Logical Foundations of Fuzzy Mathematics

Logické základy fuzzy matematiky

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Prohlašuji, že jsem disertační práci vypracoval samostatně s využitím uvedených pramenů a literatury.

I hereby declare that this dissertation is the result of my own work and that all sources have been duly acknowledged.

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Preface

This is a commentary associated with the author's PhD thesis in logic at the Faculty of Arts, Charles University in Prague. The thesis is based on papers containing results of five years of the author's research in logic-based fuzzy mathematics. The papers have been published in peer-reviewed international journals [14, 30, 34, 26, 28, 41], proceedings of international conferences [33, 31, 32, 16, 17, 19, 22, 18, 37, 43, 42, 44, 29] and other volumes [15, 21, 25, 35]. By the time of the submission of this thesis and according to the author's knowledge, the papers have been cited 25 times in peer-reviewed international journals and 15 times in edited volumes and proceedings of international conferences, excluding auto-citations and citations by co-authors. Two co-authored conference papers won the Best Paper [32] and Distinguished Student Paper [42] awards (respectively at the 11th IFSA World Congress and the 5th Conference of EUSFLAT).

The work is part of a larger project in formal fuzzy mathematics, which is still in progress (cf. the end of Section 4.1 below): several important topics in formal fuzzy mathematics are being investigated by my colleagues and myself, with results not yet complete for publication. Therefore it seemed more appropriate to present the results of this research in the form of a commented collection of papers, rather than to compile a monographic text, as at the time of submission the topic was still under permanent construction and re-construction and not yet ripe for a book-style presentation.

Due to the brevity or purely expository nature of some of the conference papers and the overlap of their topics with full journal articles, only the six journal papers [34, 30, 28, 41, 26, 14] and four of the proceedings papers [16, 19, 43, 42] have actually been included in the thesis. The author's contribution to co-authored papers is indicated in Section 4.2.

The thesis is organized as follows: In the cover text (Part I), Section 1 provides a general introduction to the area of research. A broader context and the state of the art upon which the thesis is based is described in Section 2. The main features of the approach developed in the thesis and the significance of the topic are discussed in Section 3. The author's own contribution to the topic and the papers included in the thesis are then described in Section 4. The author versions of the published papers constitute the main body of the thesis (Part II). The thesis is concluded by mandatory annexes (Part III).

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I have also benefited from discussions with many colleagues and kind advice given to me by senior researchers in the field. Among those whose remarks helped me a lot are (in alphabetical order and besides those listed above) Christian Fermüller, David Makinson, Jeff Paris, and several anonymous referees. Thanks are due to Petr Cintula, Petr Hájek, Tomáš Kroupa, and Carles Noguera for comments on a draft version of the cover chapter. Many other people have helped me during the course of my PhD study with scientific and organizational matters; I appreciate all help I have been given.

1 Introduction

Fuzzy mathematics is the study of fuzzy structures, or structures that involve *fuzziness*—i.e., such mathematical structures that at some points replace the two classical truth values 0 and 1 with a larger structure of *degrees.* Often, the real unit interval $[0, 1]$ is employed as the system of degrees, but other options are common as well—a finite set, an arbitrary lattice, an algebra of some kind or other. The degrees are intended to provide more flexibility to a fuzzy mathematical structure than the two truth degrees provide to the corresponding classical ("crisp") mathematical structure.

A simple example of a fuzzy mathematical structure is that of a *fuzzy set.* Instead of classical two-valued characteristic functions $\chi\colon X \to \{0, 1\}$, fuzzy sets employ real-valued membership functions $\mu\colon X \to [0, 1]$, where X is a fixed universe of discourse. While ordinary crisp sets clearly cut the elements of X between members and non-members, the richer system of degrees in fuzzy sets allows modeling *gradual change* between membership and non-membership.

Since the introduction of fuzzy sets by Zadeh [212] in 1965, a plethora of fuzzy mathematical structures have been proposed and investigated in the literature. The degrees that replace the classical truth values 0 and 1 (usually called *membership degrees,* as they serve as values of membership functions) can appear at various places in such fuzzy structures. For instance in *fuzzy topology,* crisp families of open fuzzy sets ("fuzzy topologies"), fuzzy families of open crisp sets ("fuzzifying topologies"), and fuzzy families of open fuzzy sets ("bifuzzy topologies") have all been investigated [55, 208, 130]. The membership degrees themselves can form various structures. For example, the original notion of $[0,1]$-valued fuzzy set was soon generalized to L-valued fuzzy sets for L an arbitrary lattice [95], and even more general (e.g., poset-valued) fuzzy sets and fuzzy structures are studied [184].

The large freedom in defining fuzzy mathematical notions correlates with freedom in interpreting the informal meaning of membership degrees. Depending on the intended interpretation, various structures of membership degrees and various definitions of fuzzy mathematical notions are appropriate. Vice versa, particular structures of membership degrees and particular definitions of fuzzy mathematical notions admit only some of all possible informal interpretations and applications of the fuzzified theory. Unfortunately, this fact is seldom reflected in the practice of the fuzzy community. The omission of such considerations can result in arbitrariness of definitions, inappropriateness of applications, and completely unclear methodology, for all of which fuzzy mathematics has often (and in many cases quite justly) been reproached and disrespected by the mainstream mathematical community.

The present work is not intended to contribute to the chaotic and methodologically confused development of the broad area of fuzzy mathematics. Instead, from the mixture of possible interpretations of membership degrees it selects one particular interpretation which has already been clarified enough to support a methodologically sound development: namely, the interpretation of membership degrees as degrees of *comparative truth,* which is studied by deductive fuzzy logic. Our approach to fuzzy mathematics can thus be characterized as *logic-based.*[1] More detailed methodological considerations justifying this approach have been presented in [26]; here we only stress the most important points.

Deductive (or *formal, symbolic, mathematical*) fuzzy logic follows the modus operandi of classical logic. Without necessarily claiming that the philosophical notion of truth as such is (or is not) many-valued, it employs semantical models that assign intermediary truth degrees to propositions. In deductive fuzzy logic, like in fuzzy mathematics in general, a richer structure of truth degrees enables to model gradual change between truth and falsity, which seems appropriate in many real-life situations. The interpretation of membership degrees in terms of truth, moreover, allows studying transmission of truth degrees in formalized arguments, in the same way as classical logic studies transmission of bivalent truth.

The study of transmission of partial truth (in the technical sense of "partial truth" as the graded quality preserved in sound arguments) is what in fact distinguishes *deductive* fuzzy logic from *traditional* fuzzy logic. Fuzzy logic in the traditional sense has emerged soon after the introduction of fuzzy sets [96], by generalizing the obvious correspondence between elementary set operations and logical connectives. If the transmission of partial truth is not taken into account, there are as many ways to define fuzzy logical connectives as there are possibilities for pointwise elementary fuzzy set operations. This makes traditional fuzzy logic subject to the same criticisms as fuzzy mathematics as a whole, especially for arbitrariness and unclear methodology. Moreover for most choices of logical

[1] Other approaches to fuzzy mathematics exist, some of them quite far developed—for example category-theoretical (see [106]) or sheaf-theoretical (see [129]). As far as they can address the methodological issues hinted at above, they provide a legitimate grounding for those branches of fuzzy mathematics that are compatible with their methodological assumptions. The logic-based approach then complements rather than rivals such approaches.

connectives, the resulting logical systems have very poor logical properties, which leads to additional criticism from the point of view of formal logic.

It can nevertheless be shown [26, 22] that if certain basic principles governing the transmission of partial truth are observed, the resulting logical systems are well-behaved and well-motivated. These principles narrow down the choice of real-valued logical connectives to a class based on left-continuous t-norms and described by deductive systems of t-norm based fuzzy logics [110, 81]. These logics not only model the gradual change from truth to falsity like other kinds of fuzzy logic, but also have a "deductive face" and belong in a well-explored and well-behaved class of substructural logics [173, p. 208].

Since Hájek's monograph [110], various propositional and first-order systems of deductive fuzzy logic have been defined and intensively studied. Nowadays the discipline is developed to the point that it is reasonable to construct and study axiomatic mathematical theories within the formal framework of deductive fuzzy logic. A systematic development of axiomatic fuzzy mathematics based on deductive fuzzy logic has been proposed as a research program in [34]; the present work can be viewed as a report on its implementation.

The strategy proposed in [34] was to utilize the similarities between deductive fuzzy logics and classical logic and employ the architecture that has proved useful in foundations of classical mathematics. The classical foundational approach consists in developing a sufficiently rich foundational theory that would harbor all (or almost all) other mathematical theories. In classical mathematics, the role of a foundational theory can be assumed, e.g., by some variant of set theory, type theory, or category theory. For the foundations of logic-based fuzzy mathematics, [34] proposes a fuzzy variant of Russell-style simple type theory that has been introduced in [30]. It can equivalently be characterized as *Henkin-style higher-order fuzzy logic* or a typed theory of cumulative fuzzy classes (i.e., Zadeh's fuzzy sets of all finite orders). The apparatus of this foundational theory, also called Fuzzy Class Theory or FCT, is described in detail in [30, 31, 32, 35]; its methodological issues are further discussed in [26, 37]. The basics of the theory of fuzzy sets and relations, which are the prerequisites of all other branches of fuzzy mathematics, are developed in [30, 28, 41, 19]. Some more advanced topics of fuzzy mathematics have already been developed, too, including the (graded) theory of fuzzy lattices [17, 15], fuzzy intervals [16, 134], aggregation operators [64, 29], fuzzy filters [149], and fuzzy topology [43, 42, 44]. In [14, 38, 25], the apparatus is applied in metamathematics of fuzzified versions of other non-classical logics.

The results achieved so far have already demonstrated that this style of development of fuzzy mathematics is viable and can facilitate generalizing known theorems as well as discovering new results. Indeed, from the point of view of formal logic the methodology and foundational structure of the theory is quite standard and straightforward. On the other hand, from the point of view of traditional fuzzy mathematics the theory presents a radical shift of paradigm, embraced till now by very few authors (for notable exceptions see Section 2). The main reasons justifying the development of logic-based fuzzy mathematics are described in Section 3 below.

There are many open questions and areas for future research in formal fuzzy mathematics, as well as problems of philosophical and methodological nature. Even though these problems are still distant from the applied practice or topics of mainstream interest, their solution can give us better understanding of the phenomenon of gradedness and its role, as well as possible applications.

For the adequate perspective on the present work with respect to the whole of fuzzy mathematics, it is necessary to keep in mind the methodological restrictions of the logic-based approach. The definite interpretation of membership degrees as degrees of truth transmitted under inference leads on the one hand to methodological clarity, but on the other hand it restricts meaningful definitions to those compatible with the deductive paradigm, and limits the scope of applicability of the results. For instance, as mentioned above and as shown in more detail in [26], the principles of deductive fuzzy logic restrict the choice of the conjunction connective on the interval $[0,1]$ to left-continuous t-norms. Consequently, the operation of fuzzy set intersection can meaningfully be defined only by means of such conjunctions.[2] Other possible notions of intersection that may be meaningful in broader fuzzy mathematics, for instance those based on aggregation operators different from left-continuous t-norms, may well be definable in a sufficiently strong higher-order fuzzy logic (e.g., higher-order logic ŁΠ), but are *ill-motivated* from the point of view of logic-based fuzzy mathematics. Thus, as argued in [26], even though the expressive power of higher-order fuzzy logic goes well beyond its intended scope, the strength of its apparatus is best manifested within the limits of its motivation. Logic-based fuzzy mathematics thus forms a specific, distinct part of fuzzy mathematics, which is based on the notion of deduction and which should not be confused with other areas of fuzzy mathematics that are based on different interpretations of membership degrees, such as degrees of uncertainty, belief, frequency, preference, etc. (cf., e.g., [90, 74, 73] for different interpretations of degrees and the concluding part of [26] for the need of their clear separation).

Another connection that should be clarified is that to the philosophy of vagueness. On the one hand, fuzzy logic is often claimed to be the logic of vague propositions, or the logic of vagueness. On the other hand, it is as often criticized by philosophers as a completely misled and inadequate theory of vagueness. Although this introduction is not a suitable place to discuss this issue in detail, it should be stressed that both claims are inaccurate and need certain qualifications. To be sure, fuzzy logic cannot claim to be *the* logic of vagueness, as vagueness is a phenomenon with many facets, most of which are not captured by deductive fuzzy logic (e.g., are not truth-functional). If anything, deductive fuzzy logic can claim to be *a* logic of a *certain kind* of vagueness, related to properties that can be understood as coming in degrees. Moreover, deductive fuzzy logic is only a *logic,* rather than a fully fledged *theory* of vagueness meeting all requirements of the philosophy of vagueness (including answers to questions not asked by logic, for instance about the objectivity of the truth degrees etc.). Still, it can be argued that deductive fuzzy logic *is* a good model of inference under (certain kinds of) vagueness and as such can serve as a *logical basis* for a (prospective) theory of vagueness, or at least can help shed light on some of its facets. The sweeping damnation of fuzzy logic by many philosophers of vagueness is therefore unjustified and is for the most part caused by the ignorance of recent advances in fuzzy logic.[3]

Finally, fuzzy mathematics is sometimes criticized by mainstream mathematicians as

[2] At least as long as we understand intersection as the operation expressing the fact that an element belongs to the first *and* the second fuzzy set, i.e., require that one can infer *both* $x \in A$ and $x \in B$ from $x \in A \cap B$ and vice versa.

[3] For instance, many criticisms are caused by an inappropriate use of weak conjunction instead of strong conjunction in Łukasiewicz fuzzy logic (which is by far the most popular fuzzy logic among philosophers of vagueness), cf., e.g., [205, §4] or [78, §3]. Bad logical properties of *some* systems of fuzzy logic which are defective from the deductive point of view (e.g., Zadeh's original system of connectives min, max, and $1-x$) induce many philosophers (e.g., [201]) to condemn fuzzy logic as a whole, without considering better options offered by present-day mathematical fuzzy logic. Further problems arise from misunderstanding the role of fuzzy logic and expecting it to be applicable to situations that are beyond its scope (e.g., related to probability, levels of belief, etc.).

giving nothing but cheap generalizations of classical results. One of the aims of the present work is to show that *indeed* large parts of fuzzy mathematics are trivial, and demonstrate their triviality by deriving them from easily provable metatheorems (e.g., [30, Th. 33–36] or theorems in [41]). This shows that unlike more traditional approaches, the deductive apparatus of higher-order fuzzy logic enables clearly to perceive the triviality of such results. At the same time it provides means for reaching less trivial theorems (cf., e.g., [28, §6–7]) and possibly for achieving higher levels of fuzzy mathematics. It can be hoped that this direction will eventually contribute to gaining a better reputation for fuzzy mathematics among mainstream mathematicians.

2 State of the art

The enterprise of logic-based fuzzy mathematics is not isolated from other areas of mathematics and logic. It is based on *formal fuzzy logic* and its metamathematics, and can be regarded as its higher-order extension. At the same time it can be regarded as a formalization, reconstruction, and further development of certain parts of *traditional fuzzy mathematics.* In a broader context it is part of *non-classical mathematics,* i.e., mathematics that uses a non-classical logic for reasoning. This section gives an overview of previous results upon which logic-based fuzzy mathematics in general and the author's contribution in particular have built, as well as main results in related areas. However, due to the breadth of the field, this section cannot give an exhaustive survey or full historical account of all important works published in this area. Works which are most relevant to particular topics of this thesis are referred to in the articles it consists of; only a brief description of the state of the discipline at the time of the current project is given here, with a focus on works relevant for formal fuzzy mathematics.

2.1 Non-classical mathematics

Non-classical mathematics can be defined as the development of mathematical theories that employ some non-classical logic for informal reasoning or formal derivations. The area of non-classical mathematics comprises several independent branches, according to the kind of underlying logic used for mathematical reasoning. Each of these branches can further be divided into many particular theories over particular logics of the respective kind.

An example of non-classical mathematics is *paraconsistent mathematics* based on some variant of paraconsistent logic. The common feature of paraconsistent logics is that contradictions are in general not explosive (i.e., A and non-A do not in general entail an arbitrary B). This fact can be used, e.g., for the development of mathematical analysis based on the (contradictory) notion of infinitesimals [165]. Another application of paraconsistent mathematics is in naive set theory with full comprehension, where Russell's paradox is not destructive thanks to paraconsistency (e.g., [47]).

Avoiding Russell's paradox is one of the most important motivations for non-classical mathematics. Besides paraconsistency, there are several alternative ways in which Russell's paradox can be eliminated by employing a non-classical logic. One option is based on the observation that the structural rule of contraction (see, e.g., [182, 175, 173, 174]) is essential for the derivation of contradiction from the definition of Russell's set. It has indeed been proved that in various contraction-free substructural logics, set theory with the unrestricted axiom scheme of comprehension is consistent, and some of such theories have indeed been developed—e.g., over variants of linear logic [195, 187].[4] In fuzzy logics (which belong among contraction-free logics, cf. [173]), the consistency of unrestricted comprehension over Łukasiewicz logic was conjectured in 1957 by Skolem ([190], according to [204]). Skolem's partial results [191] were later extended by Chang [53] and Fenstad [84], and the conjecture finally confirmed in 1979 by White [204]. The theory has recently been investigated by Hájek [115] and Yatabe [207]. It is still an open question whether the theory or some extension thereof is sufficiently strong to support non-trivial mathematics (as conjectured by Skolem): Hájek's paper [115] contains some negative results on this question regarding arithmetic.[5] Although certainly worth investigating, this style of fuzzy mathematics is very different from traditional fuzzy mathematics. Logic-based fuzzy mathematics presented in this thesis avoids Russell's paradox by different means (namely in the style of type theory) and is only remotely related to fuzzy set theories with unrestricted comprehension.[6]

Another way to avoid Russell's paradox by means of non-classical logic has been proposed by Krajíček in [146, 147], namely by adding (epistemically interpretable) modalities to the language of set theory. The resulting theory is an example of *modal mathematics,* which in general can employ various kind of modalities serving various purposes. Modal mathematics related to the phenomenon of vagueness (and thus remotely to fuzzy logic) is proposed in [136].

Probably the most influential of all branches of non-classical mathematics is *intuitionistic mathematics;* related to the latter is *constructive mathematics* which usually uses some variant of intuitionistic reasoning, plus or minus some principles considered (non)constructive. The informal development of intuitionistic mathematics by Brouwer and his followers (cf. its formalization [143]) and constructive mathematics by constructivists can be considered the first non-classical mathematics ever developed. The later development of formal theories over intuitionistic logic (e.g., [77, 183]) is of special importance for logic-based fuzzy mathematics, since deductive fuzzy logics can be characterized as prelinear contraction-free intuitionistic logics;[7] informally speaking, fuzzy logics show in general intuitionistic features (especially in the behavior of quantifiers and negation).

The most important parts of intuitionistic mathematics for the foundations of fuzzy mathematics are set theories over intuitionistic logic. Since they are directly connected with the development of set theories over fuzzy logic, they will be described together with the latter in Section 2.4.

Especially strong links exist between mathematics over intuitionistic logic and that over Gödel fuzzy logic (for which see [76, 135, 110, 4]), as Gödel logic extends intuitionistic

[4] Related theories with unrestricted comprehension were originally studied by Grishin over the logic known as (classical) logic without contraction, Grishin's logic, or Ono's $\mathrm{CFL}_{\mathrm{ew}}$ (classical full Lambek calculus with exchange and weakening, see [173]). A description and further elaboration of Grishin's work [108] can be found, e.g., in [51].

[5] Further negative results [181, 121] regard the related question (which classically is a variation of Russell's paradox or the Liar) whether a truth predicate can be added to arithmetic over Łukasiewicz logic.

[6] It should be noted that for the consistency of unrestricted comprehension, a necessary condition on the underlying logic is that no bivalent connective be definable. Consequently, fuzzy set theories with full comprehension cannot be based on Gödel or product logics (as they have bivalent negation), nor any fuzzy logic with the Baaz $\triangle$ connective. Many important fuzzy logics are therefore excluded from this style of fuzzy mathematics. Fuzzy set theory with unrestricted comprehension is thus a very specific theory rather than a universal formalization of traditional fuzzy mathematics.

[7] More precisely, the weakest deductive fuzzy logic MTL, which is arguably [26] the weakest fuzzy logic suitable for formal fuzzy mathematics, arises by adding the axiom of prelinearity to the intuitionistic calculus LJ without the rule of contraction.

logic just by Dummett's axiom of prelinearity $(\varphi \to \psi) \vee (\psi \to \varphi)$ and the first-order axiom of constant domains $(\forall x)(\varphi \vee \psi(x)) \to (\varphi \vee (\forall x)\psi(x))$. Fuzzy mathematics over Gödel logic is thus stronger (i.e., closer to classical mathematics) than intuitionistic mathematics and most results in intuitionistic mathematics are readily transferrable to Gödel fuzzy mathematics. Since the connectives of Gödel logic are available in all extensions of the fuzzy logic $\mathrm{MTL}_\triangle$, the results in Gödel fuzzy mathematics have also some relevance in general fuzzy mathematics.

Kripke semantics for predicate Gödel logics, characterized in [10] as countable linear Kripke frames for intuitionistic logic with constant domains, with a possible modal interpretation of epistemic states of the idealized mathematician (or Brouwer's creating subject, cf. [199]) provides an additional link between fuzzy, modal, and intuitionistic mathematics. The Kripke semantics can be extended to non-contractive first-order fuzzy logics [163, 164] along the lines of [174] (i.e., equipping the Kripke frame with a monoidal operation, or a ternary accessibility relation). Kripke semantics can provide another possible link, beside that based on the algebraic semantics of (linear) residuated lattices, of logic-based fuzzy mathematics to other substructural (e.g., relevant [159, 88]) mathematical theories; this option has not yet been investigated, though. Since furthermore intuitionistic logic is the inner logic of topoi (see, e.g., [98]), intuitionistic mathematics may also provide a link between the logic-based and category-theoretic or sheaf-theoretic approaches to fuzzy mathematics [106, 129]. This link, however, has not yet been investigated, either.

2.2 Formal fuzzy logic

Logic-based fuzzy mathematics could not be developed without previous sufficient advancement of formal fuzzy logic. The requisite advances in formal fuzzy logic were achieved only in the past decade,[8] even though there were some (rather isolated) predecessors to this development.

Logics now regarded as belonging to the family of fuzzy logics were defined and studied from about 1920 on by several logicians, including Łukasiewicz [154], Wajsberg [200],[9] Gödel [94], Dummett [76], Hay [125], Belluce and Chang [45], Horn [135], and others. Fuzzy logic related to Zadeh's idea of a fuzzy set first occurred in Goguen's 1969 paper [96], motivated by the obvious correspondence between elementary fuzzy set operations and logical operations on truth degrees. In subsequent years, however, the term "fuzzy logic" was used either in a very broad sense (cf. the distinction between fuzzy logic in broad and narrow sense made by Zadeh in [214]), or only in reference to the semantical truth tables defining some (often rather arbitrarily chosen) operations on truth degrees.

The first formal calculus specifically devised for fuzzy logic, later proved to be equivalent to Łukasiewicz fuzzy logic with real truth constants [110, 122], was given in 1979 by Pavelka [177]. This line of research, further pursued and extended to first-order logic by Novák [167, 166], studies the so-called *fuzzy logic with evaluated syntax*—a specific kind of labeled-deduction calculus for fuzzy logic that enjoys the so-called Pavelka-style completeness (i.e., the correspondence between syntactic provability degrees of formulae and their semantic truth degrees). The logical foundations of fuzzy mathematics presented here are, however, based on systems of fuzzy logic with traditional logical syntax rather

[8] This explains why the logic-based approach to fuzzy mathematics started to be systematically investigated only a few years ago and almost forty years after fuzzy mathematics itself.

[9] The historical papers by Łukasiewicz and Wajsberg are cited according to [110] and [172], respectively.

than evaluated syntax, and utilize its similarity to classical Boolean logic and classical foundations of mathematics.[10]

The best known fuzzy logics with traditional syntax are those that use left-continuous t-norms as the truth functions of conjunction and their residua as the truth functions of implication (see, e.g., [144] for the theory of t-norms). Members of this family of *t-norm fuzzy logics* have systematically been studied since Hájek's milestone 1998 monograph [110], in which the 'basic' fuzzy logic BL of all continuous t-norms and its most important extensions, both propositional and first-order, were described in detail. Since then, a plenitude of formal systems of t-norm logics have been defined and their basic metamathematical properties (incl. general and standard completeness of their axiomatic systems, arithmetical or computational complexity, functional representation, etc.) investigated.

Of these systems, the most important for our present investigation are the logic MTL (or its variant $\mathrm{MTL}_\triangle$) of all left-continuous t-norms [81] and the logic ŁΠ (or its variant $\text{Ł}\Pi\frac{1}{2}$) joining the three basic t-norms [83, 59]. As argued in [26], MTL is the weakest fuzzy logic suitable for the deductive style of fuzzy mathematics, thus providing the largest generality within certain reasonable constraints. The logic $\text{Ł}\Pi\frac{1}{2}$, on the other hand, is the most expressive system among best-known fuzzy logics that still possesses very good metamathematical properties: a deduction theorem [59], introduction and elimination of Skolem functions [60, 30], etc. Besides other features, its standard semantics contains all basic arithmetical operations on truth degrees; thus it is a good approximation of the needs of traditional fuzzy logic. Moreover, a broad class of propositional t-norm logics is interpretable in ŁΠ; thus it can serve as a common framework for integration of fuzzy mathematics over more specialized fuzzy logics.[11] Nevertheless, various modifications of these logics can be useful for more specific purposes within the project (for example, the involutiveness of negation was needed in [43]; therefore, $\mathrm{IMTL}_\triangle$ was employed as the ground logic). Higher-order logic and formal fuzzy mathematics can be based on any t-norm fuzzy logic, and all of them may be useful for this purpose in specific situations.

General algebraic semantics of well-behaved propositional t-norm logics consists of suitable quasivarieties of residuated lattices (possibly enriched with additional operators). Consequently, t-norm fuzzy logics belong to the family of *substructural logics,* as the latter can be identified with logics of (classes of) residuated lattices [173]. Both the theory of residuated lattices [139, 89] and substructural logics [175, 182] thus provide a broader background for the more specific study of t-norm fuzzy logics. In particular, t-norm logics fall within *contraction-free* substructural logics [174], since their local[12] consequence relation in general fails to satisfy the structural law of contraction (or the idempotence of

[10]One of the reasons for the choice of traditional rather than evaluated syntax is the necessary condition for the Pavelka-style completeness that implication be continuous, which limits fuzzy logic with evaluated syntax to variants of Łukasiewicz logic.

[11]These were the reasons why ŁΠ was chosen as the ground logic of the foundational Fuzzy Class Theory in the original paper [30], while later most of the more particular disciplines of logic-based fuzzy mathematics have for the sake of generality been developed in Fuzzy Class Theory over the logic $\mathrm{MTL}_\triangle$ (as combinations of connectives pertaining to different t-norms turned out to be used only rarely).

[12]Like in modal logics or other logics with partially ordered truth values, local and global consequence can be distinguished in t-norm fuzzy logics [117, 26]. Even though the global consequence relation (which transmits the full truth of fuzzy propositions) is more commonly studied in formal fuzzy logic, it is the *local* consequence relation between partially true premises and a partially true conclusion which is more important for formal fuzzy mathematics, as it allows deriving graded results with imperfectly true premises. In the practice of formal fuzzy mathematics, we derive theorems of the form $\varphi_1 \,\&\, \ldots \,\&\, \varphi_n \to \psi$ by the rules of global consequence, which is axiomatized by the usual systems of fuzzy logic; the latter form internalizes precisely the local consequence between the premises $\varphi_1, \ldots, \varphi_n$ and the conclusion ψ.

conjunction), while the laws of exchange and weakening do hold in t-norm fuzzy logics. Related systems that lack some of the latter structural laws, e.g., Metcalfe's uninorm logic UL which drops weakening or logics with non-commutative conjunction like pseudo-BL or the flea logic, are studied as well [157, 150, 71, 114]. With appropriate changes, higher-order logic and formal fuzzy mathematics can be developed over these related systems, too.

As contraction-free substructural logics with exchange and weakening, t-norm fuzzy logics extend Ono's logic $\mathrm{FL_{ew}}$ (full Lambek calculus with exchange and weakening, see, e.g., [173]), also known as affine multiplicative additive intuitionistic linear logic [157], Höhle's monoidal logic [127] or intuitionistic logic without contraction [1], i.e., the logic of commutative bounded integral residuated lattices. The distinctive feature of t-norm fuzzy logics among contraction-free logics is the validity of the axiom of prelinearity $(\varphi \to \psi) \vee (\psi \to \varphi)$. In [36] we argued that prelinearity (in a more general form) can be regarded as a characteristic feature of the class of *fuzzy logics* (i.e., not just t-norm based) among Cintula's weakly implicative logics [62]. A general theory of weakly implicative fuzzy logics as the logics of classes of linearly ordered logical matrices is also described in [62].

The conditions of weakly implicative fuzzy logics, however, only ensure suitable properties of implication as the principal connective in formulae true to degree 1. For the deductive style of fuzzy mathematics aimed at graded theorems that transmit partial truth, further conditions are needed that ensure that implication and conjunction respectively internalize the local consequence relation and cumulation of premises. The resulting class of *deductive fuzzy logics* [26] can be characterized as the intersection of the classes of Cintula's weakly implicative fuzzy logics and Ono's substructural logics (optionally with exchange and weakening, which will be assumed further on), or as the class of fragments or expansions of MTL (or Metcalfe's [156] uninorm logic UL when working without weakening) with all connectives congruent w.r.t. bi-implication. Deductive fuzzy logics (which include all common t-norm logics) are the intended background logics for logic-based fuzzy mathematics as studied in the present project.

Propositional fuzzy logic is of course insufficient for the development of formal fuzzy mathematics, which besides fuzzy logical connectives also needs some means for (preferably fuzzy) quantification over its individuals. Some first-order systems of particular fuzzy logics were developed already during the pre-fuzzy and early fuzzy era [125, 45, 194]. A systematic treatment of first-order variants of t-norm based fuzzy logics has started with Hájek's book [110]. Those first-order fuzzy logics that are most important for logic-based fuzzy mathematics are described in [110, 59, 81]; a comprehensive survey of first-order t-norm fuzzy logics is [118]. The basics of model theory for t-norm fuzzy logics have been developed in [119] and [118, §6]. Metamathematical properties of first-order fuzzy logics relevant to the important question of their completeness w.r.t. the standard real-valued semantics can be found in [161, 113, 116, 163].[13] Initial steps toward a general theory of first-order weakly implicative fuzzy logics are given in [60], which largely conforms with Rasiowa's general approach to first-order implicative logics [180]. Higher-order systems of fuzzy logics and axiomatic theories over first-order fuzzy logics will be mentioned in Section 2.4.

The quantifiers in all of the first-order systems studied in the papers mentioned above are the *lattice* quantifiers, corresponding to lattice conjunction and disjunction.[14] This

[13]Of the most important first-order fuzzy logics, the standard completeness holds only for MTL and Gödel logic. The standard incompleteness of other logics is usually proved by showing that the arithmetical complexity of the set of their standard real-valued tautologies is larger than Σ_1.

[14]Recall that in contraction-free substructural logics there are two different meaningful conjunctions and

is fully sufficient for the needs of formal fuzzy mathematics, since the first-order systems with just the lattice quantifiers turn out to be strong enough to enable the construction of Henkin-style higher-order logic, in which various classes of strong (multiplicative) quantifiers become definable [39, 65, 64].

First-order fuzzy logics containing a strong quantifier as a logical symbol have been studied by Montagna in [160]; cf. also another strong quantifier introduced for the real-valued semantics by Thiele [196, 197] (as described in [160]). However, these quantifiers do not coincide with the *weakest* quantifier Π that allows the inference $(\Pi x)\varphi(x) \to \mathop{\&}_{t\in M} \varphi(t)$ for any finite multiset M of terms.[15] According to our knowledge, the latter 'inferentially optimal' multiplicative quantifier has so far only been sketched in our abstract [65].

Besides the special case of multiplicative quantifiers, also a general notion of quantifier is of importance for formal fuzzy mathematics. Generalized quantifiers formally studied in a logic-based setting by Novák [168, 171] and Holčapek [132] are motivated mainly by modeling natural language (cf. generalized quantifiers in classical logic [203, 178] and linguistically-motivated fuzzy quantifiers in traditional fuzzy mathematics [215, 93]) or applications in fuzzy control. From the point of view of formal fuzzy mathematics, crisp generalized quantifiers such as *for infinitely many x*, fuzzy counting quantifiers like *for many x*, or quantifiers relativized to a fuzzy mathematical condition like *for all large numbers x* are of the greatest interest; however, these have apparently not yet been systematically studied in the framework of formal fuzzy logic, although some of the linguistic-oriented approaches mentioned above are undoubtedly applicable in this area as well. Such quantifiers are nevertheless implicit in many constructions of formal fuzzy mathematics: for example, Bandler and Kohout's "local properties" of fuzzy relations [8, 9] are instances of fuzzily relativized quantifiers. The initial studies on fuzzy quantifiers in formal fuzzy logic mentioned above mostly employ higher-order systems (Novák's fuzzy type theory [169] or our higher-order fuzzy logic [30]), since fuzzy quantifiers can be regarded as fuzzy sets of fuzzy sets. Apart from strong quantifiers mentioned above, the possibility of having generalized fuzzy quantifiers as primitives in the logical language has probably not been considered in formal fuzzy logic yet; for the development of formal fuzzy mathematics they are not indispensable, as they can be introduced internally in higher-order systems that are based on lattice quantifiers.

Perhaps even more important for formal fuzzy mathematics than various kinds of strong and generalized quantifiers is the related notion of *exponentials*. Exponentials (in our sense of [65]) can be seen as propositional counterparts of truth-functional strong quantifiers.[16] They are motivated by similar considerations as Girard's exponentials for linear logic [91], which are used generally in substructural logics (see, e.g., [175]). Exponentials studied in fuzzy logic so far include Montagna's storage operator of [160], which corresponds to his strong quantifier mentioned above, and Baaz's operator $\triangle$, introduced

disjunctions. One of the pair of connectives is in the literature variably called *weak, lattice, comparative, extensional,* or *additive* and the other one *strong, group, parallel, intensional* or *multiplicative.* For the difference between them see, e.g., [175]. The distinction can be extended to quantifiers, see [176].

[15] A multiset, since the strong conjunction & is not contractive (idempotent); therefore multiple occurrences of the same term have to be taken into account. Thiele's quantifier only ensures the property for a *set* of terms, while Montagna's quantifier additionally ensures the idempotence of $(\Pi x)\varphi(x)$ w.r.t. conjunction; the latter quantifier therefore coincides with the optimal one in extensions of BL, but not generally in extensions of MTL, as was already proved in [160], even though the optimal quantifier was only implicit there.

[16] They can be defined from the latter by dummy quantification, i.e., $\varphi^* \equiv_{\mathrm{df}} (\Pi x)\varphi$ if x is not free in φ. As propositional modifiers they are special *hedges* in the terminology of [151] (followed in [112]), or *modalities* in that of [58].

in [2] for Gödel logic, transferred to the most important fuzzy logics in [110, 81], and generalized for all weakly implicative fuzzy logics in [62]. A proof-theoretical investigation of logics expanded with certain classes of exponentials is given in [58].

The importance of exponentials for formal fuzzy mathematics stems from their role as lower estimates for the truth values of propositions $\varphi, \varphi^2, \varphi^3, \dots$ (where φ^n is the n-tuple strong conjunction of φ). Formulae of the form φ^n occur frequently in formal fuzzy mathematics due to the general non-idempotence of strong conjunction; exponentials $*$ such that $\varphi^* \to \varphi^n$ for any n then provide a common bound on the strength of φ^n for all n. Although Baaz $\triangle$ can always be taken for such an estimate, in many cases it is too strong (e.g., if there is an idempotent w.r.t. & below the truth value of φ). Montagna's storage operator provides a better alternative, but is still unnecessarily strong in some cases (cf. footnote 15). The inferentially optimal exponential φ^ω, related to the inferentially optimal multiplicative quantifier mentioned above, has by now only been sketched in [65].[17]

The general state of the art of formal fuzzy logic can be characterized as follows: Chagrov [52] distinguishes three stages in the development of a new area of non-classical logic. In the first stage, the concepts and logics of the area emerge without a clear methodology or well-developed metamathematics. In the second stage, when the methodology and metamathematics has become available, the area is systematically explored: often, many new logics are defined and their properties studied by advanced techniques. The third stage then offers a synthesizing view on the area, when common properties of whole classes of logics are obtained by generalized methods, and unifying insights are achieved by mature understanding of the area. The three phases need not be sharply separated and may chronologically overlap. As observed by Chagrov, this account, though abstracted from the particular development of modal logic, can be applied to the history of most disciplines of non-classical logic.

In formal fuzzy logic, these three stages can be found as well. The first phase commenced with the early study of Łukasiewicz and Gödel–Dummett logics in the 1920–60's, and continued by the informal development and applications of fuzzy logic since the 1970's. The second phase was announced by the first works on formal fuzzy logic since the late 1970's, especially those by Pavelka [177], Novák [166], and Gottwald [102]. The heyday of the second stage came after Hájek's 1998 monograph [110], when an explosion of new systems of fuzzy logic and their systematic metamathematical study has begun. Now we find ourselves in the maturity of the second stage and the beginning of the third, as the exploration of the fuzzy-logical landscape is far advanced (though new logics still do emerge—recently, e.g., uninorm [156, 157] and weakly cancellative [162] logics) and the properties of known fuzzy logics have already been deeply investigated (including their arithmetical [113, 116] and computational [3, 123, 124] complexity, expansions by various kinds of connectives [82, 160, 80, 58], standard completeness theorems [138, 79, 133], proof theory [57, 158], etc.). One of the first works that clearly belongs to the third stage is Cintula's [62], in which a unified metamathematical treatment of all weakly implicative fuzzy logics is given. This framework was further generalized to weakly implicational (fuzzy) logics in [66]; a narrower class of ($\triangle$-)core fuzzy logics [119] was further studied

[17] As a primitive symbol of propositional logic, the exponential $^\omega$ is axiomatizable by a straightforward infinitary rule. It can moreover be approximated by a finitary axiomatization such that the finitarily axiomatized exponential coincides with the optimal one if the latter does exist on the algebra of truth values (which in general need not be the case). The exponential $^\omega$ is of course definable in higher-order fuzzy logic, though only with the qualification that the Henkin-style axiomatization of higher-order fuzzy logic admits its non-intended models (it is nevertheless the optimal *internal* exponential in the theory).

in [63]. The second stage, however, cannot be considered completed, as is apparent for instance from the as yet insufficient investigation of exponentials in fuzzy logic.

An indispensable precondition for the development of logic-based fuzzy mathematics was to achieve at least an advanced phase of Chagrov's second stage in formal fuzzy logic. In particular, it was the extensive exploration of the logical landscape in the fuzzy area that enabled finding the most suitable systems of fuzzy logic that could support formal fuzzy mathematics (esp. the logics $\mathrm{MTL}_\triangle$, ŁΠ, and t-norm fuzzy logics in general), allowed applying their basic metamathematical properties in the development of the formalism and helped clarify the area of their applicability. The emergence of logic-based fuzzy mathematics indeed coincides with this stage of development of formal fuzzy logic. The above considerations can partly explain why it had not appeared earlier during the four decades of the existence of traditional fuzzy mathematics.

2.3 Traditional fuzzy mathematics

As fuzzy mathematics has been developed by many researchers for more than forty years, it is impossible to present an overview of all its developments in this brief survey. Therefore we shall only deal with those areas of fuzzy mathematics to which the papers included in this thesis are related, namely the theory of fuzzy sets and fuzzy relations, fuzzy topology, and fuzzy numbers. The developments in other areas of traditional fuzzy mathematics are described, e.g., in the surveys [73, 142]. A compendium of application-oriented traditional fuzzy mathematics is, e.g., [145]. For each of the relevant disciplines of fuzzy mathematics, only the works that initiated the research and recent representative books or surveys of the area will be mentioned here. Approaches that are close to logic-based fuzzy mathematics, where they exist, will also be noticed. Further details can be found in the introductions and references to the papers included in this thesis, and in the literature cited in the surveys.

The theory of *fuzzy sets* (and fuzzy mathematics as the whole) is usually considered to have started with Zadeh's 1965 paper [212], which introduced the concept of fuzzy set (and coined the term *fuzzy*), identifying fuzzy sets with membership functions from a crisp ground set to $[0, 1]$. There have, nevertheless, been several predecessors who proposed similar or identical concepts, most notably Max Black [48], Abraham Kaplan and Hermann Schott (see [73, §1.2.4]), Karl Menger (ibid.), and Dieter Klaua (see [105]). In 1967, the notion of fuzzy set was generalized to lattice-valued membership functions by Goguen [95]; since then, various structures of membership degrees have been considered.

Graded properties of fuzzy sets have been considered mainly in the setting related to or based on formal fuzzy logic (esp. by Bandler and Kohout [7] and Gottwald [99]). For axiomatic theories of fuzzy sets based on formal systems of fuzzy logic, in which graded properties of fuzzy sets appear quite naturally, see Section 2.4.

Important monographs with chapters on fuzzy sets include [166, 145, 104]. An overview of basic notions in the theory of fuzzy sets is given, e.g., in [73]. Besides the direct representation by means of membership functions, various alternative foundations for the notion of fuzzy set have been considered in the literature: category-theoretical approaches to fuzzy sets are surveyed in [131, 106], and categories of fuzzy sets are treated in detail in [206] and [172, Ch. 7]. A sheaf-theoretic foundation of fuzzy sets is described in [129]. Axiomatic theories of fuzzy sets based on formal fuzzy logic are described in more detail in Section 2.4 below.

The notion of *fuzzy relation* was defined already in Zadeh's first paper on fuzzy sets [212]. It was generalized to lattice-valued relations in Goguen's 1967 paper [95], in which several important fuzzy-relational concepts (including, e.g., sup-product compositions) were studied. Many important concepts, incl. fuzzy similarity and fuzzy ordering, were introduced in Zadeh's 1973 paper [213]. Important contributions to the theory of fuzzy relations were made by Bandler and Kohout, esp. regarding generalized relational products [6]. Further references to the vast literature on fuzzy relations can be found in the papers on fuzzy relations included in this thesis [28, 41].

Graded properties of fuzzy relations, which are fundamental in logic-based theory of fuzzy relations (cf. [28]), were first proposed by Gottwald in [101]. They are systematically studied in Gottwald's monographs [102] and [104, §18.6] and Bělohlávek's book [46]. The graded approach has been applied by Gottwald to the solvability of fuzzy relational equations in [103]. Several graded notions of fuzzy function have been studied by Demirci [70].

The discipline of *fuzzy topology* was established in the 1960's and 1970's in papers by Chang [55], Goguen [97], Lowen [152], and others.[18] It has been given a considerable attention throughout the history of fuzzy mathematics and elaborated by a number of researchers. Several approaches to fuzzy topology have been developed: besides those based on membership functions and fuzzy sets, the most prominent are the (point-free) lattice-theoretical and categorial treatments. Various definitions of fuzzy topology were surveyed by Höhle and Šostak in [130]. Many results are also surveyed in the (more recent, but somewhat self-promoting) historical overview [142, §6]. A detailed exposition based on the categorial viewpoint is given in Höhle's monograph [128].

An early example of logic-based fuzzy topology is Ying's investigation of fuzzifying and bifuzzy topologies in the early 1990's [208, 209, 210]. His definitions and proofs were based on the semantics of Łukasiewicz predicate logic (or complete residuated lattices later in [211]), which naturally led him to graded fuzzy topological notions and theorems. Graded topological notions (of compactness and connectedness) had even earlier been studied by Šostak (see the references in [70], esp. to [192]).

Two main competing approaches to *fuzzy numbers* have originally been proposed: one of them treats fuzzy numbers as fuzzy intervals (Mizumoto and Tanaka 1979, see [142, §11]), while the other regards them as (certain equivalence classes of) distribution functions (Rodabaugh 1982, see [142, §11]).[19] In the interval approach, Dubois and Prade (1980, see [142, §11]) have added further conditions of monotony and continuity. The distribution-based approach has been extensively studied in relation to the construction of fuzzy real numbers and the topology of the fuzzy real line, by Lowen, Höhle and others [153, 126].

The interval-based approach was recently criticized by Dubois and Prade [75] as representing the fuzzified notion of interval rather than number. Their proposal to define a fuzzy number as a gradual element, i.e., a function from truth values to the domain of discourse rather than vice versa, is discussed from the point of view of formal fuzzy logic in [26, §2] included in this thesis.

2.4 Formal fuzzy set theories

In this section we shall describe the state of the art in formal theories of fuzzy sets. We shall leave aside set theories with the unrestricted comprehension scheme, mentioned in Section 2.1, as these are very specific theories, unrelated to logic-based mathematics as

[18] These papers are cited according to [130].

[19] It can be observed that the approach of [16] (included in this thesis) in fact combines both approaches, since it treats fuzzy numbers as intervals between two distribution functions (or fuzzy Dedekind cuts).

presented here and constrained to a very narrow class of fuzzy logics (cf. footnote 6 on page 9 above). Instead, we shall focus on theories which are closest to our approach to fuzzy set theory, namely such theories over fuzzy logics whose axioms ensure that basic set-theoretical constructions (such as forming unions, intersections, singletons, or power sets) can be carried out. As this is exactly the motivation of the axioms of classical Zermelo–Fraenkel set theory, we shall call such theories *ZF-style fuzzy set theories.* Akin to ZF-style fuzzy set theories are *fuzzy type theories* (including Fuzzy Class Theory upon which our logic-based fuzzy mathematics is founded), since their hierarchy of types and the comprehension axioms (or the mechanism of λ-abstraction in Church-style type theories) are aimed at ensuring the availability of basic set-theoretic constructions, too (and vice versa, the set-theoretical constructions, esp. those of power set and union, guaranteed by the axioms of ZF-style set theories usually impose a cumulative structure on the universe of sets analogous to the hierarchy of types in type theories).

At least two strands can be recognized in the history of ZF-style theories of fuzzy sets.[20] One of them attempted at formal treatment of fuzzy (or many-valued) sets from the outset, while the other originated in ZF-style set theories over intuitionistic logic, whose methods were subsequently transferred to fuzzy logics (with systems close to Gödel logic as an intermediary step).

Early works in the former strand are due to Dieter Klaua (in 1965–1973, see Gottwald's survey [105]), who defined (variants of) a cumulative hierarchy of fuzzy sets using definitions based on Łukasiewicz logic. This approach was followed and further modified by Siegfried Gottwald [99, 100] who derived many results on fuzzy sets in this framework.

While Klaua's and Gottwald's fuzzy set theories were essentially based on (Łukasiewicz) fuzzy logic, other early axiomatizations of fuzzy sets were based on membership functions and classical logic. Chapin's axiomatic fuzzy set theory [56] considered a ternary membership predicate, with the third argument representing the degree of membership. An important feature of Chapin's theory was a homogeneity of its objects (which is desirable in foundational theories—cf. classical set theory, where all objects are sets), as the membership degrees were not external objects different from fuzzy sets: rather, the role of membership degrees was played by some of the fuzzy sets themselves (hence the papers' title 'Set-valued set theory'). Basic parts of the formal theory of so defined fuzzy sets were derived from the proposed axioms in the two parts of the paper (the announced third part was never published). A similar setting was presented by Weidner [202], whose system (called Zadeh–Brown set theory ZB) aimed at emending some features of Chapin's axioms; to this effect, the ordering relation between the degrees was taken as an additional primitive notion besides the ternary membership predicate. Consistency of ZB was shown by constructing a Boolean-valued model in ZF.

The construction of formal ZF-models valued in an appropriate structure of degrees was also a main motive in a series of papers, by several different authors, that originated in set theory over intuitionistic logic and subsequently shifted towards formal theories of fuzzy sets. In his 1975 paper [179], Powell constructs a syntactic interpretation (called the *inner model*) of classical ZF in a certain reformulation of ZF over intuitionistic logic (Int). To this end, he first needs to introduce and investigate various set-theoretical notions (e.g., ordinal numbers) and prove several results (e.g., the transfinite recursion theorem) within the formal intuitionistic set theory.[21] Grayson's 1979 paper [107] studies in detail various properties of ordinal numbers in a similar reformulation of ZF over Int, and shows

[20] A much more refined classification of formal theories of fuzzy sets can be found in Gottwald's recent survey [105], which also includes approaches distant from ours.

[21] A similar Heyting-valued model was much later studied by Shimoda [186].

counterexamples to some classically valid theorems on ordinals in a Heyting-valued sheaf model of the theory (which is a generalization of Boolean-valued models of Scott and Solovay). Even though the main results of those papers are metamathematical (viz, mutual interpretability of the theory and classical ZF), the theory itself is proposed as a prospective axiomatic setting for intuitionistic set theory in the sense of non-classical mathematics (see Section 2.1 above), and the theorems derived within the theory are regarded as results on intuitionistic (Heyting-valued) sets. This attitude differs from that of an earlier book [85] by Fitting, who only employs a reformulation of ZF over Int for metamathematical purposes, namely as a means for classical independence proofs by forcing. Driven mainly by this motivation, Fitting's axioms arise from the classical axioms of ZF by replacing all occurrences of $\forall$ by $\neg\exists\neg$, which yields a rather unintuitive axiomatic system from the point of view of non-classical set theory over Int.

Powell's results and methods were in 1984 adapted by Takeuti and Titani [193] for a ZF-style set theory over a variant of Gödel logic (with a rule ensuring density of truth values). Besides the main result on mutual interpretability with classical ZF by means of inner models, they developed some parts of the formal theory, incl. the properties of real numbers.[22] In 1992 the same authors [194] presented a ZF-style set theory over a richer logic which contained further connectives besides those of Gödel logic, representing the basic arithmetical operations (except division) in the standard $[0, 1]$-interpretation of the logic. Again they constructed a cumulative $[0, 1]$-valued model of their theory and proved mutual interpretability with classical ZFC. The paper also contains the construction of internal truth values (adapted in [41] for FCT over MTL) and various definitions and results within the theory.

Takeuti and Titani's definitions mostly employ Gödel connectives as primary ones, and make use of the arithmetical operations only where necessary for their metamathematical purposes (mainly in the construction of internal truth values); the theory thus retains the structure of intuitionistic and Gödel set theories of the previously mentioned papers. Titani's 1999 lattice-valued set theory of [198] is also largely based on lattice connectives in the underlying logic (although the implication connective is generalized so that it also admits a quantum-logic interpretation). The results and methods of Gödel set theory are, however, hardly transferable to other fuzzy logics, as they depend heavily on the idempotence (i.e., contractivity) of the minimum conjunction. The step to non-contractive fuzzy logics was undertaken by Hájek and Haniková in their 2003 paper [120], in which they adapt the previous methods (using also ideas from Shirahata's work on set theory over linear logic [188]) for a set theory over the logic $\mathrm{BL}_\triangle$.[23]

A (Church-style) *fuzzy type theory* FTT over the logic $\mathrm{IMTL}_\triangle$ has been introduced in 2004 by V. Novák [169]. Although it has been mainly used as a formal background for linguistic modeling [171, 170], some parts of fuzzy mathematics have necessarily been developed in its framework, too (e.g., the theory of feasible natural numbers, [170, §3.5.3]).

Fuzzy Class Theory FCT of [30], which is the foundational theory of logic-based fuzzy mathematics as studied in this thesis, can be regarded as a (Henkin-style) simple type theory (of Russell's type), too.[24] The developments of the theory by the present author

[22]Basic notions of set theory over Gödel logic, motivated by both intuitionistic and fuzzy considerations, have also been developed in the present author's master thesis [12] (in Czech; a short English summary can be found in [13]).

[23]The $\triangle$ connective is used for limiting the size of powersets, which otherwise would be inconsistently large, and ensuring full existence of postulated sets, which is justifiable by the Skolem function equivalents of the axioms.

[24]FTT and FCT (in logics over which FTT has been defined) seem to be mutually faithfully interpretable.

and his co-authors (U. Bodenhofer, P. Cintula, M. Daňková, R. Horčík, S. Saminger-Platz, and T. Kroupa) are described in detail in other parts of this thesis. Further works that contributed to the study of FCT and formal fuzzy mathematics developed within its framework are [64, 149, 148, 134] by P. Cintula, R. Horčík, and T. Kroupa.

3 Significance of the area of research

Logic-based fuzzy mathematics is a minor, rather than mainstream, current in fuzzy mathematics. This fact may raise questions about the meaningfulness of the enterprize and the significance of this area of research. In this section we shall summarize some reasons why the study of logic-based fuzzy mathematics is worthwhile.

There are general arguments in favor of the importance of any kind of non-classical mathematics (cf. also Section 2.1). Changing the logical principles that underlie mathematical reasoning may reflect some *external motivation* under which these principles are no longer valid—compare, e.g., the rejection of certain laws of classical reasoning by intuitionistic mathematics; this is also the case in fuzzy mathematics, as for instance the law of excluded middle is in general implausible for graded propositions. Non-classical mathematics can, however, also be justified independently of such 'applied' motivations, and be studied for the intrinsic reason of developing an *alternative view on classical mathematics,* as removing some assumptions of classical logic may reveal various kinds of dependencies between classical notions and present classical mathematical structures as special (or degenerate) cases of more general non-classical structures. This enables us, for instance, to compare the robustness of various mathematical definitions and theorems with respect to the changed logical assumptions.[25] The splitting of classically equivalent notions in weaker logics (in which their equivalence may no longer be provable), can shed light on classically indistinguishable aspects of the notions and provide a better understanding of the interdependencies between such aspects. (For an illustration, compare the various notions of finiteness in intuitionistic mathematics, cf. [87, §IV.6] and [77]—or, for that matter, in classical mathematics without the axiom of choice, see, e.g., [137, §4.6].) In the particular case of logic-based fuzzy mathematics, the change of the underlying logic yields linear-valued (and often continuous-valued) mathematical structures as semantical models, variants of which have been studied—mainly for such intrinsic reasons rather than for the sake of applications—since the 1960's [54, 55, 152].

Besides being a sub-area of non-classical mathematics, logic-based fuzzy mathematics is also a specific sub-area of the theory of fuzzy sets. The importance of fuzzy sets for certain kinds of engineering applications is beyond doubt. In such applications, the richer system of membership degrees allows modeling the gradual change of a property, using it as a feedback measure for fine-tuning the value of the property by approximation steps—which would not be enabled by a crisp jump from 0 to 1 without intermediate values. Giving a formal foundation to various engineering fuzzy methods was one of the original motivations for the development of formal fuzzy logic, e.g., in [110, p. 2]. Although logic-based fuzzy mathematics does not directly address all methods of engineering-applicable fuzzy mathematics (cf. [26]), it provides a unifying framework for at least some of its parts [34, 26]. A consistent application of the logic-based approach moreover yields certain

[25]For example, it is known [107] that the axiom of choice entails bivalence already in very weak set theories over intuitionistic logic, while Zorn's lemma does not do so even in rather strong intuitionistic set theories. This shows, not only that the classical theorem on their equivalence uses the law of double negation in an essential way, but also that Zorn's lemma is a more robust variant of the axiom of choice with respect to the behavior of negation.

favorable features of the resulting theory that provide further reasons for developing fuzzy mathematics in this specific manner; we shall list some of them in the next paragraphs.

One of the most important characteristics of the logic-based approach to fuzzy mathematics is the universal gradedness of defined notions. Traditional fuzzy mathematics employs classical logic for mathematical reasoning, therefore its defined concepts are by default crisp; gradedness has to be intentionally introduced in each definition. In logic-based fuzzy mathematics, on the contrary, defined notions are introduced by formulae of many-valued logic, and therefore are by default many-valued. This applies not only to fuzzy structures themselves (where adding gradedness is usual even in traditional fuzzy mathematics), but also to their *properties,* which in logic-based fuzzy mathematics are naturally fuzzy as well. Such graded properties of fuzzy structures have occasionally been studied in traditional fuzzy mathematics, too, often in partially logic-based setting. In fuzzy relations they have been first studied by Gottwald [101, 102, 104] and later by Bělohlávek [46]. Graded properties of fuzzy structures are also met in fuzzifying topology [208, 209, 211, 185]. Measures of defects of a broad range of mathematical properties, though motivated by other than logic-based considerations, were studied by Ban and Gal in [5]. A few properties like fuzzy set inclusion are commonly introduced as graded even in mainstream fuzzy mathematics [6, 7]. The idea of gradedness is also very strong in Pavelka-style fuzzy logic with evaluated syntax [177, 167, 172], which fuzzifies even the concept of provability. Full gradedness is a general feature of fuzzy mathematics based on formal fuzzy logic, be it a higher-order logic like FCT, a fuzzy type theory [169], or a formal fuzzy set theory [194, 120]. There are many reasons why graded properties of fuzzy structures are important; some of them are given in [19, §1], [35, §2.1], and [28, §1]. A reason which has not been stressed in these papers is that graded properties, like all fuzzy sets, enable one to *optimize* the property that is only imperfectly satisfied, where the degrees give a feedback for the optimization that cannot be provided by a crisp jump from 0 to 1. (Generalized fuzzy quantifiers would provide more kinds of logically meaningful measures of graded properties, thus enabling more kinds of optimization besides that on the infimum; however, a logic-based theory of generalized quantifiers is only in its beginnings, see Section 2.2.)

A related feature of logic-based fuzzy mathematics is a smooth accommodation of fuzzy sets of fuzzy sets. This is desirable in many branches of fuzzy mathematics: prototypically in fuzzy topology, as topological structures are usually formed of sets of sets (namely, sets of open sets, systems of neighborhoods, etc.), but also in other areas (consider, e.g., fuzzy sets of fuzzy numbers, of fuzzy points, of fuzzy events, etc.). Since formal fuzzy set theories axiomatize fuzzy sets of all kinds, their theorems apply as well to fuzzy sets of complex structures as to simple fuzzy sets of atomic urelements. Thus even though their semantical models are as complex as required, the syntactic logic-based apparatus that describes them is much simpler than their direct semantical description that is usual in traditional fuzzy mathematics. This demonstrates an advantage of the strict separation of syntax from semantics in the approach based on formal logic.

A related advantage of logic-based fuzzy mathematics ensues from its radical axiomatic approach, which contributes to its methodological clarity. An axiomatic approach has proved beneficial in countless fields of mathematics; it has occasionally been employed in traditional fuzzy mathematics, too (cf., for instance, de Luca and Termini's [67] axioms for fuzzy entropy or various axioms for aggregation operators—see, e.g., [145, Ch. 3]). However, grounding the axiomatic method on formal fuzzy logic offers an additional advantage for fuzzy mathematics, as the assumptions on the structure of truth degrees are then isolated and encapsulated in the logic itself, rather than re-introduced at each definition. Fuzziness is thus only introduced into the theory by the set of rules that can generally be employed for sound reasoning about fuzzy propositions.[26] By this treatment, the general properties of fuzziness are removed from a particular theory to the level of logic; the theory itself can then deal with its specific notions only, and need not be concerned at the same time with the properties of truth degrees.

By hiding truth degrees in the semantics of fuzzy logic, logic-based fuzzy mathematics also alleviates one of the criticisms of traditional fuzzy mathematics, namely the artificial over-precision of fuzzy sets: an argument often raised against fuzzy set theory points out that instead of giving less information about the membership in a vague concept, fuzzy sets provide more (indeed too much) information by specifying its value to a precise real number (or an element of another lattice of truth degrees). However, by screening off direct references to truth degrees, logic-based fuzzy mathematics avoids (in a principled way) computing with particular truth degrees: not only are such calculations absent from formulae of the theory, but the theory in fact *abstracts* from them, in consequence of the definition of validity in formal logic by generalization over all models.

Another appealing consequence of hiding fuzziness into the rules of logic is the resulting similarity of formulae of fuzzy mathematics to those of classical mathematics. Since deductive fuzzy logics are not too different from classical logic, many concepts of classical mathematics can be naturally transferred to fuzzy mathematics simply by reinterpreting the logical connectives that appear in their formal definitions (cf. [126, §5]).[27] Quite often, classical definitions reinterpreted in fuzzy logic yield useful and interesting notions of fuzzy mathematics. The meaning of fuzzy concepts obtained in this way can be clarified by taking the meaning of fuzzy connectives and quantifiers into account. For instance, fuzzy inclusion $A \subseteq B \equiv_{\mathrm{df}} (\forall x)(Ax \to Bx)$, defined by the same formula as in classical mathematics (since Ax is just an abbreviation for $x \in A$), is not just *some* measure of inclusion of fuzzy sets, as it is understood in traditional fuzzy mathematics, but is *the* strongest measure which allows for any x to infer[28] Bx from $Ax \mathbin{\&} (A \subseteq B)$, which is a transparent generalization of the same idea underlying the classical notion of inclusion. The parallel with classical logic and the meaning of fuzzy connectives thus provides additional motivation and guidance in defining concepts of fuzzy mathematics, besides the criteria of traditional fuzzy mathematics (which in practice often fail to prevent ad hoc definitions).

A further consequence of the closeness between classical and fuzzy logic is the fact that the three-layer architecture of classical mathematics (with the layers of logic, foundations, and particular theories) can be paralleled in fuzzy mathematics. (This was the leading idea of the position paper [34], included in this thesis.) The layer of foundations, provided by a sufficiently general formal theory over fuzzy logic, establishes a common language and a unifying framework for different disciplines of fuzzy mathematics. The foundational theory thus facilitates the exchange of concepts and results across the subfields of fuzzy mathematics.

As stressed above (p. 20), logic-based fuzzy mathematics directly formalizes only a limited part of traditional fuzzy mathematics. Nevertheless, its clearly isolated pre-

[26]Particular sets of inference rules—i.e., particular fuzzy logics—then reflect special assumptions on the structure of degrees.

[27]Though of course not too mechanically, as there are usually more options for finding a fuzzy counterpart to a crisp notion (e.g., if classically equivalent definitions are no longer equivalent in fuzzy logic). Some selection is needed, based on pragmatic criteria; often it leads to splitting classical notions, cf. [37, §4].

[28]I.e., to ensure that the truth degree of the consequent is at least as large as that of the antecedent. This kind of inference in deductive fuzzy logic is based on the local consequence relation, cf. [26].

theoretical assumptions, captured in the form of the axioms of the background fuzzy logic, enable it to find applications even beyond the traditional realm of fuzzy logic, namely in the areas where these extracted assumptions are applicable. An example of such application is the interpretation of deductive fuzzy logics in terms of resources or costs [11], similar to the resource-based interpretation of linear logic (cf. [92]). Under this interpretation of deductive fuzzy logic, the semantic values of formulae represent (prelinear) resources or costs rather than degrees of truth.[29] Deductions in formal fuzzy logic then preserve costs rather than partial truth, and particular fuzzy logics correspond to different ways how the costs can be summed by conjunction. The resource-based applications of deductive fuzzy logic (esp. in epistemic, deontic, and dynamic logics) are yet to be elaborated: currently they are just sketched in the present author's conference abstracts [20, 23, 25].

The latter connection between fuzzy logic and linear logic is just an instance of similar connections between fuzzy logic and other substructural logics. These links follow from the fact that deductive fuzzy logics are specific substructural logics (namely those with the law of prelinearity), and so usual motivations for having dropped structural rules apply to them as well. Logic-based fuzzy mathematics can therefore model specific (namely, prelinear) situations modeled by substructural logics. Even though substructural logics are mostly studied in their propositional forms (because of the problems with strong quantifiers, see Section 2.2), it is clear that more complex situations modeled by substructural logics would require first- or higher-order language. This motivates the need for substructural mathematics (cf. Section 2.1), of which logic-based fuzzy mathematics is a specific and important part. Possible generalizations of the methods of logic-based fuzzy mathematics to broader classes of higher-order substructural logics with a wider area of applications thus give another reason for the development of fuzzy mathematics in the logic-based setting.

The above paragraphs summarized *motivations,* i.e., "ex ante" reasons for developing logic-based fuzzy mathematics. However, there is also an "ex post" reason, namely the results already achieved in the framework of Fuzzy Class Theory. As witnessed by the papers included in this thesis (esp. [30, 41]), the logic-based approach is capable of trivializing certain parts of traditional fuzzy mathematics. This demonstrates that logic-based fuzzy mathematics is capable of providing powerful tools for traditional fuzzy set theory (which in turn is directly applicable in engineering practice).

4 Description of the author's contribution

This section provides a commentary on the papers comprising this thesis, with a special focus on several points. First, the relation of each paper to the topic of the thesis and to other papers connected with the project is explained. Second, some of the older papers are commented from the point of view of the later development of the theory. Finally, the author's contribution to the joint papers included in this thesis is indicated. (The co-authors have read the descriptions of author contribution and explicitly confirmed their accuracy by email.)

In order also to clarify the author's contribution to the project of logic-based fuzzy mathematics itself, a short history of the development of Fuzzy Class Theory is given first. Though unavoidably subjective, it tries to describe the emergence of ideas related to the project in as accurate way as possible. For the account of predecessor ideas and results upon which the project has been built see Section 2.

[29] Parts of this idea arose in discussions with Petr Cintula.

4.1 A short history of Fuzzy Class Theory

The author's master thesis [12] dealt with axiomatic set theory over Gödel logic.[30] The thesis was written shortly after the period (ca. in 2000–2001) when a small semi-regular seminar was organized in Prague by Petr Hájek, which was devoted to developing formal set theories over t-norm fuzzy logics and in which the present author actively participated.[31] The attempt at investigation of some basic disciplines of fuzzy mathematics (with fuzzified set theory and arithmetic as first choices) seemed to be a natural next step after first-order fuzzy logic and its metamathematics had advanced enough [110] to provide a meaningful machinery for such theories. The study of set theory based on Gödel logic was a reasonable choice as the latter logic extends intuitionistic logic in which successful variants of set theory had been built [179, 107, 85], and is closely related to the (slightly stronger) logic in which Takeuti and Titani's fuzzy set theory [193] had been constructed; also Takeuti and Titani's subsequent variant of fuzzy set theory [194], even though defined over a much stronger logic similar to ŁΠ, employs mainly Gödel operations in definitions, and therefore most of its constructions can be modified for set theory over Gödel logic, too. The seminar was, nevertheless, partly devoted to set theories and arithmetics over other fuzzy logics (esp. Łukasiewicz and BL), which later resulted in Hájek's study of Cantor–Łukasiewicz set theory with full comprehension over Łukasiewicz logic [115] and the construction of Hájek and Haniková's ZF-style set theory over BL [120]. The seminar laid stress on actual developing mathematics formally within the theories (in the spirit of Klaua's, Chapin's [56], and Gottwald's [99, 100] papers), and not just on the metamathematical study of their properties. Although the seminar stopped meeting in 2001, several participants continued investigating formal fuzzy set theory individually (including the present author, whose master thesis on the topic was defended in 2002). An attempt by Cintula, Hájek, and the present author to revive the seminar in 2003 led instead to the employment of the present author at the Institute of Computer Science (where the former two were working) and a close collaboration by the three on the topic, and eventually to the development of Fuzzy Class Theory and the current research project.

Fuzzy Class Theory was conceived in discussions between Petr Cintula and the present author during their research stay in Barcelona (at IIIA CSIC, Bellaterra) in October 2003. At that stage, only the first-order classes over the logic ŁΠ were considered, and the aim was to construct a common framework for the study of elementary operations and relations on fuzzy sets and fuzzy relations (such as various kinds of intersection, union, inclusion, etc.) over first-order fuzzy logic. The authors' motivations for this study, however, slightly differed from each other. P. Cintula had shortly before (in 2002) solved a problem on fuzzy orderings, presented to him by U. Bodenhofer, by means of first-order fuzzy logic (so in fact by using first-order classes) and wanted to continue the study of fuzzy orderings to see how far could the theory be developed with the limited means of elementary theory of fuzzy classes. The present author, on the other hand, had the experience from his work on Gödel set theory that a very large number of concepts of applied fuzzy mathematics can be defined and investigated just by means of first-order fuzzy classes (i.e., without considering membership of fuzzy sets in fuzzy sets). Even though elementary class theory consists for the most part just in translating the first-order predicate calculus into the set-

[30] The thesis was written in Czech; an English overview of its topic and methodological principles can be found in [13].

[31] Establishing the new focused seminar followed a series of talks on the same topic at the Seminar in Applied Mathematical Logic (an activity of the Czech Society for Cybernetics and Informatics) held at the Institute of Computer Science of the Academy of Sciences of the Czech Republic. Regular attendants were P. Cintula and Z. Haniková (then students of Petr Hájek), K. Bendová (later the supervisor of the author's master thesis related to the topic of the seminar), A. Sochor, K. Trlifajová, and several others.

theoretical language, it is actually just a theory of fuzzy *classes* which is mostly used in applied fuzzy set theory, rather than a fully fledged fuzzy *set* theory.[32] In particular, such concepts as the empty and universal class; the relations of inclusion, equality, disjointness, and compatibility; the properties of fuzziness and crispness, normality, and height; the unary class operations of complement, kernel, and support; the binary class operations of intersection, union, and difference; the properties of reflexivity, symmetry, transitivity, antisymmetry, and functionality of fuzzy relations; the operations of composition and inversion of fuzzy relations; and many other important relations and operations on fuzzy sets and fuzzy relations can all be expressed and investigated in the theory of first-order classes, not needing a theory with fuzzy sets of higher ranks (or orders).

The aim therefore was to have an axiomatic framework for the study of such concepts, with the possibility of quantification over classes (rather than just over atomic elements as in first-order fuzzy logic) and with the apparatus for handling tuples in order to internalize fuzzy relations (besides fuzzy classes). Since the tuples were intended just to represent multiple arguments of predicates with arities larger than 1, there was no need to fuzzify tuples, and the classical axioms for crisp tuples (regarded as crisp logical functions in the sense of [111]) could be adopted.[33] As fuzzy classes were to be treated in the same manner as crisp classes in classical second-order logic, axioms analogous to those of classical second-order logic could be adopted to describe them: the axiom scheme of comprehension, ensuring that each fuzzy property expressible in a fixed formal language defines a fuzzy class; and the axiom of extensionality, ensuring that a fuzzy set is uniquely determined by its members (i.e., by the truth values of membership of x in A for all x—in other words, by its membership function). The fuzzy logic ŁΠ was chosen for the background logic in order to have full arithmetic power over the system of truth degrees[34] in a system that would still enjoy good metamathematical properties. The logic was also suitable as a unified framework for the investigation of many different fuzzy set operations, by virtue of the representability of a large class of truth functions in the standard ŁΠ-algebra.

Initially, the theory was expected to provide little more than a convenient framework for easy proofs of schematic theorems on several kinds of intersection, union, inclusion, etc. However, the full potential of the theory was realized soon (before the end of 2003). The present author observed that the fragment of class theory reducible to propositional logic [30, Th. 33–36] is so large that it covers most interesting elementary theorems of traditional fuzzy set theory. Jointly we observed that by iterating the machinery for classes of higher orders, the expressive power of the resulting simple fuzzy type theory (FCT) is sufficient for a large part of traditional fuzzy mathematics, as classical higher-order theories are interpretable in FCT [30, L. 41] (so we can assume any crisp structure on the universe of discourse), and moreover such concepts of fuzzy set theory as Zadeh's extension principle become definable objects of FCT [30, Def. 39]. In this form, the theory

[32]By a class theory we mean the study of classes that contain atomic individuals from some fixed domain, but the membership of classes in classes is not considered. Set theory proper, on the other hand, is the study of sets as objects that contain other sets or objects and are themselves members of other sets.

[33]Subsumption of sorts of variables had to be introduced for convenient handling of tuples; this was done by Petr Cintula when our discussions convinced us that other possibilities would probably not be more easily implementable. Even though sorted first-order languages had been used before [110, 46], subsumption of one sort by another had not yet been considered in formal fuzzy logic.

[34]Insufficient expressive power could lead to the undefinability of various notions of traditional fuzzy mathematics. For instance in set theory over Gödel logic without $\triangle$, even such basic concepts as the normality and the crisp kernel of a fuzzy set are undefinable [12].

was presented at the FSTA conference in January 2004, where also the first version of the paper [30] was finished.

The expressive power of the theory suggested the possibility of its foundational role for fuzzy mathematics, analogous to that of Russell's simple type theory for classical mathematics. Furthermore, the fact that the underlying logic of the theory was fuzzy offered a consistent methodology of fuzzification of classical notions by a (controlled) reinterpretation of classical defining formulae in fuzzy logic (cf. [126, §5]), building upon the corresponding roles of classical and fuzzy logical symbols (cf. p. 21 above). The idea of a foundational research program based on this methodology and implemented by means of FCT emerged in a discussion between P. Cintula and the present author at an institutional workshop in March 2004. The foundational program was then described in the manifesto [34] (presented at *The Challenge of Semantics* in Vienna, July 2004) and the research program was elaborated into a grant proposal in April 2004. The grant was awarded for 2005–2007 and the grant team included, besides P. Cintula (the principal investigator) and the present author, also T. Kroupa (as the principal co-investigator) and R. Horčík. The latter two focused on the development of particular disciplines of fuzzy mathematics in FCT: R. Horčík on fuzzy intervals [134] and fuzzy quantifiers [65, 64] and T. Kroupa on fuzzy filters [149] and fuzzy topology [43, 42, 44]. The description [32] of the foundational program won the Best Paper Award at the 11th IFSA World Congress in Beijing, July 2005.

The next task after the development of the basic apparatus of FCT was to advance a formal theory of fuzzy relations within its framework, as fuzzy relations are indispensable in all disciplines of fuzzy mathematics. Following P. Cintula's previous contacts in this area, in November 2004 we started a cooperation with Ulrich Bodenhofer, focusing on basic properties of fuzzy preorders and similarities. The first joint results [33, 49] were presented at the Linz Seminar in February 2005, and the cooperation eventually led to the comprehensive paper [28], finished in 2007.

Since 2005, the investigation of particular disciplines of fuzzy mathematics has begun and the project participants turned their interests to various directions; only a sketchy description of these activities can be given here.

A different approach to basic properties of fuzzy relations, making them relative to a fuzzy relation representing indistinguishability of elements, was proposed by the present author at IPMU 2006 [19]. In 2005, the present author started working with M. Daňková on properties of fuzzy relational operations that had not been covered by his joint paper with Cintula and Bodenhofer. It was soon realized that many relational operations had a form similar to either Zadeh's [213] sup-T relational composition or Bandler and Kohout's [6] BK-product (i.e., inf-R composition) of fuzzy relations. The informal correspondence was made precise by means of internalized truth values (cf. [194]) and formal interpretations [21] by the present author, and systematically explored in a joint paper with M. Daňková [41]. The method described in the paper provides a reduction to a simpler calculus for fuzzy relational operations, in a similar manner as the metatheorems of [30] do for class operations.

The internalization of truth values described in [41] initiated later (in 2007) an investigation of graded properties of truth-value operators (e.g., t-norms, copulas, etc.) under a Czech–Austrian project on aggregation operators. The first results (by U. Bodenhofer, P. Cintula, S. Saminger-Platz, and the present author) were presented at the Linz Seminar 2008 [29]; a full paper is in preparation. In 2004–5, the first steps were also done in the logic-based theory of measures on clans of fuzzy set by T. Kroupa [148] and fuzzy Dedekind–MacNeille lattice completion and fuzzy Dedekind reals by the present author [15, 16]. An application of the formalism to the fuzzified logic of questions, sketched by the present author at the VlaPoLo workshop in Zielona Góra as early as in November 2003, was turned into a full paper [14] in 2004. Several further areas are currently under investigation; for an overview of the work in progress and future plans see the end of this section.

During the work on formal fuzzy mathematics, several peculiar features of axiomatic theories over fuzzy logic have been noticed which are not met in classical nor mainstream fuzzy mathematics. These features, due mainly to the non-idempotence of strong conjunction and thus common to mathematical theories in all contraction-free substructural logics, have been summarized in [37]. The different style of fuzzy mathematics ensuing from these peculiarities has been gradually introduced in papers since 2006, cf. [19, 28, 29, 43, 42]. This also emphasized the need of exponentials and generalized fuzzy quantifiers for fully fledged formal fuzzy mathematics (to be worked out yet, with initial results in [65, 64]) and directions for further elaboration of the basic apparatus of FCT.

The experience with fuzzy mathematics also helped to analyze fundamental differences between the fundamental assumptions of mainstream fuzzy mathematics and logic-based fuzzy mathematics. As argued by the present author in [26], logic-based fuzzy mathematics directly addresses only a very specific portion of traditional fuzzy mathematics, and even though its apparatus is powerful enough to encompass a much larger area of traditional fuzzy mathematics, the advantages of the logic-based approach are manifested best in problems close to its own principles and motivation (i.e., logical inference preserving the degrees). The scope of the logic-based approach thus should be specified more narrowly than originally in the manifesto [34]. Nevertheless, its applicability is still broad enough to make it a significant part of mainstream fuzzy mathematics, with clear methodology and interpretation.

At present, the project of logic-based foundations of fuzzy mathematics is by no means finished and continues to be under permanent progress. Among the proximate future tasks is the elaboration of the theory of fuzzy quantifiers and their application in all disciplines of logic-based fuzzy mathematics (which would include revisiting areas that have already been developed, and a thorough study of new notions defined by means of such quantifiers). Another important topic is the notion of fuzzy function, which has not yet been sufficiently investigated in FCT, either. The notion can then be employed for defining in FCT the concepts of fuzzy cardinality (based on fuzzily bijective fuzzy functions) and fuzzy morphism of fuzzy structures. Various properties of fuzzy orderings have not yet been systematically studied, for instance linearity, directedness, or well-foundedness. Fuzzy topology, fuzzy aggregation operators, and fuzzy interval arithmetic are currently under study; fuzzy lattices, measures, and metric spaces are possible candidates for forthcoming topics of research in FCT.

4.2 The papers comprising the thesis

This section comments on the papers comprising the thesis. The papers are grouped and ordered by topic rather than chronologically, in order to give an exposition of the theory proceeding in a logical way from the methodological assumptions and the basic apparatus of FCT to more advanced disciplines of fuzzy mathematics. The texts of the papers were recompiled for inclusion in the thesis, and may therefore differ from the published versions in such details as formatting, numbering of footnotes or references, etc.[35] Several typos that occurred in the published papers have also been fixed in the present version.

[35]The applicable copyright transfer agreements allow including the papers in a thesis.

L. Běhounek: *On the difference between traditional and deductive fuzzy logic* [26]. The paper analyzes methodological principles of logic-based fuzzy mathematics and demonstrates them to be fundamentally different from those of traditional fuzzy mathematics. The paper shows that even though most concepts of traditional fuzzy mathematics can be modeled in higher-order fuzzy logic (as its expressive power includes classical mathematics), the logic-based rendering of notions that are based on principles alien to deductive fuzzy logic is rather artificial and gives little advantage over studying such notions by traditional methods. Therefore, the logic-based approach is best suited to a specific area of fuzzy mathematics consonant with its methodological assumptions (namely those related to the deductive treatment of partial truth), and its foundational significance is smaller in other areas of fuzzy mathematics.

The paper was based on several years of experience with developing logic-based fuzzy mathematics; therefore it could make distinctions that had not been recognized in the Manifesto [34] written at the beginning of the research program. Even though the more precise delimitation of the scope of the foundational program could be seen as a retreat from the too optimistic tone of the Manifesto (which purported to give foundations to all fuzzy mathematics), it can on the other hand be interpreted as a clarification of the fact that traditional fuzzy mathematics actually deals with several phenomena that are too different from each other, and therefore it in fact comprises several different fields of research. The field in which the logic-based approach is most fruitful is marked by a clear interpretation of membership degrees as degrees of truth (preserved under inference), while other areas of fuzzy mathematics work with a mixture of several different conceptions of membership degree (cf. [74]), often not clarified enough. Naturally, logic-based methods apply in a less straightforward manner to such fields. The paper thus presents a more precise delimitation of the area of research, rather than a retreat from the foundational program.

Although the paper uses the term *partial truth* frequently, it was not meant to engage in the philosophical dispute on the nature of truth and its (un?)necessary bivalence:[36] the term should be understood in the technical sense of "the (gradual) quality of propositions that is preserved under the deductions in fuzzy logic". The gradual quality is in the paper called "partial truth" in analogy with the (bivalent) quality transmitted in deductions of classical logic, which is usually called—and understood as—*truth.* Whether we call the gradual quality "partial truth" or another name has no effect on the observations made in the article: the only important thesis is that, similarly as classical logic operates *salva veritatis,* deductive fuzzy logics infer their conclusions *salvo gradu*—i.e., preserving the grades assigned to propositions,[37] no matter whether the grades are interpreted as degrees of truth, a measure of the underlying attributes [140], utility values [90], costs [11, 25], or grades of any other kind.

The term *deductive fuzzy logic* is in the paper used for logic-based fuzzy mathematics in general (i.e., not only for formal fuzzy *logic* in the strict sense), since the intended audience usually employs the term *fuzzy logic* (both in Zadeh's [214] broad and narrow sense) in the broader sense of *fuzzy mathematics.* The term is in the paper additionally given a concrete mathematical meaning of the logics of linear residuated lattices, which delimits the class of logics upon which logic-based fuzzy mathematics in our sense can be built.

[36] This was not stressed in the paper, as the intended audience were researchers in traditional fuzzy logic rather than philosophers.

[37] *Preserving* should here be understood in the sense of the local consequence in substructural logics (cf. footnote 12 on page 11 above and see [26] for details), not in the sense of [50, 86].

In the paper, the structural rule of exchange (i.e., commutativity of conjunction) is assumed for deductive fuzzy logics. This assumption was based on the idea that while the absence of the rules of contraction and weakening can be motivated by considerations about truth degrees,[38] non-commutativity of conjunction is motivated by (e.g., temporal) considerations that are not related to degrees of truth. However, since fuzzy logics can [11, 20, 23, 25] also be motivated as logics of resource-aware reasoning (or logics of costs), and the rule of exchange can fail for resources (i.e., fusion of resources need not be commutative), it is more reasonable to discard in general the assumption of commutativity, too.

In [86], Josep Maria Font proposed to call the intervals $\{\beta \in \mathrm{L} \mid \beta \geq \alpha\}$ for each $\alpha \in \mathrm{L}$ truth *degrees,* as opposed to the truth *values* $\alpha \in \mathrm{L}$. Then one can say that it is a truth degree what is preserved by fully true implication in deductive fuzzy logics, rather than a truth value. Font's distinction is consonant with the considerations presented in the discussed paper and provides a better formulation of what in the discussed paper is described as "guaranteed degrees of truth", "guaranteed truth thresholds", etc.

L. Běhounek, P. Cintula: *From fuzzy logic to fuzzy mathematics: A methodological manifesto* [34]. The paper was written in June 2004 and presented at the workshop *The Challenge of Semantics* in Vienna in July 2004. The main motivation for writing the paper was to have a concise description of the methodology of logic-based fuzzy mathematics (called *Hájek's program* in the Manifesto) that could be referred to in subsequent papers. The contents of the paper arose from extensive discussions between both authors and is their joint work. The structure and actual wording of the paper was drafted by the present author and finalized by both.

At the moment of writing it was assumed that deductive fuzzy logics could provide foundations for the whole of traditional fuzzy mathematics. While this is true to some extent, the best-suited area of applicability of the approach was later clarified in [26]; see the previous paragraph on [26] for details.

A skeptical attitude towards the methodology described in the Manifesto (and towards non-classical many-valued mathematics in general) was expressed by D. Dubois in [72, p. 195–6]:

> Although some may be tempted to found new mathematics on many-valued logics [34], this grand purpose still looks out of reach if not delusive. It sounds like a paradox of its own since we use classical mathematics to formally model many-valued logic notions. What could be named "many-valued mathematics" essentially looks like an elegant way of expressing properties of many-valued extensions of Boolean concepts in a Boolean-like syntax. For instance, the transitivity property of similarity relations is valid in Łukasiewicz logic, and, at the syntactic level, exactly looks like the transitivity of equivalence relations, but should be interpreted as the triangular inequality of distances measures.

To answer the criticism, the following clarification should be given first. Formal fuzzy mathematics based on the methodology of [34] can essentially be understood in any of the following two ways:

[38] Namely, by observing that combining imperfect truths combines their imperfection, which justifies the general non-idempotence of conjunction; and that there can be degrees of full truth (e.g., in such predicates as *acute angle*)—i.e., that the residuated lattice of truth degrees need not be integral.

- In a *"traditionalist view",* logic-based fuzzy mathematics is just a methodologically advantageous treatment of traditional fuzzy mathematics, where in exchange for voluntary restrictions on the language and methods we obtain a formalism that enables us to derive theorems of certain forms more easily (cf. [30, §3.4] or [41]). Under this view, the formulae of higher-order fuzzy logic indeed describe the behavior of membership functions valued in the real unit interval or more generally in an appropriate semilinear residuated lattice.

  One of the benefits of this approach indeed comes from the fact that many notions of traditional fuzzy mathematics turn out to be expressed by formulae of exactly the same form as analogous notions of classical mathematics—e.g., T-transitivity by a formula expressing classical transitivity, only reinterpreted in many-valued logic. This enables to treat fuzzy notions in a similar way as classical notions: e.g., the proofs of theorems often just copy classical proofs. Moreover, it allows us to extrapolate this observed correspondence and find new important notions of fuzzy mathematics by reinterpreting classical definitions in fuzzy logic. Furthermore, when employing many-valued logic, all defined notions become naturally graded, which radically facilitates the study of graded properties (in the sense of [101]) of fuzzy notions.

- Alternatively, in a *"foundationalist view",* logic-based fuzzy mathematics presents a fundamental treatment of fuzzy mathematics (indeed a "new mathematics", as called by Dubois in the cited passage of [72]), based on non-classical logics. This interpretation understands fuzzy sets as a *primitive notion,* axiomatized (or governed) by the axioms and rules of the non-classical logic, in a similar manner as crisp sets are governed (and can be axiomatized) by the axioms and rules of classical logic.

  Under this approach, fuzzy sets are *not* represented or modeled by their membership functions, but are primitive objects *sui generis.* Pre-theoretical considerations (cf. [22, 117]) about (certain kinds of) vague propositions suggest that they can be assumed to be governed by the laws of the fuzzy logic MTL or some of its variations. Importantly, the justification of the logical laws governing vague propositions is pre-theoretical and independent of any model of fuzzy sets in classical mathematics. Based on the axioms and rules of fuzzy logic, a formal theory of fuzzy sets can be developed, with the intended informal semantics of *actual* fuzzy sets, i.e., unsharply delimited collections of objects—similarly as the intended informal semantics of classical sets is that of sharply delimited collections of objects. The *formal* semantics of fuzzy logic is then formed by fuzzy sets described by (a fragment of) the very same theory itself—similarly as the semantics of classical logic is formed by sets described by (a fragment of) classical set theory (i.e., the same form of 'circularity' is encountered as in the foundations of classical mathematics).

  It turns out that, *incidentally,* the theory of fuzzy sets can be *formally interpreted* in classical mathematics: this formal interpretation is what more usually is called "the many-valued semantics" of fuzzy set theory, in which fuzzy sets become interpreted by "membership functions". Although classical mathematics is thus, by means of the formal interpretation, capable of faithful modeling fuzzy mathematics, it does not establish its priority over fuzzy mathematics, as both theories can be founded independently of each other and are faithfully interpretable in each other.[39]

[39] A formal interpretation of classical mathematics in fuzzy mathematics can be done by means of the propositional connective $\triangle$—which is no wonder as the connective is intended to represent crisp propositions among fuzzy ones and is axiomatized by the laws valid for crisp sets.

> Both classical and fuzzy mathematics are therefore of equal standing as foundational theories, and precedence can be given to one of them only on the basis of some pragmatic criteria. Classical mathematics may be preferred because of our long experience with it or because of its simplicity (as it only considers crisp sets). Fuzzy mathematics, on the other hand, can be preferred in vague contexts because it renders vaguely delimited sets more directly, and because of the advantages of its apparatus in proving theorems on fuzzy sets as described under the traditionalistic view above.

It can be seen that the three points of the above criticism of many-valued mathematics from [72, pp. 195–196], namely that

1. "we use classical mathematics to formally model many-valued logic notions",

2. "what could be named 'many-valued mathematics' essentially looks like an elegant way of expressing properties of many-valued extensions of Boolean concepts in a Boolean-like syntax", and that

3. "the transitivity property of similarity relations [...] should be interpreted as the triangular inequality of distances measures",

only apply to the traditionalistic view of the non-classical theory. The second statement is explicitly admitted in the Manifesto [34, p. 643]:

> Admittedly, a formal theory over fuzzy logic is just a notational abbreviation of classical reasoning about the class of all models of the theory.

