Charles University in Prague
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MASTER THESIS

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Interval Linear Programming

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Study program: Computer science

I would like to thank to my supervisor Doc. Grygarová for full support during thesis elaboration, for new ideas and critics.

Hereby I proclaim that I created this thesis by myself and I only used listed literature. I agree with lending of this thesis.

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Abstract

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Interval linear programming means

$$\min_{M} c^T x, \qquad \text{for } c \in \boldsymbol{c},$$

where $M = \{x \in \mathbb{R}^n; Ax = b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$, $\boldsymbol{A} \subset \mathbb{R}^{m \times n}$, $\boldsymbol{b} \subset \mathbb{R}^m$, $\boldsymbol{c} \subset \mathbb{R}^n$, $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are intervals.

The first part of the master thesis introduce a new approach to interval linear programming, defining always bounded set of feasible solutions of a linear programming problem and studying its properties. The main result of this part demonstrates that the modified set of feasible solutions varies "continuously" with the entries in the matrix A and in the vector b. The second part studies the solution function continuity for an interval linear programming problem.

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Chapter 1

Introduction

1.1 Linear Programming

Linear programming problems are optimization problems in which the objective function and the constraints are all linear:

$$\min_{M} c^T x, \tag{1.1}$$

where $M = \{x \in \mathbb{R}^n; Ax = b, x \geq 0\}$ and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. M is called a feasible region or a set of feasible solutions and it is a convex polyhedral set.

Linear programming is an important field of optimization for several reasons. Many practical problems in operations research can be expressed as linear programming problems. Certain special cases of linear programming, such as network flow problems and multicommodity flow problems are considered important enough to have generated much research on specialized algorithms for their solution.

Since the objective function is linear, all local optima are automatically global optima. The linear objective function also implies that an optimal solution of (1.1) can only occur at a boundary point of the feasible region.

It is proved that a linear programming problem is solvable by the worst-case polynomial-time algorithm, although the most famous algorithm - simplex algorithm is the worst-case exponential-time algorithm.

1.2 Interval Linear Programming

Interval linear programming (as indicated by the name) is derived from linear programming. An interval linear programming problem is a linear programming problem with inexact data. A matrix A, vectors b and c are not fixed in this case, but they are from an interval:

$$\min_M c^T x \qquad \text{for } c \in \boldsymbol{c}, \tag{1.2}$$

where $M = \{x \in \mathbb{R}^n; Ax = b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$ and $\boldsymbol{A} = \langle \underline{A}, \overline{A} \rangle$, $\boldsymbol{b} = \langle \underline{b}, \overline{b} \rangle$, $\boldsymbol{c} = \langle \underline{c}, \overline{c} \rangle$, $\underline{A}$, $\overline{A} \in \mathbb{R}^{m \times n}$, $\underline{b}$, $\overline{b} \in \mathbb{R}^m$, $\underline{c}$, $\overline{c} \in \mathbb{R}^n$.

Interval linear programming has also many applications in practice, however, due to its complexity there exist only algorithms for special cases. It is proved by Rohn [2] that an interval linear programming problem is NP-hard problem. We can say that an inexact data in the objective function and in the vector b do not increase much the complexity of the problem, but inexact data in the matrix A create a significant increase in the complexity of the problem. Prof. Rohn has published wide range of the publications concerning matrixes, matrix intervals, systems of linear equations with inexact data and interval linear programming including NP-hardness problematics, e.g. [3], [4], [5]. Concerning interval linear programming, Rohn was focused on bounds of the solution function, bounds of the set of feasible solutions and methods of their estimation. His latest results in interval linear programming are [6] and [7]. This thesis gets out of Prof. Rohn's work, but is focused on the topic that Prof. Rohn has not researched. Even, this topic si not studied in any available literature on interval linear programming.

1.3 The Goals and the Contributions of the Thesis

The thesis is focused on the set of feasible solutions of problem (1.2) and its changes with a matrix A and a vector b perturbations. In this thesis it will be studied what kind of a dependency there is between changes in the entries in the matrix A and in the vector b and changes in the set of feasible solutions of (1.2). At the same time we will find out required assumptions for such relation.

The second goal of the thesis is to research possibilities, that the solution function[1] of an interval linear programming problem (1.2) is continuous and to find out required assumptions, so that the solution function is continuous in the point (A, b, c). Continuity of the solution function will be studied through the properties of the set of feasible solutions of (1.2).

The thesis wants to be a contribution to the theoretical understanding of the interval linear programming problematics, but will not study any practical applications of these theoretical results.

[1]See section Notation for the definition of the solution function.

1.4 Notation

We shall use the following notation:

Notation	Description	Most common examples
Scalar	lower-case letters with or without subscript(s), greek letters with or without a subscript, upper-case letter(s) in braces with two subscripts	i, j, k, b_i, c_i, x_i, a_{ij}, ε, ε_1, α, β, λ, $(A^{-1})_{ij}$, $(AB)_{ij}$
Vector	lower-case letters with or without an overline, an underline, a tilde, with or without a superscript	x, $\overline{x}$, $\widetilde{x}$, y, $\overline{y}$, z, $\overline{z}$, b, $\overline{b}$, $\underline{b}$, $\widetilde{b}$, c, $\overline{c}$, $\underline{c}$, $\widetilde{c}$
Vector interval	bold lower-case letter	$\boldsymbol{b}$, $\boldsymbol{c}$
Matrix	upper-case letter with or without an overline, an underline, a tilde, a subscript	A, $\overline{A}$, $\underline{A}$, $\widetilde{A}$, A_i
Matrix interval	bold upper-case letter	$\boldsymbol{A}$
Set	upper-case letter with or without a tilde, a subscript	M, M_1, M_ε, $\widetilde{M}$, B, N

Matrix $A = (a_{ij})$. For two matrixes A, B of the same size, inequalities like $A \leq B$ and $A < B$ are understood componentwise, i.e. $A < B$ if and only if $\forall_{ij} a_{ij} < b_{ij}$. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. A is called nonnegative if $0 \leq A$, A^T is transpose of A. The same notations also apply to vectors, which are always considered one-column matrixes. The norm of a vector $x = (x_1, \ldots, x_n)$ is defined as $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

We will use (A_1, A_2) for an open matrix interval, $\langle A_1, A_2\rangle$ for a closed matrix interval. For set comparison we will use $\subseteq$ for a subset and $\subset$ for a proper subset, i.e. $M_1 \subset M_2$ if and only if $M_1 \subseteq M_2$ and $M_1 \neq M_2$.

Unless said otherwise, it is always assumed that $A, \underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $b, \underline{b}, \overline{b} \in \mathbb{R}^m$ and $c, \underline{c}, \overline{c} \in \mathbb{R}^n$, where m and n are positive integers $m \leq n$. We also assume, if not said otherwise, that $\boldsymbol{A} = \langle \underline{A}, \overline{A}\rangle$, $\boldsymbol{b} = \langle \underline{b}, \overline{b}\rangle$, $\boldsymbol{c} = \langle \underline{c}, \overline{c}\rangle$, $\underline{A} < \overline{A}$, $\underline{b} < \overline{b}$, $\underline{c} < \overline{c}$ and A is a fixed matrix $\underline{A} < A < \overline{A}$ with the maximum rank, i.e. $rank(A) = m$. A system of linear equations $Ax = b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution.

For an arbitrary matrix $D \in \mathbb{R}^{m\times n}$, $m \geq 1$, $n \geq 1$ we define an open neighborhood $U(D)$ as an open matrix interval $U(D) = (D_1, D_2)$, $U(D) \subseteq \mathbb{R}^{m\times n}$, $D_1 < D < D_2$.

Definition 1. *Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The problem*

$$\textit{minimize } c^T x \tag{1.3}$$

subject to (s. t.)

$$Ax = b, x \geq 0 \tag{1.4}$$

is called a linear programming problem, *or simply a* linear program. *We shall write the problem (1.3), (1.4) briefly as*

$$\textit{Min } \left\{c^T x; Ax = b, x \geq 0\right\}. \tag{1.5}$$

Notice the use of the upper case in "Min" to denote a problem in contrast to "min" which denotes a minimum when applicable. A vector x satisfying (1.4) is called a *feasible solution* of (1.5). A problem (1.5) having a feasible solution is said to be *feasible*, and *infeasible* in the opposite case.

Definition 2. *For a given linear program (1.5) we introduce the value*

$$f(A, b, c) \equiv inf \left\{c^T x; Ax = b, x \geq 0\right\} \tag{1.6}$$

and we shall call it the optimal value *of (1.5).*

The optimal value (1.6) of a linear programming problem (1.5) can obtain the following values:

∞	if a linear programming problem (1.5) is infeasible,
$-\infty$	if a set of feasible solutions of a linear programming problem (1.5) is unbounded and contains a half-line along which the value of $c^T x$ tends to $-\infty$,
finite	every other case.

Definition 3. *Given $\boldsymbol{A} = \langle \underline{A}, \overline{A} \rangle$, $\boldsymbol{b} = \langle \underline{b}, \overline{b} \rangle$, $\boldsymbol{c} = \langle \underline{c}, \overline{c} \rangle$, $\underline{A} \leq \overline{A}$, $\underline{b} \leq \overline{b}$, $\underline{c} \leq \overline{c}$. The problem*

$$\textit{minimize } c^T x \tag{1.7}$$

subject to (s. t.)

$$Ax = b, x \geq 0, \tag{1.8}$$

where

$$A \in \boldsymbol{A},\ b \in \boldsymbol{b} \textit{ and } c \in \boldsymbol{c} \tag{1.9}$$

is called an interval linear programming problem, or simply an interval linear program. We shall write the problem (1.7), (1.8), (1.9) briefly as

$$\textit{Min } \left\{c^T x; Ax = b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\right\}. \tag{1.10}$$

Definition 4. *For an interval linear programming problem (1.10) we introduce a function $f(A,b,c) : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty, -\infty\}$ in the following way*

$$f(A,b,c) \equiv inf\left\{c^T x; Ax = b, x \geq 0\right\}, \tag{1.11}$$

where

$$A \in \boldsymbol{A},\ b \in \boldsymbol{b} \text{ and } c \in \boldsymbol{c}$$

and we shall call it the solution function *of (1.10).*

As we can see, the *solution function* in the point (A,b,c) is the *optimal value* of the linear program (1.5).

1.5 Basis Solution

Let be $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $m < n$, $rank(A) = m$ and $Ax = b$. Then columns of the matrix A and of the vector x elements can be rearranged into the form $A = (A_B A_N)$ and $x = (x_B x_N)$ such that $(A_B A_N)(x_B x_N) = b$, $A_B \in \mathbb{R}^{m\times m}$, $A_N \in \mathbb{R}^{(n-m)\times m}$, $x_B \in \mathbb{R}^m$ $x_N \in \mathbb{R}^{n-m}$ and A_B is a regular matrix. B is an index set of the columns of the matrix A which create a regular matrix A_B. B is called the *basis* of matrix A. The rest of the column indexes of the matrix A are in the set $N = \{1,\dots,n\} \setminus B$.
Thus for the matrix A_B exists an inverse matrix and the system of linear equations $Ax = b$ can be rewritten into the form

$$\begin{aligned} Ax &= b, \\ (A_B A_N)(x_B x_N) &= b, \\ A_B x_B + A_N x_N &= b, \\ x_B &= A_B^{-1} b - A_B^{-1} A_N x_N. \end{aligned} \tag{1.12}$$

Chapter 2

Set of the Feasible Solutions

In this chapter, the symbols $\widetilde{A}$, $\widetilde{b}$ and $\widetilde{c}$ will be used for an arbitrary matrix and arbitrary vectors, $\widetilde{A} \in \mathbb{R}^{m\times n}$, $\widetilde{b} \in \mathbb{R}^m$ and $\widetilde{c} \in \mathbb{R}^n$. If not said otherwise, then $\underline{A} < \widetilde{A} < \overline{A}$, $\underline{b} < \widetilde{b} < \overline{b}$ and $\underline{c} < \widetilde{c} < \overline{c}$.

