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Logika otázek

Logic of Questions

PhD Thesis

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Michal Peliš

Abstract

The thesis deals with logic of questions (erotetic logic), which is one of the branches of non-classical logic. In the introductory part we speak generally about formalization of questions and the newest approaches to questions in logic are summed up. We introduce a formalization based on sets of direct answers and point out the role of inferences with questions. The rest of the thesis consists of two parts that can be read independently.

The first part focuses on relationships among consequence relations in inferential erotetic logic (IEL). We keep the framework of original IEL, introduced by Andrzej Wiśniewski, together with the representation of questions by sets of direct answers. Answers are strictly formulas of the declarative language. The mix of interrogatives and declaratives occours just on the level of consequences. Consequence relations with questions are defined by means of multiple-conclusion entailment among sets of declarative formulas. This way, one can work with classes of models and to make transparent some properties and relationships. We provide a general study of erotetic inferences based on IEL that is open for non-classical applications.

The second part contains epistemic erotetic logic. A question is understood as a set of direct answers; however, this time the set is finite. The satisfiability of a question in a state (possible world) of an epistemic model is defined by three conditions (a questioner does not know any direct answer, each direct answer considers as possible and at least one of them must be the right one). This approach is a new one and it is suitable for generalization to every epistemic-like system by questions and common erotetic concepts (e.g., various types of answers and erotetic inferences). The goal of this study is a future application in communication theory of a group of agents. We finish this part by multi-agent public announcement logic with application of questions by answer mining in a group of agents.

Abstrakt

Práce se zabývá jedním z odvětví neklasických logik – logikou otázek (erotetickou logikou). V úvodní části se hovoří obecně o formalizovaném přístupu k otázkám v logice a současně je stručně shrnuta zejména nejnovější historie tohoto odvětví. Zde je též zdůvodňována smysluplnost zachycení úsudků, v nichž se otázky objevují, a je představena formalizace otázky založená na explicitním stanovení množiny přímých odpovědí. Zbytek práce je rozdělen na dvě části, které lze číst nezávisle.

První část se zabývá důsledkovými relacemi v inferenční erotetické logice (inferential erotetic logic). Plně zde využíváme rámec původní inferenční erotetické logiky zavedené Andrzejem Wiśniewskim. Používáme výhradně formalizaci otázek pomocí množiny přímých odpovědí, kdy přímé odpovědi jsou formule deklarativního jazyka. Protože jsou důsledkové relace s otázkami definovány pomocí klasického vícezávěrového sémantického důsledku, využíváme při důkazech přístup založený na třídách modelů. V této části nám jde o obecný přístup ovšem s omezením, kdy je deklarativní jazyk rozšířen o otázky, ale k propojení deklarativního a interogativního jazyka dojde až na úrovni důsledkových relací. Primárním zájmem je studovat vztahy mezi jednotlivými erotetickými důsledkovými relacemi.

Druhá část obsahuje epistemickou erotetickou logiku. Prvotní je epistemický rámec tvořený zvolenou epistemickou logikou. Otázka je nadále formalizována jako množina přímých odpovědí, tentokrát však uvažujeme konečnou variantu a její splněnost v možném světě epistemického modelu je vázána na platnost tří epistemických podmínek (tazatel nezná žádnou z odpovědí, každou z přímých odpovědí však považuje za možnou a alespoň jedna z přímých odpovědí musí být správná). Jde o nový přístup, který umožňuje rozšířit libovolný epistemický systém o otázky a s nimi spojenou erotetickou terminologii (různé typy odpovědí, inference s otázkami a další). Přestože i zde se snažíme o obecný systém, naším hlavním cílem je komunikace ve skupině agentů při hledání odpovědí na otázky. V závěru druhé části tak představujeme multiagentní logiku veřejného vyhlášení (public announcement logic) s otázkami a skupinovými znalostmi.

Preface

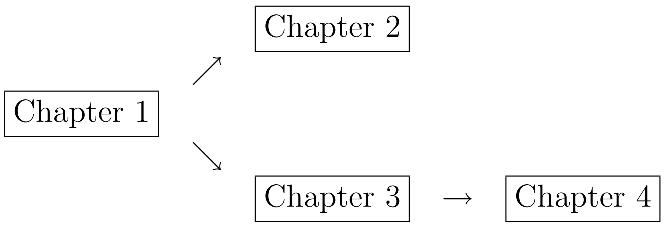
The work on this thesis started in 2003 when I met logic of questions—the branch of logic, which seemed to me very promising in possible applications and open for an extensive development. At the very beginning, I balanced between the reasons for and against having a special logic of questions. Then I was lucky to find two erotetic logics that proved their vitality in the recent years. The first one was *inferential erotetic logic* (IEL) developed by Andrzej Wiśniewski and his collaborators and the second one was an intensional approach to questions of Jeroen Groenendijk and Martin Stokhof.

My interest in non-standard consequence relations brought me to the study of inferential erotetic logic first. IEL provides consequence relations with declaratives as well as interrogatives. I decided to learn this system, to go through the relationships of inferences with questions and, inspired by the methodology of a question representation used in IEL, to suggest some generalizations and new relationships. (See chapter 2.)

During the study of IEL, I worked with my colleagues on epistemic interpretation of the relevant logic, see our results in [22] and in [2]. This and the intensional approach of Groenendijk and Stokhof were inspiration for my own extension of epistemic logic by questions. In many approaches to the formalization of interrogative sentences, epistemic terminology is used and questions are often seen as requirements of knowledge completion. I wished to formulate a completely general framework that can be used in all 'epistemic-like' systems. Such aspirations were successful, and the findings are presented in chapter 3. Naturally, if there is an epistemic system with questions, it is only one step away to its dynamic application. In the thesis I decided to use *public announcement logic* together with a group epistemic modalities and chapter 4 contains results obtained in this field.

Structure of the thesis

The thesis includes two main parts that can be read independently. The first one is chapter 2 and the second one consists of chapters 3 and 4. Chapter 1 provides common methodology for both parts. Chapter 2 can be understood as an inspiration of some erotetic concepts used in the rest of the thesis. The mentioned independence of chapter 2 and chapters 3 and 4 is also implied by the fact that these parts are based on separate papers—chapter 2 was published in [31] and a simplified version of chapters 3 and 4 appeared in [32] for the first time and [33] contains the last results from chapter 4.



Chapter 2 can be read as a full introduction to the topic. In chapters 3 and 4 it is not the case, the basic knowledge of modal logic is required.

The last chapter 5 contains some final remarks to the used set-of-answers methodology, to the main results in the second part of the thesis, and to the related approaches, and also discusses further directions.

Chapter 1: Logic and questions

The chapter briefly introduces the multi-paradigmatic situation in the methodology of erotetic logic and contains a short historical overview of this branch of logic with a special emphasis on the recent development. We introduce briefly *inferential erotetic logic*, Groenendijk-Stokhof's intensional approach, and some developments of these theories. However, the core of the chapter is devoted to a formalization of questions based on sets of answers. We justify the usefulness of the methodology in the study of consequence relations with questions as well as in an epistemic interpretation of questions.

Chapter 2: Consequence relations in inferential erotetic logic

This part is aimed to study relationships among consequence relations that were introduced by *inferential erotetic logic* (IEL). We keep the framework of IEL, but the question representation uses the methodology from chapter 1. IEL requires that declarative and interrogative formulas are not mixed on the object-language level. Also, answers are strictly declarative sentences. The defined consequence relations with questions are naturally based on the multiple-conclusion entailment among sets of declarative formulas. We add the *semantic range* of a question to the terminology of IEL and sets of declaratives are associated with classes of models. This 'model-based approach' makes proofs and properties transparent. This chapter, a technical overview of some IEL concepts and their properties, can provide a general framework and inspiration for the work with inferences among questions and declaratives.

Chapter 3: Epistemic logic with questions

The main goal is to incorporate questions in a general epistemic framework. Questions are considered to be finite sets of direct answers and their satisfiability in a state of an epistemic model is based on three conditions, which express ignorance and presuppositions of a questioner. While this definition of questions' *askability* is fully general for any 'epistemic-like' system and it is not necessary to keep the finite set-of-answers methodology, we work with the normal multi-modal propositional logic K as a background for the introduction of multi-agent epistemic logic. In this framework, a question becomes a complex modal formula. Inspired by inferential structures in IEL, we show that there are 'philosophically' similar structures based on classical implication. The rest of the chapter is devoted to answerhood conditions and the role of implication with respect to epistemic context and conjunctions of yes-no questions.

Chapter 4: A step to dynamization of erotetic logic

This chapter takes the full advantage of the multi-agent setting from chapter 3 and can be considered as an application of the introduced erotetic-epistemic approach in a dynamic framework. We define here an epistemic logic based on the modal system $\mathsf{S5}$ extended by group modalities together with public announcement modality. Askablity of questions as well as answerhood conditions are studied from the viewpoint of groups of agents. Finally we show the role of questions and group modalities in answer 'mining' among agents.

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Before I started to think about erotetic logic, Hana Skřivanová was the first one to bring me to this topic, and, a bit later, Martin Víta brought my attention to epistemic logic with group modalities. During the work, Jeroen Groenendijk sent me some of his papers in progress.

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Contents

Chapter 1

Logic and questions

1.1 Questions, answers, and inferences

In this chapter we wish to show that it is reasonable to consider questions as a part of logical study. In logic, declarative sentences usually have their formal (logical) counterparts and play an important role in argumentation. We often see logic to be primarily a study of inferences. Inferential structures are studied in formal systems, which can differ in the formalization of declaratives as well as in admitting or rejecting of some principles.

We believe that the dealing with questions in the logical framework will be justified if we show that questions can play an autonomous and important role in inferences. Perhaps this point may be considered as the most important to justify logic of questions.[1]

This introductory chapter provides a brief overview of history as well as methodology in recent approaches to erotetic logic. However, the main aim is to concentrate on a methodology used in the rest of the paper—we introduce and discuss a variant of the methodology based on sets of answers.

1.1.1 Questions and answers

Let us imagine a group of three players: Ann, Bill, and Catherine. Each of them has one card and nobody can see the cards of the others. One of the cards is the Joker and everybody knows this fact. Then

Who has the Joker?

is a reasonable sentence in this situation. We recognize it as an *interrogative sentence* because of its word order and the question mark. The hearing

[1] In this paper we use the term *logic of questions* in the same meaning as *erotetic logic*, a discussion on both terms can be found in [16].

or uttering of an interrogative is followed by intonation and interrogative pronounce.

An interrogative sentence includes a pragmatic aspect. It is a "request to an addressee to provide the speaker with certain information"—*interrogative speech act* [14, p. 1057]. Pragmatic approach emphasizes the roles of a speaker and an addressee, which seem to be outside of the interrogative context, but it seems very important in an analysis of questions. This can be the reason why some logicians argue against erotetic logic.

If we want to work with interrogatives in a formal system, we have to decide two problems:

1. What is the formal shape of questions?

2. What is the (formal) semantics of questions?

Reviewing the history of erotetic logic, there is no unique solution. There are many approaches to the formalization of questions and every approach varies according to what is considered as important. Logic of questions is considered to be multiparadigmatic. This is nicely illustrated by Harrah's examples of 'meta-axioms', see [15, pp. 25–26]. He groups them into three sets according to the acceptance by erotetic logicians.

1. The first group includes meta-axioms accepted in almost all systems. Harrah calls them *absolute axioms* and examples are:

   (a) Every question has at least one partial answer.

   (b) (In systems with negation) For every statement P, there exists a question Q whose direct answers include P and the negation of P.

   (c) Every question Q has a presupposition P such that: P is a statement, and if Q has any true direct answer, then P is true.

2. The second group, *standard axioms*, is often accepted, but not in all systems.

   (a) Every question has at least one direct answer.

   (b) Every direct answer is a statement.

   (c) Every partial answer is implied by some direct answer.

   (d) Every question is expressed by at least one interrogative.

   (e) Each interrogative expresses exactly one question.

   (f) Given an interrogative I there is an effective method for determining the direct answers to the question expressed by I.

3. The last group is called *excentric axioms*. Thus, the following examples of such axioms are accepted only in some interrogative systems.

    (a) If two questions have the same direct answers, then the two questions are identical.

    (b) Every question Q has a presupposition that is true just in case some direct answer to Q is true.

Let us notice the terminology, the difference between *interrogative (sentence)* and *question* was just introduced by standard axioms. The first term mostly refers to a type of sentence and the second one is a bit more complex. A question is expressed by an interrogative (sentence) and can be 'posed', 'asked', etc. Similarly, a proposition is expressed by a declarative (sentence), cf. [16]. Although we are used to use *interrogative* and *question* in the same meaning, if necessary, the term *interrogative sentence* is reserved for a natural-language sentence.

What seems to be common to all approaches is viewing questions as something structured and connected with answers. The relationship

question — answer(s)

is a very conspicuous sign and the meaning of questions is closely connected to answerhood conditions.

Since an answer to a question is often represented by a declarative, the starting point of many erotetic theories is a formal system for declaratives. "Any first-order language can be supplemented with a question-and-answer system" [44, p. 37]. This broadly accepted statement combines both the formal shape and the meaning of a question. Questions' autonomy depends on the chosen solution. Wiśniewski distinguishes two basic groups of erotetic theories: *reductionist* and *non-reductionist* theories. Roughly speaking, non-reductionism is characterized by questions that "are not reducible to expressions of other syntactic categories" [44, p. 40]. The boundary between both groups is vague. Perhaps only pure pragmatically oriented approaches belong to the radical reductionism with a complete rejection of questions as a specific entity. An example of such approach is commented in [30].

1.1.2 Inferences with questions

Although there is a discussion whether it is necessary to work with questions as a new specific entity, almost all theorists agree that questions play a specific role in inferences. Let us come back to our group of players. The situation, where the Joker is held by a member of the group, can raise to the question

> Q: *Who has the Joker?*

from the declarative

> *Either Ann has the Joker or Bill has the Joker or Catherine has the Joker.*

What makes this raising reasonable are answerhood conditions of Q connected to the declarative.

Another kind of inferential structure is based on declaratives as well as questions among premises. For example, from

> Q: *Who has the Joker?*

and

> Γ: *The only person from London has the Joker.*

can be inferred the question

> Q_1: *Who is from London?*

The relationship of the inferred question Q_1 and the question Q is based on their answerhood conditions again. An answer to Q can provide an answer to Q_1 with respect to the context Γ. Moreover, in this example, Q can be inferred from Q_1 and Γ as well. This shows that the relationship is structured dependently on various kinds of answerhood conditions and contexts. Let us have Q the same, but the context is

> *A person from London has the Joker.*

If two persons are from London and we gain their names in an answer to Q_1, then we receive only a *partial answer* to Q.[2] If each player (or nobody) is from London and an answer to Q_1 does not provide any help for the answering of Q, it has no sense to speak of an inferential relation between Q and Q_1 with respect to this context.

The role of answerhood conditions in inferences among questions is clearly obvious in the following example: From any (complete) answer to Q we gain a (complete) answer to the question

[2]Informally, a *partial answer* does not completely answer a question, but it eliminates some of the possible answers.

Has Ann the Joker?

as well as for the questions

Has Bill the Joker?

and

Has Catherine the Joker?

Roughly and informally speaking, answerhood conditions of the previous three questions are 'entailed' in the answerhood conditions of the question Q; they can be inferred from the answerhood conditions of Q. The question *Has Ann the Joker?* is 'entailed' by Q.

Though we do not doubt that there are inference-like structures with questions based on answerhood conditions, we have been still faced with the problem how to formalize the relationship of questions and answers. We see the convenient solution in a liberal set-of-answers methodology.

1.2 Set-of-answers methodology

In the previous section we emphasized the close connection of questions and answers in most erotetic theories as well as in inferential structures with questions. We believe that we can solve the problem of the formal shape of questions together with the problem of the questions' semantics. In this section we introduce a formalization of questions based on a set of answers. Our aim is to show that such approach can also reflect some semantic and pragmatic requirements.

1.2.1 Semantics of questions

Some theories do not admit that questions could have an independent meaning in logic. For example, questions are paraphrased by declarative sentences; the question *Who has the Joker?* may be then paraphrased by

I ask you who has the Joker.

Another way is the paraphrasing by epistemic-imperative sentences:

Bring it about that I know who has the Joker!

The propriety of both paraphrases as a complete meaning of a question is rather problematic. Although we expect to utilize the importance of a questioner and an addressee later on, now it may be second-rate from the semantic viewpoint.

Nuel Belnap formulated three methodological constraints on a theory of questions, which he used for a classification and evaluation of erotetic theories:[3]

1. **Independence** Interrogatives are entitled to a meaning of their own.
2. **Equivalence** Interrogatives and their embedded forms are to be treated on a par.
3. **Answerhood** The meaning of an interrogative resides in its answerhood conditions.

The most important is the first requirement, which is the main sign of non-reductionist theories. To accept 'independence requirement' means that we are obliged to look for a specific semantics of questions. The 'equivalence requirement' is closely related to a semantic entailment and is dependent on the chosen semantics. The 'answerhood' requires that the meaning of questions is related to the meaning of answers. Moreover, we can work with the idea that the semantics of answers forms a good background for the study of the meaning of interrogatives. The approach, where answers are crucial for the meaning of questions, is displayed in Hamblin's postulates from 1958:

1. Knowing what counts as an answer is equivalent to knowing the question.
2. An answer to a question is a statement.
3. The possible answers to a question are an exhaustive set of mutually exclusive possibilities.

Each postulate may be argued against and the detailed discussion is available in [14]. However, according to David Harrah, adopting the first one is "the giant step toward formalization often called *set-of-answers methodology*" [16, section 2]. Although there are many kinds of set-of-answers methodology (SAM, for short) in the literature, we will not make any survey here. In the next subsection we introduce an easy idea of a question representation by a set of direct answers.

[3] Belnap, N.D., 'Approaches to the semantics of questions in natural language. Part I', Pittsburgh, 1981. Cited from [13, p. 3–4].

1.2.2 Sets of answers

Generally, without any context, the question *Who has the Joker?* can be answered by expressions of the following form:

> *Ann.*
> *Ann has it.*
> *Ann has the Joker.*
> *Ann and Bill.*
> ⋮
> *Batman has the Joker.*
> ⋮
> *Your friends.*
> *People at this table.*
> ⋮
> *Nobody.*
> etc.

The question seems to be answered if a (complete) list of Joker's owners is given. We can assume that answers are sentences; thus, the first three items in the list have the same meaning in the answering of this question. From the viewpoint of propositional logic and in accordance with the first two Hamblin's postulates, we can understand every question closely connected with a set of (propositional) formulas—answers.

Of course, we can receive the following responses to the same question as well:

> *Ann hasn't the Joker.*

or

> *I don't know who has the Joker.*

Neither of them answers completely the question *Who has the Joker?*. The first one can be considered to be a partial answer; it removes some answers as impossible, e.g., all answers with Ann having the Joker. The second one appears to bear another kind of information; an addressee says to a questioner that she has the same problem and would ask the same question. We will return to this topic shortly in Section 4.4.

If we had decided to represent every question by a complete set of its answers, we would not have a clear and useful formalization of questions. Let us return to our example. Considering the context, a questioner expects one of the following responses to the question *Who has the Joker?*:

> α: *Ann has the Joker.*
> β: *Bill has the Joker.*
> γ: *Catherine has the Joker.*

In fact, the question might be reformulated to

> Q': *Who has the Joker: Ann, Bill, or Catherine?*

The question Q' is a combination of the question

> *Who has the Joker?*

and the context

> *Either Ann has the Joker or Bill has the Joker or Catherine has the Joker.*

Of course, various responses to Q' can be received again, but the 'core' answers (the term *direct answers* will be used) are α, β, and γ. The sentence

> δ: *Neither Ann nor Bill have the Joker.*

is an answer, from which γ is inferred thanks to the context; δ is a *complete answer* to Q'. Complete answers are 'solutions' of a question and the set of direct answers is a subset of the set of complete ones.

Our SAM is inspired by the syntactic representation of questions in *inferential erotetic logic* founded by Andrzej Wiśniewski.[4] We want to be very liberal and this leads us to considering questions to be sets of formulas, which play the role of direct answers. A (general) declarative language $\mathcal{L}$ is extended only by curly brackets ($\{, \}$) and question mark (?).

A question is the following structure

$$?\{\alpha_1, \alpha_2, \ldots\}$$

where $\alpha_1, \alpha_2, \ldots$ are formulas of the extended language.

No wonder that we want to impose some restrictions on direct answers to keep their exclusive position. Such restrictions are mostly combination of syntactic as well as semantic requirements. From the syntactic viewpoint and being inspired by the previous examples, we require:

1. Formulas $\alpha_1, \alpha_2, \ldots$ are syntactically distinct.

2. A set of direct answers has at least two elements.

[4] The best overview of questions' formalization in inferential erotetic logic is in the chapter 3 of the book [44]. See the article [46] as well.

Both restrictions introduce questions as 'tasks' with at least two distinct 'solutions'. Syntactical distinctness is a first step to the idea that direct answers form the 'core' of questions' meaning. In semantics we will require non-equivalence above that.

The most typical questions with only two direct answers are *yes-no questions*. The question

Has Ann the Joker?

has, in fact, the following two direct answers:

Yes. (*Ann has the Joker.*)
No. (*Ann has not the Joker.*)

Such question will be identified with the form $?\{\alpha, \neg\alpha\}$ and shortened as $?\alpha$. A yes-no question is a variant of a *whether question*, where an answer is a choice from two possibilities.[5] The role of negation is considered to be very important in SAM. Negation is always related to a background system and receiving $\neg\alpha$ can mean more than 'it is not the case that α'—it expresses something like 'strict denial of α'. Compare it with an epistemic interpretation of relevant logic in [22].

We believe that the introduced SAM is more or less successful in formalization of most types of natural language interrogatives with respect to the chosen background logical system. However, it brings more—the next subsection informally shows how to incorporate epistemic aspects.

1.2.3 Epistemic aspects of SAM

In section 1.2.1 we used the paraphrase

Bring it about that I know who has the Joker!

for the question *Who has the Joker?* as it is common in Hintikka's analysis [17]. It is natural to see epistemic aspects in the meaning of questions. At first sight, a question expresses an ignorance of a questioner delivered to an addressee. Looking for an epistemic counterpart of questions in the history of erotetic logic, the most known are *epistemic-imperative* approaches of Åqvist and Hintikka. These theories are reductionist ones, questions are translated into epistemic-imperative statements. Generally, both approaches are based on the idea of a questioner who does not know any answer to a question and who calls for a completion of knowledge.[6]

[5] For example, *Is two even or odd?* is such a question with a possible formalization $?\{\alpha_1, \alpha_2\}$.

[6] More details are in [16] and [44, chapter 2].

Epistemic analysis of questions has two important levels. The first one, personal level, works with the knowledge and ignorance of a questioner. The second one, pragmatic level, considers an exchange of information in a group of agents, which is supposed to be used in a theory of communication.

Let us return to the question

Q': *Who has the Joker: Ann, Bill, or Catherine?*

with the formalization based on the discussion in the previous section: $Q' = ?\{\alpha, \beta, \gamma\}$. Just introduced SAM makes it possible to specify expectations and presuppositions of a questioner. A questioner expresses not only the ignorance of Joker's holder, but the presupposition that the holder must be either Ann or Bill or Catherine. In the personal level, our SAM informs us what answers are considered as possible and, moreover, what is the rank of complete answers.

To employ the pragmatic level we have to indicate a questioner and an addressee. In particular, if Catherine wants to find out who has the Joker in the group of her friends-players, she could ask $?_c\{\alpha, \beta\}$ publicly among them. This will be studied in chapters 3 and 4.

1.3 Note on the recent history of erotetic logic

We are not going to present a complete survey of erotetic theories. The reader can find a comprehensive overview of the history of erotetic logic in [16]. Erotetic theories with the main influence in this field of study are described in [44, chapter 2] as well. Moreover, both papers provide a good introduction to the terminology used in logic of questions and cover enough the history of erotetic logic till the 1990s. The period from the 1950s till 1990s is mapped in [15]. Mainly linguistic viewpoint with the detailed discussion about the semantics of questions and pragmatic approaches can be found in [14].

