Charles University in Prague, Faculty of Arts
Department of Logic

Michal Dančák

ZMĚNY STAVŮ ZNALOSTÍ V DYNAMICKÉ EPISTEMICKÉ LOGICE
KNOWLEDGE CHANGES IN DYNAMIC EPISTEMIC LOGIC

Master's Thesis

2012 Supervisor: PhDr. Michal Peliš, Ph.D.

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Michal Dančák

Abstract

The Logic of Hybrid Action Models is presented in this thesis along with a sound and complete Hilbert style calculus. It is an original multimodal epistemic logic that tries to give the agents (i) a strong method of communication and (ii) a rich vocabulary to communicate about by combining action models with hybrid logics. The language of the resulting logic is rather complex and includes common knowledge, deterministic action updates and all three commonly used hybrid operators @, $\downarrow$ and E. An overview of both action models and hybrid logics is included. The Hybrid Logic with Partial Denoting Nominals is briefly described and used as a stepping stone. The semantics and the Completeness Theorem of the Logic of Hybrid Action Models form the backbone of this work.

Contents

1 Introduction

The main goal of this thesis is to introduce the notion of Hybrid Action Models (HAM) and a logic associated with them. The logic of Hybrid Action Models is a dynamic epistemic logic standing on two legs—Action Models and Hybrid Logics.

Action Models were first introduced in [1]. Their role in this paper is to generalize a Public Announcement Logic because it gives our agents a much broader scope of communication than public announcements. This is caused by the fact that agents can now communicate through actions instead of announcements. These actions can be public but they can also be semi-private (other agents know some action came to pass but are unaware which one) or completely private (other agents have no idea that some action happened). It is however important to note that these actions cannot change the facts about our world. The atomical sentences will keep their validity for example. Talking, reading a letter, showing your cards in a game or sending a mail are all different types of actions.

Hybrid Logics are the invention of Arthur Prior although they travelled a long way since then. A great summary of history of Hybrid Logics can be found in [2]. The basic idea behind them is to introduce a way to talk about particular states in models into a language itself. They introduce a second sort of atoms into a modal language, a set of nominals, that serves as a set of names for our states. This allows the agents to agree on a particular state and talk about their knowledge or beliefs relating to that state. They will also be able to refer to other states or talk about states even without knowing their names. This gives the agents a great expression power.

We will first introduce Action Models, and discuss and contrast them with another notion of dynamization—Public Announcement Logic. The definition of Hybrid Logics will follow as well as an attempt to hybridize the Action Models. We shall do this with the help of the Hybrid Logic with Partial Denoting Nominals, first presented in [3]. This mixing of logics will introduce several problems, some of which were already solved. The result of solving all of them will be the main scope of this thesis—the Logic of Hybrid Action Models, and its sound and complete axiomatics. This logic is, to the knowledge of the author, original and was never published before. A similar approach was presented in [4] but language, semantics and axiomatics used in there are quite different.

The Logic of Hybrid Action Models has no ambition to become a 'logic of reality'. The agents are infallible and logically omniscient, no misunderstanding due to ambiguity of words is possible. It does however present a closer look on the communication of these perfect agents in the form of actions.

This paper assumes a general knowledge of dynamic epistemic logics, especially the Logic of Public Announcements. Reader can find more information on them in [5] or [6] for example. Knowledge of Classical First Order Logic is also assumed.

1.1 Used Symbols

There are several symbols used throughout this work that are not defined anywhere. We list them here, along with a brief description, to prevent any confusion.

$\forall$, $\exists$	The symbols for metaquantification.
$\Longrightarrow$	The symbol for the act of updating models with a restricted modal product. $\mathcal{M} \otimes M \Longrightarrow (\mathcal{M} \otimes M)$. See also Definition 3.
$\vdash$	The relation of provability. $\Gamma \vdash \varphi$ means there exists a proof of φ from Γ in a given calculus.
$\Vdash$	The relation of satisfaction in a model-state pair or a model-state-assignment triple. $\mathcal{M}, w \Vdash \varphi$ means that φ is satisfied by a given valuation in state w of model $\mathcal{M}$.
$\vDash$	The relation of satisfaction in a whole model. $\mathcal{M} \vDash \varphi$ means $\mathcal{M}, w \Vdash \varphi$ for any state w in the domain of $\mathcal{M}$ while $\vDash \varphi$ means $\mathcal{M} \vDash \varphi$ for any model $\mathcal{M}$. We will work in the $S5$ multimodal logic. So $\vDash$ will mean $\vDash_{S5}$ unless noted otherwise.
$\Rightarrow$	The symbol for metaimplication.
&, *or*	The symbols for metaconjunction and metadisjunction.
$\equiv$, *iff*	The symbols for metaequivalence. $\equiv$ is used as an equivalence of formulas or pseudoformulas, *iff* as an equivalence of statements or metaformulas. Writing $\varphi \equiv \psi$ means $\mathcal{M}, w \Vdash \varphi$ *iff* $\mathcal{M}, w \Vdash \psi$ for any $\mathcal{M}, w$.
$\dot{\mathcal{M}}$	For a given model $\mathcal{M}$, $\dot{\mathcal{M}}$ designates its domain set.
$K_a\varphi$	Agent a knows φ; the universal modality, also known as $\Box_a\varphi$, $[R_a]\varphi$ or $[a]\varphi$.
$M_a\varphi$	Agent a admits φ; the existential modality, also known as $\Diamond_a\varphi$, $\langle R_a\varphi\rangle$ or $\langle a\rangle\varphi$. It is dual to K_a, that is $M_a\varphi \equiv \neg K_a\neg\varphi$.

$E_B\varphi$ Each agent in group B knows φ. It is equal to $\bigwedge_{a\in B} K_a\varphi$. Note that we will also use the symbol E without any index. This will however mean the existential operator as described in Section 3.1.3.

$C_B\varphi$ φ is a common knowledge for agents in group B. $C_B\varphi$ is equal to $\bigwedge_{n=0}^{\infty} E_B^n\varphi$, where $E_B^0\varphi \equiv \varphi$ and $E_B^{n+1}\varphi \equiv E_B E_B^n\varphi$.

1.2 Acknowledgements

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2 Action Models

The most common way to dynamize a given epistemic logic is with the use of public announcements. Public Announcement Logic (PAL) introduces the symbols of [and] into the language. The meaning of a formula $[\varphi]\psi$ is that 'formula ψ must hold after the public announcement of a formula φ'. This announcement is truly public—all agents must hear it and must be able to understand and comprehend it. This greatly limits what our agents can do. Imagine agents Alice, Bob and Carol in a situation where Alice wants to tell Bob she loves him but she doesn't want Carol to overhear it. The system of public announcements doesn't give her such an option. Let's say however that such a message does get through without Carol knowing it. Imagine further that everything that Carol knew before this private announcement was true. This suddenly stops being the case after the 'private announcement'. For instance Carol is still certain that Alice didn't communicate with Bob in any way or that Bob doesn't know that Alice loves him. This is however impossible to do in the system of public announcements.[1]

There is a second important limitation of Public Announcement Logic. Take the same situation as before. Let's say that Carol has stalked Alice for days and is sure that Alice didn't get a chance to talk with Bob. But one day Carol sees Alice saying something to Bob but is too far to hear them. Before this happened Carol knew that Alice didn't tell Bob she loved him. But now Carol stops knowing this. Alice and Bob may have talked about this or about something else instead. This 'semi-private' announcement increased Bob's knowledge while simultaneously decreasing that of Carol's.

These two limitations of PAL (true knowledge cannot be changed into a false one and knowledge can never decrease) are overcome with action models.[2] We will base our definitions on [6].

2.1 Action Updates

Definition 1. *Let A and P be subsets of a language $\mathcal{L}$ such that A, P are non-empty sets, A is finite, P is countably infinite and $A \cap P = \emptyset$ and all*

[1]There are many other logics of private or semi-private communication. One can, for example, replace the public announcement with a group announcement. But the author chose the system of Action Models for its generality and customizability.

[2]We will use $S5$ Action Models which do not allow the first thing to occur. This doesn't need to concern us though. Our intent is to model the communication of agents' knowledge, not beliefs. And due to an $S5$ tautology $K_a\varphi \to \varphi$ all knowledge is true. Moreover the agents will be infallible and perfect logicians. They will know all tautologies and they will be capable of both a positive and a negative introspection. However weaker action models can be used instead, see [1].

other members of $\mathcal{L}$ are members of neither A nor P. An $S5$ Action Model is any structure $M = \langle S, R_a, pre \rangle$ where S is a finite non-empty set, for any $a \in A$ R_a is an equivalence relation on S and pre: $S \rightarrow FLA$ is a total function from S to a set of all formulas of language $\mathcal{L}$. A Pointed $S5$ Action Model is an ordered couple (M, s) where M is an $S5$ Action Model and s is a member of its domain.

This is a general definition independent on the language.[3] A denotes a set of agents and P a set of propositional atoms. One can view an action model as a set of interconnected actions. Agents may or may not be able to distinguish between them, depending on their accessibility relations inside the action model.

Our choice of the modal logic will influence the properties of the action models. Whatever the requirements for the accessibility relation in epistemic models are, the same must hold for action models as well. In our case the $S5$ action model relation R_a must be an equivalence relation for all agents $a \in A$. For a given action model $M = \langle S, R_a, pre \rangle$ we will call *pre* the prerequisite or precondition function and members of S action states. In the case of a pointed action model (M, s) s will be called a designated action state. Each of these action states in S can be viewed as an action that may or may not happen. The designated action state in a pointed action model denotes the action that really came to pass. The prerequisite function defines conditions that have to be true for an action to be even considered happening.

Let's now define the language we will be using.

Definition 2. *Call $\mathcal{L}_{KC\otimes} = \{\wedge, \vee, \rightarrow, \neg, K_a, C_B, [(M, s)]\} \cup A \cup P$ an Action Model Language if $\{\wedge, \vee, \rightarrow, \neg, K_a, C_B\} \cup A$ is a multimodal epistemic language with common knowledge and (M, s) is an $S5$ pointed action model. Define formulas in the action model language in the following way*

$$\varphi \ := \ p \mid (\psi \wedge \chi) \mid (\psi \vee \chi) \mid (\psi \rightarrow \chi) \mid \neg\psi \mid K_a\psi \mid C_B\psi \mid [(M, s)]\psi$$

where ψ and χ are well-formed formulas, $p \in P$, $a \in A$, $B \subseteq A$ and (M, s) is a pointed $S5$ action model such that for all $t \in S$, $pre(t)$ is an already formed formula.

Since we allow only finite action models there is only countably many different pointed action models. Thus the language $\mathcal{L}_{KC\otimes}$ stays countable. The meaning of formula $[(M, s)]\psi$ is that the formula ψ holds after updating by

[3] The reason for this generality is that action models require some language to be defined but action model language needs a definition of action models first.

the pointed model (M, s).[4] This update takes place thanks to a metaoperator of a restricted modal product $\otimes$.

Definition 3 (Restricted Modal Product). *Let $\mathcal{M} = \langle W, \mathcal{R}_a, V\rangle$ be an $S5$ epistemic model and $M = \langle S, R_a, pre\rangle$ an $S5$ action model. A Restricted Modal Product $(\mathcal{M} \otimes M)$ is an ordered triple $\langle W', \mathcal{R}'_a, V'\rangle$ where*

$$W' = \{(w, s); w \in W \ \&\ s \in S \ \&\ \mathcal{M}, w \Vdash pre(s)\}$$

$$\forall a \in A \forall (w, s), (w', s') \in W' \ (((w, s)\mathcal{R}'_a(w', s') \text{ iff } w\mathcal{R}_a w' \ \&\ sR_a s'))$$

$$\forall (w, s) \in W' \forall p \in P \ (w, s) \in V'(p) \text{ iff } w \in V(p)$$

The basic idea is that $\otimes$ creates a cartesian product of an epistemic model with an action model, removes the impermissible world-state combinations according to the precondition function and defines the accessibility relations and valuation. It can be easily verified that the new model is again an $S5$ epistemic model (or its domain is empty) and thus can be updated again.

Lemma 1. *Let $\mathcal{M} = \langle W, \mathcal{R}_a, V\rangle$ be an $S5$ epistemic model and let $M = \langle S, R_a, pre\rangle$ an $S5$ action model. If the domain of $(\mathcal{M} \otimes M)$ is non-empty then it is an $S5$ epistemic model.*

Proof. It is obvious that V' is a valuation function. All we need to verify is that $\mathcal{R}'_a$ is an equivalence relation for any agent $a \in A$. Let's check that it is transitive and leave the reflexivity and symmetry to the reader.

Let $(w_1, s_1)\mathcal{R}'_a(w_2, s_2)$ and $(w_2, s_2)\mathcal{R}'_a(w_3, s_3)$. By definition $w_1\mathcal{R}_a w_2$ and $w_2\mathcal{R}_a w_3$. From transitivity of $\mathcal{R}_a$ we have $w_1\mathcal{R}_a w_3$. A similar reasoning gives us $s_1 R_a s_3$ as well. Thus by Definition 3 $(w_1, s_1)\mathcal{R}'_a(w_3, s_3)$. □

It is quite evident that if we didn't force the accessibility relations on action models to be equivalence relations the result of a restricted modal product need not be an $S5$ Epistemic Model. Analogous results can be stated for different epistemic logics.

Example 1. An example of the restricted modal product in work can be seen in Figure 1. $\mathcal{M}$ is an $S5$ epistemic model, M is an $S5$ action model. Relations forced by transitivity and reflexivity are omitted for better clarity.

This picture can be used for an earlier example with Alice, Bob and Carol. Let us designate the fact that Alice loves Bob with atom p. Before

[4] The common definition of updates allows for non-deterministic updating, that is updating with action models that need not have a designated action state. We will however omit this for the sake of simplicity. For more information about non-deterministic updates see [6].

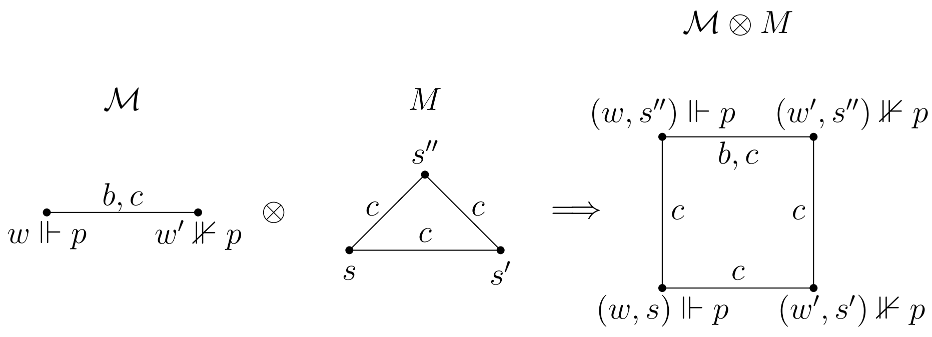


Figure 1: An example of creating a new epistemic model. $pre(s) = p$, $pre(s') = \neg p$ and $pre(s'') = p \vee \neg p$.

Alice's and Bob's meeting only Alice knows whether p holds or $\neg p$ holds. The action states s, s' and s'' correspond to actions 'Alice tells Bob that p', 'Alice tells Bob that $\neg p$' and 'Alice and Bob talk about something else' respectively. Both Alice and Bob can differentiate between all three actions while Carol cannot. The result matches our intuition. Not only Carol doesn't know whether p or $\neg p$, she doesn't know whether Bob knows either! This is because Alice and Bob could have talked about weather instead of love. If that was true Bob couldn't get to know p or $\neg p$ as well.

This example shows a crucial difference between public announcements and action updates—the resulting epistemic model can have more states than the original one. This property is the main reason behind the power of action models; $\mathcal{M} \vDash K_c \neg K_b p$ while $(\mathcal{M} \otimes M) \nvDash K_c \neg K_b p$. The update caused Carol to lose a part of her knowledge. This expansion of models is also the cause of the biggest obstacle in creating the Logic of Hybrid Action Models. More on this will be said in Section 4.2

It is interesting to note that action models can simulate public announcements. The corresponding action model for a public announcement of formula φ is a model $P_\varphi = \langle \{s\}, (\langle s, s \rangle)_{a \in A}, \{\langle s, \varphi \rangle\} \rangle$. $P_{(p \vee \neg p)}$ creates an isomorphic copy of the original model, much like a public announcement of a tautology does. This simulation lets us pronounce the Logic of Action Models (AML) not closed under substitution for free. Public Announcement Logic is not closed under substitution because $[p]p$ is a tautology while $[\varphi]\varphi$ is not. $[p]p$ is equal to $p \to p$ by one of the reduction axioms. A counterexample to $[\varphi]\varphi$ is the well-known formula $[p \wedge \neg K_a p](p \wedge \neg K_a p)$. Thus neither AML is closed under substitution.

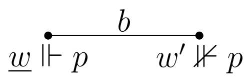
Let us present one more example before moving on to the next part. It is known as the Coordinated Attack Problem or Two Generals' Problem. It deals with the problem of communicating via an unreliable connection.

Example 2. There are two armies in this scenario. One of them is split in half by the second. Each of these halves is led by one general. They want to agree on a coordinated attack on the opposing army. If only one of the generals attacked his half of the army would be decimated. If however the generals manage to agree on a time of the attack they will have enough manpower to defeat the enemy. The only way for them to communicate is through a messenger. This messenger can be caught by the enemy when trying to travel from one general to the other. The generals will only attack if they become one hundred percent sure that the other one attacks as well. Will the generals ever agree on an attack? If so then when?

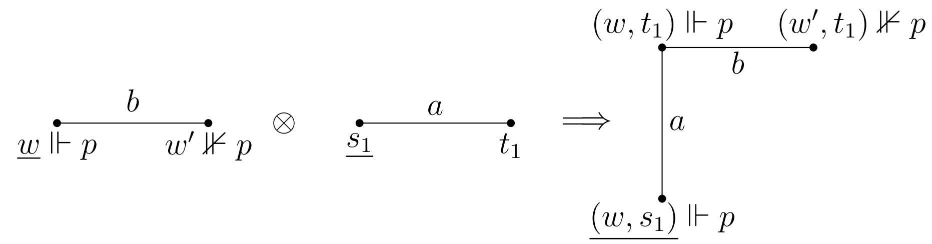
The answer is no, they will never attack. The problem lies in the unreliability of the messenger. Let's say the first general sends a message 'We will attack tomorrow at 6 a.m. Send a confirmation'. Let's also say that the messenger does get through and delivers this message to the second general. This is however not known to the first general. The second one will write a confirmation message and send it back. Again even though the messenger might have gotten through the second general cannot know this and will require a confirmation from the first one. This goes on and on until the next morning when neither of them attacks.

The reason for this is that both generals require a common knowledge of the message. The unreliable communication can never let them achieve it though. What is interesting however is that unlike the Logic of Public Announcements the Logic of Action Models allows us to model this example.

Let us denote the generals as agents a and b and the fact that the content of the message is 'We will attack tomorrow at 6 a.m.' as atom p. The situation starts with agent a knowing p and agent b not knowing it. This means we need at least two epistemic states, one satisfying p and the other not. Let us also underline the designated state for better clarity.