We will work with an interval linear programming problem

$$\text{Min } \{c^T x; Ax = b, x \geq 0, A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\}, \tag{2.1}$$

where

$$\boldsymbol{A} = \langle \underline{A}, \overline{A} \rangle,\ \boldsymbol{b} = \langle \underline{b}, \overline{b} \rangle,\ \boldsymbol{c} = \langle \underline{c}, \overline{c} \rangle,\ \underline{A} < \overline{A},\ \underline{b} < \overline{b} \text{ and } \underline{c} < \overline{c} \tag{2.2}$$

for which a basis B is given such that

$$\forall_{\widetilde{A}, \underline{A} < \widetilde{A} < \overline{A}} rank(\widetilde{A}_B) = m. \tag{2.3}$$

Let us consider an arbitrary $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, and $c \in \boldsymbol{c}$, which fulfill

$$\underline{A} < A < \overline{A},\ \underline{b} < b < \overline{b},\ \underline{c} < c < \overline{c}. \tag{2.4}$$

and a constant $h \in \mathbb{R}$, $h > 0$ sufficiently large. Meaning of this constant will be explained after Definition 6.

Observation 5. *If the matrix A does not have its maximum rank, i.e. $rank(A) < m$, then, in general, the solution function $f(A, b, c)$ is not continuous in the point (A, b, c).*

Example of such linear programming problem:

$$\text{Min } \left\{(0, -1)^T x; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x \geq 0\right\},$$

$$c = (0, -1), A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The set of feasible solutions of $Ax = b$ is the abscissa $((1,0),(0,1))$. Obviously $f(A,b,c) = -1$ and A is singular. We slightly change entries in A to create a nonsingular matrix $\widetilde{A}$:

$$\widetilde{A} = \begin{pmatrix} 1 & 1 \\ 1+\alpha & 1+\beta \end{pmatrix}, \alpha \neq \beta.$$

$\widetilde{A}x = b$ has exactly one solution $x = \left(\frac{\beta}{\beta-\alpha}, -\frac{\alpha}{\beta-\alpha}\right)$ and the solution function $f(\widetilde{A},b,c) = \frac{\alpha}{\beta-\alpha}$ (if the system of linear equations is feasible). For any small α and β with $\alpha \approx \beta$ and $\alpha \neq \beta$, $\widetilde{A}x = b$ is an ill-conditioned system and its solution is extremely dependent on changes in the matrix entries, e.g.

$$\alpha = 0.01, \beta = 0.0099, \text{ i.e. } \widetilde{A}_1 = \begin{pmatrix} 1 & 1 \\ 1.001 & 1.00099 \end{pmatrix},$$

$$\alpha = 0.001, \beta = 0.00099, \text{ i.e. } \widetilde{A}_2 = \begin{pmatrix} 1 & 1 \\ 1.0001 & 1.000099 \end{pmatrix}$$

and the only solution of both systems of linear equations $\widetilde{A}_1 x = b$ and $\widetilde{A}_2 y = b$ is $x = y = (-99, 100)$. However this is not a feasible solution, thus $f(\widetilde{A}_i, b, c) = \infty$. In this way, we can create sequence

$$\widetilde{A}_i = \begin{pmatrix} 1 & 1 \\ 1+10^{-2-i} & 1+10^{-2-i} - 10^{-4-i} \end{pmatrix}$$

with $\widetilde{A}_i \stackrel{i\to\infty}{\longrightarrow} A$. $\forall_i \widetilde{A}_i x = b$ has exactly one solution $x = (-99, 100)$. Because there is no feasible solution, $\forall_i f(\widetilde{A}_i, b, c) = \infty$ holds.

2.1 Definition of the Set M

Definition 6. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4) and a sufficiently large constant $h \in \mathbb{R}$, $h > 0$. Let be $\varepsilon \in \mathbb{R}$. We define the following sets:*

$$\begin{aligned} M &\equiv \{x \in \mathbb{R}^n; Ax = b, \forall_{i\in N} 0 \leq x_i \leq h\}, \\ M_0 &\equiv \{x \in \mathbb{R}^n; Ax = b, x \geq 0, \forall_{i\in N} x_i \leq h\}, \\ M_\varepsilon &\equiv \{x \in \mathbb{R}^n; Ax = b, x_B \geq \varepsilon, \forall_{i\in N} 0 \leq x_i \leq h\} \end{aligned}$$

and for arbitrary $\widetilde{A}$, $\widetilde{b}$, $\underline{A} < \widetilde{A} < \overline{A}$ *and* $\underline{b} < \widetilde{b} < \overline{b}$

$$\begin{aligned} \widetilde{M} &\equiv \left\{x \in \mathbb{R}^n; \widetilde{A}x = \widetilde{b}, \forall_{i\in N} 0 \le x_i \le h\right\}, \\ \widetilde{M_0} &\equiv \left\{x \in \mathbb{R}^n; \widetilde{A}x = \widetilde{b}, x \ge 0, \forall_{i\in N} x_i \le h\right\}, \\ \widetilde{M_\varepsilon} &\equiv \left\{x \in \mathbb{R}^n; \widetilde{A}x = \widetilde{b}, x_B \ge \varepsilon, \forall_{i\in N} 0 \le x_i \le h\right\}. \end{aligned}$$

Further in this text we will use the following letters and numbers as indexes of M:

- greek letters - M_α, M_β, M_λ - meaning as defined in Definition 6, e.g.

$$M_\alpha \equiv \{x; Ax = b, x_B \ge \alpha, \forall_{i\in N} 0 \le x_i \le h\},$$

- integer numbers (except 0) - M_1, M_2 - meaning of a set without any special property.

Our attention is focused on a set of feasible solutions of a linear programming problem and its reaction on changes in the entries in the matrix A and in the vector b. The set of feasible solutions is, in general, unbounded and its reaction on changes in the entries in the matrix A and in the vector b can not be bounded, too. Therefore we defined the "set of feasible solutions" M, and in this way we bounded nonbasis variables x_N by a constant h. In the following text we will study reaction of the "set of feasible solutions" M on changes in the entries in the matrix A and in the vector b.

Observation 7. *For any* A, b *it holds*

$$\varepsilon_1 \le \varepsilon_2 \Rightarrow M_{\varepsilon_1} \supseteq M_{\varepsilon_2}. \tag{2.5}$$

Lemma 8. *Let* $M \neq \emptyset$ *has its meaning from Definition 6. Then* M *is a convex, bounded and closed set.*

Proof. M is defined as

$$M = \{x; Ax = b, \forall_{i\in N} 0 \le x_i \le h\}.$$

We prove all three properties separately:

- M is convex:
  Let $x, y \in M$ and $z = \rho x + (1-\rho)\, y$, where ρ is an arbitrary real number

$0 < \rho < 1$. Because of $0 \leq x_N \leq (h, \ldots, h)$ and $0 \leq y_N \leq (h, \ldots, h)$, we have

$$\begin{aligned} z_N &= \rho x_N + (1-\rho)\, y_N \geq \rho 0 + (1-\rho)\, 0 = 0, \\ z_N &= \rho x_N + (1-\rho)\, y_N \leq \rho(h, \ldots, h) + (1-\rho)\,(h, \ldots, h) = (h, \ldots, h) \end{aligned}$$

and

$$Az = A\,(\rho x + (1-\rho)\, y) = \rho Ax + (1-\rho)\, Ay = \rho b + (1-\rho)\, b = b.$$

Thus $z \in M$.

- M is bounded:
  $0 \leq x_N \leq (h, \ldots, h)$ and
  $$x_B = A_B^{-1} b - A_B^{-1} A_N x_N,$$
  therefore $\exists_{k,l} (k, \ldots, k) \leq x_B \leq (l, \ldots, l)$ and M is bounded.

- M is closed:
  $\{x; Ax = b\}$ is a shifted vector space, thus it is a closed set. Because the intersection of a finite number of closed sets is a closed set, we proved that
  $$M = \{x; Ax = b\} \cap \bigcap_{i \in N} \{0 \leq x_i \leq h\}$$
  is closed.

□

Assertion 9. *Let $\varepsilon \in \mathbb{R}$ be arbitrary and M_ε be a set defined in Definition 6. If $M_\varepsilon \neq \emptyset$, then M_ε is a convex, bounded and closed set.*

Proof. For any $\varepsilon \in \mathbb{R}$, M_ε can be defined also as

$$M_\varepsilon = M \cap \bigcap_{i \in B} \{x; x_i \geq \varepsilon\}$$

and we prove all three properties separately:

- M_ε is convex:
  M is convex, $\{x; x_i \geq \varepsilon\}$ is also convex for $i \in B$ and the intersection of finite number of convex sets is a convex set, too.

- M_ε is bounded:
  M is bounded and the intersection of a bounded set with any set is a bounded set, too.

- M_ε is closed:
  M is a closed set, $\{x; x_i \geq \varepsilon\}$ is a closed half-space for $i \in B$ and the intersection of a finite number of closed sets is a closed set.

□

2.2 Study of M

In this section we will study a reaction of the set M from Definition 6 on changes in the entries in the matrix A and in the vector b.

Lemma 10. *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ be arbitrary matrixes. Then*

$$\forall_{\varepsilon>0}\exists_{U(A),U(B)}\forall_{\widetilde{A}\in U(A),\widetilde{B}\in U(B)}\forall_{ij}\left|(AB)_{ij} - \left(\widetilde{A}\widetilde{B}\right)_{ij}\right| < \varepsilon.$$

Proof. Let

$$\begin{aligned}
\Delta_A &\equiv \widetilde{A} - A,\\
\Delta_B &\equiv \widetilde{B} - B.
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{A}\widetilde{B} &= AB + A\Delta_B + \Delta_A B + \Delta_A\Delta_B,\\
\left|\widetilde{A}\widetilde{B} - AB\right| &= \left|A\Delta_B + \Delta_A B + \Delta_A\Delta_B\right|,\\
\left|\widetilde{A}\widetilde{B} - AB\right| &\leq \left|A\Delta_B\right| + \left|\Delta_A B\right| + \left|\Delta_A\Delta_B\right|,\\
\left|\widetilde{A}\widetilde{B} - AB\right| &\leq |A|\left|\Delta_B\right| + \left|\Delta_A\right||B| + \left|\Delta_A\right|\left|\Delta_B\right|.
\end{aligned}$$

Let $U(A) = (A - \varepsilon_A E_A, A + \varepsilon_A E_A)$ and $U(B) = (B - \varepsilon_B E_B, B + \varepsilon_B E_B)$, where $\varepsilon_A > 0$, $\varepsilon_B > 0$, $E_A \in \mathbb{R}^{m\times n}$, $E_B \in \mathbb{R}^{n\times p}$, $\forall_{ij}(E_A)_{ij} = 1$ and $\forall_{ij}(E_B)_{ij} = 1$. ε_A and ε_B will be specified further in the proof. Thus for an arbitrary choice of $\widetilde{A} \in U(A)$ and $\widetilde{B} \in U(B)$ holds

$$\begin{aligned}
\left|\Delta_A\right| &< \varepsilon_A E_A,\\
\left|\Delta_B\right| &< \varepsilon_B E_B.
\end{aligned}$$

Hence

$$\begin{aligned}\left|\widetilde{A}\widetilde{B}-AB\right| &< \varepsilon_B\left|A\right|E_B+\varepsilon_A E_A\left|B\right|+\varepsilon_A\varepsilon_B E_A E_B,\\ \left|(AB)_{ij}-\left(\widetilde{A}\widetilde{B}\right)_{ij}\right| &< \varepsilon_B\sum_k\left|a_{ik}\right|+\varepsilon_A\sum_k\left|b_{kj}\right|+\varepsilon_A\varepsilon_B n.\end{aligned}$$

