_\alpha$. Thus, in fact, $\pi(\hat{x}^i_{\alpha+1}) = \{\langle \hat{x}^l_\alpha, \{x^l_\alpha \in x^j_{\alpha+1}\}\rangle; l \in \omega\} = \hat{x}^j_{\alpha+1}$. □

We denote by $\hat{X}$ the name $\{\langle \hat{x}^i_\alpha, \mathbb{1}_\mathbb{P}\rangle; x^i_\alpha \in X\}$ and write $X_\mathbf{G}$ for $\hat{X}_\mathbf{G}$, so $X_\mathbf{G} = \{\hat{x}^i_{\alpha\mathbf{G}}; \omega \leq \alpha \leq \omega + \mathrm{n}, i \in \omega\}$. Similarly, $\mathrm{S}(\overline{g}_i)_\mathbf{G} = \{\hat{x}^j_{\alpha\mathbf{G}}; x^j_\alpha \in \mathrm{S}(\overline{g}_i)\}$.

Definition 60. *Inside* $\mathrm{M}[\boldsymbol{G}]$. $\mathrm{M}^* = \mathrm{HOD}(X_{\boldsymbol{G}})$

It is a classical result that for any A the class $\mathrm{HOD}(A)$ is a transitive model of ZF.[2]

First, we define an automorphism of the class $\mathrm{OD}(X_\mathbf{G})$. Let us recall that this class can be defined using Gödel operations in the following way:

$$\mathrm{OD}(X_\mathbf{G}) = \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{cl}(\{V_\beta; \beta < \alpha\} \cup \{X_\mathbf{G}\} \cup X_\mathbf{G}).$$

We started with a countable model; thus, externally, there exists an enumeration $\{\alpha_i\}_{i \in \mathbb{N}}$ of the class of ordinals. So

$$\mathrm{OD}(X_\mathbf{G}) = \bigcup_{i \in \mathbb{N}} \mathrm{cl}(\{V_\beta; \beta < \alpha_i\} \cup \{X_\mathbf{G}\} \cup X_\mathbf{G}).$$

Moreover, for a construction of any $s \in \mathrm{OD}(X_\mathbf{G})$ using Gödel operations we need only finitely many (in the sense of $\mathrm{M}[\mathbf{G}]$) members of $X_\mathbf{G}$. Let X_s be a finite set containing all the elements needed for some construction of s. Then there is some i, such that $X_s \subseteq \mathrm{S}(\overline{g}_i)$, as conditions are arbitrarily finitely large. It holds that $X_\mathbf{G} = \bigcup_{i \in \mathbb{N}} \mathrm{S}(\overline{g}_i)_\mathbf{G}$ and $\mathrm{S}(\overline{g}_i)_\mathbf{G} \subseteq \mathrm{S}(\overline{g}_{i+1})_\mathbf{G}$. Thus in fact

$$\mathrm{OD}(X_\mathbf{G}) = \bigcup_{i \in \mathbb{N}} \mathrm{cl}(\{V_\beta; \beta < \alpha_i\} \cup \{X_\mathbf{G}\} \cup \mathrm{S}(\overline{g}_i)_\mathbf{G}).$$

[2]The proof and other details about ordinal-definable set can be found in the Jech's book [5] in Chapter 13.

Lemma 61. *Let us have $A \subseteq B$, π_A, π_B permutations on A, B respectively, π_B extending π_A. If π_A, π_B are partial automorphisms on A, B, i.e. for any formula φ it holds that*

$$\varphi(a_1, \ldots, a_n) \Leftrightarrow \varphi(\pi_A(a_1), \ldots, \pi_A(a_n)),$$

$$\varphi(b_1, \ldots, b_n) \Leftrightarrow \varphi(\pi_B(b_1), \ldots, \pi_B(b_n)),$$

where $a_i \in A, b_i \in B$, then π_A, π_B can be extended to partial automorphisms on $\mathrm{cl}(A), \mathrm{cl}(B)$ so that π_B still extends π_A.

Proof. By induction, we extend π_A, π_B in the natural way. Let G be a Gödel operation. For π_A we define

$$\pi_A(G(x_1, x_2)) = G(\pi_A(x_1), \pi_A(x_2)), \text{ or}$$

$$\pi_A(G(x)) = G(\pi_A(x)),$$

depending on the arity of G. Similarly, we extend π_B. It is clear that π_B still extends π_A.

By induction again, we prove $\varphi(a_1, \ldots, a_n) \Leftrightarrow \varphi(\pi_A(a_1), \ldots, \pi_A(a_n))$ for the extended π_A and $a_i \in \mathrm{cl}(A)$. Assume that, for every $1 \leq i \leq n$, a_i is the value of a Gödel operation G^i on $\bar{x_i}$ ($\bar{x_i}$ contains either one or two elements, depending on the arity of G^i). The induction hypothesis implies that

$$\varphi(G^1(\bar{x_1}), \ldots, G^n(\bar{x_n})) \Leftrightarrow \varphi(G^1(\pi_A(\bar{x_1})), \ldots, G^n(\pi_A(\bar{x_n})).$$

Therefore, $\varphi(a_1, \ldots, a_n) \Leftrightarrow \varphi(\pi_A(a_1), \ldots, \pi_A(a_n))$. Similarly for π_B. □