Still, the advantages of the logic-based approach fully justify the development of logic-based fuzzy mathematics even under the traditionalistic interpretation. The possibility of the foundationalist attitude, however, shows that the non-classical theory need not be regarded just as formally modeling many-valued notions while still using classical mathematics. Rather, the non-classical notions can be regarded as primitive and independent of classical mathematics: since the theory is *syntactical,* it does not need to presuppose that classical mathematics has been developed first. And under the foundationalist approach, the transitivity of similarity relations is *not* interpreted as the triangular inequality of distance measures, but indeed as *transitivity* of unsharply delimited relations (regarded as *primitive* entities). The application of the name "transitivity" to fuzzy relations is then justified by the fact that $\mathrm{Trans}\, R \equiv_{\mathrm{df}} (\forall xyz)(Rxy \mathbin{\&} Ryz \to Rxz)$ is the necessary and sufficient graded condition ensuring that Rxz can for any instances of x, y, z be inferred[40] from Rxy and Ryz (which is exactly the property we usually call "transitivity"). Only accidentally the property coincides, when fully true, with the notion of T-transitivity that is known from traditional fuzzy mathematics and that expresses the triangle inequality of distance measures.

In sum, Dubois' criticism of [72] only applies to the traditionalist understanding of logic-based fuzzy mathematics, and not to the foundationalist one. But even under the traditionalistic view, logic-based fuzzy mathematics has undisputable advantages described above, which fully justify its development.

[40]In the graded way, i.e., preserving the truth degrees in the sense of the local consequence of deductive fuzzy logics, see footnote 12 on page 11 or [26].

L. Běhounek, P. Cintula: *Fuzzy class theory* [30]. This chronologically first paper on Fuzzy Class Theory introduced its apparatus, demonstrated its expressive power (by interpreting notions of fuzzy and classical mathematics) and hinted at benefits of its formal methods (by reducing a large part of graded elementary theory of fuzzy classes to propositional calculations).[41]

In the paper, the logic ŁΠ was used as the background logic of the theory, because of its expressive power. The aim of the paper was to construct a unified framework for most of fuzzy mathematics, which required having a large class of t-norm based propositional connectives interpretable in the underlying logic. The logic ŁΠ which interprets all finite ordinal sums of the three basic continuous t-norms (Ł, G, and Π) and many left-continuous t-norms (e.g., NM) as well as their residua while still retaining good metamathematical properties provided a suitable compromise between the expressive power and simplicity of the logic. For the sake of generality, all notions were in the paper defined relative to an arbitrary ŁΠ-representable t-norm (intended to interpret the connectives in the defining formula), and theorems and proofs were formulated schematically, with connectives indexed by ŁΠ-representable t-norms. Later the practice showed that connectives pertaining to different t-norms are seldom mixed in particular disciplines of logic-based fuzzy mathematics, and that it is therefore more convenient to work in a fragment of ŁΠ containing just the connectives needed for the particular purpose. The schematic formulae indexed by an ŁΠ-representable t-norm can then be replaced by formulae of the logic $\mathrm{MTL}_\triangle$, which is sound for any left-continuous t-norm. If needed, the logic $\mathrm{MTL}_\triangle$ can be strengthened by additional assumptions (e.g., to $\mathrm{IMTL}_\triangle$ if the involutiveness of the residual negation is required) or expanded by additional connectives (e.g., to $\mathrm{MTL}_\sim$ if an independent involutive negation is needed). Currently, therefore, logic-based fuzzy mathematics is mostly done in FCT over $\mathrm{MTL}_\triangle$ or a similarly weak logic rather than over ŁΠ.[42] It is obvious that the apparatus of FCT can straightforwardly be transferred to any deductive fuzzy logic that extends $\mathrm{MTL}_\triangle$. Fuzzy Class Theory over ŁΠ, nevertheless, remains being the common framework for the study of notions pertaining to different $\text{Ł}\Pi\frac{1}{2}$-representable t-norms in one theory, and thus (a candidate for) the common foundational theory for logic-based fuzzy mathematics.

The paper, though necessarily technical, was also aimed at the audience not specialized in formal fuzzy logic; therefore some technical details were only sketched (e.g., the apparatus of tuples) or not discussed at all (for instance, that the comprehension schema should be extended to formulae of the enriched language if new symbols are added to the language, as in Section 6 of the paper). As it was sufficient for the basic development of logic-based fuzzy mathematics, only the axioms of extensionality and comprehension (and, optionally, fuzziness) were considered in the paper, although it was already clear that advanced topics in logic-based fuzzy mathematics will sooner or later require some forms of the axiom of choice or similar principles. Since only the basics of fuzzy mathematics have been investigated by now, such a need has not arisen yet. The expected complexity of the relationships between possible variants of choice principles over fuzzy logic makes them another topic for future investigation.

[41] Theorems on fuzzy sets that are typically found on the first several dozens of pages in standard textbooks in fuzzy set theory (e.g., [166, 145]) are corollaries of the metatheorems [30, Th. 33–36] and simple theorems of propositional fuzzy logic. Since usual propositional deductive fuzzy logics are decidable, the metatheorems show that basic properties of fuzzy sets could easily be generated by a computer program. A similar comment applies to the theorems on fuzzy relations from the paper [41] described below.

[42] Working in $\mathrm{MTL}_\triangle$ is slightly more general than the schematic work in ŁΠ, since it admits interpreting the connectives by all left-continuous t-norms rather than only those representable in ŁΠ. Notice, however, that *propositional* MTL is complete w.r.t. all left-continuous t-norms representable in $\text{Ł}\Pi\frac{1}{2}$ [155].

The paper was written in January 2004; the authors' motivations for the study of the theory are described in Section 4.1. In the actual preparation of the paper, all parts were extensively discussed by both authors, and most sections are their joint work. Of particular developments that are due mainly to one of the authors, the apparatus of subsumption of sorts in first-order fuzzy logic was prepared by Petr Cintula, while the metatheorems of Section 3 were observed by the present author.

L. Běhounek, U. Bodenhofer, P. Cintula: *Relations in Fuzzy Class Theory: Initial steps* [28]. The paper treats basic properties of fuzzy relations in the graded setting of FCT. Since relations occur in all parts of mathematics, the investigation of basic properties of fuzzy relations in FCT is an indispensable prerequisite for all disciplines of logic-based fuzzy mathematics. A parallel aim of the paper was to present Fuzzy Class Theory to researchers in traditional fuzzy mathematics and introduce to them the fully graded approach in fuzzy mathematics. To this end we wanted to recast in FCT known theorems on graded properties of fuzzy relations (esp. those from Gottwald's monograph [104, Ch. 18]), and to give graded generalizations of some representative non-graded results of traditional fuzzy mathematics; a few previously unknown concepts and results were discovered along the way, too. As it was impossible to cover the whole area of fuzzy relations, the paper focused mainly on fuzzy preorders and similarities; but even with this reduction of scope, the paper could only treat a selection of their most basic properties. Further classes of fuzzy relations (e.g., fuzzy orderings or functions) still wait for a thorough investigation.

Several parts of the paper have preliminary versions in conference proceedings [33, 49, 61, 16, 27]. In order to keep the introduction to the paper short, a primer in Fuzzy Class Theory [35] was written and made freely available online as a research report.

The paper was written over the period of more than three years (2004–7), mostly during several research stays of the Czech co-authors at Johannes Kepler University in Linz. All parts of the paper were edited, discussed, and checked for correctness by all of the co-authors. Particular contributions of the co-authors (so far as they can be determined) were as follows: Ulrich Bodenhofer provided the examples and links to known results of traditional fuzzy mathematics (cf. [49]), wrote most of the Introduction, edited many passages in other sections, and collaborated on several parts of the paper (esp. in Sections 4, 6, and 7). Most of the introductory Section 2 and the Appendices were written by Petr Cintula and the present author (cf. [37, 35]); the latter is also responsible for Section 5 (on bounds, cf. [16]) and smaller parts of other sections. Section 6 (on Valverde representation, cf. [27]) is a joint work of all three co-authors. Section 7 (on partitions, cf. [61]) is mostly due to Petr Cintula, who also produced most proofs in Sections 3 and 4 (all of these proofs were presented in the paper in order to keep the exposition self-contained, even though some of the properties follow independently from more general theorems of [41]).

The clumsy proof of Corollary 4.11(I52) in the published version of the paper resulted from a trivial mistake discovered only when the final version was already submitted. The statement has in fact a trivial proof that uses just the monotony of the opening and closure operators and of min-intersection and max-union with respect to inclusion.

L. Běhounek, M. Daňková: *Relational compositions in Fuzzy Class Theory* [41]. The paper, written in 2006–2007, was originally intended to deal with properties of relational notions not covered by [28] (then under preparation) such as Cartesian products or preimages. However, it was soon realized by the present author that most of such notions are just instances of (sup-T or inf-R) relational compositions for arguments with lesser arities, and that this relationship, which had only informally been sketched in Bělohlávek's monograph [46, Rem. 6.16], can be made precise in the formal framework of FCT by means of syntactic interpretations (cf. [21]). The systematic exploitation of the correspondence (including classes of arity 0 that represent truth values, cf. [18, §2]) resulted in a systematic and uniform description of more than 30 relational notions, with many properties translated automatically from a few basic properties of relational compositions. A large class of properties of these notions furthermore turns out to be derivable from a few identities in a simple equational calculus for fuzzy relations. The method thus provides a reduction of a large fragment of the elementary theory of fuzzy relations to a much simpler calculus, comparable to the reduction of a fragment of the theory of fuzzy classes to fuzzy propositional calculus by the metatheorems of [30, §3]. The fact that the fuzziness of the relations under consideration does not play a significant role in the application of the equational calculus to the relational notions further supports the thesis that with a suitable apparatus (here, of deductive fuzzy logic), the generalization of some parts of classical mathematics to fuzzy sets is rather straightforward (cf. the end of Section 1).

Even though a larger part of the paper is due to the present author, Martina Daňková had an indispensable role in the exhaustive derivation of relational properties in the equational calculus and providing links to the applied practice (esp. Examples 5.12–13). She also prepared and presented the preliminary conference version [40] of the paper and made a search for relevant literature. All parts of the paper were discussed and checked for correctness by both co-authors.

L. Běhounek: *Extensionality in graded properties of fuzzy relations* [19]. The conference paper, presented at IPMU 2006, offers new definitions of basic graded properties of fuzzy relations, relative to a fuzzy indistinguishability relation between the objects of discourse. The approach is part of the effort to avoid hidden crispness in definitions, suggested already in the original FCT paper [30, §7]: the proposed definitions eliminate the implicit crisp identity of traditional graded definitions that is hidden in using multiple references to the same variable, and replace it by an explicit fuzzy indistinguishability relation E; the traditional definitions are then the special cases for E equal to the crisp identity relation Id. The paper gives arguments supporting the need for such definitions, answers the counter-argument referring to an infinite regress, and shows that the traditional property of extensionality of a fuzzy relation w.r.t. an indistinguishability, which in the non-graded setting has the same motivation as the new definitions, cannot substitute the new definitions in the graded setting (although it can do so in the non-graded setting). The paper furthermore offers an explanation why only some of the indistinguishability-based properties have previously been defined in the non-graded setting.

It was not mentioned in the paper, though it should have been, that also E-functionality had been defined in the traditional non-graded setting (alongside several variant definitions of a fuzzy function) by Demirci [68, 69].

The conference paper only gave results relevant to its main theses, rather than a comprehensive list of properties of indistinguishability-based properties; these will be given in a full paper, which is currently under construction.[43] The full paper will also

[43] Incidentally, all results included in the conference paper were first-order; therefore just first-order logic ($\mathrm{MTL}_\triangle$) could be employed. The higher-order setting is, nevertheless, needed for the study of

extend the game-theoretically motivated generalization sketched in Section 4 of the paper and a systematization of the properties in terms of sup-T and inf-R compositions.

L. Běhounek: *Towards a formal theory of fuzzy Dedekind reals* [16]. The conference paper, presented at EUSFLAT 2005, presents a sketch of a theory of fuzzy real numbers and fuzzy intervals based on the Dedekind completion of an underlying structure of crisp numbers. Besides purely theoretical motivations, one of the aims of the investigation was to model the traditional notion of fuzzy number (cf. Section 2.3 above, p. 16) in the logic-based framework. The (fuzzified) lattice completeness of the resulting fuzzy real numbers, construed as fuzzy Dedekind cuts, is proved, and the transition from Dedekind cuts to fuzzy intervals representing traditional fuzzy numbers is sketched.

Only fuzzy reals satisfying the defining conditions to degree 1 were considered in the paper, partly for simplicity and partly in order to adhere to the motivation that Dedekind cuts express the distribution of the fuzzy real number (which would be violated by any misbehavior of its distribution function, and therefore the condition should be satisfied to the full degree). The definitions are thus regarded as the *axioms* for fuzzy Dedekind cuts, rather than graded conditions. Results on the graded notion of fuzzy real would for a large part be obtainable by replacing $\triangle$'s by suitable exponents, but this generalization might not be very interesting for mainstream fuzzy mathematics, as traditional fuzzy reals form a crisp class, too.

Even though only crisp rationals were considered for the underlying structure in this paper (as they are sufficient for generating a structure of fuzzy reals), the results obviously hold for each dense linearly ordered field of numbers (e.g., crisp real numbers, which are more often used for a construction of fuzzy numbers in the mainstream fuzzy mathematics). Fuzzy lattice completions of crisp dense linear orders are studied in more detail in the author's workshop paper [15], where two methods of obtaining a fuzzy lattice from such crisp orderings (viz, by Dedekind cuts and MacNeille stable sets, which differ in the fuzzy setting) are described.

The fuzzy lattice completion employed in the paper differs from fuzzy lattice completions described earlier in the literature [126, 46]: while [126] and [46] study the *minimal* fuzzy lattice completions of *fuzzy* orderings (achieved by MacNeille stable fuzzy sets), the present paper is concerned with a fuzzy completion of a *crisp* ordering, which need not be minimal, but should contain all fuzzy Dedekind cuts. Despite different settings and definitions in [126, 46], some results are nevertheless similar (for more details see [15]).

A similar approach to fuzzy numbers (or intervals) has later been taken by Horčík in his paper [134] on fuzzy interval analysis, where analogous results on representation and arithmetic of fuzzy intervals have been derived.

The results of [28, §5] on suprema were originally derived by the present author for the purposes of the discussed conference paper [16]. Therefore most proofs omitted from [16] can be found in [28, §5]. A full paper on this topic is still in progress; the main obstacle to finishing it is an as yet unclarified suitable definition of multiplication of fuzzy Dedekind cuts. (Observe that Horčík [134] also defines just multiplication by a scalar, i.e., a crisp number, which is unproblematic.) The aim is to extend the operations from the underlying crisp numbers to fuzzy cuts A in such a way that Aq can be interpreted as the truth value of $A \leq q$ (or a measure of the distribution of the fuzzy real A in $(-\infty, q]$), with the ordering preserved by the extended operations (cf. [16, §4]). Zadeh's extension

preservation of the indistinguishability-based properties w.r.t. class unions, intersections, etc., which will be given in the full paper.

principle works to this effect only if the original operation on crisp numbers is monotone; a suitable definition of multiplication of fuzzy cuts therefore has to separate positive numbers from negative ones (cf. the definition of multiplication for crisp Dedekind cuts), but the details of the construction that would capture the informal motivation correctly are not yet clear enough.[44] The 'game-theoretical' considerations on the interpretation of the truth values of the operands hinted at in Horčík's paper [134] (similar to those sketched in [19, §4]) should also be taken into account. At present, a sound treatment of the extended operations on fuzzy intervals remains a subject for future work.

L. Běhounek: *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* [14]. This early (and in many respects premature) paper is included in the dissertation in order to demonstrate a possible application of logic-based fuzzy mathematics as a formal semantics for fuzzified non-classical logics. By defining a fuzzified consequence relation of a non-classical logic in Fuzzy Class Theory, the fuzzified non-classical logic gets formally interpreted in FCT (i.e., in higher-order fuzzy logic). The apparatus of FCT then provides a well-defined framework for introducing semantical notions of the non-classical logic and deriving its metamathematical properties.

The paper avoids the problem of quantification over a fuzzy domain W by requiring the crispness [14, §6] or full contractivity [14, §7] of W; an adequate account for arbitrary fuzzy logical spaces would need a better understanding of quantification over a fuzzy domain (cf. footnote 45 and comments on [43, 42] below). The paper only deals with yes–no questions, since yes–no partitions of a logical space are explicitly definable by means of negation; a logic-based theory of fuzzy partitions, needed for choice questions and first-order fuzzy erotetic logic, had not yet been developed in the time of writing the paper. A possible extension to fuzzy choice questions or to first-order fuzzy erotetic logic, generalizing the framework of [109], could use the results of [28, §7] on graded T-partitions: by [28, §7], T-partitions correspond to fuzzy equivalences (even in the graded manner); graded T-partitions thus provide a well-motivated basis for a partition semantics of fuzzy questions. This approach would enable to fuzzify the notion of question itself, by considering the fuzzy notion of T-partition. (In the discussed paper [14], the concept of fuzzy yes–no question itself is crisp, although it admits fuzzy answerhood.)

Further applications of the apparatus of FCT in the semantics of non-classical logic are sketched by the present author in the workshop paper [25] on fuzzified propositional dynamic logic (employed for modeling costs of program runs) and the Czech conference papers [23, 24] on fuzzified epistemic logic (employed for modeling feasible and vague knowledge). Full journal papers based on these conference papers are being prepared.

L. Běhounek, T. Kroupa: *Topology in Fuzzy Class Theory: Basic notions* [43]; *Interior-based topology in Fuzzy Class Theory* [42]. The conference papers [43] and [42] were presented, respectively, at the IFSA World Congress 2007 and the Conference of EUSFLAT 2007 (where the latter paper won the Distinguished Student Paper Award). The papers present the first treatment of fuzzy topology in the framework of FCT: they investigate the mutual relationships between alternative graded definitions of a fuzzy topology, namely by open or closed fuzzy sets [43, §3], fuzzy neighborhoods [43, §4], and fuzzy interior operators [42].

[44]The definition of multiplication for fuzzy cuts over the discrete domain of integers, extending a cardinality-based multiplication of natural numbers, could help to clarify the matter; however, the theory of fuzzy functions and cardinalities has to be developed first.

Only fuzzy topologies over crisp universes have been considered in [43, 42]. This restriction has rather been chosen for methodological than technical difficulties. Even though it would be technically quite straightforward to generalize the notions for fuzzy universes (in exchange for several more exponents in definitions), the meaningfulness of such definitions would need a much more thorough discussion. For example, fuzzy domains are not preserved under the usual (sup-T) composition $\circ$ of fuzzy mappings, which makes many straightforward constructions over fuzzy topological spaces ill-defined (as the mapping $F \circ G$ is defined on *another* fuzzy space, different from the domain of F) or ill-motivated. The motivational discussion needed to avoid ad hoc solutions exceeds the scope of fuzzy topology, as the questions encountered here are particular instances of more general problems of quantification over a fuzzy domain, which have not yet been satisfactorily addressed by deductive fuzzy logic.[45] The discussion of topologies over a fuzzy universe in the fully graded setting of FCT thus remains a task for future work, which can only be successfully solved after the general questions of quantification over fuzzy domains are addressed.

Although the underlying logic employed in [43] was IMTL, most results hold generally in MTL: the involutiveness of negation was only used to obtain the duality between open and closed fuzzy sets (which, moreover, seems to be inessential for fuzzy topology: cf. the successful development of constructive topology [183] where the duality fails as well).

The general approach and the definitions of basic concepts arose from joint discussions of both authors. Actual derivations of most particular properties listed in the papers have been done by T. Kroupa (some of them followed easily from his results in [149] on fuzzy filters), while the present author is responsible for most of the examples. Both authors participated in writing the papers and checking the results for correctness. P. Cintula gave us a hint on the importance of the inner exponent in the definition of $\mathrm{U_2c}$ [43, Def. 4.3]. The papers have been followed by an abstract [44] on the notions of continuity in the present setting; a full paper summarizing these results is under construction.

The present author's current view (which may not be shared by his co-author) on how logic-based fuzzy topology should further be developed differs somewhat from that presented in the above papers and is based on a more radical reading of [37, §7] on deprecating fixed preconditions in definitions. Obviously, the notions of fuzzy topology as presented in [43, 42] have to be parameterized by several indices that determine the multiplicities of conjuncts in the compound notion. The list of such indices, which is already too long, can further grow if more special properties of fuzzy topologies (like stratification [128] or separation axioms) are considered. Even the defining conditions proposed for open fuzzy topology (OTop) in our papers [43, 42] are themselves disputable as they are not independent (since $\emptyset \in \tau$ is implied by the union-closedness of τ); yet it would not be reasonable to omit the condition $\emptyset \in \tau$ in favor of the union-closedness, as the latter is much stronger and many properties only need the former. It is not at all clear which are the 'right' counterparts in fuzzy topology of the classical conditions that the empty set and the whole space are open. This suggests that the notion of fuzzy topological space is even less rigid than in classical mathematics or in traditional fuzzy mathematics (cf. the plenitude of variant kinds of topological spaces defined in both), and that there is no predetermined set of properties which together would form a well-motivated and sufficiently stable notion of fuzzy topology. Rather, there is a vague

[45] Observe that the apparatus of deductive fuzzy logic itself only considers crisp domains of discourse, and thus is best suited to modeling fuzzy structures over crisp universes. Possibly, a proper use of strong quantifiers (see Section 2.2) might provide an adequate treatment of quantification over fuzzy domains: a detailed investigation in this direction has yet to be done.

informal set of independent ‘topologically flavored’ graded properties of *arbitrary* fuzzy systems of fuzzy subsets (or of fuzzy neighborhoods), and various combinations of such properties should be studied without restricting our attention in advance to a fixed set of conditions. Under this approach, fuzzy topology would not ask the properties of a pre-defined notion of fuzzy topology, but rather proceed in a reverse manner, by deriving preconditions ensuring such ‘topologically flavored’ properties (cf. the research program of reverse mathematics, e.g., in [189, Ch. 1]). This reverse style of logic-based fuzzy topology may well be the right manner of developing logic-based fuzzy mathematics in general, as the problem of too many indices and the absence of a fixed set of defining conditions are not specific for fuzzy topology (being due just to the non-contractivity of conjunction).

References

[1] R. J. Adillon and V. Verdú. On a contraction-less intuitionistic propositional logic with conjunction and fusion. *Studia Logica*, 65:11–30, 2000.

[2] M. Baaz. Infinite-valued Gödel logic with 0-1-projections and relativisations. In P. Hájek, editor, *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*, volume 6 of *Lecture Notes in Logic*, pages 23–33. Springer-Verlag, Brno, 1996.

[3] M. Baaz, P. Hájek, F. Montagna, and H. Veith. Complexity of t-tautologies. *Annals of Pure and Applied Logic*, 113:3–11, 2002.

[4] M. Baaz, N. Preining, and R. Zach. First-order Gödel logics. *Annals of Pure and Applied Logic*, 147:23–47, 2007.

[5] A. I. Ban and S. G. Gal. *Defects of Properties in Mathematics: Quantitative Characterizations*, volume 5 of *Series on Concrete and Applicable Mathematics*. World Scientific, 2002.

[6] W. Bandler and L. J. Kohout. Fuzzy relational products and fuzzy implication operators. In *International Workshop of Fuzzy Reasoning Theory and Applications*, London, 1978. Queen Mary College, University of London.

[7] W. Bandler and L. J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[8] W. Bandler and L. J. Kohout. Special properties, closures and interiors of crisp and fuzzy relations. *Fuzzy Sets and Systems*, 26:317–332, 1980.

[9] W. Bandler and L. J. Kohout. Fast fuzzy relational algorithms. In A. Ballester, D. Cardus, and E. Trillas, editors, *Proceedings of the 2nd World Conference on Mathematics at the Service of Man*, pages 123–131, Las Palmas, Spain, 1982. Universidad Politechnica de Las Palmas.

[10] A. Beckmann and N. Preining. Linear Kripke frames and Gödel logics. *Journal of Symbolic Logic*, 72:26–44, 2007.

[11] L. Běhounek. Fuzzy logics interpreted as logics of resources. To appear in *The Logica Yearbook 2008*, King's College, 2009.

[12] L. Běhounek. Teorie množin v Gödelově logice ('Set theory over Gödel logic'). Master's thesis, Charles University, Faculty of Arts, Prague, 2002. In Czech.

[13] L. Běhounek. Axiomatic fuzzy set theories. In P. Jirků and K. Bendová, editors, *Miscellanea Logica V*, pages 31–43. Karolinum Press, Prague, 2003.

[14] L. Běhounek. Fuzzification of Groenendijk–Stokhof propositional erotetic logic. *Logique et Analyse*, 47:167–188, 2004.

[15] L. Běhounek. Fuzzy MacNeille and Dedekind completions of crisp dense linear orderings. In F. Hakl, editor, *Doktorandský den '05*, pages 5–10, Prague, 2005. ICS AS CR & Matfyzpress. Available at http://www.cs.cas.cz/hakl/doktorandsky-den/history.html.

[16] L. Běhounek. Towards a formal theory of fuzzy Dedekind reals. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 946–954, Barcelona, 2005.

[17] L. Běhounek. Two notions of fuzzy lattice completion. In V. Novák and M. Štěpnička, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 22–24, Ostrava, 2005. University of Ostrava.

[18] L. Běhounek. Dubois and Prade's fuzzy elements: A challenge for formal fuzzy logic. In S. Gottwald, P. Hájek, and M. Ojeda-Aciego, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 90–95, Málaga, 2006.

[19] L. Běhounek. Extensionality in graded properties of fuzzy relations. In *Proceedings of 11th IPMU Conference*, pages 1604–1611, Paris, 2006. Edition EDK.

[20] L. Běhounek. Fuzzy logics interpreted as expressing the cost of knowledge. In *The Book of Abstracts of Trends in Logic IV: Towards Mathematical Philosophy*, page 45, Toruń, 2006. Universitas Nicolai Copernici.

[21] L. Běhounek. Relative interpretations over first-order fuzzy logics. In F. Hakl, editor, *Doktorandský den '06*, pages 5–10, Prague, 2006. ICS AS CR / Matfyzpress. Available at http://www.cs.cas.cz/hakl/doktorandsky-den/history.html.

[22] L. Běhounek. An alternative justification of the axioms of fuzzy logics. *The Bulletin of Symbolic Logic*, 13:267, 2007.

[23] L. Běhounek. Dvě souvislosti mezi epistemickou a fuzzy logikou. In V. Kelemen, V. Kvasnička, and J. Pospíchal, editors, *Kognícia a umelý život VII*, pages 37–42. Slezská univerzita v Opavě, 2007. In Czech (Two connections between epistemic and fuzzy logic).

[24] L. Běhounek. Introspekce vágních znalostí ve fuzzy logice. In J. Kelemen, V. Kvasnička, and K. Pstružina, editors, *Kognice a umělý život VIII*, pages 43–48. Silesian University of Opava, 2008. In Czech (Introspection of vague knowledge in fuzzy logic).

[25] L. Běhounek. Modeling costs of program runs in fuzzified propositional dynamic logic. In F. Hakl, editor, *Doktorandské dny '08*, pages 11–18, Prague, 2008. Matfyzpress.

[26] L. Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159:1153–1164, 2008.

[27] L. Běhounek, U. Bodenhofer, and P. Cintula. Valverde-style representation results in a graded framework. In M. Štěpnička, V. Novák, and U. Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies: Proceedings of the 5th Eusflat Conference*, volume I, pages 153–160. University of Ostrava, 2007.

[28] L. Běhounek, U. Bodenhofer, and P. Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159:1729–1772, 2008.

[29] L. Běhounek, U. Bodenhofer, P. Cintula, and S. Saminger-Platz. Graded dominance. In E. P. Klement, S. E. Rodabaugh, and L. N. Stout, editors, *Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology. Abstracts of the 29th Linz Seminar on Fuzzy Set Theory*, pages 11–14, Linz, 2008.

[30] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[31] L. Běhounek and P. Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[32] L. Běhounek and P. Cintula. General logical formalism for fuzzy mathematics: Methodology and apparatus. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1227–1232, Beijing, 2005. Tsinghua University Press/Springer.

[33] L. Běhounek and P. Cintula. Relations in higher-order fuzzy logic I, II. In S. Gottwald, P. Hájek, U. Höhle, and E. P. Klement, editors, *Fuzzy Logics and Related Structures: Abstracts of 26th Linz Seminar on Fuzzy Set Theory*, pages 10–15, Linz, 2005. Johannes Kepler University Linz.

[34] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[35] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[36] L. Běhounek and P. Cintula. Fuzzy logics as the logics of chains. *Fuzzy Sets and Systems*, 157(5):604–610, 2006.

[37] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, 2007.

[38] L. Běhounek and P. Cintula. Modal fuzzy logics based on Scott-Montague semantics. In *Volume of Abstracts of Trends in Logic V*, pages 2–3, Canton, 2007.

[39] L. Běhounek and P. Cintula. Multiplicative quantifiers in fuzzy and substructural logics. In *Logic Colloquium 2007: Book of Abstracts*, page 36, Wrocław, 2007. Uniwersytet Wrocławski. ISBN 978–83–910055–6–9.

[40] L. Běhounek and M. Daňková. Automated proofs for composition-based fuzzy relational notions. In S. Gottwald, P. Hájek, and M. Ojeda-Aciego, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 96–102, Málaga, 2006.

[41] L. Běhounek and M. Daňková. Relational compositions in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 160:1005–1036, 2009.

[42] L. Běhounek and T. Kroupa. Interior-based topology in Fuzzy Class Theory. In M. Štěpnička, V. Novák, and U. Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies: Proceedings of the 5th Eusflat Conference*, volume I, pages 145–151. University of Ostrava, 2007.

[43] L. Běhounek and T. Kroupa. Topology in Fuzzy Class Theory: Basic notions. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 513–522. Springer, 2007.

[44] L. Běhounek and T. Kroupa. Continuous relations over topological spaces in Fuzzy Class Theory. In E. P. Klement, S. E. Rodabaugh, and L. N. Stout, editors, *Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology. Abstracts of the 29th Linz Seminar on Fuzzy Set Theory*, pages 15–18, Linz, 2008.

[45] L. P. Belluce and C. C. Chang. A weak completeness theorem on infinite valued predicate logic. *Journal of Symbolic Logic*, 28(1):43–50, 1963.

[46] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering.* Kluwer Academic/Plenum Press, New York, 2002.

[47] F. Bjørdal. There are only countably many objects. In L. Běhounek and M. Bílková, editors, *Logica Yearbook 2004*, pages 47–58. Filosofia, Prague, 2005.

[48] M. Black. Vagueness: An exercise in logical analysis. *Philosophy of Science*, 4:427–455, 1937. Reprinted in [141] pp. 69–81.

[49] U. Bodenhofer, L. Běhounek, and P. Cintula. Relations in higher-order fuzzy logic III. In S. Gottwald, P. Hájek, U. Höhle, and E. P. Klement, editors, *Fuzzy Logics and Related Structures: Abstracts of 26th Linz Seminar on Fuzzy Set Theory*, pages 20–22, Linz, 2005. Johannes Kepler University Linz.

[50] F. Bou, F. Esteva, J. M. Font, A. Gil, L. Godo, A. Torrens, and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. To appear in Journal of Logic and Computation, 2008.

[51] A. Cantini. The undecidability of Grišin's set theory. *Studia Logica*, 74:345–368, 2003.

[52] A. V. Chagrov. K voprosu ob obratnoi matematike modalnoi logiki (Russian, To the question of reverse mathematics of modal logic). *Online journal Logical Studies*, 6, 2001.

[53] C. C. Chang. The axiom of comprehension in infinite valued logic. *Mathematica Scandinavica*, 13:9–30, 1963.

[54] C. C. Chang and H. J. Keisler. *Continuous Model Theory*, volume 58 of *Annals of Mathematical Studies*. Princeton University Press, Princeton, 1966.

[55] C. L. Chang. Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, 24:182–190, 1968.

[56] E. W. Chapin, Jr. Set-valued set theory (I–II). *Notre Dame Journal of Formal Logic*, pages 15: 619–634 (1974), 16: 255–267 (1975), 1974.

[57] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in non-classical logics. In *Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science (LICS)*, pages 229–240, 2008.

[58] A. Ciabattoni, G. Metcalfe, and F. Montagna. Adding modalities to MTL and its extensions. To appear in the Proceedings of Linz Seminar 2005, 2005.

[59] P. Cintula. The ŁΠ and ŁΠ$\frac{1}{2}$ propositional and predicate logics. *Fuzzy Sets and Systems*, 124(3):289–302, 2001.

[60] P. Cintula. Weakly implicative (fuzzy) logics. Technical Report V-912, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2004.

[61] P. Cintula. Basics of a formal theory of fuzzy partitions. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 884–888, Barcelona, 2005.

[62] P. Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[63] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna, and C. Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. To appear in Annals of Pure and Applied Logic, 2008.

[64] P. Cintula and R. Horčík. Fuzzy Class Theory: Some advanced topics. In M. Štěpnička, V. Novák, and U. Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies: Proceedings of the 5th Eusflat Conference*, volume I, pages 137–143. University of Ostrava, 2007.

[65] P. Cintula, R. Horčík, and L. Běhounek. Exponentials and multiplicative quantifiers in fuzzy logic. In *Abtstracts of Trends in Logic V: Many-Valued Logic and Cognition*, pages 13–14, Canton, 2007.

[66] P. Cintula and C. Noguera. A hierarchy of implicational (semilinear) logics: The propositional case. Submitted, 2008.

[67] A. de Luca and S. Termini. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. *Information and Control*, 20:301–312, 1972.

[68] M. Demirci. Fuzzy functions and their fundamental properties. *Fuzzy Sets and Systems*, 106:239–246, 1999.

[69] M. Demirci. Fuzzy functions and their applications. *Journal of Mathematical Analysis and Applications*, 252:495–517, 2000.

[70] M. Demirci. Gradation of being fuzzy function. *Fuzzy Sets and Systems*, 119:383–392, 2001.

[71] A. Di Nola, G. Georgescu, and A. Iorgulescu. Pseudo-BL-algebras, part I–II. *Multiple-Valued Logic*, 8(5–6):673–714, 717–750, 2002.

[72] D. Dubois. On ignorance and contradiction considered as truth-values. *Logic Journal of the Interest Group of Pure and Applied Logic*, 16:195–216, 2008.

[73] D. Dubois, W. Ostasiewicz, and H. Prade. Fuzzy sets: History and basic notions. In D. Dubois and H. Prade, editors, *Fundamentals of Fuzzy Sets*, pages 21–124. Kluwer, 2000.

[74] D. Dubois and H. Prade. The three semantics of fuzzy sets. *Fuzzy Sets and Systems*, 90:141–150, 1997.

[75] D. Dubois and H. Prade. Gradual elements in a fuzzy set. *Soft Computing*, 12:165–175, 2008.

[76] M. Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 27:97–106, 1959.

[77] D. Dzierzgowski. Finite sets and natural numbers in intuitionistic TT. *Notre Dame Journal of Formal Logic*, 37:585–601, 1996.

[78] D. Edgington. Vagueness by degrees. In R. Keefe and P. Smith, editors, *Vagueness: A Reader*, pages 294–316. MIT Press, 1999.

[79] F. Esteva, J. Gispert, L. Godo, and F. Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71(2):199–226, 2002.

[80] F. Esteva, J. Gispert, L. Godo, and C. Noguera. Adding truth-constants to continuous t-norm based logics: Axiomatization and completeness results. *Fuzzy Sets and Systems*, 158:597–618, 2007.

[81] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[82] F. Esteva, L. Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[83] F. Esteva, L. Godo, and F. Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[84] J. E. Fenstad. On the consistency of the axiom of comprehension in the Łukasiewicz infinite valued logic. *Mathematica Scandinavica*, 14:65–74, 1964.

[85] M. C. Fitting. *Intuitionistic Logic Model Theory and Forcing.* Studies in Logic and the Foundations of Mathematics. North-Holland, 1969.

[86] J. M. Font. Taking degrees of truth seriously. Submitted, 2008.

[87] A. A. Fraenkel, Y. Bar-Hillel, and A. Levy. *Foundations of Set Theory*, volume 67 of *Studies in Logic and the Foundations of Mathematics.* North-Holland, Amsterdam and London, second revised edition, 1973.

[88] H. Friedman and R. K. Meyer. Whither relevant arithmetic? *Journal of Symbolic Logic*, 57:824–831, 1992.

[89] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, Amsterdam, 2007.

[90] R. Giles. The concept of grade of membership. *Fuzzy Sets and Systems*, 25:297–323, 1988.

[91] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[92] J.-Y. Girard. Linear logic: Its syntax and semantics. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic: Proceedings of the Workshop on Linear Logic, Cornell University, June 1993*, pages 1–42. Cambridge University Press, 1995.

[93] I. Glöckner. An axiomatic theory of fuzzy quantifiers in natural languages. Technical Report 2000–03, Bielefeld University, 2000.

[94] K. Gödel. Zum intuitionistischen Aussagenkalkül. *Anzieger Akademie der Wissenschaften Wien*, 69:65–66, 1932.

[95] J. A. Goguen. *L*-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18:145–174, 1967.

[96] J. A. Goguen. Logic of inexact concepts. *Synthese*, 19:325–373, 1969.

[97] J. A. Goguen. The fuzzy Tychonoff theorem. *Journal of Mathematical Analysis and Applications*, 43:734–742, 1973.

[98] R. Goldblatt. *Topoi. The Categorial Analysis of Logic*, volume 98 of *Studies in Logic and the Foundations of Mathematics.* North-Holland, 2nd revised edition, 1984.

[99] S. Gottwald. Set theory for fuzzy sets of higher level. *Fuzzy Sets and Systems*, 2:125–151, 1979.

[100] S. Gottwald. Fuzzy uniqueness of fuzzy mappings. *Fuzzy Sets and Systems*, 3:49–74, 1980.

[101] S. Gottwald. Fuzzified fuzzy relations. In R. Lowen and M. Roubens, editors, *Proceedings of 4th IFSA Congress*, volume Mathematics (ed. P. Wuyts), pages 82–86, Brussels, 1991.

[102] S. Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View.* Vieweg, Wiesbaden, 1993.

[103] S. Gottwald. Generalized solvability behaviour for systems of fuzzy equations. In V. Novák and I. Perfilieva, editors, *Discovering the World with Fuzzy Logic*, volume 57 of *Studies in Fuzziness and Soft Computing*, pages 401–430. Physica-Verlag, Heidelberg, 2000.

[104] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[105] S. Gottwald. Universes of fuzzy sets and axiomatizations of fuzzy set theory. Part I: Model-based and axiomatic approaches. *Studia Logica*, 82(2):211–144, 2006.

[106] S. Gottwald. Universes of fuzzy sets and axiomatizations of fuzzy set theory. Part II: Category theoretic approaches. *Studia Logica*, 84:23–50, 2006.

[107] R. J. Grayson. Heyting-valued models for intuitionistic set theory. In M. Fourman, C. Mulvey, and D. Scott, editors, *Application of Sheaves*, volume 743 of *Lecture Notes in Computer Science*, pages 402–414. Springer, Berlin, 1979.

[108] V. N. Grishin. Predicate and set-theoretic calculi based on logic without contractions. *Mathematics of the USSR—Izvestiya*, 18:41–59, 1982.

[109] J. Groenendijk and M. Stokhof. Questions. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 1055–1124. Elsevier and MIT Press, 1997.

[110] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[111] P. Hájek. Function symbols in fuzzy logic. In *Proceedings of the East–West Fuzzy Colloquium*, pages 2–8, Zittau/Görlitz, 2000. IPM.

[112] P. Hájek. Some hedges for continuous t-norms logics. *Neural Network World*, 12:159–164, 2002.

[113] P. Hájek. Arithmetical complexity of fuzzy predicate logics—a survey. *Soft Computing*, 9(12):935–941, 2005.

[114] P. Hájek. Fleas and fuzzy logic. *Journal of Multiple-Valued Logic and Soft Computing*, 11(1–2):137–152, 2005.

[115] P. Hájek. On arithmetic in Cantor–Łukasiewicz fuzzy set theory. *Archive for Mathematical Logic*, 44(6):763–782, 2005.

[116] P. Hájek. On arithmetical complexity of fragments of prominent fuzzy predicate logics. *Soft Computing*, 12(4):335–340, 2008.

[117] P. Hájek and L. Běhounek. Fuzzy logics among substructural logics. Presentation at The Second World School on Universal Logic, Xian 2007 (available online), 2007.

[118] P. Hájek and P. Cintula. Triangular norm based predicate fuzzy logics. To appear in the Proceedings of Linz Seminar 2005, 2005.

[119] P. Hájek and P. Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

[120] P. Hájek and Z. Haniková. A development of set theory in fuzzy logic. In M. Fitting and E. Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Physica-Verlag, Heidelberg, 2003.

[121] P. Hájek, J. Paris, and J. C. Shepherdson. The liar paradox and fuzzy logic. *Journal of Symbolic Logic*, 65(1):339–346, 2000.

[122] P. Hájek, J. Paris, and J. C. Shepherdson. Rational Pavelka logic is a conservative extension of Łukasiewicz logic. *Journal of Symbolic Logic*, 65(2):669–682, 2000.

[123] Z. Haniková. A note on the complexity of propositional tautologies of individual t-algebras. *Neural Network World*, 12(5):453–460, 2002. Special issue on SOFSEM 2002.

[124] Z. Haniková. On the complexity of propositional logic with an involutive negation. In M. Wagenknecht and R. Hampel, editors, *Proceedings of EUSFLAT Conference*, pages 636–639, Zittau, 2003.

[125] L. S. Hay. Axiomatization of the infinite-valued predicate calculus. *Journal of Symbolic Logic*, 28:77–86, 1963.

[126] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[127] U. Höhle. Commutative, residuated l-monoids. In U. Höhle and E. P. Klement, editors, *Non-Classical Logics and Their Applications to Fuzzy Subsets*, pages 53–106. Kluwer, Dordrecht, 1995.

[128] U. Höhle. *Many Valued Topology and Its Applications*. Kluwer, Boston, MA, 2001.

[129] U. Höhle. Fuzzy sets and sheaves (I–II). *Fuzzy Sets and Systems*, 158:1143–1174, 1175–1212, 2007.

[130] U. Höhle and A. Šostak. A general theory of fuzzy topological spaces. *Fuzzy Sets and Systems*, 73:131–149, 1995.

[131] U. Höhle and L. N. Stout. Foundations of fuzzy sets. *Fuzzy Sets and Systems*, 40:257–296, 1991.

[132] M. Holčapek. Monadic L-fuzzy quantifiers of the type $\langle 1^n, 1\rangle$. *Fuzzy Sets and Systems*, 159:1811–1835, 2008.

[133] R. Horčík. Standard completeness theorem for ΠMTL. *Archive for Mathematical Logic*, 44(4):413–424, 2005.

[134] R. Horčík. Solution of a system of linear equations with fuzzy numbers. *Fuzzy Sets and Systems*, 159(14):1788–1810, 2008.

[135] A. Horn. Logic with truth values in a linearly ordered Heyting algebras. *Journal of Symbolic Logic*, 34:395–408, 1969.

[136] H. Inaoka and S. Yatabe. On Evans's vague object from set theoretic viewpoint. *Journal of Philosophical Logic*, 35:423–434, 2006.

[137] T. J. Jech. *The Axiom of Choice*, volume 75 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1973.

[138] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[139] P. Jipsen and C. Tsinakis. A survey of residuated lattices. In J. Martinez, editor, *Ordered Algebraic Structures*, pages 19–56. Kluwer Academic Publishers, Dordrecht, 2002.

[140] R. Keefe. Vagueness by numbers. *Mind*, 107:565–579, 1998.

[141] R. Keefe and P. Smith, editors. *Vagueness: A Reader*. MIT Press, 1999.

[142] E. E. Kerre and J. N. Mordeson. A historical overview of fuzzy mathematics. *New Mathematics and Natural Computation*, 1:1–26, 2005.

[143] S. C. Kleene and R. E. Vesley. *The Foundations of Intuitionistic Mathematics*. North-Holland, Amsterdam, 1965. Russian translation: Nauka, Moscow 1978.

[144] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer, Dordrecht, 2000.

[145] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall, 1995.

[146] J. Krajíček. A possible modal formulation of comprehension scheme. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 33:461–480, 1987.

[147] J. Krajíček. Some results and problems in the modal set theory MST. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 34:123–134, 1988.

[148] T. Kroupa. Towards formal theory of measure on clans of fuzzy sets. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 351–356, Barcelona, 2005.

[149] T. Kroupa. Filters in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 159(14):1773–1787, 2008.

[150] J. Kühr. Pseudo-BL algebras and DRℓ-monoids. *Mathematica Bohemica*, 128:199–208, 2003.

[151] G. Lakoff. Hedges: A study in meaning criteria and the logic of fuzzy concepts. *Journal of Philosophical Logic*, 2:458–508, 1973.

[152] R. Lowen. Fuzzy topological spaces and fuzzy compactness. *Journal of Mathematical Analysis and Applications*, 56:623–633, 1976.

[153] R. Lowen. The order aspect of the fuzzy real line. *Manuscripta Mathematica*, 39:293–309, 1985.

[154] J. Łukasiewicz. O logice trójwartościowej (On three-valued logic). *Ruch filozoficzny*, 5:170–171, 1920.

[155] E. Marchioni and F. Montagna. Complexity and definability issues in $Ł\Pi\frac{1}{2}$. *Journal of Logic and Computation*, 17:311–331, 2007.

[156] G. Metcalfe. Uninorm logics. In *Proceedings of EUROFUSE Workshop on Data and Knowledge Engineering*, Warsaw, 2004.

[157] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *Journal of Symbolic Logic*, 72(3):834–864, 2007.

[158] G. Metcalfe, N. Olivetti, and D. M. Gabbay. *Proof Theory for Fuzzy Logics*, volume 36 of *Applied Logic Series*. Springer, 2009. To appear.

[159] R. K. Meyer. $\supset$E is admissible in "true" relevant arithmetic. *Journal of Philosophical Logic*, 27:327–351, 1998.

[160] F. Montagna. Storage operators and multiplicative quantifiers in many-valued logics. *Journal of Logic and Computation*, 14(2):299–322, 2004.

[161] F. Montagna. On the predicate logics of continuous t-norm BL-algebras. *Archive for Mathematical Logic*, 44:97–114, 2005.

[162] F. Montagna, C. Noguera, and R. Horčík. On weakly cancellative fuzzy logics. *Journal of Logic and Computation*, 16(4):423–450, 2006.

[163] F. Montagna and H. Ono. Kripke semantics, undecidability and standard completeness for Esteva and Godo's logic MTL$\forall$. *Studia Logica*, 71(2):227–245, 2002.

[164] F. Montagna and L. Sacchetti. Kripke-style semantics for many-valued logics. *Mathematical Logic Quarterly*, 71(2):227–245, 2002.

[165] C. Mortensen. *Inconsistent Mathematics*. Kluwer, Dordrecht, 1995.

[166] V. Novák. *Fuzzy Sets and Their Applications*. Adam Hilger, Bristol, 1989.

[167] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic part I (syntax and semantic), part II (main results). *Kybernetika*, 26:47–66, 134–154, 1990.

[168] V. Novák. Antonyms and linguistic quantifiers in fuzzy logic. *Fuzzy Sets and Systems*, 124:335–351, 2001.

[169] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149(2):235–273, 2004.

[170] V. Novák. A comprehensive theory of trichotomous evaluative linguistic expressions. *Fuzzy Sets and Systems*, 159:2939–2969, 2008.

[171] V. Novák. A formal theory of intermediate quantifiers. *Fuzzy Sets and Systems*, 159:1229–1246, 2008.

[172] V. Novák, I. Perfilieva, and J. Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.

[173] H. Ono. Substructural logics and residuated lattices—an introduction. In V. F. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pages 193–228. Kluwer, Dordrecht, 2003.

[174] H. Ono and Y. Komori. Logic without the contraction rule. *Journal of Symbolic Logic*, 50:169–201, 1985.

[175] F. Paoli. *Substructural Logics: A Primer*. Kluwer, 2002.

[176] F. Paoli. The ambiguity of quantifiers. *Philosophical Studies*, 124:313–330, 2005.

[177] J. Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.

[178] S. Peters and D. Westerståhl. *Quantifiers in Language and Logic*. Oxford University Press, USA, 2006.

[179] W. C. Powell. Extending Gödel's negative interpretation to ZF. *Journal of Symbolic Logic*, 40(2):221–229, 1975.

[180] H. Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

[181] G. Restall. Arithmetic and truth in Łukasiewicz's infinitely valued logic. *Logique et Analyse*, 36:25–38, 1995.

[182] G. Restall. *An Introduction to Substructural Logics*. Routledge, New York, 2000.

[183] G. Sambin. Some points in formal topology. *Theoretical Computer Science*, 305:347–408, 2003.

[184] B. Šešelja and A. Tepavčevič. Relational valued fuzzy sets. *Fuzzy Sets and Systems*, 52:217–222, 1992.

[185] J. Shen. Separation axiom in fuzzifying topology. *Fuzzy Sets and Systems*, 57:111–123, 1993.

[186] M. Shimoda. A natural interpretation of fuzzy sets and fuzzy relations. *Fuzzy Sets and Systems*, 128:135–147, 2002.

[187] M. Shirahata. Fixpoint theorem in linear set theory. Preprint, available at http://www.fbc.keio.ac.jp/∼sirahata/Research (accessed 6 Oct 2008), 1997.

[188] M. Shirahata. Phase-valued models of linear set theory. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 377–397. World Scientific Press, 2003. A preprint of 1999 available at http://www.fbc.keio.ac.jp/∼sirahata/Research (accessed 14 Oct 2008).

[189] S. G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic. Springer, Berlin, 1999.

[190] T. Skolem. Bemerkungen zum Komprehensionsaxiom. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 3:1–17, 1957.

[191] T. Skolem. A set theory based on a certain 3-valued logic. *Mathematica Scandinavica*, 8:71–80, 1960.

[192] A. Šostak. On compactness and connectedness degrees of fuzzy sets in fuzzy topological spaces. In *General Topology and Its Relations to Modern Analysis and Algebra*, pages 519–532. Helderman, Berlin, 1988.

[193] G. Takeuti and S. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *Journal of Symbolic Logic*, 49(3):851–866, 1984.

[194] G. Takeuti and S. Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32:1–32, 1992.

[195] K. Terui. Light affine set theory: A naive set theory of polynomial time. *Studia Logica*, 77(1):9–40, 2004.

[196] H. Thiele. On fuzzy quantifiers. In *Proceedings of the World Congress of IFSA*, pages 395–398, Seoul, 1993.

[197] H. Thiele. On T-quantifiers and S-quantifiers. In *Proceedings of the 24th ISMVL*, pages 264–270, Boston, 1994.

[198] S. Titani. A lattice-valued set theory. *Archive for Mathematical Logic*, 38:395–421, 1999.

[199] D. van Dalen. From Brouwerian counter examples to the creating subject. *Studia Logica*, 62:305–314, 1999.

[200] M. Wajsberg. Aksjomatyzacja trójwartościowego rachunku zdań. *Société des Sciences et Letters Varsovie, cl. III*, 24:126–148, 1931.

[201] B. Weatherson. True, truer, truest. *Philosophical Studies*, 123:47–70, 2005.

[202] A. J. Weidner. Fuzzy sets and Boolean-valued universes. *Fuzzy Sets and Systems*, 6:61–72, 1981.

[203] D. Westerståhl. Quantifiers in formal and natural languages. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume IV, pages 1–131. D. Reidel, Dordrecht, 1989.

[204] R. B. White. The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz. *Journal of Philosophical Logic*, 8:509–534, 1979.

[205] T. Williamson. *Vagueness*. Routledge, London, 1994.

[206] M. Winter. *Goguen Categories: A Categorial Approach to L-fuzzy Relations*, volume 25 of *Trends in Logic*. Springer, Dordrecht, 2007.

[207] S. Yatabe. Distinguishing non-standard natural numbers in a set theory within Łukasiewicz logic. *Archive for Mathematical Logic*, 46:281–287, 2007.

[208] M. Ying. A new approach for fuzzy topology (I–III). *Fuzzy Sets and Systems*, 39:302–321 (1991), 47:221–232 (1992), 55:193–202 (1993).

[209] M. Ying. Compactness in fuzzifying topology. *Fuzzy Sets and Systems*, 55:79–92, 1993.

[210] M. Ying. Fuzzifying topology based on complete residuated lattice-valued logic (I). *Fuzzy Sets and Systems*, 56:337–373, 1993.

[211] M. Ying. Fuzzy topology based on residuated lattice-valued logic. *Acta Mathematica Sinica (English Series)*, 17:89–102, 2001.

[212] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[213] L. A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.

[214] L. A. Zadeh. Fuzzy logic and approximate reasoning. *Synthese*, 30:407–428, 1975.

[215] L. A. Zadeh. A computational approach to fuzzy quantifiers in natural languages. *Computers and Mathematics*, 9:149–184, 1983.

Part II

Papers comprising the thesis

On the difference between traditional and deductive fuzzy logic

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Abstract: In three case studies on notions of fuzzy logic and fuzzy set theory (Dubois–Prade's gradual elements, the entropy of a fuzzy set, and aggregation operators), the paper exemplifies methodological differences between traditional and deductive fuzzy logic. While traditional fuzzy logic admits various interpretations of membership degrees, deductive fuzzy logic always interprets them as degrees of truth preserved under inference. The latter fact imposes several constraints on systems of deductive fuzzy logic, which need not be followed by mainstream fuzzy logic. That makes deductive fuzzy logic a specific area of research that can be characterized both methodologically (by constraints on meaningful definitions) and formally (as a specific class of logical systems). An analysis of the relationship between deductive and traditional fuzzy logic is offered.

Keywords: Deductive fuzzy logic, fuzzy elements, gradual sets, entropy of fuzzy sets, aggregation, membership degrees, methodology of fuzzy mathematics. MSC: 03B52, 03E72, 68T37.

Introduction

Lotfi Zadeh [41] has made the distinction between fuzzy logic in broad sense (FLb) and fuzzy logic in narrow sense (FLn). FLn is based on certain many-valued logics, but its agenda differs from that of formal logic: it deals with such concepts as linguistic variable, fuzzy if–then rule, defuzzification, interpolative reasoning, etc.; and FLb roughly coincides with the broad theory and applications of fuzzy sets.

In this paper we shall focus on a sub-area of FLn that studies or uses *formal deductive systems* of fuzzy logic. Prototypical examples of such systems are those centered around Hájek's basic fuzzy logic BL of continuous t-norms [22], including for instance Łukasiewicz, Gödel, and product logics [22], the logics MTL [20], ŁΠ [21], etc., both propositional and first- or higher-order [22, 35, 6]. The area also covers those parts of fuzzy mathematics (i.e., of FLb) which are built as deductive axiomatic theories based on these formal fuzzy logics (cf. [38, 25, 26, 8, 6, 7, 11], etc.). To avoid a conflict of terms, we shall call this area *deductive fuzzy logic* (FLd). Other parts of FLn and FLb will in the present paper be labeled *traditional fuzzy logic* (FLt), as the latter has a much longer tradition than the relatively newer FLd.

The aim of this paper is to point out and analyze certain fundamental differences between FLd and FLt. The differences are illustrated in three case studies, regarding respectively:

1. Dubois and Prade's notion of fuzzy element

2. The notion of entropy of fuzzy sets

3. Aggregation of fuzzy data

Since the paper is methodological rather than technical, I omit most technical details and focus on the analysis of the principles behind the approaches of FLt and FLd. I assume that the reader has a basic knowledge of some formal system of fuzzy logic, for instance Hájek's logic BL of continuous t-norms [22]. Here I only briefly recapitulate some characteristic features of deductive fuzzy logics, which will be of importance for further considerations:

- Deductive fuzzy logic is a kind of (many-valued) *logic.*[1] Therefore, like other kinds of logics, it primarily studies preservation of some quality ("truth") of propositions under inference. In the particular case of formal *fuzzy* logic, the quality is *partial truth,* i.e., the *degrees* of truth.[2] Thus, deductive fuzzy logic interprets membership degrees exclusively as degrees of truth of the membership predication. In this it differs from the rest of traditional fuzzy logic, which admits various interpretations of membership degrees [17, 16].

- As a kind of *formal* or *symbolic* logic, FLd strictly distinguishes syntax from semantics. In syntax, deductive fuzzy logic works with some fixed language composed of propositional connectives, quantifiers, predicate and function symbols, and variables. The symbols (and formulae built up from these symbols) are then interpreted in semantical models, which are composed of usual fuzzy sets and fuzzy relations of FLt. In this way the formulae of the symbolic language formally describe actual fuzzy sets.

[1]Fuzzy logic understood as part of the theory of many-valued logics is sometimes called *mathematical, symbolic,* or *formal* fuzzy logic, as it employs the methods of mathematical (symbolic, formal) logic [23]. Deductive fuzzy logic in our sense is a proper part of mathematical fuzzy logic: it will be shown that in addition to being formal systems of (mathematical, or symbolic) fuzzy logic, deductive fuzzy logics should satisfy certain principles in order to be suitable for graded logical deduction.

[2]Note that throughout the paper, "preservation of partial truth" or "graded inference" refers to the so-called *local* consequence relation. The more commonly studied *global* consequence relation expresses the preservation of *full* truth between fuzzy propositions. The global consequence relation is defined as follows: ψ globally follows from $\varphi_1, \ldots, \varphi_n$ iff the following holds: whenever all $\varphi_1, \ldots, \varphi_n$ are fully true (i.e., of truth degree 1), so is ψ. The local consequence relation, on the other hand, is defined by means of partial truth: ψ locally follows from $\varphi_1, \ldots, \varphi_n$ iff the truth degree of ψ is at least as large as the aggregation (by strong conjunction) of the truth degrees of all $\varphi_1, \ldots, \varphi_n$.

Even though it is the global consequence relation which is most often studied in current mathematical fuzzy logic, local consequence is important for actual reasoning in formal fuzzy logic, as it can be used even when premises are only partially true. Notice that usual systems of deductive fuzzy logic axiomatize the global (rather than local) consequence relation; however, the relation of local consequence between $\varphi_1, \ldots, \varphi_n$ and ψ can in these logics be defined as the (global) validity of $(\varphi_1 \mathbin{\&} \ldots \mathbin{\&} \varphi_n) \to \psi$, where & is strong conjunction and $\to$ implication. As argued in Case Study 3 below, the requirements on good behavior of local consequence and its interplay with & and $\to$ form the constitutive features of deductive fuzzy logics.

- FLd is based on the axiomatic method and works in the formal deductive way. Valid statements about fuzzy sets are derived in an axiomatic theory, through iterated application of sound rules of a particular system of deductive fuzzy logic. Since FLd employs non-classical many-valued logic, formal theories in FLd can have some peculiar features [11], which are not met in standard axiomatic theories of FLt or classical mathematics.

- Most systems of FLd impose specific constraints on some of its components. For example, most systems of formal fuzzy logic require that conjunction be realized as a left-continuous t-norm, and are much less interested in other conjunctive operators studied in FLt. A partial explanation of this selectiveness of FLd will be elaborated in the present paper. It can be shown that such restrictions largely follow from the features of FLd listed above (viz the interpretation of degrees in terms of truth, the study of partial truth preservation, formal deducibility, etc.).

A further explanation and illustration of these points, as well as an analysis of the difference between FLd and FLt which results from the above features of FLd, will be given in the following sections. I will argue that FLd is a rather sharply delimited area of FLt, and that the agenda of FLd differs significantly from that of FLt. Therefore, to avoid confusion in fuzzy set theory, we should clearly distinguish between their respective areas of competence.

Case study 1: Dubois and Prade's gradual elements

In [18], Dubois and Prade have introduced the notions of gradual element and gradual set by the following definitions:

Definition 1. Let S be a set and L a complete lattice with top 1 and bottom 0. A *fuzzy* (or *gradual*) *element* e in S is identified with a (partial) assignment function $a_e\colon L \setminus \{0\} \to S$.

Definition 2. A *gradual subset* G in S is identified with its assignment function $a_G\colon L \setminus \{0\} \to 2^S$. If S is fixed, we may simply speak of *gradual sets*.

A prototypical example of a fuzzy element is the fuzzy middle-point of a fuzzy interval A, which assigns the middle point of the α-level of A to each $\alpha \in L \setminus \{0\}$. Notice that the assignment function of a gradual element need not be monotone nor injective (cf. the middle points of certain asymmetric fuzzy intervals). Fuzzy elements of this kind are met in many real-life situations (e.g., the average salary of older people). Gradual elements and gradual sets are claimed by the authors to be a missing primitive concept in fuzzy set theory.

The authors proceed to define the fuzzy set induced by a gradual set, the membership of a gradual element in a fuzzy set, etc. As these notions are not important for our present case, I refer the reader to the original article [18]. We shall only notice that the operations proposed for gradual sets are defined cut-wise (with possible rearrangements of cuts in the case of complementation).

The declared motivation for introducing gradual elements is to distinguish *impreciseness* (i.e., intervals) from *fuzziness* (i.e., gradual change from 0 to 1). As implicit in [18, 19], a general guideline for definitions of fuzzy notions should be the following principle (we shall call it the *principle of cuts*):

Principle of cuts: *The α-cuts of a fuzzy notion FX should be instances of the corresponding crisp notion X.*

I.e., the fuzzy version FX of a crisp notion X should be defined in such a way that the α-cuts of FX's are X's. Thus the fuzzy counterpart of the notion of element is exactly the fuzzy element of Definition 1, that of the notion of set is the gradual set of Definition 2, etc.

The definitions of gradual sets and gradual elements are clearly sound and the notions will probably prove to be of considerable importance for FLt. Let us see if they can be represented in FLd as well. A more detailed analysis of this question has been done in [4]; here we extract its important parts:

Apparently there are no direct counterparts of gradual elements or sets among the primitive concepts of current propositional or first-order fuzzy logics. Nevertheless, it can be shown that gradual elements and gradual sets are representable in *higher-order fuzzy logic* [6, 9] or *simple fuzzy type theory* [35, 6]. For technical details of the representation see [4]; here we only sketch the construction:

1. By the comprehension axioms of higher-order fuzzy logic, the notions of crisp kernel, fuzzy subset, fuzzy powerset, and crisp function are definable in higher-order fuzzy logic (see [6], [4] or a freely available primer [9] for details).

2. By a standard construction (cf. [38]), an internalization of truth degrees is definable in higher-order fuzzy logic (see [4] or [12] for the details of the construction and some meta-mathematical provisos). The lattice that represents truth degrees within the theory is defined as $\mathrm{L} = \mathrm{Ker}(\mathrm{Pow}(\{a\}))$, i.e., the kernel of the powerset of the crisp singleton of any element a of the universe of discourse. (In fuzzy type theory of [35], this step can be omitted, since the set of truth values is a primitive concept there.)

3. Since Definitions 1 and 2 need no further ingredients beyond those listed in items (1)–(2), crisp functions from L to the domain of discourse or its powerset represent respectively the notions of gradual element and gradual set in higher-order fuzzy logic. By similar means, all other notions defined in [18] can be defined in higher-order fuzzy logic as well (see [4]).

In particular, the definitions of gradual elements and gradual sets in the standard framework of higher-order logic (or *fuzzy class theory* [6, 9]) run as follows:

Definition 3. A fuzzy element of S (in higher-order fuzzy logic) is any (second-order) class $\mathcal{E}$ such that

$$\mathrm{Crisp}\,\mathcal{E}\ \&\ \Delta(\mathrm{Dom}\,\mathcal{E} \subseteq \mathrm{L} \setminus \{\emptyset\})\ \&\ \Delta(\mathrm{Rng}\,\mathcal{E} \subseteq S)\ \&\ \mathrm{Fnc}\,\mathcal{E}.$$

Definition 4. A gradual subset of S in higher-order fuzzy logic is any (second-order) class $\mathcal{G}$ such that

$$\mathrm{Crisp}\,\mathcal{G}\ \&\ \Delta(\mathrm{Dom}\,\mathcal{E} \subseteq \mathrm{L} \setminus \{\emptyset\})\ \&\ \Delta(\mathrm{Rng}\,\mathcal{E} \subseteq \mathrm{Ker}\,\mathrm{Pow}\,S)\ \&\ \mathrm{Fnc}\,\mathcal{G}.$$

In this way, the FLt notions of gradual element and gradual set can also be defined in FLd of higher order. However, their rendering in FLd is not very satisfactory. First, the formal representatives in FLd of the simple FLt notions are rather complex—namely certain very special second-order predicates, whose relationship to traditional fuzzy sets (i.e., first-order predicates) is far from perspicuous.[3] Although this presents no obstacle to handling them in the formal framework of higher-order fuzzy logic, the apparatus of FLd does not much simplify working with these notions (unlike it does with traditional fuzzy sets), since they are represented by crisp functions like in their semantic treatment by FLt. Considering the fundamental role fuzzy elements are to play in Dubois and Prade's recasting of fuzzy set theory, it would certainly be desirable to have fuzzy elements and gradual sets rendered more directly in FLd—as primitive notions rather than defined complex entities, preferably of propositional or first-order rather than higher-order level. These demands, however, encounter the following deep-rooted difficulty:

The new notions represent the horizontal (cut-wise) view of a fuzzy set (construed as a system of cuts), while usual fuzzy set theory represents fuzzy sets vertically (by membership degrees of its elements). Predicates in first-order fuzzy logic only formalize the vertical view of fuzzy sets; and although the latter can also be represented by systems of cuts, all usual FLd systems of first-order fuzzy logic require that the cuts be *nested.* This requirement is already built in the propositional core of common formal fuzzy logics, all of which presuppose the following principle (further on, we shall call it the *principle of persistence*):

> **Principle of persistence:** *If a proposition φ is guaranteed to be (at least) α-true, then it is also guaranteed to be (at least) β-true for all $\beta \leq \alpha$.*

The principle is manifested, i.a., in the transitivity of implication, which is satisfied in all systems of FLd and is indispensable for multi-step logical deduction (more on this in Case Study 3 below). Since Dubois and Prade's gradual sets do not meet this requirement (the α-cuts need not be nested), the known systems of first-order fuzzy logic cannot represent them as fuzzy predicates. (Similarly, known systems of propositional fuzzy logic cannot represent them as fuzzy propositions.)

The reason why Dubois and Prade's notions depart so radically from the presuppositions of FLd resides in the conceptual difference between the approaches to fuzziness in FLd and FLt. In FLt, there are many possible interpretations of the meaning of membership degrees [16, 17]. In particular (as stressed by Dubois and Prade in [18]), fuzzy sets may in FLt represent *imprecision* and membership degrees the *gradual change.* In FLd, however, membership degrees are only interpreted as guaranteed degrees of *truth;* and fuzzy sets in FLd represent the degree of satisfaction of truth conditions rather than interval-like imprecision. Thus in FLt, membership degrees can be understood as mere indices which parameterize the membership in a fuzzy set and which allow the gradual change from 0 to 1 ("fuzziness by fibering"). In FLd, truth degrees are what is preserved in graded inference, i.e., preserved w.r.t. the *ordering* of truth values; and this enforces the principle of persistence.

It should be noticed that the principle of cuts, which motivates the distinction between gradual elements and fuzzy sets in [18], is not itself alien to FLd. On the contrary—when following a certain FLd-sound methodology, many fuzzy counterparts of crisp notions do satisfy the principle of cut. The methodology was already sketched in [27, §5] by Höhle, then elaborated in [6, §7], and proposed as a general guideline for FLd in [8]; it consists in re-interpreting the formulae of classical crisp definitions in many-valued logic. If fuzzy

[3]It can, e.g., be observed that the FLd models of Definitions 3 and 4 do not exactly follow the principle of cuts, since the crisp elements or sets are in fact *functional values* rather than α-cuts of the crisp functions that represent gradual elements and sets in higher-order FLd.

notions are defined in this natural way, then the principle of cuts is often observed: the α-cuts of fuzzy sets are crisp sets, the α-cuts of fuzzy relations are crisp relations, the α-cuts of fuzzy Dedekind or MacNeille cuts [27, 13, 3, 2] come out as crisp Dedekind–MacNeille cuts, etc. Unlike in FLt, however, in FLd the fuzzy notions have also to conform with the principle of persistence; this constrains the α-cuts to *nested* systems of the corresponding crisp objects. In the particular case of fuzzy elements, the α-cuts of an FLd fuzzy element a must not only be crisp elements (as in FLt), but also must satisfy the principle of persistence for all formulae, in particular for the formula $x = a$. The latter already necessitates that the α-cut of a equals its β-cuts for all $\beta \leq \alpha$; and since this should hold for all α, the fuzzy element a has to be constant. Thus in FLd we can only have constant fuzzy elements, which can be identified with ordinary crisp elements. Similarly, by enforcing the nesting of α-cuts, the principle of persistence reduces in FLd gradual sets to common fuzzy sets.

No doubt fuzzy elements are a natural notion, abundant in many real-life situations; therefore the above difficulties should not stop us from investigating them. There are no obstacles to investigating them in the framework of FLt. However, current FLd can only render them indirectly in a higher-order setting, since they do not conform to the principle of persistence upon which all current systems of FLd are founded. Thus even though (advanced) FLd can (clumsily) capture the new notions, they actually do not fall into its primary area; and so the way in which FLd can contribute to the investigation of these notion is rather limited.[4] This of course does not diminish the importance of the new notions for FLt and does not even exclude the usefulness of their formal counterparts in some parts of FLd. The above analysis only shows that when employing fuzzy elements in FLd, we shall have to deal with complex objects (crisp functions from the set of internalized truth values) rather than some kind of more primitive notion.

A further analysis will be needed to find out if Dubois and Prade's gradual elements and sets can be treated propositionally or as a primitive first-order notion in a *radically new* system of deductive fuzzy logic. Since a direct logical rendering of gradual sets would need to drop the principle of persistence, it would have to adopt an entirely different concept of truth preservation under inference; such a radical change would consequently affect virtually all logical notions. Unfortunately, many straightforward approaches are not viable, as they would trivialize the theory. E.g., a notion of truth preservation based on the identity (rather than order) of truth degrees would reduce truth degrees to mere indices exactly in the way FLt does; however, it would trivialize the logic to classical Boolean logic.[5] From the opposite point of view, this could be an indication that by treating membership degrees as mere indices (rather than truth degrees that should be preserved under graded inference), FLt does not in fact step out of the classical framework; it is the gradual inference what makes things genuinely fuzzy from the FLd point of view, rather than just employing some set of indices like $[0, 1]$.

[4] One of the few advantages of studying gradual elements in formal higher-order setting might be the possibility of generalizing them easily to “fuzzy gradual elements” by dropping the condition of the crispness of the function that represents a gradual element or set in Definitions 3 and 4. The apparatus of higher-order fuzzy logic then facilitates the investigation of this higher-order fuzzy notion, which could be more difficult to study in the classical models of FLt.

[5] The α-levels of fuzzy or gradual notions are crisp, therefore they follow the rules of classical logic, i.e., the logic of Boolean algebras. An α-level based definition of truth preservation would correspond to taking the direct product of Boolean algebras B_α for all levels $\alpha \in [0, 1]$. However, the direct product of Boolean algebras is a Boolean algebra, therefore the resulting logic would remain classical.

Case study 2: The entropy of a fuzzy set

Various definitions of the *entropy* of a fuzzy set have been proposed in traditional fuzzy mathematics, for instance:

- De Luca and Termini's [15] entropy $E_k(A) = D_k(A) + D_k(A^c)$
- Yager's [39] entropy $Y_p(A) = 1 - \ell^p(A, A^c)/(\ell^p(A, \emptyset))^p$
- Kaufmann's [29] entropy $K_p(A) = 2n^{-1/p} \cdot \ell^p(A, \overline{A})$
- Kosko's [31] entropy $R_p(A) = \ell^p(A, \overline{A})/\ell^p(A, \underline{A})$

where A is a finite $[0,1]$-valued fuzzy set; A^c is its additive complement, $A^c(x) = 1 - A(x)$; $\overline{A}$ is defined as $\overline{A}(x) = 1$ if $A(x) \geq 0.5$, and 0 otherwise; $\underline{A} = (\overline{A})^c$; p, k are parameters, $p \geq 1$ and $k > 0$; $D_k(A) = -k \sum_i A(x_i) \log A(x_i)$; and ℓ^p is the distance between finite fuzzy sets defined as $\ell^p(A, B) = (\sum_i |A(x_i) - B(x_i)|^p)^{1/p}$.

The common feature of all such entropy measures is that they assign the minimal (zero) entropy to crisp sets, and maximal (unit) entropy to fuzzy sets with $A(x) = 0.5$ for all x in the universe of discourse.[6]

The definition is motivated (and the name *entropy* justified) by the idea that the membership degree 0.5 tells us the least amount of information ("nothing") about the membership of x in A. In other words, that the membership degree of 0.5 gives us the same degree of "certainty" that x belongs to A as that x does not belong to A, and so it provides us with no information (knowledge) as to *whether* x belongs to A. The membership degrees of 0 and 1, on the other hand, give us full "knowledge" or "certainty" about the membership of x in A, and thus provide us with maximal information as regards the membership of x in A. The degree of fuzziness, measured by the entropy measures, thus (in FLt) expresses the informational contents contained in the fuzziness of the fuzzy set.

In FLd, on the other hand, such concepts of entropy do not have good motivation.[7] This is because in FLd, the membership degree cannot be interpreted as the degree of *knowledge* or *certainty* of whether x belongs to A or not, but only as the degree of the (guaranteed) *truth* of the statement that x belongs to A. From the FLd point of view it is not true that $A(x) = 0.5$ gives us the least information on the membership in A. On the contrary—each membership degree gives us the same (namely, full) information about the *extent* of membership in A.

The difference between the information conveyed by membership degrees in FLt and FLd can be illustrated by the following consideration. We have the following trivial observation in all usual systems of FLd that contain a well-behaved implication connective $\Rightarrow$.

Fact 5. If it is provable that $A(z) \Rightarrow \varphi(z)$ for all z, then for any membership degree α, if the truth degree of $A(x)$ is α, then the truth degree of $\varphi(x)$ is at least α.

Thus in FLd, if we know that $x \in A$ is true to degree 0.5 and that all elements of A satisfy some property φ (in the sense of FLd—i.e., that $A(z) \Rightarrow \varphi(z)$ is valid for all z), then we know that x satisfies φ at least to degree 0.5. Therefore in FLd, the truth degree

[6] In more detail, they satisfy de Luca and Termini's [15] axioms for entropy measures $E\colon [0,1]^X \to [0,1]$, namely: (i) $E(A) = 0$ iff A is crisp; (ii) $E(A) = 1$ iff $A(x) = 0.5$ for all $x \in X$; (iii) $E(A) \leq E(B)$ if for every $x \in X$ either $A(x) \leq B(x) \leq 0.5$ or $A(x) \geq B(x) \geq 0.5$; and (iv) $E(A) = E(A^c)$.

[7] At least not as measures of the informational contents of fuzziness. If definable in a particular fuzzy logic, they can only serve as measures of fuzziness, without any connection to information.

of 0.5 does *not* represent “no knowledge” or “equal possibility of both cases”. Rather, like any other membership degree, it represents a certain *guaranteed degree of participation* of x on the properties of A. In other words, *any* membership degree α of $x \in A$ tells us in FLd that the properties entailed by the membership in A will be satisfied by x at least to degree α.

From the informational point of view, in FLd (as shown by Fact 5) the membership degree 0.5 restricts the possible truth values of $\varphi(x)$, for any property φ entailed by the membership in A, to the interval $[0.5, 1]$. In this sense, the least informative membership degree should in FLd be 0 (as it does not restrict the interval at all) and the most informative degree should be 1 (as it maximally restricts the interval to the single value 1). However, 0 is also the most informative (and 1 the least informative) degree as regards the satisfaction by x of the properties of another set, namely A^c. Therefore in FLd, the informational contents of membership degrees is not determined simply by their value.

Thus from the point of view of FLd, no membership degree conveys more information than another just by its value. Therefore, no concept of entropy which assigns the least informational contents to the fuzzy set with $A(x) = 0.5$ for all x is well-motivated in FLd. Consequently we have to conclude that the notions of entropy belong to the area of FLt rather than FLd; and even though they can be defined in higher-order FLd,[8] their significance in FLd and the extent to which FLd can help investigate them is limited. This does not deny their importance and good motivation in FLt under the interpretations of membership degrees as indicated above (of knowledge, certainty, etc.); only they are not meaningful for the concept of *guaranteed truth,* which is the domain of FLd.

As stressed above, the unmotivatedness of the concept of entropy in FLd is caused by the fact that membership degrees represent in FLd the degrees of *truth* (of the statement “$x \in A$”) rather than the degrees of *knowledge* or *certainty* about $x \in A$. The *uncertainty* about $x \in A$ would not in FLd be expressed by an intermediate membership degree, but rather by an *uncertain* membership degree. The first idea how to render uncertain membership degrees in FLd is, obviously, to take a crisp or fuzzy set of possible membership degrees, like in interval-valued fuzzy sets [1] or type-2 fuzzy sets [40]. However, in the framework of FLd, this idea has to be refined: a fuzzy set of membership degrees does not in FLd represent the degree of certainty or knowledge about the membership degrees, either, but only expresses the degree of *truth* of some property of membership degrees. Thus it would be necessary to introduce some modality, e.g., “it is known that”, and interpret the fuzzy set of membership degrees α as expressing the *truth* degree of the statement “it is known that the membership degree of $x \in A$ is α”, rather than the degree of knowledge itself. This subtle difference is insignificant for atomic epistemic statements, but plays a role when considering complex epistemic statements composed by means of propositional connectives. (For more on this distinction see [24, 22].) The rendering of the uncertainty of membership in a fuzzy set, which motivates the notion of entropy in FLt, is thus in FLd much more complicated than what is expressed by simple intermediary membership degrees.

[8]For instance, Yager’s entropy Y_1 and Kosko’s entropy R_1 can be defined in the higher-order fuzzy logic ŁΠ [6], since it contains all arithmetical ingredients necessary for their definitions: additive negation $(1-x)$, product implication (i.e., division), and the Baaz Δ connective which ensures [6, §7] the definability of crisp finite sequences, needed for the inductive definition of sums of membership degrees. Any classically definable entropy measure is eventually definable in higher-order FLd by more sophisticated means, since classical mathematics is interpretable in standard higher-order fuzzy logics [6, §7]. (By definability we mean here definability in standard $[0, 1]$ models.)

Case study 3: Aggregation of fuzzy data

The exclusive interpretation of membership degrees as guaranteed degrees of truth leads to certain restrictions on admissible logical systems of FLd. Since the intended interpretation "truth at least to α" is based on an order ("at least") of truth degrees, logical systems suitable for FLd have to be among the logics of partially ordered (or at least preordered) algebras or logical matrices, i.e., among Rasiowa's implicative logics [37] or Cintula's weakly implicative logics [14].[9] The property of prelinearity, advocated in [10] as the characteristic feature of deductive *fuzzy* logics, then leads to Cintula's class of weakly implicative fuzzy logics [14]. Another condition that further constrains the class of logical systems best suitable for *deductive* fuzzy logic is the law of residuation [22, 36]. As will be shown in this section, the law of residuation and related requirements present another important difference between FLd and FLt.

One of the typical tasks of applied FLt is to gather some fuzzy data $\varphi_1, \ldots, \varphi_k$, aggregate their truth values by means of some aggregation operator $\bigodot$,[10] and draw some conclusion ψ (possibly, about the action to be performed or the answer to be given) based on $\bigodot_{i=1}^{k} \varphi_i$. In symbols, to perform an inference $(\bigodot_{i=1}^{k} \varphi_i) \to \psi$, where $\to$ is a suitable implication. We have in mind, e.g., the following kinds of applications:

Example 6. In a fuzzy controller based on if-then rules, the input data φ_i are the truth values of the evaluating expressions "X_i is Y_i" given by the measured values of linguistic variables X_i; the output of a single rule is the truth value of "X is Y", inferred from φ_i's by suitable operations $\bigodot$ and $\to$.

Example 7. A fuzzy logic based engine for answering database queries (say, for accommodation search) may ask for the degrees of the user's preferences, i.e., the weights of such variables as price, distance, etc. Based on the aggregated weighted values of these variables for particular hotels, the engine lists the hotels in descending order by their suitability for the user.

An important observation about this kind of applications is that just one inference step is performed for each set of input data:

- When a fuzzy controller performs an action based on the fuzzy inference, the values of measured variables change, and the next inference is based on the new (changed) data.
- When listing hotels in the order of the user's preferences, the evaluation of each hotel is based on the hotel's own parameters; the evaluation of the next hotel takes new (i.e., the next hotel's) data.

In such cases, therefore, the device may work in a cycle, but each iteration processes a new set of data. The modus operandi of such applications of FLt is as depicted in Figure 1.

[9] The defining conditions of (weakly) implicative logics embody the correspondence between the full truth of implication and the (pre)ordering of truth degrees. Besides the conditions of substitution-invariant Tarski consequence (common to most systems of formal logic), weakly implicative logics require the logical validity of (i) the axiom $\varphi \to \varphi$ and the rules of (ii) modus ponens (from φ and $\varphi \to \psi$ infer ψ), (iii) transitivity of implication (from $\varphi \to \psi$ and $\psi \to \chi$ infer $\varphi \to \chi$), and (iv) congruence of all connectives w.r.t. bi-implication $\varphi \to \psi$ and $\psi \to \varphi$. Weakly implicative logics in general admit multiple degrees of full truth; Rasiowa's implicative logics forbid them by the additional rule (v) of weakening (from φ infer $\psi \to \varphi$).

[10] The term "aggregation operator" is here understood in a broad sense, without requiring any fixed set of axioms for the operator.

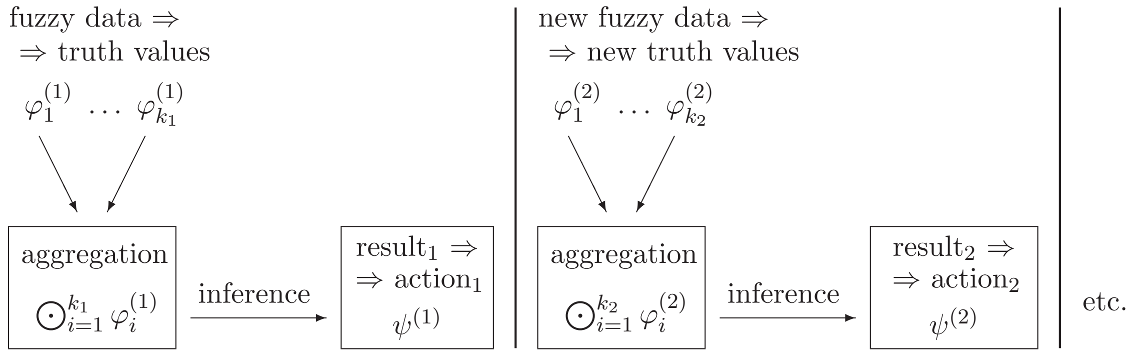

Figure 1: Modus operandi of applied FLt

Another observation is that the data that enter the aggregation and inference are usually extra-logical (measured in the real world, read from a database etc.). In particular, they usually do not contain the operators $\bigodot$ and $\to$ of the inference mechanism, and so in FLt inference one usually need not consider nested implications (the formulae expressing the inference laws are “flat”).

The operations used for aggregation of the input data vary widely among particular applications. Consequently, various classes of aggregation operators $\bigodot$ are studied in theoretical FLt, including t-norms and co-norms, uninorms, copulas, semi-copulas and quasi-copulas, various kinds of averages and means, etc. (for a brief overview see, e.g., [30, Ch. 3]).

The situation in FLd is different, as is the typical modus operandi of FLd. The formally-deductive aims of FLd require the preservation of guaranteed truth values also in successive (iterated) inferences, which are typical for multiple-step deductions. In formal derivations we often have intermediary steps and results, lemmata, partial conclusions, etc., and we want the guaranteed truth degree of a conclusion to remain coherent throughout long deductions. Therefore, a typical modus operandi of FLd is the one depicted in Figure 2.

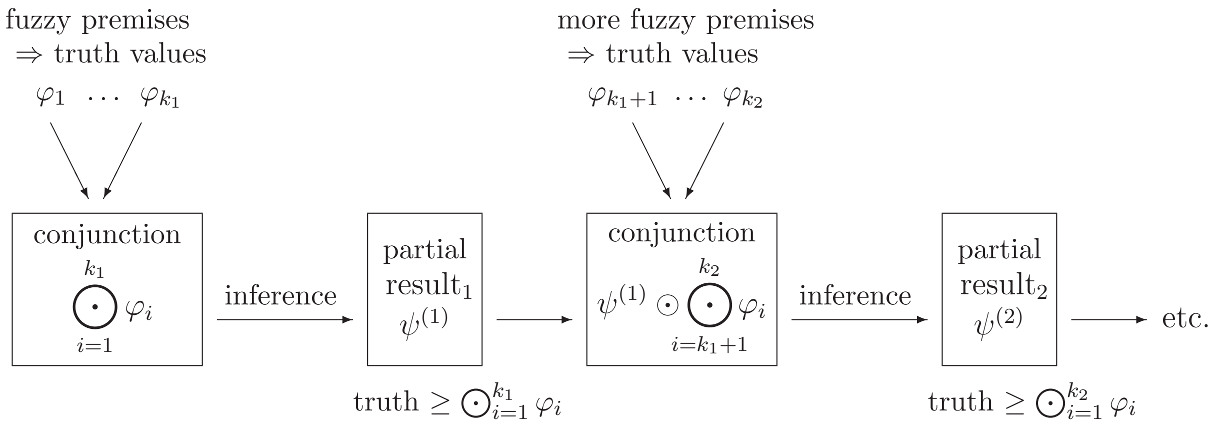

Figure 2: Modus operandi of FLd

Observe first that in the multiple-step derivations of FLd, the premises of the first steps still play a role in the following steps, since partial results enter further deductions. Furthermore, in the formally logical setting of FLd, formulae entering deductions need not be purely extralogical and can have inner logical structure, i.e., be built up from subformulae by means of logical connectives, including those used for inference, i.e., $\odot$ and $\to$. Thus unlike in FLt, the formulae in FLd inference need not be "flat" and nested implications can occur. Implication thus plays a double role in FLd deduction: it is used for making inferences, but can also occur as a connective within a formula that enters the inference as a premise or comes out as a conclusion. Similarly conjunction is used for the aggregation of premises, but can also appear as a connective inside the premises and conclusions. If both roles of the operators are to match, they have to satisfy conditions that describe the match of the roles. Namely, whenever φ_1 is a premise of implication (inference) and $\varphi_2 \to \psi$ is its conclusion, both roles of implication will accord iff φ_1 and φ_2 together (i.e., aggregated) imply ψ (since both φ_1 and φ_2 are after all premises for ψ—one in implication-as-inference and one in implication-as-connective); and vice versa, if φ_1 and φ_2 jointly imply ψ, then φ_1 alone should imply $\varphi_2 \to \psi$ (for the same reason). Similarly, φ_1 and φ_2 aggregated should imply ψ if and only if $\varphi_1 \odot \varphi_2$ implies ψ (this corresponds to the match of both roles for conjunction). Since by the earlier considerations implication-as-inference is in FLd understood as truth-preservation (i.e., the relation $\leq$), the requirement can be formulated as the condition

$$\varphi_1 \odot \varphi_2 \leq \psi \quad \text{iff} \quad \varphi_1 \leq \varphi_2 \to \psi \tag{1}$$

The general form for an arbitrary number of premises as in Figure 2 already follows from (1). This law of *residuation* is therefore required in FLd for ensuring the coherence of the guaranteed truth thresholds in multiple-step deductions with nested implications, while it need not be required in one-step inferences with flat formulae in FLt.

The principle of residuation restricts significantly the class of conjunctive operators admissible in FLd. Together with a few reasonable additional requirements (see Remark 10 below) it confines the FLd-suitable $[0,1]$-conjunctions & to *left-continuous t-norms* (or residuated uninorms, if we admit degrees of full truth) [20, 28, 33]. Other operators for fuzzy data aggregation are not meaningful in FLd, though they are both meaningful and important in FLt (as FLt need not preserve guaranteed truth degree in nested and iterated inferences).

Like in the case of fuzzy elements and the notion of entropy, many FLt conjunctive operators are still definable in systems of deductive fuzzy logic: e.g., a broad class of t-norms which are *not* left-continuous is representable in the logic ŁΠ [21, 34]. Nevertheless, as in the cases above, the apparatus of FLd is most efficient for conjunctions to which the FLd systems are tailored, i.e., which respect the above constraints.

Remark 8. The constraints on admissible conjunction connectives rule out the meaningfulness of most cut-wise definitions in FLd. Since most left-continuous t-norms are not idempotent, α-cuts are in general not preserved by conjunction in most systems of FLd (except in Gödel fuzzy logic of the minimum t-norm).[11] Thus, e.g., the cut-wise definition of the intersection of fuzzy sets is from the FLd point of view only meaningful in Gödel logic; in other systems of FLd, the cut-wise intersection (which equals the minimum-intersection) does not satisfy the defining condition of intersection that the membership degree of x in $A \cap B$ be the conjunction of the membership degrees of x in A and B, i.e., that $(A \cap B)x = Ax \,\&\, Bx$.