With an appropriate choice of ε_A and ε_B, the proof will be finished. We want the following inequality

$$\left|(AB)_{ij}-\left(\widetilde{A}\widetilde{B}\right)_{ij}\right|<\varepsilon$$

to hold for each i and j. Therefore, for example, the following conditions need to be fulfilled for each i and j:

$$\begin{aligned}\varepsilon_B\sum_k\left|a_{ik}\right| &\le \frac{\varepsilon}{3},\\ \varepsilon_A\sum_k\left|b_{kj}\right| &\le \frac{\varepsilon}{3},\\ \varepsilon_A\varepsilon_B n &\le \frac{\varepsilon}{3}.\end{aligned}$$

So

$$\begin{aligned}&\forall_i\left(\varepsilon_B\sum_k\left|a_{ik}\right|\le\frac{\varepsilon}{3}\right),\\ &\forall_i\left(\varepsilon_B\le\frac{\varepsilon}{3\sum_k\left|a_{ik}\right|}\right),\\ &\varepsilon_B\le\min_i\left\{\frac{\varepsilon}{3\sum_k\left|a_{ik}\right|}\right\},\\ &\varepsilon_B\le\frac{\varepsilon}{3\max_i\left\{\sum_k\left|a_{ik}\right|\right\}},\end{aligned}$$

respectively

$$\begin{aligned}&\forall_j\left(\varepsilon_A\sum_k\left|b_{kj}\right|\le\frac{\varepsilon}{3}\right),\\ &\forall_j\left(\varepsilon_A\le\frac{\varepsilon}{3\sum_k\left|b_{kj}\right|}\right),\\ &\varepsilon_A\le\frac{\varepsilon}{3\max_j\left\{\sum_k\left|b_{kj}\right|\right\}},\end{aligned}$$

and

$$\varepsilon_A \varepsilon_B n \leq \frac{\varepsilon}{3},$$
$$\varepsilon_A \varepsilon_B \leq \frac{\varepsilon}{3n},$$
$$\varepsilon_A \leq \sqrt{\frac{\varepsilon}{3n}},$$
$$\varepsilon_B \leq \sqrt{\frac{\varepsilon}{3n}}.$$

Summarizing these facts we finish the proof with

$$\begin{aligned} \varepsilon_A &= \min\left\{\frac{\varepsilon}{3\max_j\left\{\sum_k |b_{kj}|\right\}}, \sqrt{\frac{\varepsilon}{3n}}\right\}, \\ \varepsilon_B &= \min\left\{\frac{\varepsilon}{3\max_i\left\{\sum_k |a_{ik}|\right\}}, \sqrt{\frac{\varepsilon}{3n}}\right\}. \end{aligned}$$

□

Lemma 11. *For a regular square matrix D it holds, that the entries in the matrix D^{-1} vary continuously with the entries in D.*

Proof. It is result of a well known linear algebra formula

$$\left(D^{-1}\right)_{ij} = \frac{(-1)^{i+j} \det D^{(j,i)}}{\det(D)},$$

where $D^{(j,i)}$ is the matrix created from the matrix D by deleting j-th row and i-th column. □

Theorem 12. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4) and M has its meaning from Definition 6. Let be $M \neq \emptyset$. Then*

$$\forall_{\varepsilon>0} \exists_{U(A),U(b)} \forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)}$$
$$\left(\left(\forall_{x\in M} \exists_{\widetilde{x}\in\widetilde{M}} \|x-\widetilde{x}\| < \varepsilon\right) \wedge \left(\forall_{\widetilde{x}\in\widetilde{M}} \exists_{x\in M} \|x-\widetilde{x}\| < \varepsilon\right)\right).$$

Proof. Let us consider arbitrary $\widetilde{A}$ and $\widetilde{b}$, such that $\underline{A} < \widetilde{A} < \overline{A}$, $\underline{b} < \widetilde{b} < \overline{b}$ holds. Let $A = (A_B A_N)$, $x = (x_B x_N)$, $\widetilde{A} = (\widetilde{A}_B \widetilde{A}_N)$ and $\widetilde{x} = (\widetilde{x}_B \widetilde{x}_N)$. Then

$$\begin{aligned} x_B &= A_B^{-1} b - A_B^{-1} A_N x_N, \\ \widetilde{x}_B &= \widetilde{A}_B^{-1} \widetilde{b} - \widetilde{A}_B^{-1} \widetilde{A}_N \widetilde{x}_N. \end{aligned}$$

Obviously x and $\widetilde{x}$ are uniquely identified by x_N and $\widetilde{x}_N$. We put

$$\widetilde{x}_N = x_N.$$

Thus

$$\begin{aligned}
\|x-\widetilde{x}\| &= \sqrt{\sum_{i=1}^{n}(x_i-\widetilde{x}_i)^2},\\
\|x-\widetilde{x}\| &= \sqrt{\sum_{i\in B}(x_i-\widetilde{x}_i)^2+\sum_{i\in N}(x_i-\widetilde{x}_i)^2},\\
\|x-\widetilde{x}\| &= \sqrt{\sum_{i\in B}(x_i-\widetilde{x}_i)^2+\sum_{i\in N}0^2},\\
\|x-\widetilde{x}\| &= \sqrt{\sum_{i\in B}(x_i-\widetilde{x}_i)^2},\\
\|x-\widetilde{x}\| &= \|x_B-\widetilde{x}_B\|. \qquad (2.6)
\end{aligned}$$

Furthermore

$$\begin{aligned}
\|x_B-\widetilde{x}_B\| &= \left\|A_B^{-1}b - A_B^{-1}A_Nx_N - \left(\widetilde{A}_B^{-1}\widetilde{b} - \widetilde{A}_B^{-1}\widetilde{A}_N\widetilde{x}_N\right)\right\|,\\
\|x_B-\widetilde{x}_B\| &= \left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b} + \widetilde{A}_B^{-1}\widetilde{A}_Nx_N - A_B^{-1}A_Nx_N\right\|,\\
\|x_B-\widetilde{x}_B\| &\le \left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| + \left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\|.
\end{aligned}$$

We want to choose neighborhoods $U(A)$ and $U(b)$ so that the following inequalities

$$\left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| < \frac{\varepsilon}{2}, \qquad (2.7)$$

$$\left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\| < \frac{\varepsilon}{2} \qquad (2.8)$$

will hold.

- Ad (2.7):
  From Lemma 10 for $\frac{\varepsilon}{2\sqrt{m}}$ there exist $U_1(A_B^{-1})$ and $U(b)$ such that for all $\widetilde{A}_B^{-1} \in U_1(A_B^{-1})$, all $\widetilde{b} \in U(b)$ and for all i the inequality

$$\left|\left(A_B^{-1}b\right)_i - \left(\widetilde{A}_B^{-1}\widetilde{b}\right)_i\right| < \frac{\varepsilon}{2\sqrt{m}}$$

holds. From Lemma 11 for $U_1(A_B^{-1})$ exists $U_1(A_B)$ such that for all $\widetilde{A}_B \in U_1(A_B)$ also $\widetilde{A}_B^{-1} \in U_1(A_B^{-1})$ holds. Hence

$$\begin{aligned}
\left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| &= \sqrt{\sum_{i=1}^{m}\left(\left(A_B^{-1}b\right)_i - \left(\widetilde{A}_B^{-1}\widetilde{b}\right)_i\right)^2}, \\
\left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| &< \sqrt{\sum_{i=1}^{m}\left(\frac{\varepsilon}{2\sqrt{m}}\right)^2}, \\
\left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| &< \sqrt{\sum_{i=1}^{m}\left(\frac{\varepsilon^2}{4m}\right)}, \\
\left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| &< \frac{\varepsilon}{2}. \qquad (2.9)
\end{aligned}$$

- Ad (2.8):
  We define ε_1 as

$$\varepsilon_1 = \frac{\varepsilon}{2h\,(n-m)\,\sqrt{m}}. \qquad (2.10)$$

From Lemma 10 for ε_1 exists $U_2(A_B^{-1})$ and $U(A_N)$ such that for all $\widetilde{A}_B^{-1} \in U_2(A_B^{-1})$, all $\widetilde{A}_N \in U(A_N)$ and for all i, j

$$\left|\left(A_B^{-1}A_N\right)_{ij} - \left(\widetilde{A}_B^{-1}\widetilde{A}_N\right)_{ij}\right| < \varepsilon_1 \qquad (2.11)$$

holds. From Lemma 11 for $U_2(A_B^{-1})$ there exist $U_2(A_B)$ such that for all $\widetilde{A}_B \in U_2(A_B)$ also $\widetilde{A}_B^{-1} \in U_2(A_B^{-1})$ holds. Therefore

$$\left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\| = \sqrt{\sum_{j=1}^{m}\left(\sum_{i\in N}\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)_{ji} x_i\right)^2}.$$

The inequality (2.11) implies that

$$\left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\| < \sqrt{\sum_{j=1}^{m}\left(\sum_{i=1}^{n-m}\varepsilon_1 h\right)^2}$$

and from (2.10) we obtain

$$\begin{aligned}
\left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\| &< \sqrt{m(n-m)^2{\varepsilon_1}^2h^2}, \\
\left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\| &< \frac{\varepsilon}{2}. \qquad (2.12)
\end{aligned}$$

Now we define $U(A_B) = U_1(A_B) \cap U_2(A_B)$. Let $U(A_B) = (\underline{A_B}, \overline{A_B})$ and $U(A_N) = (\underline{A_N}, \overline{A_N})$. Then $U(A) = ((\underline{A_B A_N}), (\overline{A_B A_N}))$. So for arbitrary $\widetilde{A} \in U(A)$ and $\widetilde{b} \in U(b)$

$$\|x_B - \widetilde{x}_B\| \leq \left\|A_B^{-1}b - \widetilde{A}_B^{-1}\widetilde{b}\right\| + \left\|\left(\widetilde{A}_B^{-1}\widetilde{A}_N - A_B^{-1}A_N\right)x_N\right\|.$$

holds. From the inequalities (2.9) and (2.12) it is clear that

$$\begin{aligned} \|x_B - \widetilde{x}_B\| &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \\ \|x_B - \widetilde{x}_B\| &< \varepsilon \end{aligned}$$

and from the equality (2.6) is easy to see that

$$\|x - \widetilde{x}\| < \varepsilon.$$

For given $\varepsilon > 0$ we defined $U(A)$, $U(b)$. If arbitrary $\widetilde{A} \in U(A)$ and $\widetilde{b} \in U(b)$ is given, then for given $y \in M$ we define

$$\begin{aligned} \widetilde{y}_B &\equiv \widetilde{A}_B^{-1}\widetilde{b} - \widetilde{A}_B^{-1}\widetilde{A}_N y_N, \\ \widetilde{y}_N &\equiv y_N. \end{aligned}$$

Thus

$$\widetilde{y} \equiv (\widetilde{y}_B \widetilde{y}_N) \in \widetilde{M}.$$

And for given $\widetilde{y} \in \widetilde{M}$ we define

$$\begin{aligned} y_B &\equiv A_B^{-1}b - A_B^{-1}A_N\widetilde{y}_N, \\ y_N &\equiv \widetilde{y}_N. \end{aligned}$$

Hence

$$y \equiv (y_B y_N) \in M.$$

We have proved previously in the proof that for such defined y and $\widetilde{y}$ ($y_N = \widetilde{y}_N$) the following inequality

$$\|y - \widetilde{y}\| < \varepsilon$$

holds. □

Observation 13. *Previous Theorem 12 can be also formulated in the following way:*
Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4) and M has its meaning from Definition 6. Let be $M \neq \emptyset$. Then

$$\forall_{\varepsilon>0}\exists_{U(A),U(b)}\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)}$$
$$\left(\left(\forall_{x\in M}\exists_{\widetilde{x}\in\widetilde{M}} \|x - \widetilde{x}\| < \varepsilon \wedge x_N = \widetilde{x}_N\right) \wedge \left(\forall_{\widetilde{x}\in\widetilde{M}}\exists_{x\in M} \|x - \widetilde{x}\| < \varepsilon \wedge x_N = \widetilde{x}_N\right)\right).$$

2.3 Study of M_ε

We continue to work with an interval linear programming problem defined in (2.1), (2.2), (2.3) and (2.4). In this section we assume that

$$\exists_{\varepsilon>0} M_\varepsilon \neq \emptyset. \tag{2.13}$$

The condition (2.13) can be reformulated into

$$\exists_{x\in M_0} x_B > 0$$

and it is necessary assumption and can not be left out.