The logic of questions has, maybe surprisingly, a long history. F. Cohen and R. Carnap seem to be the first authors attempting to formalize questions in a logical framework—their attempts date back to the 1920s [16, p. 3]. The first 'boom' of logical approach to questions took place in the 1950s (Hamblin, Prior, Stahl) and continued in the 1960s (Åqvist, Harrah, Kubiński). The first comprehensive monograph on questions [1] brought into life many important terms used in erotetic logic so far. The late 1970s gave birth to influential reductionist theories: Hintikka's epistemic-imperative approach and Tichý's approach based on his *transparent intensional logic* [37].

Most influential modern logics of questions with the important role of erotetic inferences are

- intensional approach of Jeroen Groenendijk and Martin Stokhof and
- *inferential erotetic logic* (IEL) of Andrzej Wiśniewski.

Both theories appears fully developed in the 1990s and we consider them giving birth to several approaches some of which are still influential.

1.3.1 Inferential erotetic logic

Wiśniewski's IEL is a complex system dealing with various interrogative inferential structures. The influence of Belnap's and Kubiński's works is apparent. Primarily it is based on classical logic and a formalization of questions, which is very similar to the introduced SAM. Conclusion relations among questions and declaratives are defined on the metalanguage level where the role of multiple-conclusion entailment is important. The advantage of IEL is its possible generalization for non-classical logics. Chapter 2 is devoted to the slightly modified IEL. All important terms concerning interrogative inferences are introduced and studied in their mutual relationships. In this chapter the reader can find a list of relevant publications to this topic. The book [44] and article [46] contain a nicely written presentation of Wiśniewski's approach.

The complex study of erotetic inferential structures in IEL predetermines studies based on an old idea that a (principal) question can be answered by asking auxiliary questions. This searching process can be seen as a tree with a principal question (and context expressed by declaratives) in the root. Nodes bear auxiliary questions (with context). A move from node to node is justified by IEL inferences in the direction to leaves with answers. *Erotetic search scenarios* is the name for this approach, see the paper [47]. The similar idea is developed in Hintikka's *interrogative model of inquiry* reflecting the usefulness of questions in reasoning [18]. Hintikka's approach has an application in game-theoretic framework for *belief revision theory*, cf. [10].[7]

The IEL methodology makes it possible to transform the derivability of a declarative formula into a sequence of questions and produce an analytic-tableaux style calculus—*socratic proofs*, see [49] for classical propositional logic and [21] for some normal modal propositional logics.

[7] IEL used to be presented as an alternative to Hintikka's approach (cf. [46, 47]).

1.3.2 Intensional erotetic logic

Groenendijk's and Stokhof's approach can be called *intensional erotetic logic*. Intensional semantics is the background of the meaning of questions. The meaning of a declarative sentence is given by truth conditions and forms a subset of *logical space*. Logical space is understood as a set of all 'possible states' (possible worlds, indexes, situations). Intension of a declarative is then a set of states where the declarative is true (this is called *proposition*). Extension of a declarative is its truth value in a given situation. Combining intensional semantics with the full acceptance of Hamblin's postulates we obtain the meaning of a question as a *partitioning* of logical space. In accordance with the third postulate, answers to a question form exhaustive set of mutually exclusive propositions. Partitioning of logical space is the intension of a question. Extension of a question in a given situation is the answer, which is true there. Questions' representation is similar to SAM with restrictions posed by Hamblin's postulates. This approach introduces an entailment between two questions as a refinement of a partitioning:

> A question Q entails a question Q_1 iff each answer to Q implies an answer to Q_1.

(A question Q provides a refinement of the Q_1-partitioning.) Moreover, many important terms are naturally defined (partial answer, complete answer, informative value of answers, and others). See [13] and [14].

This intensional approach influenced many works in the last ten years. It is a good inspiration for epistemic representations of questions, see [6] and [29]. Very recently the logic of questions receives more attention in connection with dynamic aspects of epistemic logic and communication theory, cf. [40].

Groenendijk's and Stokhof's intensional interpretation inspired some extensional approaches. [27] brings a many-valued interpretation of declaratives and interrogatives based on bilattices and in the paper [25] Gentzen style calculus is presented.[8] [34] gives a syntactic characterization of answerhood for the partition semantics of questions and then the authors implement partition semantics in question answering algorithm based on tableaux theorem proving [35].[9]

Presented logics of questions deal with questions as having 'crisp' answers. However, we can imagine that there is a scale of answers, e.g., in case of yes-no questions, the scale

yes — rather yes — rather no — no

[8] An algebraic approach, where sets of answers form distributive lattices, is studied also in [19].

[9] Nice and brief comments are in [12].

is usual in questionnaires. Such kind of scale does not require to introduce four new answers, but it corresponds to comparative degrees of truth of the original yes-/no-answer. Truth-degrees are studied in multi-valued logics. The paper [4] presents propositional Groenendijk-Stokhof's erotetic logic with fuzzy intensional semantics based on fuzzy class theory. Although fuzzy logic seems to be suitable for the study of reasoning under vagueness, its combination with logic of questions is still rather underdeveloped.

Chapter 2

Consequence relations in inferential erotetic logic

2.1 Introduction

Inferential structures that will be introduced and studied in this chapter are based on the slightly adapted *inferential erotetic logic* (IEL). We utilize the framework of this theory and show some properties, relationships, and possible generalizations.

2.1.1 Adapted set-of-answers methodology in IEL

Inferential erotetic logic accepts only the first two Hamblin's postulates and tries to keep the maximum of the (classical) declarative logic and its consequence relation. On the syntactic level of a considered formalized language, a question is assigned to a set of sentences (direct answers). Direct answers are declarative formulas, each question has at least two direct answers, and each finite and at least two-element set of sentences is the set of direct answers to some question [46, p. 11].

Let us apply our SAM introduced in the previous chapter. We define a (general) erotetic language $\mathcal{L}_Q$. A (general) declarative language $\mathcal{L}$ is extended by curly brackets ($\{, \}$) and question mark (?).[1] In the correspondence with Section1.2 a question Q is the following structure

$$?\{\alpha_1, \alpha_2, \ldots\}$$

For the set of direct answers of a question Q we will use the symbol dQ.

[1] In this chapter we use only propositional examples in the language with common connectives $(\wedge, \vee, \rightarrow, \neg)$.

Let us repeat that direct answers are syntactically distinct and $|dQ| \geq 2$. Moreover, we require here that elements of dQ are declarative sentences.

In case of finite versions of questions $?\{\alpha_1, \ldots, \alpha_n\}$ we suppose that the listed direct answers are semantically non-equivalent. The class of finite questions corresponds to the class of *questions of the first kind* in [44]. Some of them are important in our future examples and counterexamples. Let us mention two abbreviations and terms that are very frequent in this thesis:

- *Simple yes-no questions* are of the form $?\alpha$, which is an abbreviation for $?\{\alpha, \neg\alpha\}$. If α is an atomic formula, then the term *atomic yes-no question* is used.

- A *conjunctive question* $?|\alpha, \beta|$ requires the answer whether α (and not β), or β (and not α), or neither α nor β, or both (α and β). It is an abbreviation for $?\{(\alpha \wedge \beta), (\neg\alpha \wedge \beta), (\alpha \wedge \neg\beta), (\neg\alpha \wedge \neg\beta)\}$. Similar versions are $?|\alpha, \beta, \gamma|$, $?|\alpha, \beta, \gamma, \delta|$, and so on.[2]

In the original version of IEL, questions are not identified with sets of direct answers: questions belong to an object-level language and are expressions of a strictly defined form, but the form is designed in such a way that, on the metalanguage level (and only here), the expression which occurs after the question mark designates the set of direct answers to the question. Questions are defined in such a way that sets of direct answers to them are explicitly specified. The general framework of IEL allows for other ways of formalizing questions.[3]

To avoid a misunderstanding, we will use the following metavariables in this chapter:

- small Greek letters $(\alpha, \beta, \varphi, \ldots)$ for declarative sentences,

- $Q, Q_1, \ldots$ for questions,

- capital Greek letters $(\Gamma, \Delta, \ldots)$ for sets of declaratives, and

- $\Phi, \Phi_1, \ldots$ for sets of questions.

2.1.2 Consequence relations in IEL

Consequence relations are the central point of logic. Declarative logic can be defined by its consequence relation as a set of pairs $\langle\Gamma, \Delta\rangle$, where Γ and Δ are

[2] Original IEL uses the symbol $?\pm|\alpha, \beta|$, etc.

[3] Personal communication with Andrzej Wiśniewski.

sets of (declarative) formulas and Δ is usually considered to be a singleton. Inferential erotetic logic makes one step more and adds new consequence relations mixing declaratives and interrogatives. The most important relations, which we are going to introduce, are the following:

- *Evocation* is a binary relation $\langle \Gamma, Q \rangle$ between a set of declaratives Γ and a question.
- *Erotetic implication* is a ternary relation $\langle Q, \Gamma, Q_1 \rangle$ between an initial question Q and an implied question Q_1 with respect to a set of declaratives Γ.
- *Reducibility* is a ternary relation $\langle Q, \Gamma, \Phi \rangle$ between an initial question Q and a set of questions Φ with respect to a set of declaratives Γ.

Motivations and natural-language examples of these consequence relations will be introduced in the next sections. In the literature, let us recommend texts [44, 46] for both the evocation and erotetic implication; reducibility is studied in [20, 44, 50].

Our aim is to study erotetic consequence relations in a very general manner, independently of the logic behind. The definitions of IEL consequences are based on the semantic entailment and the model approach relative to the chosen logical background.

2.1.3 Model-based approach

The following model-based approach was inspired by *minimal erotetic semantics* from [46]. Let us introduce the set of all models for a declarative language as follows:

$$\mathcal{M}_{\mathsf{L}} = \{\mathbf{M} \mid \mathbf{M} \text{ is a (semantic) } \textit{model} \text{ for } \mathcal{L}\}.$$

The term *model* varies dependently on a background logic L. If L is classical propositional logic (CPL, for short), then $\mathcal{M}_{CPL}$ is a set of all *valuations*. In case of predicate logic it is a set of all *structures* with a realizations of non-logical symbols. Because of the possibility of adding some other constraints for models we will deal with (e.g., finiteness, preferred models, etc.), let us generally use a set $\mathcal{M} \subseteq \mathcal{M}_{\mathsf{L}}$. If necessary, the background logic and restrictions posed on models will be stated explicitly.

Speaking about *tautologies* of a logic L we mean the set of formulas

$$\textsc{Taut}_{\mathsf{L}} = \{\varphi \mid (\forall \mathbf{M} \in \mathcal{M}_{\mathsf{L}})(\mathbf{M} \models \varphi)\}.$$

If a restricted set of models $\mathcal{M}$ is in use, we speak about $\mathcal{M}$-*tautologies*

$$\textsc{Taut}_{\mathsf{L}}^{\mathcal{M}} = \{\varphi \mid (\forall \mathbf{M} \in \mathcal{M})(\mathbf{M} \models \varphi)\}.$$

All semantic terms may be relativized to $\mathcal{M}$. Each declarative sentence φ (in the language $\mathcal{L}$) has its (restricted) set of models

$$\mathcal{M}^{\varphi} = \{\mathbf{M} \in \mathcal{M} \mid \mathbf{M} \models \varphi\}$$

and similarly for a set of sentences Γ

$$\mathcal{M}^{\Gamma} = \{\mathbf{M} \in \mathcal{M} \mid (\forall \gamma \in \Gamma)(\mathbf{M} \models \gamma)\}.$$

(Semantic) entailment

Let us recall the common (semantic) entailment relation. For any set of formulas Γ and any formula ψ:

$$\Gamma \models \psi \text{ iff } \mathcal{M}^{\Gamma} \subseteq \mathcal{M}^{\psi}.$$

In case $\Gamma = \{\varphi\}$ we write only $\varphi \models \psi$.

$$\varphi \models \psi \text{ iff } \mathcal{M}^{\varphi} \subseteq \mathcal{M}^{\psi}$$

Now, we introduce *multiple-conclusion* entailment (mc-entailment, for short).

$$\Gamma \mid\models \Delta \text{ iff } \mathcal{M}^{\Gamma} \subseteq \bigcup_{\delta \in \Delta} \mathcal{M}^{\delta}$$

If $\mathcal{M}^{\Gamma} = \mathcal{M}^{\Delta}$, let us write $\Gamma \equiv \Delta$.[4]

Mc-entailment is reflexive ($\Gamma \mid\models \Gamma$), but it is neither symmetric nor transitive relation:

Example 1. *Let $\Gamma \subseteq \textsc{Taut}_{\mathsf{L}}$, Δ be a set of sentences containing at least one tautology and at least one contradiction, and Σ be such that $\bigcup_{\sigma \in \Sigma} \mathcal{M}^{\sigma} \subset \mathcal{M}_{\mathsf{L}}$. Then $\Gamma \mid\models \Delta$ and $\Delta \mid\models \Sigma$, but $\Gamma \mid\not\models \Sigma$.*

Entailment is definable by mc-entailment:

$$\Gamma \models \varphi \text{ iff } \Gamma \mid\models \{\varphi\}$$

On the other hand, mc-entailment is not definable by entailment. In this context, the following theorem could be surprising at the first sight.[5]

[4] In case of the semantic equivalence of formulas φ and ψ it will be only written $\varphi \equiv \psi$. On the other hand, two different sets of models do not imply the existence of two different sets of sentences (in $\mathcal{L}$).

[5] We say that mc-entailment is compact iff for each $\Gamma \mid\models \Delta$ there are finite subsets $G \subseteq \Gamma$ and $D \subseteq \Delta$ such that $G \mid\models D$.

Theorem 1. *Entailment (for logic* L*) is compact iff mc-entailment (for* L*) is compact.*

Proof. See [44, pp. 109–110]. □

2.1.4 Basic properties of questions

After we have introduced the SAM representation of questions and the model-based approach, we can mention some basic properties of questions. First, let us introduce the term *soundness*, which is one of the most important terms in IEL.

Definition 1. *A question* Q *is* sound in $\mathbf{M}$ *iff* $\exists\alpha \in dQ$ *such that* $\mathbf{M} \models \alpha$*.*

A question is sound with respect to a model $\mathbf{M}$ whenever it has at least one direct answer true in $\mathbf{M}$. See [44, p. 113].

For all IEL consequnce relations, it is important to state the soundness of a question with respect to a set of declaratives.

Definition 2. *A question* Q *is* sound relative to Γ *iff* $\Gamma \mid\models dQ$*.*

The sum of all classes of models of each direct answer α, i.e., $\bigcup_{\alpha\in dQ}\mathcal{M}^{\alpha}$, is called *semantic range* of a question Q. Considering semantic range, our liberal approach admits some strange questions; one of them is a *completely contradictory question* that has only contradictions in its set of direct answers, its semantic range beeing just $\emptyset$. Another type is a question with a tautology among its direct answers, then the semantic range expands to the whole $\mathcal{M}$. Questions with such a range are called *safe*.[6] Of course, it need not be any tautology among direct answers for to be a safe question.

Definition 3.
- *A question* Q *is* safe *iff* $\bigcup_{\alpha\in dQ}\mathcal{M}^{\alpha} = \mathcal{M}$*.*
- *A question* Q *is* risky *iff* $\bigcup_{\alpha\in dQ}\mathcal{M}^{\alpha} \subset \mathcal{M}$*.*

Questions $?\alpha$, $?|\alpha,\beta|$ are safe in CPL, but neither is safe in Bochvar logic. If β is not equivalent to $\neg\alpha$, then $?\{\alpha,\beta\}$ is risky in CPL. Neither $?\alpha$ nor $?|\alpha,\beta|$ are safe in intuitionistic logic, but there are safe questions in this logic; just each question with at least one tautology among direct answers. Simple yes-no questions are safe in logics that accept the law of excluded middle.

It is good to emphasize that the set of direct answers of a safe question is mc-entailed by every set of declaratives. On the other hand, knowing a question to be sound relative to every set of declaratives implies its safeness.

Fact 1. Q *is safe iff* $(\forall\Gamma)(\Gamma \mid\models dQ)$*.*

Specially, safe questions are sound relative to $\Gamma = \emptyset$.

[6]This term originates from Nuel Belnap.

2.1.5 The road we are going to take

After introducing *evocation* and the term *presupposition* of a question in section 2.2, we will show the role of maximal and prospective presuppositions in the relationship to semantic range of questions. Some classes of questions will be based on it. One could be surprised that we are not going to discuss answers in this section; in fact, there is not much to say about them. It turns out that various types of answers do not play any special role in the inferential structures.

Section 2.3 is crucial from the chosen viewpoint. We investigate *erotetic implication* and *reducibility* there. An important part is devoted to a discussion of the role of an auxiliary set of declaratives. We will demonstrate some variants of erotetic implication and their properties. The chosen formal shape of questions in IEL makes it possible to compare questions in the sense of their *answerhood power*. Inspired by [13] and [44, section 5.2.3] we will examine the relationship of 'giving an answer' of one question to another, which is a generalisation of Kubiński's term 'weaker question'.

Questions will be considered as independent structures not being combined by logical connectives. Reviewing the definitions of erotetic implication and reducibility we can recognize their 'both-sidedness' and just reducibility can substitute such combination of questions.

This brings us to the last note on the use of symbols $\models$ and $\vdash$. Because of the clear border between declarative and interrogative parts of the language $\mathcal{L}_Q$ we will use them in many meanings. However, the meaning will be transparent by the context the symbols are used in. Compare the definition of evocation and various definitions of erotetic implications in the next sections.

2.2 Questions and declaratives

In this section, we introduce two terms: *evocation* and *presupposition*. The first one will provide a consequence relation between a set of declaratives and a question. The second one is an important term in almost all logics of questions and there are some classes of questions based on it in IEL.

2.2.1 Evocation

Consider the following example: after a lecture, we expect a lecturer to be ready to answer some questions that were *evoked* by his or her talk. Thus, evocation seems to be the most obvious relationship among declarative sentences and questions. (Of course, next to the connection *question—answer*.)

Almost every information can give rise to a question. What is the aim of such a question?

First, it should complete our knowledge in some direction. Asking a question we want to get more then by the conclusion based on a background knowledge. A question Q should be *informative relative to* Γ, it means, there is no direct answer to Q which is a conclusion of Γ.

Second, after answering an evoked question, no matter how, the answer must be consistent with the evoking knowledge. Moreover, *transmission of truth into soundness* is required: if an evoking set of declaratives has a model, there must be at least one direct answer of the evoked question that is true in this model. An evoked question should be sound relative to an evoking set of declaratives (see Definition 2).[7]

The definition of *evocation* is based on the previous two points (cf. [44, 46]). A question Q is evoked by a set of declaratives Γ if Q is sound and informative relative to Γ.

Definition 4. *A set of declarative sentences* Γ evokes *a question* Q *(let us write* $\Gamma \models Q$*) iff*

1. $\Gamma \mid\models dQ$,
2. $(\forall \alpha \in dQ)(\Gamma \not\models \alpha)$.

In our model-based approach we can rewrite both conditions this way:

1. $\mathcal{M}^\Gamma \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$
2. $(\forall \alpha \in dQ)(\mathcal{M}^\Gamma \not\subseteq \mathcal{M}^\alpha)$

In some special cases (e.g., dQ is finite or entailment is compact) we can define evocation without the link to mc-entailment. The first condition is of the form: there are $\alpha_1, \ldots, \alpha_n \in dQ$ such that $\Gamma \models \bigvee_1^n \alpha_i$.

This is the case of one of our introductory examples. Let us remind the group of three card players. The context

> Γ: *Either Ann has the Joker or Bill has the Joker or Catherine has the Joker.*

evokes the question

> Q: *Who has the Joker: Ann, Bill, or Catherine?*

[7] For now, as we do not discuss epistemic issues, we shall not use the word 'knowledge' but the phrases 'set of declarative(s) (sentences)' or 'database' will be used instead.

with direct answers

> α: *Ann has the Joker.*
> β: *Bill has the Joker.*
> γ: *Catherine has the Joker.*

Let Γ consist of one formula in the form of disjunction of direct answers of Q, thus, the first condition is satisfied. The second one is satisfied because no direct answer is entailed by Γ.

Evocation yields some clear and useful properties of both a set of declaratives and an evoked question. The following fact lists some of them.

Fact 2. *If* $\Gamma \models Q$*, then*

- Γ *is not a contradictory set,*
- *there is no tautology in* dQ*, and*
- Q *is not a completely contradictory question.*

However, by Fact 1, we obtain a less intuitive conclusion: every safe question is evoked by any Γ that does not entail any direct answer to it. It underlines the special position of safe questions and their semantic range. When we restrict the definition of evocation to risky questions only, we get the definition of *generation*, see [44, chapter 6].

Generation does not solve all problems with irrelevant and inefficient evoked questions either. We can accept another restriction to avoid questions that have direct answers which are incompatible with declaratives in Γ. Borrowing an example from [7], $\Gamma = \{\alpha \vee \beta, \gamma\}$ evokes also $?\{\alpha, \beta, \neg\gamma\}$. To eliminate this, the consistency of each direct answer with respect to Γ could be required, i.e., we could add the third condition to Definition 4:

$$(\forall \alpha \in dQ)(\mathcal{M}^{\Gamma} \cap \mathcal{M}^{\alpha} \neq \emptyset)$$

Some solutions of the problem of irrelevant and inefficient questions based on a semantics in the background are discussed in the just mentioned paper [7]. For our purpose, the study of consequence relations in IEL, we keep Definition 4 unchanged.

Back to safe questions, let us mention the following fact:

Fact 3. *If* $\emptyset \models Q$*, then* Q *is safe.*

As a conclusion of semantic definition of evocation we have the following expected behavior of evocation: semantically equivalent databases evoke the same questions.

Fact 4. *For every* Γ, Δ *and* Q*, if* $\Gamma \equiv \Delta$*, then* $\Gamma \models Q$ *iff* $\Delta \models Q$*.*

If Γ evokes Q, then we have to be careful of concluding that there is a subset $\Delta \subseteq \Gamma$ such that Δ evokes Q, see the first item in the following fact.

Fact 5. *If* $\Gamma \models Q$ *and* $\Delta \subseteq \Gamma \subseteq \Sigma$*, then*

- $\Delta \models Q$ *if* $\Delta \,|\!\!\models dQ$*,*
- $\Sigma \models Q$ *if* $(\forall \alpha \in dQ)(\Sigma \not\models \alpha)$*.*

The second item points out the non-monotonicity of evocation (in declaratives). Considering questions as sets of answers, evocation is non-monotonic in interrogatives as well, see section 2.3.3.

Fact 6. *If* $\Gamma \models Q$ *and the entailment is compact, then* $\Delta \models Q_1$ *for some finite subset* dQ_1 *of* dQ *and some finite subset* Δ *of* Γ*.*

These and some more properties of evocation (and generation) are discussed in the book [44].

2.2.2 Presuppositions

Many properties of questions are based on the concept of *presupposition*. Everyone who has attended a basic course of research methods in social sciences has heard of importance to consider presuppositions of a question in questionaries.

If we receive the question *Who has the Joker: Ann, Bill, or Catherine?*, we can recognize that it is presupposed that Ann has it or Bill has it or Catherine has it. What is presupposed must be valid under each answer to a question. Moreover, an answer to a question should bring at least the same information as presupposition does. The following definition (originally given by Nuel Belnap) is from [44]:

Definition 5. *A declarative formula* φ *is a* presupposition *of a question* Q *iff* $(\forall \alpha \in dQ)(\alpha \models \varphi)$*.*

A presupposition of a question is entailed by each direct answer to the question. Let us write $\mathsf{Pres}Q$ for the set of all presuppositions of Q.