Thus, e.g., Dubois and Prade's definitions of elementary operations on gradual sets proposed in [18] (which are cut-wise, as we noted in the first case study), can only be well-motivated in FLt. Similar considerations restrict the FLd-meaningfulness of many parts

[11] Most FLd conjunctions thus do not satisfy the axiom often required in FLt of aggregation operators, namely that $x \odot \cdots \odot x = x$.

of categorial (sheaf) approach to fuzzy sets, which often works cut-wise (i.e., fiber-wise) and thus belongs to FLt rather than FLd.

Again, this does not diminish the importance of cut-wise notions in FLt; only we should be aware that they are not well-motivated in FLd. In deductive fuzzy logic, many cut-wise notions can still be defined, and some of them do have some importance even in FLd. For instance, in all logics based on continuous t-norms, the minimum conjunction $\wedge$ and maximum disjunction $\vee$ are definable, and by means of these connectives one can define the cut-wise operations of min-intersection and max-union. However, their role in FLd systems is different than that of the notions based on usual (strong) conjunction &; in particular, min-conjunction cannot be used as a surrogate for strong conjunction, since both connectives have different meaning. Strong conjunction & represents the use of *both* conjuncts, while min-conjunction $\wedge$ represents the use of any *one* of the conjuncts, as can be seen from the following equivalences valid in BL and related systems:

$$[(\varphi_1 \mathbin{\&} \varphi_2) \to \psi] \quad \leftrightarrow \quad [\varphi_1 \to (\varphi_2 \to \psi)] \tag{2}$$

$$[(\varphi_1 \wedge \varphi_2) \to \psi] \quad \leftrightarrow \quad [(\varphi_1 \to \psi) \vee (\varphi_2 \to \psi)] \tag{3}$$

Since it is (2) that we need in iterated inference rather than (3), minimum conjunction cannot be used for aggregation of premises in FLd. Similarly, min-intersection does not represent membership in *both* fuzzy sets, but only in *any* of them, and cannot be used in contexts when both Ax and Bx are required. The following example demonstrates the methodological consequences of the distinction between the two conjunctions in FLd.

Example 9. The notion of antisymmetry of a fuzzy relation R w.r.t. a similarity E defined with min-conjunction, i.e., by $\inf_{xy}(Rxy \wedge Ryx \to Exy)$ as in [13] or similarly in [27], is not well-motivated in FLd, since in antisymmetry we clearly need *both* Rxy and Ryx to infer Exy (neither Rxy nor Ryx alone is sufficient for Exy in antisymmetric relations; cf. (3)). Thus in FLd, we have to define antisymmetry with strong conjunction, i.e., as $\inf_{xy}(Rxy \mathbin{\&} Ryx \to Exy)$, even though some theorems of [27, 13] will then fail. From the *deductive* point of view, min-conjunction antisymmetry is only well-motivated in Gödel logic.

Remark 10. As mentioned at the beginning of this section, the requirements on the transmission of truth in FLd lead to the defining conditions of Rasiowa's implicative logics or Cintula's weakly implicative (fuzzy) logics. However, these conditions only ensure good behavior of *fully true* implication, which then corresponds to the order of truth degrees [14]. Inferentially sound behavior of *partially true* implication and conjunction requires further axioms, including the law of residuation (as seen above), since only then implication internalizes the transmission of partial truth and conjunction internalizes the cumulation of premises in graded inference. The latter law also makes formal systems of FLd belong to the well-known and widely studied class of substructural logics (in Ono's [36] sense, i.e., the logics of residuated lattices).

Thus from the point of view of *deductive* fuzzy logic, Cintula's class of weakly implicative fuzzy logic is still too broad. Best suitable logics for FLd are only those weakly implicative fuzzy logics that satisfy residuation and several natural requirements of the internalization of local consequence (namely the logical axioms expressing the antitony resp. isotony of implication in the first resp. second argument, and associativity and commutativity of conjunction; cf. [5]). The resulting class can be understood as the *formal* mathematical delimitation of deductive fuzzy logics. The above conditions characterize them as those weakly implicative fuzzy logics which are extensions or expansions of the logic UL of residuated uninorms [33] or, if we add the law of weakening $\varphi \to (\psi \to \varphi)$, of the logic MTL of left-continuous t-norms [20].[12]

Again this does not imply that other weakly implicative fuzzy logics or logics used in FLt are deficient. However, only logics from the above defined class suit best to the motivation of FLd (i.e., transmission of guaranteed partial truth in multi-step deductions) and admit the construction of formal fuzzy mathematics in the sense of [8]. This is because their implication and conjunction respectively internalize the local consequence relation and the cumulation of premises: they have, in Ono's [36] words, a "deductive face". Other logics[13] lie outside the primary area of interest of FLd, though they may be of their own importance and interest in FLt.

Conclusions

The three case studies show that FLd differs from broader FLt in many aspects, including the area of competence, methods, motivation, formalism, etc. It should perhaps be admitted that symbolic fuzzy logicians on the one hand and researchers in "mainstream fuzzy logic" on the other hand do rather different things and work in two distinct, even though related, areas (with some non-empty intersection). Since after the years of usage there is no chance for changing the name "fuzzy logic" in either tradition, a suitable determinative adjective (like *symbolic, formal, mathematical,* or as proposed here, *deductive*) attached to the name of the narrower and younger of both areas is probably the best solution to possible terminological confusions.

It is sometimes complained that fuzzy logic does not have a clear methodology for defining its notions and the direction of research. FLd, as its very narrow and specific branch, however, does possess a rather clear methodology, inherited for a large part from the methodology of non-classical logics and classical foundations of mathematics [22, 8]. This may be a consequence of the fact that FLd has chosen and clarified *one* of all possible interpretations of membership degrees, and now studies the properties of this single clarified concept. FLt, on the other hand, admits many interpretations of membership degrees and often tries to investigate them together, without separating them properly and without clarifying carefully which of the possible interpretations is considered.[14]

A historical parallel can be seen in the early history of classical (crisp) set theory. As noticed by Kreisel in [32], Cantor's notion of set was a mixture of at least three concepts—finite sets of individuals, subsets of some domain, and properties (unbound classes). Part of the opposition against set theory was due to its confusion of these notions of set: the *crude mixture* (as Kreisel calls it) did not possess good properties, and the paradoxes of naive set theory confirmed the bad feeling. Only after one element of the crude mixture (viz iterated subsets) was clearly separated by Russell and Zermelo and shown to have

[12]The class is only slightly broader than Metcalfe and Montagna's class of "substructural fuzzy logics" [33], which in addition requires the completeness w.r.t. standard $[0, 1]$ semantics.

[13]Including Zadeh's original system with min-conjunction, max-disjunction, $(1 - x)$-negation and S-implication, as well as Łukasiewicz logic with strong conjunction replaced by min-conjunction—a system both favored and targeted by many philosophers of vagueness.

[14]For instance, general definitions (e.g., cut-wise) of operations on fuzzy sets are often given, regardless of what is the intended interpretation of membership functions. (This is also the case with the operations on gradual sets defined in [18].) However, suitable definitions may depend on the intended meaning of membership degrees (as also demonstrated by the unsuitability of many such definitions for FLd), since different underlying phenomena may have different properties (and thus also different demands on the behavior of suitable operations).

good and rich enough properties, the notion of set could start playing its foundational role in mathematics.

Similarly the theory of fuzzy sets presents a mixture of various *different* notions of fuzzy set (truth-based, possibilistic, linguistic, frequentistic, probabilistic, etc.). While FLd has distilled one element of the mixture (namely the truth-based notion of fuzzy set), FLt often continues to investigate the crude mixture as a whole, only partially aware of the distinctions needed to be made. (Not that it never distinguishes the areas of applicability of its own notions: sometimes it does; but often it forgets to do so or is not careful enough.)

The methodological success of FLd and its advances should stimulate FLt to distinguish with similar clarity the *exact* components of the crude mixture of notions of fuzzy set. Theoretical gains from their clear separation and investigation of the most promising ones would certainly be large (as was, e.g., the gain from conceptual and methodological clarification of the notion of probability); some areas of FLt besides FLd (e.g., possibility theory) already seem to be close to such clarification.

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References

[1] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[2] L. Běhounek. Towards a formal theory of fuzzy Dedekind reals. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 946–954, Barcelona, 2005.

[3] L. Běhounek. Two notions of fuzzy lattice completion. In V. Novák and M. Štěpnička, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 22–24, Ostrava, 2005. University of Ostrava.

[4] L. Běhounek. Dubois and Prade's fuzzy elements: A challenge for formal fuzzy logic. In S. Gottwald, P. Hájek, and M. Ojeda-Aciego, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 90–95, Málaga, 2006.

[5] L. Běhounek. An alternative justification of the axioms of fuzzy logics. *The Bulletin of Symbolic Logic*, 13:267, 2007.

[6] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[7] L. Běhounek and P. Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[8] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[9] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[10] L. Běhounek and P. Cintula. Fuzzy logics as the logics of chains. *Fuzzy Sets and Systems*, 157(5):604–610, 2006.

[11] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, 2007.

[12] L. Běhounek and M. Daňková. Relational compositions in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 160:1005–1036, 2009.

[13] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[14] P. Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[15] A. de Luca and S. Termini. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. *Information and Control*, 20:301–312, 1972.

[16] D. Dubois, L. Godo, H. Prade, and F. Esteva. An information-based discussion of vagueness. In H. Cohen and C. Lefebvre, editors, *Handbook of Categorization in Cognitive Science*, chapter 40, pages 892–913. Elsevier, 2005.

[17] D. Dubois and H. Prade. The three semantics of fuzzy sets. *Fuzzy Sets and Systems*, 90:141–150, 1997.

[18] D. Dubois and H. Prade. Fuzzy elements in a fuzzy set. In Y. Liu, G. Chen, and M. Ying, editors, *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 1, pages 55–60, Beijing, 2005. Tsinghua University Press/Springer.

[19] D. Dubois and H. Prade. Fuzzy intervals versus fuzzy numbers: Is there a missing concept in fuzzy set theory? In S. Gottwald, P. Hájek, U. Höhle, and E. P. Klement, editors, *Fuzzy Logics and Related Structures: Abstracts of 26th Linz Seminar on Fuzzy Set Theory*, pages 45–46, 2005.

[20] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[21] F. Esteva, L. Godo, and F. Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[22] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[23] P. Hájek. What is mathematical fuzzy logic. *Fuzzy Sets and Systems*, 157(5):597–603, 2006.

[24] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. In *Proceedings of the 11th UAI Conference*, pages 237–244, Montreal, 1995.

[25] P. Hájek and Z. Haniková. A development of set theory in fuzzy logic. In M. Fitting and E. Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Physica-Verlag, Heidelberg, 2003.

[26] P. Hájek and V. Novák. The sorites paradox and fuzzy logic. *International Journal of General Systems*, 32:373–383, 2003.

[27] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[28] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[29] A. Kaufmann. *Introduction to the Theory of Fuzzy Subsets*, volume I. Academic Press, New York, 1975.

[30] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Prentice Hall, 1995.

[31] B. Kosko. Fuzzy entropy and conditioning. *Information Sciences*, 40:165–174, 1986.

[32] G. Kreisel. Informal rigour and completeness proofs. In I. Lakatos, editor, *Problems in the Philosophy of Mathematics*, pages 138–157. North-Holland, Amsterdam, 1967.

[33] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *Journal of Symbolic Logic*, 72(3):834–864, 2007.

[34] F. Montagna and G. Panti. Adding structure to MV-algebras. *Journal of Pure and Applied Algebra*, 164(3):365–387, 2001.

[35] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149(2):235–273, 2004.

[36] H. Ono. Substructural logics and residuated lattices—an introduction. In V. F. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pages 193–228. Kluwer, Dordrecht, 2003.

[37] H. Rasiowa. *An Algebraic Approach to Non-Classical Logics.* North-Holland, Amsterdam, 1974.

[38] G. Takeuti and S. Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32:1–32, 1992.

[39] R. R. Yager. On the measure of fuzziness and negation, part I: Membership in the unit interval. *International Journal of General Systems*, 5:221–229, 1979.

[40] L. A. Zadeh. Concept of a linguistic variable and its application to approximate reasoning—I. *Information Sciences*, 8:199–249, 1975.

[41] L. A. Zadeh. Fuzzy logic and approximate reasoning. *Synthese*, 30:407–428, 1975.

From fuzzy logic to fuzzy mathematics: A methodological manifesto

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Abstract: The paper states the problem of fragmentation of contemporary fuzzy mathematics and the need of a unified methodology and formalism. We formulate several guidelines based on Hájek's methodology in fuzzy logic, which enable us to follow closely the constructions and methods of classical mathematics recast in a fuzzy setting. As a particular solution we propose a three-layer architecture of fuzzy mathematics, with the layers of formal fuzzy logic, a foundational theory, and individual mathematical disciplines developed within its framework. The ground level of logic being sufficiently advanced, we focus on the foundational level; the theory we propose for the foundations of fuzzy mathematics can be characterized as Henkin-style higher-order fuzzy logic. Finally we give some hints on the further development of individual mathematical disciplines in the proposed framework, and proclaim it a research programme in formal fuzzy mathematics.

Keywords: Non-classical logics, formal fuzzy logic, formal fuzzy mathematics, higher-order fuzzy logic. MSC: 03A05, 03B52, 03E70, 03E72.

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One of the motives for theoretical studies in fuzzy mathematics is the pursuit of formal reconstruction of the methods commonly used in applied fuzzy mathematics. The greatest success in such investigations is undoubtedly the area of formal fuzzy logic: this discipline has recently reached the point when it is reasonable to attempt to use it as a ground theory for the formalization of other branches of fuzzy mathematics.

This paper tries to provide certain guidelines for such a transition from formal fuzzy logic to formal fuzzy mathematics. The guidelines are based upon doctrines observed by the Prague workgroup on fuzzy logic founded and led by Petr Hájek. We attempt to formulate explicitly some distinct features of Petr Hájek's approach, which we reconstruct from his scattered remarks and the general direction of his papers, and implement them in the form of a research programme. We hope that Petr Hájek will find our reconstruction of his doctrine faithful enough; or else that he will enter into a fruitful dispute with his own disciples over the methodological foundations of our discipline. If the former is the case, then we deem that the best label for the enterprise described in this paper would be *Hájek's Programme* in the foundations of fuzzy mathematics.

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The cornerstone of Hájek's approach to fuzzy mathematics is the doctrine of working in a formal axiomatic theory over a fuzzy logic, rather than investigating particular models. For ease of reference, let us call it the *formalistic imperative.* Good reasons for such an approach can be found, both of a philosophical and pragmatic nature.

A philosophical reason is found in the following argument. Fuzzy logic describes the laws of truth preservation in reasoning under (a certain form of) vagueness. Its interpretation in terms of truth degrees and membership functions is just a mathematical *model*—a classical rendering of vague phenomena. Of course, the laws of fuzzy logic were originally discovered with the help of this model, and truth degrees form its principal semantics; but once we believe that the laws capture fuzzy inference correctly, we can abstract from the model that helped us to find them.

Fuzzy predicates are essentially not different from crisp predicates: the only difference is the graded boundary of fuzzy sets, due to which some of the laws of classical reasoning about them fail. The laws of inference valid for fuzzy predicates form fuzzy logic; classical logic is its limit case, applicable if by chance all predicates involved are crisp. Reasoning about fuzzy predicates therefore follows the laws of fuzzy logic in the same manner as reasoning about crisp predicates follows the laws of classical logic. In mathematics, such reasoning can be formalized into formal theories, in which the deduction follows the rules of classical logic if all predicates are crisp, or fuzzy logic if any of them are fuzzy. The mathematics of structures involving fuzziness can thus assume the form of formal theories over fuzzy logic, rather than the study of membership functions which uses classical logic. The former way is to be preferred as a genuinely fuzzy approach, while the latter is only a secondary classical model of fuzziness.

Admittedly, a formal theory over fuzzy logic is just a notational abbreviation of classical reasoning about the class of all models of the theory. Nonetheless, the axiomatic method is the general paradigm of mathematics; one of its main advantages is that the appropriate choice of the language of the formal theory screens off irrelevant features of the models. An axiomatic system is thus not only the means of generalization over all models, but rather an abstraction to their constitutive features.

Obviously, the formalistic imperative applies mainly to the development of mathematical fuzzy logic and various branches of fuzzy mathematics, not to particular applications of fuzzy sets. In an application, we are modeling particular phenomena and thus we naturally work with a particular model. For instance, some real-life problems (e.g., processing of a questionnaire with five grades between absolute yes and absolute no) may invite a definite algebra of truth values. However, having a general theory may of course help even in particular cases, since it will describe the general features of the problem. The programme of developing fuzzy mathematics in a theoretical manner stresses the priority of general theories over immediate applicational needs.

The idea that fuzzy inference cannot be reduced to a particular model able to account for its rules entails that in the investigation of fuzzy inference we should not limit ourselves to one particular fuzzy logic (e.g., Łukasiewicz). The model which underlies it—e.g., a specific t-norm—is particular, while fuzzy reasoning in general is broader. There are examples of fuzzy reasoning that follow variant inference rules, all of which are suitable for different respective contexts of real-life situations and invite explanations in terms of various individual t-norms or other semantics. The multitude of existing fuzzy logics varying both in expressive power and inference rules is not only explicable by the need of capturing of all aspects of fuzzy inference in diverse contexts, but even indispensable for this enterprise.

Similar considerations are related to Hájek's preference for fuzzy logics without truth constants in the language (except for those which are definable). First, the truth constants have little support in natural language. Second, by incorporating the truth values into the syntax, we force the logic to follow too closely a particular model of vague inference, viz that using truth values. Of course, we cannot be too dogmatic about rejecting truth constants: it turns out that in sufficiently strong theories, at least rational truth constants are definable. Sometimes, the truth degrees are useful for a particular application. However, we should be cautious of deliberately introducing them into logic and thus restricting the possible models of vague inference.

Thus, even though liberal in both the expressive power and inference rules, we—following Hájek—believe a certain style of logical systems to be a most suitable formalism for representing fuzzy inference. For brevity's sake, in what follows we shall call them *Hájek-style fuzzy logics*. Put in a nutshell, they are fuzzy logics retaining the syntax of classical logic (preferably without truth constants), defined as axiomatic systems (rather than non-axiomatizable sets of tautologies). A prototypical example is Hájek's Basic Logic BL, propositional or predicate. This certainly does not mean that other systems (for instance, with some kind of labelled formulae) have no merits of their own; only they are not preferred for the development of fuzzy mathematics by the Prague school. In the following paragraphs we give some reasons for such preferences.

There is a pragmatic motive for retaining as much of classical syntax as possible. The way of working in theories over Hájek-style fuzzy logics resembles closely the way of working in classical logic: Hájek-style fuzzy logics are often just weaker variants of Boolean logic—syntactically fully analogous, just lacking some of its laws. Therefore, many theoretical and metatheoretical methods developed for classical logic can be mimicked and employed, resulting in a quick and sound development of the theory. This feature has already been utilized in the metamathematics of fuzzy logic—the proofs of the completeness, deduction, and other metatheorems have often been obtained by adjustments of classical proofs.

To illustrate the utility of this guideline, we allege that an axiomatic theory of fuzzy sets can more easily be developed as a formal theory of binary membership predicate over some fuzzy logic than if the graded membership is rendered, e.g., as a ternary predicate between elements, sets, and truth values in the framework of classical logic. Many constructions and even proofs of the classical theory will work in the former case and need not be rediscovered (nor even reformulated). Even though both theories may turn out equivalent, the resemblance of fuzzy concepts to classical ones becomes more visible in Hájek's approach: cf. the many 'breakthrough' definitions of fuzzy set inclusion which, if put down in Hájek-style fuzzy logic have exactly the form of the classical definition of set inclusion. This is another reason for preferring the classical syntax in fuzzy logic, over non-standard logical systems.

The imperative to work deductively in a formal theory explains also our preference for axiomatic systems over non-axiomatizable sets of tautologies. The infeasibility of algorithmical recognition of valid inferences in the latter is a strong reason supporting the preference. Thus, e.g., predicate fuzzy logics are better conceived as the systems of axioms and rules for quantifiers than the sets of valid $[0, 1]$-tautologies, even though the former usually admit non-intended models.

The respect for the priority of formal theories to models can partly be seen as emphasizing the syntax against the semantics of fuzzy logic. Hájek's approach thus can be viewed as a *syntactic turn* in fuzzy logic. The accent on syntax is of course not meant to contest the fundamental rôle of semantics in logic, nor the heuristic value of the models. Nevertheless, playing up the importance of formal deduction in fuzzy logic corresponds to its motivation as a description of the rules of correct reasoning under vagueness.

Such, then, is a reconstruction of the methodological background we adopt. It has already proved worthy in the area of metamathematics of fuzzy logic. Thus it seems reasonable to apply its doctrines to other branches of fuzzy mathematics as well.

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The need for axiomatization of further areas of fuzzy mathematics besides fuzzy logic is beyond doubt. Axiomatization has always aided the development of mathematical theories. There have been many—more or less successful—attempts to formalize or even axiomatize some areas of fuzzy mathematics. However, these axiomatics are usually designed ad hoc: some concepts in a classical theory are turned fuzzy, however their selection is based on non-systematic intuitions or intended applications; seldom all is fuzzified that could be. (To fuzzify as much as possible is desirable for generality's sake; if an application requires some features to be crisp, they can be 'defuzzified' by an additional assumption of the crispness of these particular features.) Many of these axiomatics are in fact semi-classical, being founded upon the notions of truth degrees and membership functions, which are merely a classical rendering of fuzzy sets.

Further problems of contemporary fuzzy mathematics lie in its fragmentation. Even if some axiomatic theories of various parts of fuzzy mathematics exist, they use completely different sets of primitive concepts and incompatible formalisms. This makes it virtually impossible to combine any two of them into one broader theory. It would certainly be better if fuzzy mathematics as a whole could employ a unified methodology in building its axiomatic theories, because it would facilitate the exchange of results between its branches. Applying the doctrines sketched above, we propose such a unified methodology for the axiomatization of fuzzy mathematics. Obviously, in our approach it assumes the form of constructing formal theories over Hájek-style fuzzy logics.

In the axiomatic construction of classical mathematic, a three-layer architecture has proved useful, with the layers of logic, foundations, and only then individual mathematical disciplines. Individual disciplines are thus developed within the framework of a unifying formal theory, be it some variant of set theory, type theory, category theory, or another sufficiently rich and general kind of theory. In fuzzy mathematics, the level of logic seems to be developed far enough so as to support sufficiently strong formal theories. The search for a suitable foundational theory is thus the task of the day. As hinted above, the close analogy between Hájek-style fuzzy logics and classical logic gives rise to a hope that fuzzy analogues of classical foundational theories will be able to harbour all (or at least nearly all) parts of existing fuzzy mathematics.

As conceivable candidates for a foundational theory, several ZF-style fuzzy set theories have already arisen. Many of them are certainly capable of doing the job. Nevertheless, they seem to be more complex than necessary for the task. Largely this is induced by the fact that such theories have to deal with a specific set-theoretical agenda and take into account the structure of the whole set universe (expressed, e.g., by the axiom of well-foundedness). Moreover, for many of them it is not clear whether they can straightforwardly be generalized to other fuzzy logics than the one in which they have been developed; thus they are only capable of providing the foundation for a limited part of fuzzy mathematics. Besides the repertoire of ZF-style set theories, fuzzy logic also offers set theories based on naïve comprehension. Although their axiomatic system is very elegant, their consistency is limited to (certain) fuzzy logics where no bivalent operator is definable (roughly speaking, to infinite-valued Łukasiewicz logic or weaker).

If nevertheless a universal foundational theory is successfully found, the development of individual concepts of fuzzy mathematics has to proceed in a systematic way, taking into the account the dependencies between them as in classical mathematics. For example, the notion of cardinality should only be defined after the introduction and investigation of the notion of function, upon which it is based (and which in turn is based upon the concept of fuzzy equality, i.e., similarity). When more than one counterpart of a classical definition suggest themselves, the choice between them should be made according to their fruitfulness, applicability, and the practice of traditional fuzzy mathematics; in many cases more than one analogue of the classical notion will have to be introduced and studied within the theory. Defined notions should also be checked against conformity with category theory; for instance, a proposed definition of Cartesian product should accord with that of mapping (one must, however, take into account that many natural notions of morphism become fuzzy under fuzzy logic).

Only this kind of systematic approach can avoid giving ad hoc definitions of fuzzy concepts, which often suffer from arbitrariness and hidden crispness, or even references to particular crisp models of fuzziness (e.g. membership functions) which are not objects of the formal theory.

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As a concrete implementation of the general programme sketched above we propose a specific foundational theory described below. We do not claim it to be the only possible way either of doing the foundations of fuzzy mathematics, or of fulfilling our foundational programme. The methodology itself is independent of this particular solution we propose. Nevertheless, we think that our theory embodies its guidelines very well and is a viable foundation for fuzzy mathematics of the present day. Moreover, because of the simplicity of its apparatus, the work done within its framework can be of use for other possible systems via a formal interpretation.

By inspecting the existing approaches and having in mind the need for generality and simplicity, it becomes obvious that a fully fledged set theory is not necessary for the foundations of fuzzy mathematics. What is necessary is only the ability to perform within the theory the basic constructions of fuzzy mathematics. On the other hand, a great variability of the backround fuzzy logic is required in order to encompass the whole of fuzzy mathematics.

Most notions of classical mathematics can be defined within the first few levels of a simple type theory. The similarity between Hájek-style and classical theories hints that this could be true of fuzzy concepts defined in a fuzzified simple type theory as well. Indeed, many important notions can be defined already at the first level, which is in fact second-order predicate fuzzy logic. Most notably, elementary fuzzy set theory, or the axiomatization of *Zadeh's notion of fuzzy set,* is contained in second-order fuzzy logic (second-order models are exactly Zadeh's universes of fuzzy sets). Some theories (e.g., topology), however, need more levels of type hierarchy, thus we employ higher-order fuzzy logic (in the limit, logic of order ω).

Unfortunately, fuzzy higher-order logic is not recursively axiomatizable. Since we prefer axiomatic deductive theories over non-axiomatizable sets of tautologies, we choose its Henkin-style variant, even though it admits non-intended models. We thus get a *first-order theory,* axiomatized very naturally by the extensionality and comprehension axioms for each order. Moreover, the construction works for virtually all imaginable fuzzy logics (and many non-fuzzy logics as well). The bunch of foundational theories we propose thus can be called *Henkin-style higher-order fuzzy logic* (for an individual fuzzy logic of one's choice; expressively rich logics like ŁΠ seem to be sufficient for all practical purposes; nevertheless, the investigation of the fragments over weaker logics has also its own importace). Equipping the theory with the obvious axioms of tuples yields an apparatus which seems to be of enough expressive power for a great part of fuzzy mathematics, since a structure on the universe of discourse (metric, measure, etc.) can then be introduced by means of relations and higher-order predicates. Furthermore, if the background logic is sufficiently strong, there is a general method of embedding any classical theory, and even of its natural fuzzification (as well as conscious and controlled 'defuzzification' of its concepts if some of their features are to be left crisp). The details of this formalism can be found in [1].

As indicated above, elementary fuzzy set theory and some parts of the theory of fuzzy relations are already formalized within our foundational theory. Several other parts of fuzzy mathematics are currently (re-)developed in our formalism. However, the reconstruction (and expected further advance) of the whole of fuzzy mathematics is an infinite task. Everybody is therefore invited to participate in this research programme of systematic formal development of fuzzy mathematics, as well as to continue the discussion of its best foundation.

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Acknowledgements. As the reader could easily observe, this methodological programme has close links to the works of many predecessors, and in fact only applies their accomplishments to the area of fuzzy mathematics.

Our formalistic approach to mathematics is close to that of Hilbert's [6]. Our aspiration to lay down the logical foundations for fuzzy mathematics is only a derivative of the admirable enterprise of Russell and Whitehead [8]. In some (and only some) respects our programme is similar to that of Bourbaki [2], though we hope not only to reconstruct and codify, but also advance the field of our interest. The link to Vienna circle [3] which results from the circumstances of the first presentation of this manifesto is rather incidental (though in some aspects one could perhaps find distant parallels).

We cannot mention all the outstanding works of fuzzy logic upon which our programme is based. Here we mention only the most important works relevant to our approach; further citations can be found in [1]. Apparently the first monograph close in spirit to our programme was Gottwald's [4]. Hájek's [5] gave firm foundations to the kind of formal fuzzy logic we use. A great influence in the propagation of rigorous fuzzy logic in the Czech mathematical community had Novák's book [7]. And, needless to say, the whole field of fuzzy mathematics we try to formalize originated with Zadeh's [9].

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References

[1] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[2] N. Bourbaki. *Théorie des Ensembles*, volume 1 of *Éléments de Mathématique*. Hermann, Paris, 1966.

[3] R. Carnap, H. Hahn, and O. Neurath. Wissenschaftliche Weltauffassung. Der Wiener Kreis. In R. Hegselmann, editor, *Wissenschaftliche Weltauffassung, Sozialismus und Logischer Empirismus*, pages 81–101. A. Wolf, Frankfurt, 1929.

[4] S. Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View*. Vieweg, Wiesbaden, 1993.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[6] D. Hilbert. Grundlagen der Geometrie. In *Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals*, pages 1–92. B.G. Teubner, Leipzig, 1899.

[7] V. Novák. Fuzzy množiny a jejich aplikace. 1986. English version: V. Novák, Fuzzy Sets and Their Applications, Adam Hilger, Bristol, 1989.

[8] A. N. Whitehead and B. Russell. *Principia Mathematica*. Cambridge University Press, Cambridge, 1910, 1911, 1913.

[9] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

Fuzzy class theory

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Abstract: The paper introduces a simple, yet powerful axiomatization of Zadeh's notion of fuzzy set, based on formal fuzzy logic. The presented formalism is strong enough to serve as foundations of a large part of fuzzy mathematics. Its essence is elementary fuzzy set theory, cast as two-sorted first-order theory over fuzzy logic, which is generalized to simple type theory. We show a reduction of the elementary fuzzy set theory to fuzzy propositional calculus and a general method of fuzzification of classical mathematical theories within this formalism. In this paper we restrict ourselves to set relations and operations that are definable without any structure on the universe of objects presupposed; however, we also demonstrate how to add structure to the universe of discourse within our framework.

Keywords: Formal fuzzy logic, fuzzy set, foundations of fuzzy mathematics, ŁΠ logic, higher-order fuzzy logic, fuzzy type theory, multi-sorted fuzzy logic. MSC: 03B52, 03E70, 03E72.

1 Introduction

Fuzzy sets were introduced approximately 40 years ago by L.A. Zadeh [17]. During these years the notion of fuzziness spread to nearly all aspects of mathematics (fuzzy relations, fuzzy topology, fuzzy algebra etc.). There have been many (more or less successful) attempts to formalize or even axiomatize some areas of fuzzy mathematics. Very successful results were achieved in the area of fuzzy logic (in narrow sense). The work of Hájek, Gottwald, Mundici, and others established fuzzy logic as a formal theory. This success allows us to move further with the formalization of other parts of fuzzy mathematics.

Although fuzzy mathematics is nowadays very broad, the notion of fuzzy set is still a central concept. There have been several previous attempts at formalizing fuzzy sets in an axiomatic way. Early works, most notably [3] and [4], axiomatized the notion within classical logic by means of a *ternary* membership predicate, whose third argument represented the membership degree. Even though we do not follow this approach here, our motivation for the axiomatic method conforms with that of [3, pp. 623–4]:

> "This unified theory in which sets, functions, etc. are all 'fuzzy' helps to obviate some of the [...] difficulties and to clarify the nature of the others. Further, it eliminates the necessity of having a predetermined theory of ordinary sets on top of which the 'fuzzy' sets are built as a superstructure by starting out axiomatically *ab initio,* as it were, assuming only elementary logic. Further,

by developing the theory in a manner parallel to the usual development of other set theories, comparisons between this new theory and the more usual ones are facilitated."

The approach we adopt here consists in 'hiding' the third argument in the semantic meta-level of the theory and using formal fuzzy logic instead of classical logic for the background logic of the theory. The reasons for this design choice are explained in more detail in [1]. Here it suffices to say that it allows us to draw on the similarity with classical set theory even more extensively than the former approach, as the formulae of the theory become virtually the same as in classical mathematics, only governed by a weaker logic. (See also footnote 6 in Section 7 below.) Axiomatic fuzzy set theories construed in this way have already been explored by several predecessors; however, their agenda differs from ours in many respects. The papers [11] and [15] are mainly interested in metamathematical properties of fuzzified Zermelo-Fraenkel set theory, rather than developing fuzzy mathematics within its framework. The elegant theory of [16] is restricted to one particular t-norm logic, and so it cannot capture the general notion of fuzzy set. Inspecting these approaches we came to two conclusions: for the axiomatization of Zadeh's notion of fuzzy set, we *do not need* an analogue of full-fledged set theory, though we *do need* an expressively rich fuzzy logic as a logical background.

By an *analogue of full-fledged set theory* we mean a theory over fuzzy logic, which contains fuzzy counterparts of all concepts of classical set theory. We observed that real-world applications of fuzzy sets need only a small portion of set-theoretical concepts. The central notion in fuzzy sets is the membership of elements (rather than fuzzy sets) into a fuzzy set. In the classical setting, the theory of the membership of atomic objects into sets is called *elementary set theory,* or *class theory.* It is a theory with two sorts of individuals—objects and classes—and one binary predicate—the membership of objects into classes. In this paper we develop a *fuzzy class theory.* The classes in our theory correspond exactly to Zadeh's fuzzy sets.

By an *expressively rich logic* (which we need) we mean a logic of great expressive power, yet with a simple axiomatic system and good logical properties (deduction theorem, Skolem function introduction and eliminability, etc.). ŁΠ∀ seems to be the most suitable logic for our needs. In this paper we developed fuzzy class theory over the first-order logic ŁΠ, however if you examine the definitions and theorems you notice that nearly all of them will work in other fuzzy logics as well. We think that fixing the underlying logic will make important class-theoretical concepts clearer. Fuzzy class theory for a wider class of fuzzy logics can be a topic of some upcoming paper.

We show that the proposed theory is a simple, yet powerful formalism for working with elementary relations and operations on fuzzy sets (normality, equality, subsethood, union, intersection, kernel, support, etc.). By a small enhancement of our theory (adding tools to manage tuples of objects) we obtain a formalism powerful enough to capture the notion of fuzzy relation. Thus we can formally introduce the notions of T-transitivity, T-similarity, fuzzy ordering, and many other concepts defined in the literature. Finally, we extend our formalism to something which can be viewed as simple fuzzy type theory. Basically, we introduce individuals for classes of classes, classes of classes of classes etc. This allows us to formalize other parts of fuzzy mathematics (e.g., fuzzy topology). Our theory thus aspires to the status of *foundations of fuzzy mathematics* and a uniform formalism that can make interaction of various disciplines of fuzzy mathematics possible.

Of course, this paper cannot cover all the topics mentioned above. For the majority of them we only give the very basic definitions, and there is a lot of work to be done to show that the proposed formalism is suitable for them. We concentrate on the development of basic properties of fuzzy sets. In this area our formalism proved itself worthy, as it allows us to state several very general metatheorems that effectively reduce a wide range of theorems on fuzzy sets to fuzzy propositional calculus. This success is a promising sign for our formalism to be suitable for other parts of fuzzy mathematics as well.

As mentioned above, in this paper we restrict ourselves to notions that can be defined without adding a structure (similarity, metrics, etc.) to the universe of objects. Nevertheless, our formalism possesses means for adding a structure to the universe (usually by fixing a suitable class which satisfies certain axioms), which is necessary for the development of more advanced parts of fuzzy set theory. Such extensions of our theory will be elaborated in subsequent papers, for some hints see Section 6.

The proposed methodology of formal fuzzy mathematics is described in more details in our paper [1], in which we also make further references to related results. Let us just mention here that some of the roots of our approach (as well as some of the concepts we employ, like graded properties of fuzzy relations) can already be found in Gottwald's monograph [9]. The systematic way of defining fuzzy notions (see Section 7) is hinted at already in Höhle's 1987 paper [12].

2 Preliminaries

This section contains formal tools necessary for developing a theory over the multi-sorted first-order logic ŁΠ. Readers acquainted with classical multi-sorted calculi can go through this section quickly.

2.1 Propositional logic ŁΠ

Here we recall the definitions of the logic ŁΠ and some of its properties (the definition and theorems in this section are from [8] and [6]).

Definition 1. The logic ŁΠ has the following *basic connectives* (they are listed together with their standard semantics in $[0,1]$; we use the same symbols for logical connectives and the corresponding algebraic operations):

0	0	truth constant falsum
$\varphi \to_{Ł} \psi$	$x \to_{Ł} y = \min(1, 1-x+y)$	Łukasiewicz implication
$\varphi \to_{\Pi} \psi$	$x \to_{\Pi} y = \min(1, \frac{x}{y})$	product implication
$\varphi \mathbin{\&_{\Pi}} \psi$	$x \mathbin{\&_{\Pi}} y = x \cdot y$	product conjunction

The logic $Ł\Pi\frac{1}{2}$ has one additional truth constant $\frac{1}{2}$ with the standard semantics $\frac{1}{2}$. We define the following *derived connectives:*

$\neg_{Ł}\varphi$	is	$\varphi \to_{Ł} 0$	$\neg_{Ł}x = 1 - x$
$\neg_{\Pi}\varphi$		$\varphi \to_{\Pi} 0$	$\neg_{\Pi}x = 1$ if $x = 0$, otherwise 0
1		$\neg_{Ł}0$	1
$\Delta\varphi$		$\neg_{\Pi}\neg_{Ł}\varphi$	$\Delta x = 1$ if $x = 1$, otherwise 0
$\varphi \mathbin{\&_{Ł}} \psi$		$\neg_{Ł}(\varphi \to_{Ł} \neg_{Ł}\psi)$	$x \mathbin{\&_{Ł}} y = \max(0, x+y-1)$
$\varphi \oplus \psi$		$\neg_{Ł}\varphi \to_{Ł} \psi$	$x \oplus y = \min(1, x+y)$
$\varphi \ominus \psi$		$\varphi \mathbin{\&_{Ł}} \neg_{Ł}\psi$	$x \ominus y = \max(0, x-y)$
$\varphi \wedge \psi$		$\varphi \mathbin{\&_{Ł}} (\varphi \to_{Ł} \psi)$	$x \wedge y = \min(x, y)$
$\varphi \vee \psi$		$(\varphi \to_{Ł} \psi) \to_{Ł} \psi$	$x \vee y = \max(x, y)$
$\varphi \to_{G} \psi$		$\Delta(\varphi \to_{Ł} \psi) \vee \psi$	$x \to_{G} y = 1$ if $x \le y$, otherwise y

We assume the usual precedence of connectives. Occasionally we may write $\neg_G$ and $\&_G$ as synonyms for $\neg_\Pi$ and $\wedge$, respectively. We further abbreviate $(\varphi \to_* \psi) \&_* (\psi \to_* \varphi)$ by $\varphi \leftrightarrow_* \psi$ for $* \in \{G, Ł, \Pi\}$.

Definition 2. An ŁΠ-*algebra* is a structure $\mathbf{L} = (L, \oplus, \neg_Ł, \to_\Pi, \&_\Pi, 0, 1)$ such that:

- $(L, \oplus, \neg_Ł, 0)$ is an MV-algebra
- $(L, \vee, \wedge, \to_\Pi, \&_\Pi, 0, 1)$ is a Π-algebra,
- $x \&_\Pi (y \ominus z) = (x \&_\Pi y) \ominus (x \&_\Pi z)$.

Furthermore, a structure $\mathbf{L} = (L, \oplus, \neg_Ł, \to_\Pi, \&_\Pi, 0, 1, \frac{1}{2})$ where the reduct $\mathbf{L}' = (L, \oplus, \neg_Ł, \to_\Pi, \&_\Pi, 0, 1)$ is an ŁΠ-algebra and the identity $\frac{1}{2} = \neg_Ł \frac{1}{2}$ holds is called an ŁΠ$\frac{1}{2}$-*algebra.*

The *standard* ŁΠ-*algebra* $[\mathbf{0}, \mathbf{1}]$ has the domain $[0, 1]$ and the operations as stated in Definition 1 above (analogously for the standard ŁΠ$\frac{1}{2}$-algebra).

The two-valued ŁΠ algebra is denoted by $\{\mathbf{0}, \mathbf{1}\}$.

Definition 3. The logic ŁΠ is given by the following axioms and deduction rules:

(Ł)	The axioms of Łukasiewicz logic
(Π)	The axioms of product logic
(ŁΔ)	$\Delta(\varphi \to_Ł \psi) \to_Ł (\varphi \to_\Pi \psi)$
(ΠΔ)	$\Delta(\varphi \to_\Pi \psi) \to_Ł (\varphi \to_Ł \psi)$
(Dist)	$\varphi \&_\Pi (\chi \ominus \psi) \leftrightarrow_Ł (\varphi \&_\Pi \chi) \ominus (\varphi \&_\Pi \psi)$

The deduction rules are modus ponens and Δ-necessitation (from φ infer $\Delta\varphi$).

The logic ŁΠ$\frac{1}{2}$ results from ŁΠ by adding the axiom $\frac{1}{2} \leftrightarrow \neg_Ł \frac{1}{2}$. The notions of proof, derivability $\vdash$, theorem, and theory over ŁΠ and ŁΠ$\frac{1}{2}$ are defined as usual.

Theorem 4 (Completeness)**.** *Let φ be a formula of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *respectively). Then the following conditions are equivalent:*

- *φ is a theorem of* ŁΠ *(*ŁΠ$\frac{1}{2}$ *resp.)*
- *φ is an* $\mathbf{L}$*-tautology w.r.t. each* ŁΠ*-algebra (*ŁΠ$\frac{1}{2}$*-algebra resp.)* $\mathbf{L}$
- *φ is a* $[\mathbf{0}, \mathbf{1}]$*-tautology.*

The following definitions and theorems demonstrate the expressive power of ŁΠ and ŁΠ$\frac{1}{2}$. Particularly, Corollary 8 shows that each propositional logic based on an arbitrary t-norm of a certain simple form is contained in ŁΠ$\frac{1}{2}$.

Definition 5. A function $f\colon [0, 1]^n \to [0, 1]$ is called a *rational* ŁΠ-*function* iff there is a finite partition of $[0, 1]^n$ such that each block of the partition is a semi-algebraic set and f restricted to each block is a fraction of two polynomials with rational coefficients.

Furthermore, a rational ŁΠ-function f is *integral* iff all the coefficients are integer and $f(\{0, 1\}^n) \subseteq \{0, 1\}$.

Definition 6. Let f be a function $f\colon [0, 1]^n \to [0, 1]$ and $\varphi(v_1, \ldots, v_n)$ be a formula. We say that the function f is *represented by the formula* φ (φ is a representation of f) iff $e(\varphi) = f(e(v_1), e(v_2), \ldots, e(v_m))$ for each evaluation e.

The following theorem was proved in [13]:

Theorem 7 (Functional representation)**.** *A function f is an integral (rational, respectively)* ŁΠ *function iff it is represented by some formula of* ŁΠ *(*$\text{Ł}\Pi\frac{1}{2}$ *resp.).*

The following theorem was proved in [5], but it can be viewed as a corollary of the previous theorem.

Corollary 8. *Let $*$ be a continuous t-norm which is a finite ordinal sum of the three basic ones (i.e., of* G, Ł *and* Π*), and $\Rightarrow$ be its residuum. Then there are derived connectives $\&_*$ and $\rightarrow_*$ of the* $\text{Ł}\Pi\frac{1}{2}$ *logic such that their standard $[\mathbf{0},\mathbf{1}]$-semantics are $*$ and $\Rightarrow$ respectively. The logic $PC(*)$ of the t-norm $*$ (see [10]) is contained in* $\text{Ł}\Pi\frac{1}{2}$ *if $\&$ and $\rightarrow$ of $PC(*)$ are interpreted as $\&_*$ and $\rightarrow_*$. Furthermore, if φ is provable in $PC(*)$ (and a fortiori, if it is provable in Hájek's logic* BLΔ*, see [10]), then the formula φ_* obtained from φ by replacing the connectives $\&$ and $\rightarrow$ of $PC(*)$ (or* BLΔ*) by $\&_*$ and $\rightarrow_*$ is provable in* $\text{Ł}\Pi\frac{1}{2}$*.*

Convention 9. Further on, the signs $*$ and $\diamond$ will be reserved for t-norms definable in $\text{Ł}\Pi\frac{1}{2}$ (incl. G, Ł and Π), and the indexed connectives will always have the meaning introduced in the previous Corollary. However, we omit the indices of connectives whenever they are irrelevant, i.e., whenever all formulae obtained by subscripting any $*$ to such a connective are provably equivalent (for example, $\neg\neg_G\varphi$, $\Delta(\varphi \rightarrow \psi)$, etc.), or equivalently provable (e.g., the principal implication in axioms and theorems).

Corollary 10. *Let $r \in [0, 1]$ be a rational number; then there is a formula φ of* $\text{Ł}\Pi\frac{1}{2}$ *such that $e(\varphi) = r$ for any $[\mathbf{0},\mathbf{1}]$-evaluation e.*

This corollary tells us that in $\text{Ł}\Pi\frac{1}{2}$ we have a truth constant $\bar{r}$ for each rational number $r \in [0, 1]$. Using the completeness theorem we get the following corollary.

Corollary 11. *The following are theorems of the* $\text{Ł}\Pi\frac{1}{2}$ *logic:*

$$\begin{aligned} \overline{r \,\&_\Pi\, s} &= \bar{r} \,\&_\Pi\, \bar{s} \\ \overline{r \rightarrow_\Pi s} &= \bar{r} \rightarrow_\Pi \bar{s} \\ \overline{r \rightarrow_Ł s} &= \bar{r} \rightarrow_Ł \bar{s} \end{aligned}$$

where the symbols $\&_\Pi, \rightarrow_\Pi, \rightarrow_Ł$ on the left-hand side are operations in $[\mathbf{0},\mathbf{1}]$ and on the right-hand side they are logical connectives.

2.2 Multi-sorted first-order logic ŁΠ∀

In this section we deal with first-order versions of the logics ŁΠ and $\text{Ł}\Pi\frac{1}{2}$. Since the difference between ŁΠ∀ and $\text{Ł}\Pi\frac{1}{2}\forall$ is purely "propositional", we focus on the logic ŁΠ∀; the definitions and theorems for the logic $\text{Ł}\Pi\frac{1}{2}\forall$ are analogous, for details see [5]. (For general first-order fuzzy logics see [10] and for multi-sorted first order fuzzy logic see [7].)

Definition 12. A *multi-sorted predicate language* $\mathbf{\Gamma}$ for the logic ŁΠ∀ is a quintuple $(\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$, where $\mathbf{S}$ is a non-empty set of sorts, $\preceq$ is an ordering on $\mathbf{S}$ (indicating the subsumption of sorts), $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{A}$ is a function assigning to each predicate and function symbol a finite sequence of elements of $\mathbf{S}$.

Let $|\mathbf{A}(P)|$ denote the length of the sequence $\mathbf{A}(P)$. The number $|\mathbf{A}(P)|$ is called the *arity of the predicate symbol P*. The number $|\mathbf{A}(f)| - 1$ is called the *arity of the function* *symbol* f. The functions f for which $\mathbf{A}(f) = <s>$ are called the *individual constants* of sort s. If $s_1 \preceq s_2$ holds for sorts s_1, s_2 we say that s_2 *subsumes* s_1.

The *logical symbols* of ŁΠ∀ are individual variables $x^s, y^s, \ldots$ for each sort s, the logical connectives of ŁΠ, the quantifier ∀ and the identity sign =. For any variable x^s, we abbreviate $\neg_{\mathrm{L}}(\forall x^s)\neg_{\mathrm{L}}$ as $(\exists x^s)$.

Definition 13. Let $\mathbf{\Gamma} = (\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$ be a multisorted predicate language. The notion of $\mathbf{\Gamma}$-*term* is defined inductively as follows:

- Each individual variable of sort $s \in \mathbf{S}$ is a $\mathbf{\Gamma}$-term of sort s.
- Let $t_1, \ldots, t_n$ be $\mathbf{\Gamma}$-terms of respective sorts $s_1, \ldots, s_n \in \mathbf{S}$, and f be a function symbol of $\mathbf{\Gamma}$ such that $\mathbf{A}(f) = <w_1, \ldots, w_n, w_{n+1}>$, where $s_i \preceq w_i$ for $i \leq n$. Then $f(t_1, \ldots, t_n)$ is a $\mathbf{\Gamma}$-term of sort w_{n+1}.
- Nothing else is a $\mathbf{\Gamma}$-term.

Definition 14. Let $\mathbf{\Gamma} = (\mathbf{S}, \preceq, \mathbf{P}, \mathbf{F}, \mathbf{A})$ be a multisorted predicate language. Let $t_1, \ldots, t_n$ be $\mathbf{\Gamma}$-terms of respective sorts $s_1, \ldots, s_n \in \mathbf{S}$, and P be a predicate symbol of $\mathbf{\Gamma}$ such that $\mathbf{A}(P) = <w_1, \ldots, w_n>$ and $s_i \preceq w_i$ for $i \leq n$. Then $P(t_1, \ldots, t_n)$ is an *atomic* $\mathbf{\Gamma}$-*formula*. If t_1 and t_2 are $\mathbf{\Gamma}$-terms of arbitrary sorts, then $t_1 = t_2$ is also an atomic $\mathbf{\Gamma}$-formula.

The notion of $\mathbf{\Gamma}$-*formula* is defined inductively as follows:

- Each atomic $\mathbf{\Gamma}$-formula is a $\mathbf{\Gamma}$-formula.
- If $\varphi_1, \ldots, \varphi_n$ are $\mathbf{\Gamma}$-formulae and c is an n-ary propositional connective of ŁΠ, then $c(\varphi_1, \ldots, \varphi_n)$ is also a $\mathbf{\Gamma}$-formula.
- Let φ be a $\mathbf{\Gamma}$-formula and x^s a variable of sort s. Then $(\forall x^s)\varphi$ is also a $\mathbf{\Gamma}$-formula.
- Nothing else is a $\mathbf{\Gamma}$-formula.

Bound and *free* variables in a formula are defined as usual. A formula is called a *sentence* iff it contains no free variables. A set of $\mathbf{\Gamma}$-formulae is called a $\mathbf{\Gamma}$-*theory*.

Convention 15. Instead of $\xi_1, \ldots, \xi_n$ (where ξ_i's are terms or formulae and n is arbitrary or fixed by the context) we shall sometimes write just $\vec{\xi}$.

Unless stated otherwise, the expression $\varphi(x_1, \ldots, x_n)$ means that all free variables of φ are among $x_1, \ldots, x_n$. Similarly, in propositional logic the expression $\varphi(p_1, \ldots, p_n)$ will mean that all propositional variables occurring in φ are among $p_1, \ldots, p_n$.

If $\varphi(x_1, \ldots, x_n, \vec{z})$ is a formula and we substitute terms t_i for all x_i's in φ, we denote the resulting formula in the context simply by $\varphi(t_1, \ldots, t_n, \vec{z})$.

The expression $(\exists! x)_*\varphi(x, \vec{z})$ abbreviates the formula

$$(\exists x)[\varphi(x, \vec{z}) \,\&_*\, (\forall y)(\varphi(y, \vec{z}) \rightarrow_* (y = x))]$$

Definition 16. A term t of sort w is *substitutable* for the individual variable x^s in a formula $\varphi(x^s, \vec{z})$ iff $w \preceq s$ and no occurrence of any variable y occurring in t is bounded in $\varphi(t, \vec{z})$.

Definition 17. Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra. An $\mathbf{L}$-*structure* $\mathbf{M}$ for $\mathbf{\Gamma}$ has the following form: $\mathbf{M} = ((M_s)_{s\in\mathbf{S}}, (P_\mathbf{M})_{P\in\mathbf{P}}, (f_\mathbf{M})_{f\in\mathbf{F}})$, where M_s is a non-empty domain for each $s \in \mathbf{S}$ and $M_s \subseteq M_w$ iff $s \preceq w$; $P_\mathbf{M}$ is an n-ary fuzzy relation $\prod_{i=1}^{n} M_{s_i} \to \mathbf{L}$ for each predicate symbol $P \in \mathbf{P}$ such that $\mathbf{A}(P) = <s_1, \dots, s_n>$; $f_\mathbf{M}$ is a function $\prod_{i=1}^{n} M_{s_i} \to M_{s_{n+1}}$ for each function symbol $f \in \mathbf{F}$ such that $\mathbf{A}(f) = <s_1, \dots, s_n, s_{n+1}>$, and an element of M_s if f is a constant of sort s.

Definition 18. Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra and $\mathbf{M}$ be an $\mathbf{L}$-structure for $\mathbf{\Gamma}$. An $\mathbf{M}$-*evaluation* is a mapping e which assigns to each variable of sort s an element from M_s (for all sorts $s \in \mathbf{S}$).

Let e be an $\mathbf{M}$-evaluation, x a variable of sort s, and $a \in M_s$. Then $e[x \to a]$ is an $\mathbf{M}$-evaluation such that $e[x \to a](x) = a$ and $e[x \to a](y) = e(y)$ for each individual variable y different from x.

Definition 19. Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra. The *value* of a term and the *truth value* of a $\mathbf{\Gamma}$-formula in an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathbf{\Gamma}$ and an $\mathbf{M}$-evaluation e are defined as follows:

$$
\begin{aligned}
\|x\|^{\mathbf{L}}_{\mathbf{M},e} &= e(x)\\
\|f(t_1, t_2, \dots, t_n)\|^{\mathbf{L}}_{\mathbf{M},e} &= f_\mathbf{M}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},e}, \|t_2\|^{\mathbf{L}}_{\mathbf{M},e}, \dots, \|t_n\|^{\mathbf{L}}_{\mathbf{M},e})\\
\|P(t_1, t_2, \dots, t_n)\|^{\mathbf{L}}_{\mathbf{M},e} &= P_\mathbf{M}(\|t_1\|^{\mathbf{L}}_{\mathbf{M},e}, \|t_2\|^{\mathbf{L}}_{\mathbf{M},e}, \dots, \|t_n\|^{\mathbf{L}}_{\mathbf{M},e})\\
\|t_1 = t_2\|^{\mathbf{L}}_{\mathbf{M},e} &= 1 \textit{ if } \|t_1\|^{\mathbf{L}}_{\mathbf{M},e} = \|t_2\|^{\mathbf{L}}_{\mathbf{M},e} \text{ and 0 otherwise}\\
\|0\|^{\mathbf{L}}_{\mathbf{M},e} &= 0\\
\|\varphi_1 \circ \varphi_2\|^{\mathbf{L}}_{\mathbf{M},e} &= \|\varphi_1\|^{\mathbf{L}}_{\mathbf{M},e} \circ \|\varphi_2\|^{\mathbf{L}}_{\mathbf{M},e} \text{ for } \circ \in \{\to_{\text{Ł}}, \to_\Pi, \&_\Pi\}\\
\|(\forall x^s)\varphi\|^{\mathbf{L}}_{\mathbf{M},e} &= \inf_{a\in M_s} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},e[x^s\to a]}
\end{aligned}
$$

If the infimum does not exist, we take its value as undefined. We say that an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathbf{\Gamma}$ is *safe* iff $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},e}$ is defined for each $\mathbf{\Gamma}$-formula φ and each $\mathbf{M}$-evaluation e.

Definition 20. Let $\mathbf{L}$ be a linearly ordered ŁΠ-algebra and φ a $\mathbf{\Gamma}$-formula. The *truth value* of the formula φ in an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathbf{\Gamma}$ is defined as follows:

$$\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},e} \mid e \textit{ is an } \mathbf{M}\textit{-evaluation}\}$$

We say that φ is an $\mathbf{L}$-*tautology* iff $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ for each safe $\mathbf{L}$-structure $\mathbf{M}$ for $\mathbf{\Gamma}$. We say that an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathbf{\Gamma}$ is an $\mathbf{L}$-*model* of a $\mathbf{\Gamma}$-theory T iff $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ for each $\varphi \in T$.

Convention 21. For a fixed $\mathbf{L}$-model $\mathbf{M}$ and an $\mathbf{M}$-evaluation e such that $e(x_i) = a_i$ (for all i's), we shall instead of $\|\varphi(x_1, \dots, x_n)\|^{\mathbf{L}}_{\mathbf{M},e}$ write simply $\|\varphi(a_1, \dots, a_n)\|$ and speak of the truth value of $\varphi(a_1, \dots, a_n)$.

Definition 22. Let $\varphi({x_1}^{s_1}, \dots, {x_n}^{s_n})$ be a formula of ŁΠ∀ and $\mathbf{M}$ be a safe structure for the language of φ over an ŁΠ-algebra $\mathbf{L}$. The function $\chi_\varphi\colon \prod_{i=1}^{n} M_{s_i} \to \mathbf{L}$ such that $\chi_\varphi(a_1, \dots, a_n) = \|\varphi(a_1, \dots, a_n)\|^{\mathbf{L}}_{\mathbf{M}}$ is called the *characteristic function* of $\varphi(x_1, \dots, x_n)$.

Definition 23. The logic ŁΠ∀ is given by the following axioms and deduction rules:

- (P) Substitution instances of the axioms of propositional ŁΠ
- (∀1) $(\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$, where t is substitutable for x in φ
- (∀2) $(\forall x)(\chi \to_{\text{Ł}} \varphi) \to (\chi \to_{\text{Ł}} (\forall x)\varphi)$, where x is not free in χ
- (=1) $x = x$
- (=2) $(x = y) \to \Delta(\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z}))$.

The deduction rules are modus ponens, Δ-necessitation, and generalization. The notions of proof, theorem, and derivability ⊢ are defined as usual.

Instead of axiom (=2) we may use the usual axioms of congruence of identity w.r.t. all predicates and functions plus the axiom of crispness of identity, i.e. $(x = y) \vee \neg(x = y)$.

Lemma 24. *The following are theorems of* ŁΠ∀*:*

- $(x = y) \vee \neg(x = y)$
- $(x = y) \to (y = x)$
- $(x = y) \mathbin{\&_*} (y = z) \to_* (x = z)$
- $(x_1 = y_1) \mathbin{\&_*} \ldots \mathbin{\&_*} (x_n = y_n) \to_* (\varphi(x_1, \ldots, x_n, \vec{z}) \leftrightarrow_* \varphi(y_1, \ldots, y_n, \vec{z}))$.

The theorems of the next lemma will be needed in the following sections.

Lemma 25. *All formulae of the following forms are provable in* ŁΠ∀*:*

$$(\forall x)(\varphi \to_* \psi) \ \to \ [(\forall x)\varphi \to_* (\forall x)\psi] \tag{1}$$
$$(\forall x)(\varphi \to_* \psi) \ \to \ [(\exists x)\varphi \to_* (\exists x)\psi] \tag{2}$$
$$(\forall x)(\varphi \wedge \psi) \ \to \ [(\forall x)\varphi \wedge (\forall x)\psi] \tag{3}$$
$$(\exists x)(\varphi \vee \psi) \ \to \ [(\exists x)\varphi \vee (\exists x)\psi] \tag{4}$$
$$(\forall x)(\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} \varphi_k \to_* \chi) \to [(\forall x)\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} (\forall x)\varphi_k \to_* (\forall x)\chi] \tag{5}$$
$$\begin{aligned}&(\forall x)(\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} \varphi_k \to_* \chi) \to \\ &\qquad \to [(\forall x)\varphi_1 \mathbin{\&_*} \ldots \mathbin{\&_*} (\forall x)\varphi_{k-1} \mathbin{\&_*} (\exists x)\varphi_k \to_* (\exists x)\chi]\end{aligned} \tag{6}$$

Proof. In the proof we use an easy generalization of Corollary 8 to the predicate case. Parts (1)–(4) are provable in BL∀ (see [10]). Part (5) is proved by a trivial inductive generalization of the following proof in BL∀:

$$\begin{aligned}&(\forall x)(\varphi \mathbin{\&} \psi \to \chi) \\ &\quad \leftrightarrow \quad (\forall x)(\varphi \to (\psi \to \chi)) \\ &\quad \to \quad [(\forall x)\varphi \to (\forall x)(\psi \to \chi)] \\ &\quad \to \quad [(\forall x)\varphi \to ((\forall x)\psi \to (\forall x)\chi)] \\ &\quad \leftrightarrow \quad [(\forall x)\varphi \mathbin{\&} (\forall x)\psi \to (\forall x)\chi].\end{aligned}$$

Finally, part (6) is proved in the same way, only applying (2) instead of (1) when distributing $(\forall x)$ over $(\varphi_k \to \chi)$. Q.E.D.

Theorem 26 (Deduction)**.** *Let T be a theory and φ be a sentence. Then $T \vdash \Delta\varphi \to \psi$ iff $T \cup \{\varphi\} \vdash \psi$.*

Theorem 27 (Strong Completeness)**.** *Let φ be a $\mathbf{\Gamma}$-formula, T a $\mathbf{\Gamma}$-theory. Then the following are equivalent:*

- $T \vdash \varphi$
- $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ *for each* ŁΠ*-algebra* $\mathbf{L}$ *and each safe* $\mathbf{L}$*-model* $\mathbf{M}$ *of* T
- $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$ *for each* linearly ordered ŁΠ*-algebra* $\mathbf{L}$ *and each safe* $\mathbf{L}$*-model* $\mathbf{M}$ *of* T

The following theorem of [7] vindicates the introduction and elimination of function symbols. Notice the connective Δ, which is provably indispensable for the validity of this theorem.

Theorem 28. *Let $\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, y^s)$ be a $\mathbf{\Gamma}$-formula and T be a theory such that*

$$T \vdash (\forall x_1^{s_1}) \ldots (\forall x_n^{s_n})(\exists y^s)\Delta\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, y^s).$$

Let f be a new function symbol such that $\mathbf{A}(f) = <s_1, \ldots, s_n, s>$. Then the $(\mathbf{\Gamma} \cup \{f\})$-theory

$$T' = T \cup \{(\forall x_1^{s_1}) \ldots (\forall x_n^{s_n})\Delta\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, f(x_1^{s_1}, \ldots, x_n^{s_n}))\}$$

is a conservative extension of T.

Furthermore, if $T \vdash (\forall x_1^{s_1}) \ldots (\forall x_n^{s_n})(\exists! y^s)\Delta\varphi(x_1^{s_1}, \ldots, x_n^{s_n}, y^s)$ then for each $(\mathbf{\Gamma} \cup \{f\})$-formula φ there is a $\mathbf{\Gamma}$-formula φ' such that $T' \vdash \varphi \leftrightarrow \varphi'$.

3 Class theory over ŁΠ

3.1 Axioms

Fuzzy class theory FCT is a theory over ŁΠ∀ with two sorts of variables: *object variables,* denoted by lowercase letters $x, y, \ldots$, and *class variables,* denoted by uppercase letters $X, Y, \ldots$ None of the sorts is subsumed by the other.

The only primitive symbol of FCT is the binary membership predicate $\in$ between objects and classes (i.e., the first argument must be an object and the second a class; class theory takes into consideration neither the membership of classes in classes, nor of objects in objects).

The principal axioms of FCT are instances of the class comprehension scheme: for any formula φ not containing X (it may, however, contain any other object or class parameters),

$$(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi(x))$$

is an axiom of FCT. The strange Δ is neccessary for securing that the required class exists in the degree 1 (rather than being only approximated by classes satisfying the equivalence in degrees arbitrarily close to 1). The Δ is also necessary for the conservativeness of the introduction of comprehension terms[1] $\{x \mid \varphi(x)\}$ with axioms

$$y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$$

and their eliminability. In the standard recursive way one proves that φ in comprehension terms may be allowed to contain other comprehension terms.

The consistency of FCT is proved by constructing a model. Let M be an arbitrary set and $\mathbf{L}$ be a complete linear ŁΠ-algebra. The *Zadeh model* $\mathbf{M}$ *over the universe* M *and the algebra of truth-values* $\mathbf{L}$ is constructed as follows:

The range of object variables is M, the range of class variables is the set of all functions from M to $\mathbf{L}$. For any evaluation e we define $\|x \in X\|^{\mathbf{L}}_{\mathbf{M},e}$ as the value of the function $e(X)$ on $e(x)$. The value of the comprehension term $\{x \mid \varphi(x)\}$ is defined as the function taking an object a to $\|\varphi(a)\|^{\mathbf{L}}_{\mathbf{M},e}$ (in fact, the characteristic function of $\varphi(x)$ where e fixes the parameters). Then it is trivial that $\|y \in \{x \mid \varphi(x)\}\|^{\mathbf{L}}_{\mathbf{M},e} = \|\varphi(y)\|^{\mathbf{L}}_{\mathbf{M},e}$ which proves the comprehension axiom.

If $\mathbf{L} = [\mathbf{0}, \mathbf{1}]$, we call the described model *standard*.

Definition 29. Let $\mathbf{M}$ be a model and A a class in $\mathbf{M}$. The characteristic function $\chi_{x \in A}$ is denoted briefly by χ_A and also called the *membership function* of A. (Instead of $\chi_A(x)$ or $\|x \in A\|$ many papers use just Ax.)

[1] I.e., the Skolem functions of comprehension axioms, see Theorem 28.

It can be observed that the crisp formula $(\forall x)\Delta(x \in X \leftrightarrow x \in Y)$ expresses the identity of the membership functions of X and Y (as in all models $\|(\forall x)\Delta(x \in X \leftrightarrow x \in Y)\| = 1$ iff the membership functions of X and Y are identical, otherwise 0). Since our intended notion of fuzzy class is extensional, i.e., that fuzzy classes are determined by their membership functions, it is reasonable to require the *axiom of extensionality* which identifies classes with their membership functions:

$$(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$$

(the converse implication follows from the axioms for identity). The consistency of this axiom is proved by its validity in Zadeh models.

The comprehension scheme of FCT still allows classical models, as the construction of Zadeh models works for the ŁΠ-algebra $\{\mathbf{0}, \mathbf{1}\}$. Sometimes it may be desirable to exclude classical models. This can be done either by taking $\text{Ł}\Pi\frac{1}{2}$ instead of ŁΠ as the underlying logic, or equivalently by adding two constants C, c and the *axiom of fuzziness* $c \in C \leftrightarrow \neg_{\text{Ł}} c \in C$ without changing the underlying logic. In both cases there is a sentence with the value $\frac{1}{2}$ in any model, and all rational truth constants are therefore definable. The consistency of this extension follows from the fact that it holds in standard Zadeh models.

General models of FCT correspond in the obvious way to Henkin's general models of classical second-order logic, while Zadeh models correspond to full second-order models. FCT with its axioms of comprehension and extensionality thus can be viewed as a notational variant of the second-order fuzzy logic ŁΠ (monadic, in the form presented in this section; for higher arities see Section 4). Following the axiomatic method, we prefer FCT formulated in the Henkin style (as a two-sorted first-order theory, rather than a second-order logic) because of its axiomatizability. For even though (standard) Zadeh models are the intended models of FCT, the theory of Zadeh models is not arithmetically definable, let alone recursively axiomatizable. This follows from the obvious fact that classical full second-order logic (which itself is non-arithmetical) can be interpreted in the theory of Zadeh models by inscribing Δ (or $\neg\neg_{\text{G}}$) in front of every atomic formula.

3.2 Elementary class operations

Elementary class operations are defined by means of propositional combination of atomic formulae of FCT.

Convention 30. Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula and $\psi_1, \ldots, \psi_n$ be any formulae. By $\varphi(\psi_1, \ldots, \psi_n)$ we denote the formula φ in which all occurrences of p_i are replaced by ψ_i (for all $i \leq n$).

Definition 31. Let $\varphi(p_1, \ldots, p_n)$ be a propositional formula. We define the *n-ary class operation induced by φ* as

$$\text{Op}_\varphi(X_1, \ldots, X_n) =_{\text{df}} \{x \mid \varphi(x \in X_1, \ldots, x \in X_n)\}.$$

Among elementary class operations we find the following important kinds:

- *Class constants.* We denote Op_0 by $\emptyset$ and call it the *empty class,* and Op_1 by V and call it the *universal class.*

- *α-Cuts.* Let α be a truth-constant. Then we call the class $\text{Op}_{\Delta(\alpha \rightarrow p)}(X)$, i.e., $\{x \mid \Delta(\alpha \rightarrow (x \in X))\}$, the *$\alpha$-cut* of X and abbreviate it X_α. Similarly, $\text{Op}_{\Delta(\alpha \leftrightarrow p)}(X)$ is called the *α-level* of X, denoted by $X_{=\alpha}$.

- *Iterated complements,* i.e., class operations Op_φ where φ is p prefixed with a chain of negations. In ŁΠ, there are only a few such formulae that are non-equivalent. They yield the following operations (their definitions are summarized in Table 1): *involutive* and *strict complements*, the *kernel* and *support*, and the complement of the kernel. Except for the involutive complement, all of them are crisp.

- *Simple binary operations.* Some of the class operations $\mathrm{Op}_{p \circ q}$ where $\circ$ is a (primitive or derived) binary connective have their traditional names and notation, listed in Table 1 (not exhaustively).

Table 1: Elementary class operations

φ	$\mathrm{Op}_\varphi(X_1, \dots, X_n)$	Name
0	$\emptyset$	empty class
1	V	universal class
$\Delta(\alpha \to p)$	X_α	α-cut
$\Delta(\alpha \leftrightarrow p)$	$X_{=\alpha}$	α-level
$\neg_{\mathrm{G}} p$	$\backslash X$	strict complement
$\neg_{Ł} p$	$-X$	involutive complement
$\neg_{\mathrm{G}} \neg_{Ł} p$ (or Δp)	$\mathrm{Ker}(X)$	kernel
$\neg\neg_{\mathrm{G}} p$ (or $\neg\Delta\neg_{Ł} p$)	$\mathrm{Supp}(X)$	support
$p \mathbin{\&_*} q$	$X \cap_* Y$	$*$-intersection
$p \vee q$	$X \cup Y$	union
$p \oplus q$	$X \uplus Y$	strong union
$p \mathbin{\&} \neg_{\mathrm{G}} q$	$X \setminus Y$	strict difference
$p \mathbin{\&_*} \neg_{Ł} q$	$X -_* Y$	involutive $*$-difference

3.3 Elementary relations between classes

Most of important relations between classes have one of the two forms described in the following definition:

Definition 32 (Uniform and supremal relations)**.** Let $\varphi(p_1, \dots, p_n)$ be a propositional formula. The n-ary *uniform relation* between $X_1, \dots, X_n$ induced by φ is defined as

$$\mathrm{Rel}^{\forall}_{\varphi}(X_1, \dots, X_n) \equiv_{\mathrm{df}} (\forall x)\varphi(x \in X_1, \dots, x \in X_n).$$

The n-ary *supremal relation* between $X_1, \dots, X_n$ induced by φ is defined as

$$\mathrm{Rel}^{\exists}_{\varphi}(X_1, \dots, X_n) \equiv_{\mathrm{df}} (\exists x)\varphi(x \in X_1, \dots, x \in X_n).$$

Among elementary class relations we find the following important kinds (they are summarized in Table 2):

- *Equalities* $\mathrm{Rel}^{\forall}_{p \leftrightarrow_* q}$ denoted $\approx_*$. The value of $X \approx_{\mathrm{G}} Y$ is the maximal truth degree below which the membership functions of X and Y are identical. In standard $[\mathbf{0},\mathbf{1}]$-models, $1 - \|X \approx_{\mathrm{L}} Y\|$ is the maximal difference of the (values of) the membership functions of X and Y, and $\|X \approx_{\Pi} Y\|$ is the infimum of their ratios. All $\approx_*$ get value 1 iff the membership functions are identical. For crisp classes, these notions of equality coincide with classical equality.

- *Inclusions* $\mathrm{Rel}^{\forall}_{p \to_* q}$, denoted $\subseteq_*$. Their semantics is analogous to that of equalities. They get the value 1 iff the membership function of X is majorized by that of Y.

- *Compatibilities* $\mathrm{Rel}^{\exists}_{p \&_* q}$. Their strict and involutive negations may respectively be called *strict* and *involutive* $*$*-disjointness.*

- *Unary properties* of height, normality, fuzziness, and crispness.

Table 2: Class properties and relations

Relation	Notation	Name
$\mathrm{Rel}^{\exists}_{p}(X)$	$\mathrm{Hgt}(X)$	height
$\mathrm{Rel}^{\exists}_{\Delta p}(X)$	$\mathrm{Norm}(X)$	normality
$\mathrm{Rel}^{\forall}_{\Delta(p \vee \neg p)}(X)$	$\mathrm{Crisp}(X)$	crispness
$\mathrm{Rel}^{\exists}_{\neg\Delta(p \vee \neg p)}(X)$	$\mathrm{Fuzzy}(X)$	fuzziness
$\mathrm{Rel}^{\forall}_{p \to_* q}(X,Y)$	$X \subseteq_* Y$	$*$-inclusion
$\mathrm{Rel}^{\forall}_{p \leftrightarrow_* q}(X,Y)$	$X \approx_* Y$	$*$-equality
$\mathrm{Rel}^{\exists}_{p \&_* q}(X,Y)$	$X \|_* Y$	$*$-compatibility

Notice that due to the axiom of extensionality, the relation $\mathrm{Rel}^{\forall}_{\Delta(p \leftrightarrow q)}$, which is obviously equivalent to $\Delta(X \approx_* Y)$, coincides with the identity of classes. Thus it is $\Delta(X \approx_* Y)$ that guarantees intersubstitutivity *salva veritate* in all formulae (equalities generally do not).

It can be noticed that Gödel equality $\approx_{\mathrm{G}}$ is highly true only if the membership functions are identical on low truth values; product equality $\approx_{\Pi}$ is also more restrictive on lower truth values. However, this does not conform with the intuition that the difference in the *high* values (on the "prototypes") should matter more than a negligible difference on objects that almost do not belong to the classes under consideration. Equality of involutive complements, $-X \approx_* -Y$, is therefore a better measure of similarity of classes. Similarly, $-Y \subseteq_* -X$ may give a better measure of containment of X in Y than $X \subseteq_* Y$.

3.4 Theorems on elementary class relations and operations

The following metatheorems show that a large part of elementary fuzzy set theory can be reduced to fuzzy propositional calculus.

Theorem 33. *Let $\varphi, \psi_1, \dots, \psi_n$ be propositional formulae.*
Then $\vdash \varphi(\psi_1, \dots, \psi_n)$

$$\text{iff} \quad \vdash \mathrm{Rel}^{\forall}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1}, \dots, X_{1,k_1}), \dots, \mathrm{Op}_{\psi_n}(X_{n,1}, \dots, X_{n,k_n})) \tag{7}$$

$$\text{iff} \quad \vdash \mathrm{Rel}^{\exists}_{\varphi}(\mathrm{Op}_{\psi_1}(X_{1,1}, \dots, X_{1,k_1}), \dots, \mathrm{Op}_{\psi_n}(X_{n,1}, \dots, X_{n,k_n})) \tag{8}$$

Proof. The substitution of the formulae $x \in X_{i,j}$ for $p_{i,j}$ into $\psi_i(p_{i,1}, \dots, p_{i,k_i})$ everywhere in the (propositional) proof of $\varphi(\psi_1, \dots, \psi_n)$ transforms it into the proof of

$$\varphi(x \in \mathrm{Op}_{\psi_1}(X_{1,1}, \dots, X_{1,k_1}), \dots, x \in \mathrm{Op}_{\psi_n}(X_{n,1}, \dots, X_{n,k_n})).$$

Then use generalization on x to get $\mathrm{Rel}^{\forall}_{\varphi}$ and $\exists$-introduction to get $\mathrm{Rel}^{\exists}_{\varphi}$.

Conversely, given an evaluation e that refutes $\varphi(\psi_1, \dots, \psi_n)$, we construct a Zadeh model $\mathbf{M}$ refuting (7) and (8) by assigning to the class variables $X_{i,j}$ the functions $A_{i,j}$ such that $A_{i,j}(a) = e(p_{i,j})$ for every a in the universe of $\mathbf{M}$. Applying Theorems 4 and 27, the proof is done. Q.E.D.

Corollary 34. *Let φ and ψ be propositional formulae.*
If $\vdash \varphi \to \psi$ then $\vdash \mathrm{Op}_{\varphi}(X_1, \dots, X_n) \subseteq \mathrm{Op}_{\psi}(X_1, \dots, X_n)$.
If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \mathrm{Op}_{\varphi}(X_1, \dots, X_n) = \mathrm{Op}_{\psi}(X_1, \dots, X_n)$.
If $\vdash \varphi \vee \neg\varphi$ then $\vdash \mathrm{Crisp}(\mathrm{Op}_{\varphi}(X_1, \dots, X_n))$.

By virtue of Theorem 33, the properties of propositional connectives directly translate to the properties of class relations and operations. For example:

$\vdash \Delta p \to p$	proves	$\vdash \mathrm{Ker}(X) \subseteq X$
$\vdash p \to p \vee q$	"	$\vdash X \subseteq X \cup Y$
$\vdash 0 \to p$	"	$\vdash \emptyset \subseteq X$
$\vdash p \,\&\, q \to p \wedge q$	"	$\vdash X \cap_* Y \subseteq X \cap_{\mathrm{G}} Y$
$\vdash \neg_{\mathrm{G}} p \vee \neg\neg_{\mathrm{G}} p$	"	$\vdash \mathrm{Crisp}(\backslash X)$
$\vdash \Delta(\alpha \to p) \to \Delta(\beta \to p)$ for $\alpha \geq \beta$	"	$\vdash X_\alpha \subseteq X_\beta$ for $\alpha \geq \beta$, etc.

In order to translate monotonicity and congruence properties of propositional connectives to the same properties of class operations, we need another theorem:

Theorem 35. *Let $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then*

$$\vdash \mathop{\&_*}_{i=1}^{k} \varphi_i(\psi_{i,1}, \dots, \psi_{i,n_i}) \to \bigwedge_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \dots, \psi'_{i,n'_i}) \tag{9}$$

iff

$$\vdash \mathop{\&_*}_{i=1}^{k} \mathrm{Rel}^{\forall}_{\varphi_i}\left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \to$$
$$\to \bigwedge_{i=1}^{k'} \mathrm{Rel}^{\forall}_{\varphi'_i}\left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right) \tag{10}$$

Proof. Without loss of generality, the principal implications of (9) and (10) can be assumed to be $\to_*$. Replacing all propositional variables p_j in the proof of (9) by the atomic formulae $x \in X_j$ then yields the proof of

$$\mathop{\&_*}_{i=1}^{k} \varphi_i\left(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\right) \to_* \bigwedge_{i=1}^{k'} \varphi'_i\left(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \dots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\right).$$

Generalization on x and distribution of $\forall$ over all conjuncts using (1), (5) and (3) of Lemma 25 proves (10). The converse is proved exactly as in Theorem 33. Q.E.D.

Examples of direct corollaries of the theorem:

Provability in BLΔ of	Proves in FCT
$(p \to q) \to ((p \,\&_* \, r) \to (q \,\&_* \, r))$	$X \subseteq_* Y \to X \cap_* Z \subseteq_* Y \cap_* Z$
$(p \to q) \to (p \to (p \wedge q))$	$X \subseteq_* Y \to X \subseteq_* X \cap_{\mathrm{G}} Y$
$[(p \to q) \,\&\, (q \to p)] \to (p \leftrightarrow q)$	$(X \subseteq_* Y \,\&\, Y \subseteq_* X) \to X \approx_* Y$
$(p \leftrightarrow q) \to [(p \to q) \wedge (q \to p)]$	$X \approx_* Y \to (X \subseteq_* Y \wedge Y \subseteq_* X)$
$[(p \to r) \,\&\, (q \to r)] \to (p \vee q \to r)$	$(X \subseteq_* Z \,\&_* \, Y \subseteq_* Z) \to X \cup Y \subseteq_* Z$
$\Delta(p \to q) \to [\Delta(\alpha \to p) \to \Delta(\alpha \to q)]$	$\Delta(X \subseteq Y) \to X_\alpha \subseteq Y_\alpha$
transitivity of $\to$, $\leftrightarrow$	transitivity of $\subseteq_*, \approx_*$, etc.

Similarly, $Ł \vdash (\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow p)$ proves $-X \subseteq_{Ł} -Y \leftrightarrow Y \subseteq_{Ł} X$, etc.

To derive theorems about $\mathrm{Rel}^{\exists}$, we slightly modify Theorem 35:

Theorem 36. *Let $\varphi_i, \varphi'_i, \psi_{i,j}, \psi'_{i,j}$ be propositional formulae. Then*

$$\vdash \mathop{\&_*}_{i=1}^{k} \varphi_i(\psi_{i,1}, \ldots, \psi_{i,n_i}) \to \bigvee_{i=1}^{k'} \varphi'_i(\psi'_{i,1}, \ldots, \psi'_{i,n'_i}) \tag{11}$$

iff

$$\begin{aligned} \vdash \mathop{\&_*}_{i=1}^{k-1} \mathrm{Rel}^{\forall}_{\varphi_i} \Big(\mathrm{Op}_{\psi_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{i,n_i}}(\vec{X})\Big) \,\&_* \\ \&_* \, \mathrm{Rel}^{\exists}_{\varphi_k} \Big(\mathrm{Op}_{\psi_{k,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi_{k,n_k}}(\vec{X}\Big) \to \\ \to \bigvee_{i=1}^{k'} \mathrm{Rel}^{\exists}_{\varphi'_i} \Big(\mathrm{Op}_{\psi'_{i,1}}(\vec{X}), \ldots, \mathrm{Op}_{\psi'_{i,n'_i}}(\vec{X})\Big) \end{aligned} \tag{12}$$

Proof. Modify the proof of Theorem 35, using (6) of Lemma 25 instead of (5), and then (4) of the same Lemma to distribute $\exists$ over the disjuncts. Q.E.D.

Examples of direct corollaries:

Provability in BLΔ of	Proves in FCT
$p \,\&\, (p \to q) \to q$	$\mathrm{Hgt}(X) \,\&_* \, (X \subseteq_* Y) \to \mathrm{Hgt}(Y)$
$\Delta(p \vee q) \to \Delta p \vee \Delta q$	$\mathrm{Norm}(X \cup Y) \to \mathrm{Norm}(X) \vee \mathrm{Norm}(Y)$
$(p \to r) \,\&\, (p \,\&\, q) \to (q \,\&\, r)$	$X \subseteq_* Z \,\&_* \, X \|_* Y \to Y \|_* Z$, etc.

4 Tuples of objects

In order to be able to deal with fuzzy relations, we will further assume that the language of FCT contains an apparatus for forming tuples of objects and accessing their components. Such an extension can be achieved, e.g., by postulating variable sorts for any multiplicity of tuples (all of which are subsumed by the sort of objects), enriching the language with the functions for forming n-tuples of any combination of tuples and accessing its components, and adding axiom schemes expressing that tuples equal iff their respective constituents equal. The definition of Zadeh model then must be adjusted by partitioning the range of object variables and interpreting the tuples-handling functions. We omit elaborating this sort of syntactic sugar.

In what follows, the usual abbreviations of the form $\{\langle x_1, \ldots, x_n\rangle \mid \varphi\}$ for $\{z \mid (\exists x_1)\ldots(\exists x_n)(z = \langle x_1, \ldots, x_n\rangle \ \&\ \varphi)\}$ will be used.

FCT equipped with tuples of objects contains common operations for dealing with relations. We can define Cartesian products, domains, ranges and the relational operations as usual:[2]

$$\begin{aligned}
X \times_* Y &=_{\text{df}} \{\langle x, y\rangle \mid x \in X \ \&_*\ y \in Y\}\\
\text{Dom}(R) &=_{\text{df}} \{x \mid \langle x, y\rangle \in R\}\\
\text{Rng}(R) &=_{\text{df}} \{y \mid \langle x, y\rangle \in R\}\\
R \circ_* S &=_{\text{df}} \{\langle x, y\rangle \mid (\exists z)(\langle x, z\rangle \in R \ \&_*\ \langle z, y\rangle \in S)\}\\
R^{-1} &=_{\text{df}} \{\langle x, y\rangle \mid \langle y, x\rangle \in R\}\\
\text{Id} &=_{\text{df}} \{\langle x, y\rangle \mid x = y\}
\end{aligned}$$

The introduction of tuples of objects also allows an axiomatic investigation of various kinds of fuzzy relations (e.g., similarities) and fuzzy structures (fuzzy preorderings, graphs, etc.). We can define the usual properties of relations, as summarized in Table 3 (for brevity's sake, we write just Rxy for $\langle x, y\rangle \in R$).[3]

Table 3: Properties of relations

Notation	Definition	Name
$\text{Refl}(R)$	$(\forall x)(Rxx)$	reflexive
$\text{Sym}_*(R)$	$(\forall x, y)(Rxy \to_* Ryx)$	$*$-symmetric
$\text{Trans}_*(R)$	$(\forall x, y, z)(Rxy \ \&_*\ Ryz \to_* Rxz)$	$*$-transitive
$\text{Dich}(R)$	$(\forall x, y)(Rxy \vee Ryx)$	dichotomic
$\text{Quord}_*(R)$	$\text{Refl}(R) \ \&_*\ \text{Trans}_*(R)$	$*$-quasiordering
$\text{Linquord}_*(R)$	$\text{Quord}_*(R) \ \&_*\ \text{Dich}(R)$	linear $*$-quasiordering
$\text{Sim}_*(R)$	$\text{Quord}_*(R) \ \&_*\ \text{Sym}_*(R)$	$*$-similarity
$\text{Equ}_*(R)$	$\text{Sim}_*(R) \ \&_*\ (\forall x, y)(\Delta Rxy \to_* x = y)$	$*$-equality

Classical definitions of some properties of relations (e.g., antisymmetry) make use of the identity predicate on objects. One may be tempted to use the identity predicate $=$ of ŁΠ∀ in the rôle of the classical identity in these definitions. However, since $=$ is crisp, such definitions do not yield useful and genuine fuzzy notions. A fuzzy analogue of the crisp notion of identity is that of similarity or equality (see Table 3). We can therefore define these properties *relative to* a $*$-similarity or $*$-equality S. For details see the last section.

[2]Obviously for crisp arguments these operations yield crisp classes; $X \times_* Y$ is crisp iff both X and Y are crisp. Unless X and Y are crisp, the property of being a relation from X to Y is double-indexed (a $*'$-subset of the Cartesian product $X \times_* Y$). Also the definitions of usual properties (e.g., reflexivity, $*$-symmetry, etc.) of a relation on a non-crisp Cartesian product have to be defined with relativized quantifiers which bring another index. It is doubtful that definitions combining various t-norms will have any real meaning. The situation is much easier if only relations on crisp classes are considered.

[3]Following the usual mathematical terminology, $*$-similarity may also be called $*$-*equivalence*; we respect the established fuzzy set terminology here. *Weak dichotomy* $(\forall x, y)(Rxy \oplus Ryx)$ could also be defined and weak versions of the properties that contain dichotomy, e.g. weakly linear $*$-ordering.

In this way, the properties of being a $*$-antisymmetric relation, a $*$-ordering, a linear $*$-ordering, a $*$-well-ordering, a $*$-function and a $*$-bijection (w.r.t. some fuzzy $*$-equality) can be introduced. By means of $*$-bijections, the notions of $*$-subvalence, $*$-equipotence and $*$-finitude of classes (again w.r.t. some fuzzy $*$-equality) can be defined. A thorough investigation of these notions, however, exceeds the scope of this paper.

5 Higher types of classes

5.1 Second-level classes

Class theory does not contain an apparatus for dealing with families of classes. In many cases, a family of classes can be represented by a class of pairs or some other kind of 'encoding'. For instance, a relation R may be understood as representing the family of classes $X_i = \{x \mid \langle i, x\rangle \in R\}$ for all $i \in \mathrm{Dom}(R)$.

In other cases, however, no suitable class of indices can be found and such an 'encoding' is not possible. Then it is desirable to extend the apparatus of class theory by classes of the second level. This is done simply by repeating the same definitions one level higher. We introduce a new sort of variables for families of classes $\mathcal{X}, \mathcal{Y}, \ldots$, a new membership predicate between classes and families of classes $X \in \mathcal{X}$, and the comprehension scheme for families of classes

$$(\exists\mathcal{X})\Delta(\forall X)(X \in \mathcal{X} \leftrightarrow \varphi(X))$$

for all formulae φ (where φ may contain any parameters except for $\mathcal{X}$). The extensionality axiom for families of classes now reads

$$(\forall X)\Delta(X \in \mathcal{X} \leftrightarrow X \in \mathcal{Y}) \rightarrow \mathcal{X} = \mathcal{Y}.$$

Again it is possible to introduce second-level comprehension terms $\{X \mid \varphi(X)\}$, which introduction is conservative and eliminable by Theorem 28.