Observation 14. *Given an interval linear programming problem defined in (1.10) with the solution function $f(A,b,c)$ from in (1.11). Let arbitrary $A \in \boldsymbol{A}$, $rank(A) = m$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$ be given and the condition (2.13) is not fulfilled, i.e.*

$$\forall_{\varepsilon>0} M_\varepsilon = \emptyset.$$

Then, in general, the solution function is not continuous in the point (A,b,c).

Example of such linear programming problem:

$$\text{Min } \left\{(1,0)^T x; (0,1)x = 0, x \geq 0\right\}$$

$A = (0,1)$, $b = (0)$, $c = (1,0)$ and $f(A,b,c) = 0$. We define sequences

$$A_i = \left(-\frac{1}{2^i}, 1\right), \; b_i = -\frac{1}{2^i}.$$

Then $A_i \overset{i\to\infty}{\longrightarrow} A$, $b_i \overset{i\to\infty}{\longrightarrow} b$ and

$$f(A_i,b,c) = \min\left\{(1,0)^T x; \left(-\frac{1}{2^i}, 1\right) x = -\frac{1}{2^i}, x \geq 0\right\} = 1$$

for every $i \geq 0$, but $f(A,b,c) = 0$.

Definition 15. *Let $M_1 \subseteq \mathbb{R}^n$, $M_2 \subseteq \mathbb{R}^n$ be closed bounded sets and $M_1 \supseteq M_2 \supset \emptyset$ or $M_2 \supseteq M_1 \supset \emptyset$. Then we define binary operation "Set distance" $|.,.| \to \mathbb{R}$ as*

$$|M_1, M_2| \equiv \max_{x\in M_1} \min_{y\in M_2} \|x - y\|. \tag{2.14}$$

Assertion 16. *Let $M_1 \subseteq \mathbb{R}^n$, $M_2 \subseteq \mathbb{R}^n$ be closed bounded sets and $M_1 \supseteq M_2 \supset \emptyset$. Then*

$$M_1 = M_2 \Leftrightarrow |M_1, M_2| = 0.$$

Proof.

$$\begin{gathered} M_1 = M_2 \Leftrightarrow \\ \Leftrightarrow M_1 \subseteq M_2 \Leftrightarrow \\ \Leftrightarrow \forall_{x \in M_1} x \in M_2 \Leftrightarrow \\ \Leftrightarrow \forall_{x \in M_1} \min_{y \in M_2} \|x - y\| = 0 \Leftrightarrow \\ \Leftrightarrow \max_{x \in M_1} \min_{y \in M_2} \|x - y\| = 0 \Leftrightarrow \\ \Leftrightarrow |M_1, M_2| = 0. \end{gathered}$$

□

Assertion 17. *Let M_1, M_2 be closed bounded sets and $M_1 \supseteq M_2 \supset \emptyset$. Then*

$$|M_1, M_2| \le d \Leftrightarrow \forall_{x \in M_1} \exists_{y \in M_2} \|x - y\| \le d.$$

Proof.

$$\begin{gathered} |M_1, M_2| \le d \Leftrightarrow \\ \Leftrightarrow \max_{x \in M_1} \min_{y \in M_2} \|x - y\| \le d \Leftrightarrow \\ \Leftrightarrow \forall_{x \in M_1} \min_{y \in M_2} \|x - y\| \le d \Leftrightarrow \\ \Leftrightarrow \forall_{x \in M_1} \exists_{y \in M_2} \|x - y\| \le d. \end{gathered}$$

□

Observation 18. *$|M_1, M_2|$ can be imagined as the minimum distance needed to get from an arbitrary point of M_1 to the set M_2.*

Properties of the operation "Set distance":

- it is not commutative operation, $|M_1, M_2| \ne |M_2, M_1|$.

  It follows from the definition of "Set distance" (2.14) as for every $M_2 \subseteq M_1$
  $$|M_2, M_1| = \max_{y \in M_2} \min_{x \in M_1} \|x - y\| = 0.$$

- $\forall_{M_1, M_2} |M_1, M_2| \ge 0$,

  because $\forall_{x,y} \|x - y\| \ge 0$.

- From the fact that the minimum from a subset is greater or equal then the minimum from the set and that the maximum from a superset is greater or equal then the maximum from the set we obtain

  $$M_1 \supseteq M_2 \supseteq M_3 \Rightarrow |M_1, M_2| \le |M_1, M_3| \wedge |M_2, M_3| \le |M_1, M_3| . \tag{2.15}$$

- if $M_1 \supset M_2$ then

$$|M_1, M_2| = \max_{x \in M_1 \setminus M_2} \min_{y \in M_2} \|x - y\| > 0, \tag{2.16}$$

because if $x \in M_1 \setminus M_2$ then $\forall_{y \in M_2} \|x - y\| > 0$ and $|M_2, M_2| = 0$.

Lemma 19. *Let $M_1 \subseteq \mathbb{R}^n$, $M_2 \subseteq \mathbb{R}^n$, $M_3 \subseteq \mathbb{R}^n$ be closed bounded sets and $M_1 \supseteq M_2 \supseteq M_3 \supset \emptyset$. Then*

$$|M_1, M_3| \leq |M_1, M_2| + |M_2, M_3|. \tag{2.17}$$

Proof. Let $d_1 = |M_1, M_2|$ and $d_2 = |M_2, M_3|$. Then

$$|M_1, M_2| = d_1 \Rightarrow \forall_{x \in M_1} \exists_{y \in M_2} \|x - y\| \leq d_1,$$
$$|M_2, M_3| = d_2 \Rightarrow \forall_{y \in M_2} \exists_{z \in M_3} \|y - z\| \leq d_2.$$

Hence

$$\forall_{x \in M_1} \exists_{z \in M_3} \|x - z\| \leq \|x - y\| + \|y - z\| \leq d_1 + d_2,$$
$$|M_1, M_3| \leq d_1 + d_2,$$
$$|M_1, M_3| \leq |M_1, M_2| + |M_2, M_3|.$$

□

This lemma can be, with a little exaggeration, called "triangle inequality".

From this moment $\varepsilon \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and M, M_ε, M_α, M_β, M_γ, M_λ are sets as defined in Definition 6. In the following proofs specific properties of M are used.

Theorem 20. *Let $\alpha \leq \beta < \gamma$ be such that $M_\beta \supset M_\gamma \supset \emptyset$. Then*

$$|M_\alpha, M_\beta| < |M_\alpha, M_\gamma|. \tag{2.18}$$

Proof. As $\alpha \leq \beta$, it is easy to see from (2.5) that $M_\alpha \supseteq M_\beta$. From $M_\alpha \supseteq M_\beta \supseteq M_\gamma$ (2.15) yields that

$$|M_\alpha, M_\beta| \leq |M_\alpha, M_\gamma|.$$

If $M_\alpha = M_\beta$, then (according to Assertion 16) it is clear, that

$$|M_\alpha, M_\beta| = 0 \text{ and } |M_\alpha, M_\gamma| = |M_\beta, M_\gamma|$$

and because $M_\beta \neq M_\gamma$, we have

$$|M_\beta, M_\gamma| > 0.$$

Hence

$$0 = |M_\alpha, M_\beta| < |M_\alpha, M_\gamma| = |M_\beta, M_\gamma|\,.$$

If $M_\alpha \neq M_\beta$, i.e. $M_\alpha \supset M_\beta$ then from (2.16) we see that

$$|M_\alpha, M_\beta| = \max_{x\in M_\alpha \setminus M_\beta} \min_{y\in M_\beta} \|x-y\|$$

and these maximum and minimum are attained at least in one point. Let's choose arbitrary one of such points $(\overline{x}, \overline{y})$. So

$$\|\overline{x}-\overline{y}\| = \max_{x\in M_\alpha \setminus M_\beta} \min_{y\in M_\beta} \|x-y\|\,,$$

where $\overline{x} \in M_\alpha \setminus M_\beta$ and $\overline{y} \in M_\beta$.
As $\overline{x} \in M_\alpha \setminus M_\beta$ and $M_\gamma \subset M_\beta$, we have $\overline{x} \in M_\alpha \setminus M_\gamma$.
Now the following inequality

$$\min_{z\in M_\gamma} \|\overline{x}-z\| > \|\overline{x}-\overline{y}\|\,,$$

will be proved, which implies that

$$\begin{aligned}
\min_{z\in M_\gamma} \|\overline{x}-z\| &> \max_{x\in M_\alpha \setminus M_\beta} \min_{y\in M_\beta} \|x-y\|\,,\\
\max_{x\in M_\alpha \setminus M_\gamma} \min_{z\in M_\gamma} \|x-z\| &> |M_\alpha, M_\beta|\,,\\
|M_\alpha, M_\gamma| &> |M_\alpha, M_\beta|
\end{aligned}$$

and its proof by a contradiction finishes the proof of the theorem.
Let be (for the contradiction) $\overline{z} \in M_\gamma$ such that

$$\|\overline{x}-\overline{z}\| = \min_{z\in M_\gamma} \|\overline{x}-z\| \leq \|\overline{x}-\overline{y}\|\,.$$

Then we define z (on the abscissa $\overline{x}\,\overline{z} \in M_\alpha$)

$$z = \rho\overline{x} + (1-\rho)\,\overline{z},$$

where $\rho \in (0,1)$. As M_α is a convex set from Assertion 9, $z \in M_\alpha$ for any $\rho \in (0,1)$. Because $\overline{x} \in M_\alpha \setminus M_\beta$ and $\overline{z} \in M_\gamma$, i.e.

$$\begin{aligned}
\overline{x}_B &\geq (\alpha,\dots,\alpha),\ \exists_{i\in B}\overline{x}_i < \beta,\\
\overline{z}_B &\geq (\gamma,\dots,\gamma)
\end{aligned}$$

and $\alpha \leq \beta < \gamma$, we can define such ρ that $z \in M_\beta \setminus M_\gamma$, i.e.

$$z_B \geq (\beta, \ldots, \beta), \; \exists_{i \in B} z_i < \gamma.$$

As $z \in M_\beta$ and

$$\begin{aligned} \|\overline{x} - z\| &< \|\overline{x} - \overline{z}\| \leq \|\overline{x} - \overline{y}\|, \\ \|\overline{x} - z\| &< \min_{y \in M_\beta} \|\overline{x} - y\|, \end{aligned}$$

we have the contradiction and the proof is finished. □

Corollary 21. *Previous Theorem 20 will be mostly used in the form: Let* $0 \leq \beta < \gamma$ *be such that* $M_\beta \supset M_\gamma \supset \emptyset$. *Then*

$$|M_0, M_\beta| < |M_0, M_\gamma|.$$

Observation 22. *Let the assumptions of previous Theorem 20 be fulfilled. If* $M_\alpha \supset M_\beta$, *then*

$$0 < |M_\alpha, M_\beta| < |M_\alpha, M_\gamma|.$$

Theorem 23. *Let* $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\alpha \leq \beta$ *and* $M_\beta \neq \emptyset$. *Then*

$$|M_\alpha, M_\lambda| \leq \frac{1}{2} |M_\alpha, M_\beta|, \tag{2.19}$$

where $\lambda = \frac{\alpha+\beta}{2}$.