At the first sight, the set $\mathsf{Pres}Q$ could contain a lot of sentences. Let us have a question $Q = ?\{\alpha_1, \alpha_2\}$, the set of presuppositions (e.g., in CPL) contains $(\alpha_1 \vee \alpha_2)$, $(\alpha_1 \vee \alpha_2 \vee \varphi)$, $(\alpha_1 \vee \alpha_2 \vee \neg\varphi)$, etc. Looking at the very relevant member $(\alpha_1 \vee \alpha_2)$ it is useful to introduce the concept of *maximal presupposition*. Formula $(\alpha_1 \vee \alpha_2)$ entails each presupposition of the question Q.

Definition 6. *A declarative formula* φ *is a* maximal presupposition *of a question* Q *iff* $\varphi \in \mathsf{Pres}Q$ *and* $(\forall \psi \in \mathsf{Pres}Q)(\varphi \models \psi)$.

The model-theoretical view shows it in a direct way. The definition of presupposition gives $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\varphi$, for each $\varphi \in \mathsf{Pres}Q$, which means

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcap_{\varphi \in \mathsf{Pres}Q} \mathcal{M}^\varphi = \mathcal{M}^{\mathsf{Pres}Q}$$

and the set $\mathcal{M}^{\mathsf{Pres}Q}$ is a model-based counterpart to the definition of maximal presuppositions.

If the background logic has tautologies, each of them is in $\mathsf{Pres}Q$.

$$\textsc{Taut}^{\mathcal{M}}_{\mathsf{L}} \subseteq \mathsf{Pres}Q$$

Considering safe questions we get

Fact 7. *If* Q *is safe, then* $\mathsf{Pres}Q = \textsc{Taut}^{\mathcal{M}}_{\mathsf{L}}$.

This fact says that if Q is safe, then $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \mathcal{M}^{\mathsf{Pres}Q}$. In classical propositional logic the disjunction of all direct answers of a question is a presupposition of this question and if $\mathsf{Pres}Q = \textsc{Taut}^{\mathcal{M}}_{\mathsf{CPL}}$, then Q is safe. This evokes a (meta)question whether the implication from right to left is valid. If Q is not safe, then we know that $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$ is a proper subset of $\mathcal{M}$. But what about $\mathcal{M}^{\mathsf{Pres}Q}$? After introducing a class of *normal questions* (see page 25) it will be valid $\mathcal{M}^{\mathsf{Pres}Q} \subset \mathcal{M}$ as well as the implication from right to left (see Fact 9).

A presupposition can be seen as an information which is announced by asking a question, without answering it. Such information is relatively small. The semantic range of all maximal presuppositions is wider than the range of a question. Looking at finite CPL example where the disjunction of all direct answers forms just the semantic range of the question brings us to the idea of *prospective presupposition*. It is a presupposition which a question Q is sound relative to.

Definition 7. *A declarative formula* φ *is a* prospective presupposition *of a question* Q *iff* $\varphi \in \mathsf{Pres}Q$ *and* $\varphi \mid\!\models dQ$. *Let us write* $\varphi \in \mathsf{PPres}Q$.

All prospective presuppositions of a question are equivalent:

Lemma 1. *If* $\varphi, \psi \in \mathsf{PPres}Q$, *then* $\varphi \equiv \psi$.

Proof. If $\mathbf{M} \models \varphi$, then there is $\alpha \in dQ$ such that $\mathbf{M} \models \alpha$. Since $\psi \in \mathsf{Pres}Q$, $\alpha \models \psi$ and it gives $\mathbf{M} \models \psi$. We got $\varphi \models \psi$.

$\psi \models \varphi$ is proved by the same way. □

A prospective presupposition forms exactly the semantic range of a question.

$$\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{PPres}Q}$$

If Q has a prospective presupposition, it can be understood as the 'strongest' one.

Two questions with the same sets of presuppositions have the same prospective presuppositions.

Lemma 2. *If* $\mathsf{Pres}Q = \mathsf{Pres}Q_1$ *and both* $\mathsf{PPres}Q$ *and* $\mathsf{PPres}Q_1$ *are not empty, then* $\mathsf{PPres}Q = \mathsf{PPres}Q_1$*.*

Proof. We show that if $\varphi \in \mathsf{PPres}Q$ and $\psi \in \mathsf{PPres}Q_1$, then $\varphi \equiv \psi$.

$\varphi \in \mathsf{PPres}Q$ implies $\mathcal{M}^{\varphi} = \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ and $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subseteq \mathcal{M}^{\psi}$, because $\psi \in \mathsf{Pres}Q$. It gives $\mathcal{M}^{\varphi} \subseteq \mathcal{M}^{\psi}$ and $\varphi \models \psi$.

The proof that $\psi \models \varphi$ is similar. □

Presuppositions of evoked questions are entailed by the evoking set of declaratives.

Fact 8. *If* $\Gamma \models Q$*, then* $\Gamma \models \varphi$*, for each* $\varphi \in \mathsf{Pres}Q$*.*

The implication from right to left does not hold. If we only know $\mathcal{M}^{\Gamma} \subseteq \mathcal{M}^{\mathsf{Pres}Q}$, we are not sure about $\mathcal{M}^{\Gamma} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ as required by the first condition of evocation. Clearly, the informativeness must be ensured as well. Let us note that it cannot be improved by replacing of $\mathsf{Pres}Q$ by $\mathsf{PPres}Q$. We will return to this in the next subsection at the topic of normal questions. To sum up all general conditions of an evoked question (by Γ) and its presuppositions let us look at this diagram:

$$\boxed{\mathcal{M}^{\Gamma} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{PPres}Q} \subseteq \mathcal{M}^{\mathsf{Pres}Q}}$$

Classes of questions based on presuppositions

Using the term *presupposition* we can define some classes of questions. Names and definitions of the classes are from [44]. We only add the model-based approach and make transparent some results on presuppositions and evocations (chapters 4 and 5 in [44]).

Normal questions A question Q is called *normal* if it is sound relative to its set of presuppositions ($\mathsf{Pres}Q \mid\models dQ$).

- $Q \in$ NORMAL iff $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{Pres}Q}$

Model-based approach introduces normal questions as questions with semantic range delimitated by models of maximal presuppositions. Working with finite sets of direct answers and in logical systems with the 'classical' behavior of disjunction (each direct answer entails the disjunction of all direct answers) we do not leave the class NORMAL. Non-normal questions can be found in classical predicate logic.

Two announced facts follow. They continue on the discussions at Fact 7 and Fact 8.

Fact 9. *If* $\mathsf{Pres}Q = \textsc{Taut}^{\mathcal{M}}_{\mathsf{L}}$ *and* Q *is normal, then* Q *is safe.*

Let us only add the clear fact, that the class of safe questions is a subset of the class of normal questions.

$$\textsc{safe} \subseteq \textsc{normal}$$

The following fact and Fact 8 give the conditions for evocation of normal questions.[8]

Fact 10. *If* $\Gamma \models \varphi$*, for each* $\varphi \in \mathsf{Pres}Q$*, and* $\Gamma \not\models \alpha$*, for each* $\alpha \in dQ$ *of a normal question* Q*, then* $\Gamma \models Q$*.*

Regular questions Each question with the non-empty set of prospective presuppositions is *regular*.

- $Q \in$ REGULAR iff $(\exists \varphi \in \mathsf{Pres}Q)(\varphi \mid\models dQ)$

Regularity of Q gives $\mathcal{M}^{\mathsf{Pres}Q} \subseteq \mathcal{M}^{\varphi} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ and it holds

$$\textsc{regular} \subseteq \textsc{normal}$$

If entailment is compact, both classes are equal.

Normal questions are sound relative to $\mathsf{Pres}Q$ and regular questions are sound relative to $\mathsf{PPres}Q$. The following example shows an expected fact that it is still not sufficient for evocation.

Example 2 (in CPL). *Let* $Q = ?\{(\alpha \vee \beta), \alpha\}$*. This question is normal and regular, the formula* $(\alpha \vee \beta)$ *is a prospective presupposition of* Q*, but* $\mathsf{Pres}Q \not\models Q$*.*

[8] Cf. Theorem 5.23 in [44].

If there is a set of declaratives Γ such that $\Gamma \models Q$, then normal (regular) questions are sound as well as informative relative to $\mathsf{Pres}Q$ ($\mathsf{PPres}Q$). This is summed up by

Lemma 3. *Let* $\Gamma \models Q$*, for some set of declaratives* Γ*. Then*

1. $Q \in$ NORMAL *implies* $\mathsf{Pres}Q \models Q$.
2. $Q \in$ REGULAR *implies* $\varphi \models Q$*, for* $\varphi \in \mathsf{PPres}Q$.

Proof. For the first item, only informativeness (relative to $\mathsf{Pres}Q$) must be showed. But if it is not valid, then Fact 8 causes non-informativeness of Q relative to Γ.

The second item is provable by the same idea. □

Self-rhetorical questions Another special class of questions are *self-rhetorical* questions. They have at least one direct answer entailed by the set of presuppositions.

- $Q \in$ SELF-RHETORICAL iff $(\exists \alpha \in dQ)(\mathsf{Pres}Q \models \alpha)$

From this definition, it is clear that self-rhetorical questions are normal. However, do we ask such questions? This class includes such strange questions as *completely contradictory* questions that have only contradictions in the set of direct answers, and questions with tautologies among direct answers.

An evoked question is not of this kind.

Lemma 4. *If there is* Γ *such that* $\Gamma \models Q$*, then* Q *is not self-rhetorical.*

Proof. From Fact 8. □

Proper questions Normal and not self-rhetorical questions are called *proper*. Proper questions are evoked by their set of presuppositions.

- $Q \in$ PROPER iff $\mathsf{Pres}Q \models Q$

Evoked normal questions are proper (compare both Lemma 3 and Lemma 4) and this makes the class PROPER prominent. The set of all presuppositions of a question Q is believed to be a natural (declarative) context for evocation of Q.

2.3 Questions and questions

This section is devoted to inferential structures in which questions appear on both sides (*erotetic implication* and *reducibility of questions to sets of questions*) and to relations between two questions based on their sets of direct answers. The second point focuses on 'answerhood power' of questions formalized by the adapted set-of-answers methodology.

2.3.1 Erotetic implication

Now, we extend the class of inferences by 'implication' between two questions with a possible assistance of some set of declaratives. Let us start with an easy and a bit tricky example. If I ask

> Q: *What is Peter a graduate of: a faculty of law or a faculty of economy?*

then I can be satisfied by the answer

> *He is a lawyer.*

even if I did not ask

> Q_1: *What is Peter: a lawyer or an economist?*

The connection between both questions could be shown by the following set of declaratives:

> *Someone is a graduate of a faculty of law iff he/she is a lawyer.*
> *Someone is a graduate of a faculty of economy iff he/she is an economist.*

The first question Q can be formalized by $?\{\alpha_1, \alpha_2\}$ and the latter one, speaking of Peter's position, can be $?\{\beta_1, \beta_2\}$. Looking at the questions there is no connection between them. The relationship is based on the set of declaratives $\Gamma = \{(\alpha_1 \leftrightarrow \beta_1), (\alpha_2 \leftrightarrow \beta_2)\}$. Now, we say that Q *implies* Q_1 on the basis of Γ and write $\Gamma, Q \models Q_1$.

This relation is called *erotetic implication* (*e-implication*, for short) and the following definition is from [44]:[9]

Definition 8. *A question Q implies a question Q_1 on the basis of a set of declaratives Γ iff*

[9] We will write shortly $\Gamma \cup \varphi$ instead of $\Gamma \cup \{\varphi\}$.

1. $(\forall\alpha \in dQ)(\Gamma \cup \alpha \mid\models dQ_1)$,
2. $(\forall\beta \in dQ_1)(\exists\Delta \subset dQ)(\Delta \neq \emptyset$ *and* $\Gamma \cup \beta \mid\models \Delta)$.

Returning to the introductory example, both questions are even *erotetically equivalent* with respect to Γ: $\Gamma, Q \models Q_1$ as well as $\Gamma, Q_1 \models Q$.

The definition requires a little comment. The first clause should express the soundness of an implied question relative to each extension of Γ by $\alpha \in dQ$. This *transmission of truth/soundness into soundness* has the following meaning: if there is a model of Γ and a direct answer to Q, then there must be a direct answer to Q_1 that is valid in this model. If Q_1 is safe, then this condition is always valid (see Fact 1).

The second clause requires direct answers to Q_1 to be *cognitively useful* in restricting the set of direct answers of the implying question Q.

In comparison with evocation, the role of the set of declaratives is a bit different. Γ plays, especially, the auxiliary role; e-implication is monotonic in declaratives and it gives the following [44, p. 173]:

Fact 11. *If* $\Gamma, Q \models Q_1$*, then* $\Delta, \Gamma, Q \models Q_1$*, for any set of declaratives* Δ*.*

This could be called *weakening in declaratives*. From this, it is clear that $\bot, Q \models Q_1$, for each Q and Q_1.

We wil say a word or two about auxiliary sets of declaratives in the next subsection.

Pure erotetic implication

Pure e-implication is e-implication with the empty set of declaratives. In our semantic approach, Γ includes only tautologies of a chosen logical system. From Fact 11, whenever two questions are in the relation of pure e-implication, then they are in the relation of e-implication for each set of declaratives.

If one question purely e-implies another question, then both questions have the same semantic range.

Lemma 5. *If* $Q \models Q_1$*, then* $\bigcup_{\alpha\in dQ} \mathcal{M}^\alpha = \bigcup_{\beta\in dQ_1} \mathcal{M}^\beta$*.*

Proof. From the first condition of Definition 8

$$\bigcup_{\alpha\in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta\in dQ_1} \mathcal{M}^\beta$$

and from the second one

$$\bigcup_{\beta\in dQ_1} \mathcal{M}^\beta \subseteq \bigcup_{\Delta} \bigcup_{\alpha\in\Delta} \mathcal{M}^\alpha \subseteq \bigcup_{\alpha\in dQ} \mathcal{M}^\alpha.$$

□

From this we can conclude that classes of safe and risky questions are closed under pure e-implication for both implied and implying questions.[10]

Fact 12. *If $Q \models Q_1$, then Q is safe (risky) iff Q_1 is safe (risky).*

The same semantic range of questions linked together by pure e-implication does not form an equivalence relation on questions (see non-symmetry in Example 4 and non-transitivity in Example 5). On the other hand, pure e-implication has some important consequences for classes of presuppositions.[11]

Lemma 6. *If $Q \models Q_1$, then $\mathsf{Pres}Q = \mathsf{Pres}Q_1$.*

Proof. First, let us prove $\mathsf{Pres}Q \subseteq \mathsf{Pres}Q_1$. Let $\varphi \in \mathsf{Pres}Q$, so $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\varphi$. Simultaneously, we know that from the second condition of the definition of pure e-implication there is a non-empty $\Delta \subset dQ$, for each $\beta \in dQ_1$, such that $\mathcal{M}^\beta \subseteq \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$. Thus, $\mathcal{M}^\beta \subseteq \mathcal{M}^\varphi$, for each $\beta \in dQ_1$.

Second, for proving $\mathsf{Pres}Q_1 \subseteq \mathsf{Pres}Q$ suppose $\varphi \in \mathsf{Pres}Q_1$. The following inclusions are valid $\mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta \subseteq \mathcal{M}^\varphi$, for each $\alpha \in dQ$. □

The claim of Lemma 6 is not extendable to the general e-implication (cf. Example 3).

On this lemma we can base the following statement about an influence of pure e-implication on classes of normal and regular questions.

Theorem 2. *If $Q \models Q_1$, then Q is normal iff Q_1 is normal.*

Proof. If Q is normal, then

$$\bigcup_{\beta \in dQ_1} \mathcal{M}^\beta = \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \mathcal{M}^{\mathsf{Pres}Q} = \mathcal{M}^{\mathsf{Pres}Q_1}$$

The first equation is from Lemma 5, the second one is from the normality of Q, and the third one is from Lemma 6. □

It is easy to prove a similar fact for regular questions.

Theorem 3. *If $Q \models Q_1$, then Q is regular iff Q_1 is regular.*

[10] Cf. Theorem 7.29 in [44, p. 184]

[11] See the same result in [44, p. 184], Theorem 7.33.

Proof. Let us suppose $Q \models Q_1$ and Q is regular. From $Q \models Q_1$ and Lemma 5 we obtain $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \bigcup_{\beta \in dQ_1} \mathcal{M}^{\beta}$, which means that $\mathcal{M}^{\mathsf{PPres}Q} = \mathcal{M}^{\mathsf{PPres}Q_1}$. The regularity of Q implies that there is a formula $\varphi \in \mathsf{PPres}Q$. Putting it together, $\mathcal{M}^{\varphi} = \bigcup_{\beta \in dQ_1} \mathcal{M}^{\beta}$. Thus, $\varphi \in \mathsf{PPres}Q_1$. □

Both theorems have similar results we have got for safe (risky) questions in Fact 12. Classes of normal and regular questions are closed to pure e-implication. Normal (regular) questions purely imply only normal (regular) questions and they are purely implied by the same kind of questions.[12]

Concerning classes of questions in relationship with e-implication, let us add that whenever $Q \models Q_1$, then Q is completely contradicory question iff Q_1 is.

Note on auxiliary sets of declaratives in e-implication Let us remind the introductory example on page 27 to emphasize the importance of declaratives for e-implication. Similarly, the following example will point out the role of implicitly and explicitly expressed presuppositions.

Example 3 (in CPL). *If $Q_1 = ?\{\alpha, \beta, \gamma\}$ and $Q_2 = ?\{\alpha, \beta\}$ (for atomic α, β, γ), then neither $Q_1 \models Q_2$ nor $Q_2 \models Q_1$ (see the different semantic ranges of both questions). On the other hand, if we would know that it must be $(\alpha \vee \beta)$, then $(\alpha \vee \beta), Q_1 \models Q_2$.*

Keeping the context of this example: the question Q_2 is normal as well as regular, then $\mathsf{PPres}Q_2 \mid\models dQ_2$ and, in addition, there is Δ, non-empty proper subset of dQ_1, such that $\mathsf{PPres}Q_2 \mid\models \Delta$. It gives $\mathsf{PPres}Q_2, Q_1 \models Q_2$. If the set $\mathsf{PPres}Q_2$ is *explicitly* expressed, the implication from Q_1 to Q_2 is justified.

But now, back to the general approach. In the following fact we display when we can say that two questions and a set of declaratives are in the relationship of e-implication.

Fact 13. *Let us have Γ and two questions Q_1 and Q_2. In order to conclude $\Gamma, Q_1 \models Q_2$ it is sufficient to have $\Gamma \mid\models dQ_2$ and $\Gamma \mid\models \Delta$, where Δ is a non-empty proper subset of dQ_1.*

This fact can be formulated in this form: if Q_2 is sound relative to Γ and Γ gives a partial answer to Q_1, then Q_1 implies Q_2 with respect to Γ. We will add some points to this discussion in section 2.3.2 and in the last paragraph of section 2.3.4.

[12] Theorems 2 and 3 put together results in [44, pp. 185–186].

Regular erotetic implication

A special kind of e-implication arises if there is exactly one direct answer in each Δ in the second clause of Definition 8. Then we say that Q *regularly* implies Q_1 (on the basis of Γ). The following definition originates from [46, p. 26]:

Definition 9. *A question Q* regularly implies *a question Q_1 on the basis of a set of declaratives Γ iff*

1. $(\forall\alpha \in dQ)(\Gamma \cup \alpha \mid\models dQ_1)$,
2. $(\forall\beta \in dQ_1)(\exists\alpha \in dQ)(\Gamma \cup \beta \models \alpha)$.

Because of the special importance of this relation let us use the symbol $\vdash$ for it (so we write $\Gamma, Q \vdash Q_1$).

In the case of *pure regular e-implication*, both conditions are changed into the form:

1. $(\forall\alpha \in dQ)(\alpha \mid\models dQ_1)$,
2. $(\forall\beta \in dQ_1)(\exists\alpha \in dQ)(\beta \models \alpha)$.

If $Q \vdash Q_1$ such that we can answer Q_1, then we have an answer to Q. The relationship of pure regular e-implication between two questions says that the implied question is 'stronger' than the implying one in the sense of answerhood (see also section 2.3.3).

Regularity can be enforced by the minimal number of direct answers of an implying question: if $Q \models Q_1$ and $|dQ| = 2$, then $Q \vdash Q_1$.

Basic properties of erotetic implication

In this subsection, we are going to be interested in such properties as reflexivity, symmetry, and transitivity of e-implication.

Erotetic implication is a reflexive relation.

Fact 14. $\Gamma, Q \models Q$*, for each* Γ *and* Q*.*

Even if there are examples of the symmetric behavior of e-implication, it is not a symmetric relation, generally.

Example 4 (in CPL). *Let* $Q_1 = ?\{(\alpha \vee \beta), \alpha\}$ *and* $Q_2 = ?\{\alpha, \beta\}$*. Then* $Q_1 \models Q_2$*, but* $Q_2 \not\models Q_1$*.*

In this example there is no non-empty proper subset of dQ_2 for the formula $(\alpha \vee \beta)$ to fulfil the second condition in the definition of e-implication. Moreover, it is useful to add that Q_1 regularly implies Q_2.

Erotetic implication is not transitive either.

Example 5 (in CPL). $?(\alpha \wedge \beta) \vdash ?|\alpha, \beta|$ *and* $?|\alpha, \beta| \models ?\alpha$, *but* $?(\alpha \wedge \beta) \not\models ?\alpha$.

On the other hand, if we consider regular e-implication only, the following theorem is valid.

Theorem 4. *If* $Q_1 \vdash Q_2$ *and* $Q_2 \vdash Q_3$, *then* $Q_1 \vdash Q_3$.

Proof. The first condition of Definition 9 is proved by Lemma 5.

The second clause of this definition is based on regularity that gives $(\forall \gamma \in dQ_3)(\exists \alpha \in dQ_1)(\mathcal{M}^\gamma \subseteq \mathcal{M}^\alpha)$. □

We can do a cautious strengthening by the following fact:

Fact 15. *If* $\Gamma, Q_1 \vdash Q_2$ *and* $Q_2 \vdash Q_3$, *then* $\Gamma, Q_1 \vdash Q_3$.

As a final remark, let us add that presuppositions of an implied question are entailed by each direct answer of an implying question (with respect to an auxiliary set of declaratives).

Fact 16. *Let* $\Gamma, Q \models Q_1$. *Then*

1. $(\forall \alpha \in dQ)(\forall \varphi \in \mathsf{Pres}Q_1)(\Gamma \cup \alpha \models \varphi)$
2. *If e-implication is regular, then* $(\forall \beta \in dQ_1)(\forall \varphi \in \mathsf{Pres}Q)(\Gamma \cup \beta \models \varphi)$.

2.3.2 Evocation and erotetic implication

Both types of inferential structures can appear together and we are going to investigate their interaction.

As shown in the next example, e-implication does not preserve evocation. If we know $\Gamma \models Q_1$ and $Q_1 \models Q_2$, it does not mean that it must be $\Gamma \models Q_2$.

Example 6 (in CPL).

- $\{(\alpha \vee \beta)\} \models ?|\alpha, \beta|$ *and* $?|\alpha, \beta| \models ?(\alpha \vee \beta)$, *but* $\{(\alpha \vee \beta)\} \not\models ?(\alpha \vee \beta)$.
- $\{\alpha\} \models ?|\alpha, \beta|$ *and* $\{\alpha\}, ?|\alpha, \beta| \models ?\alpha$, *but there is an answer to* $?\alpha$ *in* $\{\alpha\}$.

Of course, we do not see anything pathological in this example. Knowing $(\alpha \vee \beta)$, resp. α, it is superfluous to ask $?(\alpha \vee \beta)$, resp. $?\alpha$.

Generally, this brings us back to the role of an auxiliary set of declaratives in e-implication. Due to the admissibility of *weakening in declaratives* (Fact 11) we can arrive at structures of e-implications with Γ containing (direct) answers to some of the two questions. On the other hand, there are some solutions of this problem proposed by erotetic logicians.[13]

In contrast to the previous example, we can prove that evocation carries over through a regular e-implication.