The consistency of this extension is proved by a construction of second-level Zadeh models over a linear ŁΠ-algebra $\mathbf{L}$, in which the object variables range over a universe U, the class variables over the set $\mathbf{L}^U$ of all functions from U to $\mathbf{L}$, and the second-level class variables range over the set $\mathbf{L}^{\mathbf{L}^U}$ of all functions from $\mathbf{L}^U$ to $\mathbf{L}$. The second-level class $\{X \mid \varphi(X)\}$ is again identified with the characteristic function of φ as in Section 3.1. Obviously, this construction makes both the second-level comprehension scheme and the axiom of extensionality satisfied in the model; the theory of second-level classes can thus be viewed as third-order fuzzy logic (we omit details).

All definitions of elementary class relations and operations and all theorems can directly be transferred from classes to second-level classes. Refining the language, axioms, and Zadeh models to tuples of classes is also straightforward.

It may be observed that the class operations and relations Op_φ, $\mathrm{Rel}^\forall_\varphi$, and $\mathrm{Rel}^\exists_\varphi$, which were introduced in Sections 3.2 and 3.3 as defined functors and predicates, are now individuals of the theory, viz second-level classes.

5.2 Simple fuzzy type theory

If there be need for families of families of classes, it is straightforward to repeat the whole construction once again to get third-level classes. By iterating this process, we get a simple type theory over ŁΠ, for which the class theory described in Sections 3–4 is the induction step. The comprehension schemes and Zadeh models can easily be generalized to allow membership of elements of any type less then n in classes of the n-th level.[4]

A type theory over a particular fuzzy logic (viz IMTLΔ, extended also to ŁΔ) has already been proposed by V. Novák in [14]. As mentioned in the Introduction, our theory can be built over various fuzzy logics with Δ; its variant over IMTLΔ and Novák's type theory seem to be equivalent (though radically different in notation, as Novák uses λ-terms).

Since almost all classical applied mathematics can be formalized within the first few levels of simple type theory, the formalism just described should be sufficient for all applications of fuzzy sets based on t-norms or other functions definable in ŁΠ (see Theorem 7). To illustrate this, we show the formalization of Zadeh's extension principle.

Definition 37 (Extension by Zadeh's principle)**.** A (fuzzy) binary relation[5] R between objects is *extended by Zadeh's principle (based on a t-norm* $*$*)* to a relation $\mathcal{R}_*$ between (fuzzy) classes as follows:

$$\mathcal{R}_*(X,Y) \equiv_{\mathrm{df}} (\exists x, y)(Rxy \mathbin{\&_*} x \in X \mathbin{\&_*} y \in Y)$$

Since relations between classes are classes of the second level in our simple type theory, Zadeh's extension principle in fact assigns to a first-level class R a second-level relation; such an assignment itself is an individual of the third level. Thus we can define Zadeh's principle as an *individual* of our theory—a special class $\mathcal{Z}_*$ of the third level:

Definition 38 (Zadeh's extension principle)**.** *Zadeh's extension principle based on* $*$ is a third-level function $\mathcal{Z}_*$ defined as follows (we adopt the usual functional notation for classes which are functions):

$$\mathcal{Z}_*(R) =_{\mathrm{df}} \{\langle X, Y\rangle \mid (\exists x, y)(Rxy \mathbin{\&_*} x \in X \mathbin{\&_*} y \in Y)\}$$

Generally we can extend any fuzzy relation $R^{(n+1)}$ of type $n+1$ to one of type $n+2$ by Zadeh's principle of type $n+3$ (based on a t-norm $*$). All these 'principles' are in fact individuals of our theory, whose existence follows from the comprehension scheme.

Definition 39 (Zadeh's extension principle for higher types)**.** *Zadeh's extension principle for relations of type* $n+1$ *(for* $n \geq 0$*) based on* $*$ is the function of type $n+3$ defined as follows:

$$\mathcal{Z}_*{}^{(n+3)}\left(R^{(n+1)}\right) =_{\mathrm{df}} \Big\{ \Big\langle X_1^{(n+1)}, \ldots, X_k^{(n+1)} \Big\rangle \;\Big| \\ \Big| \; (\exists W_1^{(n)}, \ldots, W_k^{(n)}) \Big(\Big\langle W_1^{(n)}, \ldots, W_k^{(n)} \Big\rangle \in R^{(n+1)} \;\&_*\; \mathop{\&_*}_{i=1}^{k} \; W_i^{(n)} \in X_i^{(n+1)} \Big) \Big\} \quad (13)$$

6 Adding structure to the domain of discourse

As we have shown, in FCT we can define many properties of individuals of our theory (objects or classes). Since our theory contains classical class theory (for classes which

[4] This is done simply by postulating that the n-th sort of variables is subsumed by the k-th sort if $n < k$. The sorts can further be refined to allow arbitrary tuples of individuals of lesser types with the appropriate tuple-forming, component-extracting, and tuple-identity axioms added. The generalization of Zadeh models is again quite straightforward.

[5] The generalization to n-ary relations is trivial.

are crisp), we can introduce arbitrary relations and functions on the universe of objects which are definable in classical class theory. As they can be described by formulae, their existence is guaranteed by the comprehension axiom. So the only thing we need to add is a constant of the appropriate sort and the instance of the comprehension axiom. The following definition is the formalization of this approach for the first-order theories.

Definition 40. Let $\mathbf{\Gamma}$ be a classical one-sorted predicate language and T be a $\mathbf{\Gamma}$-theory. For each n-ary predicate symbol P of $\mathbf{\Gamma}$ let us introduce a new constant $\bar{P}$ for a class of n-tuples, and for each n-ary function symbol F we take a new constant $\bar{F}$ for a class of $(n+1)$-tuples. We define the language FCT($\mathbf{\Gamma}$) as the language of FCT extended by the symbols $\bar{Q}$ for each symbol $Q \in \mathbf{\Gamma}$. The translation $\bar{\varphi}$ of a $\mathbf{\Gamma}$-formula φ to FCT($\mathbf{\Gamma}$) is obtained as the result of replacing all occurrences of all $\mathbf{\Gamma}$-symbols Q in φ by $\bar{Q}$.

We define the theory FCT(T) in the language FCT($\mathbf{\Gamma}$) as the theory with the following axioms:

- The axioms of FCT
- The translations $\bar{\varphi}$ of all axioms φ of T
- Crisp($\bar{Q}$) for each symbol $Q \in \mathbf{\Gamma}$ (for the definition of Crisp, see Table 2)
- $\langle x_1, \ldots x_n, y\rangle \in \bar{F} \wedge \langle x_1, \ldots x_n, z\rangle \in \bar{F} \rightarrow y = z$ for each n-ary function symbol $F \in \mathbf{\Gamma}$.

Lemma 41. *Let $\mathbf{\Gamma}$ be a classical predicate language, T a $\mathbf{\Gamma}$-theory, $\mathbf{L}$ an ŁΠ-algebra. If $\mathbf{M}$ is an $\mathbf{L}$-model of* FCT(T)*, then $\mathbf{M}^c = (M, (Q_{\mathbf{M}^c})_{Q\in\mathbf{\Gamma}})$, where $Q_{\mathbf{M}^c} = \bar{Q}_{\mathbf{M}}$ for each $Q \in \mathbf{\Gamma}$, is a model (in the sense of classical logic) of the theory T.*

Vice versa, for each model $\mathbf{M}$ of T there is an $\mathbf{L}$-model $\mathbf{N}$ of FCT(T) *such that $\mathbf{N}^c$ is isomorphic to $\mathbf{M}$.*

Therefore (in virtue of Theorem 27), $T \vdash \varphi$ iff FCT(T) $\vdash \bar{\varphi}$*, for any $\mathbf{\Gamma}$-formula φ.*

Proof. If $\mathbf{M}$ is an $\mathbf{L}$-model of FCT, then for each $Q \in \mathbf{\Gamma}$, $\bar{Q}_{\mathbf{M}}$ is crisp due to the axiom Crisp($\bar{Q}$) of FCT(T). Setting the universe of $\mathbf{M}^c$ to that of $\mathbf{M}$, and for each symbol $Q \in \mathbf{\Gamma}$, setting $Q_{\mathbf{M}^c}$ to the set whose characteristic function is $\bar{Q}_{\mathbf{M}}$, we can see that $\mathbf{M}^c$ models T, because the axioms of T, which contain only crisp predicates, are evaluated classically in $\mathbf{M}^c$.

Conversely, we define $\mathbf{M}$ as the standard Zadeh model with the universe of $\mathbf{N}$, in which $\bar{F}_{\mathbf{M}} = F_{\mathbf{N}}$ for every function symbol $F \in \mathbf{\Gamma}$, and for every predicate $P \in \mathbf{\Gamma}$, $\bar{P}_{\mathbf{M}}$ is realized as the characteristic function of $P_{\mathbf{N}}$. Then $\mathbf{M}$ obviously satisfies all axioms of FCT(T); the axioms of T are again evaluated classically in $\mathbf{M}$, as the realizations of all predicates involved are crisp. Q.E.D.

Example 42. Let R be a constant for a class of pairs. Then in each $\mathbf{L}$-model of the theory Crisp(R), Refl(R), Trans(R), $(\forall x, y)(Rxy \mathbin{\&} Ryx \rightarrow x = y)$, the constant R is represented by a crisp ordering on the universe of objects. (For the definitions of Refl and Trans, see Table 3.)

Example 43. If T is a classical theory of the real closed field, then in each $\mathbf{L}$-model $\mathbf{M}$ of the theory FCT(T), the universe of objects with $\bar{\leq}_{\mathbf{M}}, \bar{+}_{\mathbf{M}}, \bar{-}_{\mathbf{M}}, \bar{\cdot}_{\mathbf{M}}, \bar{0}_{\mathbf{M}}, \bar{1}_{\mathbf{M}}$ is a real closed field.

In Lemma 41 we speak of first-order theories only. Nevertheless, it can be extended to any theory formalizable in classical type theory. Here we present only one example.

Example 44. Let τ be a constant for a class of classes and T the theory with the axioms:

- $\mathrm{Crisp}(\tau)$
- $(\forall X)(X \in \tau \to \mathrm{Crisp}(X))$
- $(\forall \mathcal{X})(\mathrm{Crisp}(\mathcal{X})\ \&\ \mathcal{X} \subseteq \tau \to \{x \mid (\exists X)(X \in \mathcal{X}\ \&\ x \in X)\} \in \tau)$
- $(\forall X_1) \ldots (\forall X_n)(X_1 \in \tau\ \&\ \ldots\ \&\ X_n \in \tau \to X_1 \cap \ldots \cap X_n \in \tau)$ for each $n \in N$

Then in each **L**-model of the theory T, the constant τ is represented by a classical topology on the universe of objects.

7 Fuzzy mathematics

If we examine the above definitions we see the crucial rôle of the predicate Crisp. If we remove this predicate from the above definitions we get the "natural" fuzzification of the above-mentioned concepts.[6]

In order to illustrate the methodology of fuzzification, let us concentrate on the concept of ordering. If we remove the predicate Crisp from the definition, then we have to distinguish which t-norm was used in the axioms of transitivity and antisymmetry. Thus we get the concept of $*$-fuzzy ordering. This is the way this concept was introduced by Zadeh. However, some carefulness is due here not to overlook some "hidden" crispness. There is crisp identity used in the antisymmetry axiom, and also in the reflexivity axiom which can be written as $(\forall x, y)(x = y \to Rxy)$. A more general definition is therefore parameterized also by a fuzzy equality in the following way:

Example 45. Let E and R be two constants of classes of tuples. The following axioms define the concept of $(*, E)$-ordering R:

- $\mathrm{Equ}_*(E)$
- $\mathrm{Trans}_*(R)$
- $(\forall x, y)(Exy \to Rxy)$
- $(\forall x, y)(Rxy\ \&_*\ Ryx \to_* Exy)$

Observe that E is a $*$-equality, and the last two conditions can be written as $R \cap_* R^{-1} \subseteq E \subseteq R$. We thus get the notion of fuzzy ordering as defined by Bodenhoffer in [2].

In contemporary fuzzy mathematics the methodology of fuzzification of concepts is somewhat sketchy and non-consistent: usually only some features of a classical concept are fuzzified while other features are left crisp.

We would like to propose another "inductive" approach. We propose to follow the usual "inductive" development of mathematics (in some metamathematical setting—here

[6]A sketch of this method can already be found in Höhle's 1987 paper [12, Section 5]:

> "It is the opinion of the author that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two-valued logic by a multiple-valued logic. [...I]t is now clear how we can find for every mathematical notion its 'fuzzy counterpart'. Since every mathematical notion can be written as a formula in a formal language, we have only to internalize, i.e. to interpret these expressions by the given multiple-valued logic."

in simple type theory) and fuzzify "along the way". In more words: develop a fuzzy generalization of basic classical concepts (the notion of class, relation, equality—as done in this paper); then define compound fuzzy notions by taking their classical definitions and consistently replacing classical sub-concepts in the definitions by their already fuzzified counterparts. The consistency of this approach promises that no crispness will be unintentionally "left behind".

This approach is formal and sometimes may lead to too complex notions. In such cases, some features of the complex notion may *intentionally* be left crisp by retaining some of the crispness axioms. The advantage of the proposed approach is that we always know *which* features are left crisp.

The framework presented in this paper provides a unified formalism for various disciplines of fuzzy mathematics. This may enable, i.a., an interchange of results and methods between distant disciplines of fuzzy mathematics, till now separated by differences in notation and incompatibilities in definitions. It can also bring new (proof-theoretic and model-theoretic) methods to traditional fuzzy disciplines and enable their further development in both theory and applications. Finally, the axiomatization of the whole fuzzy mathematics, independent of particular $[0, 1]$-functions, can be an important step in understanding vague phenomena. Further elaboration of the proposed formalism and its application to various disciplines of fuzzy mathematics is thus a possible direction towards firm foundations of fuzzy mathematics.

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References

[1] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[2] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(5):593–610, 2000.

[3] E. W. Chapin, Jr. Set-valued set theory: Part one. *Notre Dame Journal of Formal Logic*, 15(4):619–634, 1974.

[4] E. W. Chapin, Jr. Set-valued set theory: Part two. *Notre Dame Journal of Formal Logic*, 16(2):255–267, 1975.

[5] P. Cintula. The ŁΠ and Ł$\Pi\frac{1}{2}$ propositional and predicate logics. *Fuzzy Sets and Systems*, 124(3):289–302, 2001.

[6] P. Cintula. Advances in the ŁΠ and Ł$\Pi\frac{1}{2}$ logics. *Archive for Mathematical Logic*, 42(5):449–468, 2003.

[7] P. Cintula. Weakly implicative (fuzzy) logics. Technical Report V-912, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2004.

[8] F. Esteva, L. Godo, and F. Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[9] S. Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View.* Vieweg, Wiesbaden, 1993.

[10] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[11] P. Hájek and Z. Haniková. A development of set theory in fuzzy logic. In M. Fitting and E. Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Physica-Verlag, Heidelberg, 2003.

[12] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[13] F. Montagna and G. Panti. Adding structure to MV-algebras. *Journal of Pure and Applied Algebra*, 164(3):365–387, 2001.

[14] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149(2):235–273, 2004.

[15] G. Takeuti and S. Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32:1–32, 1992.

[16] R. B. White. The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz. *Journal of Philosophical Logic*, 8:509–534, 1979.

[17] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

Relations in Fuzzy Class Theory: Initial steps

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Abstract: This paper studies fuzzy relations in the graded framework of Fuzzy Class Theory (FCT). This includes (i) rephrasing existing work on graded properties of binary fuzzy relations in the framework of Fuzzy Class Theory and (ii) generalizing existing crisp results on fuzzy relations to the graded framework. Our particular aim is to demonstrate that Fuzzy Class Theory is a powerful and easy-to-use instrument for handling fuzzified properties of fuzzy relations. This paper does not rephrase the whole theory of (fuzzy) relations; instead, it provides an illustrative introduction showing some representative results, with a strong emphasis on fuzzy preorders and fuzzy equivalence relations.

Keywords: Fuzzy Class Theory, fuzzy relation, fuzzy preorder, fuzzy equivalence relation, similarity, graded properties. MSC: 04A72, 03E72, 03E70.

1 Introduction

Fuzzy relations are of fundamental importance in almost all sub-fields of fuzzy logic and fuzzy set theory, including particularly fuzzy preference modeling, fuzzy mathematics, fuzzy inference, and many more. In the most general setting, fuzzy relations are mappings from the Cartesian product of non-empty domains $U_1 \times \cdots \times U_n$ (usually with $n \geq 2$) to the unit interval or a more general lattice of truth values L (see e.g. [38, 41, 42, 47, 52]). Clearly the motivation behind fuzzy relations is to allow more flexibility by admitting intermediate degrees of relationship [70, 61, 60, 36, 13, 58].

An important class are the so-called *binary* fuzzy relations that are used to express graded relationships between two objects coming from the same domain. Technically, they are defined as $U \times U \to L$ mappings, where U is some non-empty set and L is again the lattice of truth values we consider. There are many important sub-classes, such as, fuzzy preorders [68, 70, 18], fuzzy orders [70, 13, 47, 16], and fuzzy equivalence relations [70, 13, 68, 52, 67, 51, 20]. Interestingly, however, the traditional characterizing properties of these important types of fuzzy relations, such as, reflexivity, symmetry, transitivity, and so forth, are defined in a strictly crisp way, i.e., as properties that either hold fully or do not hold at all. One may be tempted to argue that it is somewhat peculiar to fuzzify relations by allowing intermediate degrees of relationships, but, at the same time, to still enforce strictly crisp properties on fuzzy relations. This particularly implies that all results are effective only if some assumptions are fulfilled, but say nothing at all if the assumptions are only fulfilled to a certain degree (even if they are *almost* fulfilled).

To illustrate our point, let us shortly consider a toy example. It is common in the theory of fuzzy relations to call a fuzzy relation $R\colon U \times U \to [0,1]$ *reflexive* if $R(x,x) = 1$ holds for all $x \in U$. From the reflexivity of a fuzzy relation R, we can infer

$$R \sqsubseteq R \circ_* R,$$

where $\sqsubseteq$ is the traditional crisp inclusion of fuzzy sets or relations [69],

$$R_1 \sqsubseteq R_2 \text{ if and only if } R_1(x,y) \leq R_2(x,y) \text{ for all } x,y \in U,$$

and $R \circ_* R$ is the composition of R with itself (with respect to some triangular norm $*$), i.e.,

$$(R \circ_* R)(x,y) = \sup_{z \in U} (R(x,z) * R(z,y)).$$

What, however, happens if a given fuzzy relation R is not reflexive, but *almost* reflexive? Let us consider $U = \{1,2,3\}$ and the fuzzy relation (in convenient matrix notation)

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a \end{pmatrix},$$

where $a \in [0,1]$. Using the Łukasiewicz t-norm $x *_{\mathrm{L}} y = \max(0, x+y-1)$, routine calculations show that

$$R \circ_{\mathrm{L}} R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a' \end{pmatrix},$$

where $a' = \max(0, 2a-1)$. So we confirm that only if $a = 1$, we also have $a' = 1$, and only in this case $R \sqsubseteq R \circ_{\mathrm{L}} R$ holds. What is also apparent, however, is the fact that, the closer the value a is to 1, the less R exceeds $R \circ_{\mathrm{L}} R$. Actually, in this example, this degree is

$$a - a' = a - \max(0, 2a-1) = \min(a, 1-a).$$

For example, if $a = 0.99$, we obtain $a' = 0.98$, and R exceeds $R \circ_{\mathrm{L}} R$ only by 0.01. So we see that, even if some assumptions are not fully satisfied, we may obtain some meaningful results. The classical theory of fuzzy relations, however, does not offer any concepts for handling this kind of "gradedness". We only know that the classical result is not applicable, since R is not reflexive.

It was actually S. Gottwald who first attempted to eliminate this eyesore by introducing what he called "graded properties of fuzzy relations" [39, 40, 41]. Let us shortly recall these ideas in the light of the above example. For instance, Gottwald defined the *degree of reflexivity* of a fuzzy relation R as

$$\mathrm{Refl}(R) = \inf_{x \in U} R(x,x)$$

and the *degree of inclusion* with respect to a left-continuous t-norm $*$ (originally introduced in [1]) as

$$R_1 \subseteq_* R_2 = \inf_{x,y \in U} (R_1(x,y) \Rightarrow_* R_2(x,y)),$$

where $(x \Rightarrow_* y) = \sup\{u \in [0,1] \mid x * u \leq y\}$ is the *residual implication* of $*$. Then it is straightforward to prove the following result

$$\mathrm{Refl}(R) \leq (R \subseteq_* R \circ_* R) \tag{1}$$

which perfectly confirms the results that we obtained for the above example, as we have $\text{Refl}(R) = a$ and (by $x \Rightarrow_{\text{Ł}} y = \min(1, 1 - x + y)$)

$$(R \subseteq_{\text{Ł}} R \circ_{\text{Ł}} R) = \min(1, 1 - a + a') = \min(1, 1 - a + 2a - 1) = a.$$

Even though these ideas seem meaningful and natural, Gottwald's approach unfortunately found only little resonance (exceptions are, for instance, [13, 48]). What may be the reasons? In our humble opinion, the following facts may have contributed to the reluctance of the research community to adopt and advance Gottwald's ideas: although Gottwald's syntax is geared to classical mathematics for better readability, he is not using a full-fledged axiomatic framework and is not strictly separating syntax from semantics. As in our example above, he has to refer to the operations used (t-norms, etc.) explicitly. Thus the results that he obtains are already quite difficult to prove, but still too basic to provide solid argumentation in favor of a full-fledged graded theory of fuzzy relations.

This paper aims at reviving and advancing Gottwald's highly valuable ideas, although we take a slightly different approach. We use the formal axiomatic framework of Fuzzy Class Theory (FCT), introduced in [5]. Fuzzy Class Theory is a powerful and expressive, yet easy-to-read and easy-to-handle, framework for fuzzy mathematics in which it is just natural to consider properties of fuzzy relations in a graded manner. In Fuzzy Class Theory, most notions are inspired by (and derived from) the corresponding notions of classical mathematics [6]; furthermore, the syntax of Fuzzy Class Theory is close to the syntax of classical mathematical theories; and also the proofs in Fuzzy Class Theory resemble the classical proofs of the corresponding classical theorems. Therefore, in FCT it is technically easier to handle graded properties of fuzzy relations than in Gottwald's previous works. Thus we are able to access deeper results than what was possible in Gottwald's framework.

This paper is organized as follows. In Section 2, we first highlight how to read results in FCT, as the language of Fuzzy Class Theory may be unusual for some readers. Section 3 is concerned with basic graded properties of fuzzy relations, which mainly means rephrasing existing results on graded properties of fuzzy relations in the frame of Fuzzy Class Theory. Section 4 deals with images under fuzzy relations in the graded framework, including closures and opening operators, whereas Section 5 deals with bounds, maxima, and suprema. Section 6 generalizes the classical representation theorems due to Valverde [68] to the graded framework. In Section 7, we finally generalize the well-known links between fuzzy equivalence relations and fuzzy partitions to the graded framework. Throughout the whole paper, we will highlight links between the graded approach presented here and the existing results available in the literature. Where possible and meaningful, we provide concrete non-trivial examples.

The aim of this paper is to demonstrate that Fuzzy Class Theory is a powerful and easy-to-use instrument for handling fuzzified properties of fuzzy relations. As this paper has the appellative sub-title "Initial Steps", we do not aim at rephrasing the whole theory of fuzzy relations (or the whole existing theory of crisp relations, which is even much larger). Instead, we attempt to provide a kind of illustrative kick-off by picking out some representative results, with a strong emphasis on two of the most important classes of binary fuzzy relations—fuzzy preorders and fuzzy equivalence relations.

2 Preliminaries

Fuzzy Class Theory aims at axiomatizing the notion of fuzzy set. A brief overview of FCT can be found in Appendix B, where also all necessary definitions and conventions freely used in the following sections are introduced. For a detailed account of the theory we refer the reader to the original paper [5] or a freely available primer [7]. In the present section we only give a brief dictionary explaining how formulae of FCT can be translated to a more traditional language of fuzzy set theory, and highlight some peculiar features of FCT that play a role in formal reasoning about the graded properties of fuzzy relations.

2.1 A brief dictionary

We aim this paper at researchers in the theory and applications of fuzzy relations to attract their interest in graded theories of fuzzy relations. In the traditional theory of fuzzy relations, it is not usual to separate formal syntax from semantics as it is the case in FCT. So it may be difficult for some readers who are new to FCT to follow the results. Therefore, we would like to provide the readers with a dictionary that improves understanding of the results in this paper and that demonstrates how the results would translate to the traditional language of fuzzy relations.

FCT strictly distinguishes between its syntax and semantics. This feature has two important consequences:

- To keep the distinction (and also for certain metamathematical reasons, see [7, Section 1.1]), the objects of the formal theory are called *fuzzy classes* rather than fuzzy sets. The name *fuzzy set* is reserved for membership functions in the *models* of the theory (see Appendix B). Nevertheless (in virtue of the soundness of FCT with respect to its models composed of traditional fuzzy sets), the theorems of FCT about fuzzy classes are always valid for fuzzy *sets* and fuzzy relations. Thus, whenever we speak of classes, the reader can always safely substitute usual fuzzy sets for our "classes".

- FCT screens off direct references to truth values; truth degrees belong to the *semantics* of FCT, rather than to its syntax (this ensures that FCT renders fuzzy sets as a primitive notion instead of modeling them by membership functions). Thus, there are *no variables for truth degrees* in the language of FCT. The degree to which an element x belongs to a fuzzy class A is expressed simply by the atomic formula $x \in A$ (which can alternatively be written in a more traditional way as Ax).

The algebraic structure behind the semantics of FCT are $\mathrm{MTL}_\triangle$-chains [33]. All results in this paper hold for fuzzy sets over any $\mathrm{MTL}_\triangle$-chain. As noted in Appendix A, if the domain of truth values is the unit interval $[0,1]$, $\mathrm{MTL}_\triangle$-chains are characterized as algebras

$$([0,1], *, \Rightarrow, \min, \max, 0, 1, \triangle),$$

where $*$ is a left-continuous t-norm, $\Rightarrow$ is its residual implication, and $\triangle$ is a unary operation defined as

$$\triangle x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This means that we can translate the results to the language of fuzzy relations as in Table 1, where we may specify an arbitrary universe of discourse U, a left-continuous t-norm $*$, its residuum $\Rightarrow$.

Let us now shortly consider some examples of definitions and results. For instance, the truth degree of $A \subseteq B$ (defined in FCT by the formula $(\forall x)(x \in A \rightarrow x \in B)$, see

Table 1: Dictionary of FCT terms.

FCT	**Fuzzy relations**
object variable x	element $x \in U$
(fuzzy) class variable A	fuzzy set $A \in \mathcal{F}(U)$
variable for a (fuzzy) class of (fuzzy) classes $\mathcal{A}$	fuzzy set $\mathcal{A} \in \mathcal{F}(\mathcal{F}(U))$
unary predicate symbol	fuzzy subset of U, $\mathcal{F}(U)$, $\mathcal{F}(\mathcal{F}(U))$, etc.
n-ary predicate symbol	(n-ary) fuzzy relation on U^n, $(\mathcal{F}(U))^n$, etc.
strong conjunction &	t-norm $*$
implication $\rightarrow$	residual implication $\Rightarrow$
weak conjunction $\wedge$	minimum
weak disjunction $\vee$	maximum
negation $\neg$	the function $\neg x = (x \Rightarrow 0)$
equivalence $\leftrightarrow$	bi-residuum, i.e., $\min(x \Rightarrow y, y \Rightarrow x)$
universal quantifier $\forall$	infimum
existential quantifier $\exists$	supremum
predicate $=$	crisp identity
predicate $\in$	evaluation of membership function
class term $\{x \mid \varphi(x)\}$	fuzzy set defined as $Ax = \varphi(x)$, for all $x \in U$

Definition B.5) is, in an MTL$_\triangle$-chain, computed as

$$\inf_{x \in U} (Ax \Rightarrow Bx)$$

which is a well-known concept of fuzzy inclusion (see [1, 13, 16, 40] and many more). The degree of reflexivity Refl(R), defined in Section 3 as $(\forall x)Rxx$, is nothing else but

$$\inf_{x \in U} Rxx.$$

As another example (cf. Definition B.4), it is easy to see that Ker(A) for some fuzzy set A exactly gives the crisp set of all values $x \in U$ for which $Ax = 1$ holds. Analogously (see Definition B.5), Norm(A) evaluates to 1 if and only if there exists an $x \in U$ such that $Ax = 1$ holds and to 0 otherwise.

The question remains how the theorems in the following sections can be read in a graded way (although they do not necessarily look graded at first glance). In traditional (fuzzy) logic, a theorem is read as follows:

> *If* some (non-graded) assumption is true (i.e., fully true, since non-graded),
> *then* some (non-graded) conclusion is (fully) true.

If we can prove an implication in FCT, by soundness, this implication always holds to degree 1. Now take into account that, in all MTL$_\triangle$-chains (comprising all standard MTL$_\triangle$-chains), the following correspondence holds:

$$(x \Rightarrow y) = 1 \text{ if and only if } x \leq y.$$

So an implication that we can prove in FCT can be read as follows:

> *The more* some (graded) assumption is true (even if partially),
> *the more* some (graded) conclusion is true(i.e., at least as true as the assumption).

In other words, the truth degree of an assumption is a lower bound for the truth degree of the conclusion in provable implications.

Thus, for instance, the assertion (R13) of Theorem 3.6 easily translates into our motivating example (1).

Remark 2.1. To motivate and illustrate the results in this paper, we will use a significant number of examples. In order to make them compact and readable, we will, *in examples,* deviate from our principle to keep formulae separate from their semantics. Instead of mentioning models over some logics, we will simply say that we use some standard logic, for instance, standard Łukasiewicz logic (standing for the standard $\mathrm{MTL}_\triangle$-chain induced by the Łukasiewicz t-norm; analogously for other logics). In examples, we shall furthermore not distinguish between predicate symbols and the fuzzy sets or relations that model them. Instead of saying that a certain model of a fuzzy predicate R fulfills reflexivity to a degree of 0.8, we will simply write $\mathrm{Refl}(R) = 0.8$. This is not the cleanest way of writing it, but it is short and expressive, and it should always be clear to the reader what is meant.

2.2 Some precautions

It can be observed that the defining formulae of most notions in FCT are exactly the same as the definitions of these properties for crisp relations in classical mathematics. This correlates with the motivation of fuzzy logic as generalization of classical logic to non-crisp predicates: classical mathematical notions are then fuzzified in a natural way just by interpreting the classical definitions in fuzzy logic. This methodology has been foreshadowed in [44, Section 5] by Höhle, much later formalized in [5, Section 7], and suggested as a general principle for formal fuzzy mathematics in [6].

Nevertheless, although such a translation of notions of classical mathematics into FCT is an important guideline, the method cannot be applied mechanically, as some classically equivalent definitions may no longer be equivalent in the logic $\mathrm{MTL}_\triangle$. In some cases, the most suitable version of the definition can be chosen; in other cases, a notion of classical mathematics splits into several meaningful notions in FCT. This can be exemplified by the notion of equality of fuzzy classes:

Besides the primitive crisp identity $=$ of fuzzy classes, at least two graded notions of natural fuzzy equality, $\approx$ and $\approxeq$, can be defined (see Definition B.5). Both of these notions have already appeared in the fuzzy literature. For instance Gottwald [41] uses $\approxeq$ while Bělohlávek [13] uses $\approx$ for graded equality of fuzzy classes. The two notions are not equivalent in FCT, as the following counter-example demonstrates.

Example 2.2. Let us consider a two-element set $U = \{x, y\}$ and standard Łukasiewicz logic. Let us consider two fuzzy sets $A, B \in \mathcal{F}(U)$ defined as $Ax = By = 1$ and $Ay = Bx = 0.5$. Then the truth value of $A \approx B$ is 0.5, while the truth value of $A \approxeq B$ is 0.

Only the following relationships hold between these notions.

Theorem 2.3. *The following theorems are provable in* FCT*:*

(L1) $\quad A \approx B \leftrightarrow (A \subseteq B \wedge B \subseteq A)$

(L2) $\quad A \approx^2 B \longrightarrow A \approxeq B \longrightarrow A \approx B$

(L3) $\quad \triangle(A \approx B) \longleftrightarrow \triangle(A \approxeq B) \longleftrightarrow A = B$

Proof. We give the proof of this lemma in full detail; proofs in the following sections will usually be more compressed and easy steps will be omitted.

(L1) By Definition B.5 and the rule of distribution of $\forall$ over $\wedge$ (which is provable in $\mathrm{MTL}_\triangle$), we have

$$\begin{aligned} A \approx B \quad &\longleftrightarrow \quad (\forall x)(Ax \leftrightarrow Bx) \longleftrightarrow (\forall x)((Ax \to Bx) \wedge (Bx \to Ax)) \\ &\longleftrightarrow \quad (\forall x)(Ax \to Bx) \wedge (\forall x)(Bx \to Ax) \longleftrightarrow A \subseteq B \wedge B \subseteq A. \end{aligned}$$

(L2) We have the following:

$$\begin{aligned} A \approx^2 B \quad &\longleftrightarrow \quad (\forall x)(Ax \leftrightarrow Bx) \,\&\, (\forall x)(Ax \leftrightarrow Bx) \\ &\longrightarrow \quad (\forall x)(Ax \to Bx) \,\&\, (\forall x)(Bx \to Ax) \\ &\longleftrightarrow \quad A \subseteq B \,\&\, B \subseteq A \longleftrightarrow A \approxeq B \end{aligned}$$

Moreover, $A \subseteq B \,\&\, B \subseteq A \longrightarrow A \subseteq B \wedge B \subseteq A \longleftrightarrow A \approx B$ by (L1).

(L3) The first equivalence follows from (L2) by the rule of $\triangle$-necessitation (see Appendix A) and distribution of $\triangle$ over $\to$ and &, which is provable in propositional $\mathrm{MTL}_\triangle$. The second equivalence can be proved as

$$\triangle(A \approx B) \longleftrightarrow \triangle(\forall x)(Ax \leftrightarrow Bx) \longleftrightarrow (\forall x)\triangle(Ax \leftrightarrow Bx) \longleftrightarrow A = B$$

by the axiom of extensionality (see Definition B.1). □

Let us add some comments on the meaning of the previous theorem. By definition, the "strong" bi-inclusion $A \approxeq B$ is $A \subseteq B \,\&\, B \subseteq A$; compare it with "weak" bi-inclusion $A \approx B$, which by (L1) just uses weak conjunction $\wedge$ instead of &. Indeed, by the second implication of (L2), $\approxeq$ is stronger than $\approx$. Notice further that (L2) in fact says that the truth value of $A \approxeq B$ is bounded by the truth values of $A \approx^2 B$ (a lower bound) and $A \approx B$ (an upper bound). In traditional non-graded fuzzy mathematics both notions coincide, since they are fully true under the same conditions, as shown by (L3); however, under the graded approach they differ, since in graded fuzzy mathematics we do not require them to be true to degree 1. This relationship between two related, but non-equivalent notions is quite common in graded fuzzy mathematics and will be met several times in this paper.[1]

Finally, it should be pointed out that, unlike in classical Boolean logic, in fuzzy logic it does make a difference how many times an assumption is used to prove a certain conclusion. For instance, if we have to use an assumption φ twice to prove a conclusion ψ, this means

$$\varphi \to (\varphi \to \psi).$$

So finally, by the axiom (A5a) of $\mathrm{MTL}_\triangle$ (see Appendix A), we have proved $\varphi^2 \to \psi$, but it need not be possible to prove $\varphi \to \psi$. Such situations will occur frequently in this paper. For instance, Example 2.2 shows that $A \approx B \to A \approxeq B$ indeed does not hold in FCT, even though $A \approx^2 B \to A \approxeq B$ is provable by (L2).

The warnings listed above may appear as oddities that somehow spoil the beauty and quality of FCT. Our opinion is, however, that exactly the opposite is the case. Otherwise, this paper could only reproduce and slightly generalize crisp results with analogous proofs, without creating really new results. However, due to the above features, FCT indeed allows to derive new, previously unknown results.

[1] In [8], it is shown that it occurs regularly under certain conditions in graded generalizations of non-graded theorems.

3 Basic properties of fuzzy relations

As announced above, the first item on the agenda of this paper is to embed existing results on so-called graded properties of fuzzy relations into the framework of FCT. Such properties were introduced first by S. Gottwald in 1991 [39]. Later on, he extended this research in his 1993 book [40]; his more recent book [41] contains an up-to-date review of the topic. Properties of fuzzy relations are studied in the graded manner also in Bělohlávek's book [13]. The idea of graded properties of fuzzy relations had also been followed by Jacas and Recasens [48]. In this section, we closely follow the structure and philosophy of [41, Section 18.6].

Definition 3.1. In FCT, we define basic properties of fuzzy relations as follows:

$$\begin{array}{rcll}
\mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)Rxx & \text{reflexivity} \\
\mathrm{Irrefl}(R) & \equiv_{\mathrm{df}} & (\forall x)\neg Rxx & \text{irreflexivity} \\
\mathrm{Sym}(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \to Ryx) & \text{symmetry} \\
\mathrm{Trans}(R) & \equiv_{\mathrm{df}} & (\forall x,y,z)(Rxy \mathbin{\&} Ryz \to Rxz) & \text{transitivity} \\
\mathrm{AntiSym}_{(E)}(R) & \equiv_{\mathrm{df}} & (\forall x,y)(Rxy \mathbin{\&} Ryx \to Exy) & (E)\text{-antisymmetry} \\
\mathrm{ASym}(R) & \equiv_{\mathrm{df}} & (\forall x,y)\neg(Rxy \mathbin{\&} Ryx) & \text{asymmetry}
\end{array}$$

Note that we slightly deviate from Gottwald in the definition of antisymmetry, which we generalize by defining it with respect to some relation E (usually a similarity). We adopt this idea from so-called similarity-based orderings which have turned out to be more suitable concepts of fuzzy orderings [47, 16]. Let us adopt the convention that the index E is dropped if $E = \mathrm{Id}$ (then it coincides with the concept of antisymmetry that Gottwald uses). Also note that some authors, e.g., [44, 13], use the minimum conjunction $\wedge$ in the definition of antisymmetry instead of the strong conjunction &. However, arguments can be given [4] that, from the deductive point of view, the strong conjunction is appropriate in the definition of antisymmetry and that the stronger definition with $\wedge$ does not express an intuitive notion of antisymmetry.

Remark 3.2. Obviously, all of the above properties remain unchanged if we replace R with its inverse relation R^{-1} (in case of (E)-antisymmetry, we also have to invert E). Hence, we can infer the following trivial correspondences:

$$\begin{array}{ll}
\mathrm{Refl}(R^{-1}) \leftrightarrow \mathrm{Refl}(R) & \mathrm{Irrefl}(R^{-1}) \leftrightarrow \mathrm{Irrefl}(R) \\
\mathrm{Sym}(R^{-1}) \leftrightarrow \mathrm{Sym}(R) & \mathrm{Asym}(R^{-1}) \leftrightarrow \mathrm{Asym}(R) \\
\mathrm{Trans}(R^{-1}) \leftrightarrow \mathrm{Trans}(R) & \mathrm{AntiSym}_{(E^{-1})}(R^{-1}) \leftrightarrow \mathrm{AntiSym}_{(E)}(R)
\end{array}$$

Example 3.3. Let us start with a simple example to illustrate the concepts introduced above. Consider the domain $U = \{1, \ldots, 6\}$ and the following fuzzy relation (for convenience, in matrix notation):

$$P_1 = \begin{pmatrix}
1.0 & 1.0 & 0.5 & 0.4 & 0.3 & 0.0 \\
0.8 & 1.0 & 0.4 & 0.4 & 0.3 & 0.0 \\
0.7 & 0.9 & 1.0 & 0.8 & 0.7 & 0.4 \\
0.9 & 1.0 & 0.7 & 1.0 & 0.9 & 0.6 \\
0.6 & 0.8 & 0.8 & 0.7 & 1.0 & 0.7 \\
0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0
\end{pmatrix}$$

It can be checked easily that P_1 is a fuzzy preorder with respect to the Łukasiewicz t-norm $\max(x + y - 1, 0)$, hence, taking standard Łukasiewicz logic, we obtain $\mathrm{Refl}(P_1) = 1$

and $\text{Trans}(P_1) = 1$. In this setting, one can easily infer $\text{Sym}(P_1) = 0.4$ (note that for a finite fuzzy relation R, in standard Łukasiewicz logic, $\text{Sym}(R)$ is nothing else but 1 minus the largest difference between two values Rxy and Ryx) as well as $\text{Irrefl}(P_1) = 0$ and $\text{Asym}(P_1) = 0$.

Now let us see what happens if we add some disturbances to P_1. We added normally distributed pseudo-random numbers to the above table (with zero mean and a standard deviation of 0.05) and truncated these values to the unit interval. Finally, we rounded the values to two digits and obtained the following fuzzy relation:

$$P_2 = \begin{pmatrix} 1.00 & 1.00 & 0.56 & 0.40 & 0.30 & 0.00 \\ 0.87 & 1.00 & 0.33 & 0.44 & 0.26 & 0.02 \\ 0.67 & 0.92 & 0.93 & 0.87 & 0.70 & 0.39 \\ 0.93 & 1.00 & 0.64 & 1.00 & 0.97 & 0.67 \\ 0.52 & 0.79 & 0.82 & 0.71 & 1.00 & 0.59 \\ 0.27 & 0.50 & 0.61 & 0.41 & 0.72 & 1.00 \end{pmatrix}$$

Then simple computations give the following results: $\text{Refl}(P_2) = 0.93$, $\text{Irrefl}(P_2) = 0$, $\text{Sym}(P_2) = 0.41$, $\text{Trans}(P_2) = 0.85$, and $\text{Asym}(P_2) = 0$.

Example 3.4. Now consider $U = \mathbb{R}$ and let us define the following parameterized class of fuzzy relations (with $a, c > 0$):

$$E_{a,c}xy = \min(1, \max(0, a - \tfrac{1}{c}|x - y|))$$

It is well known that, for $a = 1$, we obtain fuzzy equivalence relations with respect to the Łukasiewicz t-norm [67, 68, 25, 27], hence, using standard Łukasiewicz logic again, $\text{Refl}(E_{1,c}) = 1$, $\text{Sym}(E_{1,c}) = 1$, and $\text{Trans}(E_{1,c}) = 1$ for all $c > 0$. It is also well-known and easy to see that, for $a < 1$, reflexivity in the non-graded manner cannot be maintained. Actually, we obtain

$$\text{Refl}(E_{a,c}) = \min(1, a).$$

for all $a, c > 0$. Similarly, it is a well-known fact that, for $a > 1$, transitivity in the non-graded sense is violated. This is a fact that, in some sense, has its roots in the Poincaré paradox [62, 63]. Note that relations like $E_{a,c}$ (for $a \geq 1$) appear prominently in De Cock and Kerre's framework of *resemblance relations* [28]. Regarding graded transitivity, we obtain the following:

$$\text{Trans}(E_{a,c}) = \min(1, \max(0, 2 - a))$$

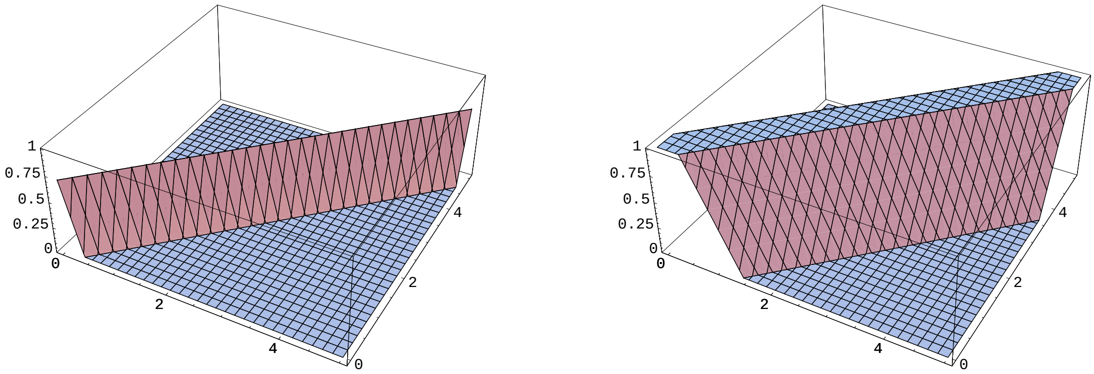
Observe that $\text{Trans}(E_{a,c})$ does not depend on c either. This is not surprising, however, because the parameter c only corresponds to a re-scaling of the domain. Finally, let us mention the following results (for all $a, c > 0$):

$$\begin{aligned} \text{Irrefl}(E_{a,c}) &= \max(0, 1 - a) \\ \text{Sym}(E_{a,c}) &= 1 \\ \text{Asym}(E_{a,c}) &= \min(1, \max(0, 2 - 2a)) \end{aligned}$$

We can conclude that the larger a, the more reflexive, but less irreflexive, asymmetric, and transitive, $E_{a,c}$ is. Figure 1 shows two examples.

The next lemma provides us with some results that will be helpful in the following. Note that it is actually a corollary of a general result [22, Theorem 3.5]. Here we give a direct proof. Some weaker variants can be obtained from [13, Lemma 4.8].

Figure 1: The fuzzy relations $E_{0.7,2}$ (left) and $E_{1.4,1}$ (right). From Example 3.4, we can infer that $\mathrm{Refl}(E_{0.7,2}) = 0.7$, $\mathrm{Trans}(E_{0.7,2}) = \mathrm{Refl}(E_{1.4,1}) = 1$, and $\mathrm{Trans}(E_{1.4,1}) = 0.6$.



Lemma 3.5. In FCT, we can prove the following:

(R1) $R \subseteq S \to (\mathrm{Refl}(R) \to \mathrm{Refl}(S))$

(R2) $S \subseteq R \to (\mathrm{Irrefl}(R) \to \mathrm{Irrefl}(S))$

(R3) $R \approxeq S \to (\mathrm{Sym}(R) \to \mathrm{Sym}(S))$

(R4) $R \subseteq S \;\&\; S \subseteq^2 R \to (\mathrm{Trans}(R) \to \mathrm{Trans}(S))$

(R5) $S \subseteq^2 R \to (\mathrm{AntiSym}_{(E)}(R) \to \mathrm{AntiSym}_{(E)}(S))$

(R6) $S \subseteq^2 R \to (\mathrm{ASym}(R) \to \mathrm{ASym}(S))$

Proof. Here we prove just (R4), the others are analogous. Obviously $S \subseteq^2 R \to (Sxy \,\&\, Syz \to Rxy \,\&\, Ryz)$. So by $\mathrm{Trans}(R)$ we get $S \subseteq^2 R \to (Sxy \,\&\, Syz \to Rxz)$, and since $R \subseteq S \to (Rxz \to Sxz)$, we get $R \subseteq S \;\&\; S \subseteq^2 R \to (Sxy \,\&\, Syz \to Sxz)$. Generalization over x, y, z and quantifier shifts then complete the proof. □

The following theorem provides us with a few basic results. Most of them are obvious translations of results that can be found in [41, Proposition 18.6.1], where (R11) has been extended to the more general concept of antisymmetry with respect to a fuzzy relation E (as noted above, this is in line with the similarity-based approach to fuzzy orderings [47, 16]) and (R13) is new in the graded framework (yet well-known in the non-graded theory of fuzzy relations).

Theorem 3.6. *The following theorems are provable in* FCT*:*

(R7) $\mathrm{Refl}(R) \leftrightarrow \mathrm{Id} \subseteq R$

(R8) $\mathrm{Irrefl}(R) \leftrightarrow \mathrm{Id} \cap R \approx \emptyset$

(R9) $\mathrm{Trans}(R) \leftrightarrow R \circ R \subseteq R$

(R10) $\mathrm{Sym}(R) \leftrightarrow R^{-1} \subseteq R$

(R11) $\mathrm{AntiSym}_{(E)}(R) \leftrightarrow R \cap R^{-1} \subseteq E$

(R12) $\mathrm{Asym}(R) \leftrightarrow R \cap R^{-1} \approx \emptyset$

(R13) $\mathrm{Refl}(R) \to R \subseteq R \circ R$

Proof. We omit the obvious and concentrate on the following non-trivial issues:

(R9) Obviously, $\langle x,y\rangle \in (R\circ R) \leftrightarrow (\exists z)(Rxz \,\&\, Rzy)$. Then, by Trans($R$) we get $(\exists z)(Rxy)$, which is just Rxy. Now let us prove the converse direction: for any x, y, we have that $(\exists z)(Rxz \,\&\, Rzy) \to Rxy$. Then the rule of quantifier shift completes the proof.

(R10) Starting from $R^{-1}xy$, i.e., Ryx, by Sym(R) we get Rxy. The other direction is trivial.

(R13) $Rxx \,\&\, Rxy \to (\exists z)(Rxz \,\&\, Rzy)$. Thus $Rxx \to (Rxy \to (\exists z)(Rxz \,\&\, Rzy))$. □

The following theorem collects several results that can be found in [41] as well (Propositions 18.6.1–18.6.5).

Theorem 3.7. *The following theorems are provable in* FCT*:*

(R14) $\mathrm{Refl}(R \sqcup \mathrm{Id})$

(R15) $\mathrm{Irrefl}(R \setminus \mathrm{Id})$

(R16) $\mathrm{Trans}(R) \to \mathrm{Trans}(R \sqcup \mathrm{Id})$

(R17) $\mathrm{Trans}(R \setminus \mathrm{Id}) \to \mathrm{Trans}(R)$

(R18) $\mathrm{Trans}(R) \,\&\, \mathrm{AntiSym}(R) \to \mathrm{Trans}(R \setminus \mathrm{Id})$

(R19) $\mathrm{AntiSym}(R) \to \mathrm{ASym}(R \setminus \mathrm{Id})$

(R20) $\mathrm{ASym}(R \setminus \mathrm{Id}) \leftrightarrow \mathrm{AntiSym}(R \setminus \mathrm{Id})$

(R21) $\mathrm{ASym}(R) \to \mathrm{AntiSym}(R \sqcup \mathrm{Id})$

(R22) $\mathrm{Trans}(R) \,\&\, \mathrm{Irrefl}(R) \to \mathrm{ASym}(R)$

(R23) $\mathrm{Trans}(R) \,\&\, \mathrm{Trans}(Q) \to \mathrm{Trans}(R \cap Q)$

Proof. For brevity, we again omit trivial and obvious parts.

(R15) $\langle x,x\rangle \in (R \setminus \mathrm{Id}) \longleftrightarrow Rxx \,\&\, x \neq x \longleftrightarrow 0$.

(R16) Observe that for $x \neq y$ we have $\langle x,y\rangle \in (R \sqcup \mathrm{Id}) \leftrightarrow Rxy$. We start from $\langle x,y\rangle \in (R \sqcup \mathrm{Id})$ and $\langle y,z\rangle \in (R \sqcup \mathrm{Id})$ and distinguish four cases: if $x = y$ and $y = z$ then $x = z$ and so $\langle x,z\rangle \in (R \sqcup \mathrm{Id})$. If $x = y$ and $y \neq z$, then we have Rxz, thus obviously $\langle x,z\rangle \in (R \sqcup \mathrm{Id})$. The case $x \neq y$ and $y = z$ is analogous. The last case is just the transitivity of R.

(R17) We start from $Rxy \,\&\, Ryz$. If $x \neq y \,\&\, y \neq z$ we get Rxz using $\mathrm{Trans}(R \setminus \mathrm{Id})$. The cases that either $x = y$ or $y = z$ are trivial.

(R18) Observe that if $x \neq y$ we have $\langle x,y\rangle \in (R \setminus \mathrm{Id}) \leftrightarrow Rxy$. Start from $\langle x,y\rangle \in (R \setminus \mathrm{Id})$ and $\langle y,z\rangle \in (R \setminus \mathrm{Id})$. Again we distinguish four cases: the only non-trivial one is $x \neq y$ and $y \neq z$. Thus we have Rxy and Ryz, observe that from AntiSym(R) we get that $z \neq x$ (because $z = x$ would give $x = y$).

(R19) $(\langle x,y\rangle \in (R \setminus \mathrm{Id})) \,\&\, (\langle y,x\rangle \in (R \setminus \mathrm{Id})) \longleftrightarrow (Rxy \,\&\, Ryx \,\&\, x \neq y) \longrightarrow (x = y \,\&\, x \neq y) \longleftrightarrow 0$ (in the second step we used AntiSym(R)).

(R22) From Trans(R) we get $Rxy \& Ryx \to Rxx$, which leads to $\neg Rxx \to \neg(Rxy \& Ryx)$. As we have $\neg Rxx$ from Irrefl(R), the proof is done.

(R23) From $Rxy \,\&\, Ryz \to Rxz$ and $Qxy \,\&\, Qyz \to Qxz$ we immediately get $Rxy \,\&\, Ryz \,\&\, Qxy \,\&\, Qyz \to Rxz \,\&\, Qxz$ which is the same as $(R \cap Q)xy \,\&\, (R \cap Q)yz \to (R \cap Q)xz$. □

Example 3.8. Consider standard Łukasiewicz logic and the following family of fuzzy relations (with $a \in [0,1]$ and $U = \mathbb{R}$):

$$L_a xy = \min(1, \max(0, a - x + y))$$

Easy computations show that the fuzzy relations L_a are transitive for all $a \in [0,1]$ (i.e., $\mathrm{Trans}(L_a) = 1$). Obviously, L_1 is also reflexive, so it is a fuzzy preorder [16], and L_0 is irreflexive, hence a typical fuzzy strict order [19, 36, 61]. Generally, we obtain $\mathrm{Refl}(L_a) = a$ and $\mathrm{Irrefl}(L_a) = 1 - a$. Therefore, we can conclude by (R22) that $\mathrm{Asym}(L_a) \geq 1 - a$ for all $a \in [0,1]$. This is only a lower bound, however. It is possible to show that

$$\mathrm{Asym}(L_a) = \min(1, \max(0, 2 - 2a))$$

holds (compare with Example 3.4). This demonstrates that under transitivity, irreflexivity is indeed a stronger requirement than asymmetry. In the non-graded framework, this is an essential fact for simplifying the definition of strict fuzzy orders [19].

Now we turn our attention to the property of extensionality of a fuzzy class with respect to a fuzzy relation. Previously, extensionality was defined as a crisp property that a given fuzzy set either had or had not [18, 51, 52, 53]. In FCT, we can generalize extensionality to the graded framework effortlessly. (See [3] for the changed role of extensionality in the fully graded theory of fuzzy relations.)

Definition 3.9. In FCT, we define the (degree of) extensionality of a fuzzy class A with respect to a fuzzy relation E as

$$\mathrm{Ext}_E(A) \;\equiv_{\mathrm{df}}\; (\forall x, y)(Exy \mathbin{\&} x \in A \to y \in A).$$

In the non-graded framework, it is well-known that inf-intersections and sup-unions of families of extensional fuzzy sets are also extensional [18, 51, 52, 53]. The following theorem states that a similar result holds in the graded framework.

Theorem 3.10. *The following theorems are provable in* FCT*:*

(R24) $(\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}) \mathbin{\&} (\forall A \in \mathcal{J})\, \mathrm{Ext}_E(A) \to \mathrm{Ext}_E(\bigcap \mathcal{J})$

(R25) $(\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}) \mathbin{\&} (\forall A \in \mathcal{J})\, \mathrm{Ext}_E(A) \to \mathrm{Ext}_E(\bigcup \mathcal{J})$.

Proof. By Lemma B.8 (L16) and (L17) we have $(\forall A \in \mathcal{J})\, \mathrm{Ext}_E(A) \longrightarrow (Exy \to (\forall A \in \mathcal{J})(x \in A \to y \in A)) \longrightarrow (Exy \to ((\forall A \in \mathcal{J})(x \in A) \to (\forall A \in \mathcal{J} \cap \mathcal{J})(y \in A)))$. Now from $\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}$ we get $A \in \mathcal{J} \to A \in \mathcal{J} \cap \mathcal{J}$, and as $(\forall A \in \mathcal{J})(x \in A)$ is exactly $x \in \bigcap \mathcal{J}$, the proof of (R24) is done. The proof of (R25) is analogous, only we use (L18) instead of (L17). □

Remark 3.11. It is easy to see that the condition $\mathcal{J} \subseteq \mathcal{J} \cap \mathcal{J}$ in the previous theorem is satisfied to degree 1 in models if and only if $A \in \mathcal{J}$ only acquires truth values that are idempotent with respect to conjunction. In particular, it is always true for crisp classes $\mathcal{J}$, and in Gödel logic for all classes. In standard Łukasiewicz logic, the condition expresses the closeness of $\mathcal{J}$ to crispness (it gets large truth values if and only if all truth values of $A \in \mathcal{J}$ are close to 0 or to 1). Thus, in Ł, the theorem expresses the fact that the property of extensionality is "almost closed" under intersections and unions of "almost crisp" families of classes. In standard product logic, the situation is similar, but the condition is much stricter in smaller truth values: it gets a large truth value if and only if $A \in \mathcal{J}$ is either equal to 0, or close to 1.

The condition of the form $X \subseteq X \cap X$ is encountered quite often in graded fuzzy mathematics (cf. for instance (C28) of Theorem 5.18 below); we could call it the *(graded) 2-contractiveness* of X.

In particular, Theorem 3.10 includes the case of crisp two-element families of fuzzy classes [18, 51, 52, 53].

Corollary 3.12. *The following theorems are provable in* FCT*:*

(R26) $\mathrm{Ext}_E(A) \wedge \mathrm{Ext}_E(B) \to \mathrm{Ext}_E(A \sqcap B)$

(R27) $\mathrm{Ext}_E(A) \wedge \mathrm{Ext}_E(B) \to \mathrm{Ext}_E(A \sqcup B)$

Example 3.13. Let us consider $U = \mathbb{R}$, standard Łukasiewicz logic, $E_{1,1}$ from Example 3.4 and the two fuzzy sets

$$Ax = \min(\tfrac{1}{2}, \max(0, -2(x-1))) \text{ and } Bx = \min(\tfrac{2}{3}, \max(0, 2(x-2))).$$

Then we obtain $\mathrm{Ext}_{E_{1,1}}(A) = \frac{3}{4}$ and $\mathrm{Ext}_{E_{1,1}}(B) = \frac{2}{3}$. The two fuzzy sets A and B are disjoint, i.e. $A \sqcap B = \emptyset$, hence, $\mathrm{Ext}_{E_{1,1}}(A \sqcap B) = \mathrm{Ext}_{E_{1,1}}(\emptyset) = 1$. This fact underlines that (R26) and (R27) provide us with lower bounds for the extensionality of intersections/unions, but these bounds need not always be very helpful.

In classical mathematics, special properties of relations are rarely studied completely independently of each other. Instead, these properties most often occur in some combinations in the definitions of special classes of relations—with (pre)orders and equivalence relations being two most fundamental examples. The same is true in the theory of fuzzy relations, where fuzzy (pre)orders and fuzzy equivalence relations are the most important classes. Compound properties of this kind are defined as conjunctions of some of the simple properties of Definition 3.1. In the non-graded case, the properties are crisp, so the conjunction we need is the classical Boolean conjunction. In FCT, however, the properties are graded, so it indeed matters which conjunction we take. Thus, besides the (more usual) combinations by strong conjunction & (corresponding to the t-norm in the standard case), we also define their weak variants combined by weak conjunction (corresponding to the minimum). In this paper, we restrict ourselves to investigation of basic properties of fuzzy preorders and similarities.[2]

Definition 3.14. In FCT we define the following compound properties of fuzzy relations:

$\mathrm{Preord}(R)$	$\equiv_{\mathrm{df}}$	$\mathrm{Refl}(R)\ \&\ \mathrm{Trans}(R)$	(strong) preorder
$\mathrm{wPreord}(R)$	$\equiv_{\mathrm{df}}$	$\mathrm{Refl}(R) \wedge \mathrm{Trans}(R)$	weak preorder
$\mathrm{Sim}(R)$	$\equiv_{\mathrm{df}}$	$\mathrm{Refl}(R)\ \&\ \mathrm{Sym}(R)\ \&\ \mathrm{Trans}(R)$	(strong) similarity
$\mathrm{wSim}(R)$	$\equiv_{\mathrm{df}}$	$\mathrm{Refl}(R) \wedge \mathrm{Sym}(R) \wedge \mathrm{Trans}(R)$	weak similarity

Example 3.15. Let us shortly revisit Example 3.3. We can conclude the following:

$\mathrm{Preord}(P_1) = 1$	$\mathrm{Preord}(P_2) = 0.78$
$\mathrm{wPreord}(P_1) = 1$	$\mathrm{wPreord}(P_2) = 0.85$
$\mathrm{Sim}(P_1) = 0.4$	$\mathrm{Sim}(P_2) = 0.19$
$\mathrm{wSim}(P_1) = 0.4$	$\mathrm{wSim}(P_2) = 0.41$

The values in the right-hand column once more demonstrate why it is justified to speak of strong and weak properties—the stronger (i.e. smaller) the conjunction, the harder a property can be fulfilled.

[2] In line with Zadeh's original work [70], we use the term *similarity (relation)* synonymously for fuzzy equivalence (relation).

For the class of fuzzy relations defined in Example 3.4, we obtain the interesting result

$$\mathrm{Preord}(E_{a,c}) = \mathrm{wPreord}(E_{a,c}) = \max(0, 1 - |1 - a|),$$

from which we can infer that $\mathrm{Preord}(E_{a,c}) = \mathrm{wPreord}(E_{a,c}) = 1$ if and only if $a = 1$. Note that $\mathrm{Sym}(E_{a,c}) = 1$, so $\mathrm{Sim}(E_{a,c}) = \mathrm{Preord}(E_{a,c})$ and $\mathrm{wSim}(E_{a,c}) = \mathrm{wPreord}(E_{a,c})$ which implies that $\mathrm{Sim}(E_{a,c}) = \mathrm{wSim}(E_{a,c}) = 1$ if and only if $a = 1$.

For the class of fuzzy relations introduced in Example 3.8, we trivially obtain the following result: $\mathrm{Preord}(L_a) = \mathrm{wPreord}(L_a) = a$ and $\mathrm{Sim}(L_a) = \mathrm{wSim}(L_a) = 0$.

Obviously $\mathrm{Preord}(R) \to \mathrm{wPreord}(R)$ and $\mathrm{Sim}(R) \to \mathrm{wSim}(R)$. From Lemma 3.5 we further obtain:

Lemma 3.16. FCT *proves:*

(R28) $R \approxeq^2 S \to (\mathrm{Preord}(R) \to \mathrm{Preord}(S))$

(R29) $R \subseteq S \mathbin{\&} S \subseteq^2 R \to (\mathrm{wPreord}(R) \to \mathrm{wPreord}(S))$

(R30) $R \approxeq^3 S \to (\mathrm{Sim}(R) \to \mathrm{Sim}(S))$

(R31) $R \subseteq S \mathbin{\&} S \subseteq^2 R \to (\mathrm{wSim}(R) \to \mathrm{wSim}(S))$

4 Images and dual images

In this section, we address images of fuzzy relations in the framework of FCT. Such operations are of central importance in fuzzy inference [10, 71], in the theory of fuzzy relational equations [23, 66], and in the study of properties of fuzzy relations, too [13, 18]. These concepts are also strongly linked with fuzzy mathematical morphology [15, 17, 56, 57].[3]

Definition 4.1. In FCT, we define the following operations:

$$\begin{aligned} R^{\uparrow} A &=_{\mathrm{df}} \{y \mid (\exists x)(x \in A \mathbin{\&} Rxy)\} \\ R^{\downarrow} A &=_{\mathrm{df}} \{x \mid (\forall y)(Rxy \to y \in A)\} \end{aligned}$$

Let us shortly clear up the terminology. In the literature, the image operator $R^{\uparrow} A$ is called *full image* [18], *direct image* [50], *conditioned fuzzy set* [10], or simply *image* of A under/with respect to R, while $R^{\downarrow} A$ appears under the names *superdirect image* [50] and *α-operation* [66]; its systematic name in [9] is *subproduct preimage*. We will simply call both operators *images*. Where necessary, we refer to $^{\downarrow}$ explicitly as *dual image*.[4]

Example 4.2. Let us consider $U = \mathbb{R}$ and the fuzzy set

$$Ax = \min(1, \max(0, \tfrac{1}{10}(x - 175))).$$

Straightforward computations then show the following (with the fuzzy relation $E_{1.5,10}$ defined as in Example 3.4):

$$\begin{aligned} (E_{1.5,10}{}^{\uparrow} A)x &= \min(1, \max(0, \tfrac{1}{10}(x - 170))) \\ (E_{1.5,10}{}^{\downarrow} A)x &= \min(1, \max(0, \tfrac{1}{10}(x - 180))) \end{aligned}$$

[3] Note that the references in this paragraph are just pointers to some important works, but do not cover all the relevant literature.

[4] The relationship between the operations $^{\uparrow}$ and $^{\downarrow}$ is in fact an instance of Morsi's duality [55] combined with the inversion duality (i.e., the duality between R and R^{-1}).

Figure 2: The fuzzy set A (middle, solid black) and the result that is obtained when applying image operators: $E_{1.5,10}{}^{\downarrow}A$ (left, light gray) and $E_{1.5,10}{}^{\uparrow}A$ (right, medium gray).

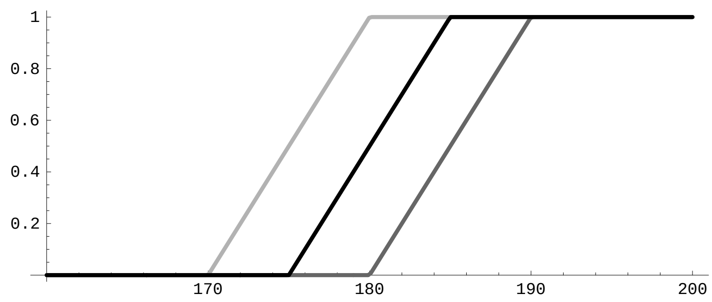


Figure 2 shows a plot of these three fuzzy sets. Note that De Cock and Kerre use the two image operators in conjunction with their resemblance relations [28] to define linguistic hedges like for instance *roughly* and *very* [29]. If we consider A as a model of *tall* (in the context of European men), we can interpret $E_{1.5,10}{}^{\uparrow}A$ as a model of *roughly tall* and $E_{1.5,10}{}^{\downarrow}A$ as a model of *very tall* according to De Cock's and Kerre's argumentation.

The next theorem clarifies some basic properties of images under fuzzy relations. Their non-graded versions are well-known and easy to prove (see e.g. [18, 40, 41]). The graded theorems (I6)–(I14) are also corollaries of more general theorems in the paper [9]; here we give their simple direct proofs.

Theorem 4.3. *The following properties of images are provable in* FCT*:*

(I1) $R^{\uparrow}\emptyset = \emptyset$

(I2) $R^{\uparrow}\mathrm{V} = \{y \mid (\exists x)(Rxy)\}$

(I3) $R^{\uparrow}\{z\} = \{y \mid Rzy\}$

(I4) $R^{\downarrow}\emptyset = \{x \mid (\forall y)(\neg Rxy)\}$

(I5) $R^{\downarrow}\mathrm{V} = \mathrm{V}$

(I6) $R^{\uparrow}(A \sqcup B) = R^{\uparrow}A \sqcup R^{\uparrow}B$

(I7) $R^{\downarrow}(A \sqcap B) = R^{\downarrow}A \sqcap R^{\downarrow}B$

(I8) $R^{\uparrow}(A \sqcap B) \subseteq R^{\uparrow}A \sqcap R^{\uparrow}B$

(I9) $R^{\downarrow}(A \sqcup B) \supseteq R^{\downarrow}A \sqcup R^{\downarrow}B$

(I10) $A \subseteq B \to R^{\uparrow}A \subseteq R^{\uparrow}B$

(I11) $A \subseteq B \to R^{\downarrow}A \subseteq R^{\downarrow}B$

(I12) $R \subseteq S \to R^{\uparrow}A \subseteq S^{\uparrow}A$

(I13) $R \subseteq S \to S^{\downarrow}A \subseteq R^{\downarrow}A$

(I14) $R^{\uparrow}A \subseteq B \leftrightarrow A \subseteq R^{\downarrow}B$

Proof. (I1)–(I5) are trivial to prove.

(I6)–(I9) are simple consequences of Lemma B.8 (L10)–(L13).

(I10) From $(Ax \to Bx) \to (Ax \,\&\, Rxy \to Bx \,\&\, Rxy)$ we obtain $A \subseteq B \to ((\exists x)(Ax \,\&\, Rxy) \to (\exists x)(Bx \,\&\, Rxy))$.

(I11) From $(Ay \to By) \to ((Rxy \to Ay) \to (Rxy \to By))$ the required statement follows by generalization and quantifier shifts.

(I12) $(Rxy \to Sxy) \to (Rxy \& Ax \to Sxy \& Ax)$. Thus $R \subseteq S \to (x \in R^{\uparrow}A \to x \in S^{\uparrow}A)$.

(I13) $(Rxy \to Sxy) \to ((Sxy \to Ay) \to (Rxy \to Ay))$, then use generalization and quantifier shifts.

(I14) $(\forall y)((\exists x)(Ax \& Rxy) \to By) \longleftrightarrow (\forall y)(\forall x)(Ax \to (Rxy \to By)) \longleftrightarrow (\forall x)(Ax \to (\forall y)(Rxy \to By))$. □

The previous theorem addressed the monotonicity of images of fuzzy relations and how these images interact with intersections and unions with respect to the weak conjunction and disjunction, respectively. The question remains how images of fuzzy relations interact with intersections with respect to the strong conjunction. The following theorem gives an answer (for its non-graded version, see [40, Proposition 2.16] or [41, Proposition 18.4.1]).

Theorem 4.4. *The following formulae are provable in* FCT*:*

(I15) $(R \cap R)^{\uparrow}(A \cap B) \subseteq (R^{\uparrow}A) \cap (R^{\uparrow}B)$

(I16) $(R^{\downarrow}A) \cap (R^{\downarrow}B) \subseteq (R \cap R)^{\downarrow}(A \cap B)$

Proof. (I15) $(\exists x)(Rxy \;\&\; Rxy \;\&\; Ax \;\&\; Bx) \longrightarrow (\exists x)(Rxy \;\&\; Ax) \;\&\; (\exists x)(Rxy \;\&\; Bx)$ by (L12) of Lemma B.8.

(I16) $(\forall y)(Rxy \to y \in A) \;\&\; (\forall y)(Rxy \to y \in B) \longrightarrow (\forall y)((Rxy \to y \in A) \;\&\; (Rxy \to y \in B)) \longrightarrow (\forall y)(Rxy \;\&\; Rxy \to y \in A \;\&\; y \in B)$ □

Remark 4.5. Theorem 4.4 intentionally cites only the first two of three assertions of [41, Proposition 18.4.1] (and, correspondingly, [40, Proposition 2.16]). If we translate the third assertion to our terminology, we obtain

$$(R^{\uparrow_G}A) \cup (R^{\uparrow_G}B) \subseteq (R \cup R)^{\uparrow_G}(A \cup B),$$

where $R^{\uparrow_G}A$ stands for the image with respect to the weak conjunction, i.e.,

$$R^{\uparrow_G}A =_{\mathrm{df}} \{y \mid (\exists x)(x \in A \wedge Rxy)\}.$$

First of all, the third assertion of [41, Proposition 18.4.1] relies on a certain concept of strong disjunction (a t-conorm in the standard case) which we cannot define in $\mathrm{MTL}_{\triangle}$ (we can do so only in FCT over stronger logics with involutive negation like $\mathrm{IMTL}_{\triangle}$ or ŁΠ). Secondly, this claim actually does not hold. Let us consider the case $U = \{1, 2\}$, standard Łukasiewicz logic (with the Łukasiewicz t-conorm $\min(1, x + y)$ as strong disjunction), and the following fuzzy relation and fuzzy sets (membership degrees in matrix/vector notation):

$$R = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.4 \end{pmatrix} \quad A = \begin{pmatrix} 0.5 \\ 0.6 \end{pmatrix} \quad B = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$$

Then easy computations show the following:

$$R^{\uparrow_G}A = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \text{ and } R^{\uparrow_G}B = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} \text{ which implies } (R^{\uparrow_G}A) \cup (R^{\uparrow_G}B) = \begin{pmatrix} 0.9 \\ 0.9 \end{pmatrix}.$$

On the other hand, we obtain

$$R \cup R = \begin{pmatrix} 1.0 & 0.8 \\ 1.0 & 0.8 \end{pmatrix} \text{ and } A \cup B = \begin{pmatrix} 0.8 \\ 1.0 \end{pmatrix} \text{ yielding } (R \cup R)^{\uparrow_G}(A \cup B) = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}.$$

So we have got a counter-example. Note that the converse inclusion does not hold either, as can be seen from the following counter-example (with analogous computations like above):

$$R' = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{pmatrix} \quad A' = \begin{pmatrix} 0.7 \\ 0.6 \end{pmatrix} \quad B' = \begin{pmatrix} 0.0 \\ 0.4 \end{pmatrix}$$

Now let us turn our attention to how image operations interact with the common special properties of fuzzy relations and the concept of extensionality.

Theorem 4.6. *The following properties of* $^{\uparrow}$ *are provable in* FCT*:*

(I17) $\mathrm{Refl}(R) \leftrightarrow (\forall A)(A \subseteq R^{\uparrow} A)$

(I18) $\mathrm{Trans}(R) \leftrightarrow (\forall A)(R^{\uparrow}(R^{\uparrow} A) \subseteq R^{\uparrow} A)$

(I19) $\mathrm{Preord}(R) \rightarrow R^{\uparrow}(R^{\uparrow} A) \approxeq R^{\uparrow} A$

(I20) $\mathrm{wPreord}(R) \rightarrow R^{\uparrow}(R^{\uparrow} A) \approx R^{\uparrow} A$

(I21) $\mathrm{Trans}(R) \leftrightarrow (\forall A)(\mathrm{Ext}_R(R^{\uparrow} A))$

(I22) $A \subseteq B \;\&\; \mathrm{Ext}_R(B) \rightarrow R^{\uparrow} A \subseteq B$

(I23) $\mathrm{Refl}(R) \;\&\; \mathrm{Ext}_R(A) \rightarrow R^{\uparrow} A \approxeq A$

(I24) $\mathrm{Refl}(R) \wedge \mathrm{Ext}_R(A) \rightarrow R^{\uparrow} A \approx A$

(I25) $R^{\uparrow} A \subseteq A \leftrightarrow \mathrm{Ext}_R(A)$

(I26) $\mathrm{Refl}(R) \rightarrow (\mathrm{Ext}_R(A) \leftrightarrow (A \approx R^{\uparrow} A)$

(I27) $\mathrm{Refl}(R) \rightarrow (\mathrm{Ext}_R(A) \leftrightarrow (A \approxeq R^{\uparrow} A))$

Proof. (I17) Left to right: obviously $Rxx \;\&\; Ax \rightarrow (\exists y)(Rxy \;\&\; Ax)$ and generalize as usual. Right to left: $(\forall A)(A \subseteq R^{\uparrow} A) \longrightarrow \{z\} \subseteq R^{\uparrow}\{z\} \longrightarrow (z = z \rightarrow Rzz)$; we used (I3) in the last step.

(I18) Left to right: From $(Rxz \rightarrow Rxy) \rightarrow (Ax \& Rxz \rightarrow Ax \& Rxy)$ we get $(\forall x)(Rxz \rightarrow Rxy) \rightarrow (z \in R^{\uparrow} A \rightarrow y \in R^{\uparrow} A)$. Next we get $(Rzy \rightarrow (\forall x)(Rxz \rightarrow Ryx)) \rightarrow (Rzy \rightarrow (z \in R^{\uparrow} A \rightarrow y \in R^{\uparrow} A))$. Thus $(\forall x)(Rzy \rightarrow (Rxz \rightarrow Rxy)) \rightarrow (Rzy \;\&\; z \in R^{\uparrow}A \rightarrow y \in R^{\uparrow}A))$. Right to left: $(\forall A)(R^{\uparrow}(R^{\uparrow}A) \subseteq R^{\uparrow}A) \longrightarrow (R^{\uparrow}(R^{\uparrow}\{z\}) \subseteq R^{\uparrow}\{z\}) \longleftrightarrow (R^{\uparrow}\{y \mid Rzy\}\} \subseteq \{y \mid Rzy\} \longrightarrow ((\exists x)(Rzx \;\&\; Rxy) \rightarrow Rzy)$, and quantifier shifts complete the proof.

(I19) and (I20) are direct consequences of (I17) and (I18).

(I21) From $(\forall x)(Ryz \rightarrow (Rxy \rightarrow Ryz))$ we get $(Ryz \rightarrow ((\exists x)(Rxy \& Ax) \rightarrow (\exists x)(Rxz \& Ax))$. The converse direction: $(\forall A)(\mathrm{Ext}_R(R^{\uparrow} A)) \longrightarrow \mathrm{Ext}_R(R^{\uparrow}\{z\}) \longrightarrow (Rzx \;\&\; Rxy \rightarrow Rzy)$.

(I22) From $A \subseteq B$ we get $Ax \& Rxy \rightarrow Bx \& Rxy$ and from $\mathrm{Ext}_R(B)$ we get $Bx \& Rxy \rightarrow By$. Thus we have $Ax \;\&\; Rxy \rightarrow By$.

(I23) and (I24) follow directly from (I22) by (I17).

(I25) $((\exists x)(Rxy \;\&\; Ax) \rightarrow Ay) \leftrightarrow (\forall x)(Rxy \;\&\; Ax \rightarrow Ay)$.

(I26) and (I27) then follow trivially. □

Theorem 4.7. *The following properties of* $^{\downarrow}$ *are provable in* FCT*:*

(I28) $\mathrm{Refl}(R) \rightarrow R^{\downarrow} A \subseteq A$

(I29) $\text{Trans}(R) \rightarrow R^{\downarrow}A \subseteq R^{\downarrow}(R^{\downarrow}A)$

(I30) $\text{Preord}(R) \rightarrow R^{\downarrow}(R^{\downarrow}A) \approxeq R^{\downarrow}A$

(I31) $\text{wPreord}(R) \rightarrow R^{\downarrow}(R^{\downarrow}A) \approx R^{\downarrow}A$

(I32) $\text{Trans}(R) \rightarrow \text{Ext}_R(R^{\downarrow}A)$

(I33) $B \subseteq A \;\&\; \text{Ext}_R(B) \rightarrow B \subseteq R^{\downarrow}A$

(I34) $\text{Refl}(R) \;\&\; \text{Ext}_R(A) \rightarrow R^{\downarrow}A \approxeq A$

(I35) $\text{Refl}(R) \wedge \text{Ext}_R(A) \rightarrow R^{\downarrow}A \approx A$

(I36) $A \subseteq R^{\downarrow}A \leftrightarrow \text{Ext}_R(A)$

(I37) $\text{Refl}(R) \rightarrow (\text{Ext}_R(A) \leftrightarrow (A \approx R^{\downarrow}A))$

(I38) $\text{Refl}(R) \rightarrow (\text{Ext}_R(A) \leftrightarrow (A \approxeq R^{\downarrow}A))$

Proof. (I28) $(\forall y)(Rxy \rightarrow Ay) \rightarrow (Rxx \rightarrow Ax)$, thus $Rxx \rightarrow (x \in R^{\downarrow}A \rightarrow Ax)$. Generalization and quantifier shifts complete the proof.

(I29) From $(Rzy \rightarrow Rxy) \rightarrow ((Rxy \rightarrow Ay) \rightarrow (Rzy \rightarrow Ay))$ we obtain $(\forall y)(Rzy \rightarrow Rxy) \rightarrow (x \in R^{\downarrow}A \rightarrow z \in R^{\downarrow}A)$. Next we get $(Rxz \rightarrow (\forall y)(Rzy \rightarrow Rxy)) \rightarrow (Rxz \rightarrow (x \in R^{\downarrow}A \rightarrow z \in R^{\downarrow}A))$. Thus $(\forall y)(Rxz \rightarrow (Rzy \rightarrow Rxy)) \rightarrow (x \in R^{\downarrow}A \rightarrow (Rxz \rightarrow z \in R^{\downarrow}A))$.

(I30) and (I31) are direct consequences of (I28) and (I29).

(I32) From $(\forall y)(Rzx \rightarrow (Rxy \rightarrow Rzy))$ we get $(Rzx \rightarrow ((\forall y)(Rzy \rightarrow Ay) \rightarrow (\forall y)(Rxy \rightarrow Ay))$.

(I33) From $B \subseteq A$ we get $(Rxy \rightarrow By) \rightarrow (Rxy \rightarrow Ax)$ and from $\text{Ext}_R(B)$ we get $Bx \rightarrow (Rxy \rightarrow By)$. Thus we have $Bx \rightarrow (Rxy \rightarrow Ay)$.

(I34) and (I35) follow directly from (I33) using (I28).

(I36) Left to right: $(Ax \rightarrow (\forall y)(Rxy \rightarrow Ay)) \rightarrow (\forall y)(Rxy \;\&\; Ax \rightarrow Ay)$. The converse direction follows from (I33).

(I37) and (I38) then follow trivially. □

Inspired by the concepts of fuzzy mathematical morphology [15, 24, 56], Bodenhofer has introduced a general concept of opening and closure operators with respect to arbitrary fuzzy relations [17]. Now we generalize these ideas to the graded framework.

Definition 4.8. We define the operations of *opening* and *closure* of A in R as

$$R^{\circ}A =_{\text{df}} R^{\uparrow}(R^{\downarrow}A)$$
$$R^{\bullet}A =_{\text{df}} R^{\downarrow}(R^{\uparrow}A)$$

Furthermore, we define two properties of fuzzy classes, *R-openness* and *R-closedness*:

$$\text{Open}_R(A) \equiv_{\text{df}} R^{\circ}A \approx A$$
$$\text{Closed}_R(A) \equiv_{\text{df}} R^{\bullet}A \approx A$$

The following lemma provides us with several properties of opening and closure operators. In particular, the question arises why R-openness and R-closedness were defined using $\approx$ rather than $\approxeq$. A clear answer to this question is given by (I40) and (I41) which state that it actually does not matter whether we use $\approx$ or $\approxeq$ in the definition of openness and closedness.

Theorem 4.9. *The following properties of relations are provable in* FCT*:*

(I39) $R^\circ A \subseteq A \subseteq R^\bullet A$

(I40) $\mathrm{Open}_R(A) \leftrightarrow R^\circ A \approxeq A$

(I41) $\mathrm{Closed}_R(A) \leftrightarrow R^\bullet A \approxeq A$

(I42) $A \subseteq B \rightarrow R^\circ A \subseteq R^\circ B$

(I43) $A \subseteq B \rightarrow R^\bullet A \subseteq R^\bullet B$

(I44) $\mathrm{Open}_R(A) \leftrightarrow (\exists B)(A \approxeq R^\uparrow B)$

(I45) $\mathrm{Closed}_R(A) \leftrightarrow (\exists B)(A \approxeq R^\downarrow B)$

(I46) $\mathrm{Open}_R(R^\circ A)$

(I47) $\mathrm{Closed}_R(R^\bullet A)$

Proof. (I39) First, we can show $y \in R^\uparrow(R^\downarrow A) \longleftrightarrow (\exists x)(Rxy \,\&\, (\forall z)(Rxz \rightarrow Az)) \longrightarrow (\exists x)(Rxy \,\&\, (Rxy \rightarrow Ay)) \longrightarrow (\exists x)Ay \longleftrightarrow Ay$. Secondly, we have $Ax \longrightarrow (Rxy \rightarrow Rxy \,\&\, Ax) \longrightarrow (Rxy \rightarrow (\exists x)(Rxy \,\&\, Ax))$. Thus $Ax \rightarrow (\forall y)(Rxy \rightarrow y \in R^\uparrow A)$.

(I40) and (I41) are then direct consequences of (I39).

(I42) and (I43) are direct consequences consequence of (I10) and (I11).

(I44) The left-to-right direction is trivial (take $B = R^\downarrow A$). The converse direction: By (I14) and (I10), $R^\uparrow B \subseteq A \longleftrightarrow B \subseteq R^\downarrow A \longrightarrow R^\uparrow B \subseteq R^\uparrow(R^\downarrow A)$. Thus $A \approxeq R^\uparrow B \longleftrightarrow A \subseteq R^\uparrow B \subseteq A \longrightarrow A \subseteq R^\uparrow B \subseteq R^\uparrow(R^\downarrow A) = R^\circ A$. Since by (I39) always $R^\circ A \subseteq A$, the proof is done.

(I45) Analogous to the proof of (I44).

(I46) and (I47) are direct consequences of (I44) and (I45), respectively. □

Note that, from (I44)–(I47), we can easily deduce the following corollaries:

(I48) $R^\circ(R^\circ A) = R^\circ A$

(I49) $R^\bullet(R^\bullet A) = R^\bullet A$

Thus, we can conclude that the two operators $^\circ$ and $^\bullet$ fulfill the most essential properties we need to require from opening and closure operators (as stated in [17] to motivate the definition of the two operators). Unlike [17], in classical mathematics (e.g. in topology), it is more usual to start from an axiomatic framework of openness and closedness (or opening and closure operators, respectively). Such general frameworks have been introduced in the fuzzy setting by Bělohlávek and Funioková [11, 12, 14]. They require that opening operators always give subsets, that closure operators always yield supersets, that both operators are monotonic with respect to the graded inclusion and that both operators are idempotent. Therefore, we can conclude that our two operators perfectly fit into the axiomatic framework of Bělohlávek and Funioková.

In many classical axiomatic frameworks (including topological ones), it is also common to represent opening and closure operators as unions of all open subsets and intersections of all closed supersets, respectively. This is well-known in the non-graded framework; the following theorem provides a generalization to the graded case.

Theorem 4.10. *The following properties of relations are provable in* FCT*:*

(I50) $R^\circ A = \bigcup\{B \mid \mathrm{Open}_R(B) \mathbin{\&} B \subseteq A\} = \bigcup\{B \mid \triangle(\mathrm{Open}_R(B) \mathbin{\&} B \subseteq A)\}$

(I51) $R^\bullet A = \bigcap\{B \mid \mathrm{Closed}_R(B) \mathbin{\&} A \subseteq B\} = \bigcap\{B \mid \triangle(\mathrm{Closed}_R(B) \mathbin{\&} A \subseteq B)\}$

Proof. To prove (I50), let us denote $\{B \mid \mathrm{Open}_R(B) \mathbin{\&} B \subseteq A\}$ as C. We shall prove that

$$R^\circ A \subseteq \bigcup \mathrm{Ker}(C) \subseteq \bigcup C \subseteq R^\circ A$$

Since $\triangle\,\mathrm{Open}_R(R^\circ A)$ and $\triangle(R^\circ A \subseteq A)$ respectively by (I46) and (I39), we obtain the first inclusion $R^\circ A \subseteq \bigcup \mathrm{Ker}(C)$ by Lemma B.8 (L8). The second inclusion is trivial. To prove the third inclusion, we use Lemma B.8 (L5): we fix B and show that $\mathrm{Open}_R(B) \mathbin{\&} B \subseteq A$ implies $B \subseteq R^\circ A$. From $\mathrm{Open}_R(B)$ we get that $R^\circ B \approx B$ and from $B \subseteq A$ we get $R^\circ B \subseteq R^\circ A$. Thus $B \subseteq R^\circ A$. The claim (I51) can be proved analogously. □

From the two representations (I50) and (I51), we can deduce how opening and closure operators interact with weak unions and weak intersections.

Corollary 4.11. *The following properties of relations are provable in* FCT*:*

(I52) $R^\circ(A \sqcup B) \supseteq R^\circ A \sqcup R^\circ B$

(I53) $R^\bullet(A \sqcap B) \subseteq R^\bullet A \sqcap R^\bullet B$

(I54) $\mathrm{Open}_R(A) \mathbin{\&} \mathrm{Open}_R(B) \to \mathrm{Open}_R(A \sqcup B)$

(I55) $\mathrm{Closed}_R(A) \mathbin{\&} \mathrm{Closed}_R(B) \to \mathrm{Closed}_R(A \sqcap B)$

Proof. (I52) From $C \subseteq A \sqcup B \leftarrow C \subseteq A \vee C \subseteq B$ we obtain $x \in C \mathbin{\&} \mathrm{Open}_R(C) \mathbin{\&} C \subseteq A \sqcup B \leftarrow (x \in C \mathbin{\&} \mathrm{Open}_R(C) \mathbin{\&} C \subseteq A) \vee (x \in C \mathbin{\&} \mathrm{Open}_R(C) \mathbin{\&} C \subseteq B)$. Generalization, quantifier distribution, Lemma B.8 (L10), and (I50) then completes the proof.

(I53) The proof is analogous, we only use (L9) and (I51) instead of (L10) and (I50).

(I54) and (I55) are then direct consequences of (I52) and (I53), respectively. □

As shown in [17], under the presence of reflexivity and/or transitivity, the results concerning opening and closure operators can be strengthened. We will see in the following that, by this way, results for images of fuzzy preorders are obtained that are well-known in the non-graded framework [18].