Proof. As $\alpha \leq \lambda \leq \beta$, then $M_\beta \subseteq M_\lambda \subseteq M_\alpha$ and

$$\begin{aligned} |M_\alpha, M_\lambda| &= \max_{x \in M_\alpha} \min_{y \in M_\lambda} \|x - y\|, \\ |M_\alpha, M_\beta| &= \max_{x \in M_\alpha} \min_{y \in M_\beta} \|x - y\|. \end{aligned}$$

Let's choose arbitrary $x \in M_\alpha$ and define $\overline{y}$

$$\|x - \overline{y}\| \equiv \min_{y \in M_\beta} \|x - y\|, \tag{2.20}$$

$$\overline{z} \equiv \frac{1}{2}(x + \overline{y}).$$

Because $x \in M_\alpha$, $\overline{y} \in M_\beta \subseteq M_\alpha$ and M_α is a convex set from Assertion 9, also $\overline{z} \in M_\alpha$. Furthermore

$$\begin{aligned} x \in M_\alpha &\Rightarrow x_N \geq 0 \wedge x_B \geq (\alpha, \ldots, \alpha), \\ \overline{y} \in M_\beta &\Rightarrow \overline{y}_N \geq 0 \wedge \overline{y}_B \geq (\beta, \ldots, \beta). \end{aligned}$$

Thus

$$\begin{aligned}\overline{z} &= \frac{1}{2}(x+\overline{y}) \geq \frac{1}{2}((\alpha,\ldots,\alpha,0,\ldots,0)+(\beta,\ldots,\beta,0,\ldots,0)) = \\ &= \left(\frac{\alpha+\beta}{2},\ldots,\frac{\alpha+\beta}{2},0,\ldots,0\right) = (\lambda,\ldots,\lambda,0,\ldots,0), \\ \overline{z} &\geq (\lambda,\ldots,\lambda,0,\ldots,0)\end{aligned}$$

and therefore $\overline{z} \in M_\lambda$.
Moreover

$$\|x-\overline{z}\| = \left\|x - \frac{1}{2}(x+\overline{y})\right\| = \frac{1}{2}\|x-\overline{y}\|.$$

From $\overline{z} \in M_\lambda$ and (2.20) we obtain

$$\min_{z\in M_\lambda}\|x-z\| \leq \|x-\overline{z}\| = \frac{1}{2}\|x-\overline{y}\| = \frac{1}{2}\min_{y\in M_\beta}\|x-y\|.$$

And as x was an arbitrary vector from M_α, we conclude the proof with

$$\begin{aligned}\forall_{x\in M_\alpha} \qquad \min_{z\in M_\lambda}\|x-z\| &\leq \frac{1}{2}\min_{y\in M_\beta}\|x-y\|, \\ \max_{x\in M_\alpha}\min_{z\in M_\lambda}\|x-z\| &\leq \frac{1}{2}\max_{x\in M_\alpha}\min_{y\in M_\beta}\|x-y\|, \\ |M_\alpha, M_\lambda| &\leq \frac{1}{2}|M_\alpha, M_\beta|.\end{aligned}$$

□

Corollary 24. *Now we state two important cases of Theorem 23, which we will use later in this chapter.*

1. *Let $\varepsilon > 0$ and $M_{2\varepsilon} \neq \emptyset$. Then*

$$|M_0, M_\varepsilon| \leq \frac{1}{2}|M_0, M_{2\varepsilon}|$$

($\alpha = 0$, $\beta = 2\varepsilon$, $\lambda = \varepsilon$).

2. *Let $\varepsilon > 0$ and $M_\varepsilon \neq \emptyset$. Then*

$$|M_{-\varepsilon}, M_0| \leq \frac{1}{2}|M_{-\varepsilon}, M_\varepsilon|$$

($\alpha = -\varepsilon$, $\beta = \varepsilon$, $\lambda = 0$).

Lemma 25. *Let be $\varepsilon > 0$ and $M_\varepsilon \neq \emptyset$. Then*

$$|M_{-\varepsilon}, M_0| \leq |M_0, M_\epsilon| .$$

Proof. Corollary 24 provides that

$$|M_{-\varepsilon}, M_0| \leq \frac{1}{2} |M_{-\varepsilon}, M_\varepsilon|$$

and Lemma 19 ("triangle inequality") provides that

$$|M_{-\varepsilon}, M_\varepsilon| \leq |M_{-\varepsilon}, M_0| + |M_0, M_\varepsilon| .$$

Summarizing these facts we obtain

$$\begin{aligned} |M_{-\varepsilon}, M_0| &\leq \frac{1}{2} |M_{-\varepsilon}, M_\varepsilon| \leq \frac{1}{2} |M_{-\varepsilon}, M_0| + \frac{1}{2} |M_0, M_\varepsilon| , \\ \frac{1}{2} |M_{-\varepsilon}, M_0| &\leq \frac{1}{2} |M_0, M_\varepsilon| , \\ |M_{-\varepsilon}, M_0| &\leq |M_0, M_\varepsilon| . \end{aligned}$$

□

Definition 26. *For $M_0 \neq \emptyset$ we define*

- *function $d_+(\varepsilon)$*

$$d_+(\varepsilon) \equiv |M_0, M_\varepsilon| \qquad \forall (\varepsilon \geq 0 \wedge M_\varepsilon \neq \emptyset) ,$$

- *function $d_-(\varepsilon)$*

$$d_-(\varepsilon) \equiv |M_\varepsilon, M_0| \qquad \forall \epsilon \leq 0,$$

- *function $d(\varepsilon)$*

$$\begin{aligned} d(\varepsilon) &\equiv d_+(\varepsilon) \qquad \textit{for } \varepsilon \geq 0, \\ d(\varepsilon) &\equiv d_-(\varepsilon) \qquad \textit{for } \varepsilon < 0. \end{aligned}$$

Observation 27. *It is not important whether is $d(0)$ defined as $d_+(0)$ or $d_-(0)$, because*

$$d_+(0) = d_-(0) = |M_0, M_0| = 0.$$

Observation 28. *The functions d_+, d_-, d basic properties:*

- *$d(\varepsilon) \geq 0$ for $\forall \varepsilon$.*

- $d_+(\varepsilon_1)$, $d_-(\varepsilon_2)$ *are defined only for* $\varepsilon_1 \geq 0$ *and* $\varepsilon_2 < 0$, *when* $M_0 \neq \emptyset$.
- *Furthermore* $d_+(\varepsilon)$ *is defined only for* $\varepsilon > 0$ *for which* $M_\varepsilon \neq \emptyset$.
- $d_+(\varepsilon)$ *is a non-decreasing function because*

$$0 \leq \varepsilon_1 < \varepsilon_2 \Rightarrow M_0 \supseteq M_{\varepsilon_1} \supseteq M_{\varepsilon_2}$$

and according to (2.15) we have

$$\begin{aligned} |M_0, M_{\varepsilon_1}| &\leq |M_0, M_{\varepsilon_2}|, \\ d_+(\varepsilon_1) &\leq d_+(\varepsilon_2). \end{aligned}$$

- *Analogously,* $d_-(\varepsilon)$ *is a non-increasing function.*
- $d_+(\varepsilon)$ *has its finite left-handed limit in* $\varepsilon = 0$

$$0 \leq \lim_{\varepsilon \to 0_+} d_+(\varepsilon) < \infty,$$

because $d_+(\varepsilon)$ *is bounded below (lower bound* 0*) and is non-decreasing.*

- $d_-(\varepsilon)$ *has its finite right-handed limit in* $\varepsilon = 0$

$$0 \leq \lim_{\varepsilon \to 0_-} d_-(\varepsilon) < \infty,$$

because $d_-(\varepsilon)$ *is bounded below (lower bound* 0*) and is non-increasing.*

Lemma 29. *Let be* $\varepsilon_0 > 0$ *such that* $M_{\varepsilon_0} \neq \emptyset$. *Then the function* $d_+(\varepsilon)$ *is upper semi-continuous in* $\varepsilon = 0$ *and the function* $d_-(\varepsilon)$ *is lower semi-continuous in* $\varepsilon = 0$.

Proof. As $\exists_{\varepsilon_0 > 0} M_{\varepsilon_0} \neq \emptyset$, Observation 28 shows us that the function $d_-(\varepsilon)$ is defined for all $\varepsilon < 0$, the function $d_+(\varepsilon)$ is defined at least on the interval $\langle 0, \varepsilon_0 \rangle$, $d_+(\varepsilon)$ has a finite left-handed limit in $\varepsilon = 0$ and $d_-(\varepsilon)$ has a finite right-handed limit in $\varepsilon = 0$. Thus for any sequence of numbers $(\varepsilon_i) \to 0_+$ must

$$\lim_{i=1,\ldots,\infty} d_+(\varepsilon_i) = \lim_{\varepsilon \to 0_+} d_+(\varepsilon)$$

hold and for any sequence of numbers $(\varepsilon_j) \to 0_-$ must

$$\lim_{j=1,\ldots,\infty} d_-(\varepsilon_j) = \lim_{\varepsilon \to 0_-} d_-(\varepsilon).$$

hold. From the assumption of the lemma follows that, there exists $\varepsilon_0 > 0$ such that $M_{\varepsilon_0} \neq \emptyset$. We define the sequence (ε_i) as

$$\varepsilon_i = \frac{\varepsilon_0}{2^{i-1}} \text{ for } i = 1, \ldots, \infty.$$

Then $\lim_{i=1,\ldots,\infty} \varepsilon_i = 0$ and therefore

$$\lim_{\varepsilon \to 0_+} d_+(\varepsilon) = \lim_{i=1,\ldots,\infty} d_+(\varepsilon_i) = \lim_{i=1,\ldots,\infty} d_+\left(\frac{\varepsilon_0}{2^{i-1}}\right).$$

Since for $i \geq 1$ $\frac{\varepsilon_0}{2^i} = \frac{0+\frac{\varepsilon_0}{2^{i-1}}}{2}$ holds and $M_{\frac{\varepsilon_0}{2^{i-1}}} \supseteq M_{\varepsilon_0} \supset \emptyset$, Theorem 23 yields that

$$\left|M_0, M_{\frac{\varepsilon_0}{2^i}}\right| \leq \frac{1}{2}\left|M_0, M_{\frac{\varepsilon_0}{2^{i-1}}}\right|.$$

In other words

$$\begin{aligned} \left|M_0, M_{\varepsilon_{i+1}}\right| &\leq \frac{1}{2}\left|M_0, M_{\varepsilon_i}\right|, \\ d_+(\varepsilon_{i+1}) &\leq \frac{1}{2} d_+(\varepsilon_i), \\ d_+(\varepsilon_{i+1}) &\leq \frac{1}{2^i} d_+(\varepsilon_0). \end{aligned}$$

Hence

$$\lim_{i=1,\ldots,\infty} d_+(\varepsilon_i) \leq \lim_{i=1,\ldots,\infty} \frac{1}{2^{i-1}} d_+(\varepsilon_0) = 0$$

and this means that

$$\lim_{\varepsilon \to 0_+} d_+(\varepsilon) = 0.$$

The function $d_+(\varepsilon)$ is upper semi-continuous in $\varepsilon = 0$.
From the proof of upper semi-continuity we know that

$$\lim_{i=1,\ldots,\infty} d_+(\varepsilon_i) = 0$$

and for any sequence of numbers $(\varepsilon_j) \to 0_-$ must

$$\lim_{j=1,\ldots,\infty} d_-(\varepsilon_j) = \lim_{\varepsilon \to 0_-} d_-(\varepsilon)$$

hold. As $\forall_{i\geq 0} M_{\varepsilon_i} \supseteq M_{\varepsilon_0} \supset \emptyset$, we obtain from Lemma 25 that

$$\left|M_{-\varepsilon_i}, M_0\right| \leq \left|M_0, M_{\varepsilon_i}\right|,$$

i.e.

$$d_-(-\varepsilon_i) \leq d_+(\varepsilon_i).$$

Hence

$$0 \le \lim_{i=1,\dots,\infty} d_-(-\varepsilon_i) \le \lim_{i=1,\dots,\infty} d_+(\varepsilon_i) = 0,$$
$$\lim_{i=1,\dots,\infty} d_-(-\varepsilon_i) = 0$$

and it is easy to see that

$$\lim_{\varepsilon \to 0_-} d_-(\varepsilon) = 0.$$

The function $d_-(\varepsilon)$ is lower semi-continuous in $\varepsilon = 0$. □

Corollary 30. *The function $d(\varepsilon)$ is continuous in $\varepsilon = 0$.*