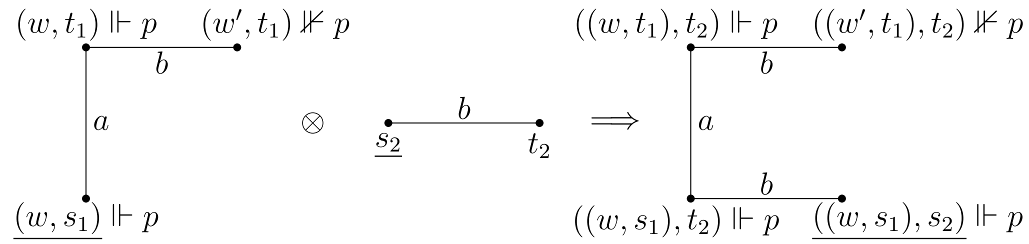


Agent a sends the message containing p. This message may or may not have been intercepted. This means our action model needs at least two action states s_1 and t_1—one corresponding to the action 'the message containing p got through' and the other one 'the message didn't get through, no matter the content' respectively. Clearly $pre(s_1) = p$ and $pre(t_1) = p \vee \neg p$. Let us see how the resulting epistemic model looks.



Now in our new designated state both generals know p but a doesn't know that b knows it. This is because a doesn't know whether the message got through or not. That's why there's an accessibility relation for a between s_1 and t_1 in the action model.

Sending the confirmation message behaves in the same way as sending a regular message, only the content of the message changes. Agent b wants to send back 'I recieved your message so I know that p'. This means that $pre(s_2) = K_b p$ and $pre(t_2) = p \vee \neg p$.



Now agent a knows that b knows that p because he just got the confirmation. But b doesn't know that a knows that b knows that p, in symbols $\neg K_b K_a K_b p$. If a wanted to send a confirmation of a confirmation, the content of this message (and precondition of the action of sending this message) will be $K_a K_b p$. But it is clear that these two agents can never achieve a common knowledge of p in this way. t_i will always have a precondition of $p \vee \neg p$ and whenever such action states are present in an action model they simply copy the original model. And since it is always accessible from s_i by some relation there is no way to (i) get rid of a state that does not satisfy p or (ii) cut the accessibility relation leading to it. Thus a state that doesn't satisfy p

will always be accessible by $(\mathcal{R}_{\{a,b\}})^*$ from everywhere in the model, where $(\mathcal{R}_{\{a,b\}})^*$ is a transitive closure of the union of relations $\mathcal{R}_a$ and $\mathcal{R}_b$ and so achieving $C_{\{a,b\}}p$ is impossible.

2.2 Semantics

We've been using the $\Vdash$ symbol without properly defining it. This doesn't need to bother us much. Its definition is the same for non-dynamic formulas as in other epistemic logics. The important part of the definition lies in formulas of the form $[(M, s)]\psi$. Let us give a full definition for further reference though.

Definition 4. *Let $\mathcal{M} = \langle W, \mathcal{R}_a, V\rangle$ be an $S5$ epistemic model in the language $\mathcal{L}_{KC\otimes}$ and $w \in W$. Let $M = \langle S, R_a, pre\rangle$ be an $S5$ action model and $s \in S$. Finally let $a \in A$, $B \subseteq A$, $p \in P$ and φ, ψ be arbitrary formulas in the language $\mathcal{L}_{KC\otimes}$. We define satisfaction relation $\Vdash$ as follows:*

$$\begin{aligned}
\mathcal{M}, w \Vdash p \quad &\textit{iff} \quad w \in V(p)\\
\mathcal{M}, w \Vdash \varphi \vee \psi \quad &\textit{iff} \quad \mathcal{M}, w \Vdash \varphi \textit{ or } \mathcal{M}, w \Vdash \psi\\
\mathcal{M}, w \Vdash \varphi \rightarrow \psi \quad &\textit{iff} \quad \mathcal{M}, w \nVdash \varphi \textit{ or } \mathcal{M}, w \Vdash \psi\\
\mathcal{M}, w \Vdash \varphi \wedge \psi \quad &\textit{iff} \quad \mathcal{M}, w \Vdash \varphi \;\&\; \mathcal{M}, w \Vdash \psi\\
\mathcal{M}, w \Vdash \neg\varphi \quad &\textit{iff} \quad \mathcal{M}, w \nVdash \varphi\\
\mathcal{M}, w \Vdash K_a\varphi \quad &\textit{iff} \quad \forall w' \in W\ (w\mathcal{R}_a w' \Rightarrow \mathcal{M}, w' \Vdash \varphi)\\
\mathcal{M}, w \Vdash C_B\varphi \quad &\textit{iff} \quad \forall w' \in W\ (w(\mathcal{R}_B)^* w' \Rightarrow \mathcal{M}, w' \Vdash \varphi)\\
\mathcal{M}, w \Vdash [(M, s)]\varphi \quad &\textit{iff} \quad \mathcal{M}, w \Vdash \textit{pre}(s) \Rightarrow (\mathcal{M} \otimes M), (w, s) \Vdash \varphi
\end{aligned}$$

where $\mathcal{R}_B = \bigcup_{b\in B} \mathcal{R}_b$ and * is a transitive closure.

Recall the condition from Definition 2 that says that for any formula of the form $[(M, s)]\varphi$ and for any $t \in S$, $pre(t)$ is an already formed formula. This is to prevent degenerated formulas from being created. Imagine a formula $[(M, s)]p$ such that $pre(s) = [(M, s)]p$. By definition $\mathcal{M}, w \Vdash [(M, s)]p$ iff $(\mathcal{M}, w \Vdash [(M, s)]p \Rightarrow (\mathcal{M} \otimes M), (w, s) \Vdash p)$. The problems become more apparent when one tries to use the reduction axiom $[(M, s)]p \leftrightarrow (pre(s) \rightarrow p)$. In this case $[(M, s)]p$ is equivalent to $[(M, s)]p \rightarrow p$, which is equivalent to $([(M, s)]p \rightarrow p) \rightarrow p \ldots$ Proof of completeness by translation would stop working because there is no way to get rid of the action update. There are even worse offenders out there, for instance a formula $[(M, s)](p \wedge \neg p)$, where $pre(s) = [(M, s)](p \wedge \neg p)$. Is this formula satisfiable?

There is a clear parallel between the definition of $\mathcal{M}, w \Vdash [(M, s)]\varphi$ in Action Model Logic and $\mathcal{M}, w \Vdash [\psi]\varphi$ in Public Announcement Logic. The crucial difference is between the way the updated models are created. Where

PAL updates models by cutting away states that don't satisfy the announced formula, AML creates a cartesian product and then cuts away the illegal states.

2.3 Axiomatics

It is interesting that in a language without common knowledge the Logic of Action Models has a sound and complete axiomatics for the same reason that Public Announcement Logic does—both logics have reduction axioms for all formulas of the form $[(M,s)]\varphi$ and $[\psi]\varphi$ respectively. The axioms of AML are very similar to those of PAL. We need one more definition before presenting the axiomatics of AML though.

Definition 5 (**Composition**). *Let $M = \langle S, R_a, pre\rangle$ and $M' = \langle S', R'_a, pre'\rangle$ be two $S5$ Action Models in language $\mathcal{L}_{KC\otimes}$. Call $(M; M') = \langle S'', R''_a, pre''\rangle$ a Composition Action Model if*

$$S'' = S \times S'$$

$$\forall a \in A \ (s, s')R''_a(t, t') \ \textit{iff} \ sR_at \ \& \ s'R_at'$$

$$pre''((s, s')) = \neg[(M, s)]\neg pre'(s')$$

Moreover given two Pointed Action Models (M, t) and (M', t') define their composition $((M, t); (M', t'))$ as a pointed action model $((M; M'), (t, t'))$.

We will now compare the axiomatics of Action Model Logic with the axiomatics of Public Announcement Logic. Let us first give the axiom schemas and derivation rules that are common for both systems. These are the axioms and rules of the system $S5_C$ or more precisely $S5^n_C$, where n corresponds to the number of agents. CPL stands for all instantiations of classical propositional tautologies in the corresponding languages.

	CPL
(A1)	$\vdash K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
(A2)	$\vdash K_a\varphi \to \varphi$
(A3)	$\vdash K_a\varphi \to K_aK_a\varphi$
(A4)	$\vdash \neg K_a\varphi \to K_a\neg K_a\varphi$
(A5)	$\vdash C_B(\varphi \to \psi) \to (C_B\varphi \to C_B\psi)$
(A6)	$\vdash C_B\varphi \to (\varphi \wedge E_BC_B\varphi)$
(A7)	$\vdash C_B(\varphi \to E_B\varphi) \to (\varphi \to C_B\varphi)$
(R1)	If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$
(R2)	If $\vdash \varphi$ then $\vdash K_a\varphi$
(R3)	If $\vdash \varphi$ then $\vdash C_B\varphi$

Adding the following axioms and rules creates the Logic of Public Announcement with common knowledge:

(A8)	$\vdash [\alpha]p \leftrightarrow (\alpha \rightarrow p)$
(A9)	$\vdash [\alpha]\neg\varphi \leftrightarrow (\alpha \rightarrow \neg[\alpha]\varphi)$
(A10)	$\vdash [\alpha](\varphi \wedge \psi) \leftrightarrow [\alpha]\varphi \wedge [\alpha]\psi$
(A11)	$\vdash [\alpha]K_a\varphi \leftrightarrow (\varphi \rightarrow K_a[\alpha]\varphi)$
(A12)	$\vdash [\alpha][\beta]\varphi \leftrightarrow [\alpha \wedge [\alpha]\beta]\varphi$
(R4)	If $\vdash \varphi$ then $\vdash [\alpha]\varphi$
(R5)	If $\vdash \chi \rightarrow [\alpha]\varphi$ and $\vdash \chi \wedge \alpha \rightarrow E_B\chi$ then $\vdash \chi \rightarrow [\alpha]C_B\varphi$

While adding the following will result in the Logic of Action Models with common knowledge:

(A8)	$\vdash [(M,s)]p \leftrightarrow (pre(s) \rightarrow p)$
(A9)	$\vdash [(M,s)]\neg\varphi \leftrightarrow (pre(s) \rightarrow \neg[(M,s)]\varphi)$
(A10)	$\vdash [(M,s)](\varphi \wedge \psi) \leftrightarrow [(M,s)]\varphi \wedge [(M,s)]\psi$
(A11)	$\vdash [(M,s)]K_a\varphi \leftrightarrow (pre(s) \rightarrow \bigwedge_{tR_as} K_a[(M,t)]\varphi)$
(A12)	$\vdash [(M,s)][(M',s')]\varphi \leftrightarrow [((M;M'),(s,s'))]\varphi$
(R4)	If $\vdash \varphi$ then $\vdash [(M,s)]\varphi$
(R5)	For a given (M,s) and formulas χ_t for all t such that $t(R_B)^*s$, if for all $a \in B$ and u such that uR_at it holds that: $\vdash \chi_t \rightarrow [(M,t)]\varphi$ and $\vdash \chi_t \wedge \mathrm{pre}(t) \rightarrow K_a\chi_u$, then $\vdash \chi_s \rightarrow [(M,s)]C_B\varphi$.

The axiomatics of AML showed above is sound and complete. The proof can be found in [6]. We will later base our axiomatics of the Logic of Hybrid Action Models on this one. The axioms and deductive rules of HAM will be given in Section 4.2.2.

The non-dynamic part is the same for both logics. But all the reduction axioms share the same difference—the announced formula α gets replaced with the precondition of the designated action state $pre(s)$ while the announcement $[\alpha]$ itself is replaced by an update with a pointed action model $[(M,s)]$. There are three more important changes concerning the reduction axioms for knowledge, iterated actions and a rule dealing with common knowledge. Each of these warrants a short commentary.

2.3.1 Knowledge

$$[(M,s)]K_a\varphi \leftrightarrow (pre(s) \to \bigwedge_{tR_as} K_a[(M,t)]\varphi)$$

The purpose of the change is that we have to take into account that agent a may not be able to distinguish between action s and other actions. Since he doesn't know whether it was action s or t that happened (if tR_as) he has to take both possibilities into consideration. Thus a simple translation of the axiom of PAL

$$[\alpha]K_a\varphi \leftrightarrow (\alpha \to K_a[\alpha]\varphi)$$

to

$$[(M,s)]K_a\varphi \leftrightarrow (pre(s) \to K_a[(M,s)]\varphi)$$

won't do. We can show this with the following counterexample.

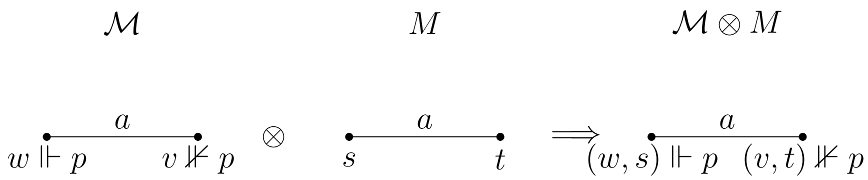


Figure 2: $pre(s) = p$ and $pre(t) = \neg p$. Agent a cannot distinguish between actions s and t and therefore he cannot distinguish between updates $[(M,s)]$ and $[(M,t)]$. $\mathcal{M}, w \Vdash (pre(s) \to K_a[(M,s)]p)$ while $\mathcal{M}, w \nVdash [(M,s)]K_ap$ which is easy to verify.

2.3.2 Composition

$$[(M,s)][(M',s')]\varphi \leftrightarrow [(M;M'),(s,s')]\varphi$$

This axiom is the main reason for the introduction of the composition operation. Its behavior is very similar to the corresponding PAL axiom. Since in PAL $(\varphi \wedge [\varphi]\psi) \leftrightarrow \langle\varphi\rangle\psi$, we get $[\alpha][\beta]\varphi \leftrightarrow [\langle\alpha\rangle\beta]\varphi$. If we were to define $\langle(M,s)\rangle\varphi$ as $\neg[(M,s)]\neg\varphi$ we would get that $pre''(s,s') = \langle(M,s)\rangle pre'(s')$ as in the definition of composition. Now notice that from the semantic point of view $\mathcal{M}, w \Vdash [\langle\alpha\rangle\beta]\varphi$ means in PAL

$$\mathcal{M}, w \Vdash \langle\alpha\rangle\beta \Rightarrow \mathcal{M}_{|\langle\alpha\rangle\beta}, w \Vdash \varphi,$$

where $\mathcal{M}_{|\langle\alpha\rangle\beta}$ is the model $\mathcal{M}$ reduced to those states that satisfy $\langle\alpha\rangle\beta$ (along with a proper reduction on accessibility relations and valuation). On the other hand $\mathcal{M}, w \Vdash [(M; M'), (s, s')]\varphi$ in AML is equal to

$$\mathcal{M}, w \Vdash \langle(M, s)\rangle pre'(s') \Rightarrow (\mathcal{M} \otimes (M; M')), (w, (s, s')) \Vdash \varphi,$$

where $(\mathcal{M} \otimes (M; M'))$ is a model whose domain is a cartesian product of $\mathcal{M}$ and $(M; M')$ reduced to those states that satisfy $\langle(M, s)\rangle pre'(s')$ (again with a corresponding reduction of accessibility relations and valuation). If the domains of both M and M' were singletons we could simplify it to

$$\mathcal{M}, w \Vdash \langle(M, s)\rangle pre'(s') \Rightarrow \mathcal{M}_{|\langle(M,s)\rangle pre'(s')}, w \Vdash \varphi,$$

which gives us a clear paralel with the corresponding PAL axiom.

2.3.3 Common Knowledge

If $\vdash \chi_t \rightarrow [(M, t)]\varphi$ and $\vdash \chi_t \wedge \mathrm{pre}(t) \rightarrow E_B\chi_u$ then $\vdash \chi_s \rightarrow [(M, s)]C_B\varphi$

The change to this rule is analogous to the change of the axiom of knowledge, described in Section 2.3.1. When talking about a satisfaction of common knowledge of group B in some state we need to look at all states accessible by a transitive closure of a union of all accissibility relations of all agents in group B, that is all states accessible by $(\bigcup \mathcal{R}_B)^*$ as defined earlier. However a similar approach must be applied to action models as well. We have to look at all the action states accessible by $(\bigcup R_B)^*$ from the designated state.

3 Hybrid Logic

The action models give our agents a strong tool for communication. They also expand what the agents can communicate about—other actions. This expansion is, with the exception of actions pertaining common knowledge, illusory since these actions can be translated to the original non-dynamic language with the use of reduction axioms. It is the Hybrid Logic that truly gives the agents something to talk about. They redefine the language of a given (modal) logic by introducing a new set of propositional atoms called nominals. These nominals behave in the same way as regular atoms but for one crucial difference. Each nominal can be satisfied in only a single state of a given model. This means that if we were given a hybrid logic model, a nominal i and a w member of the domain of the model such that $w \Vdash i$ we would immediatelly know that for all v from the same model disctinct from w it must hold that $v \nVdash i$. In this case i serves as an internal name for the state w, in the sense that it is inside the language itself. We will however allow a state to have several different names and we'll also allow nameless states. This language expansion itself permits us to define new classes of frames with modal formulas. For example the formula $i \rightarrow K_a\neg i$ holds only in frames where $\mathcal{R}_a$ is an antireflexive relation while $M_a i$ holds only in frames with a universal accessibility relation for agent a, that is $\forall w\ wR_a w$. More information on defining new frame classes can be found in [2] while other interesting examples are in [7]. Let us give a first draft of what a hybrid language and an epistemic model in such a language are.

Definition 6. *Let $\mathcal{L} = \{\wedge, \vee, \rightarrow, \neg, K_a\} \cup A \cup P$ be a multimodal epistemic language. Call $\mathcal{L}_H = \{\wedge, \vee, \rightarrow, \neg, K_a\} \cup A \cup P \cup NOM$ a Hybrid Language if $NOM \cap \mathcal{L} = \emptyset$ and NOM is countably infinite. A formula φ in language $\mathcal{L}$ is inductively defined as follows:*

$$\varphi \ := \ p \mid i \mid (\psi \wedge \chi) \mid (\psi \vee \chi) \mid (\psi \rightarrow \chi) \mid \neg\psi \mid K_a\psi$$

where ψ, χ are well-formed formulas, $p \in P$, $i \in NOM$ and $a \in A$.

As stated earlier the nominals act very much like propositional atoms. They have the same syntax and a very similar semantics. The only difference is that for any valuation V, $V \upharpoonright NOM$ is a total function that always assigns singletons.