Theorem 4.12. *The following properties of relations are provable in FCT:*

(I56) $\mathrm{Preord}(R) \to (R^\bullet A \approxeq R^\uparrow A)$

(I57) $\mathrm{wPreord}(R) \to (R^\bullet A \approx R^\uparrow A)$

(I58) $\mathrm{Preord}(R) \to (R^\circ A \approxeq R^\downarrow A)$

(I59) $\mathrm{wPreord}(R) \to (R^\circ A \approx R^\downarrow A)$

(I60) $\mathrm{Trans}(R) \to (\mathrm{Open}_R(A) \to \mathrm{Ext}_R(A))$

(I61) $\mathrm{Trans}(R) \to (\mathrm{Closed}_R(A) \to \mathrm{Ext}_R(A))$

(I62) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \to \mathrm{Open}_R(A))$

(I63) $\mathrm{Refl}(R) \to (\mathrm{Ext}_R(A) \to \mathrm{Closed}_R(A))$

(I64) $\mathrm{wPreord}(R) \to (\mathrm{Ext}_R(A) \leftrightarrow \mathrm{Open}_R(A))$

(I65) $\text{wPreord}(R) \to (\text{Ext}_R(A) \leftrightarrow \text{Closed}_R(A))$

(I66) $\text{Preord}(R) \to (\text{Open}_R(A) \leftrightarrow \text{Closed}_R(A))$

Proof. (I56) and (I57): From (I32) we know $\text{Trans}(R) \to \text{Ext}_R(R^{\downarrow}A)$. Then (I56) follows from (I23) and (I57) follows from (I24).

(I58) and (I59): From (I21) we know $\text{Trans}(R) \to \text{Ext}_R(R^{\uparrow}A)$. Then (I58) follows from (I34) and (I59) follows from (I35).

(I60) We start from $\text{Trans}(R)$. Using (I32) we get $\text{Ext}_R(R^{\downarrow}A)$, thus by (I25), $R^{\uparrow}(R^{\downarrow}A) \subseteq R^{\downarrow}A$. So by $\text{Open}_R(A)$ (i.e. $A \approx R^{\circ}A$) we obtain $A \subseteq R^{\downarrow}A$. Now we use (I36) and get $\text{Ext}_R(A)$.

(I61) Analogously to (I60), by (I21), (I36), and (I25).

(I62) From $\text{Refl}(R)$, by (I17), we get $R^{\downarrow}A \subseteq R^{\uparrow}(R^{\downarrow}A)$. From $\text{Ext}_R(A)$, we obtain by (I36) that $A \subseteq R^{\downarrow}A$. So finally, we can conclude $A \subseteq R^{\circ}A$ which, with (I39), proves $\text{Open}_R(A)$.

(I63) Analogously to (I62), using (I28), (I25), and the second inclusion of (I39).

(I64)–(I66) then follow trivially. □

Example 4.13. Let us consider standard Łukasiewicz logic and the following fuzzy set (with $U = \mathbb{R}$):

$$Ax = \begin{cases} 2x - 5 & \text{if } x \in [2.5, 3] \\ 4 - x & \text{if } x \in]3, 3.5] \\ 0.5 & \text{if } x \in]3.5, 5] \\ 10.5 - 2x & \text{if } x \in]5, 5.25] \\ 0 & \text{otherwise} \end{cases}$$

Further we consider the fuzzy relation $E_{1.2,1}$ from Example 3.4 for which we know $\text{Refl}(E_{1.2,1}) = 1$ and $\text{Trans}(E_{1.2,1}) = \text{Preord}(E_{1.2,1}) = \text{wPreord}(E_{1.2,1}) = 0.8$. Figure 3 shows plots of A, $E_{1.2,1}{}^{\uparrow}A$, and $E_{1.2,1}{}^{\bullet}A$. Basic computations show that $\text{Closed}_{E_{1.2,1}}(A) = 0.5$. Moreover, we have that $(A \approx E_{1.2,1}{}^{\uparrow}A) = 0.3$. From (I26) we can infer, therefore, that $\text{Ext}_{E_{1.2,1}}(A) = 0.3$. It also holds that $(E_{1.2,1}{}^{\bullet}A \approx E_{1.2,1}{}^{\uparrow}A) = (E_{1.2,1}{}^{\bullet}A \approxeq E_{1.2,1}{}^{\uparrow}A) = 0.8$. Figure 4 shows plots of A, $E_{1.2,1}{}^{\downarrow}A$ and $E_{1.2,1}{}^{\circ}A$. We can show that $\text{Open}_{E_{1.2,1}}(A) = 0.5$ and $(A \approx E_{1.2,1}{}^{\downarrow}A) = 0.3$. Thus, we can infer $\text{Ext}_{E_{1.2,1}}(A) = 0.3$ also via (I37). Further we can show that $(E_{1.2,1}{}^{\circ}A \approx E_{1.2,1}{}^{\downarrow}A) = (E_{1.2,1}{}^{\circ}A \approxeq E_{1.2,1}{}^{\downarrow}A) = 0.8$. If we take into account that $(0.5 \to 0.3) = (0.3 \leftrightarrow 0.5) = 0.8$, these numbers demonstrate that, in this special case, the estimations provided by Theorem 4.12 are tight.

Finally, we can formulate representations of images under fuzzy preorders. Note that the first four assertions (I67)–(I70) of the following theorem are "fuzzy representations", i.e. they do not determine the truth degree of $R^{\uparrow}A$ or $R^{\downarrow}A$ itself. We can only infer from the degree to which R is a (weak) preorder to which degree the image is guaranteed to resemble to the intersection (resp. union). The "real" (non-graded) representations (I71)–(I72), known from [17, 18], are their special cases for R being a preorder to degree 1.

Corollary 4.14. *The following properties of relations are provable in* FCT*:*

(I67) $\text{Preord}(R) \to R^{\uparrow}A \approxeq \bigcap\{X \mid A \subseteq X \mathbin{\&} \text{Ext}_R(X)\}$

(I68) $\text{Preord}(R) \to R^{\downarrow}A \approxeq \bigcup\{X \mid X \subseteq A \mathbin{\&} \text{Ext}_R(X)\}$

Figure 3: The fuzzy set A from Example 4.13 (light gray), $E_{1.2,1}{}^{\uparrow}A$ (medium gray), and $E_{1.2,1}{}^{\bullet}A$ (solid black).

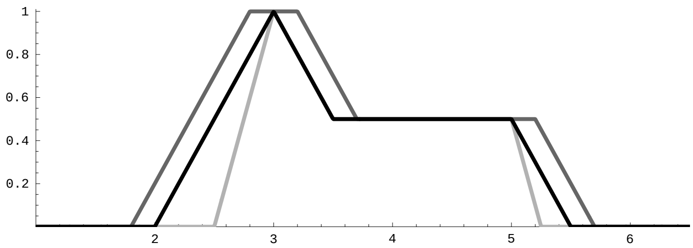
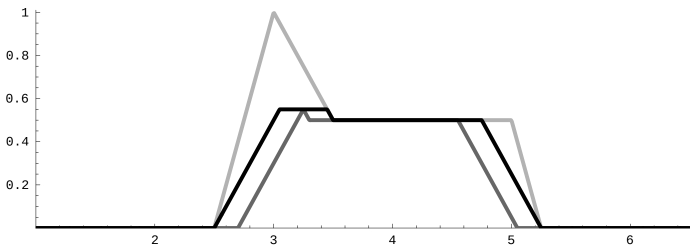


Figure 4: The fuzzy set A from Example 4.13 (light gray), $E_{1.2,1}{}^{\downarrow}A$ (medium gray), and $E_{1.2,1}{}^{\circ}A$ (solid black).



(I69) $\quad \mathrm{wPreord}(R) \to R^{\uparrow}A \approxeq \bigcap\{X \mid A \subseteq X \wedge \mathrm{Ext}_R(X)\}$

(I70) $\quad \mathrm{wPreord}(R) \to R^{\downarrow}A \approxeq \bigcup\{X \mid X \subseteq A \wedge \mathrm{Ext}_R(X)\}$

(I71) $\quad \triangle\,\mathrm{Preord}(R) \to R^{\uparrow}A = \bigcap\{X \mid \triangle(A \subseteq X \;\&\; \mathrm{Ext}_R(X))\}$

(I72) $\quad \triangle\,\mathrm{Preord}(R) \to R^{\downarrow}A = \bigcup\{X \mid \triangle(X \subseteq A \;\&\; \mathrm{Ext}_R(X))\}$

Proof. (I67) For any X such that $A \subseteq X \;\&\; \mathrm{Ext}_R(X)$ we can infer $R^{\uparrow}A \subseteq X$ from (I22). Hence, the first inclusion $R^{\uparrow}A \subseteq \bigcap\{X \mid A \subseteq X \;\&\; \mathrm{Ext}_R(X)\}$ follows by (L6) of Lemma B.8. Conversely, (I17) and (I21) imply $\mathrm{Preord}(R) \to \mathrm{Ext}_R(R^{\uparrow}A) \;\&\; A \subseteq R^{\uparrow}A$. Then (L7) completes the proof.

The proofs of (I68)–(I70) are analogous. The assertions (I71) and (I72) follow from the proofs of (I67) and (I68), respectively, if we take basic properties of $\triangle$ into account. □

Remark 4.15. At the beginning of this section, we mentioned the close relationship of images, closures and openings with concepts in fuzzy mathematical morphology. In (crisp) mathematical morphology, images are considered as crisp subsets of an Abelian group $(U, +, \mathbf{0})$ (more commonly, a linear vector space structure is assumed). Given a set A (the *image*) and a set B (the so-called *structuring element*), the four standard operations (on the image A with respect to the structuring element B) can be defined as follows:

$$\begin{array}{rcll} A \oplus B & =_{\mathrm{df}} & \{y \mid (\exists x)(Ax \;\&\; B(y-x))\} & \text{(dilation)} \\ A \ominus B & =_{\mathrm{df}} & \{x \mid (\forall y)(B(y-x) \to Ay)\} & \text{(erosion)} \\ A \bullet B & =_{\mathrm{df}} & (A \oplus B) \ominus B & \text{(closure)} \\ A \circ B & =_{\mathrm{df}} & (A \ominus B) \oplus B & \text{(opening)} \end{array}$$

The language in the definitions above has been chosen intentionally to comply fully with the language of FCT. Thus, if we consider gray level images as $U \to L$ mappings (with the standard case $L = [0, 1]$ being the natural choice), we can generalize the four morphological operations to gray level images and gray level structuring elements simply by the above formulae. In the standard case $L = [0, 1]$, the well-known t-norm based fuzzy mathematical morphology is obtained [15, 17, 56, 57]. This is not at all new, but it demonstrates that the expressive power of FCT allows rather effortless generalizations—the obvious secret is the commonality of its syntax with classical Boolean logic. As demonstrated in [17], the operations of fuzzy mathematical morphology can be embedded in the concepts of this section in the following way:

1. If we define a fuzzy relation R as $Rxy = B(y-x)$ for a given structuring element B, then the following four equalities hold:

$$A \oplus B = R^{\uparrow} A$$
$$A \ominus B = R^{\downarrow} A$$
$$A \bullet B = R^{\bullet} A$$
$$A \circ B = R^{\circ} A$$

2. If R is a shift-invariant fuzzy relation, i.e. if

$$\triangle(\forall x, y, z)(Rxy \leftrightarrow R(x+z)(y+z))$$

holds, then the above equalities are satisfied if we define the structuring element B as $Bx = R\mathbf{0}x$.

This relationship particularly implies that we can transfer all results of this section to fuzzy mathematical morphology without any restriction. For the non-graded case, most of these results are already known [17, 24], but it is worth to mention that, hereby, we have generalized fuzzy mathematical morphology to the graded framework almost effortlessly. It may be questionable whether a graded framework of fuzzy mathematical morphology is useful in image processing practice, but it is certainly interesting from a theoretical perspective.

5 Bounds, maxima, and suprema

The aim of this section is to study the lattice-like structure induced by a fuzzy relation. We follow the philosophy of Demirci's approach [31, 32]. Note that this is not a classical axiomatic approach to lattices; instead, lattice-theoretical notions are defined on the basis of a given fuzzy relation, where Demirci assumes that fuzzy relation under consideration is a similarity-based fuzzy ordering [16, 47]. As in the previous sections, we do not restrict ourselves to a particular class of fuzzy relations in advance, but we infer gradual results from the degrees to which the relation fulfills some properties (in particular, reflexivity and transitivity).

Throughout this section, assume that R denotes a binary fuzzy relation that is arbitrary, but fixed.

Definition 5.1. The properties of being an *upper* or *lower class* in X with respect to R are defined as follows:

$$\begin{aligned} \mathrm{Upper}_R^X(A) &\equiv_{\mathrm{df}} (\forall x \in X)(\forall y \in X)[Rxy \to (Ax \to Ay)] \\ \mathrm{Lower}_R^X(A) &\equiv_{\mathrm{df}} (\forall x \in X)(\forall y \in X)[Rxy \to (Ay \to Ax)] \end{aligned}$$

Let us further make the conventions $\mathrm{Upper}_R(A) \equiv_{\mathrm{df}} \mathrm{Upper}^{\mathrm{V}}_R(A)$ and $\mathrm{Lower}_R(A) \equiv_{\mathrm{df}} \mathrm{Lower}^{\mathrm{V}}_R(A)$. Further, to ease notation, we omit the lower index R unless we require special properties of R or unless a relation different from the default choice R is used.

Remark 5.2. Note that $\mathrm{Upper}_R(A)$ is in fact nothing else but $\mathrm{Ext}_R(A)$ and that $\mathrm{Lower}_R(A)$ is just $\mathrm{Ext}_{R^{-1}}(A)$. We make this terminological distinction in order to increase readability and to make explicit that we have some preorder-related notions in mind.

Remark 5.3. There is an "inversion duality" between the pairs of notions defined in this section, consisting in the observation that the second notion of each pair is just the first one applied to the inverse relation. Thus, $\mathrm{Lower}^X_R(A) \leftrightarrow \mathrm{Upper}^X_{R^{-1}}(A)$ in Definition 5.1 above, $R^{\triangledown}A = (R^{-1})^{\triangle}A$ in Definition 5.7 below, $\mathrm{Min}_R(A) = \mathrm{Max}_{R^{-1}}(A)$ in Definition 5.9, and $\mathrm{Inf}_R(A) = \mathrm{Sup}_{R^{-1}}(A)$ in Definition 5.14. As the theorems on the dual notions follow trivially by taking R^{-1} for R, we shall usually not write them down explicitly.

As a first simple result, we consider the antitony of (degrees of) upperness and lowerness.

Proposition 5.4. *The following properties are provable in* FCT*:*

(C1) $(X \subseteq Y)^2 \to (\mathrm{Upper}^Y_R(A) \to \mathrm{Upper}^X_R(A))$

(C2) $(X \subseteq Y)^2 \to (\mathrm{Lower}^Y_R(A) \to \mathrm{Lower}^X_R(A))$

Proof. $(X \subseteq Y)^2$ implies $x \in X \mathbin{\&} y \in X \to x \in Y \mathbin{\&} y \in Y$. Assuming $\mathrm{Upper}^Y_R(A)$, equivalently $(\forall x)(\forall y)(x \in X \mathbin{\&} y \in X \mathbin{\&} Rxy \to (Ax \to Ay))$, we can thus infer $(\forall x)(\forall y)$ $(x \in Y \mathbin{\&} y \in Y \mathbin{\&} Rxy \to (Ax \to Ay))$, which proves (C1). Then (C2) follows trivially by duality. □

Note that in Proposition 5.4 we need to require an assumption twice. The following simple example demonstrates that the proof of Proposition 5.4 cannot be improved in the sense that the "doubled assumption" could only be used once.

Example 5.5. Let us consider standard Łukasiewicz logic and $U = \{x, y\}$ and define fuzzy sets A, X, Y by $Xx = Xy = Ax = 1$, $Yx = Yy = 0.9$ and $Ay = 0.8$. Using the fuzzy relation R defined as $Rxx = Ryy = Ryx = 0$ and $Rxy = 1$, we obtain that $X \subseteq Y$ is true to a degree of 0.9. Furthermore, we have $\mathrm{Upper}^X_R(A) = 0.8$ and $\mathrm{Upper}^Y_R(A) = 1$; thus the truth degree of $X \subseteq Y \to (\mathrm{Upper}^Y_R(A) \to \mathrm{Upper}^X_R(A))$ is only 0.9.

As $X \subseteq \mathrm{V}$ is always true to a degree of 1, we can infer the following simple corollary on upperness from Proposition 5.4 (by the duality of Remark 5.3, we omit the same result for lowerness).

Corollary 5.6. $\mathrm{Upper}_R(A) \to (\forall X)\, \mathrm{Upper}^X_R(A)$

Like in classical mathematics, we can define the classes of all upper (and dually, lower) bounds of a class:

Definition 5.7. The *upper cone* and the *lower cone* of a class A (with respect to R) are defined as follows:

$$\begin{aligned} R^{\triangle}A &=_{\mathrm{df}} \{x \mid (\forall a \in A) Rax\} \\ R^{\triangledown}A &=_{\mathrm{df}} \{x \mid (\forall a \in A) Rxa\} \end{aligned}$$

If we do not suppose any special conditions involving R, we write just ${}^{\triangle}A$ and ${}^{\triangledown}A$ instead of $R^{\triangle}A$ and $R^{\triangledown}A$, respectively.

Note that $R^{\triangle}A$ appears in some literature as an image operator in its own right. It is called *sub-direct image* by some authors (e.g. [29]). In [9], the systematic names of the operators ${}^{\triangle}$ and ${}^{\triangledown}$are *subproduct image* and *superproduct preimage*, respectively.

Theorem 5.8. *The following properties of cones are provable in* FCT *for an arbitrarily fixed R:*

(C3) $\mathrm{Trans}(R) \to \mathrm{Upper}_R(R^{\triangle}A)$

(C4) $A \subseteq B \to {}^{\triangle}B \subseteq {}^{\triangle}A$

(C5) $A \subseteq {}^{\triangledown\triangle}A$

(C6) ${}^{\triangle\triangledown\triangle}A = {}^{\triangle}A$

(C7) ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}A \cap {}^{\triangle}B$

(C8) ${}^{\triangle}A \cup {}^{\triangle}B \subseteq {}^{\triangle}(A \cap B)$

(C9) $\displaystyle\bigcap_{A \in \mathcal{A}} {}^{\triangle}A = {}^{\triangle}\Big(\bigcup_{A \in \mathcal{A}} A\Big)$

(C10) $\displaystyle\bigcup_{A \in \mathcal{A}} {}^{\triangle}A \subseteq {}^{\triangle}\Big(\bigcap_{A \in \mathcal{A}} A\Big)$

(Converse inclusions and implications have well-known crisp counter-examples.)

Proof. (C3) $\mathrm{Trans}(R)$ implies $Rxy \to (Rax \to Ray)$, which implies $Rxy \to ((a \in A \to Rax) \to (a \in A \to Ray))$, whence we get the required assertion $Rxy \to ((\forall a \in A)\, Rax \to (\forall a \in A)Ray)$ by generalization and quantifier shifts.

(C4) The required $(\forall x \in A)(x \in B) \to (\forall y)[(\forall x \in B)Rxy \to (\forall x \in A)Rxy]$ follows by generalization and distribution of the quantifiers from $(Ax \to Bx) \to [(Bx \to Rxy) \to (Ax \to Rxy)]$.

(C5) The required $a \in A \to (\forall x)((\forall y \in A)Ryx \to Rax)$ follows by generalization from $a \in A \to ((\forall y \in A)Ryx \to Rax)$, which is a variant of the specification axiom $(\forall y)(y \in A \to Ryx) \to (a \in A \to Rax)$.

(C6) By the dual of (C5) it is proved that ${}^{\triangle}A \subseteq {}^{\triangle\triangledown}({}^{\triangle}A)$. By (C5) and (C4), it is proved that ${}^{\triangle}({}^{\triangledown\triangle}A) \subseteq {}^{\triangle}A$. By the axiom of extensionality, we are done.

(C7) and (C8) follow directly from the antitony of cones: by (C4), ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}A$ and ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}B$, therefore ${}^{\triangle}(A \cup B) \subseteq {}^{\triangle}A \cap {}^{\triangle}B$; analogously for ${}^{\triangle}(A \cap B)$.

(C9) This assertion can be proved as follows:

$$x \in \bigcap_{A \in \mathcal{A}} {}^{\triangle}A \longleftrightarrow (\forall A \in \mathcal{A})(\forall a \in A)Rax$$
$$\longleftrightarrow (\forall a)[(\exists A \in \mathcal{A})(a \in A) \to Rax] \longleftrightarrow x \in {}^{\triangle}\Big(\bigcup_{A \in \mathcal{A}} A\Big)$$

(C10) Similarly to (C9), we can infer the following:

$$x \in \bigcup_{A \in \mathcal{A}} {}^{\triangle}A \longleftrightarrow (\exists A \in \mathcal{A})(\forall a \in A)Rax$$
$$\longrightarrow (\forall a)[(\forall A \in \mathcal{A})(a \in A) \to Rax] \longleftrightarrow x \in {}^{\triangle}\Big(\bigcap_{A \in \mathcal{A}} A\Big),$$

where the middle implication follows from Lemma B.8 (L4) by generalization and appropriate quantifier shifts. □

It is worth mentioning that the following two corollaries can be inferred directly from (C9) and (C10):

(C11) $^{\vartriangle}(A \sqcup B) = {}^{\vartriangle}A \sqcap {}^{\vartriangle}B$

(C12) $^{\vartriangle}A \sqcup {}^{\vartriangle}B \subseteq {}^{\vartriangle}(A \sqcap B)$

Theorems (C4)–(C12) as well as their duals are also corollaries of more general theorems found in [9]. Now let us move closer to the lattice-theoretical notions at which this section aims. First of all, we define maxima and minima.

Definition 5.9. The classes of all *maxima* and *minima* of a class A with respect to R are defined as follows:

$$\begin{aligned} \mathrm{Max}_R\, A &=_{\mathrm{df}} A \cap (R^{\vartriangle} A) \\ \mathrm{Min}_R\, A &=_{\mathrm{df}} A \cap (R^{\triangledown} A) \end{aligned}$$

The index R is dropped under the same conditions as noted above.

Remark 5.10. Observe that Definition 5.9 is just a more compact way of expressing the usual definition of maxima and minima as those elements of A that are larger resp. smaller than all elements in A, i.e.,

(C13) $\mathrm{Max}_R\, A = \{x \in A \mid (\forall y \in A) Ryx\}$

(C14) $\mathrm{Min}_R\, A = \{x \in A \mid (\forall y \in A) Rxy\}$

Notice further that since the property of being an upper (or lower) bound is graded in FCT, maxima (minima) have to be defined as fuzzy classes (unlike in classical mathematics, where they are determined uniquely and therefore can be defined as single elements).

Example 5.11. Let us consider the fuzzy set A from Example 4.13 and standard Łukasiewicz logic again. Further consider the fuzzy relation L_1 from Example 3.8 which is a fuzzy preorder [16]. Figure 5 shows A, $L_1{}^{\vartriangle}A$ and $\mathrm{Max}_{L_1} A$, while Figure 6 shows A, $L_1{}^{\triangledown}A$ and $\mathrm{Min}_{L_1} A$. The results we obtain for the lower cone and the minimum are what one may expect intuitively. The results we obtain for the upper cone and the maximum in this case demonstrate, however, that quite peculiar results may be obtained for more unusual fuzzy sets.[5]

As the above example suggests, cones, minima and maxima may not be as intuitive and simple concepts as in classical mathematics. The following theorem demonstrates that still properties hold that one would expect intuitively.

Theorem 5.12. *The following properties of maxima are provable in* FCT*:*

(C15) $A \subseteq B \mathbin{\&} x \in \mathrm{Max}_R\, A \mathbin{\&} y \in \mathrm{Max}_R\, B \rightarrow Rxy$

[5] Although unusual, the results are nevertheless not counter-intuitive and in Figure 5 they can be explained by the shape of the membership function of A: the gradual decrease of A to the right makes the maximum subnormal (compare it with right-open crisp intervals which have no maximum at all), and the increase of the membership function in the left part induces a second peak of the maximum (as the α-cuts of A for large α have their maxima exactly there).

Figure 5: The fuzzy set A from Example 4.13 (light gray), its upper cone $L_1{}^{\triangle}A$ (medium gray), and its maximum $\mathrm{Max}_{L_1} A$ (solid black).

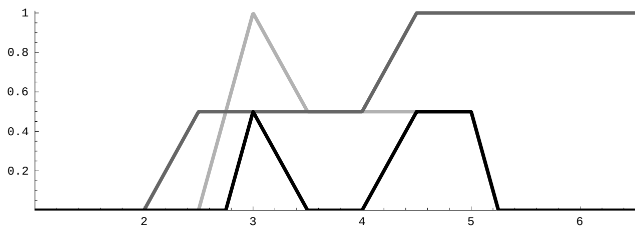
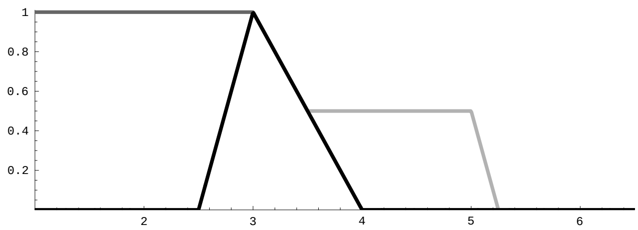


Figure 6: The fuzzy set A from Example 4.13 (light gray), its lower cone $L_1{}^{\triangledown}A$ (medium gray), and its minimum $\mathrm{Min}_{L_1} A$ (solid black).



(C16) $x \in \mathrm{Max}_R A \;\&\; y \in \mathrm{Max}_R A \to Rxy \;\&\; Ryx$

(C17) $x \in \mathrm{Max}_R A \;\&\; y \in \mathrm{Max}_R A \;\&\; \mathrm{AntiSym}_{(E)} R \to Exy$

Proof. (C15) We have to prove

$$(A \subseteq B) \;\&\; (x \in A \;\&\; x \in R^{\triangle}A) \;\&\; (y \in B \;\&\; y \in R^{\triangle}B) \to Rxy.$$

Now $A \subseteq B \;\&\; y \in R^{\triangle}B$ implies $y \in R^{\triangle}A$ by (C4) which, together with $x \in A$, implies Rxy.

(C16) To prove this, we simply have to combine the antecedents and consequents of the following two trivial assertions

$$x \in A \;\&\; y \in R^{\triangle}A \to Rxy$$
$$y \in A \;\&\; x \in R^{\triangle}A \to Ryx$$

and the proof is completed.

(C17) follows directly from (C16). □

A nonchalant interpretation of (C15) is that the larger (with respect to inclusion) A is, the larger (with respect to R) $\mathrm{Max}_R A$ is. The property (C16) can be interpreted as the fact that $\mathrm{Max}_R A$ is "fuzzily" unique up to the symmetrization of R. In the case that, in a non-graded setting, R is a fuzzy preorder, it is easily possible to show that its symmetrization is a similarity [16, 68]. Then (C16) means nothing else than that $\mathrm{Max}_R A$ is a *fuzzy point* [53]. The property (C17) generalizes this to any relation R antisymmetric (to some degree) with respect to E.

The following theorem shows that maxima are upper classes inside the fuzzy class that is considered (to the degree R is transitive).

Theorem 5.13. *The following property of maxima is provable in* FCT*:*

(C18) $\mathrm{Trans}(R) \to \mathrm{Upper}_R^A(\mathrm{Max}_R\, A)$

Proof. By (C3), $\mathrm{Trans}(R)$ implies $x \in R^{\triangle} A \,\&\, Rxy \to y \in R^{\triangle} A$ which implies the required $x \in A \,\&\, y \in A \,\&\, Rxy \,\&\, (x \in A \,\&\, x \in R^{\triangle} A) \to (y \in A \,\&\, y \in R^{\triangle} A)$. □

Now we can finally define suprema and infima. Not surprisingly, the suprema are defined as the least upper bounds, i.e., the minima of the upper cone. Again the condition of being a supremum is graded, as the notion of a bound itself is graded. Dually, the infima are defined as the greatest lower bounds.

Definition 5.14. The classes of all *suprema* and *infima* of a class A with respect to R are defined as follows:

$$\begin{aligned} \mathrm{Sup}_R\, A &=_{\mathrm{df}} \mathrm{Min}_R(R^{\triangle} A) \\ \mathrm{Inf}_R\, A &=_{\mathrm{df}} \mathrm{Max}_R(R^{\triangledown} A) \end{aligned}$$

The index R is dropped under the same conditions as noted above.

Obviously, we can rewrite the definitions in the following way:

(C19) $\mathrm{Sup}\, A = {}^{\triangle}A \cap {}^{\triangledown\triangle}A$

(C20) $\mathrm{Inf}\, A = {}^{\triangledown}A \cap {}^{\triangle\triangledown}A$

As shown by the following theorem, suprema and infima are interdefinable.

Theorem 5.15. *The following property of suprema is provable in* FCT*:*

(C21) $\mathrm{Sup}\, A = \mathrm{Inf}({}^{\triangle}A)$

Proof. By (C20) and (C6), $\mathrm{Inf}({}^{\triangle}A) = {}^{\triangledown\triangle}A \cap {}^{\triangle\triangledown\triangle}A = {}^{\triangledown\triangle}A \cap {}^{\triangle}A = \mathrm{Min}({}^{\triangle}A) = \mathrm{Sup}\, A$. □

Since suprema are a special kind of minima, the general properties of the latter hold for suprema as well; further properties of suprema hold by virtue of the properties of cones. Some of such properties of suprema are summarized in the following theorem.

Theorem 5.16. *The following properties of suprema are provable in* FCT*:*

(C22) $A \subseteq B \,\&\, x \in \mathrm{Sup}_R\, A \,\&\, y \in \mathrm{Sup}_R\, B \to Rxy$

(C23) $x \in \mathrm{Sup}_R\, A \,\&\, y \in \mathrm{Sup}_R\, A \to Rxy \,\&\, Ryx$

(C24) $x \in \mathrm{Sup}_R\, A \,\&\, y \in \mathrm{Sup}_R\, A \,\&\, \mathrm{AntiSym}_{(E)}\, R \to Exy$

(C25) $\mathrm{Trans}(R) \to \mathrm{Upper}_R^A(\mathrm{Sup}_R\, A)$

(C26) $\mathrm{Trans}(R) \to \mathrm{Lower}_R^{R^{\triangle}A}(\mathrm{Sup}_R\, A)$

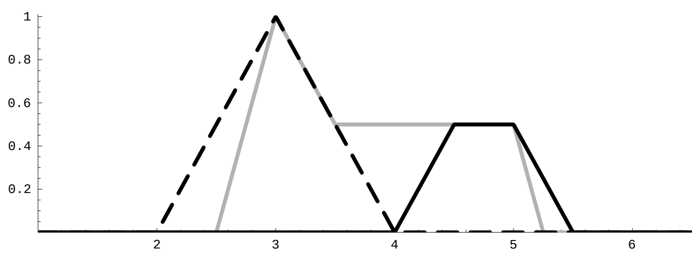
Proof. (C22) Follows from (C4) and the dual of (C15).

(C23) and (C24) follow respectively from the duals of (C16) and (C17).

(C25) By (C3), $\mathrm{Trans}(R)$ implies $x \in {}^{\triangle}A \,\&\, Rxy \to y \in {}^{\triangle}A$. Furthermore, by (C5), $y \in A \to y \in {}^{\triangledown\triangle}A$. Combining the antecedents and consequents of these implications we get the required $x \in A \,\&\, y \in A \,\&\, Rxy \,\&\, (x \in {}^{\triangle}A \,\&\, x \in {}^{\triangledown\triangle}A) \to (y \in {}^{\triangle}A \,\&\, y \in {}^{\triangledown\triangle}A)$.

(C26) Follows fr om the dual of (C18). □

Figure 7: The fuzzy set A from Example 4.13 (light gray), its infimum $\mathrm{Inf}_{L_1} A$ (dashed black), and its supremum $\mathrm{Sup}_{L_1} A$ (solid black).



Suprema differ from maxima already in crisp sets. The following example shows how the difference may look like in fuzzy sets.

Example 5.17. Let us revisit Example 5.11. Figure 7 shows the fuzzy set A along with $\mathrm{Inf}_{L_1} A$ and $\mathrm{Sup}_{L_1} A$. (Compare with Figures 5 and 6.)

The following theorem provides us with two results on how suprema and maxima are related to each other. For the precondition $A \subseteq A \cap A$ in (C28) see Remark 3.11.

Theorem 5.18. *The following interrelations between maxima and suprema are provable in* FCT*:*

(C27) $A \cap \mathrm{Max}\, A \subseteq A \cap \mathrm{Sup}\, A \subseteq \mathrm{Max}\, A$

(C28) $A \subseteq A \cap A \to \mathrm{Max}\, A \approxeq A \cap \mathrm{Sup}\, A$

Proof. (C27) Using (C5), we can infer $A \cap {}^{\triangle}A \cap A \subseteq A \cap {}^{\triangle}A \cap {}^{\triangledown\triangle}A \subseteq A \cap {}^{\triangle}A$.

(C28) $A \subseteq A \cap A \to A \cap {}^{\triangle}A \subseteq A \cap {}^{\triangle}A \cap A$ which, together with the proof of (C27), yields the converse implication to $A \cap \mathrm{Sup}\, A \subseteq \mathrm{Max}\, A$ of (C27). □

By means of suprema and infima, the notion of lattice completeness can be defined [32]. A systematic study of complete lattices and fuzzy lattice completions in FCT will be part of a subsequent paper. For some particular cases, see [2].

6 Valverde-style characterizations of preorders and similarities

This section aims at generalizing some of the most important and influential theorems in the theory of fuzzy relations to FCT—Valverde's representation theorems for fuzzy preorders and similarities [68]. In the tradition of Cantor [21], Valverde uses score functions to represent relations. Actually, he uses score functions that map into the unit interval, so these functions can also be considered as fuzzy sets. This interpretation facilitates an easy reformulation of these results in FCT.

Let us first consider the fuzzy relation R^{ℓ} defined as

$$R^{\ell}xy \equiv_{\mathrm{df}} (\forall z)(Rzx \to Rzy)$$

(for a given fuzzy relation R). This is called the *left trace* of R [35, 36]. Analogously we define the *right trace* (which will be used in Section 7) as

$$R^{r}xy \equiv_{\mathrm{df}} (\forall z)(Ryz \to Rxz).$$

Now we can formulate another characterization of graded reflexivity and transitivity besides those of Theorem 3.6.

Theorem 6.1. *The following properties hold in* FCT*:*

(V1) $\quad \mathrm{Refl}(R) \leftrightarrow R^{\ell} \subseteq R$

(V2) $\quad \mathrm{Trans}(R) \leftrightarrow R \subseteq R^{\ell}$

Proof. (V1) By definition, $R^{\ell} \subseteq R \leftrightarrow (\forall x, y)[(\forall z)(Rzx \rightarrow Rzy) \rightarrow Rxy]$. Thus to prove the first implication of (V1), we need to show that Rxy is implied by $\mathrm{Refl}(R)$ and $(\forall z)(Rzx \rightarrow Rzy)$. Specifying x for z in the latter, we get $Rxx \rightarrow Rxy$, which implies Rxy by $\mathrm{Refl}(R)$. To prove the converse implication, we can specify x for y in $(\forall x, y)[(\forall z)(Rzx \rightarrow Rzy) \rightarrow Rxy]$ and get $(\forall x)[(\forall z)(Rzx \rightarrow Rzx) \rightarrow Rxx]$, i.e. $(\forall x)(1 \rightarrow Rxx)$, i.e. $(\forall x)Rxx$.

(V2) $\quad \mathrm{Trans}(R) \longleftrightarrow (\forall z, x, y)(Rzx \,\&\, Rxy \rightarrow Rzy) \longleftrightarrow (\forall x, y)(\forall z)[Rxy \rightarrow (Rzx \rightarrow Rzy)] \longleftrightarrow (\forall x, y)[Rxy \rightarrow (\forall z)(Rzx \rightarrow Rzy)] \longleftrightarrow R \subseteq R^{\ell}$ □

Corollary 6.2. *The following is provable in* FCT*:*

(V3) $\quad \mathrm{wPreord}(R) \leftrightarrow R \approx R^{\ell}$

(V4) $\quad \mathrm{Preord}(R) \leftrightarrow R \approxeq R^{\ell},$

(V5) $\quad R \approx^2 R^{\ell} \longrightarrow \mathrm{Preord}(R) \longrightarrow R \approx R^{\ell}.$

So we have obtained graded versions of Fodor's characterizations [35, Theorems 4.1, 4.3, and Corollary 4.4]. Note that, regardless of the symmetry of R, we can replace R^{ℓ} in the above characterizations by the right trace as well (since $(R^{r})^{-1} = (R^{-1})^{\ell}$, and reflexivity and transitivity are invariant to inversion by Remark 3.2).

Remark 6.3. Observe that the following holds obviously (cf. Definitions B.7 and 5.7):

$$R^{\leftarrow}\{x\} = \{z \mid (\exists y \in \{x\})Rzy\} = \{z \mid Rzx\}$$
$$R^{\triangledown}\{x\} = \{z \mid (\forall a \in \{x\})Rza\} = \{z \mid Rzx\}$$

So we can rewrite (V3) as follows:

$$\mathrm{wPreord}(R) \;\leftrightarrow\; (\forall x, y)\big(Rxy \leftrightarrow R^{\leftarrow}\{x\} \subseteq R^{\leftarrow}\{y\}\big)$$
$$\mathrm{wPreord}(R) \;\leftrightarrow\; (\forall x, y)\big(Rxy \leftrightarrow R^{\triangledown}\{x\} \subseteq R^{\triangledown}\{y\}\big)$$

In words, a relation R is a weak preorder to the degree it coincides with graded inclusion between the cones (or preimages) of crisp singletons.

Now we have all prerequisites for formulating and proving a graded version of Valverde's representation theorem for preorders. In order to make notations more compact, let us define two graded notions of *Valverde preorder representation* (a strong one and a weak one), for a given fuzzy relation R and a fuzzy class of fuzzy classes $\mathcal{A}$:

$$\mathrm{ValP}(R, \mathcal{A}) \;\equiv_{\mathrm{df}}\; (R \approxeq \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \rightarrow Ay)\})$$
$$\mathrm{wValP}(R, \mathcal{A}) \;\equiv_{\mathrm{df}}\; (R \approx \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \rightarrow Ay)\})$$

The predicates ValP and wValP express the degree to which the fuzzy class $\mathcal{A}$ represents the relation R.

Then we can prove the following essential result for preorders and weak preorders.

Theorem 6.4. FCT *proves the following:*

(V6) $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $\mathrm{ValP}^2(R, \mathcal{A}) \longrightarrow \mathrm{Preord}(R) \longrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})$ & $\mathrm{ValP}(R, \mathcal{A}))$

(V7) $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$&$\mathrm{wValP}^3(R, \mathcal{A}) \longrightarrow \mathrm{wPreord}(R) \longrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})$&$\mathrm{wValP}(R, \mathcal{A}))$

Proof. We prove just (V6), the proof of (V7) is analogous. To show the first implication we define $S_{\mathcal{A}} =_{\mathrm{df}} \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$. If we show $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \to \mathrm{Preord}(S_{\mathcal{A}})$, then the application of (R28) and some quantifier shifts complete the proof.

Obviously $\mathrm{Refl}(S_{\mathcal{A}})$ is a theorem, now we show $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \to \mathrm{Trans}(S_{\mathcal{A}})$. First we have:

$$\begin{aligned} S_{\mathcal{A}}xy \mathbin{\&} S_{\mathcal{A}}yz &\longleftrightarrow (\forall A \in \mathcal{A})(Ax \to Ay) \mathbin{\&} (\forall A \in \mathcal{A})(Ay \to Az) \\ &\longrightarrow (A \in^2 \mathcal{A} \to (Ax \to Ay) \mathbin{\&} (Ay \to Az)) \\ &\longrightarrow (A \in^2 \mathcal{A} \to (Ax \to Az)), \end{aligned}$$

whence $((\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $S_{\mathcal{A}}xy$ & $S_{\mathcal{A}}yz) \to (A \in \mathcal{A} \to (Ax \to Az))$. Finally, by generalization we get $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $S_{\mathcal{A}}xy$ & $S_{\mathcal{A}}yz \longrightarrow (\forall A \in \mathcal{A})(Ax \to Az) \longleftrightarrow S_{\mathcal{A}}xz$, thus $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \to \mathrm{Trans}(S_{\mathcal{A}})$.

To prove the second implication, just take $\mathcal{A} = \big\{A \mid (\exists z)(A = \{x \mid Rzx\})\big\}$, observe that $\mathrm{Crisp}(\mathcal{A})$ and use (V4). □

Theorem 6.4 gives bounds for the degree to which a relation is a fuzzy preorder, depending on its Valverde-representability by a family of fuzzy classes. Notice that by a restatement of Theorem 6.4, also the degree of the Valverde-representability of a relation by a crisp family of fuzzy classes can be estimated from the degree of its being a preorder:

Corollary 6.5. $\mathrm{Preord}^2(R) \longrightarrow [(\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})$ & $\mathrm{ValP}(R, \mathcal{A}))]^2 \longrightarrow \mathrm{Preord}(R)$

Proof. Follows immediately from Theorem 6.4 by taking into account that the assertion $\mathrm{Crisp}(\mathcal{A}) \to \mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$ holds. □

Obviously, (V6) is more complicated than Valverde's original result; it is an example where the graded framework does not provide us with just a plain copy of the non-graded (or crisp) result. The following corollary gives us a result that is comparable with Valverde's original theorem.

Corollary 6.6. FCT *proves the following:*

(V8) $\triangle\,\mathrm{Preord}(R) \longleftrightarrow R = R^{\ell}$
$\longleftrightarrow (\exists\mathcal{A})(\triangle(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $\triangle\,\mathrm{ValP}(R, \mathcal{A}))$
$\longleftrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})$ & $\triangle\,\mathrm{ValP}(R, \mathcal{A}))$

Proof. The first equivalence is a simple consequence of (V4). To prove that $\triangle\,\mathrm{Preord}(R)$ implies the last formula inspect the proof of the second implication of (V6) and observe that we can show that $\triangle\,\mathrm{Preord}(R) \to \mathrm{Crisp}(\mathcal{A})$ & $\triangle\,\mathrm{ValP}(R, \mathcal{A})$ for $\mathcal{A} = \big\{A \mid (\exists z)(A = \{x \mid Rzx\})\big\}$.

The fact that the last formula implies the third one is a simple consequence of provability of $\mathrm{Crisp}(\mathcal{A}) \to \mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$.

The final implication we need to prove (that the third formula implies $\triangle\,\mathrm{Preord}(R)$) is a simple consequence of the first implication of (V6). □

Observe that also the analogous formulae with wPreord and wValP are equivalent to those in (V8).

Example 6.7. Let us shortly revisit Example 3.3 (in which we use standard Łukasiewicz logic). The fuzzy relation P_1 was actually constructed from the following crisp family of three fuzzy sets $\mathcal{A} = \{A_1, A_2, A_3\}$ that are defined as follows (for convenience, in vector notation):

$$A_1 = (0.7, 0.8, 0.2, 0.5, 0.4, 0.6)$$
$$A_2 = (0.3, 0.5, 0.6, 0.4, 0.7, 1.0)$$
$$A_3 = (1.0, 1.0, 0.6, 0.4, 0.3, 0.0)$$

Example 6.8. Consider $U = \{1, 2, 3\}$, standard Łukasiewicz logic, and the following fuzzy relation:

$$R = \begin{pmatrix} 1.00 & 1.00 & 0.60 \\ 0.00 & 1.00 & 1.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

It is easy to see that $\mathrm{Refl}(R) = 1$ and $\mathrm{Trans}(R) = 0.6$, hence, $\mathrm{Preord}(R) = \mathrm{wPreord}(R) = 0.6$. Now consider the crisp class $\mathcal{A} = \{A_1, A_2, A_3\}$, where A_1, A_2 and A_3 are fuzzy classes defined as follows:

$$A_1 = (1.00, 0.85, 0.70)$$
$$A_2 = (0.00, 1.00, 0.85)$$
$$A_3 = (0.00, 0.00, 1.00)$$

Then we obtain the following (according to the definition $S_{\mathcal{A}} =_{\mathrm{df}} \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$ from the proof of Theorem 6.4):

$$S_{\mathcal{A}} = \begin{pmatrix} 1.00 & 0.85 & 0.70 \\ 0.00 & 1.00 & 0.85 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

Obviously, $R \subseteq S_{\mathcal{A}} = 0.85$ and $S_{\mathcal{A}} \subseteq R = 0.90$, so we have $R \approxeq S_{\mathcal{A}} = 0.75$ and $R \approx S_{\mathcal{A}} = 0.85$, thus, $\mathrm{ValP}(R, \mathcal{A}) = 0.75$ and $\mathrm{wValP}(R, \mathcal{A}) = 0.85$. Note further that $\mathrm{Crisp}(\mathcal{A}) = 1$ holds, which also implies $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) = 1$. These findings demonstrate that the reverse implications

$$(\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{ValP}(R, \mathcal{A})) \longrightarrow \mathrm{Preord}(R)$$
$$(\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{wValP}(R, \mathcal{A})) \longrightarrow \mathrm{wPreord}(R)$$

both do not hold in general. Moreover, we see that the double exponent in the first formula of (V6) and the triple exponent in the first formula of (V7) cannot be improved.

Although the last formula in (V8) is a perfect copy of Valverde's non-graded representation, the corollary still contains a graded feature—note that unlike Valverde's theorem, in which a crisp family of functions is used, the class $\mathcal{A}$ in the third equivalent formula may still be a fuzzy class of fuzzy classes, if only it satisfies $\triangle(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$. Recall from Remark 3.11 that in Gödel logic, this condition is fulfilled by *all* fuzzy classes $\mathcal{A}$, and that in any logic it is satisfied by a system $\mathcal{A}$ in a model if all degrees of membership in $\mathcal{A}$ are idempotent with respect to conjunction.

The degree of $A \in \mathcal{A}$ may be considered as a weighting factor that controls the influence of a specific A on the final result. Corollary 6.6 requires all membership degrees in $\mathcal{A}$ to be idempotent to ensure that the relation represented by $\mathcal{A}$ is a fuzzy preorder, but its graded version in Theorem 6.4 also shows that (loosely speaking) it will *almost* be a fuzzy preorder if $\mathcal{A}$ *almost* satisfies $\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$ (e.g., in standard Łukasiewicz logic if it is close to crispness).

Example 6.9. Let us consider a $[0,1]$-valued fuzzy logic with the triangular norm

$$x * y = \begin{cases} \max(x + y - \frac{1}{2}, 0) & \text{if } x \in [0, \frac{1}{2}]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

i.e. a simple ordinal sum with a scaled Łukasiewicz t-norm in $[0, \frac{1}{2}]^2$ and the Gödel t-norm anywhere else. It is clear that the set of idempotent elements of this t-norm is $\{0\} \cup [\frac{1}{2}, 1]$ and that the corresponding residual implication is given as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ \max(y, \frac{1}{2} - x + y) & \text{otherwise.} \end{cases}$$

Now reconsider $U = \{1, \ldots, 6\}$ and the three fuzzy sets A_1, A_2 and A_3 from Example 6.7 and define a fuzzy class of fuzzy classes $\mathcal{A}$ such that $\mathcal{A}A_1 = 0.9$, $\mathcal{A}A_1 = 1.0$, and $\mathcal{A}A_3 = 0.8$. Since all three values are idempotent elements of $*$, we can be sure by (V8) that the construction $R_1 =_{\text{df}} \{\langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$ always gives us a fuzzy preorder in the given logic. In this particular example, we obtain the following:

$$R_1 = \begin{pmatrix} 1.0 & 1.0 & 0.2 & 0.4 & 0.3 & 0.0 \\ 0.3 & 1.0 & 0.2 & 0.4 & 0.3 & 0.0 \\ 0.3 & 0.5 & 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.2 & 1.0 & 0.4 & 0.1 \\ 0.3 & 0.5 & 0.3 & 0.4 & 1.0 & 0.2 \\ 0.3 & 0.5 & 0.2 & 0.4 & 0.4 & 1.0 \end{pmatrix}$$

If we repeat this construction and define a fuzzy relation $R_2 =_{\text{df}} \{\langle x, y \rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$ with $\mathcal{A}$ defined as above, but the connectives interpreted in standard Łukasiewicz logic, we obtain the following:

$$R_2 = \begin{pmatrix} 1.0 & 1.0 & 0.6 & 0.6 & 0.5 & 0.2 \\ 0.8 & 1.0 & 0.5 & 0.6 & 0.5 & 0.2 \\ 0.7 & 0.9 & 1.0 & 0.8 & 0.9 & 0.6 \\ 0.9 & 1.0 & 0.8 & 1.0 & 1.0 & 0.8 \\ 0.6 & 0.8 & 0.9 & 0.7 & 1.0 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0 \end{pmatrix}.$$

Straightforward calculations show that $\text{Refl}(R_2) = 1$ and $\text{Trans}(R_2) = \text{Preord}(R_2) = \text{wPreord}(R_2) = 0.8$. This is not at all contradicting to (V6) and (V7), as $\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$ holds only to a degree of 0.8 in standard Łukasiewicz logic.

In his landmark paper [68], Valverde not only considers fuzzy preorders, but also similarities (as obvious from the title of his paper). So the question naturally arises how we can modify the above results in the presence of symmetry. As will be seen next, the modifications are not as straightforward as in the non-graded case. Let us first define the fuzzy relation $R^{\ell s}$ as

$$R^{\ell s}xy =_{\text{df}} (\forall z)(Rzx \leftrightarrow Rzy)$$

(for a given fuzzy relation R). This is called the *left symmetric trace* of R.

The following lemma demonstrates how this notion is related to the defining properties of similarity. More or less unexpectedly, the result is not that straightforward for symmetry.

Theorem 6.10. *The following are theorems of* FCT*:*

(V9) $R^{\ell s} \subseteq R \leftrightarrow \mathrm{Refl}(R)$

(V10) $R \subseteq R^{\ell s} \rightarrow \mathrm{Trans}(R)$

(V11) $R \approxeq R^{\ell s} \rightarrow \mathrm{Sym}(R)$

(V12) $\mathrm{Sym}(R) \mathbin{\&} \mathrm{Trans}(R) \rightarrow R \subseteq R^{\ell s}$

Proof. (V9) Analogous to the proof of (V1).

(V10) Follows from (V2) by observation that $R^{\ell s} \subseteq R^{\ell}$.

(V11) Obviously we can get $R \subseteq R^{\ell s} \longrightarrow (Rxy \rightarrow (Ryx \leftrightarrow Ryy)) \longrightarrow (Rxy \rightarrow (Ryy \rightarrow Ryx))$. So $R \subseteq R^{\ell s} \mathbin{\&} \mathrm{Refl}(R) \rightarrow (\forall y)(Rxy \rightarrow Ryx)$. Finally (V9) completes the proof.

(V12) We need to show that $Rzx \leftrightarrow Rzy$ is implied by $\mathrm{Sym}(R)$, $\mathrm{Trans}(R)$, and Rxy. First by $\mathrm{Trans}(R)$ and Rxy we get $Rzx \rightarrow Rzy$; secondly, by $\mathrm{Sym}(R)$ and Rxy we get Ryx, whence by $\mathrm{Trans}(R)$ we get $Rzy \rightarrow Rzx$. □

The following theorem provides us with an analogue of Corollary 6.2, unfortunately, with looser bounds on the left-hand side.

Corollary 6.11. FCT *proves:*

(V13) $R \approx^4 R^{\ell s} \longrightarrow R \approxeq^2 R^{\ell s} \longrightarrow \mathrm{Sim}(R) \longrightarrow R \approxeq R^{\ell s} \longrightarrow R \approx R^{\ell s}$

(V14) $R \approx^2 R^{\ell s} \longrightarrow R \approxeq R^{\ell s} \longrightarrow \mathrm{wSim}(R)$

(V15) $\mathrm{wSim}^2(R) \longrightarrow \mathrm{Refl}(R) \wedge (\mathrm{Trans}(R) \mathbin{\&} \mathrm{Sym}(R)) \longrightarrow R \approx R^{\ell s}$

The question arises whether it is really necessary to require $\approxeq$ rather than $\approx$ in (V11). The following example tells us that this is indeed the case. It also implies that $R \approx R^{\ell s} \rightarrow \mathrm{wSim}(R)$ does *not* hold in general.

Example 6.12. Consider $U = \{1, 2\}$, standard Łukasiewicz logic, and the following fuzzy relation:

$$R = \begin{pmatrix} 0.5 & 1.0 \\ 0.0 & 0.5 \end{pmatrix}$$

It is obvious that $\mathrm{Refl}(R) = 0.5$ and $\mathrm{Sym}(R) = 0$. Moreover, routine calculations show that $\mathrm{Trans}(R) = 1$. To compute $R \approxeq R^{\ell s}$, we have to consider the truth values of $Rxy \leftrightarrow (\forall z)(Rzx \leftrightarrow Rzy)$ for all $x, y \in U$:

$$\begin{array}{ll}
x = 1, y = 1\colon & \min(\overbrace{0.5 \leftrightarrow (0.5 \leftrightarrow 0.5)}^{z=1}, \overbrace{0.5 \leftrightarrow (0.0 \leftrightarrow 0.0)}^{z=2}) = 0.5 \\
x = 1, y = 2\colon & \min(1.0 \leftrightarrow (0.5 \leftrightarrow 1.0), 1.0 \leftrightarrow (1.0 \leftrightarrow 0.5)) = 0.5 \\
x = 2, y = 1\colon & \min(0.0 \leftrightarrow (1.0 \leftrightarrow 0.5), 0.0 \leftrightarrow (0.5 \leftrightarrow 0.0)) = 0.5 \\
x = 2, y = 2\colon & \min(0.5 \leftrightarrow (1.0 \leftrightarrow 1.0), 0.5 \leftrightarrow (0.5 \leftrightarrow 0.5)) = 0.5
\end{array}$$

So, we finally obtain $R \approx R^{\ell s} = 0.5$ and $R \approxeq R^{\ell s} = 0$.

Now we can formulate a graded version of Valverde's representation theorem for similarities. Analogously to the above considerations, let us define the graded notion of

Valverde similarity representation (strong one and weak one) for a given fuzzy relation R and a fuzzy class $\mathcal{A}$ as

$$\begin{aligned} \mathrm{ValS}(R,\mathcal{A}) &\equiv_{\mathrm{df}} R \approxeq \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}, \\ \mathrm{wValS}(R,\mathcal{A}) &\equiv_{\mathrm{df}} R \approx \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}. \end{aligned}$$

In the same way as for preorders, we can prove graded versions of Valverde's representation theorem of similarities and weak similarities.

Theorem 6.13. FCT *proves the following:*

(V16) $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $\mathrm{ValS}^3(R,\mathcal{A}) \longrightarrow \mathrm{Sim}(R) \longrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})$ & $\mathrm{ValS}(R,\mathcal{A}))$

(V17) $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $\mathrm{wValS}^3(R,\mathcal{A}) \longrightarrow \mathrm{wSim}(R)$

(V18) $\mathrm{wSim}^2(R) \longrightarrow \mathrm{Refl}(R) \wedge (\mathrm{Trans}(R) \,\&\, \mathrm{Sym}(R)) \longrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \,\&\, \mathrm{wValS}(R,\mathcal{A}))$

Again, (V16) is more complicated than Valverde's original representation of similarities. In the following corollary, analogously to preorders, we can infer a result very similar to Valverde's original theorem in case that the corresponding properties are fulfilled to degree 1.

Corollary 6.14. FCT *proves the following:*

(V19) $\triangle\,\mathrm{Sim}(R) \longleftrightarrow R = R^{\ell s}$
$\longleftrightarrow (\exists\mathcal{A})(\triangle(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$ & $\triangle\,\mathrm{ValS}(R,\mathcal{A}))$
$\longleftrightarrow (\exists\mathcal{A})(\mathrm{Crisp}(\mathcal{A})$ & $\triangle\,\mathrm{ValS}(R,\mathcal{A}))$

Again, like in the case of preorders, the formulae in (V19) are equivalent to the variants with wSim and wValS. Also observe that again (V19) contains a graded ingredient—the class $\mathcal{A}$ may (under the same condition as in preorders) be a fuzzy class of fuzzy classes.

Example 6.15. Consider $U = [0,3]$, standard Łukasiewicz logic, and the following four fuzzy sets:

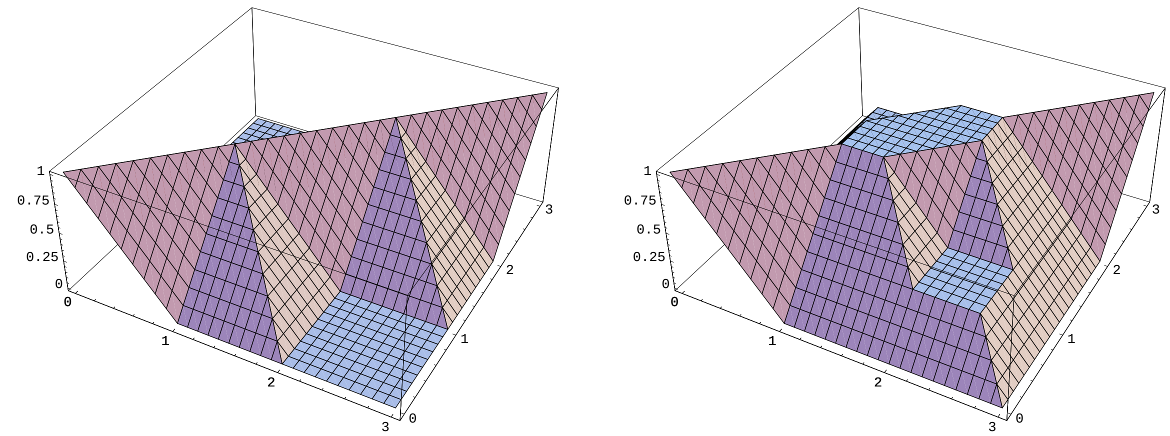
$$\begin{aligned} A_1x &= \max(0, \min(1, x)) \\ A_2x &= \max(0, \min(1, x-1)) \\ A_3x &= \max(0, \min(1, x-2)) \\ A_4x &= \max(0, \min(1, x-3)) \end{aligned}$$

Figure 8 shows plots of two fuzzy similarities that we obtain by the construction that is provided by (V19):

$$\begin{aligned} E_1xy &= (\forall A \in \mathcal{A}_1)(Ax \leftrightarrow Ay) \\ E_2xy &= (\forall A \in \mathcal{A}_2)(Ax \leftrightarrow Ay) \end{aligned}$$

where $\mathcal{A}_1 = \{A_1, A_2, A_3, A_4\}$, i.e. a crisp finite family of fuzzy sets. Hence, E_1 is the fuzzy relation obtained from Valverde's original construction. The fuzzy class $\mathcal{A}_2$, however, is defined such that $\mathcal{A}_2A_1 = \mathcal{A}_2A_3 = \mathcal{A}_2A_4 = 1$ and $\mathcal{A}_2A_2 = 0.6$, i.e. we assign a lower weight of 0.6 to the second fuzzy set. Since then $\mathcal{A}_2 \subseteq \mathcal{A}_2 \cap \mathcal{A}_2$ is true to degree 0.6, Theorem 6.13 only ensures that E_2 is a similarity to degree 0.6. In this case the bound is tight, since indeed $\mathrm{Sim}(E_2) = 0.6$.

Figure 8: Plots of the two fuzzy relations E_1 (left) and E_2 (right) from Example 6.15.



Example 6.16. Consider $U = \{1, 2, 3\}$, standard Łukasiewicz logic, and the following fuzzy relation:

$$R = \begin{pmatrix} 1.00 & 1.00 & 0.60 \\ 1.00 & 1.00 & 1.00 \\ 0.40 & 1.00 & 1.00 \end{pmatrix}$$

It is trivial to see that $\mathrm{Refl}(R) = 1$, $\mathrm{Sym}(R) = 0.8$ and $\mathrm{Trans}(R) = 0.4$, hence, $\mathrm{Sim}(R) = 0.2$ and $\mathrm{wSim}(R) = 0.4$. Now consider the crisp class $\mathcal{A} = \{A_1, A_2, A_3\}$, where A_1, A_2 and A_3 are fuzzy classes defined as follows:

$$\begin{aligned} A_1 &= (1.00, 0.75, 0.50) \\ A_2 &= (0.75, 1.00, 0.75) \\ A_3 &= (0.50, 0.75, 1.00) \end{aligned}$$

Then we obtain the following (according to the analogous definition $S'_{\mathcal{A}} = \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}$):

$$S'_{\mathcal{A}} = \begin{pmatrix} 1.00 & 0.75 & 0.50 \\ 0.75 & 1.00 & 0.75 \\ 0.50 & 0.75 & 1.00 \end{pmatrix}$$

We obtain $R \subseteq S'_{\mathcal{A}} = 0.75$ and $S'_{\mathcal{A}} \subseteq R = 0.90$, so we have $R \approxeq S'_{\mathcal{A}} = 0.65$ and $R \approx S'_{\mathcal{A}} = 0.75$, thus, $\mathrm{ValS}(R, \mathcal{A}) = 0.65$ and $\mathrm{wValP}(R, \mathcal{A}) = 0.75$. Of course, $\mathrm{Crisp}(\mathcal{A}) = 1$ and $(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) = 1$ hold, too. So, analogously to Example 6.8, we have a counterexample that demonstrates that the reverse implication

$$(\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{ValS}(R, \mathcal{A})) \longrightarrow \mathrm{Sim}(R)$$

does not hold in general. Moreover, we see that the triple exponents in (V16) and (V17) cannot be improved.

7 Similarities and partitions

The one-to-one correspondence between equivalence relations and partitions is one of the most fundamental correspondences in classical mathematics. It is clear, therefore, that fuzzy partitions have been studied intensively in connection with similarity relations. The first approach to fuzzy partitions by Ruspini [64] does not facilitate a direct correspondence with similarity relations. Only more logically oriented approaches to fuzzy partitions that were introduced more recently are able to provide a smooth interplay with similarity relations. In this section, we demonstrate how the well-accepted (non-graded) approach by De Baets and Mesiar [26] (for similar or complementary studies, see also [13, 41, 30, 45, 53, 52, 42]) can be transferred to our graded framework.

Definition 7.1. Consider a fuzzy relation R. For a given element x, we define the *afterset* of x (with respect to R) as

$$[x]_R =_{\mathrm{df}} \{y \mid Rxy\}.$$

It is clear that, if R is a similarity, $[x]_R$ can be understood as the *equivalence class* of x. Note that Gottwald, in his studies [40, 41], defines the equivalence class of x inversely as the *foreset* $\{y \mid Ryx\}$. We stick to the afterset-based definition in this section. The choice is immaterial, since the aftersets of R are the foresets of R^{-1} and vice versa, and R and R^{-1} satisfy Refl, Sym, and Trans both to the same degrees (see Remark 3.2).

The following lemma provides us with some easy-to-see links to concepts we have introduced earlier in this paper.

Lemma 7.2. *The following properties of aftersets are provable in* FCT*:*

(P1) $[x]_R = R^{\triangle}\{x\} = R^{\uparrow}\{x\}$

(P2) $[x]_R \subseteq [y]_R \longleftrightarrow (\forall z)(Rxz \rightarrow Ryz) \longleftrightarrow R^r yx$

Now we can prove some basic properties of aftersets (note that semantically equivalent results for left-continuous t-norms can be found in [41, Section 18.6]).

Theorem 7.3. *The following properties are provable in* FCT*:*

(P3) $\mathrm{Refl}(R) \leftrightarrow (\forall x)(x \in [x]_R)$

(P4) $\mathrm{Refl}(R) \leftrightarrow (\forall x, y)([x]_R \subseteq [y]_R \rightarrow Rxy)$

(P5) $\mathrm{Refl}(R) \;\&\; \mathrm{Sym}(R) \rightarrow (\forall x, y)([y]_R \subseteq [x]_R \rightarrow Rxy)$

(P6) $\mathrm{Refl}(R) \rightarrow (\forall x, y)([x]_R \approx [y]_R \rightarrow Rxy)$

(P7) $\mathrm{Refl}^2(R) \;\&\; \mathrm{Sym}(R) \rightarrow (\forall x, y)([y]_R \approxeq [x]_R \rightarrow R^2xy)$

(P8) $\mathrm{Trans}(R) \leftrightarrow (\forall x, y)(Rxy \rightarrow [y]_R \subseteq [x]_R)$

(P9) $\mathrm{Trans}(R) \;\&\; \mathrm{Sym}(R) \rightarrow (\forall x, y)(Rxy \rightarrow [x]_R \subseteq [y]_R)$

(P10) $\mathrm{Trans}(R) \;\&\; \mathrm{Sym}(R) \rightarrow (\forall x, y)(Rxy \rightarrow [x]_R \approx [y]_R)$

(P11) $\mathrm{Trans}^2(R) \;\&\; \mathrm{Sym}(R) \rightarrow (\forall x, y)(R^2xy \rightarrow [x]_R \approxeq [y]_R)$

Proof. (P3) Follows directly from the definition of $\mathrm{Refl}(R)$.

(P4) Follows from $\mathrm{Refl}(R) \leftrightarrow R^r \subseteq R$ (compare with (V1)) and (P2).

(P5) Take (P4) and apply symmetry.

(P6) Trivial consequence of (P4).

(P7) Use (P4) and (P5).

(P8) Follows from $\mathrm{Trans}(R) \leftrightarrow R \subseteq R^r$ (compare with (V2)) and (P2).

(P9) Use (P8) and symmetry.

(P10) and (P11) both follow from (P8) and (P9). □

From Theorem 7.3, we can now infer a first important result—that similarities can be represented by their aftersets (i.e., equivalence classes).

Corollary 7.4. *The following can be proved in* FCT*:*

(P12) $\mathrm{Sim}(R) \to (\forall x, y)(Rxy \leftrightarrow [x]_R \approx [y]_R)$

(P13) $\mathrm{Sim}^2(R) \to (\forall x, y)(R^2xy \leftrightarrow [x]_R \approxeq [y]_R)$

In classical mathematics, the notion of quotient set is essential for the study of the correspondence between equivalence relations and partitions. As also in previous literature, we define quotient classes in perfect analogy to the crisp case.

Definition 7.5. For a given fuzzy relation R, we define the *quotient class* V/R as the class of all aftersets (equivalence classes):

$$\mathrm{V}/R =_{\mathrm{df}} \{A \mid (\exists x)(A = [x]_R)\}$$

It is clear that the name *quotient class* is best justified if R is a similarity. Let $\mathcal{A}$ be a class of (fuzzy) classes resulting from some similarity in this way. By investigating properties of $\mathcal{A}$, we found four constituting properties: crispness, normality of its members, covering, and disjointness (in a wider sense). They are defined as follows.

Definition 7.6. Let $\mathcal{A}$ be a fuzzy class of fuzzy classes. We define the following properties of $\mathcal{A}$:

$$\begin{aligned}
\mathrm{NormM}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A \in \mathcal{A})(\exists x)\triangle Ax \\
\mathrm{Cover}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall x)(\exists A \in \mathcal{A})\triangle Ax \\
\mathrm{Disj}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A, B \in \mathcal{A})(A \parallel B \to A \approx B)
\end{aligned}$$

Correspondingly, we can define the degree to which $\mathcal{A}$ is a partition as

$$\mathrm{Part}(\mathcal{A}) \equiv_{\mathrm{df}} \mathrm{Crisp}(\mathcal{A}) \,\&\, \mathrm{NormM}(\mathcal{A}) \,\&\, \mathrm{Cover}(\mathcal{A}) \,\&\, \mathrm{Disj}(\mathcal{A})$$

The first three properties are self-explanatory, $\mathrm{Disj}(\mathcal{A})$ is a straightforward (graded) generalization of the disjointness criterion that is well-known from the literature [26, 45, 53, 52]. Without explicitly referring to this as a notion of fuzzy partition, some authors [45, 53, 52] studied the disjointness property in conjunction with normality (and crispness, as they are working in a non-graded framework). The covering property was later introduced by De Baets and Mesiar [26] and similarly studied by Demirci [30] and Bělohlávek [13]. The degree $\mathrm{Part}(\mathcal{A})$ to which a class of classes $\mathcal{A}$ is a partition is thus a straightforward (graded) generalization of the concept of *T-partition* introduced by De Baets and Mesiar [26].[6]

[6]An alternative option in Definition 7.5 is taking $\{A \mid (\exists x)(A \approx [x]_R)\}$ for the quotient class. This would yield a meaningful, fully fuzzified notion of quotient class and the results of this section would only need a slight adaptation (the $\triangle$'s in Definition 7.6 could be dropped in exchange for some more exponents in definitions and proofs). The usage of $=$ in Definition 7.5 is motivated mainly by keeping the direct correspondence with De Baets and Mesiar's notion.

Observe that the properties $\mathrm{Crisp}(\mathcal{A}), \mathrm{NormM}(\mathcal{A})$, and $\mathrm{Cover}(\mathcal{A})$ are crisp. Thus, we have

$$\mathrm{Part}(\mathcal{A}) \leftrightarrow \mathrm{Crisp}(\mathcal{A}) \wedge \mathrm{NormM}(\mathcal{A}) \wedge \mathrm{Cover}(\mathcal{A}) \wedge \mathrm{Disj}(\mathcal{A}),$$

i.e. there is no need to define a separate concept of a "weak fuzzy partition". Moreover, it follows that

$$(\mathrm{Part}(\mathcal{A}) \leftrightarrow 0) \vee (\mathrm{Part}(\mathcal{A}) \leftrightarrow \mathrm{Disj}(\mathcal{A})).$$

In other words, the truth value of $\mathrm{Part}(\mathcal{A})$ for a given $\mathcal{A}$ is either 0 or equal to the truth value of $\mathrm{Disj}(\mathcal{A})$.

Theorem 7.7. FCT *proves the following properties of the quotient* V/R*:*

(P14) $\mathrm{Crisp}(\mathrm{V}/R)$

(P15) $\triangle\,\mathrm{Refl}(R) \to \mathrm{Cover}(\mathrm{V}/R)$

(P16) $\triangle\,\mathrm{Refl}(R) \to \mathrm{NormM}(\mathrm{V}/R)$

(P17) $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$

(P18) $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R)\ \&\ \triangle\,\mathrm{Refl}(R) \to \mathrm{Part}(\mathrm{V}/R)$

Proof. (P14)–(P16) are straightforward to prove.

(P17) From $\mathrm{Trans}(R)$ and $\mathrm{Sym}(R)$, we get $Ryx\ \&\ Rzx \to Ryz$, which, using the definition, can be written as $x \in [y]_R\ \&\ x \in [z]_R \to Ryz$. Using (P8) and $\mathrm{Trans}(R)$ again, we get $x \in [y]_R\ \&\ x \in [z]_R \to [y]_R \subseteq [z]_R$. In the same way, we get $x \in [z]_R\ \&\ x \in [y]_R \to [z]_R \subseteq [y]_R$. Combining these two formulae, we get $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \to (x \in [z]_R\ \&\ x \in [y]_R \to [z]_R \approx [y]_R)$. Then applying generalization (for x), quantifier shifts, and the definition of $\|$ completes the proof.

(P18) Immediate consequence of (P14)–(P17). □

Now, after we have studied the properties of the quotient of a given fuzzy relation, the question arises how we can extract a fuzzy relation (a similarity in the ideal case) from a given fuzzy partition.

Definition 7.8. For a given fuzzy class of fuzzy classes $\mathcal{A}$ we define a fuzzy relation $R^{\mathcal{A}}$ in the following way:[7]

$$R^{\mathcal{A}} \quad =_{\mathrm{df}} \quad \{\langle x, y\rangle \mid (\exists A \in \mathcal{A})(Ax\ \&\ Ay)\}$$

This definition allows us to relate properties of partitions with properties of the induced relations in a meaningful graded manner.

Theorem 7.9. *The following properties of* $R^{\mathcal{A}}$ *are provable in* FCT*:*

(P19) $\mathrm{Sym}(R^{\mathcal{A}})$

(P20) $\mathrm{Cover}(\mathcal{A}) \to \triangle\,\mathrm{Refl}(R^{\mathcal{A}})$

(P21) $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$

(P22) $\mathrm{Part}(\mathcal{A}) \longrightarrow \triangle\,\mathrm{Sym}(R^{\mathcal{A}})\ \&\ \triangle\,\mathrm{Refl}(R^{\mathcal{A}})\ \&\ \mathrm{Trans}(R^{\mathcal{A}}) \longrightarrow \mathrm{Sim}(R^{\mathcal{A}})$

[7]Note that the definition $R^{\mathcal{A}}$ is not the only possible definition of how to "extract" a fuzzy relation from a family of subsets. Another often-used way to do that is $R = \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}$, see, e.g., [26, 52, 68].

Proof. (P19) Trivial.

(P20) $\mathrm{Cover}(\mathcal{A}) \longrightarrow (\exists X \in \mathcal{A})\triangle(x \in X) \longrightarrow \triangle(\exists X \in \mathcal{A})(x \in X \,\&\, x \in X) \longleftrightarrow \triangle R^{\mathcal{A}}xx$

(P21) From $R^{\mathcal{A}}xy$ & $R^{\mathcal{A}}yz$, we get $(\exists A, B \in \mathcal{A})(Ax \,\&\, Ay \,\&\, By \,\&\, Bz)$. Since from $\mathrm{Disj}(\mathcal{A})$ we get $(Ay \,\&\, By) \to A \approx B$, we have $(\exists A, B \in \mathcal{A})(Ax \,\&\, A \approx B \,\&\, Bz)$. As $A \approx B \,\&\, Bz \to Az$, we obtain $(\exists A \in \mathcal{A})(Ax \,\&\, Az)$, i.e., $R^{\mathcal{A}}xz$, and the proof is done.

(P22) Immediate consequence of (P19)–(P21). □

The property (P18) has told us that the quotient of a similarity is a partition. Now (P22) entails that partitions induce similarities. Note, however, that this is not yet a proof of one-to-one correspondence. We do not know yet whether these correspondences are invertible, i.e., (i) whether the quotient of a similarity induced by a partition is the same as the original partition, and (ii) whether the quotient of a given similarity induces the same similarity. The following final theorem gives answers to these questions—fortunately in a fully graded manner.

Theorem 7.10. FCT *proves the following:*

(P23) $\mathrm{Sim}(R) \to (R^{\mathrm{V}/R} \approxeq R)$

(P24) $\mathrm{Part}(\mathcal{A}) \longrightarrow \mathrm{Crisp}(\mathcal{A}) \,\&\, \mathrm{NormM}(\mathcal{A}) \,\&\, \mathrm{Disj}(\mathcal{A}) \longrightarrow (\forall A \in \mathcal{A})(\exists B \in \mathrm{V}/R^{\mathcal{A}})(A \approxeq B)$

(P25) $\mathrm{Part}(\mathcal{A}) \longrightarrow \mathrm{Crisp}(\mathcal{A}) \,\&\, \mathrm{Cover}(\mathcal{A}) \,\&\, \mathrm{Disj}(\mathcal{A}) \longrightarrow (\forall B \in \mathrm{V}/R^{\mathcal{A}})(\exists A \in \mathcal{A})(A \approxeq B)$

Proof. (P23) We shall show that $\mathrm{Sym}(R) \,\&\, \mathrm{Trans}(R) \to R^{\mathrm{V}/R} \subseteq R$ and that $\mathrm{Refl}(R) \to R \subseteq R^{\mathrm{V}/R}$. The first part is proved by the following steps:

$$
\begin{aligned}
R^{\mathrm{V}/R}xy &\longrightarrow (\exists A \in \mathrm{V}/R)(Ax \,\&\, Ay)\\
&\longrightarrow (\exists A)((\exists z)([z]_R = A) \,\&\, Ax \,\&\, Ay)\\
&\longrightarrow (\exists A)(\exists z)([z]_R = A \,\&\, Ax \,\&\, Ay)\\
&\longrightarrow (\exists A)(\exists z)([z]_R = A \,\&\, Ax \,\&\, [z]_R = A \,\&\, Ay)\\
&\longrightarrow (\exists z)(x \in [z]_R \,\&\, y \in [z]_R)\\
&\longrightarrow (\exists z)(Rzx \,\&\, Rzy)\\
&\longrightarrow (\exists z)(Rxz \,\&\, Rzy), \text{ by } \mathrm{Sym}(R),\\
&\longrightarrow Rxy, \text{ by } \mathrm{Trans}(R).
\end{aligned}
$$

The second part is proved by the following steps:

$$
\begin{aligned}
Rxy &\longrightarrow [x]_R = [x]_R \,\&\, x \in [x]_R \,\&\, y \in [x]_R, \text{ by } \mathrm{Refl}(R),\\
&\longrightarrow (\exists z)([z]_R = [z]_R \,\&\, x \in [z]_R \,\&\, y \in [z]_R)\\
&\longrightarrow (\exists A)(\exists z)([z]_R = A \,\&\, Ax \,\&\, Ay)\\
&\longrightarrow (\exists A)((\exists z)([z]_R = A) \,\&\, Ax \,\&\, Ay)\\
&\longrightarrow (\exists A \in \mathrm{V}/R)(Ax \,\&\, Ay)\\
&\longrightarrow R^{\mathrm{V}/R}xy.
\end{aligned}
$$

(P24) Let us choose a fuzzy set $A \in \mathcal{A}$. Since $\mathrm{Crisp}(\mathcal{A})$ is assumed, $\triangle A \in \mathcal{A}$ holds. Since $\mathrm{NormM}(\mathcal{A})$ holds, we know that there exists an x such that $\triangle Ax$. Now we choose $B = [x]_{R^{\mathcal{A}}}$, i.e. $By \longleftrightarrow R^{\mathcal{A}}xy \longleftrightarrow (\exists C \in \mathcal{A})(Cx \,\&\, Cy)$. Since $\triangle Ax$ and $\triangle A \in \mathcal{A}$ we get:

$$Ay \longrightarrow A \in \mathcal{A} \,\&\, Ax \,\&\, Ay \longrightarrow (\exists C \in \mathcal{A})(Cx \,\&\, Cy),$$

i.e. we have proved

$$\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{NormM}(\mathcal{A}) \rightarrow A \subseteq B \tag{2}$$

Conversely, we can prove the following:

$By \longleftrightarrow (\exists C \in \mathcal{A})(Cx \;\&\; Cy)$
$\quad \longrightarrow (\exists C \in \mathcal{A})(Ax \;\&\; Cx \;\&\; Cy)$, by $\triangle Ax$,
$\quad \longrightarrow (\exists C \in \mathcal{A})(A \approx C \;\&\; Cy)$, by $\mathrm{Disj}(\mathcal{A})$,
$\quad \longrightarrow (\exists C \in \mathcal{A})\; Ay$
$\quad \longrightarrow Ay$

So we have proved

$$\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{NormM}(\mathcal{A}) \;\&\; \mathrm{Disj}(\mathcal{A}) \rightarrow B \subseteq A. \tag{3}$$

Finally, we can join (2) and (3) to complete the proof (as the properties Crisp and NormM are crisp).

(P25) Let us consider an arbitrary $B \in \mathrm{V}/R^{\mathcal{A}}$. Since $\mathrm{Crisp}(\mathrm{V}/R^{\mathcal{A}})$ holds by (P14), we have $\triangle(B \in \mathrm{V}/R^{\mathcal{A}})$, which means that there exists an x such that $B = [x]_{R^{\mathcal{A}}} = \{y \mid R^{\mathcal{A}}xy\}$. By (P20), we have $\mathrm{Cover}(\mathcal{A}) \rightarrow \triangle\,\mathrm{Refl}(R^{\mathcal{A}})$. Hence, we have $\triangle Bx$. From $\mathrm{Cover}(\mathcal{A})$ and $\mathrm{Crisp}(\mathcal{A})$ we can deduce that we can choose an $A \in \mathcal{A}$ such that $\triangle Ax$. Hence, we can deduce the following:

$$Ay \longrightarrow A \in \mathcal{A} \;\&\; Ax \;\&\; Ay \longrightarrow (\exists C \in \mathcal{A})(Cx \;\&\; Cy) \longrightarrow By$$

So we have proved the following:

$$\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{Cover}(\mathcal{A}) \rightarrow A \subseteq B \tag{4}$$

Conversely, we can prove

$$\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{Cover}(\mathcal{A}) \;\&\; \mathrm{Disj}(\mathcal{A}) \rightarrow B \subseteq A. \tag{5}$$

completely analogously to the proof of (3) (just to get $\triangle Ax$ we use $\mathrm{Cover}(\mathcal{A})$ instead of $\mathrm{NormM}(\mathcal{A})$). Finally, we can join (4) and (5) to complete the proof. □

Nonchalantly speaking, we can say that (P24) and (P25) together mean that the more $\mathcal{A}$ is a partition, the more similar $\mathcal{A}$ and $\mathrm{V}/R^{\mathcal{A}}$ are. The question arises as to whether they are equal if $\mathcal{A}$ is a partition to a degree of 1. The next corollary gives a positive answer and lists some other well-known non-graded results [13, 26, 30] that are consequences of graded results from above.

Corollary 7.11. FCT *proves the following:*

(P26) $\triangle\,\mathrm{Sim}(R) \rightarrow \triangle\,\mathrm{Part}(\mathrm{V}/R)$

(P27) $\triangle\,\mathrm{Sim}(R) \rightarrow R^{\mathrm{V}/R} = R$

(P28) $\triangle\,\mathrm{Part}(\mathcal{A}) \rightarrow \triangle\,\mathrm{Sim}(R^{\mathcal{A}})$

(P29) $\triangle\,\mathrm{Part}(\mathcal{A}) \rightarrow \mathrm{V}/R^{\mathcal{A}} = \mathcal{A}$

Proof. The assertions (P26), (P27) and (P28) are immediate consequences of (P18), (P23) and (P22), respectively. The assertion (P29) can be proved as follows: from $\mathrm{Part}(\mathcal{A})$, we know that $\mathcal{A}$ is a crisp set and, by (P14), we know that $\mathrm{V}/R^{\mathcal{A}}$ is crisp too. Then, using $\triangle\,\mathrm{Part}(\mathcal{A})$, (P24) implies $\mathcal{A} \subseteq \mathrm{V}/R^{\mathcal{A}}$ and (P25) implies $\mathrm{V}/R^{\mathcal{A}} \subseteq \mathcal{A}$, which completes the proof. □

Let us close this section with a simple example that illustrates the above results.

Example 7.12. Let us consider $U = \{1, 2, 3, 4\}$, standard Łukasiewicz logic, and the crisp class $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, where A_1, A_2, A_3, A_4 are fuzzy sets defined in the following way:

$$\begin{aligned}
A_1 &= (1.0, 0.4, 0.3, 0.0) \\
A_2 &= (0.0, 1.0, 0.7, 0.0) \\
A_3 &= (0.1, 0.2, 1.0, 0.5) \\
A_4 &= (0.0, 0.1, 0.5, 1.0)
\end{aligned}$$

Obviously, $\text{Crisp}(\mathcal{A}) = \text{Cover}(\mathcal{A}) = \text{NormM}(\mathcal{A}) = 1$. To compute $\text{Disj}(\mathcal{A})$, we first compute the degrees of compatibility (overlapping) and equality:

$\parallel$	A_1	A_2	A_3	A_4
A_1	1.0	0.4	0.3	0.0
A_2	0.4	1.0	0.7	0.2
A_3	0.3	0.7	1.0	0.5
A_4	0.0	0.2	0.5	1.0

$\approx$	A_1	A_2	A_3	A_4
A_1	1.0	0.0	0.1	0.0
A_2	0.0	1.0	0.2	0.0
A_3	0.1	0.2	1.0	0.5
A_4	0.0	0.0	0.5	1.0

From these values, we see that the pair (A_2, A_3) is the one for which compatibility exceeds equality to the largest extent. So, we obtain

$$\text{Disj}(\mathcal{A}) = (A_2 \parallel A_3 \to A_2 \approx A_3) = (0.7 \to 0.2) = 0.5$$

which implies $\text{Part}(\mathcal{A}) = 0.5$. We can derive $R^{\mathcal{A}}$ as follows:

$$R^{\mathcal{A}} = \begin{pmatrix} 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.7 & 0.1 \\ 0.3 & 0.7 & 1.0 & 0.5 \\ 0.0 & 0.1 & 0.5 & 1.0 \end{pmatrix}$$

Obviously $\text{Refl}(R^{\mathcal{A}}) = \text{Sym}(R^{\mathcal{A}}) = 1$ (any other result would contradict our findings above). Straightforward calculations show that

$$\text{Sim}(R^{\mathcal{A}}) = \text{wSim}(R^{\mathcal{A}}) = \text{Trans}(R^{\mathcal{A}}) = 0.9.$$

Hence, we can conclude that the bounds in (P21) are not necessarily tight (which proves that the converse implication cannot generally hold).

Now let us consider the quotient $U/R^{\mathcal{A}}$. Obviously, $U/R^{\mathcal{A}} = \{B_1, B_2, B_3, B_4\}$ with

$$\begin{aligned}
B_1 &= (1.0, 0.4, 0.3, 0.0) \\
B_2 &= (0.4, 1.0, 0.7, 0.1) \\
B_3 &= (0.3, 0.7, 1.0, 0.5) \\
B_4 &= (0.0, 0.1, 0.5, 1.0)
\end{aligned}$$

and we immediately see the discrepancy between $\mathcal{A}$ and $U/R^{\mathcal{A}}$. Interestingly, we have $A_1 \subseteq B_1$, $A_2 \subseteq B_2$, $A_3 \subseteq B_3$, and $A_4 \subseteq B_4$. This is not surprising, however, if one looks at the proofs of (P24) and (P25), where we show that, for an $A \in \mathcal{A}$, we can find a $B \in \text{V}/R^{\mathcal{A}}$ such that $A \subseteq B$. Not surprisingly either, A_1 is most similar to B_1, just as A_2

is most similar to B_2, and so on. Simple calculations show that the truth values of the formulae on the right-hand sides of (P24) and (P25) are both 0.5.

If we compute $R^{U/R^{\mathcal{A}}}$, we obtain the following fuzzy relation:

$$R^{U/R^{\mathcal{A}}} = \begin{pmatrix} 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.7 & 0.1 \\ 0.3 & 0.7 & 1.0 & 0.5 \\ 0.0 & 0.1 & 0.5 & 1.0 \end{pmatrix}$$

Then routine computations show that this fuzzy relation is a similarity. So, at least in the setting of this example, successive application of computing quotients and induced similarities yields increasing degrees to which the relations are similarities and the classes of fuzzy sets are partitions.

8 Concluding remarks

In this paper, we have rephrased and generalized results on binary fuzzy relations to the graded framework of Fuzzy Class Theory (FCT). While Section 3 was more or less concerned with rewriting Gottwald's previously published results, Sections 4–7 have generalized results that were known in the non-graded framework of traditional theory of fuzzy relations to the fully graded framework of FCT. These new results hereby demonstrate that Fuzzy Class Theory is indeed a very powerful and easy-to-use framework for handling fuzzified properties of fuzzy relations.

This paper has never been intended as a comprehensive treatise that covers the whole theory of crisp or fuzzy relations. We only tried to communicate the idea of how to apply Fuzzy Class Theory to generalizing existing (and possibly discovering new) results on fuzzy relations in the fully graded framework of FCT. Obviously, much is left for future studies, and we would like to encourage everybody interested in this topic to adopt the framework and advance the results.

A First-order MTL$_\triangle$: Basic definitions

Monoidal t-norm based logic (MTL for short) was introduced by Esteva and Godo in [33] as an extension of Höhle's monoidal logic [46] by the axiom of prelinearity (i.e., the axiom (A6) below). In this appendix we recall the definitions and some of the basic properties of MTL and its expansion by the connective $\triangle$. We start with the propositional variant and then expand it to the first-order predicate variant.

The formulae of propositional logic MTL are composed from a countable set of propositional atoms by using three basic binary connectives $\rightarrow$, $\wedge$, and $\&$, and a nullary connective 0. Further connectives can be defined as:

$$\begin{aligned} \varphi \vee \psi &\equiv_{\mathrm{df}} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi), \\ \neg\varphi &\equiv_{\mathrm{df}} \varphi \rightarrow 0, \\ \varphi \leftrightarrow \psi &\equiv_{\mathrm{df}} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi), \\ 1 &\equiv_{\mathrm{df}} \neg 0. \end{aligned}$$

Convention A.1. In order to avoid unnecessary parentheses, we stipulate that unary connectives take precedence over $\wedge$, $\vee$, and $\&$, which in turn bind more closely than $\rightarrow$ and $\leftrightarrow$.