Lemma 7. *If* $\Gamma \models Q_1$ *and* $Q_1 \vdash Q_2$*, then* $\Gamma \models Q_2$*.*

Proof. The first condition requires $\mathcal{M}^\Gamma \subseteq \bigcup_{\beta \in dQ_2} \mathcal{M}^\beta$. It is valid because of the same semantic range of both questions.

Let us suppose that there is $\beta \in dQ_2$ entailed by Γ. Then $\mathcal{M}^\Gamma \subseteq \mathcal{M}^\beta$ and, thanks to regularity of e-implication, $\mathcal{M}^\Gamma \subseteq \mathcal{M}^\alpha$, for some $\alpha \in dQ_1$. But it is in contradiction with $\Gamma \models Q_1$. $\square$

Lemma 7 can be formulated not only in the version of pure regular e-implication.

Theorem 5. *If* $\Gamma \models Q_1$ *and* $\Gamma, Q_1 \vdash Q_2$*, then* $\Gamma \models Q_2$*.*

Proof. First, we prove $\Gamma \mid\models dQ_2$. Supposing it is not true, then there is a model $\mathbf{M}_0$ of Γ such that $\mathbf{M}_0 \not\models \beta$, for each $\beta \in dQ_2$. Because of $\Gamma \mid\models dQ_1$, there is $\alpha_0 \in dQ_1$ and $\mathbf{M}_0 \models \alpha_0$. From $\Gamma \cup \alpha \mid\models dQ_2$, for each $\alpha \in dQ_1$, there must be some $\beta_0 \in dQ_2$ such that $\mathbf{M}_0 \models \beta_0$ and that is a contradiction.

Secondly, let us suppose that there is $\beta_0 \in dQ_2$ and $\Gamma \models \beta_0$. Regularity and second condition of e-implication give $\Gamma \cup \beta_0 \models \alpha$ and it follows $\Gamma \models \alpha$ that is in contradiction with $\Gamma \models Q_1$. $\square$

Since regularity was used only in the second part of the proof, we get an expected fact that $\Gamma \models Q_1$ and $\Gamma, Q_1 \vdash Q_2$ gives soundness of an implied question Q_2 relative to Γ.[14]

At the first sight, it need not be $\Gamma, Q_1 \models Q_2$ or $\Gamma, Q_2 \models Q_1$ if we only know that $\Gamma \models Q_1$ as well as $\Gamma \models Q_2$.[15] Generally, neither evocation nor

[13]See, for example, the definition of *strong e-implication* given by Wiśniewski in [44]. Fact 13 includes the original inspiration for the definition of strong e-implication. The definition is the same as that of e-implication, but $\Gamma \mid\not\models \Delta$ is added into the second clause.

[14]The second part of the proof of Theorem 5 could be slightly changed and we obtain that strong e-implication carries over as well. (Andrzej Wiśniewski called my attention to this.)

[15]Let us take as an example (in CPL) the case $\{\varphi\} \models ?\alpha$ and $\{\varphi\} \models ?\beta$. Then neither $\{\varphi\}, ?\alpha \models ?\beta$ nor $\{\varphi\}, ?\beta \models ?\alpha$.

e-implication says something new about structures of engaged questions. Nevertheless, we can expect that some clearing up of the structure of sets of direct answers could be helpful for the study of inferences. This will be discussed in the next section.

2.3.3 Comparing questions: relations of questions based on direct answers

So far we have introduced inferences that can provide certain relations between questions. Moreover, it would be useful to be able to compare questions with respect to their 'answerhood power'. The chosen set-of-answers methodology brings us to a natural approach. Sets of direct answers can be purely compared or we can investigate their relationship based on entailment relation, moreover, we can control the cardinality of sets of direct answers by a mapping from one set to the other.

Let us start with relations among questions based on pure comparison of sets of direct answers.

Definition 10.

- *Two questions are* equal *($Q_1 = Q_2$) iff they have the same set of direct answers ($dQ_1 = dQ_2$).*[16]
- *A question Q_1 is* included *in a question Q_2 ($Q_1 \subset Q_2$) iff $dQ_1 \subset dQ_2$.*

This approach could be extended in a semantic way. We say that (an answer) α *gives* an answer to a question Q iff there is $\beta \in dQ$ such that $\alpha \models \beta$. Having two questions Q_1 and Q_2 we can define a relationship of 'giving answers':

Definition 11. *A question Q_1* gives *a (direct) answer to Q_2 iff $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \models \beta)$.*

In this definition the first question is considered as to be (semantically) 'stronger' than the second one. For this we use the symbol $\geq$ and write $Q_1 \geq Q_2$.

If $Q_1 = Q_2$ or $Q_1 \subset Q_2$, then $Q_1 \geq Q_2$ and, moreover, each direct answer to Q_1 not only gives an answer to Q_2 but also *is* a (direct) answer to Q_2, i.e., $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \equiv \beta)$.

[16] The original definition refers to *equivalent* questions instead of *equal* (cf. [44, p. 135]), but we use the first term for *erotetically equivalent* or semantically equivalent. In our set-of-answers methodology (questions are defined by sets of direct answers), this term is redundant.

The ordering based on the relation $\geq$ has a slightly non-intuitive consequence: a completely contradictory question is the strongest one. However, the class of evoked questions is not affected by this problem.

Let us note an expected fact—stronger questions presuppose more than weaker ones.

Fact 17. *If* $Q_1 \geq Q_2$*, then* $\mathsf{Pres}Q_2 \subseteq \mathsf{Pres}Q_1$.

This fact is not too useful. It is better to notice the relationship among maximal presuppositions. We have $\mathcal{M}^{\mathsf{Pres}Q_1} \subseteq \mathcal{M}^{\mathsf{Pres}Q_2}$. Each maximal presupposition of a stronger question entails a maximal presupposition of a weaker one, respectively, a prospective presuposition of a stronger question entails a prospective presupposition of a weaker question. The semantic range of a stronger question is included in the semantic range of a weaker question.

Fact 18. *If* $Q_1 \geq Q_2$*, then* $\bigcup_{\alpha \in dQ_1} \mathcal{M}^{\alpha} \subseteq \bigcup_{\beta \in dQ_2} \mathcal{M}^{\beta}$.

It follows that the set of safe questions is closed under weaker questions.

Fact 19. *If* Q_1 *is safe and* $Q_1 \geq Q_2$*, then* Q_2 *is safe.*

The next example shows that safeness of weaker questions is not transferred to stronger ones.

Example 7 (in CPL). $?\{\beta \wedge \alpha, \neg\beta\} \geq ?\beta$

Answerhood, evocation, and erotetic implication

We can show some results of evocation and e-implication based on properties of the $\geq$-relation. The first one is an obvious fact that an implied stronger question is regularly implied.

Lemma 8. *If* $\Gamma, Q_1 \models Q_2$ *and* $Q_2 \geq Q_1$*, then* $\Gamma, Q_1 \vdash Q_2$.

Recall what is required of the regular e-implication: $(\forall\beta \in dQ_2)(\exists\alpha \in dQ_1)(\Gamma \cup \beta \models \alpha)$. Then the lemma follows.

If the relation $\geq$ is turned, i.e., Q_1 gives an answer to Q_2, then whenever Q_1 implies Q_2, Q_2 regularly implies Q_1 (both with respect to Γ). Moreover, both questions are erotetically equivalent relative to Γ.

Theorem 6. *If* $\Gamma, Q_1 \models Q_2$ *and* $Q_1 \geq Q_2$*, then* $\Gamma, Q_2 \vdash Q_1$.

Proof. First, we need to show that $\Gamma \cup \beta \mid\models dQ_1$, for each $\beta \in dQ_2$. But it is an easy conclusion from $\Gamma, Q_1 \models Q_2$ because there is a subset $\Delta \subseteq dQ_1$ for each $\beta \in dQ_2$ such that $\Gamma \cup \beta \mid\models \Delta$.

The second condition of regular e-implication is obvious, it follows from $Q_1 \geq Q_2$. $\square$

Now, as it was stated before, we are going to study the influence of 'giving answers' on the relationship of evocation and e-implication. We know that, generally, if Γ evokes Q_1 and Q_2, it need not be that either Q_1 implies Q_2 or Q_2 implies Q_1 (with respect to Γ). If a stronger question is evoked by Γ, then every weaker question regularly implies this stronger one with respect to Γ.

Theorem 7. *If* $\Gamma \models Q_1$ *and* $Q_1 \geq Q_2$*, then* $\Gamma, Q_2 \vdash Q_1$*.*

Proof. First, $\Gamma \cup \beta \mid\models dQ_1$ is required for each $\beta \in dQ_2$. We get $\Gamma \mid\models dQ_1$ from $\Gamma \models Q_1$.

Second, from $Q_1 \geq Q_2$ we have $(\forall \alpha \in dQ_1)(\exists \beta \in dQ_2)(\alpha \models \beta)$ and it gives the second condition of regular e-implication $(\forall \alpha \in dQ_1)(\exists \beta \in dQ_2)(\Gamma \cup \alpha \models \beta)$. □

To digress for a moment, this repeated connection of $\geq$ and regular e-implication is not an accident. The definition of regular e-implication says that if $Q_1 \vdash Q_2$, then Q_2 gives an answer to Q_1, i.e., $Q_2 \geq Q_1$. However, 'giving an answer' does not produce e-implication, see the next example.

Example 8 (in CPL). $?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\} \geq ?\{\alpha, \beta\}$*, but neither* $?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\} \models ?\{\alpha, \beta\}$ *nor* $?\{\alpha, \beta\} \models ?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\}$*.*

To go back to evocation, it is clear that two equal questions are both evoked by a set of declaratives if one of them is evoked by this set. Generally, it is not sufficient to know $\Gamma \models Q_1$ and $Q_1 \geq Q_2$ to conclude $\Gamma \models Q_2$. An evoked stronger question only implies the soundness of weaker questions relative to Γ. Let us illustrate it in the case that the first question is included in the second one ($Q_1 \subset Q_2$); there could be a direct answer to Q_2 which is entailed by Γ. This reminds us of the non-monotonic behavior of evocation. Notice that $Q_2 \subset Q_1$ will not help us either. In the connection with the relation $\geq$ we have to require a version of an equality.[17]

Fact 20. *Let* $Q_1 \geq Q_2$ *and* $Q_2 \geq Q_1$*. Then*

- $\Gamma \models Q_1$ *iff* $\Gamma \models Q_2$*,*
- $Q_1 \vdash Q_2$ *as well as* $Q_2 \vdash Q_1$*.*

[17]We are not going to introduce a special name for this relationship; it is included in the erotetic equivalence.

Controlling the cardinality of sets of direct answers

The set of direct answers of a weaker question can be much larger than that of a stronger question. The book [44] introduces two relations originated from Tadeusz Kubiński that prevent this uncontrolled cardinality.

Definition 12. *A question Q_1 is* stronger *then Q_2 ($Q_1 \succeq Q_2$) iff there is a surjection $j : dQ_1 \rightarrow dQ_2$ such that for each $\alpha \in dQ_1$, $\alpha \models j(\alpha)$.*

The number of direct answers of the weaker question Q_2 does not exceed the cardinality of dQ_1, i.e., $|dQ_1| \geq |dQ_2|$. From the surjection, additionaly, we know that each direct answer of a weaker question is given by some direct answer to a stronger question. We have used the term *stronger* in a bit informal way for questions that 'give an answer' to weaker ones. It is clear that if $Q_1 \succeq Q_2$, then $Q_1 \geq Q_2$. But, unfortunately, we cannot provide any special improvement of previous results for $\succeq$-relation. In particular, Examples 7 and 8 are valid for $\succeq$-relation as well.

The other definition corresponds to a both-way relationship of 'being stronger'.

Definition 13. *A question Q_1 is* equipollent *to a question Q_2 ($Q_1 \equiv Q_2$) iff there is a bijection $i : dQ_1 \rightarrow dQ_2$ such that for each $\alpha \in dQ_1$, $\alpha \equiv i(\alpha)$.*

In this case, both sets of direct answers have the same cardinality ($|dQ_1| = |dQ_2|$). Let us add expected results gained from equipollency.

Fact 21. *If $Q_1 \equiv Q_2$, then*

- *both $Q_1 \succeq Q_2$ and $Q_2 \succeq Q_1$,*
- $\Gamma \models Q_1$ *iff* $\Gamma \models Q_2$,
- *$Q_1 \vdash Q_2$ as well as $Q_2 \vdash Q_1$.*

Of course, two equal questions are equipollent.

Partial answerhood

We declared that the study of various types of answers (generally speaking, answerhood) is not the central point of this chapter. However, we can utilize the idea evoked by the second clause of Definition 8. Narrowing down the set of direct answers of an implying question seems to be a good base for the definition of *partial answer*.

Definition 14. *A declarative* φ gives a partial answer *to a question* Q *iff there is a non-empty proper subset* $\Delta \subset dQ$ *such that* $\varphi \mid\models \Delta$.[18]

This definition allows us to cover many terms from the concept of the answerhood. Every direct answer gives a partial answer. Whenever ψ gives a (direct) answer, then ψ gives a partial answer. As a useful conclusion we obtain a weaker version of Theorem 7:

Fact 22. *If* $\Gamma \models Q_1$ *and each* $\alpha \in dQ_1$ *gives a partial answer to* Q_2*, then* $\Gamma, Q_2 \models Q_1$.

2.3.4 Questions and sets of questions

Working in the classsical logic, let us imagine we would like to know whether it is the case that α or it is the case that β. The question $?\{\alpha, \beta\}$ is posed. But there could be a problem when an entity, to which we are going to address this question, is not able to accept it. This can be caused, e.g., by a restricted language-acceptability, i.e., a device cannot 'understand' this question. However, assume that there exist two devices such that: the first one can be asked by the question $?\alpha$, and the other one is able to work with the question $?\beta$. From both machines, independently, we can get the following pairs of answers: $\{\alpha, \beta\}$, $\{\neg\alpha, \beta\}$, $\{\alpha, \neg\beta\}$ or $\{\neg\alpha, \neg\beta\}$.

Posing the question $Q = ?\{\alpha, \beta\}$ we expect that if an answer to Q is true, then there must be a true answer to each question from the set $\{?\alpha, ?\beta\}$. Thus, we obtain a soundness transmission from an initial question to a set of questions.

Generally speaking, let us suppose that there are a question Q and a set of questions $\Phi = \{Q_1, Q_2, \ldots\}$. For each model of a direct answer to Q there must be a direct answer in each Q_i valid in this model.

$$(\forall \alpha \in dQ)(\forall Q_i \in \Phi)(\alpha \mid\models dQ_i)$$

Possible states (of the world) given by answers to questions in the set Φ must be in a similar relation to the initial question. Whenever we keep a model of the choice of direct answers from each question in Φ, then there must be a direct answer to Q true in this model. For this, let us introduce a *choice function* ξ such that $\xi(Q_i)$ chooses exactly one direct answer from dQ_i. For each set of questions Φ and a choice function ξ there is a *choice set* $A^{\Phi}_{\xi} = \{\xi(Q_i) \mid Q_i \in \Phi\}$.[19] The soundness condition in the other direction (from a set to initial question) will be expressed, generally, by $(\forall A^{\Phi}_{\xi})(A^{\Phi}_{\xi} \mid\models dQ)$.

[18] Compare this definition with Definition 4.10 in [44].

[19] If the set Φ is clear from the context, we will write only A_{ξ}.

Back to our example, there are four choice sets:

$$A_{\xi_1} = \{\alpha, \beta\}$$

$$A_{\xi_2} = \{\neg\alpha, \beta\}$$

$$A_{\xi_3} = \{\alpha, \neg\beta\}$$

$$A_{\xi_4} = \{\neg\alpha, \neg\beta\}$$

But the fourth one is not in compliance with the second soundness requirement, it is in contradiction with our (prospective) presupposition $(\alpha \vee \beta)$. If we admit the additional answer $(\neg\alpha \wedge \neg\beta)$ and a question in the form $?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\}$, mutual soundness of this question and the set of questions $\{?\alpha, ?\beta\}$ will be valid. But this solution seems to be rather awkward. A questioner posing the question $?\{\alpha, \beta\}$ evidently presupposes $(\alpha \vee \beta)$. This will bring us to the definition of reducibility with respect to an auxiliary set of declaratives and the *mutual soundness* will be required in the following forms:

$$(\forall\alpha \in dQ)(\forall Q_i \in \Phi)(\Gamma \cup \alpha \mid\models dQ_i)$$

and

$$(\forall A_\xi^\Phi)(\Gamma \cup A_\xi^\Phi \mid\models dQ).$$

Our example produces more than soundness of Q relative to each A_ξ^Φ (with respect to Γ), also *efficacy* of each A_ξ^Φ with respect to a question Q is valid:

$$(\forall A_\xi^\Phi)(\exists\alpha \in dQ)(\Gamma \cup A_\xi^\Phi \models \alpha).$$

It will be reasonable to keep this strengthening. We require to obtain at least one answer to an initial question from a choice set. Whenever Γ and A_ξ^Φ describe the state of the world, there must be a direct answer to a question Q that does the same job.

Reducibility of questions to sets of questions

We can take advantage of the previous discussion for the direct definition of reducibility of a question to a set of questions. Now, we introduce *pure reducibility* that does not use any auxiliary set of declaratives.

Definition 15. *A question Q is* purely reducible *to a non-empty set of questions Φ iff*

1. $(\forall\alpha \in dQ)(\forall Q_i \in \Phi)(\alpha \mid\models dQ_i)$
2. $(\forall A_\xi^\Phi)(\exists\alpha \in dQ)(A_\xi^\Phi \models \alpha)$

3. $(\forall Q_i \in \Phi)(|dQ_i| \leq |dQ|)$

First two conditions express mutual soundness, the second one adds efficiacy, as it was discussed, and the last one requires *relative simplicity.* If Q is reducible to a set Φ, we will write $Q \gg \Phi$. The definition of pure reducibility was introduced by Andrzej Wiśniewski in [43].

Example 9 (in CPL).

- $?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\} \gg \{?\alpha, ?\beta\}$
- $?|\alpha, \beta| \gg \{?\alpha, ?\beta\}$
- $?(\alpha \circ \beta) \gg \{?\alpha, ?\beta\}$*, where* $\circ$ *is any of the connectives:* $\wedge, \vee, \rightarrow$

In the first item, there is the 'pure' version from the introductory discussion. All items display reducibilities between initial safe questions and sets of safe questions. The following theorem shows that it is not an accident.

Theorem 8. *If* $Q \gg \Phi$*, then* Q *is safe iff each* $Q_i \in \Phi$ *is safe,*

Proof. The first condition of Definition 15 can be rewritten as $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$, for each $Q_i \in \Phi$, and it gives the implication from left to right.

For the proof of the other implication, let us suppose $Q \gg \Phi$ and that each $Q_i \in \Phi$ is safe, but Q is not. It implies the existence of model $\mathbf{M}_0 \in \mathcal{M}$ such that $\mathbf{M}_0 \not\models \alpha$, for each $\alpha \in dQ$. The safeness of all Q_i gives $(\forall Q_i \in \Phi)(\exists \beta \in dQ_i)(\mathbf{M}_0 \models \beta)$. Thus, there is A^Φ_ξ made from these βs and $\mathbf{M}_0 \models A^\Phi_\xi$. But it is in contradiction with the second condition of the definition of $Q \gg \Phi$, which gives the existence of some $\alpha \in dQ$ such that $\mathbf{M}_0 \models \alpha$. □

From this we know that if there is a risky question among questions in Φ and $Q \gg \Phi$, then Q must be risky too (cf. [44, p. 197]).

The rewritten first condition of Definition 15 is of the form

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcap_i \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$$

and it brings out the relationship of semantic ranges. The semantic range of a reduced question is bounded by the intersection of all semantic ranges of Q_is.

The relation of pure reducibility is reflexive ($Q \gg \{Q\}$) and we can prove the following version of transitivity:[20]

Theorem 9. *If* $Q \gg \Phi$ *and each* $Q_i \in \Phi$ *is reducible to some set of questions* Φ_i*, then* $Q \gg \bigcup_i \Phi_i$*.*

[20] Presented Theorem 9 corresponds to Corollary 7.6 in [44].

Proof. The third condition of Definition 15 is clearly valid.

The first one is easy to prove. From $Q \gg \Phi$ we get $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^{\beta}$, for each $Q_i \in \Phi$, and from the reducibility of all Q_i in Φ to an appropriate Φ_i we have $\bigcup_{\beta \in dQ_i} \mathcal{M}^{\beta} \subseteq \bigcup_{\gamma \in dQ_j} \mathcal{M}^{\gamma}$, for each $dQ_j \in \Phi_i$. It gives together $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subseteq \bigcup_{\gamma \in dQ_j} \mathcal{M}^{\gamma}$, for each $Q_j \in \bigcup_i \Phi_i$.

For the second one we require the existence of $\alpha \in dQ$ such that $A_{\xi}^{\cup_i \Phi_i} \models \alpha$, for each $A_{\xi}^{\cup_i \Phi_i}$. From the reducibility of all Q_i in Φ to an appropriate Φ_i we have that each $A_{\xi'}^{\Phi_i}$ is a subset of some $A_{\xi}^{\cup_i \Phi_i}$. It implies that if there is any model $\mathbf{M}$ of $A_{\xi}^{\cup_i \Phi_i}$, it must be a model of some $A_{\xi'}^{\Phi_i}$. From $Q \gg \Phi$ we know that there is $\alpha \in dQ$ for each choice set $A_{\xi''}^{\Phi}$ on Φ. This choice set is made by elements of all $dQ_i \in \Phi$ which are valid in $\mathbf{M}$. It means that $A_{\xi''}^{\Phi}$ is valid in $\mathbf{M}$ as well as $\alpha \in dQ$. □

Now let us look at the relationship of pure reducibility and pure e-implication. The following example shows that it need not be that e-implication causes reducubility. Both definitions have the same first conditions, but the second condition of reducibility can fail.

Example 10 (in CPL). $?|\alpha, \beta| \models ?(\alpha \wedge \beta)$, *but* $?|\alpha, \beta| \not\gg \{?(\alpha \wedge \beta)\}$.

On the other hand, we can prove that regular e-implication implies reducibility.

Lemma 9. *Let* Φ *be a set of questions such that* $Q \vdash Q_i$*, for each* $Q_i \in \Phi$*. If* $(\forall Q_i \in \Phi)(|dQ_i| \leq |dQ|)$*, then* $Q \gg \Phi$*.*

Proof. Let us prove the second condition of Definition 15 that requires existence of $\alpha \in dQ$ such that $\mathcal{M}^{A_{\xi}^{\Phi}} \subseteq \mathcal{M}^{\alpha}$, for each A_{ξ}^{Φ}. It is known that $\mathcal{M}^{A_{\xi}^{\Phi}} \subseteq \mathcal{M}^{\beta}$, for each $\beta \in A_{\xi}^{\Phi}$. If $Q \vdash Q_i$, then for each $\beta \in dQ_i$ there is $\alpha \in dQ$ such that $\mathcal{M}^{\beta} \subseteq \mathcal{M}^{\alpha}$. Thus, $\mathcal{M}^{A_{\xi}^{\Phi}} \subseteq \mathcal{M}^{\alpha}$. □

What about if we know $Q_i \models Q$ or, even, $Q_i \vdash Q$, for each $Q_i \in \Phi$, and $(\forall Q_i \in \Phi)(|dQ_i| \leq |dQ|)$, can we conclude that $Q \gg \Phi$? Example 10 gives the negative answer to this question as well as $?(\alpha \wedge \beta) \vdash ?|\alpha, \beta|$.

Not even reducibility produces e-implication.