We have defined $\{\sigma_i\}_{i \in \mathbb{N}}$ so that σ_i is a permutation on $\mathrm{S}(\overline{g}_i)$. Using σ_i, we now define σ_i^* as a permutation on $\{V_\beta; \beta < \alpha_i\} \cup \{X_\mathbf{G}\} \cup \mathrm{S}(\overline{g}_i)_\mathbf{G}$ in the following way:

$$\sigma_i^*(X_\mathbf{G}) = X_\mathbf{G},$$

$$\sigma_i^*(V_\beta) = V_\beta,$$

$$\sigma_i^*(\hat{x}^j_{\alpha\mathbf{G}}) = (\sigma_i(\hat{x}^j_\alpha))_\mathbf{G}, \text{ for any } \hat{x}^j_{\alpha\mathbf{G}} \in \mathrm{S}(\overline{g}_i)_\mathbf{G}.$$

Lemma 62. *For any $i \in \mathbb{N}$, $x_{1\boldsymbol{G}}, \ldots, x_{k\boldsymbol{G}} \in \mathrm{S}(\overline{g}_i)_{\boldsymbol{G}}$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Ord}$ it holds that*

$$\mathrm{M}[\boldsymbol{G}] \vDash \varphi(V_{\alpha_1}, \ldots, V_{\alpha_n}, X_{\boldsymbol{G}}, x_{1\boldsymbol{G}}, \ldots, x_{k\boldsymbol{G}}) \leftrightarrow$$

$$\leftrightarrow \varphi(V_{\alpha_1}, \ldots, V_{\alpha_n}, X_{\boldsymbol{G}}, \sigma_i^*(x_{1\boldsymbol{G}}), \ldots, \sigma_i^*(x_{k\boldsymbol{G}})).$$

Proof. We prove the left-to-right direction, the opposite direction is similar. Let

$$\mathrm{M}[\mathbf{G}] \vDash \varphi(V_{\alpha_1}, \ldots, V_{\alpha_n}, X_{\mathbf{G}}, x_{1\mathbf{G}}, \ldots, x_{k\mathbf{G}}).$$

Then there is some $p \in \mathbf{G}$ such that

$$p \Vdash \varphi(\check{V}_{\alpha_1}, \ldots, \check{V}_{\alpha_n}, \hat{X}, \hat{x_1}, \ldots, \hat{x_k})$$

and some $j \in \mathbb{N}$ such that $\overline{g}_j < p$. Let $m = max(i, j)$, then $\sigma_i \subseteq \sigma_m$ and

$$\overline{g}_m \Vdash \varphi(\check{V}_{\alpha_1}, \ldots, \check{V}_{\alpha_n}, \hat{X}, \hat{x_1}, \ldots, \hat{x_k}).$$

By Lemma 58, it holds

$$\sigma_m(\overline{g}_m) \Vdash \varphi(\check{V}_{\alpha_1}, \ldots, \check{V}_{\alpha_n}, \hat{X}, \sigma_m(\hat{x_1}), \ldots, \sigma_m(\hat{x_k})),$$

as $\sigma_m(\check{V}_\alpha) = \check{V}_\alpha$ and $\sigma_m(\hat{X}) = \hat{X}$. But $\sigma_m(\overline{g}_m) = \overline{g}_m$, so

$$\overline{g}_m \Vdash \varphi(\check{V}_{\alpha_1}, \ldots, \check{V}_{\alpha_n}, \hat{X}, \sigma_m(\hat{x_1}), \ldots, \sigma_m(\hat{x_k})).$$

σ_m was chosen so that $\sigma_m(\hat{x_1}) = \sigma_i(\hat{x_1}), \ldots, \sigma_m(\hat{x_k}) = \sigma_i(\hat{x_k})$, and $\overline{g}_m \in \mathbf{G}$, thus

$$\mathrm{M}[\mathbf{G}] \vDash \varphi(V_{\alpha_1}, \ldots, V_{\alpha_n}, X_{\mathbf{G}}, \sigma_i^*(x_{1\mathbf{G}}), \ldots, \sigma_i^*(x_{k\mathbf{G}})).$$

□

$\{\sigma_i^*\}_{i\in\mathbb{N}}$ is a sequence of permutations, where σ_{i+1}^* extends σ_i^*. By Lemma 62, every σ_i^* is a partial automorphism on $\{V_\beta; \beta < \alpha_i\} \cup \{X_{\mathbf{G}}\} \cup \mathrm{S}(\overline{g}_i)_{\mathbf{G}}$. By Lemma 61, we can extend each σ_i^* to a partial automorphism on $\mathrm{cl}(\{V_\beta; \beta < \alpha_i\} \cup \{X_{\mathbf{G}}\} \cup \mathrm{S}(\overline{g}_i)_{\mathbf{G}})$. For these extensions it still holds $\sigma_i^* \subseteq \sigma_{i+1}^*$. We then put $\sigma^* = \bigcup_{i\in\mathbb{N}} \sigma_i^*$. σ^* is an external permutation of $\mathrm{OD}(X_{\mathbf{G}})$ and being the union of partial automorphisms it is in fact an automorphism of the class $\mathrm{OD}(X_{\mathbf{G}})$.

Theorem 63. *σ^* restricted to $\mathrm{HOD}(X_{\mathbf{G}})$ is an automorphism of order two.*

Proof. We have already argued that σ^* is an automorphism of the class $\mathrm{OD}(X_{\mathbf{G}})$, so

$$s \in \mathrm{HOD}(X_{\mathbf{G}}) \Leftrightarrow \sigma^*(s) \in \mathrm{HOD}(X_{\mathbf{G}})$$

and the restriction of σ^* is an automorphism of $\mathrm{HOD}(X_{\mathbf{G}})$. For every $i \in \mathbb{N}$, we proved $\sigma_i^2 = Id$. By checking the inductive definition of σ_i^*, it is clear that $\sigma_i^{*2} = Id$. Because of the choice of σ_0, σ^* is not the identity. Therefore σ^* is an automorphism of order two. □

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