The deduction rule of MTL is Modus Ponens (from φ and $\varphi \to \psi$ infer ψ) and the following formulae are the axioms of MTL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $\varphi \,\&\, \psi \to \varphi$

(A3) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$

(A4a) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$

(A4b) $\varphi \wedge \psi \to \varphi$

(A4c) $\varphi \wedge \psi \to \psi \wedge \varphi$

(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$

(A5b) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

(A7) $0 \to \varphi$

The logic MTL$_\triangle$ was introduced in [33] as an expansion of the logic MTL by a unary connective $\triangle$, the deduction rule of necessitation (from φ infer $\triangle\varphi$), and the following axioms:

($\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$

($\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$

($\triangle$3) $\triangle\varphi \to \varphi$

($\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$

($\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

Formulae derived from these axioms by means of the mentioned deduction rules are called *theorems* of MTL$_\triangle$.

Definition A.2. An MTL-*algebra* is a structure $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1)$, where

1. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice
2. $(L, *, 1)$ is a commutative monoid
3. $x \leq (y \Rightarrow z)$ if and only if $x * y \leq z$ for all $x, y, z \in L$ (residuation)
4. $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ for all $x, y \in L$ (prelinearity)

Definition A.3. A structure $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, 0, 1, \triangle)$ is called an MTL$_\triangle$-*algebra* if $(L, *, \Rightarrow, \wedge, \vee, 0, 1)$ is an MTL-algebra and if the additional connective $\triangle$ has the following properties (for all $x, y \in L$):

1. $\triangle x \vee (\triangle x \Rightarrow 0) = 1$
2. $\triangle(x \vee y) \leq (\triangle x \vee \triangle y)$
3. $\triangle x \leq x$
4. $\triangle x \leq \triangle\triangle x$

5. $\triangle(x \Rightarrow y) \leq \triangle x \Rightarrow \triangle y$

6. $\triangle 1 = 1$

If the lattice order of an MTL$_\triangle$-algebra $\mathbf{L}$ is linear, we say that $\mathbf{L}$ is an MTL$_\triangle$-*chain*. If the lattice reduct of $\mathbf{L}$ is the real unit interval with the usual order, we say that $\mathbf{L}$ is a *standard* MTL$_\triangle$-*chain*. It can be easily shown that in each MTL$_\triangle$-chain the following holds:

$$\triangle x = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

The structure $([0,1], *, \Rightarrow, \min, \max, 0, 1, \triangle)$ is a standard MTL$_\triangle$-chain if and only if $*$ is a left-continuous t-norm and $\Rightarrow$ its residuum.

Given an MTL$_\triangle$-algebra, we can evaluate formulae of MTL$_\triangle$ by assigning elements of L to propositional atoms and computing values of compound formulae using operations of $\mathbf{L}$. A formula is a *tautology* of a given MTL$_\triangle$-algebra if it always evaluates to 1.

The completeness theorem for MTL and MTL$_\triangle$ with respect to standard algebras was proved in [49]: a formula is a theorem in MTL$_\triangle$ if and only if it is a tautology of each standard MTL$_\triangle$-algebra.

Now we introduce the language of first-order MTL$_\triangle$ logic (we give a slightly simplified account, omitting the subsumption of sorts; for full details see [5]).

Definition A.4. A *predicate language* $\mathbf{\Gamma}$ is a tuple $(\mathbf{S}, \mathbf{P}, \mathbf{F}, \mathbf{a})$, where $\mathbf{S}$ is a non-empty set of sorts of variables, $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{a}$ is an *arity function* which assigns a sequence of sorts $(s_1, \ldots, s_k)$ to each predicate symbol and a sequence of sorts $(s_1, \ldots, s_k, s_{k+1})$ to each function symbol ($k \geq 0$ in both cases). Functions with arity (s_1) are called *object constants* of sort s_1. The set $\mathbf{P}$ is supposed to contain a symbol $=$ of arity (s, s) for each sort s. For each sort s, there are countably many variables $x_1^s, x_2^s, \ldots$

For the rest of this appendix, fix a predicate language $\mathbf{\Gamma}$ and an MTL$_\triangle$-chain $\mathbf{L}$.

Definition A.5. Any variable x^s of sort s is a *term* of sort s. If $F \in \mathbf{F}$ is a function symbol of arity $(s_1, \ldots, s_k, s_{k+1})$, then for any terms $t_1, \ldots, t_k$ of respective sorts $s_1, \ldots, s_k$, the expression $F(t_1, \ldots, t_k)$ is a term of sort s_{k+1}.

Atomic formulae have the form $P(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are terms of respective sorts $s_1, \ldots, s_k$ and $P \in \mathbf{P}$ is a predicate symbol of arity $(s_1, \ldots, s_k)$. Where convenient, we switch to infix notation for binary predicate symbols.

Formulae are built from atomic formulae by using the connectives of MTL$_\triangle$ and the quantifiers $\forall, \exists$ (for a formula φ and a variable x, both $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae).

Definition A.6. An occurrence of a variable x in a formula φ is *bound* if it is in the scope of a quantifier over x; otherwise it is called *free*. A formula φ is called a *sentence* if all occurrences of variables in φ are bound.

A term t is *substitutable* for the object variable x^s of sort s in a formula $\varphi(x^s)$ if and only if t is also of sort s and no variable occurring in t becomes bound in $\varphi(t)$.

Definition A.7. First-order MTL$_\triangle$ logic (with crisp identity) has the following axioms:

(P) The axioms resulting from the axioms of MTL$_\triangle$ by substituting first-order formulae for propositional formulae

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ

(∃1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ

(∀2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ

(∃2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ

(∀3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ

(=1) $x = x$

(=2) $x = y \to (\varphi(x) \leftrightarrow \varphi(y))$, where y is substitutable for x in φ

The deduction rules are those of $\mathrm{MTL}_\triangle$ and *generalization*: from φ infer $(\forall x)\varphi$.

We define the notion of a *theorem* in the same way as in the propositional case.

Definition A.8. A *theory* is a set of formulae. A formula is *provable in a theory* T if it is derivable from the axioms of first-order $\mathrm{MTL}_\triangle$ and formulae belonging to T by the deduction rules. We denote this fact by $T \vdash \varphi$.

Definition A.9. An $\mathbf{L}$-*structure* $\mathbf{M}$ has the form: $\mathbf{M} = ((M_s)_{s\in\mathbf{S}}, (P_\mathbf{M})_{P\in\mathbf{P}}, (F_\mathbf{M})_{F\in\mathbf{F}})$, where each M_s is a non-empty set; each $P_\mathbf{M}$ is a k-ary fuzzy relation $P_\mathbf{M}\colon \prod_{i=1}^k M_{s_i} \to \mathbf{L}$ for each predicate symbol $P \in \mathbf{P}$ of arity $(s_1, \ldots, s_k)$; and $F_\mathbf{M}$ is a k-ary function $F_\mathbf{M}\colon \prod_{i=1}^k M_{s_i} \to M_{s_{k+1}}$ for each function symbol $F \in \mathbf{F}$ of arity $(s_1, \ldots, s_k, s_{k+1})$. Furthermore, $=_\mathbf{M}$ is the crisp identity of the elements of M_s for each $s \in \mathbf{S}$.

In words: an $\mathbf{L}$-structure consists of (i) domains for all sorts of variables, (ii) an interpretation of all predicate symbols by $\mathbf{L}$-fuzzy relations defined on appropriate domains, and (iii) an interpretation of all function symbols by crisp functions between appropriate domains.

Definition A.10. Let $\mathbf{M}$ be an $\mathbf{L}$-structure. An $\mathbf{M}$-*evaluation* is a mapping v which assigns an element from M_s to each object variable x of sort s. For an $\mathbf{M}$-evaluation v, a variable x of sort s, and $a \in M_s$ we define the $\mathbf{M}$-evaluation $v[x \mapsto a]$ as

$$v[x \mapsto a](y) = \begin{cases} a \text{ if } y = x \\ v(y) \text{ otherwise} \end{cases}$$

Definition A.11. Let $\mathbf{M}$ be an $\mathbf{L}$-structure and v an $\mathbf{M}$-evaluation. We define the *values* of terms and the *truth values* of formulae in $\mathbf{M}$ for an $\mathbf{M}$-evaluation v as:

$$\begin{array}{rcll} \|x\|^\mathbf{L}_{\mathbf{M},v} & = & v(x) & \\ \|F(t_1,\ldots,t_n)\|^\mathbf{L}_{\mathbf{M},v} & = & F_\mathbf{M}(\|t_1\|^\mathbf{L}_{\mathbf{M},v},\ldots,\|t_n\|^\mathbf{L}_{\mathbf{M},v}) & \text{for each } F \in \mathbf{F} \\ \|P(t_1,\ldots,\ t_n)\|^\mathbf{L}_{\mathbf{M},v} & = & P_\mathbf{M}(\|t_1\|^\mathbf{L}_{\mathbf{M},v},\ldots,\|t_n\|^\mathbf{L}_{\mathbf{M},v}) & \text{for each } P \in \mathbf{P} \\ \|c(\varphi_1,\ldots,\varphi_n)\|^\mathbf{L}_{\mathbf{M},v} & = & c_\mathbf{L}(\|\varphi_1\|^\mathbf{L}_{\mathbf{M},v},\ldots,\|\varphi_n\|^\mathbf{L}_{\mathbf{M},v}) & \text{for each connective } c \\ \|(\forall x)\varphi\|^\mathbf{L}_{\mathbf{M},v} & = & \inf\limits_{a\in M} \|\varphi\|^\mathbf{L}_{\mathbf{M},v[x\to a]} & \\ \|(\exists x)\varphi\|^\mathbf{L}_{\mathbf{M},v} & = & \sup\limits_{a\in M} \|\varphi\|^\mathbf{L}_{\mathbf{M},v[x\to a]} & \end{array}$$

If an infimum or supremum does not exist, we consider its value as undefined. We say that a structure $\mathbf{M}$ is *safe* if and only if $\|\varphi\|^\mathbf{L}_{\mathbf{M},v}$ is defined for each formula φ and each $\mathbf{M}$-evaluation v. Note that, in a standard $\mathrm{MTL}_\triangle$-algebra (or more generally in any $\mathrm{MTL}_\triangle$-algebra whose lattice reduct is a complete lattice), the *safeness* of a structure is a superfluous condition, as the suprema and infima of *all* sets exist.

Definition A.12. A formula φ is *valid* in a structure $\mathbf{M}$ (denoted as $\mathbf{M} \models \varphi$) if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 1$ for each $\mathbf{M}$-evaluation v. A structure $\mathbf{M}$ is a *model* of a theory T if $\mathbf{M} \models \varphi$ for each φ in T.

Finally we present the (strong) completeness theorem which relates syntactical and semantical aspects of the first-order $\mathrm{MTL}_\triangle$ logic (see [54, 33] for a proof). Recall that the direction from provability to validity is usually called *soundness*, whereas the converse direction one is called *completeness*.

Theorem A.13. *Let T be a theory and φ a formula. Then the following are equivalent:*

1. $T \vdash \varphi$.
2. $\mathbf{M} \models \varphi$ *for each* $\mathrm{MTL}_\triangle$*-chain* $\mathbf{L}$ *and each safe* $\mathbf{L}$*-model* $\mathbf{M}$ *of* T.
3. $\mathbf{M} \models \varphi$ *for each standard* $\mathrm{MTL}_\triangle$*-chain* $\mathbf{L}$ *and each* $\mathbf{L}$*-model* $\mathbf{M}$ *of* T.

Thus by (1)⇒(2) we get that if a formula is provable in a given theory T, then it is valid in all models of T over *all* $\mathrm{MTL}_\triangle$-chains. Conversely, by (3)⇒(1) we get that if a formula is valid in all models of T over all all *standard* $\mathrm{MTL}_\triangle$-chains, then it is provable in T.

B Fuzzy Class Theory: Basic definitions

In this section, we present an overview of Fuzzy Class Theory (FCT) in order to provide the reader with the necessary background. Note that this is only a brief introduction to the most basic concepts of FCT with the aim to keep the paper self-contained. Readers who want to understand all proof details or even to make proofs in FCT themselves should not expect to find all necessary material in this appendix. Instead, they are referred to the freely available primer [7].

Fuzzy Class Theory has the aim to axiomatize the notion of fuzzy set. In the first paper [5], it was based on the logic ŁΠ [34]. In this paper, we use the logic $\mathrm{MTL}_\triangle$; obviously all definitions and basic results of [5] can be transferred from ŁΠ to $\mathrm{MTL}_\triangle$. For an introduction to $\mathrm{MTL}_\triangle$, see Appendix A (for a more extensive overview of propositional MTL, see [33]; a more detailed treatment on first-order $\mathrm{MTL}_\triangle$ with crisp equality can be found in [43]).

Definition B.1. *Fuzzy Class Theory* (over $\mathrm{MTL}_\triangle$) is a theory over multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp equality. We have sorts for individuals of the zeroth order (i.e., atomic objects) denoted by lowercase variables $a, b, c, x, y, z, \ldots$; individuals of the first order (i.e., fuzzy classes) denoted by uppercase variables $A, B, X, Y, \ldots$; individuals of the second order (i.e., fuzzy classes of fuzzy classes) denoted by calligraphic variables $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \ldots$; etc. Individuals $\xi_1, \ldots, \xi_k$ of each order can form k-tuples (for any $k \geq 0$), denoted by $\langle \xi_1, \ldots, \xi_k \rangle$; tuples are governed by the usual axioms known from classical mathematics (e.g., that tuples equal if and only if their respective constituents equal). Furthermore, for each variable x of any order n and for each formula φ there is a class term $\{x \mid \varphi\}$ of order $n + 1$.

Besides the logical predicate of identity, the only primitive predicate is the membership predicate $\in$ between successive sorts (i.e., between individuals of the n-th order and individuals of the $(n + 1)$-st order, for any n).[8] The axioms for $\in$ are the following (for variables of all orders):

[8] By this requirement, Russell's paradox is avoided in a similar fashion as in type theory [65].

(∈1) $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for each formula φ (comprehension axioms)

(∈2) $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$ (extensionality)

Notice that for proving theorems of FCT from the axioms, we have to use the logic $\mathrm{MTL}_\triangle$ rather than classical Boolean logic.

Observation B.2. Since the language of FCT is the same at each order, defined symbols of any order can be shifted to all higher orders as well. Since furthermore the axioms of FCT have the same form at each order, all theorems on FCT-definable notions are preserved by uniform upward order-shifts.

Convention B.3. For better readability, let us make the following conventions:

- We use the notations $(\forall x \in A)\varphi$, $(\exists x \in A)\varphi$ as abbreviations for $(\forall x)(x \in A \rightarrow \varphi)$ and $(\exists x)(x \in A \mathbin{\&} \varphi)$, respectively.
- The notation $\{x \in A \mid \varphi\}$ is short for $\{x \mid x \in A \mathbin{\&} \varphi\}$.
- We use $\{\langle x_1, \ldots, x_k\rangle \mid \varphi\}$ as abbreviation for

$$\{x \mid (\exists x_1) \ldots (\exists x_k)(x = \langle x_1, \ldots, x_k\rangle \mathbin{\&} \varphi)\}.$$

- The formulae $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (n times) are abbreviated φ^n; instead of $(x \in A)^n$, we can write $x \in^n A$ (analogously for other predicates).
- Furthermore, $x \notin A$ is shorthand for $\neg(x \in A)$; analogously for other binary predicates.
- We use Ax and $Rx_1 \ldots x_n$ synonymously for $x \in A$ and $\langle x_1, \ldots, x_n\rangle \in R$, respectively.
- A chain of implications $\varphi_1 \rightarrow \varphi_2, \varphi_2 \rightarrow \varphi_3, \ldots, \varphi_{n-1} \rightarrow \varphi_n$ will for short be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$; analogously for the equivalence connective.

Definition B.4. In FCT, we define the following elementary fuzzy set operations:

$\emptyset$	$=_{\mathrm{df}}$	$\{x \mid 0\}$	empty class
V	$=_{\mathrm{df}}$	$\{x \mid 1\}$	universal class
$\mathrm{Ker}(A)$	$=_{\mathrm{df}}$	$\{x \mid \triangle(x \in A)\}$	kernel
$A \cap B$	$=_{\mathrm{df}}$	$\{x \mid x \in A \mathbin{\&} x \in B\}$	intersection
$A \sqcap B$	$=_{\mathrm{df}}$	$\{x \mid x \in A \wedge x \in B\}$	min-intersection
$A \sqcup B$	$=_{\mathrm{df}}$	$\{x \mid x \in A \vee x \in B\}$	max-union
$A \setminus B$	$=_{\mathrm{df}}$	$\{x \mid x \in A \mathbin{\&} x \notin B\}$	difference

Definition B.5. Further we define in FCT the following elementary relations between fuzzy sets:

$\mathrm{Norm}(A)$	$\equiv_{\mathrm{df}}$	$(\exists x)\triangle(x \in A)$	normality
$\mathrm{Crisp}(A)$	$\equiv_{\mathrm{df}}$	$(\forall x)\triangle(x \in A \vee x \notin A)$	crispness
$A \subseteq B$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \rightarrow x \in B)$	inclusion
$A \approxeq B$	$\equiv_{\mathrm{df}}$	$(A \subseteq B) \mathbin{\&} (B \subseteq A)$	(strong) bi-inclusion
$A \approx B$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \leftrightarrow x \in B)$	weak bi-inclusion
$A \parallel B$	$\equiv_{\mathrm{df}}$	$(\exists x)(x \in A \mathbin{\&} x \in B)$	compatibility

Definition B.6. The union and intersection of a class of classes are functions defined as

$$\bigcup \mathcal{A} \quad =_{\mathrm{df}} \quad \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$$

$$\bigcap \mathcal{A} \quad =_{\mathrm{df}} \quad \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$$

Definition B.7. In FCT, we define the following operations:

$R^{\leftarrow} A$	$=_{\mathrm{df}}$	$\{x \mid (\exists y)(y \in A \,\&\, Rxy)\}$	pre-image
$R \circ S$	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid (\exists z)(Rxz \,\&\, Szy)\}$	composition
R^{-1}	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid Ryx\}$	converse relation
Id	$=_{\mathrm{df}}$	$\{\langle x, y\rangle \mid x = y\}$	identity relation

The following lemma lists a collection of results that are helpful in this paper.

Lemma B.8. *The following results are provable in* FCT*:*

(L4) $\varphi \,\&\, (\psi \to \chi) \to ((\varphi \to \psi) \to \chi)$

(L5) $\bigcup\{B \mid \varphi(B)\} \subseteq A \leftrightarrow (\forall B)(\varphi(B) \to B \subseteq A)$

(L6) $A \subseteq \bigcap\{B \mid \varphi(B)\} \leftrightarrow (\forall B)(\varphi(B) \to A \subseteq B)$

(L7) $\varphi(C) \to \bigcap\{B \mid \varphi(B)\} \subseteq C$

(L8) $\varphi(C) \to C \subseteq \bigcup\{B \mid \varphi(B)\}$

(L9) $(\forall x)(\varphi \wedge \psi) \leftrightarrow (\forall x)\varphi \wedge (\forall x)\psi$

(L10) $(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\varphi$

(L11) $(\exists x)(\varphi \wedge \psi) \to (\exists x)\varphi \wedge (\exists x)\psi$

(L12) $(\exists x)(\varphi \,\&\, \psi) \to (\exists x)\varphi \,\&\, (\exists x)\psi$

(L13) $(\forall x)\varphi \vee (\forall x)\psi \to (\forall x)(\varphi \vee \psi)$

(L14) $(\exists x)(\varphi \,\&\, \chi) \leftrightarrow (\exists x)\varphi \,\&\, \chi$*, where x is not free in χ*

(L15) $(\forall x)\varphi \,\&\, (\forall x)\psi \to (\forall x)(\varphi \,\&\, \psi)$

(L16) $(\forall x \in A)(\chi \to \psi) \to (\chi \to (\forall x \in A)\psi)$*, where x is free in χ*

(L17) $(\forall x \in A)(\varphi \to \psi) \to ((\forall x \in A)\varphi \to (\forall x \in A \cap A)\psi)$

(L18) $(\forall x \in A)(\varphi \to \psi) \to ((\exists x \in A \cap A)\varphi \to (\exists x \in A)\psi)$

The models of FCT are systems (closed under definable operations) of fuzzy sets (and fuzzy relations) of all orders over some crisp universe U, where the membership functions of fuzzy subsets take values in some $\mathrm{MTL}_{\triangle}$-chain (see [33] and Appendix A). Intended models are those which contain *all* fuzzy subsets and fuzzy relations over U (of all orders); we call such models *full*. Models in which moreover the $\mathrm{MTL}_{\triangle}$-chain is standard (i.e., given by a left-continuous t-norm on the unit interval $[0, 1]$) correspond to Zadeh's [69] original notion of fuzzy set; therefore we call them *Zadeh models*.

FCT is sound with respect to Zadeh (or full) models; thus, whatever we prove in FCT is true about real-valued (or $\mathbf{L}$-valued for any $\mathrm{MTL}_{\triangle}$-chain $\mathbf{L}$) fuzzy sets and relations. Although the theory of Zadeh models is not *completely* axiomatizable,[9] the axiomatic system of FCT approximates it very well: the comprehension axioms ensure the existence of (at least) all fuzzy sets which are *definable* (by a formula of FCT), and the axioms of extensionality ensure that fuzzy sets are determined by their membership functions. This axiomatization is sufficient for almost all practical purposes; it can be characterized as *simple type theory over fuzzy logic* (cf. [59]) or *Henkin-style higher-order fuzzy logic.*

[9]Due to Gödel's Incompleteness Theorem [37], as natural numbers are definable in Zadeh models over $\mathrm{MTL}_{\triangle}$.

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References

[1] W. Bandler and L. J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[2] L. Běhounek. Two notions of fuzzy lattice completion. In V. Novák and M. Štěpnička, editors, *Proceedings of LSC IV & 4th Workshop of the ERCIM Working Group on Soft Computing*, pages 22–24, Ostrava, 2005. University of Ostrava.

[3] L. Běhounek. Extensionality in graded properties of fuzzy relations. In *Proceedings of 11th IPMU Conference*, pages 1604–1611, Paris, 2006. Edition EDK.

[4] L. Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159:1153–1164, 2008.

[5] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[6] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[7] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[8] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, 2007.

[9] L. Běhounek and M. Daňková. Relational compositions in Fuzzy Class Theory. *Fuzzy Sets and Systems*, 160:1005–1036, 2009.

[10] R. E. Bellman and L. A. Zadeh. Decision making in a fuzzy environment. *Management Science*, 17(4):141–164, 1970.

[11] R. Bělohlávek. Fuzzy closure operators. *Journal of Mathematical Analysis and Applications*, 262:473–489, 2001.

[12] R. Bělohlávek. Fuzzy closure operators II: Induced relations, representation, and examples. *Soft Computing*, 7:53–64, 2002.

[13] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[14] R. Bělohlávek and T. Funioková. Fuzzy interior operators. *International Journal of General Systems*, 33(4):415–430, 2004.

[15] I. Bloch and H. Maître. Triangular norms as a tool for constructing fuzzy mathematical morphologies. In *Proceedings of EURASIP Workshop on Mathematical Morphology and its Applications to Signal Processing*, pages 157–161, Barcelona, 1993.

[16] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(5):593–610, 2000.

[17] U. Bodenhofer. A unified framework of opening and closure operators with respect to arbitrary fuzzy relations. *Soft Computing*, 7:220–227, 2003.

[18] U. Bodenhofer, M. De Cock, and E. E. Kerre. Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems. *International Journal of General Systems*, 32(4):343–360, 2003.

[19] U. Bodenhofer and M. Demirci. Strict fuzzy orderings in a similarity-based setting. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 297–302, Barcelona, 2005.

[20] D. Boixader, J. Jacas, and J. Recasens. Fuzzy equivalence relations: Advanced material. In D. Dubois and H. Prade, editors, *Fundamentals of Fuzzy Sets*, volume 7 of *The Handbooks of Fuzzy Sets*, pages 261–290. Kluwer Academic Publishers, Boston, 2000.

[21] G. Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen*, 46:481–512, 1895.

[22] P. Cintula and R. Horčík. Fuzzy Class Theory: Some advanced topics. In M. Štěpnička, V. Novák, and U. Bodenhofer, editors, *New Dimensions in Fuzzy Logic and Related Technologies: Proceedings of the 5th Eusflat Conference*, volume I, pages 137–143. University of Ostrava, 2007.

[23] B. De Baets. Analytical solution methods for fuzzy relational equations. In D. Dubois and H. Prade, editors, *Fundamentals of Fuzzy Sets*, volume 7 of *The Handbooks of Fuzzy Sets*, pages 291–340. Kluwer Academic Publishers, Boston, 2000.

[24] B. De Baets. Generalized idempotence in fuzzy mathematical morphology. In E. E. Kerre and M. Nachtegael, editors, *Fuzzy Techniques in Image Processing*, volume 52 of *Studies in Fuzziness and Soft Computing*, chapter 2, pages 58–75. Physica-Verlag, Heidelberg, 2000.

[25] B. De Baets and R. Mesiar. Pseudo-metrics and T-equivalences. *Journal of Fuzzy Mathematics*, 5(2):471–481, 1997.

[26] B. De Baets and R. Mesiar. T-partitions. *Fuzzy Sets and Systems*, 97(2):211–223, 1998.

[27] B. De Baets and R. Mesiar. Metrics and T-equalities. *Journal of Mathematical Analysis and Applications*, 267:331–347, 2002.

[28] M. De Cock and E. E. Kerre. On (un)suitable fuzzy relations to model approximate equality. *Fuzzy Sets and Systems*, 133(2):137–153, 2003.

[29] M. De Cock and E. E. Kerre. Fuzzy modifiers based on fuzzy relations. *Information Sciences*, 160:173–199, 2004.

[30] M. Demirci. On many-valued partitions and many-valued equivalence relations. *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems*, 11(2):235–253, 2003.

[31] M. Demirci. A theory of vague lattices based on many-valued equivalence relations—I: General representation results. *Fuzzy Sets and Systems*, 151(3):437–472, 2005.

[32] M. Demirci. A theory of vague lattices based on many-valued equivalence relations—II: Complete lattices. *Fuzzy Sets and Systems*, 151(3):473–489, 2005.

[33] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[34] F. Esteva, L. Godo, and F. Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[35] J. Fodor. Traces of fuzzy binary relations. *Fuzzy Sets and Systems*, 50(3):331–341, 1992.

[36] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.

[37] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38:173–198, 1931.

[38] J. A. Goguen. L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18:145–174, 1967.

[39] S. Gottwald. Fuzzified fuzzy relations. In R. Lowen and M. Roubens, editors, *Proceedings of 4th IFSA Congress*, volume Mathematics (ed. P. Wuyts), pages 82–86, Brussels, 1991.

[40] S. Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View*. Vieweg, Wiesbaden, 1993.

[41] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[42] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[43] P. Hájek and P. Cintula. Triangular norm based predicate fuzzy logics. To appear in the Proceedings of Linz Seminar 2005, 2005.

[44] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[45] U. Höhle. Fuzzy equalities and indistinguishability. In *Proceedings of 1st European Congress on Fuzzy and Intelligent Technologies*, volume 1, pages 358–363, Aachen, 1993.

[46] U. Höhle. Commutative, residuated l-monoids. In U. Höhle and E. P. Klement, editors, *Non-Classical Logics and Their Applications to Fuzzy Subsets*, pages 53–106. Kluwer, Dordrecht, 1995.

[47] U. Höhle and N. Blanchard. Partial ordering in L-underdeterminate sets. *Information Sciences*, 35(2):133–144, 1985.

[48] J. Jacas and J. Recasens. Fuzzified properties of fuzzy relations. In *Proceedings of the 9th IPMU Conference*, volume 1, pages 157–161, Annecy, 2002.

[49] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[50] E. E. Kerre, editor. *Introduction to the Basic Principles of Fuzzy Set Theory and Some of Its Applications*. Communication and Cognition, Gent, 1993.

[51] F. Klawonn. Fuzzy sets and vague environments. *Fuzzy Sets and Systems*, 66:207–221, 1994.

[52] F. Klawonn and J. Castro. Similarity in fuzzy reasoning. *Mathware and Soft Computing*, 3(2):197–228, 1995.

[53] R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.

[54] F. Montagna and H. Ono. Kripke semantics, undecidability and standard completeness for Esteva and Godo's logic MTL∀. *Studia Logica*, 71(2):227–245, 2002.

[55] N. N. Morsi, W. Lotfallah, and M. El-Zekey. The logic of tied implications, part 2: Syntax. *Fuzzy Sets and Systems*, 157:2030–2057, 2006.

[56] M. Nachtegael and E. E. Kerre. Classical and fuzzy approaches towards mathematical morphology. In E. E. Kerre and M. Nachtegael, editors, *Fuzzy Techniques in Image Processing*, volume 52 of *Studies in Fuzziness and Soft Computing*, chapter 1, pages 3–57. Physica-Verlag, Heidelberg, 2000.

[57] M. Nachtegael, E. E. Kerre, and A. M. Radzikowska. On links between fuzzy morphology and fuzzy rough sets. In *Proceedings of 8th IPMU Conference*, volume 3, pages 1381–1388, Madrid, 2000.

[58] H. Nguyen and E. Walker. *A First Course in Fuzzy Logic*. CRC Press, Boca Raton, 1997.

[59] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149(2):235–273, 2004.

[60] S. V. Ovchinnikov. Stucture of fuzzy binary relations. *Fuzzy Sets and Systems*, 6:169–195, 1981.

[61] S. V. Ovchinnikov. An introduction to fuzzy relations. In D. Dubois and H. Prade, editors, *Fundamentals of Fuzzy Sets*, volume 7 of *The Handbooks of Fuzzy Sets*, pages 233–259. Kluwer Academic Publishers, Boston, 2000.

[62] H. Poincaré. *La Science et l'Hypothése*. Flammarion, Paris, 1902.

[63] H. Poincaré. *La Valeur de la Science*. Flammarion, Paris, 1904.

[64] E. Ruspini. A new approach to clustering. *Information and Control*, 15:22–32, 1969.

[65] B. Russell. Mathematical logic as based on the theory of types. *American Journal of Mathematics*, 30:222–262, 1908.

[66] E. Sanchez. Solution of fuzzy equations with extended operations. *Fuzzy Sets and Systems*, 12:237–248, 1984.

[67] E. Trillas and L. Valverde. An inquiry into indistinguishability operators. In H. J. Skala, S. Termini, and E. Trillas, editors, *Aspects of Vagueness*, pages 231–256. Reidel, Dordrecht, 1984.

[68] L. Valverde. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, 17(3):313–328, 1985.

[69] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[70] L. A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.

[71] L. A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Transactions on Systems, Man, and Cybernetics*, 3(1):28–44, 1973.

Relational compositions in Fuzzy Class Theory

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Abstract: We present a method for mass proofs of theorems of certain forms in a formal theory of fuzzy relations and classes. The method is based on formal identification of fuzzy classes and inner truth values with certain fuzzy relations, which allows transferring basic properties of sup-T and inf-R compositions to a family of more than 30 composition-related operations, including sup-T and inf-R images, pre-images, Cartesian products, domains, ranges, resizes, inclusion, height, plinth, etc. Besides yielding a large number of theorems on fuzzy relations as simple corollaries of a few basic principles, the method provides a systematization of the family of relational notions and generates a simple equational calculus for proving elementary identities between them, thus trivializing a large part of the theory of fuzzy relations.

Keywords: Fuzzy relation, sup-T-composition, inf-R-composition, BK-product, fuzzy class theory, formal truth value. MSC 2000: 03E72, 03E70.

1 Introduction

The theory of fuzzy relations is a prerequisite to any other discipline of fuzzy mathematics. In this paper we show a method for mass proofs of theorems of certain forms in a formal theory of fuzzy relations. The method is based on transferring the properties of sup-T and inf-R relational compositions [42, 2] to a family of related notions in the theory of fuzzy sets and relations. We work in the formal framework of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT), introduced in [12]; we follow the methodology of [15].

Some part of the method we employ has already been briefly and informally sketched in Bělohlávek's book [18, Remark 6.16]. Our formal setting allows us to extend it to a larger family of notions and exploit the analogies between composition-related notions systematically, thus obtaining a large number of theorems on fuzzy relations for free. Furthermore, the syntactical apparatus of FCT makes it possible to show the soundness of this method by means of formal interpretations [9].

In consequence of methodological assumptions of deductive fuzzy logic explained in [10], our framework is constrained by certain requirements. First, our fuzzy sets can only take membership degrees in $\mathrm{MTL}_\triangle$-algebras [23] (possibly expanded by additional operators). In particular, if the system of membership degrees is the real unit interval

$[0,1]$, then our conjunction is bound to be a left-continuous t-norm $*$ and implication its residuum $\Rightarrow$. Thus we do not deal with more general conjunctive or implicational operators, such as mean conjunctions, Kleene–Dienes or early Zadeh implication, etc., which have also been considered for relational products [2]. Secondly, we always assume that we work over a fixed crisp ground set V. That is, our atomic objects (urelements) x are always elements of V (so *for all* x means actually *for all* $x \in V$); our fuzzy sets are always elements of the system $\mathcal{F}(V)$ of all fuzzy subsets of V (so their membership functions are defined for all $x \in V$); our n-ary fuzzy relations are elements of $\mathcal{F}(V^n)$; our second-order fuzzy sets are elements of $\mathcal{F}(\mathcal{F}(V))$, or $\mathcal{F}\big(\bigcup_{n=1}^{\infty} \mathcal{F}(V^n)\big)$ if fuzzy sets of fuzzy relations are considered; etc.

For simplicity, our exposition only deals with homogeneous binary fuzzy relations. Nevertheless, the results can easily be extended to heterogeneous binary fuzzy relations (see Remarks 4.10 and 5.6); a further extension to fuzzy relations of larger arities is hinted at in Remark 5.23.

The paper presents an application of formal methods of FCT to fuzzy relational notions; hence it is inevitably loaded with heavy formalism. Its details may therefore be hard to follow for readers unfamiliar with the apparatus of FCT or formal fuzzy logic. Nevertheless, some of the results and the general picture of interrelations between the composition-based notions might still be of interest to readers who are not interested in formalistic details. Therefore we shall first give an informal account of the methods presented in the paper and indicate which parts of the paper could be relevant for a broader audience.

The basic idea of the paper is to systematically exploit the similarity of the definitions of several fuzzy relational concepts. For instance, the definition of sup-T-composition of fuzzy relations, which in the traditional style of fuzzy mathematics reads

$$(R \circ S)xy = \bigvee_z Rxz * Szy, \tag{1}$$

is very similar to the definitions of the preimage and image of a fuzzy set under a fuzzy relation, which read, respectively,

$$(R^{\leftarrow} A)x = \bigvee_z Rxz * Az \tag{2}$$

$$(S^{\rightarrow} A)y = \bigvee_z Az * Szy \tag{3}$$

(where $*$ is a left-continuous t-norm). As observed in [18, Remark 6.16], the similarity extends to the point that many properties of sup-T-composition transfer to the properties of images and preimages. By formalization of the definitions in a suitable formal framework (viz, that of FCT), we are able to delimit a class of relational notions (listed in Tables 1–5 below) and a class of their properties that transfer automatically, without the need of separate proofs.

Obviously the reason why many properties of sup-T-compositions transfer to images and preimages is the same form of the definitions (1)–(3), the only difference being the absence of one of the variables occurring in (1) from the formulae (2) and (3). The definitions (2) and (3) can actually be reduced to instances of (1), by substituting a dummy object $\underline{0}$ for the variable to be eliminated from (1). By this trick, the fuzzy set A in the definition of preimage becomes identified with a suitable fuzzy relation S, namely such S

that $S z \underline{0} = A z$ and $S z y = 0$ for $y \neq \underline{0}$, where $\underline{0}$ is an arbitrarily chosen (but fixed) element.

It turns out to be useful to employ this representation of a fuzzy set by a suitable fuzzy relation systematically, as it will enable us to reduce several more notions to relational compositions. Thus in general we identify a fuzzy set A with the fuzzy relation $\mathbf{R}_A$ such that

$$\mathbf{R}_A xy = \begin{cases} Ax & \text{if } y = \underline{0} \\ 0 & \text{otherwise} \end{cases}$$

(in the following sections, the relation $\mathbf{R}_A$ is denoted simply by $\boldsymbol{A}$ or even just A). The operation of preimage then satisfies $\mathbf{R}_{R^{\leftarrow}A} = R \circ \mathbf{R}_A$, i.e., is represented as a special case of $R \circ S$ (for $S = \mathbf{R}_A$). Simplifying the notation, we may write simply $R^{\leftarrow} A = R \circ A$. Similarly, the operation of image satisfies $\mathbf{R}_{R^{\rightarrow}A} = R^{\mathrm{T}} \circ \mathbf{R}_A$ (or simply $R^{\rightarrow} A = R^{\mathrm{T}} \circ A$), where R^{T} is the transposition of R, i.e., $R^{\mathrm{T}} xy = Ryx$ (the transposition is needed for substituting $\underline{0}$ for the first rather than second variable in the definition of $R \circ S$, to match with the definition of $R^{\rightarrow} A$).

With the identification of A and $\mathbf{R}_A$, we can extend the compositional representation to further notions, for instance the Cartesian product of two fuzzy sets,

$$(A \times B)xy = Ax * By.$$

This is done by substituting the dummy object $\underline{0}$ for the variable z in the definition (1) of $R \circ S$ (notice that $\bigvee_z$ becomes void if z is fixed to the single element $\underline{0}$), which yields $A \times B = \mathbf{R}_A \circ \mathbf{R}_B^{\mathrm{T}}$, or simply $A \times B = A \circ B^{\mathrm{T}}$. The properties of sup-T-compositions (e.g., associativity) thus automatically transfer to Cartesian products as well.

Besides fuzzy sets, single membership degrees can also be represented by suitable fuzzy relations: namely, if both arguments in Rxy are replaced by $\underline{0}$, then the expression $R\underline{0}\underline{0}$ denotes the particular membership degree of the single pair $\underline{0}\underline{0}$ in R. Conversely, a membership degree α can be represented by a fuzzy relation $\mathbf{R}_\alpha$ defined as

$$\mathbf{R}_\alpha xy = \begin{cases} \alpha & \text{for } x = y = \underline{0} \\ 0 & \text{otherwise.} \end{cases}$$

(Again, in the following sections we simply write $\boldsymbol{\alpha}$ or just α instead of $\mathbf{R}_\alpha$.) This representation of membership degrees by fuzzy relations yields further composition-based relational notions, obtained by replacing more than one variable by the dummy object $\underline{0}$ in (1).

For instance, replacing all three variables x, y, z in (1) by $\underline{0}$ will yield the conjunction $*$ of truth degrees, as clearly $\mathbf{R}_{\alpha * \beta} = \mathbf{R}_\alpha \circ \mathbf{R}_\beta$ (notice that $\bigvee_z$ is again void as z is fixed to the single value $\underline{0}$). Similarly, by setting $x = z = \underline{0}$ (or $y = z = \underline{0}$) we obtain the operation of α-resize αA, defined as $(\alpha A)x = \alpha * Ax$ for all x, satisfying, as again the supremum over $z = \underline{0}$ is void, $\mathbf{R}_{\alpha A} = \mathbf{R}_\alpha \circ \mathbf{R}_A$, or simply $\alpha A = \alpha \circ A$. Finally, by setting $x = y = \underline{0}$ in (1) we obtain the graded relation of compatibility (or the height of intersection) of two fuzzy sets,

$$(A \parallel B) = \bigvee_z (Az * Bz),$$

with $\mathbf{R}_{A \parallel B} = \mathbf{R}_A^{\mathrm{T}} \circ \mathbf{R}_B$.

Further useful notions can be obtained, e.g., by substituting the maximal fuzzy set, i.e., the fuzzy set V such that $Vx = 1$ for all x, for some of the arguments in the above definitions. Thus, e.g., the graded property of height of a fuzzy set,

$$\operatorname{Hgt} A = \bigvee_z Az,$$

arises as $V \parallel A$, i.e., $\mathbf{R}_{\operatorname{Hgt} A} = \mathbf{R}_V^{\mathrm{T}} \circ \mathbf{R}_A$, or $\operatorname{Hgt} A = V^{\mathrm{T}} \circ A$. Similarly the domain and range of a fuzzy relation R are defined as $R^{\leftarrow}V$ and $R^{\rightarrow}V$, respectively, i.e., $\operatorname{Dom} R = R \circ V$ and $\operatorname{Rng} R = R^{\mathrm{T}} \circ V$.

Such properties of sup-T-composition that are preserved under restricting its arguments to relations of the form $\mathbf{R}_A$ or $\mathbf{R}_\alpha$ then automatically transfer to all members of the above family of notions. Among such properties are, e.g., the associativity of $\circ$, its monotony with respect to fuzzy inclusion, its invariance or monotony under unions and intersections, etc. Representing the family of notions as special cases of composition thus yields a mass proof method for their properties, as it is only necessary to prove such properties for the single notion of sup-T-composition $\circ$; their validity for the whole family of derived notions then follows automatically.

Furthermore, the associativity and transposition properties of sup-T-composition

$$(R \circ S) \circ T = R \circ (S \circ T) \tag{4}$$

$$(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}} \tag{5}$$

allow us to derive interrelations between the composition-based notions by simple equational calculations. For instance, $R^{\rightarrow}(S^{\rightarrow}A) = (S \circ R)^{\rightarrow}A$ is proved by the following identities, which just apply (4) and (5) to the derived notions:

$$R^{\rightarrow}(S^{\rightarrow}A) = R^{\mathrm{T}} \circ (S^{\mathrm{T}} \circ A) = (R^{\mathrm{T}} \circ S^{\mathrm{T}}) \circ A = (S \circ R)^{\mathrm{T}} \circ A = (S \circ R)^{\rightarrow}A.$$

The application of the simple rules (4), (5) to nested composition-based notions thus yields an infinite number of easily derivable corollaries.

The method just described for sup-T-compositions can also be applied to other kinds of fuzzy relational products—for instance, the BK-products, i.e., the inf-R-composition $\lhd$ and the related products $\rhd$ and $\square$, introduced by Bandler and Kohout in [2] and defined as

$$(R \lhd S)xy = \bigwedge_z (Rxz \Rightarrow Szy) \tag{6}$$

$$(R \rhd S)xy = \bigwedge_z (Szy \Rightarrow Rxz) \tag{7}$$

$$(R \,\square\, S)xy = \bigwedge_z ((Rxz \Rightarrow Szy) \wedge (Szy \Rightarrow Rxz)) \tag{8}$$

(where $\Rightarrow$ is the residuum of the left-continuous t-norm $*$). The elimination of some variables from (6)–(8), formally achieved by the same trick of substituting the dummy object $\underline{0}$, produces a family of notions analogous to those based on sup-T-composition. The family includes further well-known operations, such as:

- The graded inclusion of fuzzy sets $(A \subseteq B) = \bigwedge_z (Az \Rightarrow Bz)$, which can be represented as the BK-product $\mathbf{R}_A^{\mathrm{T}} \lhd \mathbf{R}_B$, or simply $A^{\mathrm{T}} \lhd B$, and thus is the BK-analogue of compatibility $(A \parallel B) = A^{\mathrm{T}} \circ B$

- The operation of plinth, $\operatorname{Plt} A = \bigwedge_z Az = V^{\mathrm{T}} \lhd A$, which is the BK-analogue of height $\operatorname{Hgt} A = V^{\mathrm{T}} \circ A$

- The implication $\Rightarrow$ itself, as $\mathbf{R}_{\alpha\Rightarrow\beta} = \mathbf{R}_\alpha \lhd \mathbf{R}_\beta$: thus by our conventions, $\alpha \Rightarrow \beta$ can also be written as $\alpha \lhd \beta$; it is the BK-analogue of the conjunction $*$.

The BK-analogues of the operations of image, preimage, Cartesian product, and α-resize are also important and appear frequently in fuzzy mathematics (see Examples 5.9–5.14 below). The present approach systematizes these notions and suggests their systematic names (e.g., $\lhd$-image, $\rhd$-preimage, etc.).

Again, the well-known properties of BK-products, such as their monotony with respect to inclusion, their invariance or monotony under unions and intersections, etc., are transferred to the whole family of BK-based notions. Furthermore, (4) and (5) jointly with the identities valid for BK-products

$$(R \lhd S)^{\mathrm{T}} = S^{\mathrm{T}} \rhd R^{\mathrm{T}}$$
$$R \lhd (S \lhd T) = (R \circ S) \lhd T$$
$$R \lhd (S \rhd T) = (R \lhd S) \rhd T$$

enable us to derive interrelations between all sup-T and BK-based notions by easy equational calculations. The resulting simple equational calculus contains more than thirty notions from both sup-T and BK families and covers a large part of the theory of fuzzy sets and fuzzy relations. The calculus thus may serve as a basis for an automated generation of a broad class of valid theorems on fuzzy sets and fuzzy relations.

The present paper carries out the above ideas in a rigorous manner within the formal framework of Fuzzy Class Theory:

Section 2 briefly introduces the apparatus of FCT over the logic $\mathrm{MTL}_\triangle$ and gives definitions of the standard notions employed in the paper. It also contains several lemmata needed later for proofs of some theorems; readers who are not interested in formal proofs can safely skip them.

Section 3 gives a formal account of the representation of fuzzy sets A and membership degrees α by the fuzzy relations $\mathbf{R}_A$ and $\mathbf{R}_\alpha$ (denoted there just $\boldsymbol{A}$ and $\boldsymbol{\alpha}$ for the sake of simplicity) and illustrates it on the matrix representation of fuzzy relations, under which fuzzy sets correspond to (file) vectors and membership degrees to scalars. For the representation of truth degrees, however, it is necessary first to internalize semantic truth values within the theory: recall that FCT has no variables for truth degrees, so a model that represents them by some FCT-defined fuzzy sets has to be constructed first. The construction of inner truth values is important for many parts of fuzzy mathematics formalized in FCT (cf. Remark 3.5). Nevertheless, the readers who are not interested in metamathematical issues can safely skip the part on the internalization and simply assume that we have the lattice L of truth values α at our disposal within the theory.

The formal definition of the family of notions based on sup-T-compositions is given in Section 4, where the reduction to compositions is also illustrated by showing how they work under the matrix and graph representations of fuzzy relations. The notions based on BK-products are then treated in Section 5; their importance for fuzzy mathematics is exemplified by Examples 5.9–5.14. The basic properties of sup-T-compositions are given in Theorem 4.2 and Corollary 4.3, and those of BK-compositions in Theorem 5.3 and Corollary 5.4. Their automatic consequents for the derived notions are listed in Corollaries 4.7–4.14 and 5.15–5.19. Independently of the formalism employed in their derivation, these corollaries may be of interest for a broader fuzzy community as a reference table listing a number of properties of fuzzy relational notions.

2 Preliminaries

Fuzzy Class Theory FCT, introduced in [12], is an axiomatization of Zadeh's notion of fuzzy set in formal fuzzy logic. Here we use its variant defined over $\mathrm{MTL}_\triangle$ [23], the logic of all left-continuous t-norms enriched with the connective $\triangle$, since it is arguably [10] the weakest fuzzy logic with good inferential properties for fully graded fuzzy mathematics and its expressive power is sufficient for our needs. The results of the present paper are readily transferable to any well-behaved extension of $\mathrm{MTL}_\triangle$(formally, to any deductive fuzzy logic in the sense of [10]), e.g., Łukasiewicz, product, or Gödel logic, Hájek's basic logic BL, etc. [30, 23].

We assume the reader's familiarity with first-order $\mathrm{MTL}_\triangle$; for details on this logic see [23, 32]. We only recapitulate its standard $[0, 1]$ semantics here:

$\&$	...	any left-continuous t-norm $*$
$\to$	...	the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$
$\wedge, \vee$	...	min, max
$\neg$	...	$\neg x =_{\mathrm{df}} x \Rightarrow 0$
$\leftrightarrow$	...	bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
$\triangle$	...	$\triangle x =_{\mathrm{df}} 1 - \mathrm{sgn}(1 - x)$
$\forall, \exists$	...	inf, sup

For reference, the following definition lists the axioms of multi-sorted first-order $\mathrm{MTL}_\triangle$ with crisp identity.

Definition 2.1. The language of *multi-sorted first-order logic* $\mathrm{MTL}_\triangle$ *with identity* consists of the binary connectives $\to$, $\&$, and $\wedge$, unary connective $\triangle$, propositional constant 0, quantifiers $\forall$ and $\exists$, binary predicate $=$, an arbitrary fixed set of predicate and function symbols of arbitrary arities, a pre-ordered set of sorts of variables, and countably many variables of each sort. There are the following defined connectives:

$$
\begin{aligned}
\varphi \vee \psi &\equiv_{\mathrm{df}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\
\neg\varphi &\equiv_{\mathrm{df}} \varphi \to 0 \\
\varphi \leftrightarrow \psi &\equiv_{\mathrm{df}} (\varphi \to \psi) \wedge (\psi \to \varphi) \\
1 &\equiv_{\mathrm{df}} \neg 0
\end{aligned}
$$

The deduction rules of first-order $\mathrm{MTL}_\triangle$ are the *modus ponens* (from φ and $\varphi \to \psi$ infer ψ), $\triangle$*-necessitation* (from φ infer $\triangle\varphi$), and *generalization* (from φ infer $(\forall x)\varphi$), for all well-formed formulae φ and ψ of the given language.

The axioms of first-order $\mathrm{MTL}_\triangle$ with crisp identity are the following, for all well-formed formulae φ, ψ, χ of the given language:

(MTL1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(MTL2) $(\varphi \,\&\, \psi) \to \varphi$
(MTL3) $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$
(MTL4a) $(\varphi \,\&\, (\varphi \to \psi)) \to (\varphi \wedge \psi)$
(MTL4b) $(\varphi \wedge \psi) \to \varphi$
(MTL4c) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(MTL5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$
(MTL5b) $((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(MTL6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(MTL7) $0 \to \varphi$

(△1)	$\triangle\varphi \vee \neg\triangle\varphi$
(△2)	$\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$
(△3)	$\triangle\varphi \to \varphi$
(△4)	$\triangle\varphi \to \triangle\triangle\varphi$
(△5)	$\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

(∀1)	$(\forall x)\varphi(x) \to \varphi(t)$ if t is substitutable for x in $\varphi(x)$
(∃1)	$\varphi(t) \to (\exists x)\varphi(x)$ if t is substitutable for x in $\varphi(x)$
(∀2)	$(\forall x)(\chi \to \varphi(x)) \to (\chi \to (\forall x)\varphi(x))$ if x is not free in χ
(∃2)	$(\forall x)(\varphi(x) \to \chi) \to ((\exists x)\varphi(x) \to \chi)$ if x is not free in χ
(∀3)	$(\forall x)(\chi \vee \varphi(x)) \to (\chi \vee (\forall x)\varphi(x))$ if x is not free in χ

(=1)	$x = x$
(=2)	$x = y \to (\varphi(x) \leftrightarrow \varphi(y))$ if y is substitutable for x in $\varphi(x)$

In (∀1)–(=2), x and y can be of any sort of variables in the given language. (Recall that in multi-sorted logics, the definition of substitutability requires the compatibility of sorts besides the usual conditions.)

By appropriate restrictions of language we get the propositional logics MTL$_\triangle$ or MTL (without $\triangle$) and the first-order logics MTL$_\triangle$ or MTL, with or without crisp identity.

Convention 2.2. In order to save some parentheses, we apply usual rules of precedence to propositional connectives of MTL$_\triangle$, namely, $\to$ and $\leftrightarrow$ have lower priority than other binary connectives, and unary connectives have the highest priority. We use the sign $\equiv$ for equivalence-by-definition. A chain of implications $\varphi_1 \to \varphi_2$, $\varphi_2 \to \varphi_3$, …, $\varphi_{n-1} \to \varphi_n$ can be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$ (and similarly for the equivalence connective).

Besides the axioms, we shall use the theorems of first-order MTL$_\triangle$ listed in [23, 16] without mention, as they are standard instruments for proving in MTL$_\triangle$ (for more details on proof techniques in MTL$_\triangle$, see [13, 16]). Furthermore we shall need the following lemmata:

Lemma 2.3. MTL$_\triangle$ *proves:*

1. $\triangle\neg\varphi \leftrightarrow \triangle(\varphi \leftrightarrow 0)$
2. $\triangle\neg\varphi \mathbin{\&} \triangle\neg\psi \to \triangle(\varphi \leftrightarrow \psi)$
3. $\varphi \mathbin{\&} (\chi \to \psi) \to (\chi \to \varphi \mathbin{\&} \psi)$
4. $\varphi \mathbin{\&} (\psi \to \chi) \to ((\varphi \to \psi) \to \chi)$
5. $(\exists y)(\forall x)\varphi \to (\forall x)(\exists y)\varphi$
6. $\chi \mathbin{\&} (\forall x)\varphi \to (\forall x)(\chi \mathbin{\&} \varphi))$, *if x is not free in χ.*

Proof. 1. By (MTL7) and $\triangle$-necessitation, $\triangle(0 \to \varphi)$ is a theorem; thus $\triangle\neg\varphi \longleftrightarrow (\triangle(\varphi \to 0) \wedge \triangle(0 \to \varphi)) \longleftrightarrow \triangle(\varphi \leftrightarrow 0)$.

2. By 1., $\triangle\neg\varphi \to \triangle(\varphi \leftrightarrow 0)$ and $\triangle\neg\psi \to \triangle(0 \leftrightarrow \psi)$, whence the statement follows by the ($\triangle$-necessitated) transitivity of $\leftrightarrow$.

3. follows from the MTL-theorems $\zeta \longleftrightarrow (1 \to \zeta) \longleftrightarrow (1\&\zeta)$ and $(\vartheta \to \varphi)\&(\psi \to \chi) \to (\vartheta \mathbin{\&} \psi \to \varphi \mathbin{\&} \chi)$ with 1 for ϑ.

4. is proved by the following chain of equivalences:
$[(\varphi \to \psi) \mathbin{\&} (\psi \to \chi) \to (\varphi \to \chi)] \longleftrightarrow [\varphi \to ((\varphi \to \psi) \mathbin{\&} (\psi \to \chi) \to \chi)] \longleftrightarrow$
$[\varphi \to ((\psi \to \chi) \to ((\varphi \to \psi) \to \chi))] \longleftrightarrow [(\varphi \mathbin{\&} (\psi \to \chi)) \to ((\varphi \to \psi) \to \chi)]$.

5. From the instance $(\forall x)\varphi \to \varphi$ of $(\forall 1)$ we get $(\forall y)((\forall x)\varphi \to \varphi)$ by generalization, whence $(\exists y)(\forall x)\varphi \to (\exists y)\varphi$ follows by quantifier distribution. Generalization over x and a quantifier shift completes the proof.

6. is proved by the following chain of equivalences and implications:
$(\forall x)(\chi \,\&\, \varphi \to \chi \,\&\, \varphi) \longleftrightarrow (\forall x)(\chi \to (\varphi \to \chi \,\&\, \varphi)) \longleftrightarrow [\chi \to (\forall x)(\varphi \to \chi \,\&\, \varphi)] \longrightarrow$
$[\chi \to ((\forall x)\varphi \to (\forall x)(\chi \,\&\, \varphi))] \longleftrightarrow [\chi \,\&\, (\forall x)\varphi \to (\forall x)(\chi \,\&\, \varphi)]$
by (MTL5a,b), $(\forall 2)$, and quantifier distribution. □

Lemma 2.4. *The following shifts of relativized quantifiers (cf. Convention 2.6 below) are provable in first-order* MTL *(with or without* $\triangle$*), if* x *is not free in* χ *and* y *is not free in* ϑ*:*

1. $(\exists y)(\chi \,\&\, (\forall x)(\vartheta \to \varphi)) \to (\forall x)(\vartheta \to (\exists y)(\chi \,\&\, \varphi))$
2. $(\forall x)(\varphi \to (\chi \to \psi)) \leftrightarrow (\chi \to (\forall x)(\varphi \to \psi))$
3. $(\forall x)(\varphi \to (\psi \to \chi)) \leftrightarrow ((\exists x)(\varphi \,\&\, \psi) \to \chi)$
4. $(\exists x)(\varphi \,\&\, (\chi \to \psi)) \to (\chi \to (\exists x)(\varphi \,\&\, \psi))$
5. $(\exists x)(\varphi \,\&\, (\psi \to \chi)) \to ((\forall x)(\varphi \to \psi) \to \chi)$

Proof. 1. is proved by the following chain of implications based respectively on Lemma 2.3, statements $(6, 5, 3)$, and the shift of $\exists$ over implication:
$(\exists y)(\chi \,\&\, (\forall x)(\vartheta \to \varphi)) \longrightarrow (\exists y)(\forall x)(\chi \,\&\, (\vartheta \to \varphi)) \longrightarrow (\forall x)(\exists y)(\chi \,\&\, (\vartheta \to \varphi)) \longrightarrow$
$(\forall x)(\exists y)(\vartheta \to \chi \,\&\, \varphi) \longrightarrow (\forall x)(\vartheta \to (\exists y)(\chi \,\&\, \varphi))$.

2. follows from the following chain of equivalences:

$$(\forall x)(\varphi \to (\chi \to \psi)) \longleftrightarrow (\forall x)(\chi \to (\varphi \to \psi)) \longleftrightarrow (\chi \to (\forall x)(\varphi \to \psi))$$

3.–5. follow in a similar way from (MTL5a,b), Lemma 2.3(3) and Lemma 2.3(4), respectively, by usual quantifier shifts. □

We now proceed to the definition of the apparatus of Fuzzy Class Theory (i.e., Henkin-style higher-order fuzzy logic) over $\mathrm{MTL}_\triangle$.

Definition 2.5. *Fuzzy Class Theory* FCT is a formal theory over a multi-sorted first-order deductive fuzzy logic (in this paper, $\mathrm{MTL}_\triangle$), with the sorts of variables for

- *Atomic objects* ('urelements'), denoted by lowercase letters $x, y, \ldots$
- *Fuzzy classes of atomic objects* (uppercase letters $A, B, \ldots$)
- *Fuzzy classes of fuzzy classes of atomic objects* (calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$)
- Etc., in general for *fuzzy classes of the n-th order* ($X^{(n)}, Y^{(n)}, \ldots$)

Besides the crisp *identity predicate* $=$, the language of FCT contains:

- The *membership predicate* $\in$ between objects of successive sorts
- *Class terms* $\{x \mid \varphi\}$ of order $n+1$, for any formula φ and any variable x of any order n
- Symbols $\langle x_1, \ldots, x_k \rangle$ for *k-tuples* of individuals $x_1, \ldots, x_k$ of any order

FCT has the following axioms (for all formulae φ and variables of any order):

- The logical axioms of multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp identity

- The *tuple-identity axioms* (for all k): $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_k \rangle \to x_1 = y_1 \,\&\, \ldots \,\&\, x_k = y_k$

- The *comprehension axioms*: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$

- The *extensionality axioms*: $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$ □

The models of FCT are systems (closed under definable operations) of fuzzy sets of all orders over a fixed crisp universe of discourse, with truth degrees taking values in an $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$ (e.g., the interval $[0, 1]$ equipped with a left-continuous t-norm). Thus all theorems on fuzzy classes provable in FCT are true statements about $\mathbf{L}$-valued fuzzy sets, for any $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$.

For details on the apparatus of FCT we refer the reader to [12, 14] or a freely available primer [16]. Peculiar properties of fuzzy mathematics axiomatized over formal fuzzy logic are described in [17]. The following features of FCT are worth mentioning here:

- In FCT, fuzzy sets are rendered as a *primitive notion,* rather than modeled by membership functions. In order to capture this distinction, the objects of FCT are called *fuzzy classes* rather than fuzzy sets; the name *fuzzy set* is reserved for membership functions in the models of the theory.[1] Nevertheless, since FCT is sound w.r.t. models formed of *all* fuzzy subsets, the reader can always safely substitute fuzzy sets for our classes.

- Not only the membership predicate $\in$, but all defined notions of FCT are in general fuzzy (unless they are defined as provably crisp). FCT thus presents a fully graded approach to fuzzy mathematics. The importance of full gradedness in fuzzy mathematics is explained in [16, 11, 8]: its main merit is in that it allows inferring relevant information even when a property of fuzzy sets is not fully satisfied.

- Since FCT is a formal theory over the fuzzy logic $\mathrm{MTL}_\triangle$, its theorems have to be derived by the rules of $\mathrm{MTL}_\triangle$ rather than classical Boolean logic which is used in usual mathematical theories. For details on proving theorems in FCT see [16] or [13].

- Since the language and axioms of FCT have the same form for all orders of fuzzy classes, it is sufficient to formulate conventions, definitions, and theorems only for the lowest order, as they can be propagated to all higher orders automatically.

Convention 2.6. In formulae of FCT, we employ usual abbreviations known from classical and fuzzy mathematics, including the following ones:

[1]The difference between fuzzy sets and classes is not just terminological: due to the first-order axiomatization, some fuzzy subsets may be missing from a model of FCT. An extreme example is provided by models consisting only of *crisp* subsets: it can be observed that they satisfy all axioms of FCT over $\mathrm{MTL}_\triangle$. Such non-intended models can be excluded by additional axioms ensuring the existence of non-crisp classes.

$$
\begin{array}{rll}
Ax & \equiv_{\mathrm{df}} & x \in A \\
x_1 \ldots x_k & =_{\mathrm{df}} & \langle x_1, \ldots, x_k \rangle \\
x \notin A & \equiv_{\mathrm{df}} & \neg(x \in A), \text{ and similarly for } \neq \\
(\forall x \in A)\varphi & \equiv_{\mathrm{df}} & (\forall x)(x \in A \to \varphi) \\
(\exists x \in A)\varphi & \equiv_{\mathrm{df}} & (\exists x)(x \in A \mathbin{\&} \varphi) \\
\{x \in A \mid \varphi\} & =_{\mathrm{df}} & \{x \mid x \in A \mathbin{\&} \varphi\} \\
(\forall \tau)\varphi & \equiv_{\mathrm{df}} & (\forall z)(z = \tau \to \varphi), \\
 & & \qquad \text{for any term } \tau \text{ of the same sort as } z, \text{ and } z \text{ not free in } \varphi \\
(\exists \tau)\varphi & \equiv_{\mathrm{df}} & (\exists z)(z = \tau \mathbin{\&} \varphi), \qquad \text{''} \\
\{\tau \mid \varphi\} & =_{\mathrm{df}} & \{z \mid z = \tau \mathbin{\&} \varphi\}, \qquad \text{''} \\
\{x_1, \ldots, x_n\} & =_{\mathrm{df}} & \{z \mid z = x_1 \vee \ldots \vee z = x_n\} \\
t_1 = \cdots = t_n & \equiv_{\mathrm{df}} & (t_1 = t_2) \mathbin{\&} \ldots \mathbin{\&} (t_{n-1} = t_n) \\
\varphi^n & \equiv_{\mathrm{df}} & \varphi \mathbin{\&} \ldots \mathbin{\&} \varphi \ (n \text{ times}) \\
y = F(x) & \equiv_{\mathrm{df}} & Fxy, \text{ if } \triangle(\forall xyy')(Fxy \mathbin{\&} Fxy' \to y = y') \text{ is proved or assumed} \\
\bigcup_{\varphi} \tau & =_{\mathrm{df}} & \bigcup\{\tau \mid \varphi\} \text{ for any term } \tau, \\
 & & \qquad \text{and similarly for } \bigcap \text{ (see Definition 2.12 for } \bigcup, \bigcap)
\end{array}
$$

Convention 2.7. Let φ be a propositional formula and let all propositional variables that occur in φ be among $p_1, \ldots, p_k$. The result of substitution of first-order formulae $\psi_1, \ldots, \psi_k$ respectively for the variables $p_1, \ldots, p_k$ in $\varphi(p_1, \ldots, p_k)$ will be symbolized by $\varphi(\psi_1, \ldots, \psi_k)$.

Definition 2.8. In FCT, we define the following class constants and operations:

$$
\begin{array}{rlll}
\emptyset & =_{\mathrm{df}} & \{x \mid 0\} & \textit{empty class} \\
\mathrm{V} & =_{\mathrm{df}} & \{x \mid 1\} & \textit{universal class} \\
\mathrm{Ker}\, A & =_{\mathrm{df}} & \{x \mid \triangle Ax\} & \textit{kernel} \\
-A & =_{\mathrm{df}} & \{x \mid \neg Ax\} & \textit{complement} \\
A - B & =_{\mathrm{df}} & \{x \mid Ax \mathbin{\&} \neg Bx\} & \textit{difference} \\
A \cap B & =_{\mathrm{df}} & \{x \mid Ax \mathbin{\&} Bx\} & \textit{(strong) intersection} \\
A \cap_{\wedge} B & =_{\mathrm{df}} & \{x \mid Ax \wedge Bx\} & \textit{min-intersection} \\
A \cup_{\vee} B & =_{\mathrm{df}} & \{x \mid Ax \vee Bx\} & \textit{max-union}
\end{array}
$$

Generally for any propositional formula $\varphi(p_1, \ldots, p_k)$ of $\mathrm{MTL}_{\triangle}$ we define the corresponding class operation

$$
\mathrm{Op}_{\varphi}(A_1, \ldots, A_k) =_{\mathrm{df}} \{x \mid \varphi(A_1x, \ldots, A_kx)\}
$$

Example 2.9. $A \cap B = \mathrm{Op}_{p\&q}(A, B)$, $-A = \mathrm{Op}_{\neg p}(A)$, $\mathrm{Ker}\, A = \mathrm{Op}_{\triangle p}(A)$, $\emptyset = \mathrm{Op}_0$, etc.

Definition 2.10. In FCT, we define the following elementary relations between fuzzy classes:

$$
\begin{array}{rlll}
A \subseteq B & \equiv_{\mathrm{df}} & (\forall x)(Ax \to Bx) & \textit{inclusion} \\
A \approx B & \equiv_{\mathrm{df}} & (\forall x)(Ax \leftrightarrow Bx) & \textit{weak bi-inclusion} \\
A \subseteq^{\triangle} B & \equiv_{\mathrm{df}} & (\forall x)\triangle(Ax \to Bx) & \textit{crisp inclusion} \\
A \parallel B & \equiv_{\mathrm{df}} & (\exists x)(Ax \mathbin{\&} Bx) & \textit{compatibility} \\
\mathrm{Hgt}(A) & \equiv_{\mathrm{df}} & (\exists x)Ax & \textit{height} \\
\mathrm{Crisp}(A) & \equiv_{\mathrm{df}} & (\forall x)\triangle(Ax \vee \neg Ax) & \textit{crispness}
\end{array}
$$

Generally for any propositional formula $\varphi(p_1, \ldots, p_k)$ of $\mathrm{MTL}_{\triangle}$ we define two induced elementary relations between fuzzy classes

$$
\begin{array}{rll}
\mathrm{Rel}^{\forall}_{\varphi}(A_1, \ldots, A_k) & \equiv_{\mathrm{df}} & (\forall x)\varphi(A_1x, \ldots, A_kx) \\
\mathrm{Rel}^{\exists}_{\varphi}(A_1, \ldots, A_k) & \equiv_{\mathrm{df}} & (\exists x)\varphi(A_1x, \ldots, A_kx)
\end{array}
$$

Example 2.11. $(A \subseteq B) \equiv \mathrm{Rel}^{\forall}_{p\to q}(A,B)$ and $\mathrm{Hgt}(A) \equiv \mathrm{Rel}^{\exists}_{p}(A)$ by definition, and $(A = B) \leftrightarrow \mathrm{Rel}^{\forall}_{\triangle(p\leftrightarrow q)}(A,B)$ by the axiom of extensionality.

Metatheorems of [12, §3.4] reduce proofs of a broad class of theorems on elementary operations and relations between fuzzy classes to simple propositional calculations. In the present paper we shall freely use corollaries of these metatheorems (like $A \cap B \subseteq A$, $\mathrm{Ker}\, A \subseteq A$, etc.), as their direct proofs in FCT are easy anyway.

Definition 2.12. In FCT, we define the following higher-order fuzzy class operations:

$$\begin{array}{rcll} \bigcup \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\exists A \in \mathcal{A})(x \in A)\} & \textit{class union} \\ \bigcap \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\forall A \in \mathcal{A})(x \in A)\} & \textit{class intersection} \\ \mathrm{Pow}\, A & =_{\mathrm{df}} & \{X \mid X \subseteq A\} & \textit{power class} \end{array}$$

Definition 2.13. In FCT, we define the following relational operations:

$$\begin{array}{rcll} A \times B & =_{\mathrm{df}} & \{xy \mid Ax \,\&\, By\} & \textit{Cartesian product} \\ \mathrm{Dom}(R) & =_{\mathrm{df}} & \{x \mid Rxy\} & \textit{domain} \\ \mathrm{Rng}(R) & =_{\mathrm{df}} & \{y \mid Rxy\} & \textit{range} \\ R^{\rightarrow} A & =_{\mathrm{df}} & \{y \mid (\exists x)(Ax \,\&\, Rxy)\} & \textit{image} \\ R^{\leftarrow} B & =_{\mathrm{df}} & \{x \mid (\exists y)(By \,\&\, Rxy)\} & \textit{pre-image} \\ R^{\mathrm{T}} & =_{\mathrm{df}} & \{xy \mid Ryx\} & \textit{transposition} \\ \mathrm{Id} & =_{\mathrm{df}} & \{xy \mid x = y\} & \textit{identity relation} \\ A^{n} & =_{\mathrm{df}} & \{x_1 \ldots x_n \mid Ax_1 \,\&\, \ldots \,\&\, Ax_n\} & \textit{Cartesian power} \end{array}$$

In particular, V^n is the class of all n-tuples of atomic objects. Subclasses of V^n are called *n-ary fuzzy relations*; the condition that a class R is an n-ary relation is expressed by the formula $R \subseteq^{\triangle} \mathrm{V}^n$. Instead of "unary relations" we usually speak simply of fuzzy classes, unless we want to stress the distinction from the general meaning of the term "class", which includes relations of arities larger than one.[2]

Since all classes in FCT are in principle fuzzy, we often omit the word "fuzzy" and speak simply of *classes* and *relations*, meaning "fuzzy (including possibly crisp) classes or relations". Since crisp classes are just a special kind of fuzzy classes, we do not distinguish operations on crisp relations from their counterparts operating on fuzzy relations (unlike certain traditions in the theory of fuzzy relations), and use the same symbols for both kinds of arguments; if necessary, the crispness of arguments can explicitly be expressed in the formula by means of the predicate Crisp introduced in Definition 2.10.

The operation of transposition (see Definition 2.13) applied to R yields its *converse* relation R^{T}. The following simple properties of transposition will be needed in subsequent sections:

Proposition 2.14. FCT *proves:*

1. $R^{\mathrm{TT}} = R$
2. $R \subseteq^{\triangle} \mathrm{Id} \to R^{\mathrm{T}} = R$

[2] Formally, we should explicitly mark the arities of variables in all formulae. We omit the arity marks for better readability, since usually the arities are either immaterial or determined by the context. If needed, the arity of a variable can be expressed by the formula $x \in \mathrm{V}^n$ if x is a variable just for n-tuples of objects, or $x \in \mathrm{V}$ if x is a variable for objects of any arity. The lowercase variables in Definitions 2.8–2.13 are universal (i.e., represent any tuples of objects), the defined notions can therefore be applied to fuzzy relations as well as classes.

3. For any propositional formula $\varphi(p_1,\ldots,p_n)$*,*

$$\begin{aligned}\mathrm{Rel}^{\forall}_{\varphi}(R_1^{\mathrm{T}},\ldots,R_n^{\mathrm{T}}) &\leftrightarrow \mathrm{Rel}^{\forall}_{\varphi}(R_1,\ldots,R_n)\\ \mathrm{Rel}^{\exists}_{\varphi}(R_1^{\mathrm{T}},\ldots,R_n^{\mathrm{T}}) &\leftrightarrow \mathrm{Rel}^{\exists}_{\varphi}(R_1,\ldots,R_n)\end{aligned}$$

In particular, $R \subseteq S \leftrightarrow R^{\mathrm{T}} \subseteq S^{\mathrm{T}}$ *and* $R = S \leftrightarrow R^{\mathrm{T}} = S^{\mathrm{T}}$*.*

4. $(\mathrm{Op}_{\varphi}(R_1,\ldots,R_n))^{\mathrm{T}} = \mathrm{Op}_{\varphi}(R_1^{\mathrm{T}},\ldots,R_n^{\mathrm{T}})$ *for any propositional formula* $\varphi(p_1,\ldots,p_n)$*. In particular,* $(R\cap S)^{\mathrm{T}} = R^{\mathrm{T}}\cap S^{\mathrm{T}}$*,* $(-R)^{\mathrm{T}} = -(R^{\mathrm{T}})$*,* $\emptyset^{\mathrm{T}} = \emptyset$*, etc.*

5. $\bigcup\limits_{R\in\mathcal{A}} R^{\mathrm{T}} = \Big(\bigcup\limits_{R\in\mathcal{A}} R\Big)^{\mathrm{T}}$*,* $\bigcap\limits_{R\in\mathcal{A}} R^{\mathrm{T}} = \Big(\bigcap\limits_{R\in\mathcal{A}} R\Big)^{\mathrm{T}}$

Proof. 1. By definition, $xy \in R^{\mathrm{TT}} \longleftrightarrow yx \in R^{\mathrm{T}} \longleftrightarrow xy \in R$; therefore, by the axiom of extensionality, $R^{\mathrm{TT}} = R$.

2. For arbitrary x, y we take the following crisp cases:[3] if $x = y$, then $Rxy \leftrightarrow Ryx$ by the axiom of identity (=2); if $x \neq y$, then $\triangle\neg Rxy \,\&\, \triangle\neg Ryx$ by the assumption $R \subseteq^{\triangle} \mathrm{Id}$, hence $Rxy \leftrightarrow Ryx$ by Lemma 2.3(2). In both cases we have $Rxy \leftrightarrow R^{\mathrm{T}}xy$, so by $\triangle$-necessitation, generalization, and the axiom of extensionality we get $R = R^{\mathrm{T}}$.

3. By renaming bound variables we get $(\forall xy)\varphi(R_1yx,\ldots,R_nyx) \leftrightarrow (\forall yx)\varphi(R_1xy,\ldots,R_nxy)$, and similarly for $\mathrm{Rel}^{\exists}_{\varphi}$.

4. By expanding the definitions we get $xy \in (\mathrm{Op}_{\varphi}(R_1,\ldots,R_n))^{\mathrm{T}} \longleftrightarrow \varphi(R_1yx,\ldots,R_nyx) \longleftrightarrow \varphi(R_1^{\mathrm{T}}xy,\ldots,R_n^{\mathrm{T}}xy) \longleftrightarrow xy \in \mathrm{Op}_{\varphi}(R_1^{\mathrm{T}},\ldots,R_n^{\mathrm{T}})$.

5. $xy \in \bigcup_{R\in\mathcal{A}} R^{\mathrm{T}} \longleftrightarrow (\exists R\in\mathcal{A})(yx \in R) \longleftrightarrow yx \in \bigcup_{R\in\mathcal{A}} R \longleftrightarrow xy \in \big(\bigcup_{R\in\mathcal{A}} R\big)^{\mathrm{T}}$, and analogously for $\bigcap$. □

3 Representation of fuzzy classes and truth values by fuzzy relations

Fuzzy classes and truth values can be represented as fuzzy relations of a certain form, as described below. This representation will allow us straightforwardly to apply the properties of fuzzy relational compositions to many derived concepts which involve fuzzy classes or truth values.

The identification of fuzzy classes and truth values with certain fuzzy relations will in this paper be described only informally. It can, nevertheless, be carried out in a rigorous formal way by means of syntactic interpretations of formal theories in FCT. We do not elaborate the apparatus of interpretations here as it would make the paper too much loaded with formalism, and simpler methods are sufficient for theorems stated in this paper. Technical details on syntactic interpretations in FCT, including the interpretations used for the identifications made in this paper, can be found in [9].

Convention 3.1. Let $\underline{0}$ be an arbitrarily chosen, but fixed, atomic object (i.e., an element of V^1). The fuzzy class $\{\underline{0}\}$ (i.e., the crisp singleton of the urelement $\underline{0}$) will be denoted by $\underline{1}$.

[3] Recall that the soundness of proofs by cases follows from the provability of $(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$ in MTL.

Convention 3.2. A fuzzy class $A \subseteq^{\triangle} \mathrm{V}^1$ will be identified with the fuzzy relation $A \times \underline{1} = \{\langle x, \underline{0}\rangle \mid x \in A\}$. When representing the fuzzy class A, the fuzzy relation $A \times \underline{1}$ will be written as $\boldsymbol{A}$ (the same letter in boldface).

Obviously, the relation $A \times \underline{1}$ is isomorphic in a very natural sense to the original fuzzy class A: each of the original elements x got replaced by a pair $x\underline{0}$, but its membership degree has not changed ($\boldsymbol{A}x\underline{0} \equiv Ax$); thus the *structure* of the fuzzy class has been preserved. Consequently, all of its properties that do not refer to the actual names of its elements have been preserved as well under this identification. Furthermore, the original class A can uniquely be reconstructed from the relation $A \times \underline{1}$ as $A = \{x \mid \langle x, \underline{0}\rangle \in A \times \underline{1}\}$. Also the identity of classes is preserved under the translation, since $A = B$ iff $A \times \underline{1} = B \times \underline{1}$ (which follows easily from $\langle x, \underline{0}\rangle = \langle y, \underline{0}\rangle \leftrightarrow x = y$, one of the axioms for tuples). The relations of the form $A \times \underline{1}$ thus faithfully represent fuzzy classes among fuzzy relations.[4]

This identification is quite natural and well-known. If the universe of discourse is finite, consisting of elements $x_1, \ldots, x_n$, fuzzy relations can be represented by $(n \times n)$-matrices of truth values, $R = (Rx_ix_j)_{ij}$:

$$R = \begin{pmatrix} Rx_1x_1 & Rx_1x_2 & \cdots & Rx_1x_n \\ Rx_2x_1 & Rx_2x_2 & \cdots & Rx_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ Rx_nx_1 & Rx_nx_2 & \cdots & Rx_nx_n \end{pmatrix}$$

Assume that $\underline{0}$ denotes the element x_1. The fuzzy class A is then identified with the relation

$$\boldsymbol{A} = \begin{pmatrix} A\underline{0} & 0 & \cdots & 0 \\ Ax_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Ax_n & 0 & \cdots & 0 \end{pmatrix}$$

which by the usual convention of linear algebra can be written as the (file) vector $n \times 1$,

$$\boldsymbol{A} = \begin{pmatrix} A\underline{0} \\ Ax_2 \\ \vdots \\ Ax_n \end{pmatrix}$$

Notice that Convention 3.2 just extends this representation in a formal way to arbitrary (not only finite) fuzzy classes.

A similar trick will allow us to represent truth values as certain relations. First observe that truth values can be internalized in FCT as subclasses of an arbitrary crisp singleton, e.g., of $\underline{1}$, in the following way:

- The truth value of a formula φ is represented by the class $\overline{\varphi} =_{\mathrm{df}} \{\underline{0} \mid \varphi\}$. Then by definition, $\overline{\varphi} \subseteq^{\triangle} \underline{1}$ and $\varphi \leftrightarrow (\underline{0} \in \overline{\varphi})$.

[4] In the language of formal interpretations we can describe this fact rigorously by observing that $A \mapsto A \times \underline{1}$ is a faithful interpretation of the theory of fuzzy classes $\mathrm{FCT}_{2,2}$ (i.e., a fragment of FCT containing only variables for atomic individuals and fuzzy classes) in the theory of binary fuzzy relations $\mathrm{FCT}_{2,3}$ (i.e., a fragment of FCT containing only variables for atomic individuals, pairs of atomic individuals, and fuzzy classes). The interpretation provides a faithful translation between the properties of fuzzy classes and the corresponding fuzzy relations. For details see [9].

- Vice versa, every $\alpha \subseteq^{\triangle} \underline{1}$ represents the truth value of a formula—e.g., of $\underline{0} \in \alpha$, since $(\forall\alpha \subseteq^{\triangle} \underline{1})(\overline{\underline{0} \in \alpha} = \alpha)$ by Proposition 3.4(1) below.

The truth values are thus represented by subclasses of $\underline{1}$, where the truth value represented is the degree of membership of $\underline{0}$ in the subclass. We shall therefore call the elements of $\operatorname{Ker}\operatorname{Pow}\underline{1}$ the *inner* (or *formal*) *truth values* and denote them by lowercase Greek letters $\alpha, \beta, \ldots$ The system of formal truth values will for brevity's sake be denoted by L:

$$\mathrm{L} =_{\mathrm{df}} \operatorname{Ker}\operatorname{Pow}\underline{1}$$

The ordering of truth values is represented by the relation $\subseteq^{\triangle}$ between their formal counterparts: by Proposition 3.4(2) below, $(\varphi \to \psi) \leftrightarrow (\overline{\varphi} \subseteq \overline{\psi})$ and $(\varphi \leftrightarrow \psi) \leftrightarrow (\overline{\varphi} \approx \overline{\psi})$ for any formulae φ and ψ. Furthermore, there is the following correspondence between the propositional connectives and class operations on L:

$$\begin{aligned}
\overline{\varphi \mathbin{\&} \psi} &= \overline{\varphi} \cap \overline{\psi} \\
\overline{\varphi \wedge \psi} &= \overline{\varphi} \cap_{\wedge} \overline{\psi} \\
\overline{\varphi \vee \psi} &= \overline{\varphi} \cup_{\vee} \overline{\psi} \\
\overline{\neg\varphi} &= \underline{1} \setminus \overline{\varphi} \\
\overline{0} &= \emptyset, \quad \text{etc., in general:} \\
\overline{c(\psi_1, \ldots, \psi_n)} &= \underline{1} \cap \mathrm{Op}_{c(p_1,\ldots,p_n)}(\overline{\psi_1}, \ldots, \overline{\psi_n})
\end{aligned}$$

for any definable n-ary propositional connective c, by Proposition 3.4(3) below. Using this correspondence, we can also denote the operations $\cap, \cap_{\wedge}, \cup_{\vee}, \ldots$ on L by $\overline{\&}, \overline{\wedge}, \overline{\vee}, \ldots$ and call them *formal connectives* on inner truth values.

Since inner truth values represent the semantical concept of truth value within the theory, we shall occasionally use the lattice-theoretical notation $\bigvee_{\alpha\in\mathcal{A}} \alpha$ and $\bigwedge_{\alpha\in\mathcal{A}} \alpha$ instead of $(\exists\alpha \in \mathcal{A})(\underline{0} \in \alpha)$ and $(\forall\alpha \in \mathcal{A})(\underline{0} \in \alpha)$, respectively, for $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$. Proposition 3.4(4) below shows that $\bigvee_{\alpha\in\mathcal{A}} \alpha$ and $\bigwedge_{\alpha\in\mathcal{A}} \alpha$ respectively correspond to the union and intersection of the class $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$.

Remark 3.3. It should be noticed that in an **L**-valued model $\mathcal{M}$ of FCT (for an $\mathrm{MTL}_{\triangle}$-chain **L**), the lattice L of inner truth values need not coincide with the lattice **L** of semantic truth values, but can be a proper sublattice of **L**: in general, only those elements of **L** are represented in L which are the truth values of FCT-formulae in $\mathcal{M}$. Thus, for instance, in any *standard* model of FCT the system **L** of semantic truth values is the real unit interval $[0, 1]$; however, *crisp* standard models of FCT (cf. footnote 1 on page 161) have only two inner truth values, $\emptyset$ and $\underline{1}$.

It can also be observed that by the axioms of comprehension, $\bigvee_{\alpha\in\mathcal{A}} \alpha$ and $\bigwedge_{\alpha\in\mathcal{A}} \alpha$ exist for any class $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$; thus FCT proves that L is a complete lattice, even though the system **L** of semantic truth values need not in general be complete: recall that only the *safeness* of the structure is required in the semantics of first-order fuzzy logic, i.e., the existence of all suprema and infima that are the truth values of formulae (see [30] for details). The difference is again due to the fact that the existence of suprema and infima is only ensured for such subsets $\mathcal{A}$ of L which are represented in the model, rather than *all* subsets of L.

Nevertheless, in the intended full models of FCT, i.e., those formed by *all* fuzzy subsets, inner truth values correspond exactly to the semantical ones.

Now we give proofs of the statements mentioned above:

Proposition 3.4. FCT *proves:*

1. $(\forall\alpha \subseteq^{\triangle} \underline{1})(\alpha = \{\underline{0} \mid \underline{0} \in \alpha\})$

2. $(\varphi \to \psi) \leftrightarrow (\overline{\varphi} \subseteq \overline{\psi})$ *for any formulae* φ *and* ψ

3. $\overline{\varphi(\psi_1, \ldots, \psi_n)} = \underline{1} \cap \mathrm{Op}_{\varphi}(\overline{\psi_1}, \ldots, \overline{\psi_n})$, *for any propositional formula* $\varphi(p_1, \ldots, p_n)$

4. $\overline{\bigvee_{\alpha\in\mathcal{A}} \alpha} = \bigcup_{\alpha\in\mathcal{A}} \alpha$ *and* $\overline{\bigwedge_{\alpha\in\mathcal{A}} \alpha} = \bigcap_{\alpha\in\mathcal{A}} \alpha$ *for any* $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$

Proof. 1. It is sufficient to prove $x \in \alpha \leftrightarrow (x = \underline{0} \mathbin{\&} \underline{0} \in \alpha)$ from the assumption $\alpha \subseteq \{\underline{0}\}$; the result then follows by $\triangle$-necessitation and generalization. Now $(x = \underline{0} \mathbin{\&} \underline{0} \in \alpha) \to x \in \alpha$ follows directly from the identity axioms, and $x \in \alpha \to (x = \underline{0} \mathbin{\&} \underline{0} \in \alpha)$ follows (by taking crisp cases $x = \underline{0}$ and $x \neq \underline{0}$) from the assumption $(\forall x \in \alpha)(x = \underline{0})$.

2. By definitions, $\overline{\varphi} \subseteq \overline{\psi} \longleftrightarrow \{\underline{0} \mid \varphi\} \subseteq \{\underline{0} \mid \psi\} \longleftrightarrow$
$\{x \mid x = \underline{0} \mathbin{\&} \varphi\} \subseteq \{x \mid x = \underline{0} \mathbin{\&} \psi\} \longleftrightarrow (\forall x)((x = \underline{0} \mathbin{\&} \varphi) \to (x = \underline{0} \mathbin{\&} \psi))$;
thus it is sufficient to prove

$$(\varphi \to \psi) \;\leftrightarrow\; (\forall x)((x = \underline{0} \mathbin{\&} \varphi) \to (x = \underline{0} \mathbin{\&} \psi)) \tag{9}$$

Now $(\varphi \to \psi) \to ((x = \underline{0} \mathbin{\&} \varphi) \to (x = \underline{0} \mathbin{\&} \psi))$, from which the left-to-right direction of (9) follows by generalization; vice versa, by specifying $\underline{0}$ for x in (9) we get: (9) $\longrightarrow ((\underline{0} = \underline{0} \mathbin{\&} \varphi) \to (\underline{0} = \underline{0} \mathbin{\&} \psi)) \longleftrightarrow (\varphi \to \psi)$.

3. By definitions,

$$\underline{1} \cap \mathrm{Op}_{\varphi}(\overline{\psi_1}, \ldots, \overline{\psi_n}) = \{x \mid (x = \underline{0}) \mathbin{\&} \varphi((x = \underline{0} \mathbin{\&} \psi_1), \ldots, (x = \underline{0} \mathbin{\&} \psi_n))\}$$

Denote the latter class by A and take crisp cases on x: if $x \neq \underline{0}$, then $Ax \leftrightarrow 0$ since $(x = \underline{0}) \leftrightarrow 0$; if $x = \underline{0}$, then $Ax \leftrightarrow (x = \underline{0}) \mathbin{\&} \varphi(\psi_1, \ldots, \psi_n)$ since $(x = \underline{0} \mathbin{\&} \psi_i) \leftrightarrow \psi_i$ for all i. Thus in both cases $Ax \leftrightarrow (x = \underline{0}) \mathbin{\&} \varphi(\psi_1, \ldots, \psi_n)$, i.e., $A = \{\underline{0} \mid \varphi(\psi_1, \ldots, \psi_n)\} = \overline{\varphi(\psi_1, \ldots, \psi_n)}$.