Definition 7. *A Hybrid Model in a hybrid language $\mathcal{L}_H$ is an ordered triple $\mathcal{M} = \langle W, \mathcal{R}_a, V\rangle$ where W is a non-empty set, $\forall a \in A$ R_a is a binary relation and $V : P \cup NOM \rightarrow \wp(W)$ is a partial function such that for all nominals i it holds that $|V(i)| = 1$.*

3.1 Semantics

The assignment of nominals to states is provided by the valuation function in the same way as assignment of atoms to states. The only difference is that for any $i \in NOM$ $|V(i)| = 1$.[5] The definition of the satisfaction relation $\Vdash$ is as usual with the addition of

$$\mathcal{M}, w \Vdash i \text{ iff } w \in V(i)$$

There is a great synergy between Hybrid and Epistemic Logics, especially the system $S5$. Knowing the name of a state means knowing everything about it. When $\mathcal{M}, w \Vdash K_a i$, the only state accessible by $\mathcal{R}_a$ from w is w itself; there are no other states that agent a finds possible. And when $\mathcal{M}, w \Vdash K_a i$, agent a also knows all truths that hold in that state—not only atomic facts but also all epistemic formulas. For example he may know that another agent b finds it possible that $\neg i$ and therefore has less information about the current state of affairs than a. Agent a has in that state *complete information* about the state. It is not surprising then that if agents a and b have complete information about a state, their knowledge coincides, that is $K_a i \wedge K_b i \rightarrow (K_a \varphi \leftrightarrow K_b \varphi)$ is a tautology.

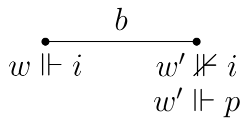


Figure 3: $\mathcal{M}, w \Vdash K_a i$ implies that all information about the state is known to a in w, such as $K_a M_b p$ or $K_a M_b K_a \neg i$.

There exists no translation from a hybrid language to a corresponding epistemic language. There are hybrid formulas that cannot be expressed in the non-hybrid language. The easiest example would be formula i for $i \in NOM$ which defines a class of all frames with a single state. There is a simple proof of this.

[5]Since NOM is countably infinite it will be very common for a state to have infinitely many names. We will omit most of them in our examples in the same way that we omit most atoms that are satisfied in these states. It is also important to note that unlike propositional atoms it is required for all nominals and every model that there must be a state that satisfies that nominal. This condition will come into play later on.

Lemma 2. *Let $i \in NOM$ and $\mathcal{F}$ be a hybrid frame. Then $\mathcal{F} \vDash i$ iff $|\dot{\mathcal{F}}| = 1$.*

Proof.
($\Leftarrow$) i must be satisfied in some state in every model no matter the valuation. If $\mathcal{F}$ has only a single state it has to satisfy i.
($\Rightarrow$) Let $\mathcal{F} \vDash i$ and let $|\dot{\mathcal{F}}| > 1$. Because i cannot be satisfied in two distinct states regardless of valuation, there must exist a state that doesn't satisfy i in every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$. Thus $\mathcal{M} \nvDash i$ and $\mathcal{F} \nvDash i$. □

We will show another example of a formula defining new a frame class, the antireflexivity that was mentioned earlier. Let the underlaying logic be K instead of $S5$ for the following lemma.

Lemma 3. *Let $i \in NOM$ and $\mathcal{F}$ be a hybrid frame. Then $\mathcal{F} \vDash i \rightarrow K_a \neg i$ iff $\mathcal{R}_a$ in $\mathcal{F}$ is antireflexive, that is $\forall w\ \neg w \mathcal{R}_a w$.*

Proof.
($\Rightarrow$) Let $\mathcal{F} \vDash i \rightarrow K_a \neg i$ and let w be any state in $\dot{\mathcal{F}}$. We will show that $\neg w \mathcal{R}_a w$. $\mathcal{F} \vDash i \rightarrow K_a \neg i$ means it holds for any valuation; let's choose the valuation V such that $V(i) = \{w\}$. Since $\langle \mathcal{F}, V \rangle, w \Vdash i \rightarrow K_a \neg i$ we get $\langle \mathcal{F}, V \rangle, w \Vdash K_a \neg i$, that is all accessible states must not be members of $V(i) = \{w\}$. Thus w must not be accessible by $\mathcal{R}_a$ from itself.
($\Leftarrow$) Now let $\mathcal{R}_a$ be antireflexive and let V be any valuation on $\mathcal{F}$ and w any state in $\dot{\mathcal{F}}$. If $\langle \mathcal{F}, V \rangle, w \nVdash i$ it trivially holds that $\langle \mathcal{F}, V \rangle, w \Vdash i \rightarrow K_a \neg i$ so let $\langle \mathcal{F}, V \rangle, w \Vdash i$. Because $i \in NOM$, w is the only state that satisfies i. So all we need to check is that w doesn't access itself with $\mathcal{R}_a$ but that is true due to the assumption that $\mathcal{R}_a$ is antireflexive. □

The so called pure formulas, formulas without any propositional atoms, are very important from the frame defining point of view. Details can be found in Section 3 of [2].

The addition of nominals allows agents to truly agree on a state. An agent can know all the truths that hold in a given state but he can never be sure there aren't any other states that hold the same truths. This uncertainty disappears once he learns the name of the state.

We will add three new operators as well: the reference operator @, the naming operator ↓ along with state variables and the existential operator E. Let us discuss each of these operators separately.

3.1.1 Reference Operator

Agents may ask not only what is true in the current state. They might also be interested in other states. Let's look at the following example.

$$w \overset{\bullet}{\Vdash} p, i \qquad \begin{array}{l} w' \overset{\bullet}{\nVdash} p \\ w' \Vdash j \end{array}$$

In state w agent a knows p. He also knows i and $\neg j$. He might be curious whether p also holds in a state named j (as mentioned earlier such a state must exist somewhere). However he cannot check this directly because the state named j is inaccessible; $\mathcal{M}, w \Vdash C_{\{a\}} \neg j$. But this is what the reference operator allows him to ask. At state named j the atom p does not hold; in symbols $\neg @_j p$. And agent a knows this in the whole model.

When checking whether $@_i \varphi$ holds in a state we look for the state named i and ask whether φ holds there. This intuition gives us a fairly straightforward extension of the definition of $\Vdash$.

$$\mathcal{M}, w \Vdash @_i \varphi \text{ iff } \exists w' \in V(i)\ \mathcal{M}, w' \Vdash \varphi$$

It is easy to see that the existential quantifier in the above definition can be replaced with a universal one without changing the meaning of the definition. This is caused by $V(i)$ always being a singleton. This gives the reference operator an interesting property—it is auto-dual, that is $@_i \varphi \equiv \neg @_i \neg \varphi$. Another interesting fact is that for any model $\mathcal{M}$ and state w it holds that $\mathcal{M}, w \Vdash @_i \varphi$ iff $\mathcal{M} \vDash @_i \varphi$. It is also a normal modal operator, that is $\vDash @_i(\varphi \to \psi) \to (@_i \varphi \to @_i \psi)$ and if $\vDash \varphi$ then $\vDash @_i \varphi$.

Note that formulas $@_i \varphi$ and $i \to \varphi$ behave very similarly but there is a curious difference between them. It holds that for any model $\mathcal{M} \vDash @_i \varphi$ iff $\mathcal{M} \vDash i \to \varphi$. This is not true once we fix a state though—$\mathcal{M}, w \Vdash @_i \varphi$ implies $\mathcal{M}, w \Vdash i \to \varphi$ but not the other way around.

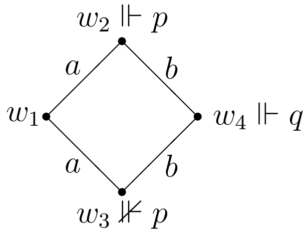
3.1.2 Binder Arrow Operator

The binder arrow operator $\downarrow$ allows us to give an internal name to the current state for further reference. It works as a quantifier. The proper syntax for it is $\downarrow x\ \varphi$ where x is a state variable—a member of a new parametre set called $SVAR$. The meaning of such a formula is 'Name the current state x and ask whether a formula φ holds in it'. This allows us to create complex formulas that talk about accessibility relations, especially with the help of the reference operator. Consider the following formula:

$$\downarrow x\ (M_a(p \wedge M_b \downarrow y\ (q \wedge @_x(M_a(\neg p \wedge M_b y)))))$$ [6]

One of the models that satisfies such a formula (in state w_1) is:

$\mathcal{M}$



The question is: how do we assign these variables to states? If we used the valuation function we would get some unwanted results. Let us return to the previous model and fix the state variables.

$\mathcal{M}'$

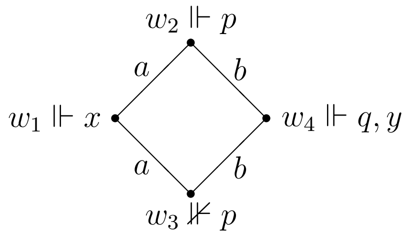


Figure 4: The earlier model with state variables fixed by valuation.

It is still true that

$$\mathcal{M}', w_1 \Vdash \downarrow x\ (M_a(p \wedge M_b \downarrow y\ (q \wedge @_x(M_a(\neg p \wedge M_b y)))))$$

but

$$\mathcal{M}', w_1 \nVdash \downarrow y\ (M_a(p \wedge M_b \downarrow z\ (q \wedge @_y(M_a(\neg p \wedge M_b z))))).$$

This would defeat the purpose of variables. We do not want their names to matter. They are only agents' labels, not true names of the states. This

[6]Name the current state x. It can access via $\mathcal{R}_a$ a state that both satisfies p and can access via $\mathcal{R}_b$ a state, called y, that satisfies q. Looking back at the state x we also see another state via $\mathcal{R}_a$ which doesn't satisfy p but this one can also access state y via $\mathcal{R}_b$.

is the reason for having a new function, the assignment function g, such that $g\colon SVAR \to \dot{\mathcal{M}}$ which is independent on a model. Writing $\mathcal{M}, w \Vdash \varphi$ will then mean $\mathcal{M}, w, g \Vdash \varphi$ for any assignment function g. And fixing the name of the current state with the binder arrow simply means changing the assignment function to one that assigns the variable x to the current state.

$$\mathcal{M}, w, g \Vdash \downarrow x\ \varphi \text{ iff } \mathcal{M}, w, g' \Vdash \varphi$$

where $g'(x) = w$ and $g'(y) = g(y)$ for all $y \neq x$.

Syntactically, state variables behave in a similar way as nominals. x and $@_x p$ are both well-formed formulas. Their semantics will be given in Section 3.2. We can distinguish between bound and free state variables, similarly to variables in the first order logic, but this distinction is mostly unimportant for this thesis.

Our assignment function is partial. It doesn't need to assign anything until we force it to.[7] The binder arrow is also a normal modal operator.

3.1.3 Existential Operator

The existential operator E allows us to jump all over the model regardless of accessibility relations and names. It works like a diamond-style modality for an agent who can access any state. Its definition is pretty straightforward.

$$\mathcal{M}, w \Vdash E\varphi \text{ iff } \exists w' \in \dot{\mathcal{M}}\ \mathcal{M}, w' \Vdash \varphi$$

It is interesting to note that $E(i \wedge \varphi) \equiv @_i\varphi$. Thus, in some sense, the existential operator is stronger than the reference one since E can define $@$.

Same as for the reference operator, for any model $\mathcal{M}$ and state w it holds that $\mathcal{M}, w \Vdash E\varphi$ iff $\mathcal{M} \vDash E\varphi$. The existential operator is not a normal modal operator however.[8] Let us show a counterexample for the formula $E(\varphi \to \psi) \to (E\varphi \to E\psi)$.

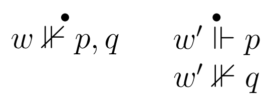


Figure 5: In the whole model $E(p \to q)$ and Ep hold while Eq does not.

[7]There is little difference between having a partial or total assignment function in Hybrid Logics. It will become important later though. See Section 4.1 for more details.

[8]But its dual $U\varphi \equiv \neg E \neg \varphi$ is normal.

3.2 Bringing Them All Together

We can now proceed to the full definition of a hybrid language and well-formed formulas.

Definition 8. *Call*

$$\mathcal{L}_{H(@,\downarrow,E)} = \{\wedge, \vee, \rightarrow, \neg, K_a, @_u, \downarrow, E\} \cup A \cup P \cup NOM \cup SVAR,$$

where $u \in NOM \cup SVAR$, $x \in SVAR$ a Hybrid Language, if $\{\wedge, \vee, \rightarrow, \neg, K_a\} \cup A \cup P \cup NOM$ is a hybrid language, all the symbols $@_u$, $\downarrow$ and E are new and SVAR is a countably infinite set disjoint from $\mathcal{L}_H$ and does not contain the symbols $@_u$, $\downarrow$ nor E. A formula φ in language $\mathcal{L}_{H(@,\downarrow,E)}$ is inductively defined as follows:

$$\varphi \;:=\; p \mid u \mid (\psi \wedge \chi) \mid (\psi \vee \chi) \mid (\psi \rightarrow \chi) \mid \neg\psi \mid K_a\psi \mid @_u\psi \mid \downarrow x\ \psi \mid E\psi$$

where ψ, χ are well-formed formulas, $p \in P$, $a \in A$, $i \in NOM$, $x \in SVAR$ and $u \in NOM \cup SVAR$.

We can define the notions of free and bounded variables in the same way as in quantified logic with $\downarrow$ being the only quantifier.

The definition of satisfaction in a model is slightly complicated by the assignment function. Let us base our definition on [3].

Definition 9. *Let $\mathcal{M} = \langle W, R_a, V\rangle$ be a hybrid model in a hybrid language $\mathcal{L}_{H(@,\downarrow,E)}$ and let g be an assignment function on $\mathcal{M}$. The satisfaction relation $\Vdash$ is inductively defined as follows*

$$\begin{array}{rll}
\mathcal{M}, w, g \Vdash p & \textit{iff} & w \in V(p) \\
\mathcal{M}, w, g \Vdash i & \textit{iff} & w \in V(i) \\
\mathcal{M}, w, g \Vdash x & \textit{iff} & x \in dom(g)\ \&\ g(x) = w \\
\mathcal{M}, w, g \Vdash \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w, g \Vdash \varphi\ \&\ \mathcal{M}, w, g \Vdash \psi \\
\mathcal{M}, w, g \Vdash \varphi \vee \psi & \textit{iff} & \mathcal{M}, w, g \Vdash \varphi\ \textit{or}\ \mathcal{M}, w, g \Vdash \psi \\
\mathcal{M}, w, g \Vdash \varphi \rightarrow \psi & \textit{iff} & \mathcal{M}, w, g \nVdash \varphi\ \textit{or}\ \mathcal{M}, w, g \Vdash \psi \\
\mathcal{M}, w, g \Vdash \neg\varphi & \textit{iff} & \mathcal{M}, w, g \nVdash \varphi \\
\mathcal{M}, w, g \Vdash K_a\varphi & \textit{iff} & \forall w' \in W\ (w\mathcal{R}_a w' \Rightarrow \mathcal{M}, w', g \Vdash \varphi) \\
\mathcal{M}, w, g \Vdash @_i\varphi & \textit{iff} & \exists w' \in V(i)\ \mathcal{M}, w', g \Vdash \varphi \\
\mathcal{M}, w, g \Vdash @_x\varphi & \textit{iff} & x \in dom(g)\ \&\ \mathcal{M}, g(x), g \Vdash \varphi \\
\mathcal{M}, w, g \Vdash \downarrow x\varphi & \textit{iff} & \mathcal{M}, w, g' \Vdash \varphi \\
\mathcal{M}, w, g \Vdash E\varphi & \textit{iff} & \exists w' \in W\ \mathcal{M}, w', g \Vdash \varphi
\end{array}$$

where $p \in P$, $a \in A$, $w \in W$, $x \in SVAR$ and g' is such that $g'(x) = w$ and $g'(y) = g(y)$ for all $y \neq x$.

We mentioned that E can define @. This means that the language $\mathcal{L}_{H(\downarrow,E)}$ is as expressive as $\mathcal{L}_{H(@,\downarrow,E)}$ and $\mathcal{L}_{H(E)}$ is as expressive as $\mathcal{L}_{H(@,E)}$.

An interesting application of the existential operator in conjunction with the binder arrow operator is in creating the backwards modalities. $K_a\varphi$ means that all the $\mathcal{R}_a$ accessible states satisfy φ. But what if we wanted to say that every $\mathcal{R}_a$ predecessor satisfies φ instead? Such a question makes no sense in the $S5$ system due to the symmetry condition of all accessibility relations. But this predecessor modality would be very useful in other systems though.

Lemma 4. *For any agent a, model $\mathcal{M}$ without any constrains on the accessibility relation $\mathcal{R}_a$, state $w \in \dot{\mathcal{M}}$, assignment function g and any φ not containing x*

$$\mathcal{M}, w, g \Vdash \downarrow x\ U(M_a x \rightarrow \varphi) \textit{ iff } \forall w'(w'\mathcal{R}_a w \Rightarrow \mathcal{M}, w', g \Vdash \varphi)$$

Proof. Note that $\mathcal{M}, w, g \Vdash \varphi$ iff $\mathcal{M}, w, g' \Vdash \varphi$. This is because the only difference between g and g' is in the value of the state variable x and φ does not contain x.
($\Rightarrow$) The following statements are equivalent:

$$\mathcal{M}, w, g \Vdash \downarrow x\ U(M_a x \rightarrow \varphi)$$

$$\mathcal{M}, w, g' \Vdash U(M_a x \rightarrow \varphi)$$

$$\forall v\ \mathcal{M}, v, g' \Vdash M_a x \rightarrow \varphi$$

$$\forall v\ (\mathcal{M}, v, g' \nVdash M_a x \textit{ or } \mathcal{M}, v, g' \Vdash \varphi) \tag{1}$$

Now let w' be any state such that $w'\mathcal{R}_a w$. Since by definition $g'(x) = w$ we know that $\mathcal{M}, w, g' \Vdash x$. It follows that $\mathcal{M}, w', g' \Vdash M_a x$. This, along with (1), gives us $\mathcal{M}, w', g' \Vdash \varphi$ and thus $\mathcal{M}, w', g \Vdash \varphi$ as we wanted.
($\Leftarrow$) All we need to prove is the statement (1). So let us have any state v such that $\mathcal{M}, v, g' \Vdash M_a x$ and we'll show that $\mathcal{M}, v, g' \Vdash \varphi$. The following are again equivalent:

$$\mathcal{M}, v, g' \Vdash M_a x$$

$$\exists v'\ (v\mathcal{R}_a v'\ \&\ \mathcal{M}, v', g' \Vdash x)$$

$$\exists v'\ (v\mathcal{R}_a v'\ \&\ x \in \mathit{dom}(g')\ \&\ g'(x) = v')$$

Let us fix any such state v'. Since $g'(x) = w$, we know that $w = v'$. By the assumption $\forall w'(w'\mathcal{R}_a w \Rightarrow \mathcal{M}, w', g \Vdash \varphi)$ we know that $\mathcal{M}, v, g \Vdash \varphi$ and therefore $\mathcal{M}, v, g' \Vdash \varphi$. □

Notice that the limitation of φ not containing x needs not bother us. $SVAR$ is infinite so there are always state variables not occuring in φ so if φ did contain x, we could just pick one of the infinitely many variables y not in φ and replace all occurances of x in φ with y.

Unlike for the Logic of Action Models we will not present the sound and complete axiomatics of Hybrid Logic. It is not important for this work because we will base our system on a somewhat different axiomatics in a different language created by Jens Ulrik Hansen in [3]. This system will be shown in Section 4.1.