Corollary 31. *From the proof of previous Lemma 29, we conclude that if $\varepsilon > 0$ and $M_\varepsilon \neq \emptyset$, then $d_+(\varepsilon) \ge d_-(-\varepsilon)$ i.e.*

$$\forall_{\varepsilon>0, M_\varepsilon \neq \emptyset}\ d(\varepsilon) \ge d(-\varepsilon).$$

Assertion 32. *Let $\exists_{\varepsilon_0>0} M_{\varepsilon_0} \neq \emptyset$ and $\exists_{\overline{x} \in M_0} \exists_{i \in B} \overline{x}_i = 0$. Then*

$$\forall_{0<\varepsilon\le\varepsilon_0} d(\varepsilon) > 0.$$

Proof. As $\overline{x}_i = 0$ and $i \in B$, then $\forall_{0<\varepsilon\le\varepsilon_0} \overline{x} \notin M_\varepsilon$, because $\forall_{x\in M_\varepsilon}\ x_B \ge (\varepsilon, \dots, \varepsilon)$. Hence $M_0 \supset M_\varepsilon$ and therefore from Assertion 16 we obtain $|M_0, M_\varepsilon| > 0$, i.e. $d(\varepsilon) > 0$. □

Theorem 33. *Let M has its meaning from Definition 6 and $\exists_{\varepsilon_0>0} M_{\varepsilon_0} \neq \emptyset$. Then*

$$\forall_{\varepsilon>0} \exists_{\varepsilon_1>0, \varepsilon_2\ge 0} \big((\varepsilon_1 + \varepsilon_2 < \varepsilon) \wedge$$
$$\wedge (\forall x \in M_0 \setminus M_{\varepsilon_1}\ \exists y \in M_{\varepsilon_1}\ \|x - y\| \le \varepsilon_2) \wedge$$
$$\wedge (\forall x \in M_{-\varepsilon_1} \setminus M_0\ \exists y \in M_0\ \|x - y\| \le \varepsilon_2) \big).$$

Proof. We divide the proof into 2 independent cases. Each M, which fulfill the assumptions of the theorem, is included in exactly one case.

1. $\exists_{\varepsilon_0>0}\ M = M_{\varepsilon_0}$, i.e.

$$\exists_{\varepsilon_0>0}\ \forall_{x\in M}\ x_B \ge (\varepsilon_0, \dots, \varepsilon_0).$$

2. $\forall_{\varepsilon>0}\ M \neq M_\varepsilon$, i.e.

$$\exists_{x\in M}\ x_B \not> (0, \dots, 0).$$

We are given arbitrary $\varepsilon > 0$.

Ad 1)
From the definition of the set M we have $M \supseteq M_0 \supseteq M_{\varepsilon_0}$ and from the assumption of this case we have $M = M_{\varepsilon_0}$. Therefore $M = M_0 = M_{\varepsilon_0}$. We define ε_1 and ε_2 as

$$\varepsilon_2 \equiv \varepsilon_1 \equiv \min\left\{\frac{\varepsilon_0}{2}; \frac{\varepsilon}{3}\right\} > 0.$$

(In the place of $\frac{\varepsilon_0}{2}$ can be any number less than ε_0 and in the place $\frac{\varepsilon}{3}$ can be any number less than $\frac{\varepsilon}{2}$.)
Hence

$$\varepsilon_1 + \varepsilon_2 \leq \frac{2}{3}\varepsilon < \varepsilon.$$

As $0 < \varepsilon_1 < \varepsilon_0$ then $M_0 \supseteq M_{\varepsilon_1} \supseteq M_{\varepsilon_0}$ and $M_0 = M_{\varepsilon_0}$. Thus $M_0 = M_{\varepsilon_1}$. We also see that $M_{-\varepsilon_1} = M_0$, because

$$\forall_{x \in M}\ x_B \geq (\varepsilon_0, \ldots, \varepsilon_0) > (0, \ldots, 0).$$

Therefore $M_0 \setminus M_{\varepsilon_1} = \emptyset$, $M_{-\varepsilon_1} \setminus M_0 = \emptyset$ and the proof in this case is finished.

Ad 2)
From the theorem assumption exists $\varepsilon_0 > 0$ such that $M_{\varepsilon_0} \neq \emptyset$. Thus $d_-(\varepsilon)$ is defined for all $\varepsilon < 0$ and $d_+(\varepsilon)$ is defined at least on the interval $< 0, \varepsilon_0 >$. Let be $0 < \varepsilon_1 \leq \varepsilon_0$ for now, ε_1 will be defined exactly later in the proof. For ε_1 we define ε_2 as

$$\varepsilon_2 \equiv \max\{d(\varepsilon_1), d(-\varepsilon_1)\} > 0,$$

but since $M_{\varepsilon_1} \supseteq M_{\varepsilon_0} \supset \emptyset$ holds, we obtain from Corollary 31 that $d(\varepsilon_1) \geq d(-\varepsilon_1)$, which means that $\varepsilon_2 = d(\varepsilon_1)$.
As $d(\varepsilon)$ is continuous in $\varepsilon = 0$ and $d(0) = 0$, we see that

$$\varepsilon_2 = d(\varepsilon_1) \overset{\varepsilon_1 \to 0_+}{\longrightarrow} 0.$$

Hence we can define ε_1 with the following inequality

$$\exists_{\varepsilon_1, 0 < \varepsilon_1 \leq \varepsilon_0}\ d(\varepsilon_1) + \varepsilon_1 < \varepsilon,$$
$$\exists_{\varepsilon_1, 0 < \varepsilon_1 \leq \varepsilon_0, \varepsilon_2 \geq 0}\ \varepsilon_1 + \varepsilon_2 < \varepsilon.$$

Furthermore

$$d_+(\varepsilon_1) = d(\varepsilon_1) = \varepsilon_2 \quad \text{and} \quad d_-(-\varepsilon_1) \leq d_+(\varepsilon_1) = \varepsilon_2,$$
$$d_+(\varepsilon_1) = \varepsilon_2 \quad \text{and} \quad d_-(-\varepsilon_1) \leq \varepsilon_2.$$

That is the definition of the functions d_+ and d_- (Definition 26)

$$|M_0, M_{\varepsilon_1}| = \varepsilon_2 \quad \text{and} \quad |M_{-\varepsilon_1}, M_0| \leq \varepsilon_2.$$

Because of Assertion 17, we can rewrite the last line into

$$\begin{array}{ccc} \forall_{x\in M_0}\exists_{y\in M_{\varepsilon_1}} \|x-y\| \leq \varepsilon_2 & \text{and} & \forall_{x\in M_{-\varepsilon_1}}\exists_{y\in M_0} \|x-y\| \leq \varepsilon_2, \\ \forall_{x\in M_0\setminus M_{\varepsilon_1}}\exists_{y\in M_{\varepsilon_1}} \|x-y\| \leq \varepsilon_2 & \text{and} & \forall_{x\in M_{-\varepsilon_1}\setminus M_0}\exists_{y\in M_0} \|x-y\| \leq \varepsilon_2. \end{array}$$

(Note that $M_{-\varepsilon_1} \setminus M_0$ can be the empty set in some interval linear programming problems for some/all ε_1. However $M_0 \setminus M_{\varepsilon_1}$ is never $\emptyset$, because of the case 2 assumption that $M \neq M_\varepsilon$ for every $\varepsilon > 0$.) □

2.4 Study of M_0

Theorem 34. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4). M_0, $\widetilde{M}_0$ have their meaning from Definition 6. Let $\exists_{x\in M_0} x_B > 0$. Then*

$$\forall_{\varepsilon>0}\exists_{U(A),U(b)}\forall_{\widetilde{A}\in U(A),\widetilde{(b)}\in U(b)}$$
$$\left(\left(\forall_{x\in M_0}\exists_{y\in\widetilde{M_0}} \|x-y\| < \varepsilon\right) \wedge \left(\forall_{y\in\widetilde{M_0}}\exists_{x\in M_0} \|x-y\| < \varepsilon\right)\right). \quad (2.21)$$

Proof. Through the proof we will also use the sets M, $\widetilde{M}$, M_ε and $\widetilde{M}_\varepsilon$ as defined in Definition 6.
As $\exists_{x\in M_0} x_B > 0$, then $\exists_{\varepsilon>0} M_\varepsilon \neq \emptyset$ and $M \neq \emptyset$. From previous Theorem 33 we obtain $\varepsilon_1 > 0$ and $\varepsilon_2 \geq 0$ such that

$$\begin{aligned} (\varepsilon_1 + \varepsilon_2 < \varepsilon) \wedge \ & (\forall x \in M_0 \setminus M_{\varepsilon_1}\ \exists y \in M_{\varepsilon_1} \|x-y\| \leq \varepsilon_2) \ \wedge \\ \wedge \ & (\forall x \in M_{-\varepsilon_1} \setminus M_0\ \exists y \in M_0 \|x-y\| \leq \varepsilon_2) \end{aligned} \quad (2.22)$$

and from Theorem 12 for a given interval linear programming problem, where $\varepsilon = \varepsilon_1$, we obtain

$$\begin{aligned} \exists_{U(A),U(b)}\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)} \ & \left(\forall_{x\in M}\exists_{\widetilde{x}\in\widetilde{M}} \|x-\widetilde{x}\| < \varepsilon_1\right) \ \wedge \\ \wedge \ & \left(\forall_{\widetilde{x}\in\widetilde{M}}\exists_{x\in M} \|x-\widetilde{x}\| < \varepsilon_1\right). \end{aligned} \quad (2.23)$$

$U(A), U(b)$ will be our wanted intervals. We will prove that for any $\widetilde{A} \in U(A)$ and $\widetilde{b} \in U(b)$ the inequality in (2.21) holds.
Let arbitrary $\widetilde{A} \in U(A)$ and $\widetilde{b} \in U(b)$ be given.

1. Let arbitrary $x \in M_0$ be given.

- If $x \in M_{\varepsilon_1}$.
  Because $M_{\varepsilon_1} \subseteq M$, we obtain from (2.23) and Observation 13

$$\exists_{\widetilde{x} \in \widetilde{M}} \|x - \widetilde{x}\| < \varepsilon_1 < \varepsilon \quad \text{and} \quad x_N = \widetilde{x}_N.$$

  As $x \in M_{\varepsilon_1}$, then $x_B \geq (\varepsilon_1, \ldots, \varepsilon_1)$ and therefore $\widetilde{x}_B \geq (0, \ldots, 0)$ and $\widetilde{x} \in \widetilde{M}_0$.

- If $x \in M_0 \setminus M_{\varepsilon_1}$.
  Because $M_0 \subseteq M$, we obtain from (2.23) and Observation 13

$$\exists_{\widetilde{x} \in \widetilde{M}} \|x - \widetilde{x}\| < \varepsilon_1 < \varepsilon \quad \text{and} \quad x_N = \widetilde{x}_N.$$

  However $\widetilde{x} \geq 0$ is not ensured. Therefore from (2.22) for $x \in M_0 \setminus M_{\varepsilon_1}$ there exists $y \in M_{\varepsilon_1}$ such that $\|x - y\| \leq \varepsilon_2$. As $y \in M_{\varepsilon_1}$, then there exists $\widetilde{y} \in \widetilde{M}_0$ such that $\|y - \widetilde{y}\| < \varepsilon_1$ and

$$\|x - \widetilde{y}\| \leq \|x - y + y - \widetilde{y}\| \leq \|x - y\| + \|y - \widetilde{y}\|.$$

  As $\|x - y\| \leq \varepsilon_2$ and $\|y - \widetilde{y}\| < \varepsilon_1$, we see from (2.22) that

$$\|x - \widetilde{y}\| < \varepsilon_1 + \varepsilon_2 < \varepsilon.$$

Thus for arbitrary $x \in M_0$ we have

$$\exists_{\widetilde{x} \in \widetilde{M}_0} \|x - \widetilde{x}\| < \varepsilon.$$

2. Let arbitrary $\widetilde{x} \in \widetilde{M}_0$ be given.

- If $\widetilde{x} \in \widetilde{M}_{\varepsilon_1}$.
  Analogously to the previous case, because of (2.23) and Observation 13, we obtain

$$\exists_{x \in M} \|x - \widetilde{x}\| < \varepsilon_1 < \varepsilon \quad \text{and} \quad x_N = \widetilde{x}_N.$$

  As $\widetilde{x} \in \widetilde{M}_{\varepsilon_1}$, then $\widetilde{x}_B \geq (\varepsilon_1, \ldots, \varepsilon_1)$ and therefore $x_B \geq (0, \ldots, 0)$ and $x \in M_0$.