4. If $x = \underline{0}$, then $(\exists\alpha \in \mathcal{A})(x \in \alpha) \leftrightarrow x = \underline{0} \mathbin{\&} (\exists\alpha \in \mathcal{A})(x \in \alpha)$; if $x \neq \underline{0}$, then $(\exists\alpha \in \mathcal{A})(x \in \alpha) \leftrightarrow 0$, since $\alpha \in \mathcal{A} \mathbin{\&} x \in \alpha \longrightarrow \alpha \in \mathrm{L} \mathbin{\&} x \in \alpha \longrightarrow x = \underline{0}$ by $\mathcal{A} \subseteq^{\triangle} \mathrm{L}$ and $(\forall\alpha \in \mathrm{L})(\alpha \subseteq^{\triangle} \{\underline{0}\})$. In both cases we have $(\exists\alpha \in \mathcal{A})(x \in \alpha) \leftrightarrow x = \underline{0} \mathbin{\&} (\exists\alpha \in \mathcal{A})(x \in \alpha)$, thus

$$\bigcup_{\alpha\in\mathcal{A}} \alpha = \{x \mid (\exists\alpha \in \mathcal{A})(x \in \alpha)\} = \{x \mid x = \underline{0} \mathbin{\&} (\exists\alpha \in \mathcal{A})(x \in \alpha)\} = \overline{\bigvee_{\alpha\in\mathcal{A}} \alpha}$$

The proof for $\bigwedge$ is analogous. □

Remark 3.5. Inner truth values are an important construction in FCT (and generally in any formal theory of fuzzy sets), neither limited to nor motivated by the purposes of the present paper. The construction presented here is rather standard (cf., e.g., [41]) and shows, i.a., that FCT is strong enough to internalize its own semantics. By means of inner truth values, usual semantical notions like membership functions can be defined and investigated within the formal theory. However, since this is not the aim of the present paper, we leave this topic aside and turn back to the representation of truth values by fuzzy relations.

Now as the truth values are represented by special *fuzzy classes* (viz, the subclasses of $\underline{1}$), they can be identified with certain fuzzy relations by Convention 3.2. Namely, an inner truth value $\alpha \subseteq^{\triangle} \underline{1}$ is identified with the fuzzy relation $\alpha \times \underline{1} = \{\langle \underline{0}, \underline{0} \rangle \mid \underline{0} \in \alpha\}$. By the same convention, when representing the truth value α, the fuzzy relation $\alpha \times \underline{1}$ can be denoted by boldface $\boldsymbol{\alpha}$.

Again, if the universe of discourse is finite and consists of elements $\underline{0}, x_2, \ldots, x_n$, an inner truth value α is identified with the relation

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha\underline{0} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \alpha\underline{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which by usual conventions of linear algebra can be identified with the (1×1)-matrix (or *scalar*) $\big(\alpha\underline{0}\big)$. (Recall that $\alpha\underline{0}$, i.e., $\underline{0} \in \alpha$, has the truth value that is represented by α. In the informal matrix expressions, we shall write just α instead of $\alpha\underline{0}$ further on.)

It can again be noticed that the apparatus of Fuzzy Class Theory employed here just extends the usual correspondence between fuzzy relations, sets, and truth values on the one hand and matrices, vectors, and scalars of truth values on the other hand, to arbitrary (not only finite) fuzzy relations and classes, and provides a uniform way of formal handling thereof. In particular, the reduction of fuzzy classes and truth values to fuzzy relations will allow us to extend the apparatus of sup-T and inf-R compositions of fuzzy relations to fuzzy classes and truth values, apply the results on compositions to a rich variety of derived notions, and get the proofs of their properties for free.

We conclude this section with some conventions and observations that will be useful later.

Convention 3.6. Unless explicitly said otherwise, we shall always assume that R, S, or T (possibly subscripted) denote fuzzy relations $\subseteq^{\triangle} \mathrm{V}^2$; A, B, or C (possibly subscripted) denote unary classes $\subseteq^{\triangle} \mathrm{V}^1$; and α, β, γ (possibly subscripted) denote inner truth values $\subseteq^{\triangle} \underline{1}$.

Proposition 3.7. FCT *proves that* $(\forall \alpha \subseteq^{\triangle} \underline{1})(\boldsymbol{\alpha} \subseteq^{\triangle} \mathrm{Id})$*; therefore,* $(\forall \alpha \subseteq^{\triangle} \underline{1})(\boldsymbol{\alpha}^{\mathrm{T}} = \boldsymbol{\alpha})$ *by Proposition 2.14(2).*

Proof. From $(x \in \alpha \to x = \underline{0}) \to (x \in \alpha \mathbin{\&} y = \underline{0} \to x = y)$, which follows from the axioms of identity, we get by generalization and distribution of quantifiers $(\forall x)(x \in \alpha \to x = \underline{0}) \to (\forall xy)(x \in \alpha \mathbin{\&} y = \underline{0} \to x = y)$, i.e., $\alpha \subseteq \underline{1} \to \{x\underline{0} \mid x \in \alpha\} \subseteq \{xy \mid x = y\}$. Then $\triangle$-necessitation finishes the proof. □

4 Sup-T-composition and derived notions

The usual definition of composition of fuzzy relations R and S is as follows:

Definition 4.1. $R \circ S =_{\mathrm{df}} \{xy \mid (\exists z)(Rxz \mathbin{\&} Szy)\}$

Since & is interpreted by a (left-continuous) t-norm and $\exists$ by the supremum, $\circ$ is also called the *sup-T-composition* of R and S. It generalizes Zadeh's original definition [42] of max-min-composition to infinite domains and arbitrary left-continuous t-norms. Notice that the defining formula is the same as the defining formula of the relational composition in classical mathematics, the fuzziness being introduced only by the semantics of the logical symbols ∃ and &. This makes it the "default" definition of fuzzy relational composition according to the methodology of [15].

The following properties of sup-T-compositions are well-known (see, e.g., [21], [18], etc.). We repeat them here for reference and give their proofs in FCT.

Theorem 4.2. FCT *proves the following properties of sup-T-compositions:*

1. *Transposition:* $(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$
2. *Monotony:* $R_1 \subseteq R_2 \to R_1 \circ S \subseteq R_2 \circ S$
3. *Union:* $\Big(\bigcup_{R\in\mathcal{A}} R\Big) \circ S = \bigcup_{R\in\mathcal{A}} (R \circ S)$
4. *Intersection:* $\Big(\bigcap_{R\in\mathcal{A}} R\Big) \circ S \subseteq \bigcap_{R\in\mathcal{A}} (R \circ S)$

   (The converse inclusion has well-known crisp counter-examples.)
5. *Associativity:* $(R \circ S) \circ T = R \circ (S \circ T)$

Proof. 1. $(R\circ S)^{\mathrm{T}} = \{xy \mid (\exists z)(Ryz \mathbin{\&} Szx)\} = \{xy \mid (\exists z)(S^{\mathrm{T}}xz \mathbin{\&} R^{\mathrm{T}}zy)\} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$.

2. $(R_1xz \to R_2xz) \longrightarrow (R_1xz \to R_2xz) \mathbin{\&} (Szy \to Szy) \longrightarrow ((R_1xz \mathbin{\&} Szy) \to (R_2xz \mathbin{\&} Szy))$, followed by generalization and distribution of quantifiers.

3. $(\exists z)[(\exists R \in \mathcal{A})(Rxz) \mathbin{\&} Szy] \longleftrightarrow (\exists z)(\exists R \in \mathcal{A})(Rxz \mathbin{\&} Szy) \longleftrightarrow$ $(\exists R \in \mathcal{A})(\exists z)(Rxz \mathbin{\&} Szy)$.

4. The claim is proved by the following chain of implications (see Lemma 2.4 for the shifts of relativized quantifiers needed here):

$$(\exists z)[(\forall R \in \mathcal{A})(Rxz) \mathbin{\&} Szy] \longrightarrow (\exists z)(\forall R \in \mathcal{A})(Rxz \mathbin{\&} Szy) \longrightarrow (\forall R \in \mathcal{A})(\exists z)(Rxz \mathbin{\&} Szy) \quad (10)$$

The existence of crisp counter-examples to the converse inclusion follows from the fact that even though the first implication in (10) can be converted in classical logic, the second one cannot (the quantifiers do not commute).

5. $\{xy \mid (\exists w)((\exists z)(Rxz \mathbin{\&} Szw) \mathbin{\&} Twy)\} = \{xy \mid (\exists z)(Rxz \mathbin{\&} (\exists w)(Szw \mathbin{\&} Twy))\}$ □

Corollary 4.3. *By Theorem 4.2(1) and Proposition 2.14(1, 3, 5),* FCT *proves the mirror variants of Theorem 4.2(2,3,4), too:*

1. $S_1 \subseteq S_2 \to R \circ S_1 \subseteq R \circ S_2$
2. $R \circ \bigcup_{S\in\mathcal{A}} S = \bigcup_{S\in\mathcal{A}} (R \circ S)$
3. $R \circ \bigcap_{S\in\mathcal{A}} S \subseteq \bigcap_{S\in\mathcal{A}} (R \circ S)$*, with crisp counter-examples to the converse inclusion.*

By means of the identification of fuzzy classes with fuzzy relations by Convention 3.2, the statements of Theorem 4.2 and Corollary 4.3 can be transferred to further relational notions besides sup-T-composition, by the following method.

Comparing, e.g., the (equivalent variant of the) definition of the preimage of a fuzzy class A under a fuzzy relation R with the definition of relational composition,

$$R^{\leftarrow}A =_{\mathrm{df}} \{x \mid (\exists z)(Rxz \mathbin{\&} Az)\}$$
$$R \circ S =_{\mathrm{df}} \{xy \mid (\exists z)(Rxz \mathbin{\&} Szy)\}$$

one can recognize the same pattern of the defining expression: the only difference is that in the definition of the preimage, the second argument as well as the result are unary rather than binary (the variable y is missing). However, our identification of the fuzzy classes A and $R^{\leftarrow}A$ with the fuzzy relations $\boldsymbol{A} = A \times \underline{1}$ and $(R^{\leftarrow}A) \times \underline{1}$, respectively, reduces the definition of preimage exactly to that of composition, by supplying the dummy argument $\underline{0}$ for the missing variable y:

$$(R^{\leftarrow}A) \times \underline{1} = \{x\underline{0} \mid (\exists z)(Rxz \,\&\, Az)\} = \{x\underline{0} \mid (\exists z)(Rxz \,\&\, (A \times \underline{1})z\underline{0})\} = R \circ (A \times \underline{1})$$

Thus $B = R^{\leftarrow}A$ iff $\boldsymbol{B} = R \circ \boldsymbol{A}$.[5]

Consequently, the properties of compositions stated in Theorem 4.2(2–4) and Corollary 4.3 automatically translate to properties of preimages:

$$\begin{array}{rcl} R_1 \subseteq R_2 & \rightarrow & R_1{}^{\leftarrow}A \subseteq R_2{}^{\leftarrow}A \\ A_1 \subseteq A_2 & \rightarrow & R^{\leftarrow}A_1 \subseteq R^{\leftarrow}A_2 \\ \left(\bigcup_{R\in\mathcal{A}} R\right)^{\leftarrow}A & = & \bigcup_{R\in\mathcal{A}}(R^{\leftarrow}A) \\ R^{\leftarrow}\bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}}(R^{\leftarrow}A) \\ \left(\bigcap_{R\in\mathcal{A}} R\right)^{\leftarrow}A & \subseteq & \bigcap_{R\in\mathcal{A}}(R^{\leftarrow}A) \\ R^{\leftarrow}\bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{R\in\mathcal{A}}(R^{\leftarrow}A) \end{array}$$

Again, the converse inclusions for intersection are not generally valid even for crisp relations and classes, since there are crisp counter-examples even with relations of the form $A \times \underline{1}$.

For a proof of the properties, one only needs to realize that the predicates involved $(\subseteq, =)$ are invariant under the transformation $\cdot \times \underline{1}$ as well as under its inverse, the operations involved $(\bigcup, \bigcap)$ commute with both of these transformations, and that $(R^{\leftarrow}A) \times \underline{1}$ is $R \circ \boldsymbol{A}$, to which Theorem 4.2 applies. Another proof consists in the observation that the proof of Theorem 4.2 remains sound when deleting all occurrences of the variable y. A general method for proving the invariance of theorems of certain forms under translations like our identification of A with $A \times \underline{1}$ is available, in virtue of theorems on formal interpretations of theories over fuzzy logic (cf. footnote 4 and see [9]). Here we shall take these results for granted, since the method of inspecting the proofs and verifying their invariance under the substitution of $\underline{0}$ for some variables is always available and sufficiently simple for all theorems listed in this paper.

In the same manner, the notion of image of a fuzzy class under a fuzzy relation, $R^{\rightarrow}A =_{\mathrm{df}} \{y \mid (\exists z)(Az \,\&\, Rzy)\}$, is obtained by substituting $\underline{0}$, only this time for x rather than y, in the definition of fuzzy relational composition, as

$$\begin{array}{rcl} (R^{\rightarrow}A) \times \underline{1} & = & \{y\underline{0} \mid (\exists z)(Az \,\&\, Rzy)\} \\ & = & \{y\underline{0} \mid (\exists z)(\boldsymbol{A}z\underline{0} \,\&\, Rzy)\} \\ & = & \{y\underline{0} \mid (\exists z)(R^{\mathrm{T}}yz \,\&\, \boldsymbol{A}z\underline{0})\} \\ & = & R^{\mathrm{T}} \circ \boldsymbol{A} \end{array}$$

Thus $B = R^{\rightarrow}A$ iff $\boldsymbol{B} = R^{\mathrm{T}} \circ \boldsymbol{A}$, so the image of A under R can simply be equated with $R^{\mathrm{T}} \circ \boldsymbol{A}$. Again the above properties of compositions translate into those of images.

[5]Having adopted Convention 3.6, we could abandon the distinction between A and $\boldsymbol{A}$ altogether and simply equate $R^{\leftarrow}A = R \circ A$, since the convention ensures that A is a unary class even if $R \circ A$ is written out of any context. We keep the distinction here only for the sake of clarity.

(Notice that this time, we also need to employ Proposition 2.14(5) to get the preservation of unions and intersections under images, since R is transposed in $R^{\mathrm{T}} \circ \boldsymbol{A}$.)

As mentioned in the Introduction, the method of transferring the results on relational compositions to related notions like images or preimages has already been suggested in [18, Remark 6.16]. In our formal setting we can exploit the method systematically:

There are three variables in the definition of sup-T-composition and each of them can be replaced by the dummy value $\underline{0}$. This yields seven relational operations derived from sup-T-composition of fuzzy relations: they are summarized in Table 1.

	$\{xy \mid (\exists z)(Rxz \mathbin{\&} Szy)\}$	$= R \circ S$	...	*composition*	$R \circ S$
$x = \underline{0}$	$\{\underline{0}y \mid (\exists z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \mathbin{\&} Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \circ R)^{\mathrm{T}}$	$= R^{\mathrm{T}} \circ \boldsymbol{A}$	...	*image*	$R^{\rightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\exists z)(Rxz \mathbin{\&} \boldsymbol{A}z\underline{0})\}$	$= R \circ \boldsymbol{A}$	...	*pre-image*	$R^{\leftarrow} A$
$z = \underline{0}$	$\{xy \mid (\exists \underline{0})(\boldsymbol{A}x\underline{0} \mathbin{\&} \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$= \boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$	...	*Cartesian product*	$A \times B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\exists z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \mathbin{\&} \boldsymbol{B}z\underline{0})\}$	$= \boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$	...	*compatibility*	$A \mathbin{\overline{\|}} B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\exists \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \mathbin{\&} \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \circ \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$= \boldsymbol{A} \circ \boldsymbol{\alpha}$	...	*α-resize*	αA
$y, z = \underline{0}$	$\{x\underline{0} \mid (\exists \underline{0})(\boldsymbol{A}x\underline{0} \mathbin{\&} \boldsymbol{\alpha}\underline{00})\}$	$= \boldsymbol{A} \circ \boldsymbol{\alpha}$	...	*α-resize*	αA
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\exists \underline{0})(\boldsymbol{\alpha}\underline{00} \mathbin{\&} \boldsymbol{\beta}\underline{00})\}$	$= \boldsymbol{\alpha} \circ \boldsymbol{\beta}$	...	*conjunction*	$\alpha \mathbin{\overline{\&}} \beta$

Table 1: Operations derived from the sup-T-composition

We shall comment on the notions in the table. The first three lines have been described in detail above. The image and preimage have also been called the *inclusive afterset* and *inclusive foreset,* respectively, by Bandler and Kohout [5].

The fourth notion, arising from setting z to $\underline{0}$, is the usual *Cartesian product* of the classes A and B. Notice that fixing $z = \underline{0}$ makes the quantification over z void, so the comprehension term indeed equals $\{xy \mid Ax \mathbin{\&} By\}$. The resulting term $\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$ just reflects the valid equation $A \times B = (A \times \underline{1}) \circ (\underline{1} \times B)$.

Setting both x and y to $\underline{0}$ in the fifth line of Table 1 makes the result a fuzzy singleton—a class to which only the pair $\langle \underline{0}, \underline{0} \rangle$ belongs to the degree $(\exists z)(Az \& Bz)$. The latter formula expresses the *compatibility* $A \parallel B$ of the fuzzy properties (or classes) A and B, i.e., the height of their intersection. Since fuzzy singletons internalize truth values,[6] the resulting expression represents the truth value of $A \parallel B$; thus $\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B} = \overline{A \parallel B} = \overline{\mathrm{Hgt}}(A \cap B)$. We shall denote the operation $\overline{\|}$, since the result is a *formal* truth value—the fuzzy singleton—rather than the semantical truth value of $A \parallel B$.

The sixth notion in Table 1, which for the lack of an established name we call the *α-resize* of A and denote by αA, is derived from composition by fixing $x, z = \underline{0}$ (notice that the same notion is obtained also by fixing $y, z = \underline{0}$). The operation is widely applicable in fuzzy set theory and often is used implicitly or without notice (see Examples 5.13 and 5.14 below).

Finally, fixing all x, y, z to $\underline{0}$ yields the operation of *formal conjunction* of two formal truth values (i.e., the intersection of the two fuzzy singletons that represent them).

Remark 4.4. It has already been observed by Zadeh in [42] that in the finite case, the sup-T-composition of fuzzy relations is computed in the same manner as the product of the corresponding matrices, only performing & instead of multiplication and taking the supremum (∃) instead of the sum: $(\,\|(R \circ S)x_i x_j\|\,)_{ij} = (\,\|(\exists x_k)(Rx_i x_k \mathbin{\&} Sx_k x_j)\|\,)_{ij} =$

[6]According to the conventions of Section 3, fuzzy truth values are represented by fuzzy singletons $\alpha \subseteq^{\triangle} \{\underline{0}\}$, which classes we have identified with fuzzy relations $\boldsymbol{\alpha} = \alpha \times \underline{1} \subseteq^{\triangle} \{\langle \underline{0}, \underline{0} \rangle\}$. Thus among fuzzy relations, formal truth values are indeed represented by fuzzy singletons of $\underline{00}$.

$(\sup_k(\|Rx_ix_k\| * \|Sx_kx_j\|))_{ij}$. The calculation is represented by the following diagram:[7]

$$\begin{array}{c|c} \circ & \begin{pmatrix} Sx_1x_1 & \cdots & Sx_1x_n \\ \vdots & \ddots & \vdots \\ Sx_nx_1 & \cdots & Sx_nx_n \end{pmatrix} \\ \hline \begin{pmatrix} Rx_1x_1 & \cdots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \cdots & Rx_nx_n \end{pmatrix} & \begin{pmatrix} (R\circ S)x_1x_1 & \cdots & (R\circ S)x_1x_n \\ \vdots & \ddots & \vdots \\ (R\circ S)x_nx_1 & \cdots & (R\circ S)x_nx_n \end{pmatrix} \end{array}$$

Because of this correspondence, the sup-T-composition is by some authors also called the *sup-T-product* of fuzzy relations. The correspondence extends to the derived notions (since after all, file and row vectors as well as scalars are just special cases of matrices). Thus, e.g., taking the pre-image of a fuzzy class A in a fuzzy relation R can in the finite case be calculated as the sup-T-product of the matrix $(Rx_ix_j)_{ij}$ and the vector $(Ax_j)_j$:

$$\begin{array}{c|c} \circ & \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} \\ \hline \begin{pmatrix} Rx_1x_1 & \cdots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \cdots & Rx_nx_n \end{pmatrix} & \begin{pmatrix} (R^{\leftarrow}A)x_1 \\ \vdots \\ (R^{\leftarrow}A)x_n \end{pmatrix} \end{array}$$

Similarly, the α-resize of a class A is the product of the $(n \times 1)$-vector $\boldsymbol{A}$ and the scalar $\boldsymbol{\alpha}$:

$$\begin{array}{c|c} \circ & (\ \alpha\ ) \\ \hline \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} & \begin{pmatrix} (\alpha A)x_1 \\ \vdots \\ (\alpha A)x_n \end{pmatrix} \end{array}$$

By the usual convention, we write transposed file vectors as row vectors; thus, e.g., for a fuzzy class A over a finite domain we can write $\boldsymbol{A}^{\mathrm{T}} = (Ax_1, \ldots, Ax_n)$. The difference between the Cartesian product $\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$ and the compatibility $\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$ illustrates the importance of distinguishing transposed classes from non-transposed ones:

$$\begin{array}{c|c} \circ & \begin{pmatrix} Bx_1 & \cdots & Bx_n \end{pmatrix} \\ \hline \begin{pmatrix} Ax_1 \\ \vdots \\ Ax_n \end{pmatrix} & \begin{pmatrix} (A\times B)x_1x_1 & \cdots & (A\times B)x_1x_n \\ \vdots & \ddots & \vdots \\ (A\times B)x_nx_1 & \cdots & (A\times B)x_nx_n \end{pmatrix} \end{array} \qquad \begin{array}{c|c} \circ & \begin{pmatrix} Bx_1 \\ \vdots \\ Bx_n \end{pmatrix} \\ \hline \begin{pmatrix} Ax_1 & \cdots & Ax_n \end{pmatrix} & (A \parallel B) \end{array}$$

Notice that compatibility corresponds to the *scalar (sup-T-)product* of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Finally, conjunction is the product of two scalars,

$$\begin{array}{c|c} \circ & (\ \alpha\ ) \\ \hline (\beta) & (\alpha \,\&\, \beta) \end{array}$$

[7] The element in the i-th row and j-th file in the resulting matrix is obtained as the supremum over the values (for all k) of the conjunction of the k-th element in the row and the k-th element in the file, respectively. The diagram just shows the usual way of calculating the matrix product, in which we now take suprema and conjunctions instead sums and products.

Obviously, infinite matrices can be considered as well as finite ones (matrices of arbitrary cardinalities have been used, e.g., in [4]). Thus it can be seen that the apparatus of FCT just formalizes the natural correspondence between fuzzy relations, classes, and truth values on the one hand and (finite or infinite) matrices, vectors, and scalars on the other hand. This will be reflected by the following convention:

Convention 4.5. For the sake of convenience, we shall sometimes employ the matrix terminology and even in the formal theory of FCT call the relations of the form $A \times \underline{1}$ *(file) vectors,* $\underline{1} \times A$ *row vectors,* and fuzzy singletons $\boldsymbol{\alpha} \subseteq^{\triangle} \{\underline{00}\}$ *scalars,* for arbitrary (not only finite) classes A and α. We shall sometimes speak of the *type* of a fuzzy relation, meaning one of these four categories which the relation belongs to.

Remark 4.6. In the graph-theoretical representation of fuzzy relations, a binary fuzzy relation R is identified with a (possibly infinite) weighted node graph, where nodes represent the elements of the domain V^1 of R, and weighted arrows between the nodes indicate the truth values of the relation R between pairs of the elements. Our representation $\boldsymbol{A}$ of a fuzzy class A among fuzzy relations can thus be visualized as a (possibly infinite) graph with arrows from elements x of V^1 to $\underline{0}$ weighted by the values of Ax, and all other arrows weighted by 0 (see Figure 1). Similarly, the transposed class $\boldsymbol{A}^{\mathrm{T}}$ is represented by a graph with arrows from $\underline{0}$ to the elements of V^1 weighted by Ax. Inner truth values are represented by graphs with the only non-zero arrow between $\underline{0}$ and itself, weighted with the truth value it represents.

Sup-T-compositions of the derived notions then work as expected in such node graphs. For instance it can be seen in Figure 1 that the composition of $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{B}$ is an arrow from $\underline{0}$ to $\underline{0}$ aggregating all values $Ax \,\&\, Bx$, which indeed represents the compatibility of A and B, while the composition of $\boldsymbol{A}$ and $\boldsymbol{B}^{\mathrm{T}}$ is a relation between all pairs xy weighted by $Ax \,\&\, By$ (as the only non-zero path from x to y goes through $\underline{0}$), which represents the Cartesian product $A \times B$.

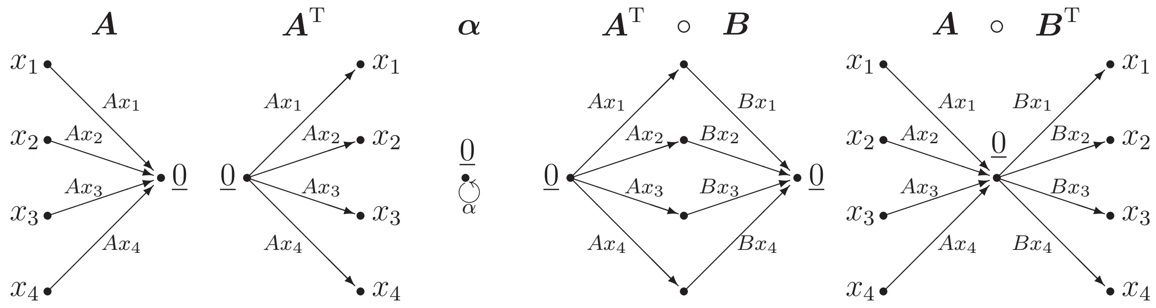

Figure 1: Graph representations of $\boldsymbol{A}$, $\boldsymbol{A}^{\mathrm{T}}$, $\boldsymbol{\alpha}$, $\boldsymbol{A} \circ \boldsymbol{B}^{\mathrm{T}}$, and $\boldsymbol{A}^{\mathrm{T}} \circ \boldsymbol{B}$ (zero-weighted arrows not indicated)

Besides the operations listed in Table 1, further important relational operations are definable from compositions—e.g., by taking the universal class V for an argument in some of the derived notions. Some of such derived notions are listed in Table 2.

Indeed, our conventions identify the domain $\operatorname{Dom} R = \{x \mid (\exists z)Rxz\}$ of a fuzzy relation R with the vector $\{x\underline{0} \mid (\exists z)(Rxz \,\&\, 1)\} = \{x\underline{0} \mid (\exists z)(Rxz \,\&\, \mathbf{V}z\underline{0})\} = R \circ \mathbf{V}$, and similarly for Rng.

domain	$\mathrm{Dom}\, R$	$=$	$R^{\leftarrow}\mathrm{V}$	$\ldots$	$R \circ \mathbf{V}$
range	$\mathrm{Rng}\, R$	$=$	$R^{\rightarrow}\mathrm{V}$	$\ldots$	$R^{\mathrm{T}} \circ \mathbf{V}$
height	$\overline{\mathrm{Hgt}}\, A$	$=$	$A \mathbin{\overline{\|}} \mathrm{V} = \mathrm{V} \mathbin{\overline{\|}} A$	$\ldots$	$\boldsymbol{A}^{\mathrm{T}} \circ \mathbf{V} = \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$

Table 2: Further operations derived from sup-T-compositions

The third operation in Table 2 yields the formal truth value of the *height* $\mathrm{Hgt}\, A \equiv_{\mathrm{df}} (\exists z) Az$ of a fuzzy class A, which our conventions indeed identify with the scalar $\{\underline{00} \mid (\exists z)(\mathbf{V} z \underline{0} \mathbin{\&} \boldsymbol{A} z \underline{0})\} = \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$. In other words, $(\underline{00} \in \mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}) \leftrightarrow \mathrm{Hgt}\, A$, and therefore we can equate the height of A with the scalar $\mathbf{V}^{\mathrm{T}} \circ \boldsymbol{A}$. Like with $\overline{\|}$ or $\overline{\&}$, we denote the operation by $\overline{\mathrm{Hgt}}$ (overlined) as it yields an *inner* truth value (i.e., a fuzzy singleton) and needs to be formally distinguished from Hgt (which is a defined *predicate* and evaluates to *semantic* truth values in a model).

The point of the reduction of the above notions to compositions is of course that the properties of sup-T-compositions automatically transfer to all of them. Thus we now get dozens of theorems on fuzzy relational operations entirely for free.

First we apply Theorem 4.2(2) and Corollary 4.3(1) to the derived notions:

Corollary 4.7. FCT *proves the monotony of all notions listed in Tables 1 and 2 w.r.t. inclusion. In particular,*

$$
\begin{array}{rcl@{\qquad}rcl}
R_1 \subseteq R_2 & \to & R_1 \circ S \subseteq R_2 \circ S & S_1 \subseteq S_2 & \to & R \circ S_1 \subseteq R \circ S_2 \\
R_1 \subseteq R_2 & \to & R_1{}^{\rightarrow} A \subseteq R_2{}^{\rightarrow} A & A_1 \subseteq A_2 & \to & R^{\rightarrow} A_1 \subseteq R^{\rightarrow} A_2 \\
R_1 \subseteq R_2 & \to & R_1{}^{\leftarrow} A \subseteq R_2{}^{\leftarrow} A & A_1 \subseteq A_2 & \to & R^{\leftarrow} A_1 \subseteq R^{\leftarrow} A_2 \\
A_1 \subseteq A_2 & \to & A_1 \times B \subseteq A_2 \times B & B_1 \subseteq B_2 & \to & A \times B_1 \subseteq A \times B_2 \\
A_1 \subseteq A_2 & \to & (A_1 \parallel B \to A_2 \parallel B) & B_1 \subseteq B_2 & \to & (A \parallel B_1 \to A \parallel B_2) \\
A_1 \subseteq A_2 & \to & \alpha A_1 \subseteq \alpha A_2 & (\alpha_1 \to \alpha_2) & \to & \alpha_1 A \subseteq \alpha_2 A \\
(\alpha_1 \to \alpha_2) & \to & (\alpha_1 \mathbin{\&} \beta \to \alpha_2 \mathbin{\&} \beta) & (\beta_1 \to \beta_2) & \to & (\alpha \mathbin{\&} \beta_1 \to \alpha \mathbin{\&} \beta_2)
\end{array}
$$

$$
\begin{array}{rcl}
R_1 \subseteq R_2 & \to & \mathrm{Rng}\, R_1 \subseteq \mathrm{Rng}\, R_2 \\
R_1 \subseteq R_2 & \to & \mathrm{Dom}\, R_1 \subseteq \mathrm{Dom}\, R_2 \\
A_1 \subseteq A_2 & \to & (\mathrm{Hgt}\, A_1 \to \mathrm{Hgt}\, A_2)
\end{array}
$$

Some comments (which apply to subsequent corollaries as well) are in order here:

Remark 4.8. Notice that, as usual in FCT, the theorems have the form of *provable implications.* Thus they are effective even if the antecedent is only partially valid: due to the semantics of implication, they express the fact that the consequent is *at least as true* as the antecedent. Therefore the theorems are stronger than the assertions of the form "if the antecedent is fully true (to degree 1), then so is the consequent", which are more usual in traditional fuzzy mathematics. The traditional theorems, which in formal fuzzy logic would have the form $\triangle\varphi \to \triangle\psi$ rather than $\varphi \to \psi$, follow from those proved in FCT as their special cases with the antecedents true to degree 1. Recall further that in FCT, not only the membership predicate $\in$, but all defined predicates are in general fuzzy (unless they are defined as provably crisp). Thus, e.g., $A \subseteq B$ does not express the fact that the membership function of B majorizes that of A (although this is the meaning of its being true to degree 1): according to its definition, $A \subseteq B$ yields the truth value of the formula $(\forall x)(Ax \to Bx)$, i.e., the infimum of all values $Ax \to Bx$. This kind of *gradual* inclusion has already been considered by Klaua in the 1960's (as reported in [29]) with Łukasiewicz implication; by Bandler and Kohout [3] with a broader class of implicational operators; and by many authors afterwards.

Remark 4.9. Many of the particular theorems listed here are known, even in their gradual forms (see esp. [27, 18]), and all of them have rather simple proofs in FCT. Therefore the main contribution of the present approach is rather the systematic method by which these propositions can be proved *all at once,* as corollaries of the simple statements of Theorem 4.2.

Remark 4.10. Although we present our methods for homogeneous relations only, they can be extended to heterogeneous relations in the following way. Heterogeneous fuzzy relations $R \subseteq^{\triangle} X \times Y$ (for crisp $X, Y \subseteq^{\triangle} \mathrm{V}$) can always be understood as homogeneous fuzzy relations $R \subseteq^{\triangle} \mathrm{V} \times \mathrm{V}$ by taking for V the disjoint union of the two domains X, Y and defining Rxy as 0 outside the domain $X \times Y$ of R. Since 0 is neutral w.r.t. $\exists$, the values of sup-T-compositions are not changed by this extension to V^2. The result of composition $R \circ S \subseteq^{\triangle} \mathrm{V}^2$ of heterogeneous fuzzy relations $R \subseteq^{\triangle} X \times Y$ and $S \subseteq^{\triangle} Y \times Z$ can then again be interpreted as the heterogeneous fuzzy relation $R \circ S \subseteq^{\triangle} X \times Z$, since it is easily proved in FCT that $R \subseteq^{\triangle} X \times Y \;\&\; S \subseteq^{\triangle} Y \times Z \rightarrow R \circ S \subseteq^{\triangle} X \times Z$. Although the theory of heterogeneous relations is not exhausted by this reduction to homogeneous relations (as, i.a., the domains of relations are lost by the reduction), at least it enables to apply the results of the present paper to heterogeneous fuzzy relations.

The following two remarks regard formal and notational aspects of the presented results. Readers that are not interested in formalistic details can safely skip them.

Remark 4.11. We translate the theorems directly into their variants with fuzzy classes A and inner truth values α rather than their relational counterparts $\boldsymbol{A}, \boldsymbol{\alpha}$, although the latter are more direct corollaries of Theorem 4.2. The translation is made possible by the "isomorphism" of A and $A \times \underline{1}$ mentioned in Section 3 and can be made precise by the methods of faithful formal interpretations described in [9]. We do not elaborate on these details here since for the theorems listed in the present paper, their preservation under the translation is perspicuous enough in each particular case.

Remark 4.12. We use the operations Hgt, $\|$ in Corollary 4.7, although more direct corollaries of Theorem 4.2 would contain their counterparts operating on inner truth values ($\overline{\mathrm{Hgt}}, \overline{\|}$). This is allowed by the fact that they directly correspond to each other, as $(\underline{00} \in \overline{\mathrm{Hgt}}\, A) \leftrightarrow \mathrm{Hgt}\, A$, and similarly for other scalar notions. Consequently, by Proposition 3.4(2), the inclusion $\overline{\mathrm{Hgt}}\, A_1 \subseteq \overline{\mathrm{Hgt}}\, A_2$ translates to implication $\mathrm{Hgt}\, A_1 \rightarrow \mathrm{Hgt}\, A_2$ (and similarly for $\overline{\|}$ and other scalar operations).

In particular, formal conjunction (i.e., the intersection of fuzzy singletons, $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$) translates into usual conjunction (as $\underline{00} \in \boldsymbol{\alpha} \cap \boldsymbol{\beta} \leftrightarrow \varphi \,\&\, \psi$ for $\alpha = \overline{\varphi}$ and $\beta = \overline{\psi}$), and similarly inclusion of formal truth values translates into implication (both by Proposition 3.4). The monotony of $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ w.r.t. inclusion thus expresses the monotony of conjunction w.r.t. implication—a theorem that can of course be proved in a much simpler way (even propositionally). We include it here for the sake of completeness, and to show that FCT "knows" the formal counterpart of this propositional law (i.e., that its internalization operating on inner truth values is provable in FCT).

Now we shall continue listing (some of) the corollaries of Theorem 4.2 and Corollary 4.3 for the derived notions.

Corollary 4.13. FCT *proves the following relational properties w.r.t. unions and intersections:*

$$
\begin{array}{rclrcl}
(\bigcup_{R\in\mathcal{A}} R)\circ S & = & \bigcup_{R\in\mathcal{A}}(R\circ S) & R\circ\bigcup_{S\in\mathcal{A}} S & = & \bigcup_{S\in\mathcal{A}}(R\circ S)\\
(\bigcup_{R\in\mathcal{A}} R)^{\rightharpoonup} A & = & \bigcup_{R\in\mathcal{A}}(R^{\rightharpoonup} A) & R^{\rightharpoonup}\bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}}(R^{\rightharpoonup} A)\\
(\bigcup_{R\in\mathcal{A}} R)^{\leftharpoonup} A & = & \bigcup_{R\in\mathcal{A}}(R^{\leftharpoonup} A) & R^{\leftharpoonup}\bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}}(R^{\leftharpoonup} A)\\
(\bigcup_{A\in\mathcal{A}} A)\times B & = & \bigcup_{A\in\mathcal{A}}(A\times B) & A\times\bigcup_{B\in\mathcal{A}} B & = & \bigcup_{B\in\mathcal{A}}(A\times B)\\
(\bigcup_{A\in\mathcal{A}} A)\parallel B & \leftrightarrow & (\exists A\in\mathcal{A})(A\parallel B) & A\parallel\bigcup_{B\in\mathcal{A}} B & \leftrightarrow & (\exists B\in\mathcal{A})(A\parallel B)\\
\alpha\bigcup_{A\in\mathcal{A}} A & = & \bigcup_{A\in\mathcal{A}}(\alpha A) & (\bigvee_{\alpha\in\mathcal{A}}\alpha)A & = & \bigcup_{\alpha\in\mathcal{A}}(\alpha A)\\
(\bigvee_{\alpha\in\mathcal{A}}\alpha)\mathbin{\&}\beta & \leftrightarrow & \bigvee_{\alpha\in\mathcal{A}}(\alpha\mathbin{\&}\beta) & \alpha\mathbin{\&}\bigvee_{\beta\in\mathcal{A}}\beta & \leftrightarrow & \bigvee_{\beta\in\mathcal{A}}(\alpha\mathbin{\&}\beta)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{Dom}(\bigcup_{R\in\mathcal{A}} R) & = & \bigcup_{R\in\mathcal{A}}\mathrm{Dom}\,R\\
\mathrm{Rng}(\bigcup_{R\in\mathcal{A}} R) & = & \bigcup_{R\in\mathcal{A}}\mathrm{Rng}\,R\\
\mathrm{Hgt}(\bigcup_{A\in\mathcal{A}} A) & \leftrightarrow & (\exists A\in\mathcal{A})(\mathrm{Hgt}\,A)
\end{array}
$$

$$
\begin{array}{rclrcl}
(\bigcap_{R\in\mathcal{A}} R)\circ S & \subseteq & \bigcap_{R\in\mathcal{A}}(R\circ S) & R\circ\bigcap_{S\in\mathcal{A}} S & \subseteq & \bigcap_{S\in\mathcal{A}}(R\circ S)\\
(\bigcap_{R\in\mathcal{A}} R)^{\rightharpoonup} A & \subseteq & \bigcap_{R\in\mathcal{A}}(R^{\rightharpoonup} A) & R^{\rightharpoonup}\bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{A\in\mathcal{A}}(R^{\rightharpoonup} A)\\
(\bigcap_{R\in\mathcal{A}} R)^{\leftharpoonup} A & \subseteq & \bigcap_{R\in\mathcal{A}}(R^{\leftharpoonup} A) & R^{\leftharpoonup}\bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{A\in\mathcal{A}}(R^{\leftharpoonup} A)\\
(\bigcap_{A\in\mathcal{A}} A)\times B & \subseteq & \bigcap_{A\in\mathcal{A}}(A\times B) & A\times\bigcap_{B\in\mathcal{A}} B & \subseteq & \bigcap_{B\in\mathcal{A}}(A\times B)\\
(\bigcap_{A\in\mathcal{A}} A)\parallel B & \rightarrow & (\forall A\in\mathcal{A})(A\parallel B) & A\parallel\bigcap_{B\in\mathcal{A}} B & \rightarrow & (\forall B\in\mathcal{A})(A\parallel B)\\
\alpha\bigcap_{A\in\mathcal{A}} A & \subseteq & \bigcap_{A\in\mathcal{A}}(\alpha A) & (\bigwedge_{\alpha\in\mathcal{A}}\alpha)A & \subseteq & \bigcap_{\alpha\in\mathcal{A}}(\alpha A)\\
(\bigwedge_{\alpha\in\mathcal{A}}\alpha)\mathbin{\&}\beta & \rightarrow & \bigwedge_{\alpha\in\mathcal{A}}(\alpha\mathbin{\&}\beta) & \alpha\mathbin{\&}\bigwedge_{\beta\in\mathcal{A}}\beta & \rightarrow & \bigwedge_{\beta\in\mathcal{A}}(\alpha\mathbin{\&}\beta)
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{Dom}(\bigcap_{R\in\mathcal{A}} R) & \subseteq & \bigcap_{R\in\mathcal{A}}\mathrm{Dom}\,R\\
\mathrm{Rng}(\bigcap_{R\in\mathcal{A}} R) & \subseteq & \bigcap_{R\in\mathcal{A}}\mathrm{Rng}\,R\\
\mathrm{Hgt}(\bigcap_{A\in\mathcal{A}} A) & \rightarrow & (\forall A\in\mathcal{A})(\mathrm{Hgt}\,A)
\end{array}
$$

The converse inclusions and implications have (well-known) crisp counter-examples, except those with the Cartesian product, resize, and conjunction, which only have fuzzy counter-examples in MTL *and do hold in stronger fuzzy logics like Gödel or Łukasiewicz.*

Proof. Since all of the inclusions and implications are direct corollaries of Theorem 4.2(3,4) and Corollary 4.3(2,3), we only need to prove the claim about counter-examples to converse inclusions and implications:

As can be seen from the proof of Theorem 4.2(4), the crisp counter-examples can be found whenever the quantification over z in formula (10) in the proof is not void, which (by definitions in Tables 1–2) is the case for all operations in Tables 1–2 except the resize, $\times$, and &. For the latter three operations, the second implication in (10) can be converted (thus they do not have crisp counter-examples), but still the converse to the first implication of (10) is not generally valid in MTL.[8] The first implication of (10) is nevertheless convertible (and so the converse inclusions and implications do hold for the resize, $\times$, and &) in stronger logics like Łukasiewicz or Gödel. □

Relational operations can also be nested, whenever the types of their results permit. The associativity and transposition properties of sup-T-compositions proved in Theorem 4.2(1,5), Proposition 2.14(1), Proposition 3.7, and Lemma 4.15 (below) then yield an infinite number of identities between expressions composed of the operations from Tables 1

[8] A (well-known) counter-example in MTL is, e.g., a $[0,1]$-model with $\alpha = 0.5$, $\beta_n = 0.5 + \frac{1}{n}$ for all natural n, and the nilpotent minimum [25] for &; then $\alpha \mathbin{\&} \bigwedge \beta_n$ is 0, while $\bigwedge(\alpha \mathbin{\&} \beta_n)$ is 0.5. (The counter-examples for the resize and Cartesian product are similar.)

and 2: some of these are listed in the following corollary. We abandon the distinction between A and $\boldsymbol{A}$ here in order to make the chains of identities more compact (cf. footnote 5); similarly we do not distinguish scalar operations from the defined predicates they represent, e.g., $\overline{\mathrm{Hgt}}$ from Hgt (cf. Remark 4.12).

Corollary 4.14. FCT *proves the following identities:*

$$\begin{array}{rcll}
(A \times B)^{\mathrm{T}} & = & (A \circ B^{\mathrm{T}})^{\mathrm{T}} = B \circ A^{\mathrm{T}} & = B \times A \\
R \circ (A \times B) & = & R \circ (A \circ B^{\mathrm{T}}) = (R \circ A) \circ B^{\mathrm{T}} & = (R^{\leftarrow} A) \times B \\
(A \times B) \circ R & = & A \circ B^{\mathrm{T}} \circ R = A \circ (R^{\mathrm{T}} \circ B)^{\mathrm{T}} & = A \times (R^{\rightarrow} B) \\
A \times \alpha B & = & A \circ (B \circ \alpha)^{\mathrm{T}} = A \circ \alpha \circ B^{\mathrm{T}} & = \alpha A \times B \\
A \times \mathrm{Rng}\, R & = & A \circ (R^{\mathrm{T}} \circ \mathrm{V})^{\mathrm{T}} = A \circ \mathrm{V}^{\mathrm{T}} \circ R & = (A \times \mathrm{V}) \circ R
\end{array}$$

$$\begin{array}{rcll}
R^{\rightarrow}(S^{\rightarrow} A) & = & R^{\mathrm{T}} \circ (S^{\mathrm{T}} \circ A) = (S \circ R)^{\mathrm{T}} \circ A & = (S \circ R)^{\rightarrow} A \\
R^{\rightarrow} \alpha A & = & R^{\mathrm{T}} \circ A \circ \alpha & = \alpha(R^{\rightarrow} A) \\
R^{\rightarrow} \mathrm{Rng}\, S & = & R^{\mathrm{T}} \circ S^{\mathrm{T}} \circ \mathrm{V} = (S \circ R)^{\mathrm{T}} \circ \mathrm{V} & = \mathrm{Rng}(S \circ R) \\
(A \times B)^{\rightarrow} C & = & (A \circ B^{\mathrm{T}})^{\mathrm{T}} \circ C = B \circ A^{\mathrm{T}} \circ C & = (A \parallel C) B \\
R^{\leftarrow}(S^{\leftarrow} A) & = & R \circ S \circ A & = (S \circ R)^{\leftarrow} A \\
R^{\leftarrow} \alpha A & = & R \circ A \circ \alpha & = \alpha(R^{\leftarrow} A) \\
R^{\leftarrow} \mathrm{Dom}\, S & = & R \circ S \circ \mathrm{V} & = \mathrm{Dom}(R \circ S) \\
\alpha(\beta A) & = & (A \circ \beta) \circ \alpha = A \circ (\alpha \circ \beta) & = (\alpha \mathbin{\&} \beta) A \\
\alpha(\mathrm{Dom}\, R) & = & R \circ \mathrm{V} \circ \alpha & = R^{\rightarrow} \alpha \mathrm{V} \\
\alpha(\mathrm{Rng}\, R) & = & R^{\mathrm{T}} \circ \mathrm{V} \circ \alpha & = R^{\leftarrow} \alpha \mathrm{V} \\
\mathrm{Dom}(A \times B) & = & A \circ B^{\mathrm{T}} \circ \mathrm{V} & = (\mathrm{Hgt}\, B) A \\
\mathrm{Rng}(A \times B) & = & (A \circ B^{\mathrm{T}})^{\mathrm{T}} \circ \mathrm{V} = B \circ A^{\mathrm{T}} \circ \mathrm{V} & = (\mathrm{Hgt}\, A) B
\end{array}$$

$$\begin{array}{rcll}
A \parallel B & = & A^{\mathrm{T}} \circ B = (A^{\mathrm{T}} \circ B)^{\mathrm{T}} = B^{\mathrm{T}} \circ A & = B \parallel A \\
\alpha \mathbin{\&} \beta & = & (\alpha \circ \beta)^{\mathrm{T}} = \beta^{\mathrm{T}} \circ \alpha^{\mathrm{T}} & = \beta \mathbin{\&} \alpha \\
A \parallel (R^{\rightarrow} B) & = & A^{\mathrm{T}} \circ R^{\mathrm{T}} \circ B = (R \circ A)^{\mathrm{T}} \circ B & = (R^{\leftarrow} A) \parallel B \\
A \parallel \alpha B & = & A^{\mathrm{T}} \circ B \circ \alpha & = \alpha \mathbin{\&} (A \parallel B) \\
A \parallel \mathrm{Dom}\, R & = & A^{\mathrm{T}} \circ R \circ \mathrm{V} = (R^{\mathrm{T}} \circ A)^{\mathrm{T}} \circ \mathrm{V} & = \mathrm{Hgt}(R^{\rightarrow} A) \\
A \parallel \mathrm{Rng}\, R & = & A^{\mathrm{T}} \circ R^{\mathrm{T}} \circ \mathrm{V} = (R \circ A)^{\mathrm{T}} \circ \mathrm{V} & = \mathrm{Hgt}(R^{\leftarrow} A) \\
\alpha \mathbin{\&} (\beta \mathbin{\&} \gamma) & = & \alpha \circ \beta \circ \gamma & = (\alpha \mathbin{\&} \beta) \mathbin{\&} \gamma \\
\mathrm{Hgt}\, \alpha A & = & \mathrm{V}^{\mathrm{T}} \circ A \circ \alpha & = \alpha \mathbin{\&} \mathrm{Hgt}\, A \\
\mathrm{Hgt}\, \mathrm{Dom}\, R & = & \mathrm{V}^{\mathrm{T}} \circ R \circ \mathrm{V} = (R^{\mathrm{T}} \circ \mathrm{V})^{\mathrm{T}} \circ \mathrm{V} & = \mathrm{Hgt}\, \mathrm{Rng}\, R
\end{array}$$

Corollary 4.14 actually lists provable identities between almost all terms with two nested sup-T-operations: it only omits some uninteresting cases like $(A \parallel B)^{\mathrm{T}} = A \parallel B$, formal artifacts like $\mathrm{Hgt}(\mathrm{Hgt}\, A) = \mathrm{Hgt}\, A$, and identities easily reducible to those above by the commutativity of $\parallel$ and & or the interdefinability $R^{\rightarrow} A = (R^{\mathrm{T}})^{\leftarrow} A$ and $\mathrm{Rng}\, R = \mathrm{Dom}\, R^{\mathrm{T}}$. Identities between more complex terms composed of sup-T-operations can be derived by similar simple calculations like those above. For proving some of them, also the following lemma is needed:

Lemma 4.15. FCT *proves the following identities:*

1. $\mathbf{V}^{\mathrm{T}} \circ \mathbf{V} = \underline{\mathbf{1}}$

2. $\boldsymbol{A} \circ \underline{\mathbf{1}} = \boldsymbol{A}, \quad \boldsymbol{\alpha} \circ \underline{\mathbf{1}} = \boldsymbol{\alpha}$

Proof. 1. $\mathbf{V}^{\mathrm{T}} \circ \mathbf{V} = \{\underline{00} \mid (\exists z)(\mathbf{V}^{\mathrm{T}} \underline{0} z \mathbin{\&} \mathbf{V} z \underline{0})\} = \{\underline{00} \mid (\exists z)(\mathrm{V}z \mathbin{\&} \mathrm{V}z)\} = \{\underline{00} \mid 1\} = \underline{\mathbf{1}}$.

2. follows similarly from the provability in MTL of $\alpha \mathbin{\&} 1 \leftrightarrow 1$ and $(\exists z)1 \leftrightarrow 1$. □

Example 4.16. By Lemma 4.15 we get $(A \times \mathrm{V})^{\leftarrow} \alpha \mathrm{V} = A \circ \mathrm{V}^{\mathrm{T}} \circ \mathrm{V} \circ \alpha = A \circ \underline{1} \circ \alpha = A \circ \alpha = \alpha A$.

5 BK-products and derived notions

Besides sup-T-composition, many other products of fuzzy relations have been defined in the literature. Perhaps the most notable among these is the relational product which can be called *inf-R-composition,* as it replaces the supremum in the definition of composition by infimum and the t-norm by its residuum.[9] It has been introduced by Bandler and Kohout in [1] for crisp relations and generalized to fuzzy relations in [2]; referring to the initials of the authors, inf-R-composition is also known as the *BK-product* of fuzzy or crisp relations. Depending on the direction of the residual implication (left-to-right, right-to-left, or both) we get three variants of BK-products:

Definition 5.1. We define the following three products of fuzzy relations R, S:

$$\begin{array}{llll} R \lhd S & =_{\mathrm{df}} & \{xy \mid (\forall z)(Rxz \to Szy)\} & \ldots \textit{BK-subproduct} \\ R \rhd S & =_{\mathrm{df}} & \{xy \mid (\forall z)(Rxz \leftarrow Szy)\} & \ldots \textit{BK-superproduct} \\ R \,\square\, S & =_{\mathrm{df}} & \{xy \mid (\forall z)(Rxz \leftrightarrow Szy)\} & \ldots \textit{BK-squareproduct} \end{array}$$

The prefix BK may be omitted if no confusion can arise. By the *BK-product* (simpliciter) we shall mean the BK-subproduct.

For the motivation and utility of BK-products see [35, 36]. In this paper we give further illustrations of their importance and ubiquity in the theory of fuzzy relations.

Remark 5.2. BK-products have some properties that are felt undesirable in certain kinds of applications of fuzzy relations. As an especially problematic property is by many authors seen the fact that $(R \lhd S)xy$ is 1 whenever $(\exists z)(Rxz)$ is 0. To avoid this particular feature of BK-products, De Baets and Kerre proposed a redefinition of the same notion in [21]: in our notation, De Baets and Kerre's modified definition of $R \lhd S$ reads $\{xy \mid (\exists z)(Rxz) \;\&\; (\forall z)(Rxz \to Szy)\}$, and similarly for $\rhd$ and $\square$. Following De Baets and Kerre's paper, some authors when speaking about BK-products refer to the modified definition rather than Bandler and Kohout's original definition. As this may lead to confusion, we need to stress that in the present paper, we always refer to the *original definitions* by Bandler and Kohout (i.e., those of Definition 5.1), and never to the modification by De Baets and Kerre.

Our sticking to Bandler and Kohout's original definition is justified not only by the suitability for our needs, but also by the fact that De Baets and Kerre's elimination of the "useless pairs" from the product is only suitable in certain applications of fuzzy relational products. In other areas of fuzzy mathematics (e.g., the theory of fuzzy orderings, as shown below), the original notion of BK-product is well-motivated, and the "useless pairs" play important roles in various manifestations of BK-products throughout the theory. This suggests that the emended definition by De Baets and Kerre should not *replace* the original definition by Bandler and Kohout, but only complement it; from this point of view it seems unfortunate that the authors of [21] chose to overload the definition and notation of BK-products rather than to use a modified name and symbols.

In what follows, we shall need the following (well-known) properties of BK-products.

Theorem 5.3. FCT *proves the following properties of BK-products:*

1. *Transposition:* $(R \lhd S)^{\mathrm{T}} = S^{\mathrm{T}} \rhd R^{\mathrm{T}}$

[9] The relationship between sup-T and inf-R composition is an instance of Morsi's duality [40].

2. *Monotony:* $R_1 \subseteq R_2 \to R_2 \lhd S \subseteq R_1 \lhd S, \quad S_1 \subseteq S_2 \to R \lhd S_1 \subseteq R \lhd S_2$

3. *Intersection:* $\bigcap_{R \in \mathcal{A}} (R \lhd S) = \left(\bigcup_{R \in \mathcal{A}} R\right) \lhd S, \quad \bigcap_{S \in \mathcal{A}} (R \lhd S) = R \lhd \bigcap_{S \in \mathcal{A}} S$

4. *Union:* $\bigcup_{R \in \mathcal{A}} (R \lhd S) \subseteq \left(\bigcap_{R \in \mathcal{A}} R\right) \lhd S, \quad \bigcup_{S \in \mathcal{A}} (R \lhd S) \subseteq R \lhd \bigcup_{S \in \mathcal{A}} S$
   (Converse inclusions have crisp counter-examples.)

5. *Residuation:* $R \lhd (S \lhd T) = (R \circ S) \lhd T$

6. *Exchange:* $R \lhd (S \rhd T) = (R \lhd S) \rhd T$

7. *Interdefinability:* $R \mathbin{\square} S = (R \lhd S) \cap_\wedge (R \rhd S)$

Proof. Claims 1–3 are proved similarly as the corresponding statements of Theorem 4.2 (for the shifts of relativized quantifiers needed here, see [16] and Lemma 2.4). The two inclusions of claim 4 are respectively proved by the following chains of implications:

$$(\exists R \in \mathcal{A})(\forall z)(Rxz \to Szy) \longrightarrow (\forall z)(\exists R \in \mathcal{A})(Rxz \to Szy) \longrightarrow (\forall z)[(\forall R \in \mathcal{A})Rxz \to Szy] \tag{11}$$

$$(\exists S \in \mathcal{A})(\forall z)(Rxz \to Szy) \longrightarrow (\forall z)(\exists S \in \mathcal{A})(Rxz \to Szy) \longrightarrow (\forall z)[Rxz \to (\exists S \in \mathcal{A})Szy] \tag{12}$$

The existence of crisp counter-examples to the converse inclusions follows from the fact that the first implications in (11)–(12) cannot be converted in classical logic (as the quantifiers do not commute), while the second implications can.

5. $xy \in R \lhd (S \lhd T) \longleftrightarrow (\forall z)(Rxz \to (\forall t)(Szt \to Tty)) \longleftrightarrow (\forall zt)(Rxz \to (Szt \to Tty)) \longleftrightarrow (\forall zt)(Rxz \,\&\, Szt \to Tty) \longleftrightarrow (\forall t)((\exists z)(Rxz \,\&\, Szt) \to Tty) \longleftrightarrow xy \in (R \circ S) \lhd T$,
and similarly for 6.

7. $xy \in R \mathbin{\square} S \longleftrightarrow (\forall z)(Rxz \leftrightarrow Szy) \longleftrightarrow (\forall z)[(Rxz \to Szy) \wedge (Rxz \leftarrow Szy)] \longleftrightarrow (\forall z)(Rxz \to Szy) \wedge (\forall z)(Rxz \leftarrow Szy) \longleftrightarrow xy \in (R \lhd S) \cap_\wedge (R \rhd S)$. □

By transposition of the statements of Theorem 5.3 we get the following properties of BK-products:

Corollary 5.4. FCT *proves:*

1. *Transposition:* $(R \rhd S)^{\mathrm{T}} = S^{\mathrm{T}} \lhd R^{\mathrm{T}}, \quad (R \mathbin{\square} S)^{\mathrm{T}} = S^{\mathrm{T}} \mathbin{\square} R^{\mathrm{T}}$

2. *Monotony:* $R_1 \subseteq R_2 \to R_1 \rhd S \subseteq R_2 \rhd S, \quad S_1 \subseteq S_2 \to R \rhd S_2 \subseteq R \rhd S_1$

3. *Intersection:* $\bigcap_{R \in \mathcal{A}} (R \rhd S) = \left(\bigcap_{R \in \mathcal{A}} R\right) \rhd S, \quad \bigcap_{S \in \mathcal{A}} (R \rhd S) = R \rhd \bigcup_{S \in \mathcal{A}} S$

4. *Union:* $\bigcup_{R \in \mathcal{A}} (R \rhd S) \subseteq \left(\bigcup_{R \in \mathcal{A}} R\right) \rhd S, \quad \bigcup_{S \in \mathcal{A}} (R \rhd S) \subseteq R \rhd \bigcap_{S \in \mathcal{A}} S$
   (Converse inclusions have crisp counter-examples.)

5. *Residuation:* $(R \rhd S) \rhd T = R \rhd (S \circ T)$

	$\{xy \mid (\forall z)(Rxz \to Szy)\}$	$= R \lhd S$	… $\lhd$-*product*	$R \lhd S$
$x = \underline{0}$	$\{\underline{0}y \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \to Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \lhd R)^{\mathrm{T}}$	$= R^{\mathrm{T}} \rhd \boldsymbol{A}$	… $\lhd$-*image*	$R^{\lhd\!\leftrightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\forall z)(Rxz \to \boldsymbol{A}z\underline{0})\}$	$= R \lhd \boldsymbol{A}$	… $\lhd$-*pre-image*	$R^{\leftarrow\!\lhd} A$
$z = \underline{0}$	$\{xy \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \to \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$= \boldsymbol{A} \lhd \boldsymbol{B}^{\mathrm{T}}$	… *Cartesian* $\lhd$-*product*	$A \times_{\lhd} B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \to \boldsymbol{B}z\underline{0})\}$	$= \boldsymbol{A}^{\mathrm{T}} \lhd \boldsymbol{B}$	… *inclusion*	$A \subseteq B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\forall \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \to \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \lhd \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$= \boldsymbol{A} \rhd \boldsymbol{\alpha}$	… *left* α-*resize*	$\alpha_{\rightharpoondown}A$
$y, z = \underline{0}$	$\{x\underline{0} \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \to \boldsymbol{\alpha}\underline{00})\}$	$= \boldsymbol{A} \lhd \boldsymbol{\alpha}$	… *right* α-*resize*	$A_{\rightharpoondown}\alpha$
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\forall \underline{0})(\boldsymbol{\alpha}\underline{00} \to \boldsymbol{\beta}\underline{00})\}$	$= \boldsymbol{\alpha} \lhd \boldsymbol{\beta}$	… *implication*	$\alpha \to \beta$

$\lhd$-*range*	$\mathrm{Rng}^{\lhd} R = R^{\lhd\!\leftrightarrow}\mathrm{V}$	…	$R^{\mathrm{T}} \rhd \mathbf{V}$
plinth	$\mathrm{Plt}\, A = \mathrm{V} \subseteq A$	…	$\mathbf{V}^{\mathrm{T}} \lhd \boldsymbol{A}$

Table 3: Operations derived from the BK-subproduct

	$\{xy \mid (\forall z)(Rxz \leftarrow Szy)\}$	$= R \rhd S$	… $\rhd$-*product*	$R \rhd S$
$x = \underline{0}$	$\{\underline{0}y \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftarrow Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \rhd R)^{\mathrm{T}}$	$= R^{\mathrm{T}} \lhd \boldsymbol{A}$	… $\rhd$-*image*	$R^{\rhd\!\rightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\forall z)(Rxz \leftarrow \boldsymbol{A}z\underline{0})\}$	$= R \rhd \boldsymbol{A}$	… $\rhd$-*pre-image*	$R^{\leftrightarrow\!\rhd} A$
$z = \underline{0}$	$\{xy \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftarrow \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$= \boldsymbol{A} \rhd \boldsymbol{B}^{\mathrm{T}}$	… *Cartesian* $\rhd$-*product*	$A \times_{\rhd} B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftarrow \boldsymbol{B}z\underline{0})\}$	$= \boldsymbol{A}^{\mathrm{T}} \rhd \boldsymbol{B}$	… *converse inclusion*	$A \supseteq B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\forall \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \leftarrow \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \rhd \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$= \boldsymbol{A} \lhd \boldsymbol{\alpha}$	… *right* α-*resize*	$A_{\rightharpoondown}\alpha$
$y, z = \underline{0}$	$\{x\underline{0} \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftarrow \boldsymbol{\alpha}\underline{00})\}$	$= \boldsymbol{A} \rhd \boldsymbol{\alpha}$	… *left* α-*resize*	$\alpha_{\rightharpoondown}A$
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\forall \underline{0})(\boldsymbol{\alpha}\underline{00} \leftarrow \boldsymbol{\beta}\underline{00})\}$	$= \boldsymbol{\alpha} \rhd \boldsymbol{\beta}$	… *converse implication*	$\alpha \leftarrow \beta$

$\rhd$-*domain*	$\mathrm{Dom}^{\rhd} R = R^{\leftrightarrow\!\rhd}\mathrm{V}$	…	$R \rhd \mathbf{V}$

Table 4: Operations derived from the BK-superproduct

Applying the identifications of the previous section to BK-products in the same way as we did to sup-T-products, we get the derived notions listed in Tables 3–5. We write just $\subseteq, \to, \leftrightarrow, \mathrm{Plt}$, instead of the more correct $\overline{\subseteq}, \overline{\rightrightarrows}, \overline{\leftrightarrows}, \overline{\mathrm{Plt}}$ (cf. Remark 4.12).

Remark 5.5. Notice that some of the analogues of notions based on sup-T-compositions are omitted from Tables 3–5 due to their triviality. The BK-subdomain $\mathrm{Dom}^{\lhd} R = R^{\leftarrow\!\lhd}\mathrm{V}$, i.e., $R \lhd \mathrm{V}$, is always equal to V (similarly for $\rhd$-range) and the superproduct analogue of height or plinth always equals $\underline{1}$. Therefore, by Theorem 5.3(7), the squareproduct analogue of Dom is in fact $\mathrm{Dom}^{\rhd}$, the squareproduct analogue of Rng is $\mathrm{Rng}^{\lhd} R$, and the squareproduct analogue of plinth is just plinth.

Remark 5.6. Unlike in sup-T-compositions, where the behavior of 0 w.r.t. & ensured the right type (in the sense of Convention 3.6) of the result of products for subclasses of $\mathrm{V} \times \underline{1}$ and $\underline{1} \times \underline{1}$ (e.g., that $R \circ \boldsymbol{A} \subseteq^{\triangle} \mathrm{V} \times \underline{1}$), in BK-products this is not automatic (since $0 \to 0$ is 1 rather than 0). For BK-compositions, the right type of the result has to be explicitly controlled by intersecting it with $\mathrm{V} \times \underline{1}$ or $\underline{1} \times \underline{1}$, according to the types of operands: for instance, the correct definition of $R^{\lhd\!\leftrightarrow} A$ is $(R^{\mathrm{T}} \lhd \boldsymbol{A}) \cap (\mathrm{V} \times \underline{1})$ rather than just $R^{\mathrm{T}} \lhd \boldsymbol{A}$, and for $\mathrm{Plt}\, A$ it is $(\mathbf{V}^{\mathrm{T}} \lhd \boldsymbol{A}) \cap (\underline{1} \times \underline{1})$ rather than just $\mathbf{V}^{\mathrm{T}} \lhd \boldsymbol{A}$. We omit the intersection in the definitions, since the right type is already indicated by Convention 3.6 and the properties studied in this paper are obviously preserved by the intersection controlling the type; thus the values of BK-compositions outside their target domain $\mathrm{V} \times \underline{1}$ or $\underline{1} \times \underline{1}$ can safely be ignored. A similar adjustment (by defining Rxy as 1 rather than 0 outside the domain $X \times Y$ of R) has to be made when using BK-compositions of heterogeneous rather than homogeneous relations (cf. Remark 4.10).

	$\{xy \mid (\forall z)(Rxz \leftrightarrow Szy)\}$	$= R \,\square\, S$	… *$\square$-product*	$R \,\square\, S$
$x = \underline{0}$	$\{\underline{0}y \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftrightarrow Rzy)\}^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}} \,\square\, R)^{\mathrm{T}}$	$= R^{\mathrm{T}} \,\square\, \boldsymbol{A}$	… *$\square$-image*	$R^{\square\!\!\rightarrow} A$
$y = \underline{0}$	$\{x\underline{0} \mid (\forall z)(Rxz \leftrightarrow \boldsymbol{A}z\underline{0})\}$	$= R \,\square\, \boldsymbol{A}$	… *$\square$-pre-image*	$R^{\leftarrow\!\!\square} A$
$z = \underline{0}$	$\{xy \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftrightarrow \boldsymbol{B}^{\mathrm{T}}\underline{0}y)\}$	$= \boldsymbol{A} \,\square\, \boldsymbol{B}^{\mathrm{T}}$	… *Cartesian $\square$-product*	$A \times_{\square} B$
$x, y = \underline{0}$	$\{\underline{00} \mid (\forall z)(\boldsymbol{A}^{\mathrm{T}}\underline{0}z \leftrightarrow \boldsymbol{B}z\underline{0})\}$	$= \boldsymbol{A}^{\mathrm{T}} \,\square\, \boldsymbol{B}$	… *weak bi-inclusion*	$A \approx B$
$x, z = \underline{0}$	$\{\underline{0}y \mid (\forall \underline{0})(\boldsymbol{\alpha}^{\mathrm{T}}\underline{00} \leftrightarrow \boldsymbol{A}^{\mathrm{T}}\underline{0}y)\}^{\mathrm{T}} = (\boldsymbol{\alpha}^{\mathrm{T}} \,\square\, \boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$	$= \boldsymbol{A} \,\square\, \boldsymbol{\alpha}$	… *left-right α-resize*	$\alpha_{\leftrightarrow}A$
$y, z = \underline{0}$	$\{x\underline{0} \mid (\forall \underline{0})(\boldsymbol{A}x\underline{0} \leftrightarrow \boldsymbol{\alpha}\underline{00})\}$	$= \boldsymbol{A} \,\square\, \boldsymbol{\alpha}$	… *left-right α-resize*	$\alpha_{\leftrightarrow}A$
$x, y, z = \underline{0}$	$\{\underline{00} \mid (\forall \underline{0})(\boldsymbol{\alpha}\underline{00} \leftrightarrow \boldsymbol{\beta}\underline{00})\}$	$= \boldsymbol{\alpha} \,\square\, \boldsymbol{\beta}$	… *equivalence*	$\alpha \leftrightarrow \beta$

Table 5: Operations derived from the BK-squareproduct

Corollary 5.7. *By Theorem 5.3(1) and the definitions of Tables 3 and 4 we have the following interdefinability between derived BK-notions:*

$$\begin{array}{rclcl} A \times_{\rhd} B &=& A \rhd B^{\mathrm{T}} = (B \lhd A^{\mathrm{T}})^{\mathrm{T}} &=& (B \times_{\lhd} A)^{\mathrm{T}} \\ R^{\leftarrow\!\!\rhd} A &=& R \rhd A = R^{\mathrm{TT}} \rhd A &=& R^{\mathrm{T}\,\lhd\!\!\rightarrow} A \\ \mathrm{Dom}^{\rhd} R &=& R \rhd \mathrm{V} = R^{\mathrm{TT}} \rhd \mathrm{V} &=& \mathrm{Rng}^{\lhd} R^{\mathrm{T}} \end{array}$$

Corollary 5.8. *By Theorem 5.3(7), the squareproduct notions are definable in terms of the corresponding subproduct and superproduct notions by means of min-intersection (or min-conjunction):*

$$\begin{array}{rcl} R^{\square\!\!\rightarrow} A &=& (R^{\lhd\!\!\rightarrow} A) \cap_{\wedge} (R^{\rhd\!\!\rightarrow} A) \\ R^{\leftarrow\!\!\square} A &=& (R^{\leftarrow\!\!\lhd} A) \cap_{\wedge} (R^{\leftarrow\!\!\rhd} A) \\ A \times_{\square} B &=& (A \times_{\lhd} B) \cap_{\wedge} (A \times_{\rhd} B) \\ A \approx B &=& (A \subseteq B) \wedge (B \subseteq A) \\ \alpha_{\leftrightarrow} A &=& (\alpha_{\rightarrow} A) \cap_{\wedge} (A_{\rightarrow} \alpha) \\ \alpha \leftrightarrow \beta &=& (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha) \end{array}$$

The importance of the ten sup-T-based operations studied in the previous section is beyond doubt. The following examples show that the BK-related notions abound in fuzzy mathematics as well. Thus the present section can also be viewed as the systematization of these miscellaneous notions and their properties.

Example 5.9. The operation of *subproduct preimage* $R^{\leftarrow\!\!\lhd} A$ appears frequently in the theory of fuzzy relations [27, 18, 19]. In [26], $R^{\leftarrow\!\!\lhd}$ is called the *right backward strong powerset operator* of R; the operation is denoted by $\downarrow$ in [27]. It is also a quantifier construction in fuzzy description logic [31], where it is written as $(\forall R.A)$. Further graded properties of this operation besides those studied here can be found in [27, 11]. The *superproduct image* $R^{\rhd\!\!\rightarrow}$ is studied, e.g., in [26] where it is called the *right forward strong powerset operator* of R.

Example 5.10. In the theory of fuzzy orderings, the *subproduct image* $R^{\lhd\!\!\rightarrow} A$ and *superproduct preimage* $R^{\leftarrow\!\!\rhd} A$ denote the fuzzy set of all upper resp. lower bounds of the fuzzy set A w.r.t. a fuzzy ordering R (also called the upper and lower *cone* of A w.r.t. R). The operations $R^{\lhd\!\!\rightarrow}$ and $R^{\leftarrow\!\!\rhd}$, respectively, are called the *exclusive image* and *exclusive preimage* in [5] and the *left forward* and *left backward strong powerset operator of* R in [26]. The operation ${}^{\lhd\!\!\rightarrow}$ has also appeared in [22] and has been used for fuzzy inference in [20].

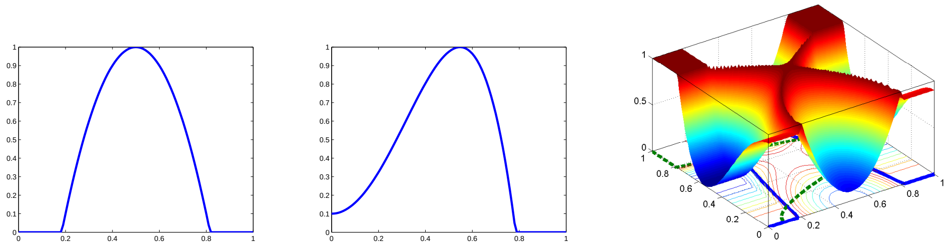

Figure 2: Fuzzy sets A and B and their Cartesian squareproduct $A \times_{\square} B$ under the Łukasiewicz t-norm

Example 5.11. For some applications of *BK-products* $\triangleleft, \triangleright, \square$ themselves see [35, 36]. Besides the practically oriented applications, their theoretical importance comes from the fact that many other relational notions can be expressed by means of BK-products. For example, fuzzy preorders can be characterized in terms of BK-products [7] by

$$\begin{aligned} \operatorname{Refl} R &\leftrightarrow R^{\mathrm{T}} \triangleleft R \subseteq R \\ \operatorname{Trans} R &\leftrightarrow R \subseteq R^{\mathrm{T}} \triangleleft R \end{aligned}$$

The operation $R^{\mathrm{T}} \triangleleft R$ and its dual $R \triangleleft R^{\mathrm{T}}$ are sometimes called the *left* resp. *right trace* of R and are of their own importance [24, 11].

Example 5.12. The *Cartesian products* $\times, \times_{\triangleleft}, \times_{\triangleright}, \times_{\square}$ are used to model sets of fuzzy rules:

$$\begin{array}{rlcll} & \{(x \text{ is } \mathrm{A}_i) & \texttt{and} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \ \bigcup_{i\in I}(A_i \times B_i) \\ \{\texttt{if} & (x \text{ is } \mathrm{A}_i) & \texttt{then} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \ \bigcap_{i\in I}(A_i \times_{\triangleleft} B_i) \\ & \{(x \text{ is } \mathrm{A}_i) & \texttt{whenever} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \ \bigcap_{i\in I}(A_i \times_{\triangleright} B_i) \\ & \{(x \text{ is } \mathrm{A}_i) & \texttt{iff} & (y \text{ is } \mathrm{B}_i)\}_{i\in I} & \ldots \ \bigcap_{i\in I}(A_i \times_{\square} B_i) \end{array}$$

The first three operations are used in many applications of fuzzy control theory, even though $\times$ is often misinterpreted as "implication" [38] rather than the Cartesian product based on strong *conjunction*. The Cartesian squareproduct $\times_{\square}$ is rather neglected in the fuzzy literature, even though in many approximation problems it is more appropriate than $\times_{\triangleleft}$ and $\times_{\triangleright}$, as it captures fuzzy equivalence between input and output fuzzy sets, expressing that "x is A to a similar degree as y is B" (see Figure 2).

Example 5.13. The *α-resizes* $\alpha A, \alpha_{\rightharpoondown} A, A_{\rightharpoondown}\alpha, \alpha_{\leftrightarrow} A$ occur in fuzzy control applications. There are two competing approaches to approximate inference over a knowledge base formalized as a set of fuzzy rules. The classical approach is *FATI* (first aggregate then infer). The *FITA* (first infer then aggregate) method of activation degrees was first used by Holmblad and Ostergaard [33] in a fuzzy control algorithm for a cement kiln. It can briefly be described as follows [28]:

For each actual input fuzzy set A and each input–output data pair (A_k, B_k) one determines a modification B_k^* of the "local" output B_k, and aggregates the modified "local" outputs into one global output: $B^* = \bigcup_{i\in I} B_i^*$. The particular choice by Holmblad and Ostergaard for B_k^* was $B_k^*(y) = \operatorname{Hgt}(A \cap_{\wedge} A_k) \cdot B_k(y)$, which is in fact the $\operatorname{Hgt}(A \cap_{\wedge} A_k)$-resize of B_k under the product t-norm.

To take another example, if Zadeh's compositional rule of inference is applied to a knowledge formalized by $\times$, which in our formalism reads $\left(\bigcup_{i\in I}(A_i \times B_i)\right)^{\rightharpoondown} A$, it can be simplified by using α-resizes in virtue of the identity

$$\left(\bigcup_{i\in I}(A_i \times B_i)\right)^{\rightharpoondown} A = \bigcup_{i\in I}(A \parallel A_i) B_i$$

which follows from Corollaries 4.13 and 4.14. Analogously, the authors of [39] speak about the consequent dilatation rule proposed in [37], where the degrees of subsethood $A \subseteq A_i$ for $i \in I$ are used to compute the final output which is in our notation written as $B^* = \bigcap_{i \in I} (A \subseteq A_i)_{\rightarrow} B_i$ (cf. the appropriate identities from Corollaries 5.16 and 5.17).

The main argument in favor of practical applications of α-resizes is the speed of computations. It is much faster to resize and then aggregate than to use the FATI approach because the values for the resizes are computed only once and then used multiple times.

Example 5.14. In theoretical investigation of fuzzy relations, α-resizes appear for instance in the following contexts: closedness under S_K-intersections for a set K of designated truth values [18, Def. 7.4] is equivalent [18, Th. 7.6] to closedness under intersections of "K-shifted" sets $(\alpha_{\rightarrow} A)$; furthermore, $A_{\rightarrow}\alpha$ and αA have been used to characterize a system of closed sets of a similarity space in [18, Th. 7.62]; the system of all extensional fuzzy sets can be characterized by means of $\alpha_{\rightarrow} A$, $A_{\rightarrow}\alpha$ and αA [34, Th. 3.2]; the α-properties of binary fuzzy relations studied in [6] are related to α-resizes of a relation [6, Th. 4.24]; etc.

The above list of applications of inf-R-compositional notions is by no means exhaustive. Like with the notions based on the sup-T-composition, the point of our construction is the possibility of applying Theorem 5.3 and Corollary 5.4 to all notions defined in Tables 3–5. Thus we are given the following corollaries entirely for free (Remarks 4.8–4.12 apply to these corollaries as well):

Corollary 5.15. *In consequence of Theorem 5.3(2) and Corollary 5.4(2),* FCT *proves:*

$$
\begin{array}{rcl@{\qquad}rcl}
R_1 \subseteq R_2 & \rightarrow & R_1 {}^{\lhd\!\!\leftrightarrow} A \subseteq R_2 {}^{\lhd\!\!\leftrightarrow} A & A_1 \subseteq A_2 & \rightarrow & R {}^{\lhd\!\!\leftrightarrow} A_2 \subseteq R {}^{\lhd\!\!\leftrightarrow} A_1 \\
R_1 \subseteq R_2 & \rightarrow & R_2 {}^{\leftarrow\!\!\lhd} A \subseteq R_1 {}^{\leftarrow\!\!\lhd} A & A_1 \subseteq A_2 & \rightarrow & R {}^{\leftarrow\!\!\lhd} A_1 \subseteq R {}^{\leftarrow\!\!\lhd} A_2 \\
A_1 \subseteq A_2 & \rightarrow & A_2 \times_{\lhd} B \subseteq A_1 \times_{\lhd} B & B_1 \subseteq B_2 & \rightarrow & A \times_{\lhd} B_1 \subseteq A \times_{\lhd} B_2 \\
A_1 \subseteq A_2 & \rightarrow & (A_2 \subseteq B \rightarrow A_1 \subseteq B) & A_1 \subseteq A_2 & \rightarrow & (B \subseteq A_1 \rightarrow B \subseteq A_2) \\
A_1 \subseteq A_2 & \rightarrow & \alpha_{\rightarrow} A_1 \subseteq \alpha_{\rightarrow} A_2 & (\alpha_1 \rightarrow \alpha_2) & \rightarrow & (\alpha_2)_{\rightarrow} A \subseteq (\alpha_1)_{\rightarrow} A \\
A_1 \subseteq A_2 & \rightarrow & (A_2)_{\rightarrow}\alpha \subseteq (A_1)_{\rightarrow}\alpha & (\alpha_1 \rightarrow \alpha_2) & \rightarrow & (A_{\rightarrow}\alpha_1 \rightarrow A_{\rightarrow}\alpha_2) \\
(\alpha_1 \rightarrow \alpha_2) & \rightarrow & [(\alpha_2 \rightarrow \beta) \rightarrow (\alpha_1 \rightarrow \beta)] & (\beta_1 \rightarrow \beta_2) & \rightarrow & [(\alpha \rightarrow \beta_1) \rightarrow (\alpha \rightarrow \beta_2)]
\end{array}
$$

$$
\begin{array}{rcl}
R_1 \subseteq R_2 & \rightarrow & \mathrm{Rng}^{\lhd} R_1 \subseteq \mathrm{Rng}^{\lhd} R_2 \\
A_1 \subseteq A_2 & \rightarrow & (\mathrm{Plt}\, A_1 \rightarrow \mathrm{Plt}\, A_2)
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
R_1 \subseteq R_2 & \rightarrow & R_2 {}^{\rhd\!\!\rightarrow} A \subseteq R_1 {}^{\rhd\!\!\rightarrow} A & A_1 \subseteq A_2 & \rightarrow & R {}^{\rhd\!\!\rightarrow} A_1 \subseteq R {}^{\rhd\!\!\rightarrow} A_2 \\
R_1 \subseteq R_2 & \rightarrow & R_1 {}^{\leftrightarrow\!\!\rhd} A \subseteq R_2 {}^{\leftrightarrow\!\!\rhd} A & A_1 \subseteq A_2 & \rightarrow & R {}^{\leftrightarrow\!\!\rhd} A_2 \subseteq R {}^{\leftrightarrow\!\!\rhd} A_1 \\
A_1 \subseteq A_2 & \rightarrow & A_1 \times_{\rhd} B \subseteq A_2 \times_{\rhd} B & B_1 \subseteq B_2 & \rightarrow & A \times_{\rhd} B_2 \subseteq A \times_{\rhd} B_1
\end{array}
$$

$$
R_1 \subseteq R_2 \quad \rightarrow \quad \mathrm{Dom}^{\rhd} R_1 \subseteq \mathrm{Dom}^{\rhd} R_2
$$

Corollary 5.16. *By Theorem 5.3(3, 4) and Corollary 5.4(3, 4),* FCT *proves:*

$$
\begin{array}{rcl@{\qquad}rcl}
\bigcap_{R \in \mathcal{A}} (R {}^{\lhd\!\!\leftrightarrow} A) & = & \left(\bigcap_{R \in \mathcal{A}} R\right) {}^{\lhd\!\!\leftrightarrow} A & \bigcap_{A \in \mathcal{A}} (R {}^{\lhd\!\!\leftrightarrow} A) & = & R {}^{\lhd\!\!\leftrightarrow} \bigcup_{A \in \mathcal{A}} A \\
\bigcap_{R \in \mathcal{A}} (R {}^{\leftarrow\!\!\lhd} A) & = & \left(\bigcup_{R \in \mathcal{A}} R\right) {}^{\leftarrow\!\!\lhd} A & \bigcap_{A \in \mathcal{A}} (R {}^{\leftarrow\!\!\lhd} A) & = & R {}^{\leftarrow\!\!\lhd} \bigcap_{A \in \mathcal{A}} A \\
\bigcap_{A \in \mathcal{A}} (A \times_{\lhd} B) & = & \left(\bigcup_{A \in \mathcal{A}} A\right) \times_{\lhd} B & \bigcap_{B \in \mathcal{A}} (A \times_{\lhd} B) & = & A \times_{\lhd} \bigcap_{B \in \mathcal{A}} B \\
(\forall A \in \mathcal{A})(A \subseteq B) & \leftrightarrow & \left(\bigcup_{A \in \mathcal{A}} A\right) \subseteq B & (\forall B \in \mathcal{A})(A \subseteq B) & \leftrightarrow & A \subseteq \bigcap_{B \in \mathcal{A}} B \\
\bigcap_{\alpha \in \mathcal{A}} (\alpha_{\rightarrow} A) & = & \left(\bigvee_{\alpha \in \mathcal{A}} \alpha\right)_{\rightarrow} A & \bigcap_{A \in \mathcal{A}} (\alpha_{\rightarrow} A) & = & \alpha_{\rightarrow} \bigcap_{A \in \mathcal{A}} A \\
\bigcap_{A \in \mathcal{A}} (A_{\rightarrow}\alpha) & = & \left(\bigcup_{A \in \mathcal{A}} A\right)_{\rightarrow}\alpha & \bigcap_{\alpha \in \mathrm{A}} (A_{\rightarrow}\alpha) & = & A_{\rightarrow} \bigwedge_{\alpha \in \mathcal{A}} \alpha \\
\bigwedge_{\alpha \in \mathcal{A}} (\alpha \rightarrow \beta) & \leftrightarrow & \left(\bigvee_{\alpha \in \mathcal{A}}\right) \rightarrow \beta & \bigwedge_{\beta \in \mathcal{A}} (\alpha \rightarrow \beta) & \leftrightarrow & \left(\alpha \rightarrow \bigwedge_{\beta \in \mathcal{A}} \beta\right)
\end{array}
$$

$$
\begin{aligned}
\textstyle\bigcap_{R\in\mathcal{A}} \operatorname{Rng}^{\triangleleft} R &= \operatorname{Rng}^{\triangleleft} \textstyle\bigcap_{R\in\mathcal{A}} R \\
(\forall A\in\mathcal{A})(\operatorname{Plt} A) &\leftrightarrow \operatorname{Plt} \textstyle\bigcap_{A\in\mathcal{A}} A
\end{aligned}
$$

$$
\begin{aligned}
\textstyle\bigcap_{R\in\mathcal{A}}(R\,{}^{\triangleright\!\!\rightarrow} A) &= \big(\textstyle\bigcup_{R\in\mathcal{A}} R\big)\,{}^{\triangleright\!\!\rightarrow} A & \textstyle\bigcap_{A\in\mathcal{A}}(R\,{}^{\triangleright\!\!\rightarrow} A) &= R\,{}^{\triangleright\!\!\rightarrow} \textstyle\bigcap_{A\in\mathcal{A}} A \\
\textstyle\bigcap_{R\in\mathcal{A}}(R\,{}^{\leftrightarrow\!\!\triangleright} A) &= \big(\textstyle\bigcap_{R\in\mathcal{A}} R\big)\,{}^{\leftrightarrow\!\!\triangleright} A & \textstyle\bigcap_{A\in\mathcal{A}}(R\,{}^{\leftrightarrow\!\!\triangleright} A) &= R\,{}^{\leftrightarrow\!\!\triangleright} \textstyle\bigcup_{A\in\mathcal{A}} A \\
\textstyle\bigcap_{A\in\mathcal{A}}(A\times_{\triangleright} B) &= \big(\textstyle\bigcap_{A\in\mathcal{A}} A\big)\times_{\triangleright} B & \textstyle\bigcap_{B\in\mathcal{A}}(A\times_{\triangleright} B) &= A\times_{\triangleright} \textstyle\bigcup_{B\in\mathcal{A}} B
\end{aligned}
$$

$$
\textstyle\bigcap_{R\in\mathcal{A}} \operatorname{Dom}^{\triangleright} R = \operatorname{Dom}^{\triangleright} \textstyle\bigcap_{R\in\mathcal{A}} R
$$

$$
\begin{aligned}
\textstyle\bigcup_{R\in\mathcal{A}}(R\,{}^{\triangleleft\!\!\leftrightarrow} A) &\subseteq \big(\textstyle\bigcup_{R\in\mathcal{A}} R\big)\,{}^{\triangleleft\!\!\leftrightarrow} A & \textstyle\bigcup_{A\in\mathcal{A}}(R\,{}^{\triangleleft\!\!\leftrightarrow} A) &\subseteq R\,{}^{\triangleleft\!\!\leftrightarrow} \textstyle\bigcap_{A\in\mathcal{A}} A \\
\textstyle\bigcup_{R\in\mathcal{A}}(R\,{}^{\leftarrow\!\!\triangleleft} A) &\subseteq \big(\textstyle\bigcap_{R\in\mathcal{A}} R\big)\,{}^{\leftarrow\!\!\triangleleft} A & \textstyle\bigcup_{A\in\mathcal{A}}(R\,{}^{\leftarrow\!\!\triangleleft} A) &\subseteq R\,{}^{\leftarrow\!\!\triangleleft} \textstyle\bigcup_{A\in\mathcal{A}} A \\
\textstyle\bigcup_{A\in\mathcal{A}}(A\times_{\triangleleft} B) &\subseteq \big(\textstyle\bigcap_{A\in\mathcal{A}} A\big)\times_{\triangleleft} B & \textstyle\bigcup_{B\in\mathcal{A}}(A\times_{\triangleleft} B) &\subseteq A\times_{\triangleleft} \textstyle\bigcup_{B\in\mathcal{A}} B \\
(\exists A\in\mathcal{A})(A\subseteq B) &\rightarrow \big(\textstyle\bigcap_{A\in\mathcal{A}} A\big)\subseteq B & (\exists B\in\mathcal{A})(A\subseteq B) &\rightarrow A\subseteq \textstyle\bigcup_{B\in\mathcal{A}} B \\
\textstyle\bigcup_{\alpha\in\mathcal{A}}(\alpha_{\rightarrow} A) &\subseteq \big(\textstyle\bigwedge_{\alpha\in\mathcal{A}} \alpha\big)_{\rightarrow} A & \textstyle\bigcup_{A\in\mathcal{A}}(\alpha_{\rightarrow} A) &\subseteq \alpha_{\rightarrow} \textstyle\bigcup_{A\in\mathcal{A}} A \\
\textstyle\bigcup_{A\in\mathcal{A}}(A_{\rightarrow}\alpha) &\subseteq \big(\textstyle\bigcap_{A\in\mathcal{A}} A\big)_{\rightarrow}\alpha & \textstyle\bigcup_{\alpha\in\mathrm{A}}(A_{\rightarrow}\alpha) &\subseteq A_{\rightarrow} \textstyle\bigvee_{\alpha\in\mathcal{A}} \alpha \\
\textstyle\bigvee_{\alpha\in\mathcal{A}}(\alpha\rightarrow\beta) &\rightarrow \big(\big(\textstyle\bigwedge_{\alpha\in\mathcal{A}} \alpha\big)\rightarrow\beta\big) & \textstyle\bigvee_{\beta\in\mathcal{A}}(\alpha\rightarrow\beta) &\rightarrow \big(\alpha\rightarrow \textstyle\bigvee_{\beta\in\mathcal{A}} \alpha\big)
\end{aligned}
$$

$$
\begin{aligned}
\textstyle\bigcup_{R\in\mathcal{A}} \operatorname{Rng}^{\triangleleft} R &\subseteq \operatorname{Rng}^{\triangleleft} \textstyle\bigcup_{R\in\mathcal{A}} R \\
(\exists A\in\mathcal{A})(\operatorname{Plt} A) &\rightarrow \operatorname{Plt} \textstyle\bigcup_{A\in\mathcal{A}} A
\end{aligned}
$$

$$
\begin{aligned}
\textstyle\bigcup_{R\in\mathcal{A}}(R\,{}^{\triangleright\!\!\rightarrow} A) &\subseteq \big(\textstyle\bigcap_{R\in\mathcal{A}} R\big)\,{}^{\triangleright\!\!\rightarrow} A & \textstyle\bigcup_{A\in\mathcal{A}}(R\,{}^{\triangleright\!\!\rightarrow} A) &\subseteq R\,{}^{\triangleright\!\!\rightarrow} \textstyle\bigcup_{A\in\mathcal{A}} A \\
\textstyle\bigcup_{R\in\mathcal{A}}(R\,{}^{\leftrightarrow\!\!\triangleright} A) &\subseteq \big(\textstyle\bigcup_{R\in\mathcal{A}} R\big)\,{}^{\leftrightarrow\!\!\triangleright} A & \textstyle\bigcup_{A\in\mathcal{A}}(R\,{}^{\leftrightarrow\!\!\triangleright} A) &\subseteq R\,{}^{\leftrightarrow\!\!\triangleright} \textstyle\bigcap_{A\in\mathcal{A}} A \\
\textstyle\bigcup_{A\in\mathcal{A}}(A\times_{\triangleright} B) &\subseteq \big(\textstyle\bigcup_{A\in\mathcal{A}} A\big)\times_{\triangleright} B & \textstyle\bigcup_{B\in\mathcal{A}}(A\times_{\triangleright} B) &\subseteq A\times_{\triangleright} \textstyle\bigcap_{B\in\mathcal{A}} B
\end{aligned}
$$

$$
\textstyle\bigcup_{R\in\mathcal{A}} \operatorname{Dom}^{\triangleright} R \subseteq \operatorname{Dom}^{\triangleright} \textstyle\bigcup_{R\in\mathcal{A}} R
$$

The converse implications and inclusions have crisp counter-examples.