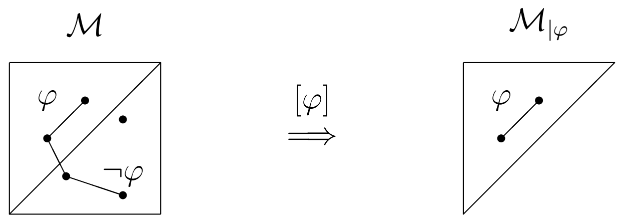
4 Logic of Hybrid Action Models

Let's briefly describe the Hybrid Logic with Partial Denoting Nominals before presenting the Logic of Hybrid Action Models. This detour will be useful for introducing some ideas behind HAM.

4.1 Hybrid Logic with Partial Denoting Nominals

The Hybrid Logic with Partial Denoting Nominals was created by Jens Ulrik Hansen and was first published in [3]. The purpose of this logic was to create a working combination of Hybrid Logic with Public Announcement Logic. It will serve us as a stepping stone towards the Logic of Hybrid Action Models. Mixing HL with PAL is a more difficult task than it might seem at first glance. That is because the very basis of Hybrid Logic is compromised.

Publicly announcing a formula φ basically cuts a model in two halves—one where φ holds and the other where it doesn't. The updated model looks just like the original but some states and accessibility relations may have gone missing (states that didn't satisfy φ and relations that led to those states).



But what if one of those states had an internal name, say i? The updated model wouldn't satisfy i anywhere which is in direct conflict with the requirement that all nominals have to be assigned. We could force the valuation of the updated model to satisfy i somewhere else, anywhere, but that would lead to many undesirable results. Another alternative is to cut only accessibility relations and keep all the states. This however introduces other problems, such as $[p]p$ not being a tautology anymore.

Hansen's solution was to relax the condition $\forall i \in NOM\ |V(i)| = 1$. It no longer needs to hold; $\forall i \in NOM\ |V(i)| \leq 1$ will suffice, that is $V \upharpoonright NOM$ becomes a partial function.[9] This small change introduces several problems

[9] A similar reasoning leads to the need of partial assignment functions as well.

Hansen managed to solve. The reference operator turned out to be the most problematic. Recall its two possible definitions:

$$\mathcal{M}, w \Vdash @_i\varphi \text{ iff } \exists w' \in V(i)\ \mathcal{M}, w' \Vdash \varphi$$

$$\mathcal{M}, w \Vdash @_i\varphi \text{ iff } \forall w' \in V(i)\ \mathcal{M}, w' \Vdash \varphi$$

Both of them are equal in Hybrid Logic thanks to the fact that for any nominal i, $V(i)$ is always a singleton. This stops being the case once we relax the condition as Hansen did. What if $V(i) = \emptyset$? The existential version of the definition would be satisfied nowhere in the model while the universal version would be satisfied everywhere. We have to choose one of these definitions for $@_i$ and Hansen chose the existential one. Now let us look what the universal definition corresponds to.

$$\begin{array}{rll} \mathcal{M}, w \Vdash \neg\varphi & \text{iff} & \mathcal{M}, w \nVdash \varphi \\ \mathcal{M}, w \Vdash @_i\neg\varphi & \text{iff} & \exists w' \in V(i)\ \mathcal{M}, w' \nVdash \varphi \\ \mathcal{M}, w \Vdash \neg @_i\neg\varphi & \text{iff} & \forall w' \in V(i)\ \mathcal{M}, w' \Vdash \varphi \end{array}$$

This means that the reference operator stops being auto-dual although the formula $@_i\varphi \leftrightarrow \neg @_i\neg\varphi$ will still hold in those models where $i \in dom(V)$. Hansen labelled the dual $\overline{@}_i$. The axiomatics had to be changed accordingly to stay sound and complete.

It is still true that knowing the name of a state means knowing everything about the state. Thus it also means that such a name must be present in the model somewhere—$K_a i \rightarrow Ei$ is a tautology in the $S5$ multimodal variant of Hybrid Logic with Partial Denoting Nominals. This implies that $K_a i \rightarrow (@_i i \leftrightarrow \overline{@}_i i)$ is also a tautology.

The following 12 axiomatic schemas and 5 rules compose the calculus of the Hybrid Logic with Partially Denoting Nominals (and assignments).

	CPL
(A1)	$\vdash K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
(A2)	$\vdash \overline{@}_u(\varphi \to \psi) \to (\overline{@}_u\varphi \to \overline{@}_u\psi)$
(A3)	$\vdash @_u\varphi \to \overline{@}_u\varphi$
(A4)	$\vdash \overline{@}_u u$
(A5)	$\vdash @_u @_v\varphi \to @_v\varphi$
(A6)	$\vdash u \to (\varphi \leftrightarrow @_u\varphi)$
(A7)	$\vdash M_a @_u\varphi \to @_u\varphi$
(A8)	$\vdash (@_u M_a v \wedge @_v\varphi) \to @_u M_a\varphi$
(A9)	$\vdash @_u\varphi \to @_u u$
(A10)	$\vdash @_u u \to (\overline{@}_u\varphi \to @_u\varphi)$
(A11)	$\vdash \overline{@}_u(\downarrow x\varphi \leftrightarrow \varphi[x := u])$, where$[x := u]$ means substituting all free occurrences of x with u in a given formula. Free occurrences are those that are not in the scope of any binder arrow.
(A12)	$\vdash @_i i \to Ei$
(R1)	If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$
(R2)	If $\vdash \varphi$ then $\vdash K_a\varphi$
(R3)	If $\vdash \varphi$ then $\vdash \overline{@}_u\varphi$
(R4)	If $\vdash \overline{@}_u\varphi$ then $\vdash \varphi$, if u does not occur in φ
(R5)	If $\vdash (@_u M_a v \wedge @_v\varphi) \to \psi$ then $\vdash @_u M_a\varphi \to \psi$, if $u \neq v$ and v occurs in neither φ nor ψ

where $u, v \in NOM \cup SVAR$, $x \in SVAR$, $i \in NOM$ and $a \in A$.

This axiomatic is sound and complete with respect to the kripke semantics. The completeness proof can be found in [3]. We will prove the soundness as a part of the soundness proof of the axiomatics of the Logic of Hybrid Action Models in Section 4.3.1.

4.2 Hybridizing Action Models

Action models can simulate public announcement as was shown earlier. This means that once we hybridize them we will have to deal with the same problem as Hansen. We will use his solution and assign names and variables only partially. There is one crucial difference between public announcements and action models however—the latter can not only reduce the size of models, it can also expand them as shown in earlier examples. This problem cannot be solved by relaxing the condition $|V(i)| \leq 1$ for any nominal i even further. If we allowed more states to have the same internal name we would just get a second sort of atoms, no different from the first one. The original intent was to give the agents a way for agreeing on a state without any doubt.

For example we still want a public announcement of a nominal to reduce a model to a single state (if such a nominal is present in the model at all). The problem lies in the definition of the restricted modal product $\otimes$. Given an epistemic model $\mathcal{M}$ and an action model M their update $\mathcal{M} \otimes M$ can have up to $|\mathcal{M}| \cdot |M|$ states in total. The valuation of atoms depends only on the epistemic model and its valuation function. If we were to introduce nominals and keep this definition as it is we would get just the undesirable results we mentioned. This is shown in the following example.

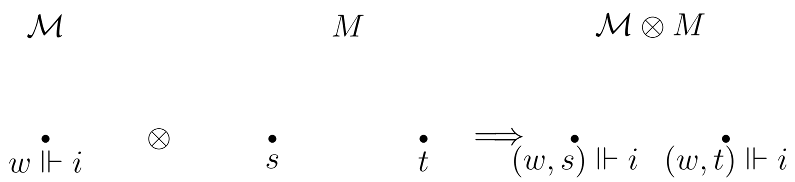


Figure 6: $pre(s) = pre(t) = p \vee \neg p$.

The solution presented in this thesis is to not only change the definition of $\otimes$ but also that of an action model. We have given internal names to epistemic states so why not give internal names to action states as well? It is a sensible thing to do from the philosophical point of view. Agents can fully describe a state of the world by pronouncing its name; now they can fully describe actions by giving their names too.[10] The definition of action models will now include a naming function for action states. Language will also have to be expanded by another set, that of action nominals NOM_A. And the set of epistemic nominals will have to change to include complex names. These complex names will either be simple names or ordered pairs where the first member is an already formed complex name and the second member is an action name.

[10] We will have to be careful with this though. We do not allow action models to stand alone as a formula and we won't allow the same for action names. The proper syntax will become clear in Definition 12.

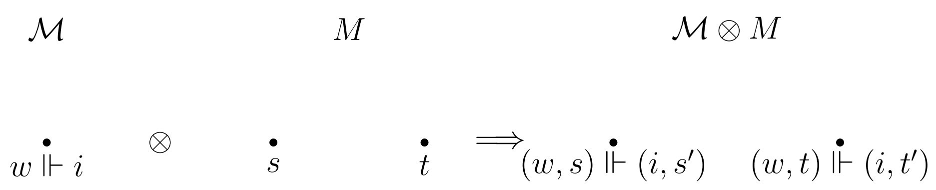


Figure 7: A proposed solution to the problem presented in Figure 6. In this example action states s and t have action names s' and t' respectively.

4.2.1 Semantics

We have many disjoint sets of parametres mixed in with several different sets of metanames. Let us agree on the following denotation to prevent any confusion. We will denote agents with letters a, b, c, propositional atoms with letters p, q, r, simple nominals with letters i, j, k, complex nominals with capitalized I, J, K, state variables with x, y, z and action names with s, t, u.[11] with indexes and primes when necessary. Metanames for epistemic states will be denoted w, v as usual. Let's define the complex name set now.

Definition 10. *Let NOM and NOM_A be two countably infinite disjoint sets. Define a set of Complex Nominals CNOM as the smallest set satisfying the following two conditions:*

$$NOM \subseteq CNOM$$

$$\forall I, s \ (I \in CNOM \ \& \ s \in NOM_A \Rightarrow (I, s) \in CNOM)$$

This means that the members of *CNOM* will be simple names or ordered pairs where the second member is an action name and the first member is an already formed complex name. This will create a nested structure of names where the rightmost action name corresponds to the latest action state with which we updated. Since both *NOM* and NOM_A sets are countably infinite, *CNOM* is countably infinite as well.

We will use the symbols for complex names $I, J, K, \dots$ in a similar way that we use the symbols for formulas $\varphi, \psi, \chi, \dots$ They will symbolize names whose internal structure is either irrelevant or unknown to us or both.

Similarly to how we defined a language and action models in Section 2.1 we need to give a general definition of an action model independent on language first and only then define the language we will be using.

[11] The action names coincide with their metanames. This needs not bother us. Context will make it clear whether we mean a metaname or an internal name in most cases.

Definition 11. *Let $\mathcal{L}$ be any language which contains sets A, P, $CNOM$, $SVAR$, NOM_A as subsets such that A is finite, P, $CNOM$, $SVAR$ and NOM_A are countably infinite and all five sets are pairwise disjoint. An $S5$ Hybrid Action Model is any structure $M = \langle S, R_a, pre, N\rangle$ where S is a finite non-empty set, for any $a \in A$, the relation R_a is an equivalence relation on S, $pre\colon S \to FLA$ is a total function from S to a set of all formulas of language $\mathcal{L}$ and $N\colon NOM_A \to S$ is a total function on S. A Pointed $S5$ Hybrid Action Model is an ordered couple (M, s) where M is an $S5$ hybrid action model and s is a member of its domain.*

The Logic of Hybrid Action Models will have two separate classes of models, same as the Logic of Action Models. Epistemic models will change only slightly (we expand the domains of valuations from subsets of $P \cup NOM$ to subsets of $P \cup CNOM$) while the change to action models is described in Definition 11.

The naming function N is defined in such a way that all action names are assigned to an action state and all action states have a name. Note that it is total, unlike $V \restriction CNOM$. This turned out to be the least problematic way to assign names to action states. Another difference between N and V is that the naming function doesn't range over sets of states but rather over states themselves; $rng(V) \subseteq \wp(\dot{\mathcal{M}})$ while $rng(N) = \dot{M}$. This might seem like a technicality but it will prove useful later. The reason is that we will sometimes want to know the internal name of the designated action state and use it while updating. For example we will want a formula $[(M, N(s))]\varphi$, where $s \in NOM_A$ to be well-formed while keeping $[(M, s)]$ for $s \in \dot{M}$ the only syntactically correct way to update a formula.

Definition 12. *Call*

$$\mathcal{L}^{KC\otimes}_{H(@,\downarrow,E)} = \{\wedge, \vee, \to, \neg, K_a, C_B, [(M, s)], @_u, \downarrow, E\} \\ \cup A \cup P \cup CNOM \cup NOM_A \cup SVAR$$

a Hybrid Action Model Language if $\{\wedge, \vee, \to, \neg, K_a, C_B\} \cup A \cup P$ is a multimodal epistemic language with common knowledge, (M, s) is a pointed $S5$ hybrid action model, $\{\wedge, \vee, \to, \neg, K_a, @_u, \downarrow, E\} \cup A \cup P \cup NOM \cup SVAR$ is a hybrid language and if $CNOM$ and NOM_A are both countably infinite sets disjoint from $\{\wedge, \vee, \to, \neg, K_a, C_B, [(M, s)], @_u, \downarrow, E\} \cup A \cup P \cup SVAR$ and $CNOM$ is a set of complex nominals, created from NOM and NOM_A as per Definition 10. Define well-formed formulas in the hybrid action model language in the following way:

$$\begin{array}{rcl} \varphi & := & p \mid (\psi \wedge \chi) \mid (\psi \vee \chi) \mid (\psi \to \chi) \mid \neg\psi \mid K_a\psi \mid C_B\psi \mid \\ & & u \mid @_u\psi \mid \downarrow x\ \psi \mid E\psi \mid [(M, s)]\psi \end{array}$$

where ψ and χ are well-formed formulas, $p \in P$, $a \in A$, $B \subseteq A$, (M, s) is a pointed $S5$ hybrid action model, $u \in CNOM \cup SVAR$ and $x \in SVAR$.

These are the formulas we will be working with in our Logic of Hybrid Action Models. The action names will be accessible to agents only indirectly—as a part of a complex name. As we hinted at before we will need to change the definition of a restricted modal product $\otimes$ too. The new definition will be very similar to the old one but we will need to deal with the (complex) nominal part of the new valuation function. As was stated earlier this part has to be defined in a different way than the atomic part in order to make sure the resulting structure is a model.

Definition 13. *Let $\mathcal{M} = \langle W, \mathcal{R}_a, V \rangle$ be an $S5$ hybrid epistemic model and $M = \langle S, R_a, pre, N \rangle$ an $S5$ hybrid action model. A Restricted Modal Product $(\mathcal{M} \otimes M)$ is an ordered triple $\langle W', \mathcal{R}'_a, V' \rangle$ where*

$$W' = \{(w, s); w \in W \ \& \ s \in S \ \& \ \mathcal{M}, w \Vdash pre(s)\}$$

$$\forall a \in A \forall (w, s), (w', s') \in W' \ (w, s)\mathcal{R}'_a(w', s') \text{ iff } w\mathcal{R}_a w' \ \& \ sR_a s'$$

$$\forall (w, s) \in W' \forall p \in P \ (w, s) \in V'(p) \text{ iff } w \in V(p)$$

and for any $(w, s) \in W'$, any $I \in CNOM$ and any $t \in NOM_A$

$$(w, s) \in V'((I, t)) \text{ iff } w \in V(I) \ \& \ s = N(t)$$

Again, as long as the domain of the restricted modal product is non-empty, $(\mathcal{M} \otimes M)$ is an $S5$ hybrid action model.

The definition of satisfaction for complex nominals stays the same as for simple nominals, that is

$$\mathcal{M}, w, g \Vdash I \text{ iff } w \in V(I).$$

One might ask: 'If we have a state named (i, s) in a model, does that mean that this model is a result of updating?' Not necessarily. We know that $\mathcal{M}, w, g \Vdash (i, s)$ iff $w \in V((i, s))$ but that is all. One of the side effects of this is that there is nothing that prevents names of different complexity to coexist in the same model. This may seem peculiar at the first glance but it is not that strange. States with names of different complexity can dwell in the same model just as people with names of different complexity live in the same world.

The definition of the satisfaction relation will copy the earlier definitions. We will need to be careful around assignment functions though. For example we will want $\downarrow x \ [(M, s)]x$ to be a tautology. x is an agent's label for the current state. That state should not be relabelled by performing actions; actions that the agent might not even have a clue that are happening. But $g(x)$ is a metaname and metanames do change when updating. So if $g(x) = w$ in $\mathcal{M}$ and we update with (M, s), we want g to be modified in such a way that $g(x) = (w, s)$ in $(\mathcal{M} \otimes M)$.

This behaviour will contrast with the behaviour of nominals. Those will change by performing updates. And that is exactly what we want to happen. Imagine that the current state has an internal name of *Prague*. An agent performs an action named *Moving to Brno*. We should not expect the new state to be still named *Prague*. The name will instead be (*Prague, Moving to Brno*) which contains the original name and the performed action. Thus $I \to [(M, s)]I$ won't be a tautology while $x \to [(M, s)]x$ will be.

Definition 14. *Let $\mathcal{M} = \langle W, \mathcal{R}_a, V \rangle$ be an $S5$ hybrid epistemic model and g an assignment function. A satisfaction relation $\Vdash$ is defined as follows:*

$$
\begin{array}{rll}
\mathcal{M}, w, g \Vdash p & \textit{iff} & w \in V(p) \\
\mathcal{M}, w, g \Vdash I & \textit{iff} & w \in V(I) \\
\mathcal{M}, w, g \Vdash x & \textit{iff} & x \in \textit{dom}(g) \;\&\; g(x) = w \\
\mathcal{M}, w, g \Vdash \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w, g \Vdash \varphi \;\&\; \mathcal{M}, w, g \Vdash \psi \\
\mathcal{M}, w, g \Vdash \varphi \vee \psi & \textit{iff} & \mathcal{M}, w, g \Vdash \varphi \textit{ or } \mathcal{M}, w, g \Vdash \psi \\
\mathcal{M}, w, g \Vdash \varphi \to \psi & \textit{iff} & \mathcal{M}, w, g \nVdash \varphi \textit{ or } \mathcal{M}, w, g \Vdash \psi \\
\mathcal{M}, w, g \Vdash \neg\varphi & \textit{iff} & \mathcal{M}, w, g \nVdash \varphi \\
\mathcal{M}, w, g \Vdash K_a\varphi & \textit{iff} & \forall w' \in W \;(w\mathcal{R}_a w' \Rightarrow \mathcal{M}, w', g \Vdash \varphi) \\
\mathcal{M}, w, g \Vdash C_B\varphi & \textit{iff} & \forall w' \in W \;(w(\mathcal{R}_B)^* w' \Rightarrow \mathcal{M}, w', g \Vdash \varphi) \\
\mathcal{M}, w, g \Vdash @_I\varphi & \textit{iff} & \exists w' \in V(I) \; \mathcal{M}, w', g \Vdash \varphi \\
\mathcal{M}, w, g \Vdash @_x\varphi & \textit{iff} & x \in \textit{dom}(g) \;\&\; \mathcal{M}, g(x), g \Vdash \varphi \\
\mathcal{M}, w, g \Vdash \downarrow x \; \varphi & \textit{iff} & \mathcal{M}, w, g' \Vdash \varphi \\
\mathcal{M}, w, g \Vdash E\varphi & \textit{iff} & \exists w' \in W \; \mathcal{M}, w', g \Vdash \varphi \\
\mathcal{M}, w, g \Vdash [(M, s)]\varphi & \textit{iff} & \mathcal{M}, w, g \Vdash \textit{pre}(s) \Rightarrow (\mathcal{M} \otimes M), (w, s), g'' \Vdash \varphi
\end{array}
$$

where ψ and χ are well-formed formulas, $p \in P$, $a \in A$, $B \subseteq A$, $I \in CNOM$, $x \in SVAR$, (M, s) is a pointed $S5$ hybrid action model, $g'(x) = w$, for all $y \neq x$ is $g'(y) = g(y)$ and g'' is an assignment function on $\mathcal{M} \otimes M$ such that

$$\textit{dom}(g'') = \textit{dom}(g) \restriction \{x \in \textit{SVAR}; \mathcal{M}, g(x), g \Vdash \textit{pre}(s)\}$$

and

$$\forall z \in \textit{dom}(g'') \forall v \in W \;\; (g''(z) = (v, s) \textit{ iff } g(z) = v).$$

Just like in Hybrid Public Announcement Logic, the reference operator is not auto-dual. We will use Hansen's labelling $\overline{@}_I\varphi \equiv \neg @_I \neg\varphi$.