Example 11 (in CPL). $?(\alpha \wedge \beta) \gg \{?\alpha, ?\beta\}$*, but* $?(\alpha \wedge \beta)$ *is not implied neither by* $?\alpha$ *nor by* $?\beta$ *and* $?(\alpha \wedge \beta)$ *does not imply neither* $?\alpha$ *nor* $?\beta$*.*

In the next subsection we will study some special cases of links between reducibility and e-implication.

So far we have worked only with the pure reducibility. It could be useful to introduce the general term of *reducibility* with respect to a context given by a set of declaratives. The definition is almost the same as Definition 15, but the mutual soundness and efficacy conditions are supplemented by an auxiliary set of declaratives Γ (cf. [20]). We will write $\Gamma, Q \gg \Phi$.

Definition 16. *A question Q is* reducible *to a non-empty set of questions Φ with respect to a set of declaratives Γ iff*

1. $(\forall\alpha \in dQ)(\forall Q_i \in \Phi)(\Gamma \cup \alpha \,||\!\!= dQ_i)$
2. $(\forall A_\xi^\Phi)(\exists\alpha \in dQ)(\Gamma \cup A_\xi^\Phi \models \alpha)$
3. $(\forall Q_i \in \Phi)(|dQ_i| \leq |dQ|)$

The introductory discussion is displayed in this example:

Example 12 (in CPL). $(\alpha \vee \beta), ?\{\alpha, \beta\} \gg \{?\alpha, ?\beta\}$

As it is expected, the role of Γ is similar to the role of an auxiliary set of declaratives in e-implication:

Fact 23. *If $Q \gg \Phi$, then $\Gamma, Q \gg \Phi$, for each Γ.*

So we can speak of *weakening in declaratives* and it enables us to generalize Lemma 9.

Theorem 10. *If $\Gamma, Q \vdash Q_i$, for each $Q_i \in \Phi$, and $(\forall Q_i \in \Phi)(|dQ_i| \leq |dQ|)$, then $\Gamma, Q \gg \Phi$.*

Proof. The third and the first conditions of Definition 16 are obvious.

The second one requires that for each A_ξ^Φ there is $\alpha \in dQ$ such that $\mathcal{M}^{\Gamma \cup A_\xi^\Phi} \subseteq \mathcal{M}^\alpha$. From the construction of choice sets we know that for each A_ξ^Φ and $Q_i \in \Phi$ there is $\beta \in dQ_i$ (member of A_ξ^Φ) such that $\mathcal{M}^{\Gamma \cup A_\xi^\Phi} \subseteq \mathcal{M}^{\Gamma \cup \beta}$. The regular e-implication provides that there is $\alpha \in dQ$ for each $\beta \in dQ_i$ such that $\mathcal{M}^{\Gamma \cup \beta} \subseteq \mathcal{M}^\alpha$. □

We close this subsection by reversing the 'direction' of the reducibility relation. Let us suppose that we have generated a set of questions Φ that are evoked by a set of declaratives Γ. Can we conclude that Γ evokes such a complex question which is reducible to the set Φ? Generally, not. But if we know that the complex question gives an answer to some question from Φ, the answer is positive.

Theorem 11. *If Γ evokes each question from a set Φ, $Q \gg \Phi$, and there is a question $Q_i \in \Phi$ such that $Q \geq Q_i$, then $\Gamma \models Q$.*

Proof. Soundness of Q relative to Γ requires the existence of an answer $\alpha \in dQ$ for each model $\mathbf{M} \models \Gamma$. From the evocation of each $Q_i \in \Phi$ we have $(\forall \mathbf{M} \models \Gamma)(\forall Q_i \in \Phi)(\exists \beta \in dQ_i)(\mathbf{M} \models \beta)$. So, each model of Γ produces some choice set such that $(\forall \mathbf{M} \models \Gamma)(\exists A_\xi^\Phi)(\mathbf{M} \models A_\xi^\Phi)$. Together with reducibility, where it is stated that $(\forall A_\xi^\Phi)(\exists \alpha \in dQ)(A_\xi^\Phi \models \alpha)$, we get $\Gamma \mid\models dQ$.

Informativness of Q with respect to Γ is justified by $\geq$-relation for some question $Q_i \in \Phi$. If $\Gamma \models \alpha$, for some $\alpha \in dQ$, then it gives a contradiction with $\Gamma \models Q_i$. □

Given the conditions of Theorem 11 are met, we obtain:

- $\Gamma, Q \models Q_i$, for each $Q_i \in \Phi$, and
- $\Gamma, Q_i \vdash Q$, for $Q \geq Q_i$.

The first item is based on Fact 13 and the second one is given by the help of Theorem 7.

Reducibility and sets of yes-no questions

The concept of reducibility is primarily devoted to a transformation of a question to a set of 'less complex' questions. The introductory discussion and its formalization in Example 12 evoke interesting questions:

- If we have an initial question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ with, at worse, a countable list of direct answers, is it possible to reduce it to a set of yes-no questions based only on direct answers of Q?
- Moreover, could we require the e-implication relationship between Q and questions in the set Φ?

We can find an easy solution to these problems under condition that yes-no questions are safe and we have an appropriate set of declaratives. We will require Q to be sound with respect to Γ.

Theorem 12. *Let us suppose that yes-no questions are safe in the background logic. If a question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ is sound with respect to a set Γ, then there is a set of yes-no questions Φ such that $\Gamma, Q \gg \Phi$ and $\Gamma, Q \models Q_i$, for each $Q_i \in \Phi$.*

Proof. First, we define the set of yes-no questions Φ based on the initial question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ such that

$$\Phi = \{?\alpha_1, ?\alpha_2, \ldots\}.$$

Secondly, we prove the condition that is common for both reducibility and e-implication. The safeness of members of Φ implies that $\mathcal{M}^{\alpha} \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^{\beta}$, for each $\alpha \in dQ$ and $Q_i \in \Phi$. This gives $\mathcal{M}^{\Gamma \cup \alpha} \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^{\beta}$.

To prove reducibility we have to justify the second condition of Definition 16. We need to find an $\alpha \in dQ$ for each A^{Φ}_{ξ} such that $\Gamma \cup A^{\Phi}_{\xi} \models \alpha$. Two cases will be distinguished.

1. If there is α from both A^{Φ}_{ξ} and dQ, then choose this direct answer.

2. If there is no direct answer $\alpha \in dQ$ in A^{Φ}_{ξ}, then $\mathcal{M}^{\Gamma \cup A^{\Phi}_{\xi}} = \emptyset$ and we can take any α from dQ.

The final step is the proof of e-implication. We have to show that for each Q_i and each direct answer $\beta \in dQ_i$ there is a non-empty subset $\Delta \subset dQ$ such that $\Gamma \cup \beta \mid\models \Delta$. For this, we use the shape of questions in Φ.

1. If $\beta \in dQ$, then Δ could be $\{\beta\}$ and $\Gamma \cup \beta \mid\models \{\beta\}$.

2. If $\beta \notin dQ$ and $\Gamma \cup \beta$ has at least one model, we recognize that $\mathcal{M}^{\Gamma \cup \beta} \subseteq \mathcal{M}^{\Gamma}$. Simultaneously, β must be of the form $\neg\alpha_j$ and Δ can be defined as $dQ \backslash \{\alpha_j\}$. Together with soundness of initial question Q with respect to Γ, which means $\mathcal{M}^{\Gamma} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$, we get $\mathcal{M}^{\Gamma \cup \beta} \subseteq \bigcup_{\alpha \in \Delta} \mathcal{M}^{\alpha}$.

□

This theorem enables us to work with classes of questions which are known to be sound relative to sets of their presuppositions. (Normal and regular questions are the obvious example.) Whenever we know that the initial question is evoked by a set of declaratives, we get the following conclusion.

Fact 24. *Working in logics where yes-no questions are safe, if a question* $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ *is evoked by a set of declaratives* Γ*, then there is a set of yes-no questions* Φ *such that* $\Gamma, Q \gg \Phi$ *and* $\Gamma, Q \models Q_i$*, for each* $Q_i \in \Phi$.

This fact corresponds to main lemma in the paper [20, pp. 104–5] where a bit different definition of reducibility is used, but the results are the same.

If dQ is finite or the entailment is compact, the set Φ is finite set of yes-no questions. Simultaneously, it is useful to emphasize that the proof of Theorem 12 shows how to construct such a set.[21]

In logics with risky yes-no questions, the first condition of reducibility as well as e-implication can fail. It need not be $\mathcal{M}^{\Gamma,\alpha} \subseteq (\mathcal{M}^{\beta} \cup \mathcal{M}^{\neg\beta})$, for each $\alpha \in dQ$ and each $?\beta \in \Phi$. More generally, we can ask for a help the auxiliary

[21] The same result is provided by theorems 7.49–7.51 in [44].

set of declaratives again. Let us remind Fact 13 and put soundness of each $Q_i \in \Phi$ with respect to Γ. Going through the proof of Theorem 12, the rest is valid independently of safeness of yes-no questions. As a conclusion we get

Fact 25. *If a question* $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ *is sound relative to a set* Γ*, and there is a set of yes-no questions* $\Phi = \{?\alpha_1, ?\alpha_2, \ldots\}$ *(based on* Q*) such that* $\Gamma \mid\models dQ_i$*, for each* $Q_i \in \Phi$*, then* $\Gamma, Q \gg \Phi$ *and* $\Gamma, Q \models Q_i$*, for each* $Q_i \in \Phi$*.*

The construction of yes-no questions provided by Theorem 12 does not prevent the high complexity of such yes-no questions. Observing the last item of Example 9, it seems worthwhile to enquire whether it is possible to follow this proces and to reduce a question (with respect to an auxiliary set of declaratives) to a set of atomic yes-no questions based on subformulas of the initial question (cf. [50]). The first restriction is clear, yes-no questions must be safe. But it is not all, the second clause of pure reducibility (Definition 15) requires the *truth-functional* connection of subformulas. Then the answer is positive. We can use repeatedly a cautious extension of Theorem 9:

Fact 26. *If* $\Gamma, Q \gg \Phi$ *and each* $Q_i \in \Phi$ *is reducible to some set of questions* Φ_i*, then* $\Gamma, Q \gg \bigcup_i \Phi_i$*.*

There is a similar concept in literature, called *erotetic search scenarios*, based on properties of the classical logic and erotetic implication (see [46, 47, 48] and [39]). The idea is that there is an initial question (and a context) and we can get an answer to it by searching through answers of some operative (auxiliary) questions. Scenarios are trees, an initial question (and a context) is the root and branching is based on direct answers of auxiliary questions. The relationship between interrogative nodes is given by erotetic implication. Some scenarios work with the descending 'complexity' of questions from the root to leafs. If we recall non-transitivity of e-implication (Example 5), we can recognize the 'truth-functional' auxiliary role of the question $?|\alpha, \beta|$ as an interlink between $?(\alpha \wedge \beta)$ and questions $?\alpha$ and $?\beta$.

2.4 Final remarks

IEL is introduced as a very general theory of erotetic inferences dealing with a general language $\mathcal{L}$ extended by questions. Following the IEL-philosophy and in accordance with our first chapter we decided to introduce questions as sets of declaratives, which are in the role of direct answers. Inferences with questions are based on multiple-conclusion entailment that makes it possible to work with sets of declaratives in premises as well as in conclusions. Thus, questions are kept as special objects of the language $\mathcal{L}_Q$, they are not combined by logical constants as declaratives are. All relationships among questions are based on inferences and a comparison of their sets of direct answers. The main modification of the original IEL can bee seen in the chosen SAM and in the model-based approach; the term *semantic range* of a question made the semantic work easier and transparent.

In this chapter we studied many properties and relationships, but the central point was whether we can formulate some relationships (meta-rules) for erotetic consequence relations in IEL. Our general view did not tend to provide any axiomatization. The discussed properties and relationships vary with the chosen background system, especially, they depend on semantics. IEL opens many possibilities for working with questions in various logical systems, see [7] as a nice example, and serves as an inspiration for the work in erotetic logic.

Chapter 3

Epistemic logic with questions

3.1 Introduction

Communication is essentially connected with an exchange of information. The basic way to complete someone's knowledge is the posing of questions. Although questions differ from declaratives, they play a similar role in reasoning. In the recent history of logic of questions we can see a success in finding the desirable position of the formal approach to questions and in a study of inferences based on them. Let us recall Wiśniewski's inferential erotetic logic and the intensional approach of Groenendijk and Stokhof. Such approaches made it possible to incorporate questions within some formal systems and not to lose their specific position in inferences. This brings us to an important point, on the one hand, questions are specific entities and they bear some special properties, on the other hand, they are used in formal systems that are the framework of reasoning.

In Section 1.2 we introduced a formalization of questions called set-of-answers methodology. It utilizes the close connection of questions and their answerhood conditions. Simultaneously we mentioned there that it is very natural to use the epistemic terms in speaking of questions.

The approach we are going to present in this chapter works with epistemic logic as a background. Roughly speaking, epistemic logic and its semantics are used for the modeling of knowledge and epistemic possibilities of agents and groups of them. Epistemic logic in use will be the normal multi-modal logic. At the very beginning we do not impose any conditions on it; thus, the working system will be the normal modal (epistemic) logic K with its standard relational semantics (Kripke frames and models). However, the way of incorporating questions will be of such generality that it can be applied in all epistemic-like logics.

Moreover, the process of communication is a dynamic matter and epistemic logic is open to enrichment by dynamic aspects, see, e.g., [42]. These aspects will be studied later on in chapter 4 where we accept the 'standard' model of knowledge based on the modal system S5. Now let us briefly mention what a question and its epistemic meaning are.

In the introductory chapter we use the example with three friends holding cards. Let us suppose Catherine wants to find out where the Joker-card is. Then she can ask

Who has the Joker: Ann, or Bill?

From our set-of-answers methodological viewpoint, the question has a two element set of direct answers:

- Ann has the Joker.
- Bill has the Joker.

In this situation we can recognize the question as 'reasonable'. Asking it, Catherine expresses that she

1. does not know what is the right answer to the question,
2. considers the answers to be possible, and, moreover,
3. presupposes what is implicitly included in the answers, i.e., it must be the case that just either Ann has the Joker or Bill has it.

An agent-questioner provides the information of her ignorance (item 1) as well as the expected way to complete her knowledge: 2 says that there are two possibilities and 3 that one of the possibilities is expected. A question is a store of information of agent's epistemic state. An asked question means that listeners can form their partial picture of questioner's knowledge structure, which is an important part in communication and solving of problems in groups.[1] As it was mentioned at the beginning, the exchange of information is a basis of communication. The main aim of communication in a group is to share data and to solve problems. Very typical example is a group of scientists trying to find an answer to their scientific problem. Only by sharing of their knowledge and ignorance they can reach a solution.

Communication will be studied in the next chapter, here we prepare an 'erotetic epistemic framework'. First, we introduce propositional single-agent

[1] Jeroen Groenendijk says that "assertions may provide new *data*, questions may provide new *issues*" [12].

(normal modal) epistemic logic and extend it by questions. We fully apply our set-of-answers methodology and allow to mix declaratives and interrogatives. Questions will be a natural part of inference relations based on the background logic. Then we discuss answerhood conditions in a relationship with conditions posed on a 'reasonable' question. And, finally, the role of epistemic context as well as sets of questions are studied.

3.2 Single-agent propositional epistemic logic and questions

Our approach to epistemic logic is very liberal. The language of classical propositional logic $\mathcal{L}_{cpl}$ is extended by modalities $[i]$ and $\langle i \rangle$. The first one can be interpreted as 'agent i knows that…', 'agent i believes that…', etc. The other one is an 'epistemic possibility'. Thus, we get a language $\mathcal{L}_{cpl}^{K}$ with a subset of signs for atomic formulas $\mathcal{P} = \{p, q, \ldots\}$ and formulas defined as follows:

$$\varphi ::= p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \psi_1 \rightarrow \psi_2 \mid \psi_1 \leftrightarrow \psi_2 \mid [i]\psi \mid \langle i \rangle \psi$$

Modality $\langle i \rangle$ is understood as a dual to $[i]$:

$$\langle i \rangle \varphi \equiv \neg[i]\neg\varphi$$

In multi-agent variants of epistemic logic we presuppose that there is a finite set of agents $\mathcal{A} = \{1, \ldots, m\}$, where numbers $1, \ldots, m$ are names for agents. This section deals with a single-agent variant for the sake of simplicity in introducing questions and their basic properties. Moreover, we do not restrict the interpretation of $[i]$ to 'knowledge' of an agent i. Now, roughly and vaguely said, it is just 'epistemic necessity' of an agent without restrictions to knowledge conditions or belief conditions.

Semantics is based on Kripke-style models. *Kripke frame* is a relational structure $\mathcal{F} = \langle S, R_i \rangle$ with a set of states (points, indices, possible worlds) S and an accessibility relation $R_i \subseteq S^2$. *Kripke model* $\mathbf{M}$ is a pair $\langle \mathcal{F}, v \rangle$ where v is a valuation of atomic formulas. The satisfaction relation $\models$ is defined by a standard way:

1. $(\mathbf{M}, s) \models p$ iff $(\mathbf{M}, s) \in v(p)$
2. $(\mathbf{M}, s) \models \neg\varphi$ iff $(\mathbf{M}, s) \not\models \varphi$
3. $(\mathbf{M}, s) \models \psi_1 \vee \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ or $(\mathbf{M}, s) \models \psi_2$

4. $(\mathbf{M}, s) \models \psi_1 \wedge \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ and $(\mathbf{M}, s) \models \psi_2$
5. $(\mathbf{M}, s) \models \psi_1 \rightarrow \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ implies $(\mathbf{M}, s) \models \psi_2$
6. $(\mathbf{M}, s) \models [i]\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$, for each s_1 such that $sR_i s_1$

We do not put any restrictions on accessibility relation, thus, we have semantics for the modal system K.

3.2.1 Incorporating questions

We extend epistemic language $\mathcal{L}^K_{cpl}$ by brackets $\{,\}$ and the question mark $?_i$ for a question of an agent i. We get the language $\mathcal{L}^{KQ}_{cpl}$. For interrogative formulas metavariables Q^i, Q^i_1, etc. will be used.

Generally, a question Q^i is any formula of the form

$$?_i\{\alpha_1, \ldots, \alpha_n\},$$

where $dQ^i = \{\alpha_1, \ldots, \alpha_n\}$ is the set of direct answers to a question Q. Direct answers are formulas of our extended epistemic language $\mathcal{L}^{KQ}_{cpl}$ and questions can be among direct answers as well. We suppose that dQ^i is finite with at least two syntactically distinct elements. In accordance with our set-of-answers methodology the intended reading of a question Q^i is:

> Is it the case that α_1 or is it the case that α_2 . . . or is it the case that α_n?

Whenever I ask such question, I presuppose that at least one of the direct answers is the case. Whenever I hear such question, I know that a questioner presupposes the same, i.e., at least one of the direct answers is the case. This brings us to an important term *presupposition*, which is studied in the next subsection. On the contrary to our liberal SAM we require dQ^i to be finite. Working in propositional logics we want to keep direct answers as clear epistemic possibilities, this seems to be useful in a communication processes. Simultaneously, it makes easier the concept of 'presupposing'.

Presuppositions

Taking an inspiration in *inferential erotetic logic* (see Section 2.2.2) we define presuppositions of questions as formulas that are implied by each direct answer. A presupposition is a 'consequence' of each direct answer, no matter which answer is right.[2]

[2]Let us make a symbol convention: if it is not necessary to use the index i, we will omit it.

Definition 17. *A formula* φ *is a* presupposition *of a question* Q *iff* $(\alpha \to \varphi)$ *is valid for each* $\alpha \in dQ$. *We write* $\varphi \in \mathsf{Pres}Q$.

The set of presuppositions of a question is full of redundant formulas. This leads to the definition of *maximal presuppositions.* Maximal presuppositions imply every presupposition.

Definition 18. *A formula* φ *is a* maximal presupposition *iff* $\varphi \in \mathsf{Pres}Q$ *and* $(\varphi \to \psi)$ *is valid for each* $\psi \in \mathsf{Pres}Q$.

Example 13. *A formula* $(\alpha_1 \vee \ldots \vee \alpha_n)$ *is a maximal presupposition of a question* $?\{\alpha_1, \ldots, \alpha_n\}$.

The theory of questions in IEL introduces one more term—*prospective presupposition*. The truth of a prospective presupposition at a state of a model gives the truth of some direct answer at this state. This can be a modal reformulation of the original IEL definition.

Definition 19. *A formula* φ *is a* prospective presupposition *of a question* Q *iff* $\varphi \in \mathsf{Pres}Q$ *and, for all models* $\mathbf{M}$ *and states* s*, if* $(\mathbf{M}, s) \models \varphi$*, then there is a direct answer* $\alpha \in dQ$ *such that* $(\mathbf{M}, s) \models \alpha$. *We write* $\varphi \in \mathsf{PPres}Q$.

A formula $(\alpha_1 \vee \ldots \vee \alpha_n)$ is a prospective presupposition of a question $?\{\alpha_1, \ldots, \alpha_n\}$ as well.

Because of working with finite sets of direct answers in a system extending classical propositional logic, things are easier. We need not distinguish between maximal and prospective presuppositions.[3]

Theorem 13. *The set of prospective presuppositions is equal to the set of maximal presuppositions of a question* Q.

Proof. First, let $\varphi \in \mathsf{PPres}Q$ but φ is not maximal. Since φ is not maximal, there must be $(\mathbf{M}, s)$ and $\psi \in \mathsf{Pres}Q$ such that $(\mathbf{M}, s) \models \varphi$ and $(\mathbf{M}, s) \not\models \psi$. $(\mathbf{M}, s) \models \varphi$ implies the existence of a direct answer α satisfied in $(\mathbf{M}, s)$. Each presupposition is implied by every direct answer, so is ψ in $(\mathbf{M}, s)$ and it gives $(\mathbf{M}, s) \models \psi$, which is a contradiction.

Second, let φ be maximal, but not prospective. In our (finite) case we can suppose that Q has at least one prospective presupposition. If φ is not prospective, then there is a state $(\mathbf{M}, s) \models \varphi$ and $(\mathbf{M}, s) \not\models \alpha$, for each $\alpha \in dQ$. All presuppositions are satisfied in the state $(\mathbf{M}, s)$, so is prospective presuppositions, but it is in contradiction with the fact that no α is valid in $(\mathbf{M}, s)$. $\square$

[3] In IEL, prospective presuppositions are maximal, but not vice versa, see [44, Corollary 4.10].

In the next theorem we show the same result we obtained for IEL (see Lemma 1): All prospective presuppositions of a question are equivalent.

Theorem 14. *If* $\varphi, \psi \in \mathsf{PPres}Q$*, then* $\varphi \equiv \psi$.

Proof. For proving semantic equivalence we have to prove $\varphi \models \psi$ as well as $\psi \models \varphi$.

If $(\mathbf{M}, s) \models \varphi$, then there is $\alpha \in dQ$ such that $(\mathbf{M}, s) \models \alpha$. Since $\psi \in \mathsf{Pres}Q$, then $(\mathbf{M}, s) \models (\alpha \rightarrow \psi)$ gives $(\mathbf{M}, s) \models \psi$. We have obtained $\varphi \models \psi$.

The other case is similar. □

Thus, the symbol $\mathsf{PPres}Q$ will be used for a formula representing prospective presuppositions of a question Q modulo the semantic equivalence.

Note on presupposing and context In the example with card players we mentioned 'reasonable' Catherine's question

Who has the Joker: Ann, or Bill?

In item 3 (see p. 48) we wrote that *either Ann or Bill has the Joker* is Catherine's presupposition, i.e., it must be the case that Ann has the Joker (and Bill not), or it must be the case that Bill has the Joker (and Ann not). This is indicated by comma in the interrogative sentence as well. Catherine's presupposition is under influence of the context given by the rules of the card 'game':

> Just one Joker is distributed among the agents Ann, Bill, and Catherine.

Now, let us concern the following question:

What is Peter: a lawyer or en economist?