- If $\widetilde{x} \in \widetilde{M}_0 \setminus \widetilde{M}_{\varepsilon_1}$.
  Because $\widetilde{M}_0 \subseteq \widetilde{M}$, we obtain from (2.23) and Observation 13

$$\exists_{x \in M} \|x - \widetilde{x}\| < \varepsilon_1 < \varepsilon \quad \text{and} \quad x_N = \widetilde{x}_N.$$

  As $\widetilde{x} \in \widetilde{M}_0$, then $\widetilde{x}_B \geq (0, \ldots, 0)$. Therefore $x_B \geq (-\varepsilon_1, \ldots, -\varepsilon_1)$ and $x \in M_{-\varepsilon_1}$. We see from (2.22) for $x \in M_{-\varepsilon_1} \setminus M_0$ that

$$\exists_{y \in M_0} \|x - y\| \leq \varepsilon_2,$$

so

$$\|y-\widetilde{x}\| = \|y-x+x-\widetilde{x}\| \leq \|y-x\| + \|x-\widetilde{x}\|.$$

As $\|x-y\| \leq \varepsilon_2$ and $\|x-\widetilde{x}\| < \varepsilon_1$, (2.22) implies that

$$\|y-\widetilde{x}\| < \varepsilon_1 + \varepsilon_2 < \varepsilon.$$

Thus for arbitrary $\widetilde{x} \in \widetilde{M_0}$ we have

$$\exists_{x \in M_0} \|x-\widetilde{x}\| < \varepsilon.$$

□

Chapter 3

Continuity of the Solution Function

3.1 Definition of the Set $\mathbb{M}$

Definition 35. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4). Let M and M_0 have their meaning from Definition 6, i.e.*

$$\begin{aligned} M &= \{x \in \mathbb{R}^n; Ax = b, \forall_{i\in N} 0 \leq x_i \leq h\}, \\ M_0 &= \{x \in \mathbb{R}^n; Ax = b, x \geq 0, \forall_{i\in N} x_i \leq h\}. \end{aligned}$$

We define the following sets:

$$\begin{aligned} \mathbb{M} &\equiv \{x \in \mathbb{R}^n; Ax = b, x_N \geq 0\}, \\ \mathbb{M}_0 &\equiv \{x \in \mathbb{R}^n; Ax = b, x \geq 0\} \end{aligned}$$

and for arbitrary $\widetilde{A}$, $\widetilde{b}$, $\underline{A} < \widetilde{A} < \overline{A}$ and $\underline{b} < \widetilde{b} < \overline{b}$

$$\begin{aligned} \widetilde{\mathbb{M}} &\equiv \left\{x \in \mathbb{R}^n; \widetilde{A}x = \widetilde{b}, x_N \geq 0\right\}, \\ \widetilde{\mathbb{M}}_0 &\equiv \left\{x \in \mathbb{R}^n; \widetilde{A}x = \widetilde{b}, x \geq 0\right\}. \end{aligned}$$

From the definition, it is obvious that $\mathbb{M} \supseteq M$ and $\mathbb{M}_0 \supseteq M_0$ because M and M_0 have additional conditions in their definition. Analogously, $\widetilde{\mathbb{M}} \supseteq \widetilde{M}$ and $\widetilde{\mathbb{M}}_0 \supseteq \widetilde{M_0}$ for the same $\widetilde{A}$ and $\widetilde{b}$.

3.2 Study of $\mathbb{M}_0$

Lemma 36. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4). $\mathbb{M}_0$, $\widetilde{\mathbb{M}}_0$ have the meaning from Definition 35. M_0, $\widetilde{M_0}$*

have the meaning from Definition 6. Let $\exists_{x\in\mathbb{M}_0} x_B > 0$*. If* $\mathbb{M}_0$ *is bounded, then*

$$M_0 = \mathbb{M}_0$$

and

$$\exists_{U(A),U(b)} \forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)} \widetilde{M_0} = \widetilde{\mathbb{M}}_0.$$

Proof. We set h from Definition 6 as

$$h \equiv 5\max\left\{1, \max_{x_1,x_2\in\mathbb{M}_0} \|x_1 - x_2\|, \max_{x\in\mathbb{M}_0}\max_{i\in N} x_i\right\}. \tag{3.1}$$

Because $\mathbb{M}_0$ is a bounded set from the theorem assumptions, $h < \infty$. $M_0 \subseteq \mathbb{M}_0$ from the definition of $\mathbb{M}_0$ and

$$\forall_{x\in\mathbb{M}_0} x_N \leq \left(\frac{h}{5}, \ldots, \frac{h}{5}\right) < \left(\frac{h}{4}, \ldots, \frac{h}{4}\right) \tag{3.2}$$

from the definition of h. That immediately leads to

$$M_0 = \mathbb{M}_0.$$

From the theorem assumption there exists $\overline{x} \in \mathbb{M}_0$ such that $\overline{x}_B > 0$. We define

$$\varepsilon_0 \equiv \min_{i\in B} \overline{x}_i. \tag{3.3}$$

Obviously $\varepsilon_0 > 0$. We define our wanted intervals as $U(A)$, $U(b)$, which are provided by Theorem 12 for $\varepsilon = \varepsilon_0$:

$$\begin{aligned} \exists_{U(A),U(b)} \forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)} \quad & \Big(\left(\forall_{x\in M} \exists_{\widetilde{x}\in\widetilde{M}} \|x - \widetilde{x}\| < \varepsilon_0\right) \quad \wedge \\ & \wedge \quad \left(\forall_{\widetilde{x}\in\widetilde{M}} \exists_{x\in M} \|x - \widetilde{x}\| < \varepsilon_0\right) \Big). \end{aligned} \tag{3.4}$$

Let's have arbitrary $\widetilde{A} \in U(A)$ and $\widetilde{b} \in U(b)$ and corresponding $\widetilde{\mathbb{M}}_0$. We will prove that

$$\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \widetilde{x}_N \leq \left(\frac{h}{2}, \ldots, \frac{h}{2}\right). \tag{3.5}$$

That will implicate that $\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \widetilde{x} \in \widetilde{M_0}$ or in other words

$$\widetilde{\mathbb{M}}_0 = \widetilde{M_0}.$$

We will prove (3.5) in two steps, first we will prove

$$\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \left(\left(\widetilde{x}_N \leq \left(\frac{h}{2}, \ldots, \frac{h}{2}\right) \right) \vee \left(\exists_{i\in N} \widetilde{x}_i > h\right) \right). \tag{3.6}$$

For the contradiction, we assume that there exists $\widetilde{y} \in \widetilde{\mathbb{M}}_0$ such that $\widetilde{y}_N \leq (h, \ldots, h)$ and $\exists_{k \in N} \widetilde{y}_k > \frac{h}{2}$. Because $\widetilde{y}_N \leq (h, \ldots, h)$ and $\widetilde{y} \in \widetilde{\mathbb{M}}_0$, also $\widetilde{y} \in \widetilde{M}$. Therefore for $\widetilde{y}$ from (3.4) must exists $y \in M$, for which $\|y - \widetilde{y}\| < \varepsilon_0$ and $y_N = \widetilde{y}_N$ (Observation 13).
Furthermore $y \notin \mathbb{M}_0$, because $y_k = \widetilde{y}_k > \frac{h}{2}$ and from (3.2) we know that $\forall_{x \in \mathbb{M}_0} x_N < \left(\frac{h}{4}, \ldots, \frac{h}{4}\right)$. Thus $y_B \ngeq 0$.
But as $\|y - \widetilde{y}\| < \varepsilon_0$ and $\widetilde{y}_B \geq 0$, it holds that $y_B > (-\varepsilon_0, \ldots, -\varepsilon_0)$. We define z as follows

$$z \equiv \frac{1}{2}\overline{x} + \frac{1}{2}y.$$

Because $\overline{x} \in M$ and $y \in M$, from M convexity (Lemma 8), we have $z \in M$. Let's summarize what we know about $\overline{x}$ and y:

- $\overline{x}_B \geq (\varepsilon_0, \ldots, \varepsilon_0) > 0$ and $0 \leq \overline{x}_N < \left(\frac{h}{4}, \ldots, \frac{h}{4}\right)$ from (3.2) and (3.3).
- $y_B > (-\varepsilon_0, \ldots, -\varepsilon_0)$, $0 \leq y_N \leq (h, \ldots, h)$ and $y_k \geq \frac{h}{2}$.

Hence

$$\begin{aligned} z_k &= \frac{1}{2}\overline{x}_k + \frac{1}{2}y_k \geq \frac{1}{2}0 + \frac{1}{2}\frac{h}{2}, \\ z_k &\geq \frac{h}{4}, \end{aligned}$$

$$\begin{aligned} z_N &\geq \frac{1}{2}\overline{x}_N + \frac{1}{2}y_N \geq \frac{1}{2}0 + \frac{1}{2}0 \geq 0, \\ z_N &= \frac{1}{2}\overline{x}_N + \frac{1}{2}y_N \leq \frac{1}{2}\left(\frac{h}{4}, \ldots, \frac{h}{4}\right) + \frac{1}{2}(h, \ldots, h), \\ z_N &< (h, \ldots, h) \end{aligned}$$

and

$$\begin{aligned} z_B &= \frac{1}{2}\overline{x}_B + \frac{1}{2}y_B \geq \frac{1}{2}(\varepsilon_0, \ldots, \varepsilon_0) + \frac{1}{2}(-\varepsilon_0, \ldots, -\varepsilon_0), \\ z_B &\geq 0. \end{aligned}$$

This means that $z \in M_0$ and $z_k \geq \frac{h}{4}$. It is the contradiction to (3.2)and we have proven (3.6).