4.2.2 Axiomatics

The following 29 axiom schemas (plus all instantiations of CPL tautologies in the language $\mathcal{L}^{KC\otimes}_{H(@,\downarrow,E)}$) and 8 deductive rules form the axiomatics of the Logic of Hybrid Action Models. Although we should say infinitely many deductive rules to be more precise. It is quite possible that this axiomatics is not minimal; there may be axioms or rules that can be left out and we would still arrive at the same set of provable formulas. Let us first define a few formula shortcuts to be used in rule $(\mathrm{R8})^n$:

$$[(M,s)]^0\varphi \equiv \varphi$$

$$[(M,s)]^n\varphi \equiv [(M_1,s_1)][(M_2,s_2)]\cdots[(M_n,s_n)]\varphi$$

$$(\chi_{t_i})^n \equiv \chi_{t_1} \wedge [(M_1,t_1)]\chi_{t_2} \wedge \cdots \wedge [(M,t)]^{n-1}\chi_{t_n}$$

$$(pre_i(t_i))^n \equiv pre_1(t_1) \wedge [(M_1,t_1)]pre_2(t_2) \wedge \cdots \wedge [(M,t)]^{n-1}pre_n(t_n)$$

where $n \geq 1$ is a natural number.

The 29 axioms can be separated into three parts—the $S5$ part, the hybrid part and the dynamic part.

The axiom schemas (A1)—(A7) and rules (R1)—(R3) compose the $S5$ part of the axiomatics:

	CPL
(A1)	$\vdash K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
(A2)	$\vdash K_a\varphi \to \varphi$
(A3)	$\vdash K_a\varphi \to K_aK_a\varphi$
(A4)	$\vdash \neg K_a\varphi \to K_a\neg K_a\varphi$
(A5)	$\vdash C_B(\varphi \to \psi) \to (C_B\varphi \to C_B\psi)$
(A6)	$\vdash C_B(\varphi \to E_B\varphi) \to (\varphi \to C_B\varphi)$
(A7)	$\vdash C_B\varphi \to (\varphi \wedge E_BC_B\varphi)$
(R1)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
(R2)	If $\vdash \varphi$ then $\vdash K_a\varphi$
(R3)	If $\vdash \varphi$ then $\vdash C_B\varphi$

The axiom schemas (A8)—(A19) and rules (R4)—(R6) compose the hybrid part of the axiomatics:

(A8) $\vdash \overline{@}_u(\varphi \to \psi) \to (\overline{@}_u\varphi \to \overline{@}_u\psi)$
(A9) $\vdash @_u\varphi \to \overline{@}_u\varphi$
(A10) $\vdash \overline{@}_u u$
(A11) $\vdash @_u @_v\varphi \to @_v\varphi$
(A12) $\vdash u \to (\varphi \leftrightarrow @_u\varphi)$
(A13) $\vdash M_a @_u\varphi \to @_u\varphi$
(A14) $\vdash (@_u M_a v \wedge @_v\varphi) \to @_u M_a\varphi$
(A15) $\vdash @_u\varphi \to @_u u$
(A16) $\vdash @_u u \to (\overline{@}_u\varphi \to @_u\varphi)$
(A17) $\vdash \overline{@}_u(\downarrow x\ \varphi \leftrightarrow \varphi[x := u])$
(A18) $\vdash @_u\psi \to E\psi$
(A19) $\vdash (@_u M_a v \wedge @_u K_a\varphi) \to @_v\varphi$
(R4) If $\vdash \varphi$ then $\vdash \overline{@}_u\varphi$
(R5) If $\vdash \overline{@}_u\varphi$ then $\vdash \varphi$, if u is not in φ.
(R6) If $\vdash (@_u M_a v \wedge @_v\varphi) \to \psi$ then $\vdash @_u M_a\varphi \to \psi$, where $u \neq v$ and v is contained in neither φ nor ψ.

And the axiom schemas (A20)—(A29) and rules (R7) and $(R8)^n$ compose the dynamic part of the axiomatics:

(A20) $\vdash [(M,s)]p \leftrightarrow (pre(s) \to p)$
(A21) $\vdash [(M,N(s'))](I,s') \leftrightarrow (pre(N(s')) \to I)$
(A22) $\vdash [(M,s)]x \leftrightarrow (pre(s) \to x)$
(A23) $\vdash [(M,s)]\neg\varphi \leftrightarrow (pre(s) \to \neg[(M,s)]\varphi)$
(A24) $\vdash [(M,s)](\varphi \wedge \psi) \leftrightarrow ([(M,s)]\varphi \wedge [(M,s)]\psi)$
(A25) $\vdash [(M,s)]K_a\varphi \leftrightarrow (pre(s) \to \bigwedge_{sR_at} K_a[(M,t)]\varphi)$
(A26) $\vdash [(M,N(s'))]@_{(I,s')}\varphi \leftrightarrow (pre(N(s')) \to @_I(pre(N(s')) \wedge [(M,N(s'))]\varphi))$
(A27) $\vdash [(M,s)]@_x\varphi \leftrightarrow (pre(s) \to @_x(pre(s) \wedge [(M,s)]\varphi))$
(A28) $\vdash [(M,s)] \downarrow x\ \varphi \leftrightarrow \downarrow x\ [(M,s)]\varphi$, if x is not contained in $pre(s)$
(A29) $\vdash [(M,s)]E\varphi \leftrightarrow (pre(s) \to E(pre(s) \wedge [(M,s)]\varphi))$
(R7) If $\vdash \varphi$ then $\vdash [(M,s)]\varphi$
$(R8)^n$ For given pointed action models $(M_1,s_1),\dots,(M_n,s_n)$, formulas $\chi_{t_1},\dots,\chi_{t_n}$ for all $t_i \in \dot{M}_i$ such that $t_i(R_B)^*s_i$, and for any $a \in B$ and any $r_1,\dots,r_n$ such that $r_1R_at_1$ & ... & $r_nR_at_n$ it holds that
If $\vdash ((\chi_{t_i})_1^n \to [(M,t)]^n\varphi)$ and $\vdash (((\chi_{t_i})_1^n \wedge (pre_i(t_i))_1^n) \to K_a(\chi_{r_i})_1^n)$ then $\vdash (\chi_{s_i})_1^n \to [(M,s)]^n C_B\varphi$

Where $p \in P$, $a \in A$, $B \subseteq A$, $I \in CNOM$, $u,v \in CNOM \cup SVAR$, $s \in \dot{M}$ and $s' \in NOM_A$.

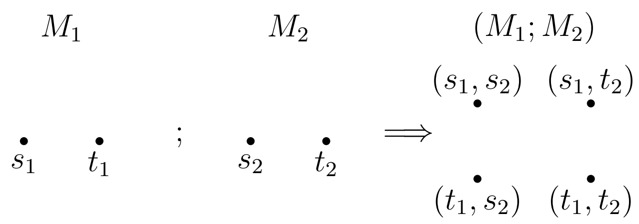
Axiom (A28) has the condition that x must not be contained in $pre(s)$. To see why this is necessary let $pre(s) = x$, $\varphi \equiv \neg x$ and $x \notin dom(g)$. Then $[(M, s)] \downarrow x\ \varphi$ would be satisfied while $\downarrow x\ [(M, s)]\varphi$ wouldn't be.

One of the conditions of rule (R6) is that v must be contained in neither φ nor ψ. This means that not only is it not a member of a set $sub(\varphi) \cup sub(\psi)$, where *sub* denotes a set of all subformulas.[12] But it must not be a part of a complex name that is a subformula of either of φ or ψ as well. If $\psi \equiv [(M, N(s))](I, s)$ then I is not a subformula of ψ; yet the valuation of ψ does depend on the valuation of I. This has to be prevented in order for the rule to be sound. For more details, see the Soundness proof for rule (R6), Lemma 7.

Notice the missing composition axiom

$$[(M, s)][(M', s')]\varphi \leftrightarrow [((M; M'), (s, s'))]\varphi$$

or any variation of it. This has a very good reason. Imagine a situation with the following action models M_1 and M_2.



And let's say that the corresponding naming functions N_1 and N_2 are completely arbitrary (as long as M_1 and M_2 stay $S5$ hybrid action models). How should we define the naming function N_3 of the composition model? It has to be total and a surjection if we want $(M_1; M_2)$ to be a hybrid action model. There are two ways to deal with this problem, none of them too attractive. Either we need to devise a way to define N_3 based on N_1 and N_2. And this definition has to be general. And we have to deal with the fact that φ on the left side of the composition axiom is updated twice while φ on the right side only once which becomes problematic in case φ contains any nominals. Or we can redefine the set of action names in a similar fashion to how we redefined epistemic names—by introducing complex action names. But this would greatly complicate the already complicated matters.

[12]A subformula is defined in the usual way. Note that I is not a subformula of (I, s). On the other hand u is a subformula of $@_u\psi$ and x is a subformula of $\downarrow x\ \psi$.

There is an alternative though. Yanjing Wang in an article [10] explores different sound and complete axiomatizations of PAL. Several of the possible axiomatizations skip the composition axiom $[\alpha][\beta]\varphi \leftrightarrow [\alpha \wedge [\alpha]\beta]\varphi$ and instead introduce different axioms or rules. This article was an inspiration to redefining the axiomatics in such a way as to leave the composition axiom out.

4.3 Soundness and Completeness

4.3.1 Soundness

We will prove the soundness of the presented calculus in the usual way: by showing that all axioms hold in all models and that deductive rules preserve tautologicity. We will not give the whole proof however. Some of the steps are well known, those of a multimodal logic of $S5$ with common knowledge for example, and it's not necessary to repeat them.

Fact 1. All instances of all classical tautologies in the language of HAM, the axiomatic schemas (A1)—(A7) are HAM tautologies. Rules (R1), (R2) and (R3) preserve tautologicity.

Let us have an arbitrary $S5$ hybrid epistemic model $\mathcal{M}$, a state w in it and any assignment function g throughout the rest of this section. We will show that all the axiom schemas are satisfied in $\mathcal{M}, w, g$.

Lemma 5. *The axiomatic schemas (A8)—(A19) are HAM tautologies.*

Proof.

(A8)

$$\overline{@}_u(\varphi \rightarrow \psi) \rightarrow (\overline{@}_u\varphi \rightarrow \overline{@}_u\psi)$$

If u is not present in the model under the current valuation and assignment then $\overline{@}_u\psi$ holds trivially. If u is present then the corresponding state must satisfy $\varphi \rightarrow \psi$ and φ and therefore also ψ.

(A12)

$$u \rightarrow (\varphi \leftrightarrow @_u\varphi)$$

Let $\mathcal{M}, w, g \Vdash u$. If $\mathcal{M}, w, g \Vdash \varphi$ it must also be that $\mathcal{M}, w, g \Vdash @_u\varphi$ and vice versa.

(A13)

$$M_a @_u \varphi \to @_u \varphi$$

$\mathcal{M}, w, g \Vdash M_a @_u \varphi$ iff $\exists w' (w \mathcal{R}_a w' \wedge \mathcal{M}, w', g \Vdash @_u \varphi)$. However changing w to w' does not affect the satisfaction of $@_u \varphi$ in any way.

(A14)

$$(@_u M_a v \wedge @_v \varphi) \to @_u M_a \varphi$$

We'll do the case for $u \equiv I$ a nominal and $v \equiv x$ a state variable. Other cases are analogous. Let $\mathcal{M}, w, g \Vdash @_I M_a x \wedge @_x \varphi$. Then there must exist $w' \in V(I)$ such that $\mathcal{M}, w', g \Vdash M_a x$, which means that there exist w', w'', such that $(w' \in V(I) \wedge w' R_a w'' \wedge \mathcal{M}, w'', g \Vdash x)$, that is $x \in dom(g) \wedge g(x) = w''$. From the second conjunct $@_x \varphi$ we have $\mathcal{M}, g(x), g \Vdash \varphi$, that is $\mathcal{M}, w'', g \Vdash \varphi$. Since $w' R_a w''$ we know that $\mathcal{M}, w', g \Vdash M_a \varphi$. And because $w' \in V(I)$ we finally have $\mathcal{M}, w, g \Vdash @_I M_a \varphi$.

(A17)

$$\overline{@}_u (\downarrow x\ \varphi \leftrightarrow \varphi[x := u])$$

Again we'll only examine the case for $u \equiv I$. If $V(I) = \emptyset$ all formulas of the form $\overline{@}_I \psi$ hold in the whole model so let $V(I) = \{w'\}$. We want to show that $\mathcal{M}, w', g \Vdash \downarrow x\ \varphi$ iff $\mathcal{M}, w', g \Vdash \varphi[x := I]$. $\mathcal{M}, w', g \Vdash \downarrow x \varphi$ iff $\mathcal{M}, w', g' \Vdash \varphi$ where g' is the same as g except for $g'(x) = w'$. This means that all the free occurances of x in φ refer to the state w'. On the other hand since $V(I) = \{w'\}$ then all the occurances of I in $\varphi[x := I]$, that is all the free occurances of x in φ, refer to w' as well.

(A19)

$$(@_u M_a v \wedge @_u K_a \varphi) \to @_v \varphi$$

Let $\mathcal{M}, w, g \Vdash @_u M_a v$ and $\mathcal{M}, w, g \Vdash @_u K_a \varphi$. The state named v must be accessible from state named u. All states accessible from u must satisfy φ. Thus state named v must satisfy φ. If u is not assigned, the implication holds trivially. If v is not assigned then u must not be assigned either because $\mathcal{M}, w, g \Vdash @_u M_a v$.

The rest of the schemas are trivial or analogous to already shown proofs. $\square$

We will deal with the dynamic part of the axiomatic system now.

Lemma 6. *The axiomatic schemas (A20)—(A29) are HAM tautologies.*

Proof.

(A20)

$$[(M,s)]p \leftrightarrow (pre(s) \to p)$$

The following lines are all equivalent:

$$\mathcal{M}, w, g \Vdash [(M,s)]p$$
$$(\mathcal{M}, w, g \Vdash pre(s) \Rightarrow \mathcal{M} \otimes M, (w,s), g'' \Vdash p)$$
$$(\mathcal{M}, w, g \Vdash pre(s) \Rightarrow \mathcal{M}, w, g \Vdash p)$$
$$\mathcal{M}, w, g \Vdash (pre(s) \to p)$$

The equivalence of the second and third line follows from the fact that the valuation of atoms in the updated model copies the valuation in the original model.

(A21)

$$[(M, N(s))](I,s) \leftrightarrow (pre(N(s)) \to I)$$

($\Rightarrow$) Let $\mathcal{M}, w, g \Vdash [(M, N(s))](I,s)$ and $\mathcal{M}, w, g \Vdash pre(N(s))$. This means $(\mathcal{M} \otimes M), (w, N(s)), g'' \Vdash (I,s)$, so $(w, N(s)) \in V_{(\mathcal{M}\otimes M)}(I,s)$ and thus from the definition of $\otimes$ we have $w \in V(I)$ which gives us $\mathcal{M}, w, g \Vdash I$.
($\Leftarrow$) If $\mathcal{M}, w, g \nVdash pre(N(s))$ then trivially $\mathcal{M}, w, g \Vdash [(M, N(s))](I,s)$. So let $\mathcal{M}, w, g \Vdash pre(N(s))$ which also means $\mathcal{M}, w, g \Vdash I$. Again from the definition of $\otimes$ we get $(\mathcal{M} \otimes M), (w, N(s)), g'' \Vdash (I,s)$ and thus $\mathcal{M}, w, g \Vdash [(M, N(s))](I,s)$.

(A22)

$$[(M,s)]x \leftrightarrow (pre(s) \to x)$$

The following lines are equivalent:

$$\mathcal{M}, w, g \Vdash [(M,s)]x$$
$$(\mathcal{M}, w, g \Vdash pre(s) \Rightarrow (\mathcal{M} \otimes M), (w,s), g'' \Vdash x)$$
$$(\mathcal{M}, w, g \Vdash pre(s) \Rightarrow (x \in dom(g'') \;\&\; g''(x) = (w,s))) \tag{2}$$
$$(\mathcal{M}, w, g \Vdash pre(s) \Rightarrow (x \in dom(g) \;\&\; g(x) = w)) \tag{3}$$
$$\mathcal{M}, w, g \Vdash (pre(s) \to x)$$

The implication from (3) to (2) utilizes the fact that $\mathcal{M}, g(x), g \Vdash pre(s)$ to conclude that $x \in dom(g'')$. If it didn't then since $g(x) = w$ we'd have $\mathcal{M}, w, g \nVdash pre(s)$ and thus $\mathcal{M}, w, g \Vdash [(M,s)]x$ trivially.

(A25)

$$[(M,s)]K_a\varphi \leftrightarrow (pre(s) \rightarrow \bigwedge_{sR_at} K_a[(M,t)]\varphi)$$

Again, the case for $\mathcal{M}, w, g \nVdash pre(s)$ is trivial so let's assume otherwise. Then the following lines are equivalent:

$$\mathcal{M}, w, g \Vdash [(M,s)]K_a\varphi$$

$$(\mathcal{M} \otimes M), (w,s), g'' \Vdash K_a\varphi$$

$$\forall (w',s')\ ((w,s)\mathcal{R}'_a(w',s') \Rightarrow (\mathcal{M} \otimes M), (w',s'), g'' \Vdash \varphi)$$

Looking at the right side of the equation, the following lines are again all equivalent:

$$\mathcal{M}, w, g \Vdash \bigwedge_{sR_at} K_a[(M,t)]\varphi$$

$$\forall t\ (sR_at \Rightarrow \mathcal{M}, w, g \Vdash K_a[(M,t)]\varphi)$$

$$\forall v, t\ (w\mathcal{R}_av\ \&\ sR_at \Rightarrow \mathcal{M}, v, g \Vdash [(M,t)]\varphi)$$

$$\forall v, t\ (w\mathcal{R}_av\ \&\ sR_at\ \&\ \mathcal{M}, v, g \Vdash pre(t) \Rightarrow (\mathcal{M} \otimes M), (v,t), g'' \Vdash \varphi)$$

Since $(w,s)\mathcal{R}'_a(w',s')$ iff $(w\mathcal{R}_aw',\ sR_as'$ and $\mathcal{M}, w', g \Vdash pre(s'))$, we get the desired equivalence.