If there is no supplementary context, the question bears a presupposition that Peter is at least one of the two possibilities (maybe, both of them). However, the formalization of both questions would be almost the same, it is expected the form

$$?\{\alpha, \beta\}.$$

The role of context will be studied later on, viz. subsection Relativized askability in 3.2.2, especially.

Askable questions

In semantics for a majority of logical systems we speak about truth or falsity of a formula (in a particular state of a particular model). It is clear that it makes little sense to speak about truth/falsity of a question. We introduce instead a concept of *askability* of a question. Askability is based on our idea of a 'reasonable' question in a certain situation. 'Reasonability' corresponds to the three conditions we informally mentioned at the introductory example. Let us repeat and name them:

1. **Non-triviality** It is not reasonable to ask a question if the answer is known.
2. **Admissibility** Each direct answer is considered as possible.
3. **Context** At least one of the direct answers must be the right one.

Whenever an agent-questioner poses a question, she does not know any (direct) answer to a question, but, simultaneously, she considers all (direct) answers possible and she is aware of what is presupposed—she knows the prospective presupposition of a question. The formal definition follows.

Definition 20. *It holds for a question* $Q^i = ?_i\{\alpha_1, \dots, \alpha_n\}$ *that*

$$(\mathbf{M}, s) \models Q^i$$

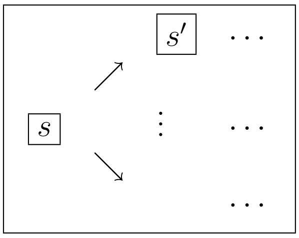
iff

1. $(\mathbf{M}, s) \not\models [i]\alpha$, *for each* $\alpha \in dQ^i$
2. $(\mathbf{M}, s) \models \langle i \rangle\alpha$, *for each* $\alpha \in dQ^i$
3. $(\mathbf{M}, s) \models [i]\mathsf{PPres}Q^i$

Then we say that Q^i *is* askable *in the state* $(\mathbf{M}, s)$ *(by an agent* i*).*

As we can see, the freedom in the syntactical form of questions was compensated by restrictions in their semantics. We say that a question is (generally) *askable* iff there is a model and a state where the question is askable (by an agent). Askable questions include neither contradiction nor tautology among their direct answers. The former is excluded by the second condition and the latter by the first one. A question Q^i is *askable relative to a model* $\mathbf{M}$ by an agent i (let us write $\mathbf{M} \models Q^i$) iff $(\mathbf{M}, s) \models Q^i$ for each $s \in S$. The definition of $\models Q^i$ is straightforward, but there are no 'tautological' questions in K. A question is not askable in a state without successors. In our version at least two successors are needed. If we work in systems extending classical logic, the first condition is equal to $(\mathbf{M}, s) \models \neg[i]\alpha$, i.e., $(\mathbf{M}, s) \models \langle i \rangle \neg\alpha$, for each $\alpha \in dQ^i$. We can see the questioner as admitting the possibility of $\neg\alpha$ for each direct answer α to a question Q^i. In these systems, questions are complex modal formulas. However, Definition 20 is meant in a full generality without the intention of reduction of questions to the epistemic language.

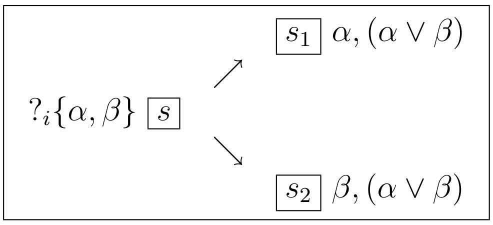
The epistemic semantic viewpoint represents agent's 'knowledge' in a state s as an afterset sR_i given by the states related to s by an accessibility relation R_i, i.e., $sR_i = \{s' : sR_is'\}$.



Let us return to the following question:

What is Peter: a lawyer or an economist?

This question can be formalized by a formula $?_i\{\alpha, \beta\}$. Its askability in a state $(\mathbf{M}, s)$ requires a substructure on sR_i consisting of (at least) two accessible states, one of them satisfies α (and does not satisfy β) and the other one β (and does not satisfy α). All states in sR_i must satisfy the prospective presupposition $(\alpha \vee \beta)$, because of the context condition.



Of course, this is a minimal requirement given by askability conditions, the complete afterset structure can contain other states, some of them may satisfy both α and β, but none of them satisfies $\neg\alpha$ and $\neg\beta$: the question $?_i\{\alpha, \beta\}$ does not consider the answer *neither* α *nor* β as possible (context condition)—such answer would be accepted, e.g., by the question

$$?_i\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\}$$

States in the afterset sR_i are understood as epistemic possibilities. In accordance with the non-triviality condition neither α nor β can be true in all of them. Finally, admissibility condition requires that there must be at least one 'α-state' and at least one 'β-state' in sR_i.

3.2.2 Some important classes of questions

In this section we introduce some classes of questions with their semantic behavior. Yes-no questions and conjunctive questions were introduced in the previous chapters. In the included subsection we suggest to link conditional and hypothetical questions with the role of a context, i.e., an auxiliary set of formulas. The names of classes originate from IEL (cf. [44]).

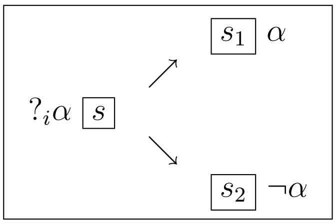
The very basic questions in their syntactical as well as semantical form are *yes-no questions*.

Is Prague the capital of the Czech Republic?

is a question requiring one of the following answers:

- (Yes,) Prague is the capital of the Czech Republic.
- (No,) Prague is not the capital of the Czech Republic.

with the formalization $?_i\{\alpha, \neg\alpha\}$, which is shortly written as $?_i\alpha$. This question is askable in a state s if there are (at least) two different states available from s, one satisfies α and the other one $\neg\alpha$. The afterset sR_i is supposed to have this form:



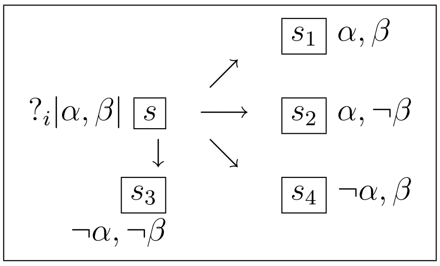
Reviewing the askability conditions for $?_i\alpha$ we can see that this question is equivalent to a formula $\langle i\rangle\alpha \wedge \langle i\rangle\neg\alpha$. In our system, yes-no questions can be seen as a 'contingency modality'. The same requirements are posed by askability conditions for $?_i\neg\alpha$. In our case, both $?_i\alpha$ and $?_i\neg\alpha$ are equivalent. Yes-no questions always form a partitioning on aftersets and their presuppositions are tautologies. Questions with presuppositions, which are all tautological, are called *safe*.

Definition 21. *A question Q^i is* safe *iff* $\mathsf{PPres}Q^i$ *is valid.*

Questions that are not safe, will be called *risky.*[4]

Another example of safe questions are *conjunctive questions*. The shortest one is $?_i\{(\alpha \wedge \beta), (\neg\alpha \wedge \beta), (\alpha \wedge \neg\beta), (\neg\alpha \wedge \neg\beta)\}$ asking for a full description based on α and β. We write it as $?_i|\alpha, \beta|$ and the following figure shows the required substructure on the afterset.

[4]The original concepts of safety and riskiness of questions come from Nuel Belnap. See chapter 2 as well.



Similarly to yes-no questions they have the exhaustive set of direct answers and direct answers are *mutually exclusive*. In section 3.4 we will deal with some restrictions posed on direct answers, which will enable us to clarify answerhood conditions.

The question

What is Peter: a lawyer or an economist?

with a formalization $?_i\{\alpha, \beta\}$ is a risky one. However, asking this question in a state an agent does not admit that she can see states where $\mathsf{PPres}(?_i\{\alpha, \beta\})$ is not satisfied. Let us define this kind of local safeness:

Definition 22. *A question Q is* safe in a state $(\mathbf{M}, s)$ *(for an agent i) iff* $(\mathbf{M}, s_1) \models \mathsf{PPres}Q^i$*, for each $s_1 \in sR_i$.*

Thus, an askable question in a state (for an agent) is safe in this state (for this agent).

Relativized askability

Let us consider the following question:

Did you stop smoking?

At first sight, it is a yes-no question, but seeing both answers it seems, there can be something more what is presupposed:

- *Yes, I did* can mean *I had smoked and stopped.*
- *No, I didn't* can mean *I smoked and go on.*

Both of them presuppose the smoking in the past. Such question is an example of a *conditional yes-no question* with a formalization $?_i\{(\alpha \wedge \beta), (\alpha \wedge \neg\beta)\}$. Generally, *conditional questions* are of the form

$$?_i\{\alpha \wedge \beta_1, \alpha \wedge \beta_2, \ldots, \alpha \wedge \beta_n\}$$

The askability of a conditional question in a state s requires the validity of α in each accessible state, i.e., an agent 'knows' α in s. The notion of an askable conditional question can be generalized with respect to a set of formulas. This leads to *relativized askability.*[5]

Definition 23. *A question Q is askable (by an agent i) in $(\mathbf{M}, s)$ with respect to a set of formulas Γ iff $(\mathbf{M}, s) \models [i]\Gamma$ and $(\mathbf{M}, s) \models Q^i$. By $[i]\Gamma$ we abbreviate the set $\{[i]\gamma \mid \gamma \in \Gamma\}$. Then we write $(\mathbf{M}, s) \models (\Gamma, Q)_i$.*

A conditional question $?_i\{\alpha \wedge \beta_1, \alpha \wedge \beta_2, \ldots, \alpha \wedge \beta_n\}$ is askable in s if and only if $?_i\{\beta_1, \ldots, \beta_n\}$ is askable there with respect to the auxiliary set (knowledge database) $\{\alpha\}$.

The term *relativized askability* will be mostly used for pointing out the importance of a set Γ. Every question askable at a state is askable with respect to its set of (prospective) presuppositions. It has an expected consequence:

Fact 27. *If $(\mathbf{M}, s) \models (\Gamma, Q)_i$, then $(\mathbf{M}, s) \models (\Delta, Q)_i$, for each $\Delta \subseteq \Gamma$.*

However, relativized askability is not 'monotonic' in knowledge databases. If $(\mathbf{M}, s) \models (\Gamma, Q)_i$, then it need not be $(\mathbf{M}, s) \models (\Delta, Q)_i$, for $\Delta \supset \Gamma$.

Relativized askability of this kind can be used for an explicit expressing of the knowledge structure. Catherine's question

Who has the Joker: Ann, or Bill?

can be formalized by

$$(\{\neg\alpha \vee \neg\beta\}, ?\{\alpha, \beta\})_c$$

In addition, IEL introduces one more term, *hypothetical question*, which is a bit similar to the previous one. A natural language example of hypothetical yes-no question might be

If you open the door, will you see a bedroom?

with a formalization $?_i\{(\alpha \to \beta), (\alpha \to \neg\beta)\}$. A general hypothetical question is then

$$?_i\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\}$$

Again, the askability of such questions can be understood as based on agent's hypothetical knowledge. Our interpretation is: if α is known, then it is to be decided whether β_1, or β_2, etc. Using a generalization similar to Definition 23 we obtain

[5]This term corresponds to *question in an information set* introduced in [13].

Definition 24. *A question Q is askable (by an agent i) in $(\mathbf{M}, s)$ with respect to a set of hypotheses Γ iff $(\mathbf{M}, s) \models [i]\Gamma$ implies $(\mathbf{M}, s) \models Q^i$. By $[i]\Gamma$ we abbreviate the set $\{[i]\gamma \mid \gamma \in \Gamma\}$. Let us write $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q$*

Askability of a conditional question ensures the askability of a hypothetical one.

Fact 28. *If $(\mathbf{M}, s) \models (\Gamma, Q)_i$, then $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q$.*

The only difference of both definitions lies in the words *and* and *implies*. We will return to these concepts in Section 3.5.

3.3 Epistemic erotetic implication

Erotetic inference is implicitly based on the (standard) implication. We say that a question Q_1 implies Q_2 (in a state s, for an agent i) whenever askability of Q_1 (in s, for i) implies askability of Q_2 (in s, for i).

$$(\mathbf{M}, s) \models Q_1^i \to Q_2^i \text{ iff } (\mathbf{M}, s) \models Q_1^i \text{ implies } (\mathbf{M}, s) \models Q_2^i$$

We have mentioned that questions $?_i\alpha$ and $?_i\neg\alpha$ have the same askability conditions. The eqivalence of both questions is a theorem in our system based on modal logic K. Let us omit the index i for now.

Example 14. *Both $(?\alpha \to ?\neg\alpha)$ and $(?\neg\alpha \to ?\alpha)$ are valid.*

The informal meaning of epistemic erotetic implication is very transparent. Whenever an agent asks Q, then she can ask every question implied by Q. A question in antecedent is 'more complex' then the implied one. Implied question's required substructure on the afterset must be a substructure of that required by an implying one. The question

What is Peter: a lawyer or an economist?

implies

Is Peter a lawyer?

as well as

Is Peter an economist?

This can be generalized:

Example 15. $?\{\alpha_1, \ldots, \alpha_n\} \to ?\alpha_j$ *is valid, for each* $j \in \{1, \ldots, n\}$.

The following example shows a special position of conjunctive questions in implications.

Example 16. *The following implications are valid:*

1. $?|\alpha_1, \ldots, \alpha_n| \to ?\alpha_j$, *for each* $j \in \{1, \ldots, n\}$.
2. $?|\alpha, \beta| \to ?(\alpha \circ \beta)$, *where* $\circ$ *is any truth-functional constant.*
3. $?|\alpha, \beta| \to ?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\}$
4. $?|\alpha, \beta, \gamma| \to ?|\alpha, \beta|$

Let us notice that conjunctive questions are safe and they imply safe questions again. We can prove that it is a rule.

Theorem 15. *If* Q_1 *is safe and* $Q_1 \to Q_2$ *valid, then* Q_2 *is safe.*

Proof. Let us suppose, Q_1 is safe and $(Q_1 \to Q_2)$ is valid, but Q_2 is not safe. We take a model $\mathbf{M}$ and a state $(\mathbf{M}, s)$ where $(\mathbf{M}, s) \models Q_1$, then $(\mathbf{M}, s) \models Q_2$. Let us slightly change the afterset of s. We add a single point s_1 accessible from s where the prospective presupposition of Q_2 is invalid. $(\mathbf{M}, s_1) \not\models [i] \bigvee_1^l \beta_j$, for $dQ_2 = \{\beta_1, \ldots \beta_l\}$. In this model the askability of Q_1 is not violated at s_1, $(\mathbf{M}, s_1) \models Q_1$, but Q_2 is not askable here. This leads to a contradiction. □

We have to be careful when speaking about 'complexity' of questions. Is question $?\{\alpha, \beta\}$ more complex than $?|\alpha, \beta|$? It is easy to check that neither $?|\alpha, \beta| \to ?\{\alpha, \beta\}$ nor $?\{\alpha, \beta\} \to ?|\alpha, \beta|$ are valid. In both cases there is a problem with the context condition; a risky question $?\{\alpha, \beta\}$ requires the validity of the disjunction $(\alpha \vee \beta)$ on the afterset. The next two examples emphasize the importance of context condition again.

Example 17. $\not\models ?\{\alpha, \beta\} \to ?\{\neg\alpha, \neg\beta\}$ *as well as* $\not\models ?\{\neg\alpha, \neg\beta\} \to ?\{\alpha, \beta\}$

Example 18. $\not\models ?\{\alpha, \beta, \gamma\} \to ?\{\alpha, \beta\}$

The question $?_i\{\alpha, \beta, \gamma\}$ requires $[i](\alpha \vee \beta \vee \gamma)$, but $?_i\{\alpha, \beta\}$ requires 'only' $[i](\alpha \vee \beta)$, which can fail in the structure sufficient for the askability of the first question.

An implying question shares presuppositions with the implied one.

Fact 29. *If* $Q_1 \to Q_2$ *is valid, then if* $\varphi \in \mathsf{Pres}Q_1$, *then* $\varphi \in \mathsf{Pres}Q_2$.

Epistemic erotetic implication has the expected property—transitivity:

Fact 30. *If* $(\mathbf{M}, s) \models Q_1 \to Q_2$ *and* $(\mathbf{M}, s) \models Q_2 \to Q_3$, *then* $(\mathbf{M}, s) \models Q_1 \to Q_3$.

3.4 Askability and answerhood

Epistemic erotetic implication creates a relationship among questions. If a question is askable at a state, so is every implied one. In our system, $Q_1 \to Q_2$ if and only if $\neg Q_2 \to \neg Q_1$. If an implied question is inaskable, so is the implying one. The askability of a question consists of the validity of three conditions (non-triviality, admissibility, and context) and inaskability is a result of the violation of at least one of them. Let us imagine that we know that $Q_1 \to Q_2$ and we have the answer to Q_2 (non-triviality condition fails), then we are sure that Q_1 is inaskable, but does it mean to have an answer to Q_1? What if Q_1 is inaskable because of failing context condition? In this section we will just deal with such violations of askability conditions, properties of inaskable questions from various classes of questions, and answerhood conditions—complete and partial answers will be introduced.

To break the non-triviality condition means that there is a direct answer which is 'known' by an agent (in a state of a model). In fact, an agent knows a direct answer even if she knows a formula that is equivalent to a direct answer or she knows a formula from which some direct answer follows. Such formula is called *complete answer*.[6]

Let us define the concept *a question is answered at a state*:[7]

Definition 25. *A question* $Q^i = ?_i\{\alpha_1, \ldots, \alpha_n\}$ *is* answered *in* $(\mathbf{M}, s)$ *(for an agent i) iff* $(\mathbf{M}, s) \models \bigvee_{\alpha_j \in dQ^i}([i]\alpha_j)$. *We write* $(\mathbf{M}, s) \models A_i Q$.

The case of invalid admissibility condition is a bit different. Let us imagine that our agent knows α in a state s. Then, even if she does not know an answer to a question $?|\alpha, \beta|$ in that state, it is not right to ask this question. All possibilities required by the admissibility condition are not available, in particular, accessible states with $\{\neg\alpha, \neg\beta\}$ and $\{\neg\alpha, \beta\}$ are missing. Some answers to the question $?|\alpha, \beta|$ give information that is superfluous in the state of agent's knowledge. Formula α is a *partial answer* to a question $?|\alpha, \beta|$. Partial answer excludes some of the (direct) answers.

Definition 26. *A question* $Q^i = ?_i\{\alpha_1, \ldots, \alpha_n\}$ *is* partially answered *in* $(\mathbf{M}, s)$ *(for an agent i) iff* $(\mathbf{M}, s) \models \bigvee_{\alpha_j \in dQ^i}([i]\neg\alpha_j)$. *We write* $(\mathbf{M}, s) \models P_i Q$.

In fact, if the admissibility condition fails, there is a direct answer which is not considered as possible, i.e., $(\exists\alpha \in dQ^i)((\mathbf{M}, s) \not\models \langle i \rangle \alpha)$, which is

[6] In [13] we can find two terms: *to be* and *to give* a semantic answer. This distinction is not necessary here.

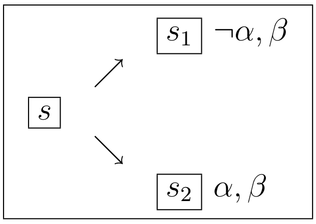
[7] We do not discuss a dynamic approach just now. An answered question is a 'potentially' answered question, in fact, an answer need not be uttered among agents.

equivalent to $(\exists\alpha \in dQ^i)((\mathbf{M}, s) \models [i]\neg\alpha)$ in our system. This means that our agent can answer the question $?\alpha$ in $(\mathbf{M}, s)$: $(\exists\alpha \in dQ^i)((\mathbf{M}, s) \models A_i?\alpha)$. This and example 15 give the proof of the following lemma.

Lemma 10. *If* $(\mathbf{M}, s) \models P_iQ$*, then there is a formula* φ *such that* $(\mathbf{M}, s) \models A_i?\varphi$ *and* $Q^i \to ?_i\varphi$ *is valid.*

Let us suppose a question Q is answered. Does it mean that Q is partially answered? Surprisingly, it is not true that if $(\mathbf{M}, s) \models A_iQ$, then $(\mathbf{M}, s) \models P_iQ$. See the next example.

Example 19.



In the structure given by Example 19 the question $?\{\alpha, \beta\}$ is answered in s (the agent knows β), but it is not partially answered; the agent is not able to get the knowledge of either $\neg\alpha$ or $\neg\beta$, both α and β are still possible.

A_iQ implies P_iQ for *questions with pairs of mutually exclusive direct answers*. It means that for each direct answer there is another one such that both of them cannot be true. Yes-no questions as well as conjunctive questions are examples from this class. Their sets of direct answers satisfy a more strict condition, they have mutually exclusive direct answers—the 'truth' of a direct answer (in a state) means that no other direct answer is satisfied there. However, both conditions are of a semantic nature, it can be caused by a model and a state. Recall Catherine's question

Who has the Joker: Ann, or Bill?

with the context, which is important here: it does not admit the afterset substructure in Example 19. Mutual exclusiveness is not preserved by implication. We cannot prove anything similar to Theorem 15, in particular, the answers of the question $?_i\{\alpha \wedge \beta, \alpha \wedge \neg\beta, \neg\alpha \wedge \beta\}$ are mutually exclusive, but this question implies $?_i\{\alpha, \beta\}$, which is not in this class, generally.

Fact 31. *If a question* Q^i *has the set* dQ^i *with pairs of mutually exclusive direct answers, then* $A_iQ \to P_iQ$ *is valid.*

The last option for inaskability is the violation of context condition. Then an agent does not know (believe) the prospective presupposition of a question. For example, the question

> *Which town is the capital of the Czech Republic: Prague, or Brno?*

can be inaskable, although an agent does not know either complete or partial answer, but she admits that there is another town, which could be the capital of the Czech Republic, i.e., neither Prague nor Brno might be the right answer. Then we say that a question is *weakly presupposed* by an agent.

Definition 27. *A question* Q^i *is* weakly presupposed *in* $(\mathbf{M}, s)$ *(by an agent* i*) iff* $(\mathbf{M}, s) \not\models [i]\mathsf{PPres}Q^i$*. We write* $(\mathbf{M}, s) \models W_i Q$*.*

An askable question in a state (for an agent i) satisfies at least the 'safeness in a state' (see Definition 22). If a question Q is safe in $(\mathbf{M}, s)$, then Q cannot be weakly presupposed in $(\mathbf{M}, s)$.

Theorem 16. *If a question with pairs of mutually exclusive direct answers* Q^i *is safe in* $(\mathbf{M}, s)$*, then the following conditions are equivalent:*

1. $(\mathbf{M}, s) \models \neg Q^i$
2. $(\mathbf{M}, s) \models P_i Q$
3. *There is a formula* φ *such that* $(\mathbf{M}, s) \models A_i ?\varphi$ *and* $Q^i \to ?_i\varphi$ *is valid.*

Proof. (2⇒1) is clear and (2⇒3) is from Lemma 10.

(1⇒2) If $(\mathbf{M}, s) \models \neg Q^i$, then there are three possibilities: $(\mathbf{M}, s) \models A_i Q$ or $(\mathbf{M}, s) \models P_i Q$ or $(\mathbf{M}, s) \models W_i Q$. The last one is impossible because of the safeness of Q^i in $(\mathbf{M}, s)$. If $(\mathbf{M}, s) \models A_i Q$, then $(\mathbf{M}, s) \models P_i Q$ (from Fact 31).

(3⇒1) Let us suppose that there is a formula φ such that the question $?_i\varphi$ is answered in $(\mathbf{M}, s)$. From $Q^i \to ?_i\varphi$ we know that if $(\mathbf{M}, s) \not\models ?_i\varphi$, then $(\mathbf{M}, s) \not\models Q^i$. □

Partial answerhood of a question Q^i in some state is equivalent to the existence of a yes-no question, which is answered in that state and implied by Q^i. From the validity of $Q^i \to ?_i\varphi$ we know that inaskability of $?_i\varphi$[8] implies inaskability of Q^i and, therefore, φ (as well as $\neg\varphi$) implies either some $\alpha \in dQ^i$ or $\neg\alpha$ (for $\alpha \in dQ^i$).