Now we are going to prove (3.5). For the contradiction let

$$\exists_{\widetilde{y} \in \widetilde{\mathbb{M}}_0} \exists_{j \in N} \widetilde{y}_j > \frac{h}{2}.$$

Applying (3.6), it must hold that

$$\exists_{\widetilde{y}\in\widetilde{\mathbb{M}}_0}\exists_{j\in N}\widetilde{y}_j > h.$$

For $\overline{x} \in M$ from (3.4) must exists $\widetilde{x} \in \widetilde{M}$, for which $\|\overline{x} - \widetilde{x}\| < \varepsilon_0$. As $\overline{x}_N = \widetilde{x}_N$ and $\overline{x}_B \geq (\varepsilon_0, \ldots, \varepsilon_0)$, it is evident that $\widetilde{x}_B \geq 0$. Thus $\widetilde{x} \in \widetilde{M_0} \subseteq \widetilde{\mathbb{M}}_0$. Whole abscissa $(\widetilde{x}\,\widetilde{y})$ is a part of $\widetilde{\mathbb{M}}_0$, because $\widetilde{\mathbb{M}}_0$ is a convex set. As $\widetilde{x}_N < (\frac{h}{4}, \ldots, \frac{h}{4})$ and $\widetilde{y}_j > h$, on this abscissa it must exist a point $\widetilde{z} \in \widetilde{\mathbb{M}}_0$ such that $\widetilde{z}_N \leq (h, \ldots, h)$ and $\exists_{i\in N}\widetilde{z}_i > \frac{h}{2}$. That is the contradiction to (3.6). Hence

$$\forall_{y\in\widetilde{\mathbb{M}}_0}\forall_{i\in N} y_i \leq \frac{h}{2}$$

and

$$\widetilde{\mathbb{M}}_0 = \widetilde{M_0}$$

holds for arbitrary $\widetilde{A} \in U(A)$ and $\widetilde{b} \in U(b)$. □

3.3 Continuity of the Solution Function

Theorem 37. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4). $\mathbb{M}_0$, $\widetilde{\mathbb{M}}_0$ have the meaning from Definition 35. Let $\exists_{x\in\mathbb{M}_0} x_B > 0$. If $\mathbb{M}_0$ is bounded, then*

$$\forall_{\varepsilon>0}\exists_{U(A),U(b)}\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{y\in\widetilde{\mathbb{M}}_0}\|x-y\| < \varepsilon\right) \wedge \left(\forall_{y\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0}\|x-y\| < \varepsilon\right)\right).$$

Proof. Arbitrary $\varepsilon > 0$ is given. The assumptions of previous Lemma 36 are fulfilled, so we obtain

$$M_0 = \mathbb{M}_0$$

and $U_1(A)$, $U_1(b)$ such that

$$\forall_{\widetilde{A}\in U_1(A),\widetilde{b}\in U_1(b)}\widetilde{\mathbb{M}}_0 = \widetilde{M_0}.$$

Theorem 34 provides us with $U_2(A)$ and $U_2(b)$ for ε such that

$$\forall_{\widetilde{A}\in U_2(A),\widetilde{b}\in U_2(b)}$$
$$\left(\left(\forall_{x\in M_0}\exists_{y\in\widetilde{M_0}}\|x-y\| < \varepsilon\right) \wedge \left(\forall_{y\in\widetilde{M_0}}\exists_{x\in M_0}\|x-y\| < \varepsilon\right)\right).$$

We define $U(A) = U_1(A) \cap U_2(A)$, $U(b) = U_1(A) \cap U_2(b)$ and conclude

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{y\in\widetilde{\mathbb{M}}_0}\|x-y\| < \varepsilon\right) \wedge \left(\forall_{y\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0}\|x-y\| < \varepsilon\right)\right).$$

□

Lemma 38. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4).* $\mathbb{M}_0$, $\widetilde{\mathbb{M}}_0$ *have the meaning from Definition 35. Let* $\exists_{x\in\mathbb{M}_0} x_B > 0$. *If* $\mathbb{M}_0$ *is bounded, then*

$$\forall_{\varepsilon>0}\exists_{U(A),U(b),U(c)}\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b),\widetilde{c}\in U(c)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{\widetilde{x}\in\widetilde{\mathbb{M}}_0}\left|c^Tx-\widetilde{c}^T\widetilde{x}\right|<\varepsilon\right)\wedge\left(\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0}\left|c^Tx-\widetilde{c}^T\widetilde{x}\right|<\varepsilon\right)\right).$$

Proof. The style of this proof is the same as the style of the proof of Lemma 10. Therefore the proof will not be done into details, like exact definition of ε_c and ε_x.
Arbitrary $\varepsilon > 0$ is given. Let

$$\begin{aligned}\Delta_c &\equiv \widetilde{c}-c,\\ \Delta_x &\equiv \widetilde{x}-x.\end{aligned}$$

Then

$$\begin{aligned}\left|c^Tx-\widetilde{c}^T\widetilde{x}\right| &= \left|c^Tx-(c+\Delta_c)^T(x+\Delta_x)\right|,\\ \left|c^Tx-\widetilde{c}^T\widetilde{x}\right| &= \left|c^T\Delta_x+\Delta_c^Tx+\Delta_c^T\Delta_x\right|,\\ \left|c^Tx-\widetilde{c}^T\widetilde{x}\right| &\le \left|c^T\right|\left|\Delta_x\right|+\left|\Delta_c^T\right|\left|x\right|+\left|\Delta_c^T\right|\left|\Delta_x\right|.\end{aligned} \tag{3.7}$$

We want

$$\left|c^Tx-\widetilde{c}^T\widetilde{x}\right|<\varepsilon.$$

to hold. One of more possibilities is

$$\begin{aligned}\left|c^T\right|\left|\Delta_x\right| &< \frac{\varepsilon}{3},\\ \left|\Delta_c^T\right|\left|x\right| &< \frac{\varepsilon}{3},\\ \left|\Delta_c^T\right|\left|\Delta_x\right| &< \frac{\varepsilon}{3}.\end{aligned}$$

As $x \in \mathbb{M}_0$ and $\mathbb{M}_0$ is bounded, then there exists a constant $H \in \mathbb{R}$ such that $|x| < (H,\ldots,H)$. Hence

$$\left|c^T\right|\left|\Delta_x\right| < \frac{\varepsilon}{3}, \tag{3.8}$$

$$\left|\Delta_c^T\right|(H,\ldots,H) < \frac{\varepsilon}{3}, \tag{3.9}$$

$$\left|\Delta_c^T\right|\left|\Delta_x\right| < \frac{\varepsilon}{3}. \tag{3.10}$$

Let $U(c) = (c-\varepsilon_c e, c+\varepsilon_c e)$, where $\varepsilon_c > 0$, $e \in \mathbb{R}^n$ and $\forall_i e_i = 1$. An exact value of ε_c can be calculated similarly as in Lemma 10 from (3.9) and (3.10).

From (3.8) and (3.10) we also define ε_x, where $\varepsilon_x > 0$, $|\Delta_x| < \varepsilon_x e$. Theorem 37 provides us with $U(A)$ and $U(b)$ for $\varepsilon = \varepsilon_x$

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \|x-\widetilde{x}\| < \varepsilon_x\right) \wedge \left(\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0} \|x-\widetilde{x}\| < \varepsilon_x\right)\right).$$

Hence

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} |x-\widetilde{x}| < \Delta_x\right) \wedge \left(\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0} |x-\widetilde{x}| < \Delta_x\right)\right).$$

That together with (3.7)-(3.10) yield that

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b),\widetilde{c}\in U(c)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \left|c^Tx-\widetilde{c}^T\widetilde{x}\right| < \varepsilon\right) \wedge \left(\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0} \left|c^Tx-\widetilde{c}^T\widetilde{x}\right| < \varepsilon\right)\right).$$

□

Theorem 39. *Given an interval linear programming problem defined in (2.1), (2.2), (2.3), (2.4). The solution function f is defined in (1.11), $\mathbb{M}_0$ has the meaning from Definition 35. Let $\exists_{x\in\mathbb{M}_0} x_B > 0$ and $\mathbb{M}_0$ be bounded. Then the function f is continuous in the point (A,b,c).*

Proof. We want to prove that

$$\forall_{\varepsilon>0}\exists_{U(A),U(b),U(c)}\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b),\widetilde{c}\in U(c)} \left|f(A,b,c)-f(\widetilde{A},\widetilde{b},\widetilde{c})\right| < \varepsilon.$$

Arbitrary $\varepsilon > 0$ is given. We can rewrite an interval linear programming problem (2.1) as

$$\text{Min }\left\{c^Tx; x\in\mathbb{M}_0, A\in\boldsymbol{A}, b\in\boldsymbol{b}, c\in\boldsymbol{c}\right\}.$$

From previous Lemma 38 we obtain $U(A)$, $U(b)$ and $U(c)$ for $\varepsilon > 0$ so that

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b),\widetilde{c}\in U(c)}$$
$$\left(\left(\forall_{x\in\mathbb{M}_0}\exists_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \left|c^Tx-\widetilde{c}^T\widetilde{x}\right| < \varepsilon\right) \wedge \left(\forall_{\widetilde{x}\in\widetilde{\mathbb{M}}_0}\exists_{x\in\mathbb{M}_0} \left|c^Tx-\widetilde{c}^T\widetilde{x}\right| < \varepsilon\right)\right).$$

Hence

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b),\widetilde{c}\in U(c)} \left|\min_{x\in\mathbb{M}} c^Tx - \min_{\widetilde{x}\in\widetilde{\mathbb{M}}_0} \widetilde{c}^T\widetilde{x}\right| < \varepsilon.$$

In another words

$$\forall_{\widetilde{A}\in U(A),\widetilde{b}\in U(b),\widetilde{c}\in U(c)} \left|f(A,b,c)-f(\widetilde{A},\widetilde{b},\widetilde{c})\right| < \varepsilon.$$

□

3.4 Note on the Assumption of Theorem 39

Theorem 39 will not hold without the assumption, that a set of feasible solutions $\mathbb{M}_0$ is bounded. However this assumption is not necessary, only sufficient. Example of a linear programming problem with unbounded $\mathbb{M}_0$ is

$$\text{Min } \left\{(0,-1)^T x; (0,1)x = 1, x \geq 0\right\}.$$

For the interval linear programming problem

$$\text{Min } \left\{(0,-1)^T x; ((-1,1),1)x = 1, x \geq 0\right\},$$

the solution function $f(A,b,c)$ is not continuous in the point $(A,b,c) = ((0,-1),(0,1),1)$. Another example of a linear programming problem with unbounded $\mathbb{M}_0$ is

$$\text{Min } \left\{(0,1)^T x; (1,-1)x = 0, x \geq 0\right\}.$$

For the interval linear programming problem

$$\text{Min} \left\{((0.9,1.1),(0.9,1.1))^T x; ((0.9,1.1),(-1.1,-0.9))x = (-0.1,0.1), x \geq 0\right\}$$

the solution function is continuous in every point of the intervals.

Chapter 4

Conclusion

The goal of the thesis was reached.

Theorem 34 is the main result of the work and is proved in Chapter 2. This theorem has two necessary assumptions - "$\widetilde{A}_B$ matrix regularity" (2.3) and $\exists_{x \in M_0} x_B > 0$ (M_0 has its meaning from Definition 6). The first assumption about the maximum rank of matrixes in the given interval was easy to see. "Regularity" is an important property of matrixes and was the first to check. But finding of the second assumption, the existence of $x \in M_0$ with $x_B > 0$, required much more effort (actually it was the most difficult part of the whole thesis), and a new view on interval linear programming problems needed to be introduced. This view clearly separates basis and nonbasis variables and it is based on the first assumption of the matrixes maximum rank. This approach also slightly modified the set of the feasible solutions[1] by bounding nonbasis variables. (Therefore the modified set of feasible solutions of (1.2) is bounded.) Stated approach leads gradually to Theorem 12, Theorem 33 and finally to Theorem 34.

Theorem 12 does not require the second assumption yet and it says that the set M varies "continuously" with the entries in the matrix A and in the vector b. However, the introduction of the nonnegativity conditions on the basis variables stopped any straightforward attempt to prove Theorem 34. Therefore an additional assumption had to be introduced and Theorem 33 about a dependency between M_0 and M_ε needed to be proved.

Theorem 34 says that the set M (the modified set of feasible solutions) varies "continuously" with the entries in the matrix A and in the vector b. This theorem would implicate that the solution function was continuous, if the set of feasible solutions was defined as in Definition 6.

[1]The modified sets of feasible solutions are M, M_0 (Definition 6). The sets of feasible solutions as known from linear programming are $\mathbb{M}$, $\mathbb{M}_0$ (Definition 35).

However, the set of feasible solutions in linear programming is defined as in Definition 35 and what was unexpected, the continuity of the solution function could not be proved without an additional assumption, that $\mathbb{M}_0$ is bounded. This assumption is different from the previous assumptions, as it is only a sufficient assumption, not necessary one (see the last section of Chapter 2 for examples). Defining exactly necessary assumptions for Theorem 39 is one of the topics for a future research, as well as a study of continuity of the solution function on an interval. These topics need to be studied in order to achieve a practical use of the theoretical results of this thesis.

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