(A26)

$$[(M,N(s))]@_{(I,s)}\varphi \leftrightarrow (pre(N(s)) \rightarrow @_I(pre(N(s)) \wedge [(M,N(s))]\varphi))$$

Again let $\mathcal{M}, w, g \Vdash pre(N(s))$. Then all the following lines are equivalent.

$$\mathcal{M}, w, g \Vdash [(M,N(s))]@_{(I,s)}\varphi$$

$$(\mathcal{M} \otimes M), (w, N(s)), g'' \Vdash @_{(I,s)}\varphi$$

$$\exists (v,t)\ ((v,t) \in V_{(\mathcal{M}\otimes M)}(I,s)\ \&\ (\mathcal{M} \otimes M), (v,t), g'' \Vdash \varphi)$$

$$\exists (v,t)\ ((v,t) \in V_{(\mathcal{M}\otimes M)}(I,s)\ \&\ \mathcal{M}, v, g \Vdash pre(t)\ \&\ (\mathcal{M}\otimes M), (v,t), g'' \Vdash \varphi)$$

$$\exists v, t\ (v \in V(I)\ \&\ t = N(s)\ \&\ \mathcal{M}, v, g \Vdash pre(t)\ \&\ (\mathcal{M}\otimes M), (v,t), g'' \Vdash \varphi)$$

$$\exists v\ (v \in V(I)\ \&\ \mathcal{M}, v, g \Vdash pre(N(s))\ \&\ (\mathcal{M} \otimes M), (v, N(s)), g'' \Vdash \varphi)$$

$$\exists v\ (v \in V(I)\ \&\ \mathcal{M}, v, g \Vdash pre(N(s))\ \&\ \mathcal{M}, v, g \Vdash [(M,N(s))]\varphi)$$

$$\exists v\ (v \in V(I)\ \&\ \mathcal{M}, v, g \Vdash pre(N(s)) \wedge [(M,N(s))]\varphi)$$

$$\mathcal{M}, w, g \Vdash @_I(pre(N(s)) \wedge [(M,N(s))]\varphi)$$

(A27)

$$[(M,s)]@_x\varphi \leftrightarrow (pre(s) \rightarrow @_x(pre(s) \wedge [(M,s)]\varphi))$$

Again let $\mathcal{M}, w, g \Vdash pre(s)$. Then all the following lines are equivalent.

$$\mathcal{M}, w, g \Vdash [(M,s)]@_x\varphi$$

$$(\mathcal{M} \otimes M), (w,s), g'' \Vdash @_x\varphi$$

$$x \in dom(g'') \ \&\ (\mathcal{M} \otimes M), g''(x), g'' \Vdash \varphi$$

$$x \in dom(g) \ \&\ \mathcal{M}, g(x), g \Vdash pre(s) \ \&\ (\mathcal{M} \otimes M), g''(x), g'' \Vdash \varphi$$

$$x \in dom(g) \ \&\ \mathcal{M}, g(x), g \Vdash pre(s) \ \&\ (\mathcal{M} \otimes M), (g(x), s), g'' \Vdash \varphi$$

$$x \in dom(g) \ \&\ \mathcal{M}, g(x), g \Vdash pre(s) \wedge [(M,s)]\varphi$$

$$\mathcal{M}, w, g \Vdash @_x(pre(s) \wedge [(M,s)]\varphi)$$

(A28)

$$[(M,s)] \downarrow x\ \varphi \leftrightarrow \downarrow x\ [(M,s)]\varphi$$

First note that since $pre(s)$ does not contain x it holds that

$$\mathcal{M}, w, g \Vdash pre(s) \text{ iff } \mathcal{M}, w, g' \Vdash pre(s),$$

where g' is defined as in Definition 14. This means that we may once again ignore the part where $\mathcal{M}, w, g \nVdash pre(s)$.
Expanding the left side of the equation gives us

$$(\mathcal{M} \otimes M), (w,s), (g'')' \Vdash \varphi$$

while expanding the right side gives us

$$(\mathcal{M} \otimes M), (w,s), (g')'' \Vdash \varphi.$$

So all we need to do is prove that

$$(g'')' = (g')''.$$

We first show that their domains coincide. Let $z \in SVAR$. We prove that $z \in dom((g'')')$ iff $z \in dom((g')'')$. There are two options—either $z = x$ or not. First let $z = x$. $x \in dom((g'')')$ trivially. On the other had $x \in dom((g')'')$ iff $(x \in dom(g') \ \&\ \mathcal{M}, g'(x), g' \Vdash pre(s))$ but this is always true as well since $g'(x) = w$. The case of $z \neq x$ is also easy. By definition $g'(z) = g(z)$. So the following lines are equivalent:

$$z \in dom((g')'')$$

$$(z \in dom(g') \ \& \ \mathcal{M}, g'(z), g' \Vdash pre(s))$$
$$(z \in dom(g) \ \& \ \mathcal{M}, g(z), g \Vdash pre(s))$$
$$z \in dom(g'')$$
$$z \in dom((g'')')$$

Now let us show that these two assignment functions always assign the same state. We are again left with two options.
Let $z = x$. $g'(x) = w$ so by definition $(g')''(x) = (w, s)$. Although x does not have to be in $dom(g'')$, it is in $dom((g'')')$. And since we started with state w, updated with (M, s), our current state is (w, s) at the time of evaluation of $\downarrow x\ \varphi$. This means that $((g'')')(x) = (w, s)$.
Finally let $z \neq x$ and let $g(z) = v$. This yields $g'(z) = v$ and

$$\mathcal{M}, g'(z), g' \Vdash pre(s) \Rightarrow (g')''(z) = (v, s).$$

Similarly
$$\mathcal{M}, g(z), g \Vdash pre(s) \Rightarrow g''(z) = (v, s)$$

and so
$$\mathcal{M}, g(z), g \Vdash pre(s) \Rightarrow (g'')'(z) = (v, s).$$

Which means we just need to verify that

$$\mathcal{M}, g'(z), g' \Vdash pre(s) \text{ iff } \mathcal{M}, g(z), g \Vdash pre(s).$$

But this follows from $g(z) = g'(z)$ and from the fact that $pre(s)$ does not contain x which is the only state variable that can differentiate g from g'.

Again, the proofs that were not mentioned are either trivial or analogous.
□

And finally let's tackle the deductive rules.

Lemma 7. *Deductive rules (R4)—(R8)^n preserve tautologicity.*

Proof.

(R6)
$$\text{If } \vdash (@_u M_a v \wedge @_v \varphi) \rightarrow \psi \text{ then } \vdash @_u M_a \varphi \rightarrow \psi,$$

where $u \neq v$ and v is not in φ or ψ. We will prove the soundness of rule (R6) by contradiction. Let $v \equiv I$ nominal and let $(@_u M_a I \wedge @_I \varphi) \rightarrow \psi$

be a tautology and let's have a model-state-assignment triple $\mathcal{M}, w, g$ such that $\mathcal{M}, w, g \Vdash @_u M_a \varphi$ while $\mathcal{M}, w, g \nVdash \psi$. Since

$$\mathcal{M}, w, g \Vdash (@_u M_a I \wedge @_I \varphi) \to \psi$$

we get that

$$\mathcal{M}, w, g \nVdash @_u M_a I \wedge @_I \varphi.$$

From $\mathcal{M}, w, g \Vdash @_u M_a \varphi$ we know that there must exist a state named u that can access a state satisfying φ via $\mathcal{R}_a$.

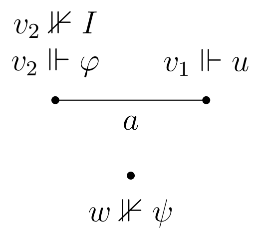


Figure 8: A substructure of a model $\mathcal{M}$

Now construct a model $\mathcal{M}'$ such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in valuation:

$$V' = (V \setminus \{\langle I, v\rangle ; v \in W\}) \cup \{\langle I, v_2\rangle\}.$$

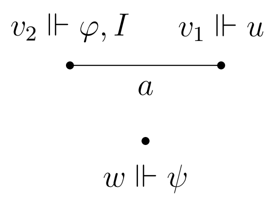


Figure 9: The same substructure of a model $\mathcal{M}'$

ψ must be satisfied in $\mathcal{M}', w, g$ because $\mathcal{M}', w, g \Vdash @_u M_a I \wedge @_I \varphi$. $\mathcal{M}, w, g \nVdash \psi$ while $\mathcal{M}', w, g \Vdash \psi$ but the only other difference between $\mathcal{M}$ and $\mathcal{M}'$ is whether v_2 satisfies I or not. This means that regardless of the structure of the formula, ψ has to depend on the satisfaction of I in v_2. But I is not contained in ψ which means that we arrive at a contradiction.

The case of $v \equiv x$ state variable is even easier—instead of creating a new model $\mathcal{M}'$ one can take a different assignment function and work analogously.

(R8)^n

$$\text{If } \vdash ((\chi_{t_i})_1^n \rightarrow [(M,t)]^n\varphi) \text{ and } \vdash (((\chi_{t_i})_1^n \wedge (pre_i(t_i))_1^n) \rightarrow K_a(\chi_{r_i})_1^n)$$

$$\text{then } \vdash (\chi_{s_i})_1^n \rightarrow [(M,s)]^n C_B\varphi$$

For given pointed action models $(M_1, s_1), \ldots, (M_n, s_n)$, given formulas $\chi_{t_1}, \ldots, \chi_{t_n}$ for all $t_i \in \dot{M}_i$ such that $t_i(R_B)^* s_i$, and for any $a \in B$ and any $r_1, \ldots, r_n$ such that $r_1 R_a t_1$ & $\ldots$ & $r_n R_a t_n$.

This proof is quite lengthy and doesn't bring much new to this work. It does incorporate the idea of paths but we will look into that more thoroughly in the Truth Lemma. The proof of soundness of $(\text{R8})^1$ can be found in [6] (Proposition 6.37) and while van Ditmarsch et al. work in the Logic of Action Models with Common Knowledge, the proof can be taken verbatim and applied to the Logic of Hybrid Action Models. The proof of soundness of $(\text{R8})^n$ for greater n uses the very same steps as the proof of $(\text{R8})^1$.

The rest are again easy to prove. □

Theorem 1 (**Soundness**). *All the axioms of the Logic of HAM are tautologies and all deductive rules preserve tautologicity.*

Proof. See Lemmas 5, 6 and 7 and Fact 1. □

4.3.2 Completeness

There are two complications when proving a completeness of this system. First—the logic is not compact. We cannot hope for a strong completeness and have to settle with the weak one: if $\vDash \varphi$ then $\vdash \varphi$. This is caused by the operator of common knowledge. The frequently repeated counterexample to $\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$ is $\Gamma = \{E_B^n p; n \in \omega\} \cup \{\neg C_B p\}$. Γ is contradictory since $C_B p \equiv \bigwedge_{i=0}^{\infty} E_B^i p$ but a contradiction can never be proved from Γ in a finite number of steps.

The second complication is caused by hybridization. The common way to prove completeness with respect to kripke semantics is through the construction of the canonical model. But in hybrid logics the canonical model needs not be a model at all. We will explain the reason for this with the following sketch of a proof. Start with $\nvdash \varphi$. Expand $\{\neg\varphi\}$ to a maximal consistent set Γ with the help of Lindenbaum lemma. Create a canonical model where states are maximal consistent sets and where satisfaction corresponds to belonging to a set. Then since $\neg\varphi \in \Gamma$ and Γ is maximal and consistent, $\varphi \notin \Gamma$ which means $\mathcal{M}^c, \Gamma \nVdash \varphi$ and thus we have a model and a state that doesn't satisfy φ in all states, that is from $\nvdash \varphi$ we proved $\nvDash \varphi$. Now let's examine a situation for $\varphi \equiv \neg i$. We want to expand $\neg\neg i$, that is i, to a maximal consistent set Γ. In the canonical model $\mathcal{M}^c, \Gamma \Vdash i$. However there may exist many other maximal consistent sets that contain i as their member. All these sets are members of the canonical model, that satisfy i. This means that there are many distinct states that satisfy i in $\mathcal{M}^c$ and thus the canonical model is not a model since $|V^c(i)| > 1$.

We will solve the first problem by limiting ourselves to consistent sets that are maximal in a closure of a given formula. This closure can be defined in virtually any way, as long as it is finite and contains the original formula, all its subformulas and a specifically selected list of other formulas. Since this closure is always finite, all consistent sets that are maximal in it will be finite too and non-compactness won't bother us. The second problem will require us to redefine canonical models in such a way that states will become equivalence classes of members of $NOM \cup SVAR$ that satisfy certain conditions instead of maximal consistent sets of formulas. The proof will incorporate proofs of completeness of the Logic of Action Models with Common Knowledge from [6] and completeness of the Hybrid Logic with Partial Denoting Nominals from [3].

Definition 15 (**Formula Closure**). *Let φ be an arbitrary formula. A set of formulas $cl(\varphi)$ will be called a closure of formula φ iff it is the smallest set satisfying the following*

- $\varphi \in cl(\varphi)$
- $\psi \in cl(\varphi) \Rightarrow sub(\psi) \subseteq cl(\varphi)$*, where $sub(\psi)$ is a set of all subformulas of ψ*
- $\psi \in cl(\varphi)$ *&* ψ *is not of the form* $\neg\chi \Rightarrow \neg\psi \in cl(\varphi)$
- $C_B\psi \in cl(\varphi) \Rightarrow \{K_a C_B \psi; a \in B\} \subseteq cl(\varphi)$
- $[(M,s)]p \in cl(\varphi) \Rightarrow (pre(s) \to p) \in cl(\varphi)$
- $[(M,N(s))](I,s) \in cl(\varphi) \Rightarrow (pre(N(s)) \to I) \in cl(\varphi)$
- $[(M,s)]x \in cl(\varphi) \Rightarrow (pre(s) \to x) \in cl(\varphi)$
- $[(M,s)]\neg\psi \in cl(\varphi) \Rightarrow (pre(s) \to \neg[(M,s)]\psi) \in cl(\varphi)$
- $[(M,s)](\psi \wedge \chi) \in cl(\varphi) \Rightarrow ([(M,s)]\psi \wedge [(M,s)]\chi) \in cl(\varphi)$
- $[(M,s)]K_a\psi \in cl(\varphi) \Rightarrow \{pre(s) \to K_a[(M,t)]\psi; sR_a t\} \subseteq cl(\varphi)$

- $[(M,s)]^n C_B\psi \in cl(\varphi) \Rightarrow (\{[(M,t)]^n\psi; s_i(R_B^i)^*t_i\} \subseteq cl(\varphi)$ & $\{K_a[(M,t)]^n C_B\psi; a \in B$ & $s_i(R_B^i)^*t_i\} \subseteq cl(\varphi))$, *where* (R_a^i) *is an accessibility relation in action model* M_i *for* $1 \leq i \leq n$.
- $[(M,N(s))]@_{(I,s)}\psi \in cl(\varphi) \Rightarrow$
$$(pre(N(s)) \rightarrow @_I(pre(N(s)) \wedge [(M,N(s))]\psi)) \in cl(\varphi)$$
- $[(M,s)]@_x\psi \in cl(\varphi) \Rightarrow (pre(s) \rightarrow @_x(pre(s) \wedge [(M,s)]\psi)) \in cl(\varphi)$
- $[(M,s)] \downarrow x\ \psi \in cl(\varphi) \Rightarrow \downarrow x\ [(M,s)]\psi \in cl(\varphi)$
- $[(M,s)]E\psi \in cl(\varphi) \Rightarrow (pre(s) \rightarrow E(pre(s) \wedge [(M,s)]\psi)) \in cl(\varphi)$
- $@_u M_a v \in cl(\varphi) \Rightarrow @_v M_a u \in cl(\varphi)$
- $(@_u M_a v \in cl(\varphi)$ & $@_v\psi \in cl(\varphi)) \Rightarrow @_u M_a\psi \in cl(\varphi)$
- $(u \in cl(\varphi)$ & $\downarrow x\ \psi \in cl(\varphi)) \Rightarrow \psi[x := u] \in cl(\varphi)$

Furthermore given a closure of a formula $cl(\varphi)$, *we define an extended closure* $cl_E(\varphi)$ *in this way:*

Let $cl(\varphi) = \{\psi_1, \ldots, \psi_m\}$ *and let* $(I_n)_{n\in\omega}$ *be any enumeration of all complex nominals CNOM such that* $\{I_1, \ldots, I_{m+1}\} \cap cl(\varphi) = \emptyset$. *Then* $cl_E(\varphi)$ *is the smallest set satisfying the following*

- $cl(\varphi) \cup \{I_{m+1}\} \subseteq cl_E(\varphi)$
- $\psi \in cl(\varphi) \Rightarrow @_{I_{m+1}}\psi \in cl_E(\varphi)$
- $\psi \in cl_E(\varphi) \Rightarrow sub(\psi) \subseteq cl_E(\varphi)$
- ψ_k *is of the form* $@_u M_a\psi \Rightarrow \{I_k, @_u M_a I_k, @_{I_k}\psi\} \subseteq cl_E(\varphi)$
- $\{u, v, \psi\} \subseteq cl_E(\varphi)$ & ψ *is not of the form* $@_w\chi \Rightarrow @_u @_v\psi \in cl_E(\varphi)$

Since an extended closure of a formula is not uniquely defined (it depends on the enumerations), we will work with an arbitrary one.

It can be proven that an extended closure of any formula φ is finite. This can be done by first proving that $cl(\varphi)$ is finite by the induction on φ and then proving that $cl_E(\varphi)$ is finite directly from the definition of $cl_E(\varphi)$. The reason for extending the closure is because (i) we need to be sure that we have a nominal at hand to keep the canonical model non-empty and (ii) all formulas of the form $@_u M_a\psi$ need witnessing. And when showing that any consistent subset of $cl(\varphi)$ can be extended to a maximal consistent set of $cl(\varphi)$, this maximal consistent set won't be a subset of $cl(\varphi)$ but instead of $cl_E(\varphi)$. And this is a correct procedure because $cl_E(\varphi)$ is finite.