[8] $(\mathbf{M}, s) \models \neg ?_i\varphi$ iff $(\mathbf{M}, s) \models A_i ?\varphi$.

3.5 Context

In subsection **Relativized askability** (page 56) we introduced askability with respect to sets of formulas. While both kinds are understood as variants of conditional or hypothetical questions, in some situations it can be useful to display and emphasize the role of a context. Especially if it has an important position in reasoning. Let us recall the example from Section 2.3.1, where we discussed erotetic implication in IEL. An agent asking

> Q_1: *What is Peter a graduate of: a faculty of law or a faculty of economy?*

can be satisfied by the answer

> *He is a lawyer.*

even if she did not ask

> Q_2: *What is Peter: a lawyer or an economist?*

The connection between both questions could be established by the following knowledge base Γ:

> *Someone is a graduate of a faculty of law iff he/she is a lawyer.*
> *Someone is a graduate of a faculty of economy iff he/she is an economist.*

Relativized askability helps us to express that Q_1 implies Q_2 with respect to an auxiliary set of formulas Γ, i.e., $(\Gamma, Q_1)_i \rightarrow Q_2^i$. In the example, Q_1 can be formalized by $?\{\alpha_1, \alpha_2\}$, Q_2 by $?\{\beta_1, \beta_2\}$, and $\Gamma = \{(\alpha_1 \leftrightarrow \beta_1), (\alpha_2 \leftrightarrow \beta_2)\}$, then

$$(\{(\alpha_1 \leftrightarrow \beta_1), (\alpha_2 \leftrightarrow \beta_2)\}, ?\{\beta_1, \beta_2\})_i \rightarrow ?_i\{\alpha_1, \alpha_2\}$$

is valid. Moreover, the questions Q_1 and Q_2 are equivalent with respect to Γ: $(\Gamma, Q_1)_i \rightarrow Q_2^i$ as well as $(\Gamma, Q_2)_i \rightarrow Q_1^i$ is valid.

The prime reason for introducing of the structures $(\Gamma, Q)_i$ is to keep the importance of a context in inferences with questions. $(\Gamma, Q)_i$ can be considered as a generalization of conditional questions in our system. (Generalized) conditional questions consist of two parts: conditional part (context) and query part. As an easy conclusion of Fact 27 we receive that a conditional question implies its query part: $?_i\{\alpha \wedge \beta_1, \ldots, \alpha \wedge \beta_n\} \rightarrow ?_i\{\beta_1, \ldots, \beta_n\}$.

Fact 32. *$(\Gamma, Q)_i \rightarrow Q^i$ is valid formula.*

Hypothetical questions consist of such two parts as well, but it is not valid that $?_i\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \to ?_i\{\beta_1, \ldots, \beta_n\}$.

In some sense, we can see (generalized) hypothetical questions, $\Gamma \xrightarrow{i} Q$, as a counterpart of evocation in IEL. Similarly, (generalized) conditional questions in the interplay with implication, $(\Gamma, Q_1)_i \to Q_2^i$, seem to be a counterpart of general erotetic implication in IEL. While the correspondence could be seen in 'philosophy', these structures differs in properties from IEL ones.

Now, let us list some properties appearing by combination of conditional and hypothetical questions with implication. Most of them are expected. For example, the following fact points out the cumulativity of explicitly expressed presuppositions.

Fact 33. *If* $(\mathbf{M}, s) \models (\Gamma, Q_1)_i \to Q_2^i$ *and* $(\mathbf{M}, s) \models (\Delta, Q_2)_i \to Q_3^i$, *then* $(\mathbf{M}, s) \models (\Gamma \cup \Delta, Q_1)_i \to Q_3^i$.

Relativized askability is transfered by implication; if $(\mathbf{M}, s) \models (\Gamma, Q_1)_i$ and $(\mathbf{M}, s) \models (Q_1^i \to Q_2^i)$, then $(\mathbf{M}, s) \models (\Gamma, Q_2)_i$. And it is easy to check the following generalization:

Lemma 11. *Whenever* $(\mathbf{M}, s) \models (\Gamma, Q_1)_i$ *and* $(\mathbf{M}, s) \models (\Gamma, Q_1)_i \to Q_2^i$, *then* $(\mathbf{M}, s) \models (\Gamma, Q_2)_i$.

The same result can be proved for generalized hypothetical questions: If $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q_1$ and $(\mathbf{M}, s) \models (Q_1^i \to Q_2^i)$, then $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q_2$.

Finally, we obtain

Theorem 17. *If* $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q_1$ *and* $(\mathbf{M}, s) \models (\Gamma, Q_1)_i \to Q_2^i$, *then* $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q_2$.

Proof. Let us suppose $(\mathbf{M}, s) \not\models \Gamma \xrightarrow{i} Q_2$, i.e., $(\mathbf{M}, s) \models [i]\Gamma$ and $(\mathbf{M}, s) \not\models Q_2^i$. From $(\mathbf{M}, s) \models [i]\Gamma$ and $(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q_1$ we get $(\mathbf{M}, s) \models Q_1^i$ and $(\mathbf{M}, s) \models (\Gamma, Q)_i$. Because of $(\Gamma, Q_1)_i \to Q_2^i$ we gain $(\mathbf{M}, s) \models Q_2^i$, which is a contradiction. □

It is necessary to point out that both variants of relativized askability were introduced for the explicit expression of the context conditions of a question. This approach is useful; however, (pure) questions and implication are of prime importance in our setting.

3.6 Implied questions

If questions are in the implicational relationship, a transmission of askability conditions from an implying question to the implied one is justified. An implied question is understood as 'less complex' in its requirements posed on the afterset substructure. Let us recall Example 15[9] where $Q \rightarrow ?\alpha$ is valid for each $\alpha \in dQ$, thus

$$Q \rightarrow \bigwedge_{\alpha \in dQ} (?\alpha)$$

On the one hand, each question $?\alpha$ is a yes-no question with many good properties. On the other hand, they may be 'worse' in the description of an agent's knowledge/ignorance structure than the initial question Q. Might it be 'better' if we consider the whole set of implied questions based on the set of direct answers to an initial question?

If we consider the set of questions $\Phi = \{?\alpha_1, ?\alpha_2, \ldots\}$, then inaskability of some $?\alpha$ means that the answer is either α or $\neg\alpha$. In the first case Q is answered as well, in the second one Q is partially answered. This can be understood as a form of 'sufficiency' condition of the set Φ: answerability of its members implies at least partial answerability of the initial question Q. It means, Φ does not include 'useless' questions. Moreover, we receive one more property, Φ is 'complete' in a way: if Q is partially answered, then there must be a question $Q_j \in \Phi$ that is answered. We can understand it as a form of 'reducibility' of an initial question to a set of yes-no questions. In comparison with IEL, this reducibility is purely based on implication.

Example 16 gives a similar result for conjunctive questions. The set of yes-no questions can be formed by constituents of their direct answers:

$$?|\alpha_1, \ldots, \alpha_n| \rightarrow \bigwedge_{j=1}^{n} (?\alpha_j)$$

In this case, we arrive to really less complex questions, but having a partial answer to $?|\alpha_1, \ldots, \alpha_n|$ does not give neither answer nor partial answer to some α_j. It could be useful in some cases. An agent can ask questions from the set $\Phi = \{?\alpha_1, \ldots, ?\alpha_n\}$ and complete her knowledge step by step. The most important property is that the set Φ does not include useless questions. Generally speaking, in some communication processes it is useful to conceal some knowledge or ignorance of a questioner—a criminal investigation is a nice example. An agent can ask questions from the set Φ without completely revealing her knowledge structure. Asking a conjunctive question

[9] Let us omit the index i in this section.

$?|\alpha_1, \ldots, \alpha_n|$ publicly, everybody is informed that the agent-questioner does not know anything with respect to $\alpha_1, \ldots, \alpha_n$.

Chapter 4

A step to dynamization of erotetic logic

4.1 Introduction

At the beginning of the previous chapter we introduced the language for multi-agent propositional epistemic logic $\mathcal{L}^K_{cpl}$ with the set of agents $\mathcal{A} = \{1, \ldots, m\}$. If we add an accessibility relation for each agent to Kripke frames, $\mathcal{F} = \langle S, R_1, \ldots, R_m \rangle$, we will obtain multi-modal system K with box-like modalities $[1]$, …, $[m]$. Although we often used 'epistemic' terminology, especially in motivations, this system is not intended for knowledge representation. In fact, there are many discussions about the best representation of knowledge as well as belief in the philosophy of logic. Nowadays, such discussions are brought into life again in studies of substructural logics. The term *knowledge* is often subjected to new interpretations based on a background system.[1]

The very aim in introducing questions in epistemic-like systems was to provide an interpretation of questions, which agrees with the interplay of the idea of representing the knowledge and ignorance structure of a questioner in the process of asking. The interpretation of questions should be mostly independent of a background system. In our philosophy, 'knowledge structure' and its representation is considered to be primary. This chapter is devoted to multi-agent epistemic logic with questions based on modal system $\mathsf{S5}$ and its dynamic extension—*public announcement logic*.

We often referred to the importance of questions in communication processes. This is understood as an information exchange among agents in a group. The delivering of information in a group has the benefit of (public)

[1] See, e.g., the definition of knowledge modality inside relevant logic in [2].

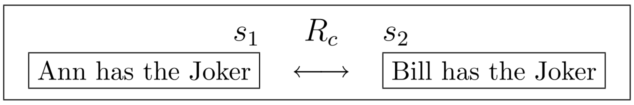
announcements and, as a result, there is a change of epistemic states of group members.

S5 represents standard epistemic logic, cf. [42, 9], where knowledge is factive and fully introspective (positively as well as negatively). Of course, this system is subjected to criticism, see [9, 8] where the 'logical omniscience problem' seems to be the most criticized aspect. However, the goal of this chapter is to show the role of questions in a formal dynamic-epistemic system. So, we are not going to solve any problem of this formal epistemic representation nor to follow philosophical discussions on it.

First of all we extend the erotetic epistemic framework by group questions and group epistemic modalities (group knowledge, common knowledge, and distributed knowledge). This makes it possible to speak about answerhood conditions for groups of agents. Then we apply the public announcement modality in the process of answer mining among agents.

4.2 Multi-agent propositional epistemic logic with questions

We have just said that our epistemic framework would be based on multi-modal logic S5. Being still in language $\mathcal{L}^K_{cpl}$ we only have to add that each accessibility relation R_i is *equivalence*, i.e., reflexive, transitive, and symmetric relation. Accessibility relations seem to play a bit different role now. In logic K we understand the accessible states as (epistemic) 'alternatives' for an actual state, an agent can see 'possibilities'. The equivalence relation makes connected states indistinguishable, an agent considers them as having the same 'value'. Let us recall the group of three friends and just one Joker-card distributed among them. From Catherine's viewpoint both possibilities—either *Ann has the Joker* or *Bill has the Joker*—are indistinguishable:



4.2.1 Group epistemic modalities

So far we have worked with personal knowledge. However, in multi-agent systems we are obliged to introduce new modalities to reflect epistemic states in groups of agents. The language $\mathcal{L}^K_{cpl}$ will be extended by symbols E_G, C_G, and D_G, where $G \subseteq \mathcal{A}$ is a group of agents.

Group knowledge $E_G\varphi$ means

> Each agent (from G) knows φ.

and is fully definable by personal knowledges of G-members:

$$E_G\varphi \leftrightarrow \bigwedge_{i\in G}[i]\varphi$$

We shall call it *group knowledge*. Let us stress that E_G does not guarantee that a member of the group G knows that she shares the same information with some other members of the group.

Common knowledge The group modality C_G is stronger in the following sense: $C_G\varphi$ requires not only that φ is a group knowledge, but also that this fact is reflected by everybody in G.

> Each agent (from G) knows φ and each agent knows that each agent knows φ and each agent knows that each agent knows that each agent knows φ and ...

C_G is called *common knowledge* and expresses that the knowledge is maximally shared by everybody in G, each agent is aware of this sharing. C_G can be seen as an infinite conjunction of all finite iterations of the group knowledge E_G:

$$C_G\varphi \leftrightarrow E_G\varphi \wedge E_GE_G\varphi \wedge E_GE_GE_G\varphi \wedge \dots$$

The system S5C is obtained by adding the operator C_G and the semantic clause:

- $(\mathbf{M}, s) \models C_G\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$ for each s_1 such that $s \left(\bigcup_{i\in G} R_i\right)^* s_1$

$\left(\bigcup_{i\in G} R_i\right)^*$ is a reflexive and transitive closure of $\bigcup_{i\in G} R_i$ and it means that s_1 is accessible from s by each R_i ($i \in G$) in k steps, for any $k \geq 0$.

As we said, E_G is definable in the language $\mathcal{L}^K_{cpl}$, so adding group knowledge is just a conservative extension of the background multimodal epistemic logic S5. Both languages $\mathcal{L}^K_{cpl}$ and $\mathcal{L}^{KE}_{cpl}$ have the same expressivity. However, this is not the case of common knowledge. Multi-modal epistemic logic with common knowledge S5C is not compact, as it is indicated in the definition of C_G and there exists formula in language $\mathcal{L}^{KC}_{cpl}$, which can distinguish two models that are indistinguishable in language $\mathcal{L}^K_{cpl}$, see [42, p. 227].

The relationship of introduced epistemic modalities is the following:

Fact 34. *$C_G\varphi \to E_G\varphi \to [i]\varphi$ is valid in* S5C *for each $i \in G$.*

Common knowledge is essential for collective behavior and coordination of collective actions. In game theory it is often presupposed that rules of a game are shared by players. It is important to know rules, to know that the other players know the same rules, to know that they know that we know it, and so on. In case of questions we considered a question to be (partially) answered for an agent if she knows a fact based on a direct answer. However, when we say that a question is (partially) answered for a group of agents? An answer must not be only known by all members, but it must be generally known that it is known. Just common knowledge is a good candidate for group answerhood conditions.

Definition 28.
- *A question Q is* answered *in $(\mathbf{M}, s)$ for a group G iff there is $\alpha \in dQ$ such that $(\mathbf{M}, s) \models C_G\alpha$.*
- *A question Q is* partially answered *in $(\mathbf{M}, s)$ for a group G iff there is $\alpha \in dQ$ such that $(\mathbf{M}, s) \models C_G(\neg\alpha)$.*

Distributed knowledge The last group modality is a bit of another kind. Let us remind the group of three friends and suppose that Ann has the Joker. Although neither Catherine nor Bill know it, the knowledge of the Joker-owner is implicitly contained in the group. If the agents can communicate, they easily reach the hidden fact that Ann has the card. The standard meaning of $D_G\varphi$ is given by the semantic clause:

- $(\mathbf{M}, s) \models D_G\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$ for each s_1 such that $s \left(\bigcap_{i\in G} R_i\right) s_1$

φ is true in all states that are accessible for every member in G. D_G is called *distributed* or *implicit knowledge*. The term *distributed knowledge* coincides with the idea of pooling agents' knowledge together. Let us imagine that a solution of some problem can be obtained by the collecting of particular data from each member of a group of agents. The crucial data are distributed among agents, but nobody can solve the problem alone because of the need of the other data.

If an agent knows φ, then φ is distributed knowledge in every agent's group:

Fact 35. *$[i]\varphi \to D_G\varphi$ is valid in* S5CD *for each $i \in G$.*

The accessibility relation based on D_G is a subset of each R_i. Adding D_G to language $\mathcal{L}^K_{cpl}$ does not increase its expressivity.[2]

[2] Axioms and properties can be found in [23].

If there is distributed knowledge for a group of agents, then it is distributed knowledge for every bigger group.

Fact 36. *$D_G\varphi \rightarrow D_{G'}\varphi$ is valid in* S5CD *for $G \subseteq G'$.*

Again, this nicely shows the idea of a hidden information; we can obtain it by a communication of (only) some agents in the group G'. The role of distributed knowledge in answerhood will be discussed in the next section.

4.2.2 Group questions and answerhood

We introduced questions in a form of an agent's personal task that fits in her knowledge structure. Whenever she wants to find an answer to a question, she has to communicate the question and find someone who can answer it. Being in a group of colleagues she asks the question and, in the best case, there is somebody who knows an answer. A worse case is that nobody can answer the question—the question is the task for them as well. Such question is askable by each member of a group G and we shall call it *group question*.

Definition 29. *A question Q is an* askable group question *in $(\mathbf{M}, s)$ (for a group of agents G) iff $(\forall i \in G)((\mathbf{M}, s) \models Q^i)$. Let us write $(\mathbf{M}, s) \models Q^G$.*

Whenever there is a question which is (partially) answered by at least one agent in a group, then we can see how to reach a (partial) answer. A question must be publicly posed and the answer is a result of a communication.

Group questions seem to be a worse problem, there is no agent with a (partial) answer to it in a group. If an answer should be sought inside the group, there is only one chance to find it. Again communication is important for to discover 'hidden' information, i.e., an answer is present in the group as distributed knowledge.

Let us have a group of two agents a and b. The following example shows their knowledge structure:

Example 20.

	a		b	
s_1	$\longleftrightarrow$	s_2	$\longleftrightarrow$	s_3
α		α		$\neg\alpha$
β		$\neg\beta$		$\neg\beta$

Agent a cannot distinguish states s_1 and s_2 and knows α, agent b cannot distinguish s_2 and s_3 and knows $\neg\beta$. However, neither of them is able to (partially) answer the yes-no question $?(\alpha \rightarrow \beta)$, it is their group question $?_{\{a,b\}}(\alpha \rightarrow \beta)$. If they communicate, they recognize the state s_2 to be common for them. This brings us to the term *implicitly (partially) answered question.*

Definition 30.
- *A question Q is* implicitly answered *in $(\mathbf{M}, s)$ by a group of agents G iff $(\exists\alpha \in dQ)((\mathbf{M}, s) \models D_G\alpha)$.*
- *A question Q is* implicitly partially answered *in $(\mathbf{M}, s)$ by a group of agents G iff $(\exists\alpha \in dQ)((\mathbf{M}, s) \models D_G\neg\alpha)$.*

Back to the example, what should agents a and b communicate to gain an answer to $?(\alpha \to \beta)$? We can find an inspiration in implied yes-no questions. The question $?(\alpha \to \beta)$ implies the disjunction of questions $?\alpha$ and $?\beta$:

$$?(\alpha \to \beta) \to (?\alpha \vee ?\beta)$$

Moreover, the agent a can completely answer $?\alpha$ and the other one can completely answer the question $?\beta$. So, it would be useful to communicate questions $?\alpha$ and $?\beta$ in the group.

Generally, we can prove that if there is a set of questions, their disjunction is implied by an initial question, and each question from the set can be (partially) answered by some agent from a group G, then the initial question is implicitly (partially) answered.

Theorem 18. *If there is a set of questions Φ such that each $Q_k \in \Phi$ is (partially) answered in $(\mathbf{M}, s)$ by some agent $i \in G$ and $(\mathbf{M}, s) \models Q \to \bigvee_{Q_k \in \Phi} Q_k$, then Q is implicitly (partially) answered in $(\mathbf{M}, s)$.*

Proof. If Q is not sound group question, then $\Phi = \{Q\}$ and there is an agent having (partial) answer to Q. From Facts 34 and 35 we get that it is implicit (partial) answer in a group G.

Let us suppose Q is a sound group question in $(\mathbf{M}, s)$, then (partial) answers to questions from Φ are distributed among agents from G. From $(\mathbf{M}, s) \models Q \to \bigvee_{Q_k \in \Phi} Q_k$ we get $(\mathbf{M}, s) \models \bigwedge_{Q_k \in \Phi} \neg Q_k \to \neg Q$. Unsoundness of Q in $(\mathbf{M}, s)$ cannot be caused by violating of context condition because of its status of sound group question. Now, let us introduce a new agent a, which pools knowledge of all agents in a group G together. $R_a = \bigcap_{i \in G} R_i$, if $R_a = \emptyset$, then everything is distributed knowledge. Let R_a be nonempty. All questions $Q_k \in \Phi$ are unsound for a in $(\mathbf{M}, s)$, so is Q, and a's knowledge of (partial) answer is in afterset sR_a. □

The content of the theorem is based on the S5CD-valid rule

$$\frac{(\psi_1 \wedge \ldots \wedge \psi_m) \to \varphi}{([l_1]\psi_1 \wedge \ldots \wedge [l_m]\psi_m) \to D_{\{l_1,\ldots,l_m\}}\varphi} \tag{4.1}$$

which expresses the mentioned idea of pooling agents' knowledge together for getting their distributed knowledge.

A question, which is posed among agents, can be (partially) answered only if it is at least implicitly (partially) answered by a group. The next section shows one of the ways of communication formalization in the role of 'answer mining'.

4.3 Public announcement

Let us return to the group of three friends—Ann, Bill, and Catherine. It is group knowledge that each of them has one card and nobody knows the cards of the others and that one of the cards is the Joker. Ann received the Joker, but neither Bill nor Catherine know which of the other two friends, has it. In particular, both of them are not able to distinguish between the states where Ann has the Joker and where she has not. If Ann publicly announces

"I've got the Joker.",

everybody in the group learns this fact. Possible worlds (states) where Ann does not have the Joker are excluded from the (epistemic) models of both Bill and Catherine.

Our example gives a typical situation represented in the public announcement logic—after a public announcement of a statement φ ("I've got the Joker"), some other statement ψ holds, e.g., *Bill knows Ann has the Joker and Catherine knows Ann has the Joker*. In fact, the author of an announced statement is irrelevant in our framework. The statement is understood as information coming to each member of a group in the same way. From this viewpoint Ann's announcement has the same effect as if an external observer announces *Ann has the Joker*.

Formally we introduce logic of public announcement as an extension of the system $\mathsf{S5}$, cf. [42]. We define a box-like operator $[\ ]$, such that the intended meaning of $[\varphi]\psi$ is:

After the public announcement of φ, it holds that ψ.

The semantics of the new announcement operator is given by the following clause:

- $(\mathbf{M}, s) \models [\varphi]\psi$ iff $(\mathbf{M}, s) \models \varphi$ implies $(\mathbf{M}|_\varphi, s) \models \psi$

where $\mathbf{M}|_\varphi = \langle\langle S', R'_1, \ldots, R'_m\rangle, v'\rangle$ is defined as follows:

$$\begin{aligned} S' &= \{s \in S \mid s \models \varphi\} \\ R'_i &= R_i \cap S'^2 \\ v'(p) &= v(p) \cap S' \end{aligned}$$

The model $\mathbf{M}|_\varphi$ is obtained from $\mathbf{M}$ by deleting of all states where φ is not true and by the corresponding restrictions of accessibility relations and the valuation function. Again we can introduce a dual operator $\langle\ \rangle$ defined in a standard way as $\langle\varphi\rangle\psi$ iff $\neg[\varphi]\neg\psi$. If we rewrite the corresponding semantic clause, we obtain

- $(\mathbf{M}, s) \models \langle\varphi\rangle\psi$ iff $(\mathbf{M}, s) \models \varphi$ and $(\mathbf{M}|_\varphi, s) \models \psi$

The intended meaning of the dual operator is 'after a *truthful* announcement of φ, it holds that ψ'. It is easy to see that the diamond-like operator is stronger:

Lemma 12. *$\langle\varphi\rangle\psi \to [\varphi]\psi$ is valid.*

The language $\mathcal{L}_{cpl}^{K[]}$ has the same expressive power as the language $\mathcal{L}_{cpl}^{K}$. This is demonstrated by the following lemma, which provides a reduction of formulas with the public announcement operator to the epistemic ones. The corresponding equivalences give, in fact, an axiomatization of the announcement operator in the public announcement epistemic logic without common knowledge, cf. [42, p. 81].