Lemma 8 (**Lindenbaum**). *Let φ be an arbitrary formula, $cl_E(\varphi)$ its extended closure and $\Gamma \subseteq cl(\varphi)$ a consistent set. Then there exists $\Gamma' \subseteq cl_E(\varphi)$ such that*

1. *Γ' is consistent*
2. *$\Gamma \subseteq \Gamma'$*
3. *$\Gamma' \subseteq cl_E(\varphi)$*
4. *for any $\psi \in cl(\varphi)$ either $\psi \in \Gamma'$ or $\Gamma' \cup \{\psi\}$ is inconsistent*
5. *$I_{m+1} \in \Gamma'$*
6. *for any member of $\Gamma' \cap cl(\varphi)$ of the form $@_u M_a \psi$ there exists a nominal $J \notin cl(\varphi)$ such that $@_u M_a J \in \Gamma'$ and $@_J \psi \in \Gamma'$*

Proof. Let us have an arbitrary formula φ and $\Gamma \subseteq cl(\varphi)$ a consistent set. Label $|cl_E(\varphi)| = m$. Enumerate the extended closure $cl_E(\varphi)$ set in such a way that there exists a k such that $cl(\varphi) = \{\psi_1, \ldots, \psi_k\}$ and $(cl_E(\varphi) \setminus cl(\varphi)) = \{\psi_{k+1}, \ldots, \psi_m\}$. We will define a sequence of sets $(\Gamma_i)_{0 \leq i \leq m}$ in such a way that Γ_m will be the desired set Γ'.

$\Gamma_0 = \Gamma \cup \{I_{k+1}\}$
For $n < k$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{n+1}\} & \text{if } \Gamma_n \cup \{\psi_{n+1}\} \text{ is consistent and} \\ & \psi_{n+1} \text{ is not of the form } @_u M_a \psi \\ \Gamma_n \cup \{\psi_{n+1}, @_u M_a I_{n+1}, @_{I_{n+1}} \psi\} & \text{if } \Gamma_n \cup \{\psi_{n+1}\} \text{ is consistent} \\ & \text{and } \psi_{n+1} \text{ is of the form } @_u M_a \psi \\ \Gamma_n & \text{otherwise.} \end{cases}$$

And for $n \geq k$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{n+1}\} & \text{if } \Gamma_n \cup \{\psi_{n+1}\} \text{ is consistent} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

$\Gamma' = \Gamma_m$.

Most of the things we need to check follow trivially from the construction of Γ'. The only non-trivial step of the proof is checking that Γ' is consistent. But this part of the proof can be taken verbatim from [3], proof of Lemma 2.4 and needs not be repeated here.

□

We will make use of the following five provable formulas. The proofs can be found in [3], Lemma 2.5.

Fact 2. The following formulas are provable in the axiomatic system of the Logic of Hybrid Action Models.

1. $@_u v \to (\overline{@}_u \varphi \leftrightarrow @_u \varphi)$
2. $@_u v \to @_v u$
3. $(@_u u \wedge @_v v) \to (@_v \varphi \leftrightarrow @_u @_v \varphi)$
4. $@_u v \to (@_u \varphi \leftrightarrow @_v \varphi)$
5. $(@_u v \wedge @_v w) \to @_u w$

We foreshadowed a change of definition of a canonical model to prevent nominals and state variables from being satisfied in more states than one. So instead of taking all maximal consistent sets in a closure of a given formula and letting each one of those represent a state, we'll take only one maximal consistent set and states will be equivalence classes of all nominals and variables that occur in the maximal consistent set.

Definition 16 (Canonical Model). *Let φ be an arbitrary formula and Γ a subset of $cl_E(\varphi)$ that satisfies all the conditions of the Lindenbaum lemma. Define $\mathcal{N}_\Gamma = \{u; u \in CNOM \cup SVAR \;\&\; @_u u \in \Gamma\}$ and a binary relation $\sim$ on $\mathcal{N}_\Gamma^2$ such that $u \sim v$ iff $@_u v \in \Gamma$ and an equivalence class $[u]_\sim = \{v; u \sim v\}$. Call $\mathcal{M}_\Gamma^C = \langle W^C, \mathcal{R}_a^C, V^C \rangle$ a canonical model and g_Γ^C a canonical assignment if they satisfy these conditions:*

$$\begin{aligned} W^C &= \{[u]; @_u u \in \Gamma\} \\ \mathcal{R}_a^C &= \{\langle [u],[v]\rangle; @_u M_a v \in \Gamma\} \\ V^C(p) &= \{[u]; @_u p \in \Gamma\} \\ V^C(I) &= \{[u]; @_u I \in \Gamma\} \\ g_\Gamma^C(x) &= [x] \textit{ for all } x \in SVAR \cap \mathcal{N}_\Gamma \end{aligned}$$

Thus defined canonical model is indeed a model of our logic as stated in the following lemma.

Lemma 9. *Let $\mathcal{M}_\Gamma^C$ be defined as in Definition 16. Then $\mathcal{M}_\Gamma^C$ is an epistemic model of the Logic of Hybrid Action Models.*

Proof. We need to verify that (i) $\sim$ is an equivalence relation, (ii) for any nominal I there are no two distinct states that satisfy it, (iii) the same condition holds for any state variable x and (iv) that for any agent a, $\mathcal{R}_a^C$ is an equivalence relation.We prove parts (ii) and (iii) by contradiction, parts (i) and (iv) have direct proofs.

(i): Let u, v and w be all members of $\mathcal{N}_\Gamma$. $u \sim u$ follows trivially from the definition of $\mathcal{N}_\Gamma$, $u \sim v \Rightarrow v \sim u$ follows from item 2 of Fact 2 and $u \sim v \ \& \ v \sim w \Rightarrow u \sim w$ follows from item 5 of Fact 2.
(ii): Let I be any nominal and let us have two distinct states $[u]$ and $[v]$ such that $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash I$ and $\mathcal{M}_\Gamma^C, [v], g_\Gamma^C \Vdash I$. By the definition of satisfaction relation both $[u]$ and $[v]$ are members of $V^C(I)$ which means that $@_u I \in \Gamma$ and $@_v I \in \Gamma$. By item 2 of Fact 2 we get $@_I v \in \Gamma$ and by item 5 of the same Fact $@_u v \in \Gamma$. This means that $u \sim v$ and $[u] = [v]$ which is a contradiction.
(iii): Let x be any state variable and let $[u]$ and $[v]$ be two distinct states such that $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash x$ and $\mathcal{M}_\Gamma^C, [v], g_\Gamma^C \Vdash x$. This by definition of satisfaction relation means that $x \in dom(g_\Gamma^C)$ and $g_\Gamma^C(x) = [u]$ and $g_\Gamma^C(x) = [v]$. Thus $[u] = [v]$ and we arrive at a contradiction.
(iv): Symmetry of $\mathcal{R}_a^C$ follows from axiom (A19):

$u \to K_a M_a u$	instance of an $S5$ provable formula	1
$\overline{@}_u(u \to K_a M_a u)$	(R4) on line 1	2
$\overline{@}_u u \to \overline{@}_u K_a M_a u$	(A8), (R1) on line 2	3
$\overline{@}_u K_a M_a u$	(A10), (R1) on line 3	4
$@_u u$	assumption	5
$@_u K_a M_a u$	(A16), (R1) on lines 4 and 5	6
$@_u M_a v$	assumption	7
$@_v M_a u$	(A19), (R1) on lines 6 and 7	8

From $@_u u$ and $@_u M_a v$ we proved $@_v M_a u$. So for any $[u], [v] \in W^C$ such that $[u]\mathcal{R}_a^C[v]$ we proved that $[v]\mathcal{R}_a^C[u]$.

Transitivity follows from the axiom (A14) and from $M_a\varphi \leftrightarrow M_a M_a \varphi$ being a provable formula of $S5$.

Reflexivity follows from symmetry and transitivity.

□

The most important part of the completeness proof is, as usual, the Truth Lemma. However since states are not maximal consistent sets, satisfaction won't correspond to belonging to a set. Instead we'll work with formulas of the form of $@_u\varphi$. $@_u\varphi \in \Gamma$ indicates that at the state $[u]$, φ holds.

Lemma 10 (**Truth**). *Let χ be an arbitrary formula and let $\varphi \in cl(\chi)$ and $\Gamma \subseteq cl_E(\chi)$ a maximal consistent set in the closure of χ. Then*

$$\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash \varphi \textit{ iff } @_u \in \Gamma$$

We will prove the Truth Lemma by the induction on complexity of the formula φ but first we have to define the complexity of a formula and action models. This definition is taken from [2] and while taken without almost any change, it will play a crucial role in the proof of the Truth Lemma and that's why it's worth repeating it here.

Definition 17 (**Complexity**). *For any formulas φ and ψ and any action model M, the complexity function c, ranging over natural numbers, is inductively defined as follows:*

$$\begin{array}{rcl}
c(p) & = & 1 \\
c(I) & = & 1 \\
c(x) & = & 1 \\
c(\neg\varphi) & = & 1 + c(\varphi) \\
c(\varphi \wedge \psi) & = & 1 + max\{c(\varphi), c(\psi)\} \\
c(K_a\varphi) & = & 1 + c(\varphi) \\
c(C_B\varphi) & = & 1 + c(\varphi) \\
c(@_u\varphi) & = & 1 + c(\varphi) \\
c(\downarrow x\ \varphi) & = & 1 + c(\varphi) \\
c(E\varphi) & = & 1 + c(\varphi) \\
c(M) & = & max\{c(pre(s)); s \in \dot{M}\} \\
c([(M,s)]\varphi) & = & (4 + c(M)) \cdot c(\varphi)
\end{array}$$

Proof of the Truth Lemma. By induction on the complexity of a given formula φ.

Base case

Cases of $\varphi \equiv p$ or $\varphi \equiv I$ follow directly from the definition of V^C so let $\varphi \equiv x$.

($\Rightarrow$) By definition $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash x$ iff ($x \in dom(g_\Gamma^C)$ & $g_\Gamma^C(x) = [u]$). Since $g_\Gamma^C(x) = [x]$, we get $[u] = [x]$, that is $u \sim x$ and thus $@_u x \in \Gamma$.

($\Leftarrow$) Let $@_u x \in \Gamma$. By axiom (A12) $@_u u \in \Gamma$ which means that $[u] \in W^C$. By item 2 of Fact 2 we also get that $@_x u \in \Gamma$ and $@_x x \in \Gamma$. Thus $x \in \mathcal{N}_\Gamma \cap SVAR$. This means that $g_\Gamma^C(x) = [x]$. But since $@_u x \in \Gamma$, $u \sim x$ which means that $[u] = [x]$. And by the definition of satisfaction it follows that $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash x$.

Induction Step

Case of $\varphi \equiv \neg\psi$

$\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash \neg\psi$ iff $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \nVdash \psi$ iff by the induction hypothesis $@_u\psi \notin \Gamma$ iff $\neg @_u\psi \in \Gamma$ iff $@_u\neg\psi \in \Gamma$ by axioms (A9), (A15) and (A16).

Case of $\varphi \equiv \psi \wedge \chi$

$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash \psi \wedge \chi$ iff by induction hypothesis $@_u\psi \in \Gamma$ and $@_u\chi \in \Gamma$ iff $@_u(\psi \wedge \chi) \in \Gamma$. The last step follows from a provable equivalence $@_u(\psi \wedge \chi) \leftrightarrow (@_u\psi \wedge @_u\chi)$. This equivalence utilizes the axioms (A8), (A9), (A15) and (A16), rule (R4) and Classical Propositional Logic.

Cases of $\varphi \equiv \psi \vee \chi$ and $\varphi \equiv \psi \rightarrow \chi$

See the cases for negation and conjunction.

Case of $\varphi \equiv M_a\psi$

$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash M_a\psi$ iff $\exists v \in W^C$ $([u]\mathcal{R}^C_a[v]$ & $\mathcal{M}^C_\Gamma, [v], g^C_\Gamma \Vdash \psi)$ iff by induction hypothesis $\exists v$ $(@_v v \in \Gamma$ & $@_u M_a v \in \Gamma$ & $@_v\psi \in \Gamma)$. This, by axiom (A14) implies that $@_u M_a\psi \in \Gamma$. On the other hand, let $@_u M_a\psi \in \Gamma$. By Lindenbaum Lemma there must exist a nominal J such that $@_u M_a J \in \Gamma$ and $@_J\psi \in \Gamma$. This implies that both $@_u u$ and $@_J J$ are in Γ so both $[u]$ and $[J]$ are in W^C. Since $@_u M_a J \in \Gamma$ we have that $[u]\mathcal{R}^C_a[J]$ and from $@_J\psi \in \Gamma$ by induction hypothesis $\mathcal{M}^C_\Gamma, [J], g^C_\Gamma \Vdash \psi$. From this it follows that $\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash M_a\psi$.

Case of $\varphi \equiv K_a\psi$

Since $K_a\psi \equiv \neg M_a\neg\psi$ we can refer to earlier cases.

Case of $\varphi \equiv @_I\psi$

$(\Leftarrow)$ $@_u @_I\psi \in \Gamma$ implies by axiom (A11) that $@_I\psi \in \Gamma$ and this by axiom (A15) that $@_I I \in \Gamma$. This, along with the induction hypothesis gets us $\mathcal{M}^C_\Gamma, [I], g^C_\Gamma \Vdash \psi$. By the definition of the canonical valuation $@_I I \in \Gamma$ implies $[I] \in V^C(I)$ which means that $\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash @_I\psi$.
$(\Rightarrow)$ Let $\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash @_I\psi$. By definition of satisfaction

$$\exists u' \ ([u'] \in V^C(I) \ \& \ \mathcal{M}^C_\Gamma, [u'], g^C_\Gamma \Vdash \psi).$$

This yields $@_{u'} I \in \Gamma$ and $@_{u'}\psi \in \Gamma$. By Fact 2, item 4 we get $@_I\psi \in \Gamma$. Since $@_u u \in \Gamma$ (by assumption) and $@_I I \in \Gamma$ (by axiom (A15)), and Fact 2, item 3 allows us to finish this case because $@_u @_I\psi \in \Gamma$.

Case of $\varphi \equiv @_x\psi$

Very similar to the previous case.

Case of $\varphi \equiv \downarrow x \ \psi$

First note that $\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash \downarrow x \ \psi$ iff $\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash \psi[x := u]$.
$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash \psi[x := u]$ iff by induction hypothesis $@_u\psi[x := u] \in \Gamma$ iff by axioms (A9), (A15) and (A16) $(@_u u \in \Gamma$ & $\overline{@}_u\psi[x := u] \in \Gamma)$ iff by axioms (A8) and (A17) $(@_u u \in \Gamma$ & $\overline{@}_u \downarrow x \ \psi \in \Gamma)$ iff by axioms (A9), (A15) and (A16) $@_u \downarrow x \ \psi \in \Gamma$.

Case of $\varphi \equiv E\psi$

($\Rightarrow$) $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash E\psi$ iff $\exists u'$ ($[u'] \in W^C$ & $\mathcal{M}_\Gamma^C, [u'], g_\Gamma^C \Vdash \psi$). These by definition of the canonical model and by induction hypothesis yield $@_u u \in \Gamma$, $@_{u'} u' \in \Gamma$ and $@_{u'}\psi \in \Gamma$. By Fact 2, item 3 $@_u @_{u'} \psi \in \Gamma$ and by axioms (A8), (A16), (A18) and rule (R4) we get $@_u E\psi \in \Gamma$. ($\Leftarrow$) We can freely add new members to our set of agents A, as long as it stays finite. So let's add a universal agent U whose accessibility relation $\mathcal{R}_U$ is universal; that is $\forall w, w'\ w\mathcal{R}_U w'$ and follow the lines of the case of $\varphi \equiv M_a\psi$. And because $E\psi \equiv M_U\psi$ we can conclude this case.

Case of $\varphi \equiv C_B\psi$

($\Leftarrow$) Let $@_u C_B\psi \in \Gamma$. We'll verify that an arbitrary $[v] \in W^C$ that is accessible from $[u]$ via $(\mathcal{R}_B^C)^*$ satisfies ψ. Since model $\mathcal{M}_\Gamma^C$ is finite there exists a finite sequence of states $[u_0], \ldots, [u_n]$ such that $[u] = [u_0]$, $[v] = [u_n]$ and for any $0 \leq i < n$, $[u_i]\mathcal{R}_B^C[u_{i+1}]$. Let $a \in B$ be any agent such that $[u_0]\mathcal{R}_a^C[u_1]$. Because $C_B\psi \to K_a C_B\psi$ is a provable formula for an arbitrary $a \in B$ and since $C_B\psi \in cl(\chi)$ implies that $K_a C_B\psi \in cl(\chi)$ as well, we get $@_u K_a C_B\psi \in \Gamma$. By definition of $\mathcal{R}_a^C$, $@_{u_0} M_a u_1 \in \Gamma$. We defined $[u_0]$ as $[u]$ so $@_u M_a u_1$ is also a member of Γ by item 4, Fact 2. As stated earlier $@_u K_a C_B\psi \in \Gamma$ so by axiom (A19) $@_{u_1} C_B\psi \in \Gamma$. Because $[v]$ is reachable from $[u]$ in n steps, all we need to do is repeat this argument $(n-1)$-times and arrive at $@_v C_B\psi \in \Gamma$ from which it follows that $@_v\psi \in \Gamma$ by axiom (A7) and by induction hypothesis $\mathcal{M}_\Gamma^C, [v], g_\Gamma^C \Vdash \psi$.

($\Rightarrow$) We will work along the lines of the proof of completeness as stated in [6]. However we cannot simply copy the proof because there are two major differences: (i) states in the canonical model are classes of equivalence on all nominals and variables mentioned in Γ instead of maximal consistent sets and (ii) from this following fact that we built the canonical model based on one maximal consistent set instead on all maximal consistent sets of $cl_E(\chi)$. The proof in [6] utilizes the fact that $\vdash \bigvee_{\Gamma \in cl(\chi)} \bigwedge_{\gamma \in \Gamma} \gamma$ in non-hybrid systems. This is however not true in HAM.[13] What we have to use instead is (9) on page 55. Assume that $\mathcal{M}_\Gamma^C, [u], g_\Gamma^C \Vdash C_B\psi$. We'll show that $@_u C_B\psi \in \Gamma$. Define

$$W_{B,\psi}^C = \{[v] \in W^C; \mathcal{M}_\Gamma^C, [v], g_\Gamma^C \Vdash C_B\psi\}$$

[13]Every maximal consistent set of $cl_E(\chi)$ must contain a nominal $I_{|cl(\chi)|+1}$ which would make this nominal a part of every disjunct. Thus the whole disjunction implies $I_{|cl(\chi)|+1}$ which is not a tautology. Yet if the whole disjunction were provable, the nominal would have to be provable as well. A contradiction with the soundness of the calculus.

and define

$$\xi = \bigvee_{[v]\in W^C_{B,\psi}} \bigwedge_{@_v\alpha\in\Gamma} \alpha.$$

We'll prove the following three statements about ξ:

$$\Gamma \vdash @_u\xi \tag{4}$$

$$\vdash \xi \to \varphi \tag{5}$$

$$\Gamma \vdash \xi \to E_B\xi \tag{6}$$

(4): From the provable equivalence

$$\vdash @_v(\alpha_1 \vee \cdots \vee \alpha_n) \leftrightarrow (@_v\alpha_1 \vee \cdots \vee @_v\alpha_n) \tag{7}$$

we get

$$@_u\xi \equiv \bigvee_{[v]\in W^C_{B,\psi}} @_u \bigwedge_{@_v\alpha\in\Gamma} \alpha.$$

Because $[u] \in W^C_{B,\psi}$ we get $\Gamma \vdash @_u\xi$.
(5): Take an arbitrary $[v] \in W^C_{B,\psi}$. By definition $\mathcal{M}^C_\Gamma, [v], g^C_\Gamma \Vdash C_B\psi$. This yields by semantics and induction hypothesis $@_v\psi \in \Gamma$. This means that for any $[v] \in W^C_{B,\psi}$,

$$\vdash \bigwedge_{@_v\alpha\in\Gamma} \alpha \to \psi.$$

Thus $\vdash \xi \to \psi$.
(6): We'll prove $\Gamma \vdash \xi \to E_B\xi$ by contradiction so let $\Gamma \cup \{\xi \wedge \neg E_B\xi\}$ be a consistent set of formulas. This means that there must exist a state $[v] \in W^C_{B,\psi}$ and an agent $a \in B$ such that

$$\Gamma \cup \{ \bigwedge_{@_v\alpha\in\Gamma} \alpha \wedge M_a\neg\xi\}$$

is consistent, in other words

$$\Gamma \cup \{ \bigwedge_{@_v\alpha\in\Gamma} \alpha \wedge M_a\neg \bigvee_{[w]\in W^C_{B,\psi}} \bigwedge_{@_w\beta\in\Gamma} \beta\} \tag{8}$$

is consistent. Every maximal consistent set in $cl_E(\chi)$ must contain the nominal $I_{|cl(\chi)|+1}$ which means that specifically $\Gamma \vdash I_{|cl(\chi)|+1}$. Note that

$$\Gamma \vdash I_{|cl(\chi)|+1} \to \bigvee_{[w]\in W^C} \bigwedge_{@_w\beta\in\Gamma} \beta$$

holds. This is because $I_{|cl(\chi)|+1}$ is a member of W^C and because for any formula $\vartheta \in cl(\chi)$ it holds that

$$@_{I_{|cl(\chi)|+1}}\vartheta \in \Gamma \text{ iff } \vartheta \in \Gamma.$$

So

$$\Gamma \vdash \bigvee_{[w]\in W^C} \bigwedge_{@_w\beta\in\Gamma} \beta \tag{9}$$

and from (8) and (9) it follows that

$$\Gamma \cup \{ \bigwedge_{@_v\alpha\in\Gamma} \alpha \wedge M_a \bigvee_{[w]\in(W^C\setminus W^C_{B,\psi})} \bigwedge_{@_w\beta\in\Gamma} \beta\}$$

is consistent. By

$$\vdash M_a(\alpha_1 \vee \cdots \vee \alpha_n) \leftrightarrow (M_a\alpha_1 \vee \cdots \vee M_a\alpha_n)$$

we get that the following set is consistent as well.

$$\Gamma \cup \{ \bigwedge_{@_v\alpha\in\Gamma} \alpha \wedge \bigvee_{[w]\in(W^C\setminus W^C_{B,\psi})} M_a \bigwedge_{@_w\beta\in\Gamma} \beta\}$$

We assumed that $[v] \in W^C$ which means that $@_v v \in \Gamma$. It follows that

$$\Gamma \cup \{v \wedge \bigvee_{[w]\in(W^C\setminus W^C_{B,\psi})} M_a w\}$$

is consistent as well. From axiom (A12) and (7) we get that

$$\Gamma \cup \{v \wedge \bigvee_{[w]\in(W^C\setminus W^C_{B,\psi})} @_v M_a w\}$$

must be consistent too. So there must exist a state $[w'] \in (W^C \setminus W^C_{B,\psi})$ such that $[v]\mathcal{R}^C_a[w']$ by the definition of $\mathcal{R}^C_a$. Thus we finally arrive at a contradiction because $C_B\psi$ is satisfied in $[v]$ but not in $[w']$ yet $[v]\mathcal{R}^C_a[w']$.

After verifying that ξ satisfies all three conditions (4), (5) and (6) we can proceed to the final part of the proof of this case. From (6) and axiom (A6) we get $\Gamma \vdash \xi \to C_B\xi$, from (5), rule (R3) and axiom (A5) $\Gamma \vdash C_B\xi \to C_B\psi$. By Classical Propositional Logic $\Gamma \vdash \xi \to C_B\psi$. By axioms (A8), (A9), (A16) and rule (R4) and fact that $@_u u \in \Gamma$ we obtain $\Gamma \vdash @_u\xi \to @_u C_B\psi$, so $\Gamma \vdash @_u C_B\psi$. Because $@_u C_B\psi \in cl(\chi)$, deductive closure of Γ yields at last $@_u C_B\psi \in \Gamma$.

Case of $\varphi \equiv [(M, s)]p$

$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash [(M, s)]p$ iff by semantics $\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash \mathit{pre}(s) \to p$. The formula $\mathit{pre}(s) \to p$ has a lower complexity than $[(M, s)]p$ so we can apply the induction hypothesis to obtain $@_u(\mathit{pre}(s) \to p) \in \Gamma$ which, by axioms (A8), (A15), (A16), (A20) and rule (R4) gives us $@_u[(M, s)]p \in \Gamma$.

Case of $\varphi \equiv [(M, N(s))](I, s)$

$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash [(M, N(s))](I, s)$ iff by semantics

$$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash \mathit{pre}(N(s)) \to I.$$

This formula has again a lower complexity so by the induction hypothesis $@_u(\mathit{pre}(N(s)) \to I) \in \Gamma$ iff by axioms (A8), (A15), (A16), (A21) and rule (R4) yields $@_u[(M, N(s))](I, s) \in \Gamma$.

Cases of $\varphi \equiv [(M, s)]\alpha$

where $\alpha \equiv \neg\psi$, $\alpha \equiv \psi \wedge \chi$, $\alpha \equiv \psi \vee \chi$, $\alpha \equiv \psi \to \chi$, $\alpha \equiv K_a\psi$, $\alpha \equiv @_{I,s}\psi$, $\alpha \equiv @_x\psi$ or $\alpha \equiv E\psi$ are very similar to the earlier cases.

Case of $\varphi \equiv [(M, s)]C_B\psi$

This case is very similar to the $\varphi \equiv C_B\psi$ case. We will not however use axiom (A5) to finish this case but instead rule (R8)[1]. We will also put the notion of paths to use. The case of $\varphi \equiv C_B\psi$ was easy enough not to need this notion. We don't need it here either but the proof would become rather incomprehensible without it. So define a $BMst$-path of length n in the model $\mathcal{M}^C_\Gamma$ as a sequence of states $[u_0], \ldots, [u_n]$ and action states $s_0, \ldots, s_n$ from $\dot{M}$ such that $s_0 = s$, $s_n = t$, for any $k < n$ there exists an agent $a \in B$ such that $[u_k]\mathcal{R}^C_a[u_{k+1}]$ and $s_k R_a s_{k+1}$ and finally for any $k \le n$, the formula $@_{u_k}\mathit{pre}(s_k)$ is a member of Γ. First note that the following equation holds:

$$\mathcal{M}^C_\Gamma, [u], g^C_\Gamma \Vdash [(M, s)]C_B\psi$$

iff

$\forall t \in \dot{M}$, every $BMst$-path from state $[u]$ ends in a state, that satisfies $[(M, t)]\psi$.

The following picture may make it clear why the equation holds:

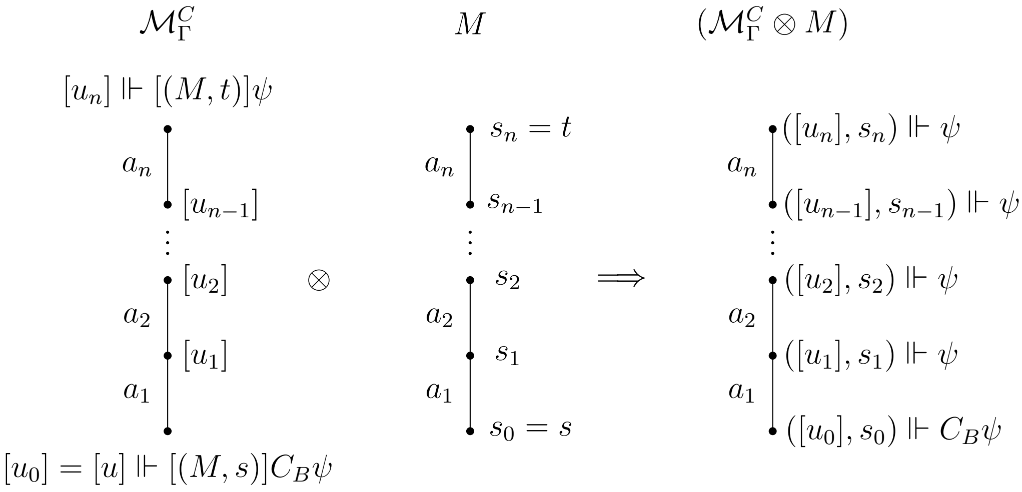


Figure 10: $\mathcal{M}_\Gamma^C, [u_0], g_\Gamma^C \Vdash [(M,s)]C_B\psi$ forces $([u_0], s_0)$ to satisfy $C_B\psi$ which means that $([u_n], s_n)$ satisfies ψ and so $\mathcal{M}_\Gamma^C, [u_n], g_\Gamma^C \Vdash [(M,t)]\psi$. On the other hand $[u]$ lies on a $BMss$-path of length 1 so it must satisfy $[(M,s)]\psi$. The similar holds for any state that is reachable from $[u]$ by a $BMst$-path for any $t \in \dot{M}$. The resulting state in the update model must satisfy ψ so all states in the updated model reachable from $([u_0], s_0)$ satisfy ψ. Thus $\mathcal{M}_\Gamma^C, [u_0], g_\Gamma^C \Vdash [(M,s)]C_B\psi$.

Thus all we need to prove is that

$\forall t \in \dot{M}$, every $BMst$-path from state $[u]$ ends in a state, that satisfies
$$[(M,t)]\psi$$
$$\text{iff}$$
$$@_u[(M,s)]C_B\psi \in \Gamma.$$

Observe that $[(M,t)]\psi$ has a lower complexity than $[(M,s)]C_B\psi$ so we know that $@_v[(M,t)]\psi \in \Gamma$ iff $\mathcal{M}_\Gamma^C, [v], g_\Gamma^C \Vdash [(M,t)]\psi$.

($\Leftarrow$) Let $@_u[(M,s)]C_B\psi \in \Gamma$, we'll show that every $BMst$-path from $[u]$ ends in a state (call it $[v]$) that satisfies $[(M,t)]\psi$ and such that $@_v[(M,t)]C_B\psi \in \Gamma$. We'll work by induction on the length of the path. The base case follows easily from $\vdash C_B\psi \to \psi$ and rule (R6).

The induction step: Let us have a $BMst$-path $[u_0], \dots, [u_{n+1}]$, such that $[u] = [u_0]$ This means that there must exist an agent $a \in B$ such that

$[u_n]\mathcal{R}_a^C[u_{n+1}]$ and $s_n R_a s_{n+1}$, where s_i is an action state corresponding to the state $[u_i]$ on the $BMst$-path. $[u_n]\mathcal{R}_a^C[u_{n+1}]$ yields

$$@_{u_n} M_a u_{n+1} \in \Gamma. \tag{10}$$

From $\vdash C_B\psi \to K_a C_B\psi$, rule (R7) and the fact that

$$\vdash [(M,s)](\psi_1 \to \psi_2) \to ([(M,s)]\psi_1 \to [(M,s)]\psi_2)$$

we get

$$\vdash [(M,s_n)]C_B\psi \to [(M,s_n)]K_a C_B\psi$$

and from the reduction axiom for knowledge (A25),

$$\vdash [(M,s_n)]K_a C_B\psi \to (pre(s_n) \to K_a[(M,s_{n+1})]C_B\psi).$$

By rule (R4) and axiom (A16) and $@_{u_n} u_n \in \Gamma$,

$$\Gamma \vdash @_{u_n}[(M,s_n)]K_a C_B\psi \to (@_{u_n} pre(s_n) \to @_{u_n} K_a[(M,s_{n+1})]C_B\psi).$$

By the induction hypothesis $@_{u_n}[(M,s_n)]C_B\psi \in \Gamma$ since $[u_n]$ lies on a $BMst$-path from $[u]$ of length n. And from $@_{u_n} pre(s_n) \in \Gamma$ we get

$$@_{u_n} K_a[(M,s_{n+1})]C_B\psi \in \Gamma. \tag{11}$$

From equations (10) and (11) along with axiom (A19) we obtain that $@_{u_{n+1}}[(M,s_{n+1})]C_B\psi \in \Gamma$ and so $@_{u_{n+1}}[(M,s_{n+1})]\psi \in \Gamma$ as well and from induction hypothesis $\mathcal{M}_\Gamma^C, [u_{n+1}], g_\Gamma^C \Vdash [(M,s_{n+1})]\psi$ which was to be proven.

($\Rightarrow$) We won't go into as many details as in the case of $\varphi \equiv C_B\psi$ because of the similarities between them. The major differences include the redefinition of $W_{B,\psi}^C$ set and the change to ξ. So define

$$W_{B,M,t,\psi}^C = \{[v]; \forall t' \ (t(R_B)^* t' \Rightarrow \mathcal{M}_\Gamma^C, [v], g_\Gamma^C \Vdash [(M,t')]\psi)\}$$

and define

$$\xi_t \equiv \bigvee_{[v] \in W_{B,M,t,\psi}^C} \bigwedge_{@_v \alpha \in \Gamma} \alpha.$$

The following three statements hold for $a \in B$ and tR_at':

$$\Gamma \vdash @_u\xi_s \tag{12}$$

$$\vdash \xi_t \to [(M,t)]\psi \tag{13}$$

$$\Gamma \vdash (\xi_t \wedge pre(t)) \to K_a\xi_{t'}. \tag{14}$$

We won't prove their validity because all the steps are analogous to the proves of statements (4), (5) and (6). By applying the rule $(R8)^1$ to the equations (13) and (14) we obtain

$$\Gamma \vdash \xi_s \to [(M,s)]C_B\psi$$

and following the same steps as in the case of $C_B\psi$ will, with the help of equation (12), yield $@_u[(M,s)]C_B\psi \in \Gamma$.

Case of $\varphi \equiv [(M,s)][(M',s')]\psi$

If ψ is of the form of $[(M,s)]^n\delta$ for any $n \geq 0$ and δ is *not* of the form $C_B\vartheta$, we can use the same procedure as in cases of $\varphi \equiv [(M,s)]\alpha$, where $\alpha \equiv p$, $\alpha \equiv (\beta\wedge\gamma)$ etc. Apply the corresponding reduction axiom on δ. This lowers the complexity of the formula so use the induction hypothesis and reapply the reduction axiom.

If however ψ is of the form of $[(M,s)]^nC_B\vartheta$, follow the lines of the case of $\varphi \equiv [(M,s)]C_B\psi$ but instead of using rule $(R8)^1$, use rule $(R8)^{n+2}$. The set $W^C_{B,M,t,\varphi}$ and formulas ξ_t have to be changed accordingly. We will however leave this open as it is still a work in progress.

□

Having proven the Truth Lemma and the Lindenbaum Lemma, nothing stands in our way to finally prove the Completeness Theorem itself.

Theorem 2 (Completeness). *Let φ be an arbitrary formula. Then*

$$\vDash \varphi \Rightarrow \vdash \varphi$$

Proof. Let $\nvdash \varphi$; we'll show that $\nvDash \varphi$. Since $\nvdash \varphi$, the set $\{\neg\varphi\}$ is consistent. Let us create an extended closure of formula $\neg\varphi$ by following the procedure in the definition of closure. Extend the set $\{\neg\varphi\}$ to a maximal consistent set Γ in such a way that Γ satisfies all the items of the Lindenbaum Lemma. Construct a canonical model $\mathcal{M}^C_\Gamma$ and a canonical assignment g^C_Γ. Since $\neg\varphi \in \Gamma$ and Γ is consistent, $\varphi \notin \Gamma$. By the construction of Γ we know that it contains at least one nominal, call it I, and a formula $@_I I$. By rule (R4) $\overline{@}_I\varphi \notin \Gamma$ and by axiom (A16) $@_I\varphi \notin \Gamma$. By the Truth Lemma $\mathcal{M}^C_\Gamma, [I], g^C_\Gamma \not\Vdash \varphi$ and thus we have found a model-state-assignment triple which doesn't satisfy φ so $\not\models \varphi$.

□

5 Open Questions

All the logics in this work were defined using Kripke semantics. However algebraic semantic can be hybridized as well. It might be interesting to check how the logic of HAM looks in algebraic semantic.

A Hilbert style calculus works well for explanation purposes but it's not too useful for finding or proving tautologies. A sequent calculus for the Logic of HAM would be most interesting. Along with all the proof theory that it brings.

There exists a translation from the hybrid language to the language of Classical First Order Predicate Logic. It is simply an extension of the standard translation and it can be found in [2]. However there also exists a backwards hybrid translation from a classical predicate language to the hybrid one, to be found in [2] as well. Hybrid action model language is however more expressive than that of Hybrid Logic. How do we have to extend the classical predicate language to get one corresponding to the language of HAM? How would a corresponding logic look?

6 Conclusion

We have shown a brand new Logic of Hybrid Action Models in the language of $\mathcal{L}^{KC\otimes}_{H(@,\downarrow,E)}$ and its sound and complete axiomatics. This logic is rather complicated and, admittedly, cumbersome. However it adds new options to look at the agents' communications.

While action models are an exciting new way to view the world of these ideal agents and how they communicate, hybridization gives knowledge its true power. From allowing agents to truly 'know' a state to applying backwards modalities to check whether the current state is accessible from another one. Both parts of the Logic of HAM are very moldable; we can remove the common knowledge and the axiom $K_a\varphi \to \varphi$ to get a logic of belief. We can remove the binder arrow and existential operators and only allow agents to refer to other states. Or we can strip the logic to its very basis—no restrains on accessibility relations, no common knowledge and no added hybrid operators. All these variants bring easily predictable changes to the axiomatics and the completeness proof.

There's been a lot of original research done in this work. The whole Logic of Hybrid Action Models is original—both its semantics and axiomatics along with solutions to most of the problems that arised from hybridizing action models. Namely the problem of having one name satisfied in several states which was overcome by introducing action and complex names. Or the unique cooperation of updates and nominals/variables. Although our solution borrowed heavily from [3] it is still a mostly original creation. Adding common knowledge to Hybrid Logic itself is an original work.

The proof of completeness is not in its final version. As it currently stands, taking rule $(R8)^1$ instead of rules $(R8)^n$ will yield a sound and complete axiomatics with respect to the Logic of Hybrid Action Models with the limitation that any formula that contains $[(M,s)]^n C_B\psi$ as a subformula for any $n \geq 2$ is forbidden. This limitation is planned to be removed in one of the upcoming articles.

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