Lemma 13. *The following equivalences are valid in* S5 *with public announcement modality (where $\circ \in \{\wedge, \vee, \to\}$):*

$$\begin{array}{rcl} [\varphi]p & \leftrightarrow & (\varphi \to p) \\ [\varphi]\neg\psi & \leftrightarrow & (\varphi \to \neg[\varphi]\psi) \\ [\varphi](\psi \circ \chi) & \leftrightarrow & ([\varphi]\psi \circ [\varphi]\chi) \\ [\varphi][i]\psi & \leftrightarrow & (\varphi \to [i][\varphi]\psi) \\ [\varphi][\psi]\chi & \leftrightarrow & [\varphi \wedge [\varphi]\psi]\chi \end{array}$$

For common knowledge there is no such reduction (axiom), the language $\mathcal{L}_{cpl}^{KC[]}$ is more expressive than $\mathcal{L}_{cpl}^{KC}$ [42, p. 232]. We only have a rule describing the relationship between the public announcement and common knowledge, e.g., the one introduced in [42, p. 83]:

$$\frac{(\chi \wedge \varphi) \to [\varphi]\psi \wedge E_G\chi}{(\chi \wedge \varphi) \to [\varphi]C_G\psi} \tag{4.2}$$

From now, our formal work will proceed in the rich propositional language $\mathcal{L}_{cpl}^{KECDQ[]}$ with formulas defined as follows:

$$\begin{array}{rcl} \varphi & ::= & p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \psi_1 \to \psi_2 \mid \psi_1 \leftrightarrow \psi_2 \mid \\ & & [i]\psi \mid E_G\psi \mid C_G\psi \mid D_G\psi \mid \\ & & ?_i\{\psi_1, \ldots, \psi_n\} \mid ?_G\{\psi_1, \ldots, \psi_n\} \mid \\ & & [\psi_1]\psi_2 \end{array}$$

We obtain public announcement logic with common knowledge and questions PACQ.

4.3.1 Updates and questions

Let us return to our example. As we said, members of a group learn what was announced. In particular, if Ann says

"I've got the Joker.",

the announced fact becomes *commonly known* in the group of players {Ann, Bill, Catherine}. This seems to suggest that a publicly announced proposition becomes common knowledge . But what if Ann says:

"You don't know it yet, but I've got the Joker."?

This announcement can be formalized by

$$(J_a \wedge \neg[b](J_a) \wedge \neg[c](J_a)),$$

where J_a means *Ann has the Joker*. Although the formula is true in the moment of announcement, it is evident that its epistemic part (*you don't know it yet*) becomes invalid after it is announced. So the formula $(J_a \wedge \neg[b](J_a) \wedge \neg[c](J_a))$ becomes false after the announcement.

A formula, which becomes false after it is truthfully announced (as in our example), is called an *unsuccessful update*; if it becomes true, we call it a *successful update*.

Definition 31.

- *Formula* φ *is a* successful update *in* $(\mathbf{M}, s)$ *iff* $(\mathbf{M}, s) \models \langle\varphi\rangle\varphi$.
- *Formula* φ *is an* unsuccessful update *in* $(\mathbf{M}, s)$ *iff* $(\mathbf{M}, s) \models \langle\varphi\rangle\neg\varphi$.

If a formula is an unsuccessful update, it cannot be commonly known in the updated model. Using the soundness proof of the rule (4.2) we can prove that a formula is true after a public announcement if and only if it gets common knowledge after the announcement (see [42, p. 83 and 86]).

Lemma 14. $[\varphi]\psi$ *is valid iff* $[\varphi]C_G\psi$ *is valid.*

As a consequence we get

Lemma 15. $[\varphi]\varphi$ *is valid iff* $[\varphi]C_G\varphi$ *is valid.*

If a formula $[\varphi]\varphi$ is valid, we call it a *successful formula*.

Definition 32. *Formula φ is a* successful formula *iff $[\varphi]\varphi$ is valid, otherwise it is an* unsuccessful formula

From Lemma 15 we know that publicly announced successful formulas are commonly known. Atoms, $K_i\varphi$, and $\neg K_i\varphi$ (for every φ) are examples of successful formulas.

Successful formulas true in a state are successful updates there:

Lemma 16. *If $[\varphi]\varphi$ is valid formula and $(\mathbf{M}, s) \models \varphi$, then $(\mathbf{M}, s) \models \langle\varphi\rangle\varphi$.*

It is easy to verify that in our S5 background system questions are successful formulas, i.e.,

Fact 37. *$[Q^i]Q^i$ is valid.*

In S5-models a question Q^i, which is askable in a state s, is askable in all states from the equivalence class sR_i. No 'cutting' of states in the model $\mathbf{M}$ forced by the public announcement of Q^i results in $(\mathbf{M}, s) \models Q^i$ and $(\mathbf{M}|_{Q^i}, s) \not\models Q^i$. Thus, a publicly announced question is commonly known (see Lemma 15). In other words there is no model and state such that $(\mathbf{M}, s) \not\models [Q^i]Q^i$.

Successful formulas have an important property: they do not bring anything new if they are announced repeatedly.

Lemma 17. *Let φ be a successful formula. $[\varphi][\varphi]\psi \leftrightarrow [\varphi]\psi$ is valid.*

Proof. $[\varphi][\varphi]\psi$ is equivalent to $[\varphi \wedge [\varphi]\varphi]\psi$ (Lemma 13), which is equivalent to $[\varphi]\psi$, because of the validity of $[\varphi]\varphi$ (φ is successful). □

It is no surprise that askable questions (in a state) are successful updates; it follows from Lemma 16 and Fact 37.

Fact 38. $(\mathbf{M}, s) \models Q^i$ *iff* $(\mathbf{M}, s) \models \langle Q^i\rangle Q^i$.

Whenever an agent publicly asks a question, it does not cause any change in her epistemic model, it remains askable until she gets some new information.

4.3.2 Public announcement and answerhood

Whenever a question is (partially) answerable in a state, then there is a formula φ such that after a public announcement of φ the question becomes inaskable there. In our example, Ann has the Joker, but neither Bill nor Catherine know it. If Catherine publicly asks

"Who has got the Joker?",

Bill can infer:

"I have not the Joker and Catherine does not know who has it, therefore Ann has it."

Catherine's question was *informative* for Bill, it caused that the question *Who has got the Joker?*, which was askable for Bill, became inaskable after Catherine had asked it, even if her question was not (partially) answered. This leads us to the definition of *informative formula*.

Definition 33. *A formula φ is* informative *for an agent i with respect to a question Q in $(\mathbf{M}, s)$ iff $(\mathbf{M}, s) \models Q^i \wedge \langle\varphi\rangle\neg Q^i$.*

Contrary to partial answerhood (see Theorem 16) there need not be any logical connection between an informative formula and direct answers to a question. The informativeness can be forced by the shape of a particular model.

However, it is a clear conclusion of Definition 33 that whenever there is an askable question in a state for an agent, then after an announcement of an informative formula the agent obtains at least partial answer to the question.

Lemma 18. *If a formula φ is informative in $(\mathbf{M}, s)$ for an agent i with respect to a question Q, then there is $\alpha \in dQ$ such that $(\boldsymbol{M}|_\varphi, s) \models [i]\alpha$ or $(\boldsymbol{M}|_\varphi, s) \models [i]\neg\alpha$.*

Proof. From the informativeness of φ we obtain $(\mathbf{M}|_\varphi, s) \models \neg Q^i$, which means $(\mathbf{M}|_\varphi, s) \models A_i Q$ or $(\mathbf{M}|_\varphi, s) \models P_i Q$, because of the safeness of Q in $(\mathbf{M}, s)$. □

If an informative formula is 'strong' enough (i.e., it implies (partial) answer to a question)[3], then the (partial) answer is commonly known among agents in the updated model.

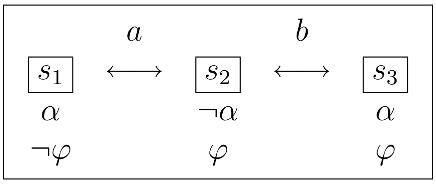
Fact 39. *If a formula φ is informative in $(\mathbf{M}, s)$ for an agent $i \in G$ with respect to a question Q and there is $\alpha \in dQ$ such that $(\varphi \rightarrow \alpha)$ or $(\varphi \rightarrow \neg\alpha)$ is valid, then $(\boldsymbol{M}|_\varphi, s) \models C_G\alpha$ or $(\boldsymbol{M}|_\varphi, s) \models C_G\neg\alpha$.*

The core of this fact is that the formula φ is true in each state in an updated model $\mathbf{M}|_\varphi$, so is α or $\neg\alpha$. The role of informativeness is minor, it 'only' informs us that a question Q was askable for an agent i in s and that φ can be truthfully announced there.

[3]We could say that an informative formula 'gives' a (partial) answer to a question.

We have to be careful; informativeness of a formula for an agent does not imply its informativeness for any other one. The next example shows a structure where φ is informative for the agent a with respect to the question $?\alpha$, but it is not informative for b with respect to the same question.

Example 21.



4.3.3 Answer mining in a group

In Example 20 we displayed two agents a and b. Neither of them can answer the question $?(\alpha \to \beta)$—it is their group question. However, the question is implicitly answered. We can imagine their cooperative communication. The question $?(\alpha \to \beta)$ is askable for the agent a and she wants to answer it. Her colleague b may help. So, a can directly and publicly ask the question $?(\alpha \to \beta)$ and reveal her ignorance. A question is a successful formula, thus, it is commonly known. In cooperative communication agents are supposed to announce what they know with respect to their group question, i.e., $[a]\alpha$ and $[b]\neg\beta$. They come to the (complete) answer $\neg(\alpha \to \beta)$, which is, moreover, common knowledge. We obtained a sequence of (truthfully) publicly announced agents' knowledge leading to a (commonly known) answer:

$$\langle [a]\alpha \rangle \langle [b]\neg\beta \rangle \neg(\alpha \to \beta),$$

resp.,

$$[[a]\alpha][[b]\neg\beta] C_{\{a,b\}} \neg(\alpha \to \beta).$$

So our group was successful in seeking an answer using just 'internal' resources. They reached an answer after a series of announcements of facts they know.

This inspires the following idea of question's 'solvability'. In general, we can say that a question is (partially) *solvable*, if there is a finite series of truthful announcements of agents' knowledge after which a (partial) answer is a result.

Definition 34. *A question Q is*

- solvable *for a group G in a state $(\mathbf{M}, s)$ iff there is a set of formulas $\{\psi_1, \ldots, \psi_k\}$ and a direct answer $\alpha \in dQ$ such that $(\mathbf{M}, s) \models \langle [l_1]\psi_1 \rangle \ldots \langle [l_k]\psi_k \rangle \alpha$, where $l_j \in G$,*

- partially solvable *for a group G in a state $(\mathbf{M}, s)$ iff there is a set of formulas $\{\psi_1, \dots, \psi_k\}$ and a direct answer $\alpha \in dQ$ such that $(\mathbf{M}, s) \models \langle [l_1]\psi_1 \rangle \dots \langle [l_k]\psi_k \rangle \neg\alpha$, where $l_j \in G$.*

Solvability follows the idea of the rule (4.1) describing the pooling agents' scattered knowledge together. This rule inspired an alternative definition of distributed knowledge as well, see [11]:

- $(\mathbf{M}, s) \models D^*_G\varphi$ iff $\{\psi \mid (\exists i \in G)((\mathbf{M}, s) \models [i]\psi)\} \models \varphi$

D^*_G is not equivalent to D_G and the following is proved in [11]:

Theorem 19. *If $(\mathbf{M}, s) \models D^*_G\varphi$, then $(\mathbf{M}, s) \models D_G\varphi$, but not vice versa.*

However, solvability is based on finite sets. Our system with common knowledge is not compact and this brings us to a finite version of D^*_G:

- $(\mathbf{M}, s) \models D^+_G\varphi$ iff there are $\psi_1, \dots, \psi_k$ such that $(\mathbf{M}, s) \models [l_1]\psi_1 \wedge \dots \wedge [l_k]\psi_k$ and formula $(\psi_1 \wedge \dots \wedge \psi_k) \rightarrow \varphi$ is valid.

Whenever a (partial) answer of a question is distributed knowledge in the new sense of D^+_G, then the question is solvable.

Lemma 19. *If $(\mathbf{M}, s) \models D^+_G\varphi$, then $(\mathbf{M}, s) \models \langle [l_1]\psi_1 \rangle \dots \langle [l_k]\psi_k \rangle \varphi$.*

Proof. From $(\mathbf{M}, s) \models D^+_G\varphi$ we obtain that each $[l_j]\psi_j$ is successful formula true in $(\mathbf{M}, s)$. Lemma 16 says that they are successful updates in $(\mathbf{M}, s)$ and, simultaneously, they are commonly known. Let us write $\mathbf{M}|_{\dots}$ for the updated model after the series of public announcements $\langle [l_1]\psi_1 \rangle \dots \langle [l_k]\psi_k \rangle$. It follows that $(\mathbf{M}|_{\dots}, s) \models \psi_j$ and due to the validity of $(\psi_1 \wedge \dots \wedge \psi_k) \rightarrow \varphi$ we have $(\mathbf{M}|_{\dots}, s) \models \varphi$. □

A formula, which is distributed knowledge in a state, is a successful update there.

Theorem 20. *If $(\mathbf{M}, s) \models D^+_G\varphi$, then $(\mathbf{M}, s) \models \langle \varphi \rangle \varphi$.*

Proof. Let us suppose that $(\mathbf{M}, s) \models D^+_G\varphi$, but $(\mathbf{M}, s) \not\models \langle \varphi \rangle \varphi$. Then either $(\mathbf{M}, s) \not\models \varphi$ or $(\mathbf{M}|_{\varphi}, s) \not\models \varphi$.

1. $(\mathbf{M}, s) \not\models \varphi$ is not possible: $s \in \left(s \bigcap_{i \in G} R_i\right)$ in $\mathsf{S5}$ and $(\mathbf{M}, s) \models D^+_G\varphi$ implies $(\mathbf{M}, s) \models D_G\varphi$ (Theorem 19).

2. If $(\mathbf{M}|_{\varphi}, s) \not\models \varphi$, then $\mathbf{M}|_{\varphi}$ must be different from the model $\mathbf{M}|_{\dots}$ we have talked about in the proof of Lemma 19. Thus, there must be a state s_0 which makes the difference.

(a) It is impossible that $s_0 \in S^{\mathbf{M}|\dots}$ and $s_0 \notin S^{\mathbf{M}|\varphi}$: each ψ_j is true in $(\mathbf{M}, s_0)$ and $(\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow \varphi$ is valid, then $(\mathbf{M}, s_0) \models \varphi$.

(b) Let $s_0 \notin S^{\mathbf{M}|\dots}$ and $s_0 \in S^{\mathbf{M}|\varphi}$. $s_0 \in S^{\mathbf{M}|\varphi}$ means that $(\mathbf{M}, s_0) \models \varphi$ and from Theorem 19 we obtain $s_0 \in \left(s \bigcap_{i \in G} R_i\right)$. However, each ψ_j is known in s by an agent from G, i.e., $(\mathbf{M}, s) \models D_G \psi_j$ (Fact 35), thus, $(\mathbf{M}, s_0) \models \psi_j$ and it is not possible that $s_0 \notin S^{\mathbf{M}|\dots}$.

□

The theorem says an important thing: if an answer to a question is accessible by a communication of agents in a group, then it is successful update. A successful update becomes common knowledge in the updated model:

Fact 40. *If* $(\mathbf{M}, s) \models \langle\varphi\rangle\varphi$, *then* $(\mathbf{M}|_\varphi, s) \models C_G \varphi$.

As a result we obtain that a question, whose answer is accessible by a communication of agents, is answered for a group of agents in the updated model, cf. Definition 28.

4.4 Final remarks

This chapter combines questions in epistemic framework and communication based on public announcement logic. The background system is $\mathsf{S5}$, which can be understood as 'introspective'. Next to positive and negative introspection we can recognize that an agent 'knows' questions askable for her: formula

$$Q^i \leftrightarrow [i]Q^i$$

is valid. If $(\mathbf{M}, s) \models Q^i$, then the askability of Q^i for an agent i holds in each state in the afterset sR_i, i.e., in the equivalence class of s.

In a multi-agent epistemic approach we can understand a question as a 'task' (or a 'problem') to be solved by a particular group of agents. Communication is one of the basic tools of a group searching for a solution to a problem (e.g., an answer to a question) and asking questions is one of the essential parts of this communication. Our liberal SAM makes it possible to mix knowledge and questions. The question

Who has got the Joker?

is mostly seen as a question about facts. However, we can receive the answer

I don't know.

In our setting this answer is not a complete answer to the question *Who has got the Joker?*. It solves another kind of questions. Let us return to the example of three card players. Bill can ask Catherine

Do you know who has got the Joker, Catherine?

This question is primarily asking for Catherine's knowledge about the card holder. Expected direct answers are

I know who has got the Joker.
I don't know who has got the Joker.

The first answer indicates that Catherine can completely answer the question *Who has got the Joker?*. The second one indicates that Catherine cannot completely answer it, but such answer does not reject the possibility that Catherine knows a partial answer. Whenever Bill wants to find out whether *Who has got the Joker?* is a task for Catherine, he should ask

Would you ask the question 'Who has got the Joker?', Catherine?

Bill asks Catherine whether the question *Who has got the Joker?* is a reasonable (askable) question for her. It is a yes-no question formalized by the formula

$$?_b\{?_c\{J_b, J_a\}, \neg ?_c\{J_b, J_a\}\}$$

The first direct answer means that Catherine would ask *Who has got the Joker?*, i.e., the question $?_c\{J_b, J_a\}$ is askable for her. The second one means, this question is not askable for her, which according to Theorem 16 means that Catherine can (at least partially) answer that question.

Multi-agent approach with group knowledge modalities makes it possible to speak on levels of answerhood. An agent's personal level of answerhood conditions is based on agent's knowledge. In case of questions posed in a group of agents we consider commonly known (partial) answer as a right solution of a question with respect to a group. If an answer to a question is sought by a communication inside a group, then an answer must be known by some member, or it must at least be included as a distributed knowledge.

Chapter 5

Conclusion

Although we can consider the thesis to consist of two almost independent parts, some points are common to both of them. Working in inferential erotetic logic as well as in epistemic erotetic logic I primarily wanted to provide tools for the development of both branches of erotetic logic. Even if I do not want to contribute to a philosophical debate on what a question is, it seems to me that a word or two should be said about the chosen methodology in both parts of the thesis.

The main inspiration came from the original IEL. It inspired the set-of-answers methodology introduced in the first chapter as well as the emphasis posed on inferences with questions. SAM is liberal enough to be used in the presented approaches. It is open for additional restrictions given by both syntax and semantics. Of course, many objections can be raised against it, especially, whenever we want to analyze all kinds of natural language questions. Our intention was to work with propositional logic and to keep maximum from the logic of declaratives. In the first chapter we showed that our SAM is convenient for the execution of erotetic inferences and that the epistemic variant is very natural. Coming back to additional restrictions, chapter 2 presents SAM containing direct answers as declaratives only. It is in correspondence with the second Hamblin's postulate (cf. subsection 1.2.1). On the contrary, our epistemic erotetic logic (chapters 3 and 4) admits to have not only declaratives among direct answers.

The first part of the thesis (chapter 2) is fully developed in the IEL framework. Questions and declaratives are mixed only on the level of consequence relations and the main goal of the chapter is to study relationships among IEL consequences. Our general approach showed that some relations must be supported by additional relations among direct answers, cf. results obtained for regular e-implication, or based on the relationship of 'strongness' between two questions.

The second part of the thesis is different in its substance. Epistemic framework is the prime and questions correspond to certain states of knowledge, ignorance, and presuppositions of an agent. This approach is a novelty inspired by Groenendijk-Stokhof's intensional erotetic logic together with our SAM. Questions become a part of epistemic language and they can be considered as satisfied in an epistemic state (in a model). The semantic work with questions has almost the same flavor as it is with (epistemic) declaratives. Only the questions' satisfiability (*askability*) in a state is more complex, being based on three conditions (non-triviality, admissibility, and context). Although we introduced a general approach, in the rest of the thesis we use finite SAM. This makes the work with the context condition and presuppositions much easier. Also the definition of *askability* of a question in a state is of such a generality that it can be used for any epistemic-like logic. In the thesis we presented epistemic logics K and $\mathsf{S5}$ extended by questions. Questions are representable by modal formulas there. Thus, these systems can be considered as 'reductionist' ones—of course, it was not intentional. Being inspired by IEL and Groenendijk-Stokhof's approach we wanted to deal with inferences with questions. In this epistemic case, inferences are based on classical implication. However, we can explicitly work with epistemic contexts and obtain similar structures (on the object-language level) that are introduced in IEL. The conditions required for askability of a question nicely correspond to natural answerhood conditions. Discovering them we were faced with the problem of some restrictions required for the chosen SAM.

Moreover, all works well with group modalities as well as with public announcement. If we compare the contents of the chapters, we may find another division of the thesis. Chapters 2 and 3 can be called 'logic of questions', they study inferences with questions and the relationships of questions and declaratives. On the other hand, chapter 4 introduces questions as a part of communication. Dynamic approaches can mostly bear the name 'logic of inquiry'. Chapter 4 shows that questions behave well together with updates and that they play the expected role in the context of distributed knowledge. The framework public announcement logic is based on $\mathsf{S5}$. Publicly asked questions are successful formulas and an askable question in a state is a successful update there. As a final result we presented the correspondence of a finite version of distributed knowledge with cooperative communication aimed at finding a commonly known (partial) answer to a group question.

5.1 Related works and future directions

Publications related to inferential erotetic logic were mentioned in chapter 2 and subsection 1.3.1 where we list papers having something to do with an 'inquiry' aspect of IEL, which was not studied in chapter 2. Moreover, we added two papers based on Hintikka's approach.

The historical part of the first chapter includes many publications referring to intensional erotetic logic of Groenendijk and Stokhof. Let us point out the cited dynamic application from [40]. This paper makes the best of dynamic logic developed in publications of Johan van Benthem and his collaborators. The usual epistemic model is enriched by a new equivalence relation of indistinguishability ('abstract issue relation'). Roughly speaking, similarly to updates of models based on public announcements, there are updates for asking yes-no questions.

Closely related to this approach is *inquisitive semantics* developed by Jeroen Groenendijk and his collaborators. The generalized version with its associated logic can be found in [5]. The original idea behind inquisitive semantics is common to dynamic approaches. A cooperative communication is a raising and resolving issues. Propositions are seen as proposals how to update the common state. "If a proposition consists of two or more possibilities, it is *inquisitive*: it invites the other participants to provide information such that at least one of the proposed updates may be established." [5, p. 112]

As the recent publications indicate, the combination of epistemic and dynamic aspects seems to be a good framework for erotetic logic. The goal of our epistemic logic of questions was just to prepare such a general framework. It opens many directions of further work, let us mention the most obvious:

- To use another epistemic logic. Relevant epistemic logic proposed by Ondrej Majer and myself, cf. [2, 22], can be took into account. Relevant implication provides a good background for erotetic implication. It is necessary to develop a multi-agent version of relevant epistemic logic.
- To combine our approach with 'logics of communication'. This should be based on the recent boom of dynamization, cf. [41, 40, 42]. One possibility is to apply generalized common knowledge $C_G(\varphi, \psi)$, which is, in fact, equivalent to $[\varphi]C_G\psi$. A correspondence with our notion *(partially) solvable question* can take advantage of a generalization: $C_G(\vec{\varphi}, \psi)$ iff $[\varphi_1] \ldots [\varphi_n]C_G\psi$.
- To develop a predicate version. This item brings us to an extensive study of types of answers.

- To apply non-monotonic approaches. Inferential erotetic logic invites to non-monotonic applications (e.g., ordering on dQ, preferred models, default rules based on questions).

- To fuzzify presented approaches. Recall the paper [4] mentioned in section 1.3.2 as an inspiration.

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