Proof. We only need to prove the claim about converse inclusions and implications, as the rest are direct corollaries of the indicated theorems. The existence of crisp counter-examples follows from the fact that neither of implications in the proof of Theorem 5.3(4) is in general convertible in classical logic. In the case of $\times_{\triangleleft}$, $\times_{\triangleright}$, $_{\rightarrow}$, and $\rightarrow$, for which the quantification over z in formulae (11)–(12) in the proof is void, the only crisp counter-examples are with $\mathcal{A}=\emptyset$. For non-empty $\mathcal{A}$, the latter converses hold in those extensions of MTL in which the law of double negation $\neg\neg\varphi\rightarrow\varphi$ is valid (i.e., the extensions of IMTL, e.g., Łukasiewicz logic), since the second implications in formulae (11)–(12) are convertible under double negation (but not generally in MTL). □

Corollary 5.17. *By Theorem 5.3(5, 6) and Corollary 5.4(5),* FCT *proves, i.a., the following identities:*

$$
\begin{aligned}
(R\,{}^{\leftarrow} A)\times_{\triangleleft} B &= R\triangleleft(A\times_{\triangleleft} B) && \textit{by}\quad (R\circ A)\triangleleft B^{\mathrm{T}} = R\triangleleft(A\triangleleft B^{\mathrm{T}}) \\
(\operatorname{Dom} R)\times_{\triangleleft} A &= R\triangleleft(\mathrm{V}\times_{\triangleleft} A) && \phantom{\textit{by}}\quad (R\circ \mathrm{V})\triangleleft A^{\mathrm{T}} = R\triangleleft(\mathrm{V}\triangleleft A^{\mathrm{T}}) \\
(\alpha A)\times_{\triangleleft} B &= A\times_{\triangleleft}(\alpha_{\rightarrow} B) && \phantom{\textit{by}}\quad (\alpha\circ A)\triangleleft B^{\mathrm{T}} = A\triangleleft(\alpha\triangleleft B^{\mathrm{T}}) \\
A\times_{\triangleleft}(R\,{}^{\triangleright\!\!\rightarrow} B) &= (A\times_{\triangleleft} B)\triangleright R && \phantom{\textit{by}}\quad A\triangleleft(R^{\mathrm{T}}\triangleleft B)^{\mathrm{T}} = A\triangleleft(B^{\mathrm{T}}\triangleright R) = (A\triangleleft B^{\mathrm{T}})\triangleright R \\
A\times_{\triangleleft}(R\,{}^{\triangleleft\!\!\leftrightarrow} B) &= (A\times B)\triangleleft R && \phantom{\textit{by}}\quad A\triangleleft(R^{\mathrm{T}}\triangleright B)^{\mathrm{T}} = A\triangleleft(B^{\mathrm{T}}\triangleleft R) = (A\circ B^{\mathrm{T}})\triangleleft R \\
A\times_{\triangleleft}(\operatorname{Rng}^{\triangleleft} R) &= (A\times \mathrm{V})\triangleleft R && \phantom{\textit{by}}\quad A\triangleleft(R^{\mathrm{T}}\triangleright \mathrm{V})^{\mathrm{T}} = A\triangleleft(\mathrm{V}^{\mathrm{T}}\triangleleft R) = (A\circ \mathrm{V}^{\mathrm{T}})\triangleleft R \\
A\times_{\triangleleft}(B_{\rightarrow}\alpha) &= (A_{\rightarrow}\alpha)\times_{\triangleright} B && \phantom{\textit{by}}\quad A\triangleleft(B\triangleleft\alpha)^{\mathrm{T}} = A\triangleleft(\alpha\triangleright B^{\mathrm{T}}) = (A\triangleleft\alpha)\triangleright B^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{array}{rcll}
(R \circ S)^{\leftarrow\!\lhd} A & = & R^{\leftarrow\!\lhd}(S^{\leftarrow\!\lhd} A) & \textit{by} \quad (R \circ S) \lhd A = R \lhd (S \lhd A) \\
(R \lhd S)^{\lhd\!\leftrightarrow} A & = & S^{\lhd\!\leftrightarrow}(R^{\rightarrow} A) & \quad (R \lhd S)^{\mathrm{T}} \rhd A = (S^{\mathrm{T}} \rhd R^{\mathrm{T}}) \rhd A = S^{\mathrm{T}} \rhd (R^{\mathrm{T}} \circ A) \\
(R \rhd S)^{\lhd\!\leftrightarrow} A & = & S^{\rhd\!\rightarrow}(R^{\lhd\!\leftrightarrow} A) & \quad (R \rhd S)^{\mathrm{T}} \rhd A = (S^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd A = S^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd A) \\
(A \times B)^{\leftarrow\!\lhd} C & = & A_{\rightarrow}(B \subseteq C) & \quad (A \circ B^{\mathrm{T}}) \lhd C = A \lhd (B^{\mathrm{T}} \lhd C) \\
(A \times_{\lhd} B)^{\lhd\!\leftrightarrow} C & = & (A \parallel C)_{\rightarrow} B & \quad (A \lhd B^{\mathrm{T}})^{\mathrm{T}} \rhd C = (B \rhd A^{\mathrm{T}}) \rhd C = B \rhd (A^{\mathrm{T}} \circ C) \\
(A \times_{\rhd} B)^{\lhd\!\leftrightarrow} C & = & B_{\rightarrow}(C \subseteq A) & \quad (A \rhd B^{\mathrm{T}})^{\mathrm{T}} \rhd C = (B \lhd A^{\mathrm{T}}) \rhd C = B \lhd (A^{\mathrm{T}} \rhd C) \\
R^{\leftarrow\!\lhd}(A_{\rightarrow}\alpha) & = & (R^{\leftarrow} A)_{\rightarrow}\alpha & \quad R \lhd (A \lhd \alpha) = (R \circ A) \lhd \alpha \\
\alpha_{\rightarrow}(\mathrm{Rng}^{\lhd} R) & = & R^{\lhd\!\leftrightarrow}(\alpha \mathrm{V}) & \quad \alpha \lhd (R^{\mathrm{T}} \rhd \mathrm{V}) = (R^{\mathrm{T}} \rhd \mathrm{V}) \rhd \alpha = R^{\mathrm{T}} \rhd (\alpha \circ \mathrm{V}) \\
\alpha_{\rightarrow}(R^{\lhd\!\leftrightarrow} A) & = & R^{\lhd\!\leftrightarrow}(\alpha A) & \quad \alpha \lhd (R^{\mathrm{T}} \rhd A) = (R^{\mathrm{T}} \rhd A) \rhd \alpha = R^{\mathrm{T}} \rhd (\alpha \circ A) \\
\alpha_{\rightarrow}(R^{\leftarrow\!\lhd} A) & = & R^{\leftarrow\!\lhd}(\alpha_{\rightarrow} A) & \quad \alpha \lhd (R \lhd A) = R \lhd (\alpha \lhd A) \\
\alpha_{\rightarrow}(\beta_{\rightarrow} A) & = & (\alpha \mathbin{\&} \beta)_{\rightarrow} A & \quad \alpha \lhd (\beta \lhd A) = (\alpha \circ \beta) \lhd A \\
\alpha_{\rightarrow}(A_{\rightarrow}\beta) & = & A_{\rightarrow}(\alpha \rightarrow \beta) & \quad \alpha \lhd (A \lhd \beta) = A \lhd (\alpha \lhd \beta) \\
A_{\rightarrow}(\alpha \rightarrow \beta) & = & (\alpha A)_{\rightarrow}\beta & \quad A \lhd (\alpha \lhd \beta) = (A \circ \alpha) \lhd \beta \\
\mathrm{Rng}^{\lhd}(R \lhd S) & = & S^{\lhd\!\leftrightarrow}(\mathrm{Rng}\, R) & \quad (R \lhd S)^{\mathrm{T}} \rhd \mathrm{V} = (S^{\mathrm{T}} \rhd R^{\mathrm{T}}) \rhd \mathrm{V} = S^{\mathrm{T}} \rhd (R^{\mathrm{T}} \circ \mathrm{V}) \\
\mathrm{Rng}^{\lhd}(R \rhd S) & = & S^{\rhd\!\rightarrow}(\mathrm{Rng}^{\lhd} R) & \quad (R \rhd S)^{\mathrm{T}} \rhd \mathrm{V} = (S^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd \mathrm{V} = S^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd \mathrm{V}) \\
\mathrm{Rng}^{\lhd}(A \times_{\lhd} B) & = & (\mathrm{Hgt}\, A)_{\rightarrow} B & \quad (A \lhd B^{\mathrm{T}})^{\mathrm{T}} \rhd \mathrm{V} = (B \rhd A^{\mathrm{T}}) \rhd \mathrm{V} = B \rhd (A^{\mathrm{T}} \circ \mathrm{V}) \\
\mathrm{Rng}^{\lhd}(A \times_{\rhd} B) & = & B_{\rightarrow}(\mathrm{Plt}\, A) & \quad (A \rhd B^{\mathrm{T}})^{\mathrm{T}} \rhd \mathrm{V} = (B \lhd A^{\mathrm{T}}) \rhd \mathrm{V} = B \lhd (A^{\mathrm{T}} \rhd \mathrm{V})
\end{array}
$$

$$
\begin{array}{rcll}
A \subseteq (R^{\leftarrow\!\lhd} B) & = & (R^{\rightarrow} A) \subseteq B & \textit{by} \quad A^{\mathrm{T}} \lhd (R \lhd B) = (A^{\mathrm{T}} \circ R) \lhd B = (R^{\mathrm{T}} \circ A)^{\mathrm{T}} \lhd B \\
A \subseteq (R^{\lhd\!\leftrightarrow} B) & = & B \subseteq (R^{\leftrightarrow\!\rhd} A) & \quad A^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd B) = (A^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd B = (R \rhd A)^{\mathrm{T}} \rhd B \\
(\alpha A) \subseteq B & = & \alpha \rightarrow (A \subseteq B) & \quad (\alpha \circ A)^{\mathrm{T}} \lhd B = (\alpha \circ A^{\mathrm{T}}) \lhd B = \alpha \lhd (A^{\mathrm{T}} \lhd B) \\
A \subseteq (\alpha_{\rightarrow} B) & = & \alpha \rightarrow (A \subseteq B) & \quad A^{\mathrm{T}} \lhd (\alpha \lhd B) = \alpha \lhd (A^{\mathrm{T}} \lhd B) \\
A \subseteq (B_{\rightarrow}\alpha) & = & (A \parallel B) \rightarrow \alpha & \quad A^{\mathrm{T}} \lhd (B \lhd \alpha) = (A^{\mathrm{T}} \circ B) \lhd \alpha \\
\alpha \rightarrow (\beta \rightarrow \gamma) & = & (\alpha \mathbin{\&} \beta) \rightarrow \gamma & \quad \alpha \lhd (\beta \lhd \gamma) = (\alpha \circ \beta) \lhd \gamma \\
\alpha \rightarrow (\beta \rightarrow \gamma) & = & \beta \rightarrow (\alpha \rightarrow \gamma) & \quad \alpha \lhd (\beta \lhd \gamma) = (\alpha \circ \beta) \lhd \gamma = (\beta \circ \alpha) \lhd \gamma = \beta \lhd (\alpha \lhd \gamma) \\
\mathrm{Plt}(R^{\lhd\!\leftrightarrow} A) & = & A \subseteq (\mathrm{Dom}^{\rhd} R) & \quad \mathrm{V}^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd A) = (\mathrm{V}^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd A = (R \rhd \mathrm{V})^{\mathrm{T}} \rhd A \\
\mathrm{Plt}(R^{\leftarrow\!\lhd} A) & = & (\mathrm{Rng}\, R) \subseteq A & \quad \mathrm{V}^{\mathrm{T}} \lhd (R \lhd A) = (\mathrm{V}^{\mathrm{T}} \circ R) \lhd A = (R^{\mathrm{T}} \circ \mathrm{V})^{\mathrm{T}} \lhd A \\
\mathrm{Plt}(\alpha_{\rightarrow} A) & = & \alpha \rightarrow \mathrm{Plt}\, A & \quad \mathrm{V}^{\mathrm{T}} \lhd (\alpha \lhd A) = \alpha \lhd (\mathrm{V}^{\mathrm{T}} \lhd A) \\
\mathrm{Plt}(\alpha_{\rightarrow} A) & = & \alpha \mathrm{V} \subseteq A & \quad \mathrm{V}^{\mathrm{T}} \lhd (\alpha \lhd A) = (\mathrm{V}^{\mathrm{T}} \circ \alpha) \lhd A = (\alpha \circ \mathrm{V})^{\mathrm{T}} \lhd A \\
\mathrm{Plt}(A_{\rightarrow}\alpha) & = & (\mathrm{Hgt}\, A) \rightarrow \alpha & \quad \mathrm{V}^{\mathrm{T}} \lhd (A \lhd \alpha) = (\mathrm{V}^{\mathrm{T}} \circ A) \lhd \alpha \\
\mathrm{Plt}(\mathrm{Rng}^{\lhd} R) & = & \mathrm{Plt}(\mathrm{Dom}^{\rhd} R) & \quad \mathrm{V}^{\mathrm{T}} \lhd (R^{\mathrm{T}} \rhd \mathrm{V}) = (\mathrm{V}^{\mathrm{T}} \lhd R^{\mathrm{T}}) \rhd \mathrm{V} = (R \rhd \mathrm{V})^{\mathrm{T}} \rhd \mathrm{V}
\end{array}
$$

Remark 5.18. Some of the identities of Corollary 5.17 express important theorems on fuzzy relations. For instance, the identity $(A \subseteq (R^{\leftarrow\!\lhd} B)) \leftrightarrow ((R^{\rightarrow} A) \subseteq B)$ entails the equivalence of two characterizations of the property of *extensionality* of a fuzzy class A w.r.t. a fuzzy relation R defined as $\mathrm{Ext}_R\, A \equiv_{\mathrm{df}} (\forall xy)(Rxy \mathbin{\&} Ax \rightarrow Ay)$, since the latter can be expressed as $(R^{\rightarrow} A) \subseteq A$. The next identity $(A \subseteq (R^{\lhd\!\leftrightarrow} B)) \leftrightarrow (B \subseteq (R^{\leftrightarrow\!\rhd} A))$ expresses a graded theorem on fuzzy preorders (cf. Example 5.10) that all elements of A are upper bounds of B iff all elements of B are lower bounds of A. These theorems are well-known in the non-graded setting; here we get their graded variants (i.e., also for partially valid inclusions) for free.

Corollary 5.19. *Furthermore, by Corollary 5.7,* FCT *proves the following identities dual to Corollary 5.17 for superproduct notions:*

$$
\begin{aligned}
(R \circ S)^{\rhd\!\!\rightarrow} A &= S^{\rhd\!\!\rightarrow}(R^{\rhd\!\!\rightarrow} A) \\
(R \triangleright S)^{\leftarrow\!\!\rhd} A &= R^{\leftarrow\!\!\rhd}(S^{\leftarrow} A) \\
(R \triangleleft S)^{\leftarrow\!\!\rhd} A &= R^{\leftarrow\!\!\lhd}(S^{\leftarrow\!\!\rhd} A) \\
(A \times B)^{\rhd\!\!\rightarrow} C &= B_{\rightarrow}(A \subseteq C) \\
(A \times_{\triangleleft} B)^{\leftarrow\!\!\rhd} C &= A_{\rightarrow}(C \subseteq B) \\
(A \times_{\triangleright} B)^{\leftarrow\!\!\rhd} C &= (B \parallel C)_{\rightarrow} A \\
R^{\rhd\!\!\rightarrow}(A_{\rightarrow}\alpha) &= (R^{\rightarrow} A)_{\rightarrow}\alpha \\
\alpha_{\rightarrow}(R^{\rhd\!\!\rightarrow} A) &= R^{\rhd\!\!\rightarrow}(\alpha_{\rightarrow} A) \\
\alpha_{\rightarrow}(R^{\leftarrow\!\!\rhd} A) &= R^{\leftarrow\!\!\rhd}(\alpha A) \\
\alpha_{\rightarrow}(\mathrm{Dom}^{\triangleright} R) &= R^{\leftarrow\!\!\rhd}(\alpha \mathrm{V}) \\
\mathrm{Dom}^{\triangleright}(R \triangleleft S) &= R^{\leftarrow\!\!\lhd}(\mathrm{Dom}^{\triangleright} S) \\
\mathrm{Dom}^{\triangleright}(R \triangleright S) &= R^{\leftarrow\!\!\rhd}(\mathrm{Dom}\, S) \\
\mathrm{Dom}^{\triangleright}(A \times_{\triangleleft} B) &= A_{\rightarrow}(\mathrm{Plt}\, B) \\
\mathrm{Dom}^{\triangleright}(A \times_{\triangleright} B) &= (\mathrm{Hgt}\, B)_{\rightarrow} A
\end{aligned}
$$

$$
\begin{aligned}
A \times_{\triangleright} (R^{\rightarrow} B) &= (A \times_{\triangleright} B) \triangleright R \\
A \times_{\triangleright} (\mathrm{Rng}\, R) &= (A \times_{\triangleright} \mathrm{V}) \triangleright R \\
A \times_{\triangleright} (\alpha B) &= (\alpha_{\rightarrow} A) \times_{\triangleright} B \\
(R^{\leftarrow\!\!\lhd} A) \times_{\triangleright} B &= R \triangleleft (A \times_{\triangleright} B) \\
(R^{\leftarrow\!\!\rhd} A) \times_{\triangleright} B &= R \triangleright (A \times B) \\
(\mathrm{Dom}^{\triangleright} R) \times_{\triangleright} B &= R \triangleright (\mathrm{V} \times B)
\end{aligned}
$$

$$
\begin{aligned}
A \subseteq (R^{\rhd\!\!\rightarrow} B) &= (R^{\leftarrow} A) \subseteq B \\
\mathrm{Plt}(R^{\rhd\!\!\rightarrow} A) &= (\mathrm{Dom}\, R) \subseteq A \\
\mathrm{Plt}(R^{\leftarrow\!\!\rhd} A) &= A \subseteq (\mathrm{Rng}^{\triangleleft} R)
\end{aligned}
$$

Although not used in the previous corollaries, the following lemma is needed for some more complex identities between BK-based terms:

Lemma 5.20. FCT *proves:*

1. $\mathbf{V} \triangleleft \boldsymbol{A}^{\mathrm{T}} = \mathbf{V} \circ \boldsymbol{A}^{\mathrm{T}}, \quad \mathbf{V} \triangleleft \boldsymbol{\alpha} = \mathbf{V} \circ \boldsymbol{\alpha}$

2. $\boldsymbol{A} \triangleleft \underline{\mathbf{1}} = \mathbf{V}, \quad \boldsymbol{A} \triangleright \underline{\mathbf{1}} = \boldsymbol{A}, \quad \boldsymbol{\alpha} \triangleleft \underline{\mathbf{1}} = \underline{\mathbf{1}}, \quad \boldsymbol{\alpha} \triangleright \underline{\mathbf{1}} = \boldsymbol{\alpha}$

Proof. $\mathbf{V} \triangleleft \boldsymbol{A}^{\mathrm{T}} = \{xy \mid \mathbf{V}x\underline{0} \rightarrow \boldsymbol{A}\underline{0}y\} = \{xy \mid \boldsymbol{A}\underline{0}y\} = \{xy \mid \mathbf{V}x\underline{0} \,\&\, \boldsymbol{A}\underline{0}y\} = \mathbf{V} \circ \boldsymbol{A}^{\mathrm{T}}$, and analogously for the other identities. □

Example 5.21. The following identities are among corollaries of Lemma 5.20:

$$
\begin{array}{rcll}
R^{\leftarrow\!\!\lhd}\mathrm{V} &=& \mathrm{V} & \textit{by}\quad R \triangleleft \mathrm{V} = R \triangleleft (\mathrm{V} \triangleleft \underline{1}) = (R \circ \mathrm{V}) \triangleleft \underline{1} \\
(A \subseteq \mathrm{V}) &=& \underline{1} & \quad A^{\mathrm{T}} \triangleleft \mathrm{V} = A^{\mathrm{T}} \triangleleft (A \triangleleft \underline{1}) = (A^{\mathrm{T}} \circ A) \triangleleft \underline{1} \\
\alpha_{\rightarrow}\mathrm{V} &=& \mathrm{V} & \quad \mathrm{V} \triangleright \alpha = (\mathrm{V} \triangleleft \underline{1}) \triangleright \alpha = \mathrm{V} \triangleleft (\underline{1} \triangleright \alpha) = \mathrm{V} \triangleleft \underline{1} \\
(\mathrm{V} \times \mathrm{V})^{\leftarrow\!\!\rhd} A &=& \mathrm{V} & \quad (\mathrm{V} \circ \mathrm{V}^{\mathrm{T}}) \triangleright A = (\mathrm{V} \triangleleft \mathrm{V}^{\mathrm{T}}) \triangleright A = \\
 & & & \quad = \mathrm{V} \triangleleft (\mathrm{V}^{\mathrm{T}} \triangleright A) = \mathrm{V} \triangleleft (A^{\mathrm{T}} \triangleleft \mathrm{V})^{\mathrm{T}} = \mathrm{V} \triangleleft \underline{1}
\end{array}
$$

Remark 5.22. The corollaries in this and the previous section show that a fairly large fragment of the elementary theory of fuzzy relations can be reduced to identities provable by several simple equational rules, namely those of Propositions 2.14(1) and 3.7, Theorems 4.2(1,5) and 5.3(1,5,6), and Lemmata 4.15 and 5.20. These rules can be viewed as axioms of an equational calculus for proving identities between fuzzy relational operations. It seems to be an open problem if there are elementary theorems on fuzzy relations expressible as identities in the language of $\circ$, $^{\mathrm{T}}$, V, $\underline{1}$, BK-products, and the notions listed in Tables 1–4, which are not provable from these equational rules (possibly extended by some missing identities), though provable in FCT (and, for that matter, if there are any such identities in which the elementary theories of fuzzy and *crisp* relations differ).

Remark 5.23. Sup-T-compositions and BK-products operate on binary fuzzy relations, i.e., fuzzy classes of ordered pairs of elements xy. The inner structure of these elements x, y can be arbitrary: if they are, for instance, themselves ordered pairs x_1x_2 and y_1y_2, then relational products are in fact operating on ordered quadruples. Composition-based notions with class operands (e.g., $\subseteq$) are thus applicable to binary fuzzy relations as well. In this way, inclusion of fuzzy relations $R \subseteq S$ can be regarded as the BK-product $(R')^{\mathrm{T}} \lhd S'$, where for a binary relation R and *quaternary* relations P, Q we define

$$\begin{aligned} P \lhd Q &=_{\mathrm{df}} \{x_1x_2y_1y_2 \mid (\forall z_1z_2)(Px_1x_2z_1z_2 \to Qz_1z_2y_1y_2)\} \\ P^{\mathrm{T}} &=_{\mathrm{df}} \{y_1y_2x_1x_2 \mid Px_1x_2y_1y_2\} \\ R' &=_{\mathrm{df}} \{xy\underline{00} \mid Rxy\} \end{aligned}$$

The corollaries shown above thus apply to inclusion, compatibility, Cartesian products, etc., not only of unary fuzzy classes, but also fuzzy relations of arbitrary arities. In this way, many further notions of the theory of fuzzy relations are reducible to sup-T- and BK-compositions: e.g., symmetry of a fuzzy relation R is expressible as $(R')^{\mathrm{T}} \lhd (R^{\mathrm{T}})'$; cf. also Example 5.11 for transitivity and reflexivity and Remark 5.18 for extensionality. The machinery demonstrated above thus can be used also for proving properties of such relational notions.

6 Conclusions

We have shown a method for mass proofs of theorems of certain forms in the theory of fuzzy relations. Its soundness is based on the notion of relative interpretation between theories over fuzzy logics, which allows a representation of fuzzy classes and formal truth values as certain kinds of fuzzy relations. This expands the applicability of simple properties of sup-T-compositions and BK-products of fuzzy relations to a larger language (of more than 30 operations) which includes many important concepts of the theory of fuzzy sets and fuzzy relations. Consequently, a large number of theorems of the latter theory are reduced to corollaries of a few simple properties of relational products, thus becoming verifiable by simple equational computations.

Among all possible kinds of fuzzy relational compositions, in this paper we have restricted our attention only to the sup-T-composition and BK-products, because they generate the most interesting families of derived notions, which occur most often in fuzzy mathematics. Similar investigation of notions based on other kinds of relational products is a topic left for future work.

Besides the practical consequences (e.g., for automated proofs of relational theorems) the results show that using a suitable formal apparatus provided by first-order and higher-order fuzzy logic enables exploitation of formal syntactic methods that can trivialize a large part of fuzzy mathematics. Together with the metatheorems of [12, §3.4] on fuzzy class operations, the methods presented here effectively reduce elementary fuzzy set theory and a large part of fuzzy relational theory to calculations in propositional fuzzy logic and simple relational algebra. Moreover they show that for a certain class of results, the fuzziness of fuzzy relations does not present an additional difficulty to the usual theory of *crisp* relations: it can be observed that Theorems 4.2 and 5.3, upon which all of the corollaries are based, hold equally for fuzzy and crisp relations. Thus a large part of the theory of crisp relations generalizes straightforwardly to fuzzy relations if a suitable framework of formal fuzzy logic is employed.

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References

[1] W. Bandler and L. J. Kohout. Mathematical relations, their products and generalized morphisms. Technical Report EES-MMS-REL 77-3, Man–Machine Systems Laboratory, Department of Electrical Engineering, University of Essex, Essex, Colchester, 1977.

[2] W. Bandler and L. J. Kohout. Fuzzy relational products and fuzzy implication operators. In *International Workshop of Fuzzy Reasoning Theory and Applications*, London, 1978. Queen Mary College, University of London.

[3] W. Bandler and L. J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[4] W. Bandler and L. J. Kohout. Fast fuzzy relational algorithms. In A. Ballester, D. Cardus, and E. Trillas, editors, *Proceedings of the 2nd World Conference on Mathematics at the Service of Man*, pages 123–131, Las Palmas, Spain, 1982. Universidad Politechnica de Las Palmas.

[5] W. Bandler and L. J. Kohout. On the general theory of relational morphisms. *International Journal of General Systems*, 13(1):47–66, 1986.

[6] W. Bandler and L. J. Kohout. Special properties, closures and interiors of crisp and fuzzy relations. *Fuzzy Sets and Systems*, 26:317–331, 1988.

[7] W. Bandler and L. J. Kohout. On the universality of the triangle superproduct and the square product of relations. *International Journal of General Systems*, 25:399–403, 1997.

[8] L. Běhounek. Extensionality in graded properties of fuzzy relations. In *Proceedings of 11th IPMU Conference*, pages 1604–1611, Paris, 2006. Edition EDK.

[9] L. Běhounek. Relative interpretations over first-order fuzzy logics. In F. Hakl, editor, *Doktorandský den '06*, pages 5–10, Prague, 2006. ICS AS CR / Matfyzpress. Available at http://www.cs.cas.cz/hakl/doktorandsky-den/history.html.

[10] L. Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159:1153–1164, 2008.

[11] L. Běhounek, U. Bodenhofer, and P. Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159:1729–1772, 2008.

[12] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[13] L. Běhounek and P. Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[14] L. Běhounek and P. Cintula. General logical formalism for fuzzy mathematics: Methodology and apparatus. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1227–1232, Beijing, 2005. Tsinghua University Press/Springer.

[15] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[16] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[17] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, 2007.

[18] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[19] U. Bodenhofer, M. De Cock, and E. E. Kerre. Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems. *International Journal of General Systems*, 32(4):343–360, 2003.

[20] M. Daňková. On approximate reasoning with graded rules. *Fuzzy Sets and Systems*, 158:652–673, 2007.

[21] B. De Baets and E. Kerre. Fuzzy relational compositions. *Fuzzy Sets and Systems*, 60(1):109–120, 1993.

[22] M. De Cock and E. E. Kerre. Fuzzy modifiers based on fuzzy relations. *Information Sciences*, 160:173–199, 2004.

[23] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[24] J. Fodor. Traces of fuzzy binary relations. *Fuzzy Sets and Systems*, 50(3):331–341, 1992.

[25] J. Fodor. Contrapositive symmetry of fuzzy implications. *Fuzzy Sets and Systems*, 69(2):141–156, 1995.

[26] A. Frascella and C. Guido. Transporting many-valued sets along many-valued relations. *Fuzzy Sets and Systems*, 159:1–22, 2008.

[27] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[28] S. Gottwald. Some general considerations on the evaluation of fuzzy rule systems. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 639–644, Barcelona, 2005.

[29] S. Gottwald. Universes of fuzzy sets and axiomatizations of fuzzy set theory. Part I: Model-based and axiomatic approaches. *Studia Logica*, 82(2):211–144, 2006.

[30] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[31] P. Hájek. Making fuzzy description logic more general. *Fuzzy Sets and Systems*, 154(1):1–15, 2005.

[32] P. Hájek and P. Cintula. Triangular norm based predicate fuzzy logics. To appear in the Proceedings of Linz Seminar 2005, 2005.

[33] L. P. Holmblad and J. J. Ostergaard. Control of a cement kiln by fuzzy logic. In M. M. Gupta and E. Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 389–399. Amsterdam, 1982.

[34] F. Klawonn and J. Castro. Similarity in fuzzy reasoning. *Mathware and Soft Computing*, 3(2):197–228, 1995.

[35] L. J. Kohout and W. Bandler. Fuzzy relational products in knowledge engineering. In V. N. (et al.), editor, *Fuzzy Approach to Reasoning and Decision Making*, pages 51–66. Academia / Kluwer, Prague / Dordrecht, 1992.

[36] L. J. Kohout and E. Kim. The role of BK-products of relations in soft computing. *Soft Computing*, 6:92–115, 2002.

[37] P. Magrez and P. Smets. Fuzzy modus ponens: A new model suitable for applications in knowledge-based systems. *International Journal of Intelligent Systems*, 4:181–200, 1989.

[38] E. H. Mamdani and S. Assilian. An experiment in linguistic synthesis with a fuzzy logic controller. *International Journal of Man–Machine Studies*, 7:1–13, 1975.

[39] N. N. Morsi and A. A. Fahmy. On generalized modus ponens with multiple rules and a residuated implication. *Fuzzy Sets and Systems*, 129:267–274, 2002.

[40] N. N. Morsi, W. Lotfallah, and M. El-Zekey. The logic of tied implications, part 2: Syntax. *Fuzzy Sets and Systems*, 157:2030–2057, 2006.

[41] G. Takeuti and S. Titani. Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, 32:1–32, 1992.

[42] L. A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.

Extensionality in graded properties of fuzzy relations

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Abstract: New definitions of graded reflexivity, symmetry, transitivity, antisymmetry, and functionality of fuzzy relations are proposed which are relative to an indistinguishability relation E on the universe of discourse. It is shown that if considered non-graded (i.e., either fully present or else fully absent), the new definitions reduce to the usual ones under full extensionality of the relation w.r.t. E. However, if *graded* properties of R (e.g., transitivity to some degree) are taken into account, the new definitions have to be distinguished from the conjunction of the original property and E-extensionality of R. Some arguments and results are given which suggest that the new concepts are well-motivated.

Keywords: Fuzzy relations, extensionality, similarity, graded properties.

1 Graded properties of fuzzy relations

In traditional fuzzy mathematics, fuzzy relations are defined as binary functions from some universe of discourse U to $[0, 1]$ (or another suitable lattice L of truth values). The usual properties of fuzzy relations are then defined as follows:

Definition 1.1. Let T be a (left-continuous) t-norm. We say that a fuzzy relation R is *reflexive* iff $R(x, x) = 1$ for all x; *symmetric* iff $R(x, y) \leq R(y, x)$ for all x, y; T*-transitive* iff $T(R(x, y), R(y, z)) \leq R(x, z)$ for all x, y, z; etc.

These conditions, formulated in ordinary mathematics over classical logic, can also be expressed by certain formulae of fuzzy logic. Let us work in the first-order fuzzy logic MTLΔ with crisp identity predicate $=$, or in any of its extensions.[1] In its usual semantics, binary predicates of its formal language are interpreted as fuzzy relations over the domain of discourse. A suitable defining formula for the reflexivity of R is then $\forall x Rxx$; for symmetry, $\forall xy(Rxy \to Ryx)$; for transitivity, $\forall xyz(Rxy \mathbin{\&} Ryz \to Rxz)$; etc. Each of these formulae has the truth value 1 iff the respective condition of Definition 1.1 is satisfied.

[1] MTL, introduced in [5], is the logic of left-continuous t-norms; see [9] for its most important extensions and an exposition of the semantics of first-order fuzzy logic. We use extensions that contain the Δ connective as we need to express the full truth of some statements. The crisp identity predicate is inessential in this paper and is only used for expository purposes: it will be consistently replaced by a fuzzy predicate E.

If the conditions of Definition 1.1 are not satisfied, then the property of R simply does not hold (its truth value is 0). The defining formulae in first-order fuzzy logic, however, may even in such cases yield meaningful non-zero truth values. For instance, if $R(x, x) = 0.999$ for all x, then the truth value of $\forall x Rxx$ is 0.999. It is clear that such a relation is "almost reflexive" (all pairs xx are almost fully in R), even though it is not reflexive according to Definition 1.1. Since furthermore the formula $\forall x Rxx$ has the same form as the formula which defines reflexivity in classical mathematics, it is natural to take its truth value for the degree of *graded reflexivity* of R, and say that R is 0.999-reflexive. (Similarly for symmetry, transitivity, and other properties of fuzzy relations.)

The graded properties of fuzzy relations have been introduced in Gottwald's paper [6] and systematically studied in his monograph [7]; more recently they have been elaborated in Gottwald's [8, §18.6], Bělohlávek's [3, §4.1], and Jacas and Recasens' [12]. The graded approach to the properties of fuzzy relations is important for several reasons:

- Graded properties generalize the traditional (non-graded) ones: R is reflexive (in the traditional sense) iff the truth value of graded reflexivity is exactly 1. In all other cases, the graded properties provide a fine-grained scale of the degrees of their validity, while the non-graded properties are then simply false.

- The graded approach allows to infer relevant information when the traditional conditions are almost, but still not completely, fulfilled. E.g., in the example above, R is 0.999-reflexive: if we prove that (graded) reflexivity of R implies (in the sense of fuzzy logic) some property φ, we shall know that φ holds at least to the degree 0.999. On the contrary, from the non-graded reflexivity of Definition 1.1 we cannot infer anything as it is simply false.

- Graded properties can easily be handled by first-order fuzzy logic: valid inferences about them can be proved by the formal rules of fuzzy logic. The semantics of fuzzy logic (relative to a particular t-norm) then translates the formal theorems into the laws valid for "real" fuzzy relations.

- Graded properties are "fuzzier" than their non-graded counterparts: if we take seriously the idea of general fuzziness of concepts, there is no reason to presuppose that the properties of fuzzy relations should only be crisp (i.e., either true or false as in Definition 1.1).

In the rest of this paper we shall always work with *graded* properties of fuzzy relations. Suspending Definition 1.1, we now define (graded) reflexivity, symmetry, transitivity, antisymmetry,[2] and functionality in the first-order logic MTL as follows:

Definition 1.2.

$$\begin{aligned}
\mathrm{Refl}\, R &\equiv \forall x Rxx \\
\mathrm{Sym}\, R &\equiv \forall xy(Rxy \to Ryx) \\
\mathrm{Trans}\, R &\equiv \forall xyz(Rxy \,\&\, Ryz \to Rxz) \\
\mathrm{Asym}\, R &\equiv \forall xy(Rxy \,\&\, Ryx \to x{=}y) \\
\mathrm{Fnc}\, R &\equiv \forall xyy'(Rxy \,\&\, Rxy' \to y{=}y')
\end{aligned}$$

[2]Even though many authors (e.g., [3], [11]) use min-conjunction in the definition of antisymmetry, arguments can be given that strong conjunction is in order here.

These definitions can be combined (by strong conjunction), yielding more complex graded notions of proximity, similarity (fuzzy equivalence), fuzzy preorder, and fuzzy order:

Definition 1.3.

$$\begin{aligned}
\mathrm{Prox}\, R &\equiv \mathrm{Refl}\, R \,\&\, \mathrm{Sym}\, R \\
\mathrm{Sim}\, R &\equiv \mathrm{Prox}\, R \,\&\, \mathrm{Trans}\, R \\
\mathrm{Preord}\, R &\equiv \mathrm{Refl}\, R \,\&\, \mathrm{Trans}\, R \\
\mathrm{Ord}\, R &\equiv \mathrm{Preord}\, R \,\&\, \mathrm{Asym}\, R
\end{aligned}$$

It can be observed that the defining formulae in Definition 1.2 are exactly the same as the definitions of these properties for crisp relations in classical mathematics. This correlates with the motivation of fuzzy logic as the generalization of classical logic to non-sharp predicates: classical mathematical notions are then fuzzified in a natural way just by interpreting the classical definitions in fuzzy logic. This methodology has been foreshadowed in [11, §5] by Höhle, much later formalized in [1, §7], and suggested as an important guideline for formal fuzzy mathematics in [2].[3]

2 Indistinguishability-relative properties

The adoption of graded properties of fuzzy relations can be viewed as part of the pursuit of a full-blown (rather than half-way) fuzzification of classical notions: the semi-classical (bivalent) notions of Definition 1.1 have been replaced by fuzzy notions of Definition 1.2. In general, according to the methodology of [2], one wants to fuzzify as much as one can, and find and eliminate hidden crispness in definitions wherever possible.

A case of such hidden crispness can be descried in the above definitions of antisymmetry and functionality: they refer to the (crisp) identity predicate =. In the fuzzy world, we should be ready to admit that not only crisp identity, but also a fuzzy similarity relation can play the role here.[4] The corrected definitions of these two notions will therefore replace = with a similarity relation E:

Definition 2.1.

$$\begin{aligned}
\mathrm{Asym}_{(E)}\, R &\equiv \forall xy(Rxy \,\&\, Ryx \to Exy) \\
\mathrm{Fnc}_{(E)}\, R &\equiv \forall xyy'(Rxy \,\&\, Rxy' \to Eyy')
\end{aligned}$$

Indeed, such definitions of E-antisymmetry and E-functionality can be found in the literature (e.g., [3], [4], [11]).

These two cases of "hidden" crispness were patent—the crisp identity was explicitly present in the formula. What I want to propose in this paper is to avoid another, less explicit case of hidden crispness present in the definitions of properties of fuzzy relations. The kind of hidden crispness I address is caused by multiple occurrences of the same variable in the defining formula: in such cases, a hidden identity predicate is present, which should again be eliminated by replacing it with fuzzy similarity.

[3]Of course, the method cannot be applied mechanically: but due to the motivation of fuzzy logical connectives and quantifiers, it often yields intuitive notions, and only occasionally a deeper analysis is required; an example of the latter situation are the new definitions presented in this paper.

[4]The intuitions behind the definitions of antisymmetry and functionality will be preserved especially if the similarity is interpreted as the indistinguishability of individuals.

Consider reflexivity, $\forall x Rxx$. If we suppose that there is a relation E which measures the degree of *indistinguishability* of individuals, we find that the formula $\forall x Rxx$ is no longer adequate for the intuitive notion of reflexivity. The reason is that it only takes into account R on pairs xx, even though Rxy should also be taken into consideration on reflexivity if y is indistinguishable from x (i.e., on condition Exy). The need for this is often obvious: if the value of R is, for example, obtained by some independent measurements on its two arguments, we may often fail to recognize whether the two arguments independently presented to us (e.g., by Nature) are indeed identical or just indistinguishable. Thus we should rather define:

$$\mathrm{Refl}_E\, R \quad \equiv \quad \forall xy(Exy \to Rxy) \tag{1}$$

From the formal point of view, the reason why the original definition ceased to be adequate in the presence of indistinguishability was that the double occurrence of x in $\forall x Rxx$ contained a hidden identity predicate: it was in fact $\forall xy(x{=}y \to Rxy)$, in which (1) has replaced $=$ by E, just as did Definition 2.1.

The same considerations can be carried out for other properties of fuzzy relations, and the hidden crispness caused by multiple occurrences of variables in the defining formulae be cured in the same way: by first making the hidden identity predicates explicit, and then replacing them with the (fuzzy) indistinguishability relation E. This leads to the following definitions:

Definition 2.2.

$$\begin{aligned}
\mathrm{Refl}_E\, R &\equiv \forall xx'(Exx' \to Rxx') \\
\mathrm{Sym}_E\, R &\equiv \forall xx'yy'(Exx' \,\&\, Eyy' \,\&\, Rxy \to Ry'x') \\
\mathrm{Trans}_E\, R &\equiv \forall xx'yy'zz'(Exx' \,\&\, Eyy' \,\&\, Ezz' \,\&\, Rxy \,\&\, Ry'z \to Rx'z') \\
\mathrm{Asym}_E\, R &\equiv \forall xx'yy'(Exx' \,\&\, Eyy' \,\&\, Rxy \,\&\, Ry'x' \to Exy) \\
\mathrm{Fnc}_E\, R &\equiv \forall xx'yy'(Exx' \,\&\, Rxy \,\&\, Rx'y' \to Eyy') \\
\mathrm{Preord}_E\, R &\equiv \mathrm{Refl}_E\, R \,\&\, \mathrm{Trans}_E\, R \\
\mathrm{Ord}_E\, R &\equiv \mathrm{Preord}_E\, R \,\&\, \mathrm{Asym}_E\, R \\
\mathrm{Prox}_E\, R &\equiv \mathrm{Refl}_E\, R \,\&\, \mathrm{Sym}_E\, R \\
\mathrm{Sim}_E\, R &\equiv \mathrm{Preord}_E\, R \,\&\, \mathrm{Sym}_E\, R
\end{aligned}$$

Generally we do not impose any restriction on E in this definition: so any assumptions regarding the properties of E will always be explicitly stated in theorems. By convention, the index E can be dropped if E is the identity (this accommodates Definitions 1.2 and 1.3).

It can be objected that the main motivation of these definitions is not yet (and generally can never be) accomplished: the formulae in Definition 2.2 still contain two occurrences of each variable, and by the same argument as above we cannot be sure whether the individuals denoted by them are indeed identical or just E-indistinguishable. In order to eliminate the double occurrences in the new definitions, we would have to make the same trick again, ending up in an infinite regress:

$$\begin{aligned}
\mathrm{Refl}_{0E}\, R &\equiv \forall x Rxx \\
\mathrm{Refl}_{1E}\, R &\equiv \forall xy(Exy \to Rxy) \\
\mathrm{Refl}_{2E}\, R &\equiv \forall xx'yy'(Exx' \,\&\, Eyy' \,\&\, Exy \to Rx'y') \\
&\ldots
\end{aligned}$$

There are at least three possible answers to this objection:[5]

First, in the formula $\mathrm{Refl}_{1E}\,R$, each variable occurs only *once under* R. Conceivably, establishing the truth value of the indistinguishability E can be much easier than the measurement of R (e.g., E can be obvious, intuitive, etc.). Thus in some cases it may only be necessary to distinguish the arguments of R, not E.

Second, observe that $\mathrm{Refl}_{2E}\,R \leftrightarrow \mathrm{Refl}_{1E'}\,R$, where[6]

$$E'xy \equiv \exists x'y'(Ex'x \,\&\, Ey'y \,\&\, Ex'y') \tag{2}$$

Thus the iterated E-properties have the same form as the non-iterated ones (only with a different E'). The *theory* of iterated properties (abstracting from particular E's) is therefore the same as that of non-iterated ones.

Finally, under the reasonable assumption that E is a similarity (to degree 1), the iterated notions coincide with the non-iterated ones:[7]

Lemma 2.3. $\Delta\,\mathrm{Sim}\,E \to E' = E$

Proof. Observe that by (2),[8]

$$E' = E^{-1} \circ E \circ E$$

By known results (see, e.g., [3] or [8]) which can be transferred to MTLΔ, if E is fully symmetric, then $E^{-1} = E$; and if E is fully reflexive and fully transitive, then $E = E \circ E$. Thus if $\Delta\,\mathrm{Sim}\,E$, then $E' = E$. □

Corollary 2.4. $\Delta\,\mathrm{Sim}\,E \to (\mathrm{Refl}_{2E}\,R \leftrightarrow \mathrm{Refl}_{1E}\,R)$
(Similarly for Sym_{2E}*,* Trans_{2E}*,* Asym_{2E}*, and* Fnc_{2E}*.)*

This ensures that under the assumption that the indistinguishability relation is a (full) similarity, all of the iterated notions coincide with those of Definition 2.2.

Remark 2.5 By the same argument as above, one should prefer $\Delta\,\mathrm{Sim}_E\,E$ as the precondition for Lemma 2.3 and Corollary 2.4. However, by Proposition 3.5 below,

$$\Delta\,\mathrm{Sim}_E\,E \leftrightarrow \Delta\,\mathrm{Sim}\,E$$

Thus the simpler precondition $\Delta\,\mathrm{Sim}\,E$ is sufficient.

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[5] We present them for the case of reflexivity; for other properties they are fully analogous.

[6] Since by the rules of MTL,

$$\begin{aligned} &\forall xx'yy'(Exx' \,\&\, Eyy' \,\&\, Exy \to Rx'y') \\ \leftrightarrow\;& \forall x'xy'y(Ex'x \,\&\, Ey'y \,\&\, Ex'y' \to Rxy) \\ \leftrightarrow\;& \forall xy(\exists x'y'(Ex'x \,\&\, Ey'y \,\&\, Ex'y') \to Rxy) \end{aligned}$$

[7] Recall that we work formally in the logic MTLΔ or some of its extensions; therefore, by stating a lemma or a theorem in this paper we mean that it is provable in MTLΔ.

[8] E^{-1} is the inverse relation and $\circ$ denotes relational composition:

$$\begin{aligned} E^{-1}xy &\equiv Eyx \\ (R \circ S)xy &\equiv \exists z(Rxz \,\&\, Szy) \end{aligned}$$

The identity of fuzzy relations is defined as the identity of their membership functions (which ensures their intersubstitutivity *salva veritate*):

$$R = S \equiv \forall xy\Delta(Rxy \leftrightarrow Sxy)$$

Remark 2.6 In a completely graded approach to fuzzy relations we should not be satisfied with the non-graded results of Lemma 2.3 and Corollary 2.4 (as they do not allow to infer anything unless E is a similarity *to degree* 1). Graded variants of Lemma 2.3 and Corollary 2.4 can indeed be derived by a more careful proof.[9] For instance,

$$\begin{aligned}
\mathrm{Sim}^2 E &\rightarrow (\mathrm{Refl}_{2E}\, R \leftrightarrow \mathrm{Refl}_{1E}\, R) \\
\mathrm{Sim}^4 E &\rightarrow (\mathrm{Sym}_{2E}\, R \leftrightarrow \mathrm{Sym}_{1E}\, R) \\
\mathrm{Sim}^6 E &\rightarrow (\mathrm{Trans}_{2E}\, R \leftrightarrow \mathrm{Trans}_{1E}\, R), \text{ etc.,}
\end{aligned}$$

where $\mathrm{Sim}^n E$ stands for $\mathrm{Sim}\, E \mathbin{\&} \ldots \mathbin{\&} \mathrm{Sim}\, E$ (n times). (The same abbreviations for multiple conjunctions will also be used for Refl, Sym, etc.)[10]

3 E-relative properties vs. extensionality w.r.t. E

In the non-graded approach, the motivation of our E-properties leads to the notion of *extensionality* of a relation R w.r.t. a relation (usually a similarity) E. Indeed, the definition of extensionality expresses the same idea of the congruence of R w.r.t. E. The *graded* definition of extensionality (of which the non-graded version is obtained by requiring its 1-validity) reads as follows:

Definition 3.1.

$$\mathrm{Ext}_E\, R \equiv \forall x x' y y' (E x x' \mathbin{\&} E y y' \mathbin{\&} R x y \rightarrow R x' y')$$

It can be shown that in the non-graded approach, extensionality is a sufficient substitute for E-properties (see Corollary 3.4 below). However, if graded properties are taken into account, E-properties can only partially be reduced to the conjunction of the usual properties and extensionality:

Theorem 3.2. *1.* $\mathrm{Refl}^2 E \mathbin{\&} \mathrm{Ext}_E\, R \rightarrow (\mathrm{Refl}_E\, R \leftrightarrow \mathrm{Refl}\, R)$

2. $\mathrm{Prox}^2 E \mathbin{\&} \mathrm{Ext}_E\, R \rightarrow (\mathrm{Sym}_E\, R \leftrightarrow \mathrm{Sym}\, R)$

3. $\mathrm{Prox}^3 E \mathbin{\&} \mathrm{Ext}^2_E\, R \rightarrow (\mathrm{Trans}_E\, R \leftrightarrow \mathrm{Trans}\, R)$

4. $\mathrm{Prox}^2 E \mathbin{\&} \mathrm{Ext}_E\, R \rightarrow (\mathrm{Asym}_E\, R \leftrightarrow \mathrm{Asym}_{(E)}\, R)$

5. $\mathrm{Prox}\, E \mathbin{\&} \mathrm{Ext}_E\, R \rightarrow (\mathrm{Fnc}_E\, R \leftrightarrow \mathrm{Fnc}_{(E)}\, R)$

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[9] From graded variants of the statements used in the proof of Lemma 2.3, see [8, Prop. 18.6.1] or [3, L. 4.21], it follows that $\mathrm{Sim}^2 E \rightarrow (E' x y \leftrightarrow E x y)$. This is then used once for each variable occurring in the defining formula of Refl_{1E}, Sym_{1E}, etc.

In fact, the precondition $\mathrm{Sim}^2 E$ can be weakened a bit: it suffices if $\mathrm{Refl}^2 E \mathbin{\&} \mathrm{Sym}\, E \mathbin{\&} \mathrm{Trans}^2 E$, i.e., if E is a *preorder similarity,* $\mathrm{Sim}\, E \mathbin{\&} \mathrm{Preord}\, E$. (Notice that in the graded approach, notions like *transitive similarity* or *reflexive preorder* are meaningful and strengthen non-trivially the respective conditions.)

[10] By [10], $\varphi \mathbin{\&} \varphi$ can be interpreted as "very φ". Thus informally, $\mathrm{Refl}^2 E$ can be understood as requiring E to be *very* reflexive, $\mathrm{Refl}^3 E$ even more reflexive, etc. One must, however, be careful here, since $\varphi \mathbin{\&} \varphi$ is not the only possible interpretation of "very", and the meaning of "very" in natural language usually differs from this particular one. Therefore this kind of reading of the exponents can only be understood as a rough, 'heuristic' aid.

Proof. We shall show e.g. the proof for antisymmetry, the other proofs are analogous.

First we prove in first-order MTL that

$$\mathrm{Refl}^2\, E \to (\mathrm{Asym}_E\, R \to \mathrm{Asym}_{(E)}\, R)$$

By specifying x for x' and y for y' in $\mathrm{Asym}_E\, R$ we get $Exx \,\&\, Eyy \,\&\, Rxy \,\&\, Ryx \to Exy$. Detaching Exx and Eyy by double use of $\mathrm{Refl}\, E$, we get $\mathrm{Asym}_{(E)}\, R$ by generalization on xy.

Next we prove that

$$\mathrm{Sym}^2\, E \,\&\, \mathrm{Ext}_E\, R \to (\mathrm{Asym}_{(E)}\, R \to \mathrm{Asym}_E\, R)$$

Clearly $Ex'x \,\&\, Ey'y \,\&\, Ry'x'$ implies Ryx by $\mathrm{Ext}_E\, R$, which together with Rxy implies Exy by $\mathrm{Asym}_{(E)}\, R$. Thus by $\mathrm{Sym}\, E \,\&\, Exx' \to Ex'x$ and $\mathrm{Sym}\, E \,\&\, Eyy' \to Ey'y$ we have $\mathrm{Sym}^2\, E \,\&\, \mathrm{Ext}_E\, R \,\&\, \mathrm{Asym}_{(E)}\, R \to (Exx' \,\&\, Eyy' \,\&\, Rxy \,\&\, Ry'x' \to Exy)$, whence the required formula follows by generalization. □

Notice that for the reduction of $\mathrm{Trans}_E\, R$ to $\mathrm{Trans}\, R$ we needed $\mathrm{Ext}_E\, R$ twice. The following counter-example shows that single $\mathrm{Ext}_E\, R$ is not sufficient.

Example 3.3. Let the universe of discourse comprise six elements a, a', b, b', c, c' with
$Eaa = Ea'a' = Ebb = Eb'b' = Ecc = Ec'c' = 1$,
$Eaa' = Ea'a = Ebb' = Eb'b = Ecc' = Ec'c = 0.9$,
$Rab = Rb'c = 1$,
$Rab' = Ra'b = Rac = Rbc = Rb'c' = 0.8$,
$Ra'b' = Rac' = Ra'c = Rbc' = 0.7$,
$Ra'c' = 0.5$, and $Exy = Rxy = 0$ otherwise.
Then for the Łukasiewicz t-norm, the truth value of $\mathrm{Prox}\, E$ is 1, that of $\mathrm{Ext}_E\, R$ is 0.9, and that of $\mathrm{Trans}\, R$ is 1; thus the truth value of $\mathrm{Prox}^3\, E \,\&\, \mathrm{Ext}_E\, R \,\&\, \mathrm{Trans}\, R$ is 0.9, while that of $\mathrm{Trans}_E\, R$ is only 0.8.

Similarly, single $\mathrm{Ext}_E\, R$ is not enough for complex notions like similarity or preorder, where we must sum up the exponents; thus, e.g., $(\mathrm{Sim}_E\, R \leftrightarrow \mathrm{Sim}\, R) \leftarrow \mathrm{Refl}^7\, E \& \mathrm{Sym}^5\, E \& \mathrm{Ext}^3_E\, R$.

As a corollary to Theorem 3.2 we get the reduction of E-properties to the usual ones by E-extensionality for non-graded notions:

Corollary 3.4. *Let $\Delta\, \mathrm{Prox}\, E \,\&\, \Delta\, \mathrm{Ext}_E\, R$. Then the following equivalences are 1-valid:*

$$\begin{aligned}
\mathrm{Refl}_E\, R &\leftrightarrow \mathrm{Refl}\, R\\
\mathrm{Sym}_E\, R &\leftrightarrow \mathrm{Sym}\, R\\
\mathrm{Trans}_E\, R &\leftrightarrow \mathrm{Trans}\, R\\
\mathrm{Asym}_E\, R &\leftrightarrow \mathrm{Asym}_{(E)}\, R\\
\mathrm{Fnc}_E\, R &\leftrightarrow \mathrm{Fnc}_{(E)}\, R\\
\mathrm{Preord}_E\, R &\leftrightarrow \mathrm{Preord}\, R\\
\mathrm{Prox}_E\, R &\leftrightarrow \mathrm{Prox}\, R\\
\mathrm{Sim}_E\, R &\leftrightarrow \mathrm{Sim}\, R
\end{aligned}$$

Proof. By Δ-necessitation applied to Theorem 3.2 and the appropriate distribution of the Δ's. □

As a corollary we can see that if E is a full similarity, it is already a full similarity w.r.t. itself:

Proposition 3.5. $\Delta \operatorname{Sim} E \leftrightarrow \Delta \operatorname{Sim}_E E$

Proof. From Corollary 3.4 and the following Lemma 3.6. □

Lemma 3.6. $\operatorname{Trans}^2 E \,\&\, \operatorname{Sym} E \to \operatorname{Ext}_E E$

Proof. $Exx' \,\&\, Exy \,\&\, Eyy' \to Ex'y'$ by applying symmetry to the first conjunct and then transitivity twice. □

Corollary 3.4 explains why most of the E-properties have not yet been defined in the fuzzy literature: in the (prevalent) non-graded approach, in order to satisfy the natural idea of congruence w.r.t. E it is sufficient for R to be (fully) E-extensional, provided E is a (full) proximity (or even similarity).

There have been three notable exceptions to the absence of E-properties from the literature: the (E)-notions of Definition 2.1 (e.g., in [3], [4], [11]), and E-reflexivity whose non-graded variant sometimes occurs as one of the axioms of similarity-based fuzzy ordering (e.g., in [4]). Corollary 3.4 sheds some light on why this is so:

First, notice that full extensionality reduces $\operatorname{Asym}_E R$ and $\operatorname{Fnc}_E R$ only to $\operatorname{Asym}_{(E)} R$ resp. $\operatorname{Fnc}_{(E)} R$; thus the two notions of Definition 2.1 are indispensable even in the non-graded approach.[11]

Second, the E-reflexivity has been explained in the non-graded theory of fuzzy orders as a *combination* of ordinary reflexivity and extensionality, to which it is indeed equivalent under certain conditions:

Theorem 3.7. $\Delta \operatorname{Prox} E \,\&\, \Delta \operatorname{Trans} R \to (\Delta \operatorname{Refl}_E R \leftrightarrow \Delta \operatorname{Refl} R \,\&\, \Delta \operatorname{Ext}_E R)$

Proof. By Δ-necessitation from the following easy lemma, which shows how the situation changes under gradedness; its proof is similar to that of Theorem 3.2. □

Lemma 3.8. *1.* $\operatorname{Prox} E \,\&\, \operatorname{Trans}^2 R \to (\operatorname{Refl}^2_E R \to \operatorname{Ext}_E R \,\&\, \operatorname{Refl} R)$

2. $\operatorname{Refl} E \to (\operatorname{Ext}_E R \,\&\, \operatorname{Refl} R \to \operatorname{Refl}_E R)$

Thus, since the preconditions of Theorem 3.7 are always presupposed in the non-graded definition of similarity-based fuzzy ordering, $\operatorname{Refl}_E R$ indeed plays the role of both reflexivity and E-extensionality there; and since therefore E-extensionality is already ensured by $\operatorname{Refl}_E R$, it is not necessary to introduce it into the definition of transitivity in the non-graded theory of fuzzy orderings.

This explains why E-transitivity and E-symmetry[12] have not been defined in the theory of fuzzy relations, even though Refl_E, $\operatorname{Asym}_{(E)}$, and $\operatorname{Fnc}_{(E)}$ have. Theorem 3.2 further shows that in *graded* properties of fuzzy relations, E-notions have nevertheless to be distinguished from the simple presence of E-extensionality.

[11] In [3] they are already generalized to their *graded* versions.

[12] $\Delta \operatorname{Refl}_E R$ has also been used in the non-graded definition of E-extensional *similarity,* in which the preconditions of Theorem 3.7 are satisfied as well; therefore it supplies the definition with the needed $\Delta \operatorname{Ext}_E R$, and thus it is not necessary to build extensionality into the definitions of symmetry or transitivity there, either.

4 Some generalizations

In this paper we restricted our attention to reflexivity, symmetry, transitivity, antisymmetry, and functionality (and some combinations thereof), as they are the most usual properties of fuzzy relations found in the literature. The definitions and results can, however, be easily extended to a wider class of graded properties of fuzzy relations.[13]

A further generalization of Fnc_E might consist in considering different indistinguishability relations on the domain and codomain of the fuzzy relation. Thus we could define $\mathrm{Fnc}_{E_1,E_2} R \equiv \forall x x' y y' (E_1 x x' \,\&\, Rxy \,\&\, Rx'y' \to E_2 y y')$. However, this definition can always be reduced to Definition 2.2 by taking the disjoint union of E_1 and E_2 on the disjoint union of (the supports of) the domain and codomain of R.[14]

Finally, the E-relative properties treated in this paper require that indistinguishable individuals behave uniformly, just as if they were equal. In some situations, however, it may be sufficient that *any* (rather than all) objects among those indiscernible are in relation R. Thus we can define the 'existential' versions of E-relative properties as follows:

Definition 4.1.

$$
\begin{aligned}
\mathrm{Refl}^{\exists}_E R &\equiv \forall x \exists x' (Exx' \,\&\, Rxx') \\
\mathrm{Sym}^{\exists}_E R &\equiv \forall xy (Rxy \to \exists x'y' (Exx' \,\&\, Eyy' \,\&\, Ry'x')) \\
\mathrm{Trans}^{\exists}_E R &\equiv \forall xyz (Rxy \,\&\, \exists y' (Eyy' \,\&\, Ry'z) \to \exists x'z' (Exx' \,\&\, Ezz' \,\&\, Rx'z'))
\end{aligned}
$$

These notions are weaker than those of Definition 2.2 if E is reflexive enough:

Observation 4.2.

$$
\begin{aligned}
\mathrm{Refl}\, E &\leftrightarrow (\mathrm{Refl}_E R \to \mathrm{Refl}^{\exists}_E R) \\
\mathrm{Refl}^2\, E &\leftrightarrow (\mathrm{Sym}_E R \to \mathrm{Sym}^{\exists}_E R) \\
\mathrm{Refl}^2\, E &\leftrightarrow (\mathrm{Trans}_E R \to \mathrm{Trans}^{\exists}_E R)
\end{aligned}
$$

The differences between both variants of E-properties are shown by their characterizations in terms of relational compositions:[15]

Observation 4.3.

$$
\begin{aligned}
\mathrm{Refl}_E R &\leftrightarrow E \subseteq R \\
\mathrm{Refl}^{\exists}_E R &\leftrightarrow \mathrm{I} \subseteq R \circ E^{-1} \\
\mathrm{Sym}_E R &\leftrightarrow E^{-1} \circ R^{-1} \circ E \subseteq R \\
\mathrm{Sym}^{\exists}_E R &\leftrightarrow R \subseteq E \circ R^{-1} \circ E^{-1} \\
\mathrm{Trans}_E R &\leftrightarrow E^{-1} \circ R \circ E \circ R \circ E \subseteq R \\
\mathrm{Trans}^{\exists}_E R &\leftrightarrow R \circ E \circ R \subseteq E \circ R \circ E^{-1}
\end{aligned}
$$

[13] The methods shown here work at least for properties given by formulae $\forall x_1 \dots x_n (\&\varphi_i \to \psi)$, where all φ_i and ψ are atoms of the form Rx_kx_l or $x_k{=}x_l$.

[14] Still, Fnc_{E_1,E_2} can sometimes be a convenient notation; e.g., $\mathrm{Fnc}_{(E)}$ of Definition 2.1 is in fact $\mathrm{Fnc}_{=,E}$.

[15] $R \subseteq S$ is defined as $\forall xy(Rxy \to Sxy)$ and $\mathrm{I}xy \equiv x{=}y$. Notice that even the notion of relational composition should be made E-relative under the presence of indistinguishability E, namely

$$(R \circ_E S)xy \equiv \exists z z' (Rxz \,\&\, Ezz' \,\&\, Sz'y)$$

Obviously $R \circ_E S = R \circ E \circ S$.

The choice of the appropriate variant of an E-property depends on the context; in particular, whether the objects are given to us (e.g., by Nature) or we can choose them; or alternatively, whether indistinguishable objects must all behave as required, or only one object suffices to witness the property.

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References

[1] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[2] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[3] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[4] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(5):593–610, 2000.

[5] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[6] S. Gottwald. Fuzzified fuzzy relations. In R. Lowen and M. Roubens, editors, *Proceedings of 4th IFSA Congress*, volume Mathematics (ed. P. Wuyts), pages 82–86, Brussels, 1991.

[7] S. Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View*. Vieweg, Wiesbaden, 1993.

[8] S. Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[9] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[10] P. Hájek. On very true. *Fuzzy Sets and Systems*, 124(3):329–333, 2001.

[11] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[12] J. Jacas and J. Recasens. Fuzzified properties of fuzzy relations. In *Proceedings of the 9th IPMU Conference*, volume 1, pages 157–161, Annecy, 2002.

Towards a formal theory of fuzzy Dedekind reals

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Abstract: In the framework of Henkin style higher-order fuzzy logic $\text{Ł}\Pi_\omega$ we construct fuzzy real numbers as fuzzy Dedekind cuts over crisp rationals, and show some of their properties provable in $\text{Ł}\Pi_\omega$. The definitions of algebraic operations and a theory of fuzzy intervals are sketched.

Keywords: Fuzzy Dedekind completion, fuzzy real numbers, higher-order fuzzy logic.

1 Introduction

In [1], Henkin-style higher-order fuzzy logic ŁΠ has been introduced and proposed as a unified foundational theory for fuzzy mathematics. This paper contributes to the programme of developing fuzzy mathematics within its framework by introducing the structure of fuzzy real numbers. A solid theory of fuzzy reals is indispensable for the more advanced disciplines of unified formal fuzzy mathematics, such as fuzzy measure theory or fuzzy probability.

The approach adopted in this paper conforms to the methodology of the Manifesto [2]. Real numbers and other concepts are therefore constructed in full analogy with classical mathematics, taking advantage of the similarity of both formalisms.

The method of construction of real numbers applied here is certainly not the only possible one, even within the framework of Henkin-style higher-order fuzzy logic. Another readily available method consists in implanting the first-order axioms of the real closed field in higher-order fuzzy logic. The systematic development of alternative notions of real number within higher-order fuzzy logic and their careful comparison, especially from the point of view of real-life applicability, is part of a broader long-term programme. Although the usability of the present notion for applications cannot yet be predicted, it nevertheless seems capable of capturing many features of fuzzy numbers already used in applied fuzzy mathematics, and furthermore shows many properties of independent mathematical interest. As sketched in Section 6, it can serve as a basis for a formal theory of fuzzy intervals, which is very close to applied practice.

We are going to construct fuzzy real numbers as fuzzy Dedekind cuts over crisp rationals. The reason why we use crisp rather than fuzzy rationals reflects the usual definitions of fuzzy numbers as fuzzy sets of (some kind of) common crisp numbers. However, unlike most definitions of fuzzy numbers, Dedekind cuts do not express the 'density' of the fuzzy number across the underlying crisp numbers, but rather its distribution (cumulative density), similar to the probabilistic distribution function. Intuitively, the membership $q \in A$ of a rational number q in a Dedekind fuzzy real number A expresses (the truth value of) the fact that q majorizes the fuzzy real.

In somewhat different settings, fuzzy Dedekind completion has already been studied in [7] and [3]. Dedekind reals in an axiomatic fuzzy set theory (over a slightly different logic) appear also in [8].

In [5], Dubois and Prade require of fuzzy reals that they be objects whose every α-cut is a (crisp) real. Interestingly, Dedekind fuzzy reals do meet this requirement, since every α-cut of a fuzzy Dedekind cut is a crisp Dedekind cut, i.e., a crisp real (represented by the cut). We will see in Section 4 that (unlike the proposal of [5]) a monotonicity condition $\alpha \leq \beta \rightarrow A_\alpha \leq A_\beta$ is met here, which seems essential for some of the motivational aspects of fuzzy notions rendered horizontally (as sets of cuts); a thorough discussion of these requirements is yet to be carried out.

2 Preliminaries

For the ease of reference, we repeat here the definitions and axioms of Henkin-style higher-order fuzzy logic ŁΠ, which will be our framework in the rest of the paper. For details, see [1].

Definition 2.1. The logic ŁΠ (introduced in [6]) has the following *primitive connectives* (listed here with their standard $[0,1]$-semantics):

The truth constant falsum	$0 = 0$
Product conjunction	$x \mathbin{\&_\Pi} y = x \cdot y$
Product implication	$x \rightarrow_\Pi y = \min(1, y/x)$, where $0/0 = 1$
Łukasiewicz implication	$x \rightarrow_Ł y = \min(1, 1 - x + y)$

We define various *derived connectives* of ŁΠ:

1	is $\neg_Ł 0$,	i.e. 1
$\neg_Ł x$	is $x \rightarrow_Ł 0$,	i.e. $1 - x$
$\neg_\Pi x$	is $x \rightarrow_\Pi 0$,	i.e. $0/x$
Δx	is $\neg_\Pi \neg_Ł x$,	i.e. $\Delta x = 1$ if $x = 1$, else 0
$x \mathbin{\&_Ł} y$	is $\neg_Ł(x \rightarrow_Ł \neg_Ł y)$,	i.e. $\max(0, x + y - 1)$
$x \wedge y$	is $x \mathbin{\&_Ł} (x \rightarrow_Ł y)$,	i.e. $\min(x, y)$
$x \vee y$	is $(x \rightarrow_Ł y) \rightarrow_Ł y$,	i.e. $\max(x, y)$
$x \oplus y$	is $\neg_Ł x \rightarrow_Ł y$,	i.e. $\min(1, x + y)$
$x \ominus y$	is $x \mathbin{\&_Ł} \neg_Ł y$,	i.e. $\max(0, x - y)$
$x \rightarrow_G y$	is $\Delta(x \rightarrow_Ł y) \vee y$,	i.e. 1 if $x \leq y$, else y

Bi-implications $\leftrightarrow_Ł$, $\leftrightarrow_\Pi$, and $\leftrightarrow_G$ are defined as usual. Furthermore, for any t-norm $*$ representable in ŁΠ, the connectives $\&_*$, $\rightarrow_*$, $\neg_*$, and $\leftrightarrow_*$ can be defined. We employ the usual precedence of connectives.

Convention 2.2. We omit the t-norm indices of connectives and other defined symbols whenever they do not matter, i.e., whenever the substitution of any other t-norm index would yield a formula provably equivalent (or, in case of axioms and theorems, just equiprovable) to the original one. An index subscripted to a closing parenthesis distributes to all connectives and other indexed symbols within its scope that do not have their index explicitly marked.

Definition 2.3. The *propositional logic* ŁΠ has the following axioms:

(Ł) The axioms of Łukasiewicz logic
(Π) The axioms of Product logic
($Ł_\Pi$) $\Delta(\varphi \to_Ł \psi) \to_Ł (\varphi \to_\Pi \psi)$
($\Pi_Ł$) $\Delta(\varphi \to_\Pi \psi) \to_Ł (\varphi \to_Ł \psi)$
(D) $(\varphi \,\&_\Pi\, (\chi \ominus \psi)) \leftrightarrow_Ł ((\varphi \,\&_\Pi\, \chi) \ominus (\varphi \,\&_\Pi\, \psi))$

The deduction rules of ŁΠ are modus ponens and Δ-necessitation (from φ infer $\Delta\varphi$).

Definition 2.4. The *first-order logic* ŁΠ [4] adds the deduction rule of generalization and the following axioms for quantifiers and (crisp) identity:

(∀1) $(\forall x)\varphi(x) \to \varphi(t)$ if t is substitutable for x in ψ
(∀2) $(\forall x)(\chi \to_Ł \varphi) \to (\chi \to_Ł (\forall x)\varphi)$ if x is not free in χ
(=1) $x = x$
(=2) $x = y \to \Delta(\varphi(x) \leftrightarrow \varphi(y))$

The symbol $(\exists x)$ is an abbreviation for $\neg_Ł(\forall x)\neg_Ł$.

Definition 2.5. The *Henkin-style second-order logic* ŁΠ is a theory in the multi-sorted first-order logic ŁΠ, with sorts for objects (lowercase variables) and classes (uppercase variables). Both of the sorts subsume subsorts of n-tuples, for all $n \geq 1$. Apart from the obvious necessary function symbols and axioms for tuples (tuples equal iff their respective constituents equal), the only primitive symbol is the membership predicate $\in$ between objects and classes. The axioms for $\in$ are (i) the comprehension axioms

$$(\exists X)\Delta(\forall x)(x \in X \leftrightarrow \varphi),$$

for all φ not containing X, which enable the (eliminable) introduction of comprehension terms $\{x \mid \varphi\}$ with axioms $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (where φ may be allowed to contain other comprehension terms); and (ii) the extensionality axiom

$$(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \to X = Y.$$

Convention 2.6. Formulae $(\forall x)(x \in X \to_* \varphi)$, $(\exists x)(x \in X \,\&_*\, \varphi)$ are abbreviated $(\forall x \in X)_*\varphi$ and $(\exists x \in X)_*\varphi$, resp.; $x \notin_* X$ stands for $\neg_*(x \in X)$; alternatively we write Ax and $Rx_1 \ldots x_n$ for $x \in A$ and $\langle x_1, \ldots, x_n \rangle \in R$, resp.

Definition 2.7. The *Henkin-style logics* ŁΠ *of higher orders* are obtained by repeating the previous definition on each level of the type hierarchy. Obviously, all defined symbols of any type can then be shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types.

Henkin-style ŁΠ of order n will be denoted by $Ł\Pi_n$, the whole hierarchy by $Ł\Pi_\omega$. The types of terms are either denoted by a superscripted parenthesized number (e.g., $X^{(3)}$), or understood from the context.

Definition 2.8. In $Ł\Pi_2$, we define the following relations and operations:

$$
\begin{array}{rcl}
\emptyset & =_{\mathrm{df}} & \{x \mid 0\} \\
\mathrm{Ker}(X) & =_{\mathrm{df}} & \{x \mid \Delta(x \in X)\} \\
X_\alpha & =_{\mathrm{df}} & \{x \mid \Delta(\alpha \to x \in X)\} \\
\backslash_* X & =_{\mathrm{df}} & \{x \mid x \notin_* X\} \\
X \cap_* Y & =_{\mathrm{df}} & \{x \mid x \in X \mathbin{\&_*} x \in Y\} \\
X \cup Y & =_{\mathrm{df}} & \{x \mid x \in X \vee x \in Y\} \\
\mathrm{Crisp}(X) & \equiv_{\mathrm{df}} & (\forall x)\Delta(x \in X \vee x \notin X) \\
\mathrm{Fuzzy}(X) & \equiv_{\mathrm{df}} & \neg\mathrm{Crisp}(X) \\
X \subseteq_* Y & \equiv_{\mathrm{df}} & (\forall x)(x \in X \to_* x \in Y) \\
X \approx_* Y & \equiv_{\mathrm{df}} & (\forall x)(x \in X \leftrightarrow_* x \in Y) \\
X \times_* Y & =_{\mathrm{df}} & \{\langle x, y\rangle \mid x \in X \mathbin{\&_*} y \in Y\} \\
R^{-1} & =_{\mathrm{df}} & \{\langle x, y\rangle \mid Ryx\} \\
\mathrm{Id} & =_{\mathrm{df}} & \{\langle x, y\rangle \mid x = y\}
\end{array}
$$

We shall freely use all elementary theorems on these notions which follow from the metatheorems proved in [1], and thus can be checked by simple propositional calculations.

Definition 2.9. In $Ł\Pi_2$, we can also define the usual properties of relations:

$$
\begin{array}{rrcl}
\text{Reflexivity} & \mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)(Rxx) \\
\text{Dichotomy} & \mathrm{Dich}(R) & \equiv_{\mathrm{df}} & (\forall x, y)(Rxy \vee Ryx) \\
*\text{-Symmetry} & \mathrm{Sym}_*(R) & \equiv_{\mathrm{df}} & (\forall x, y)(Rxy \to_* Ryx) \\
*\text{-Transitivity} & \mathrm{Trans}_*(R) & \equiv_{\mathrm{df}} & (\forall x, y, z)(Rxy \mathbin{\&} Ryz \to Rxz)_* \\
*\text{-Antisymmetry} & \mathrm{Asym}_{E,*}(R) \text{ (w.r.t. } E) & \equiv_{\mathrm{df}} & (\forall x, y)(Rxy \mathbin{\&} Ryx \to Exy)_* \\
*\text{-Quasi-ordering} & \mathrm{QOrd}_*(R) & \equiv_{\mathrm{df}} & (\mathrm{Refl}(R) \mathbin{\&} \mathrm{Trans}(R))_* \\
*\text{-Ordering} & \mathrm{Ord}_{E,*}(R) \text{ (w.r.t. } E) & \equiv_{\mathrm{df}} & (\mathrm{QOrd}(R) \mathbin{\&} \mathrm{Asym}_E(R))_* \\
*\text{-Linear ordering} & \mathrm{LOrd}_{E,*}(R) \text{ (w.r.t. } E) & \equiv_{\mathrm{df}} & (\mathrm{Ord}_E(R) \mathbin{\&} \mathrm{Dich}(R))_* \\
*\text{-Function} & \mathrm{Fnc}_{E,*}(R) \text{ (w.r.t. } E) & \equiv_{\mathrm{df}} & (\forall x, y, z)(Rxy \mathbin{\&} Rxz \to Eyz)_*
\end{array}
$$

We adopt the convention that the index E can be dropped if $\Delta(E = \mathrm{Id})$. If $\Delta\mathrm{Fnc}_*(F)$, we can write $y = F(x)$ instead of ΔFxy.

Definition 2.10. The class union and class intersection are the functions $\bigcup_*^{(n+3)}$ and $\bigcap_*^{(n+3)}$, respectively, assigning a class $A^{(n+1)}$ to a class of classes $\mathcal{A}^{(n+2)}$ and defined as follows:

$$
\begin{array}{rcl}
\bigcup_* \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\exists A \in \mathcal{A})_*(x \in A)\} \\
\bigcap_* \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\forall A \in \mathcal{A})_*(x \in A)\}
\end{array}
$$

3 Formal theory of suprema and infima

The notions defined in this section are most meaningful for (quasi)orderings. Nevertheless, the definitions can be formulated for just any relation and most of the results hold regardless of any properties of the relation involved.

Definition 3.1. The *upper* and *lower $*$-cone* of a class A w.r.t. $\leq$ is defined as follows:

$$
\begin{array}{rcl}
A^{\uparrow_*} & =_{\mathrm{df}} & \{x \mid (\forall a \in A)_*(a \leq x)\} \\
A^{\downarrow_*} & =_{\mathrm{df}} & \{x \mid (\forall a \in A)_*(x \leq a)\}
\end{array}
$$

Let us fix some relation $\leq$ and denote its converse as usual by $\geq$. The usual definition of suprema and infima as least upper bounds and greatest lower bounds can then be formulated as follows (notice that they are *fuzzy classes,* since the property of being a supremum is graded):

Definition 3.2. The classes of $*$-*suprema* and $*$-*infima* of a class A w.r.t. $\leq$ are defined as

$$\begin{aligned} \leq\text{-}\operatorname{Sup}_* A &=_{\text{df}} A^{\uparrow_*} \cap_* A^{\uparrow_* \downarrow_*} \\ \leq\text{-}\operatorname{Inf}_* A &=_{\text{df}} A^{\downarrow_*} \cap_* A^{\downarrow_* \uparrow_*} \end{aligned}$$

Example 3.3. $\bigcup_* \mathcal{A}$ is a $*$-supremum of $\mathcal{A}$ w.r.t. $\subseteq_*$. Similarly, $\bigcap_* \mathcal{A} \in \subseteq_*\text{-}\operatorname{Inf}_* \mathcal{A}$.

The following lemmata on suprema and infima, needed for the formal theory of Dedekind reals, are mostly known in the algebraic setting (see e.g. [3]); here we reconstruct them in the formal theory $\text{Ł}\Pi_\omega$. In the rest of this section we drop the $\leq$ sign in $\leq\text{-}\operatorname{Sup}_*$ and $\leq\text{-}\operatorname{Inf}_*$, and assume all formulae indexed by $*$. We formulate the lemmata only for suprema, omitting their dual versions.

Lemma 3.4. $\operatorname{Sup} A = \operatorname{Inf} A^{\uparrow}$

Lemma 3.5. $(x \in \operatorname{Sup} A \;\&\; y \in \operatorname{Sup} A) \to (x \leq y \;\&\; y \leq x)$

Corollary 3.6. *The $*$-suprema w.r.t. $\subseteq_*$ are $\approx_*$-unique. By the extensionality axiom, the element of the kernel of $\subseteq_*\text{-}\operatorname{Sup}_* \mathcal{A}$ is unique w.r.t. identity. (Generally,* 1*-true suprema w.r.t. R are E-unique if R is antisymmetric w.r.t. E.)$_*$*

Lemma 3.7. $(A \subseteq B \;\&\; x \in \operatorname{Sup} A \;\&\; y \in \operatorname{Sup} B) \to x \leq y$

4 Fuzzy Dedekind reals

In [1] it is shown that any classical nth-order theory can be interpreted in $\text{Ł}\Pi_n$ by adding the axioms of crispness of all predicates and functions in the language of the theory. Thus we may assume that in $\text{Ł}\Pi_\omega$ we have at our disposal a theory of crisp natural numbers (obtained e.g. by the interpretation of 1st- or 2nd-order Peano arithmetic or any sufficiently strong theory of natural numbers in $\text{Ł}\Pi_\omega$). By the standard construction we get integers and rationals as certain pairs of natural numbers, with the usual crisp ordering and operations. Further on we shall therefore presuppose the existence of the class Q of crisp rational numbers, equipped with all usual relations and operations. We shall freely use any classical theorem of the classical theory of rational numbers, as they are provable in $\text{Ł}\Pi_\omega$ due to Lemma 41 of [1].

We require the following axioms of Dedekind cuts $A \subseteq \mathrm{Q}$ (which will represent Dedekind reals):

1. $(\forall p, q \in \mathrm{Q})[(p \leq q \to (p \in A \to q \in A)]$
2. $(\forall p \in \mathrm{Q})[(\forall q \in \mathrm{Q})(q > p \to q \in A) \to p \in A]$

The first axiom (which says that A is an upper set) reflects the intuitive motivation (see Section 1) that the membership $p \in A$ of a rational p in the Dedekind fuzzy real A expresses (the truth value of) the fact that p *majorizes* the fuzzy real: thus if $q \geq p$, then *a fortiori* q majorizes A at least in the degree p does.

The second axiom (the right-continuity of the membership function of A) is aimed at excluding the "left-continuous" doppelgangers of cuts with discontinuous membership functions. The reason for this requirement is the same as in classical mathematics, where the set of all cuts must similarly be pruned. Keeping the *left*-closed cuts corresponds to the choice of the informal meaning of $q \in A$ as "$A \leq q$" (rather than "$A < q$").

Definition 4.1. The (second order) class R of *fuzzy Dedekind reals* is the class of all $A \subseteq Q$ that satisfy both axioms 1 and 2 above. (It exists by the comprehension axiom of $\text{Ł}\Pi_3$.)

Crisp cuts in R correspond to (all and only) classical real numbers. A crisp cut with the least element q can be identified with the rational number q itself; if the distinction is necessary, we denote the cut by $\overline{q}$. Crisp cuts lacking the least element represent classical irrational numbers; those which are definable can be given the same names as in classical mathematics, e.g. $\sqrt{2} =_{\text{df}} \{q \in \mathrm{Q} \mid q^2 > 2\}$. We denote the empty cut $\emptyset$ by $+\infty$ and the whole Q by $-\infty$.

Zadeh's extension principle does not yield a useful notion of ordering for cumulative distributions (e.g., we would have $A \leq B$ for any crisp $A, B \neq +\infty$, as surely $(\exists p, q \in \mathrm{Q})(Ap \mathbin{\&} Bq \mathbin{\&} p \leq q)$). On the other hand, the usual definition of ordering as inclusion (reversed, as we chose the upper cuts) used in classical Dedekind completions is well-motivated and works well:

Definition 4.2. Let $A, B \in \mathrm{R}$, then

$$A \leq_* B \;\equiv_{\text{df}}\; B \subseteq_* A$$

Obviously, $\leq_*$ extends the order on Q, i.e., $(\forall p, q \in \mathrm{Q})(\overline{p} \leq \overline{q} \leftrightarrow p \leq q)$. Moreover, it embodies our original motivation of interpreting $q \in A$ as "$A \leq q$", since it can be proved that for $q \in \mathrm{Q}$ and $A \in \mathrm{R}$,

$$q \in A \;\leftrightarrow\; A \leq_* \overline{q}. \tag{1}$$

It follows immediately from the properties of inclusion that $\leq_*$ is an $(\approx_*, *)$-ordering, though not linear. Like in classical mathematics, $+\infty$ is the greatest and $-\infty$ the least real.

There are several candidates for the definition of strict ordering $<$ on R. Here we only give one of the strongest $<$-like notions, which is analogous to the intuitionistic relation of apartness:

Definition 4.3. For $A, B \in \mathrm{R}$,

$$A \ll B \;\equiv_{\text{df}}\; (\exists q)(\Delta Aq \mathbin{\&} \Delta\neg Bq)$$

Reals A such that $-\infty \ll A \ll +\infty$ are bounded, and thus can be called *proper* reals.

Like in classical mathematics, the chief merit of the Dedekind completion is the existence of all suprema and infima:

Theorem 4.4. $\mathcal{A} \subseteq \mathrm{R} \rightarrow \bigcap_* \mathcal{A} \in \mathrm{R}$

From Example 3.3 and Corollary 3.6 it follows that $\bigcap_* \mathcal{A}$ is the unique 1-true $*$-supremum w.r.t. $\leq_*$. On the contrary, $\bigcup_* \mathcal{A}$ need not be in R (it is an upper subset of Q, but not necessarily left-closed). Nevertheless, due to Lemma 3.4, all infima exist in R as well. We shall denote the unique element of $\mathrm{Ker}(\leq_*\text{-}\mathrm{Sup}_* \mathcal{A})$ by $\sup_* \mathcal{A}$ (and similarly for $\inf_* \mathcal{A}$). The suprema and infima that already existed in Q are obviously (since all sets involved are crisp) preserved.

5 Algebraic operations

We only sketch the definitions of addition and multiplication of fuzzy reals.

Since addition of rationals is monotonous w.r.t. $\leq$, Zadeh's principle yields a well-motivated extension of + to fuzzy reals: if defined as

$$q \in A +_* B \equiv_{\mathrm{df}} (\exists a \in A)_* (\exists b \in B)_* (q = a + b)$$

then $q \in A +_* B$ (i.e. $A +_* B \leq_* q$) is true just as much as Aq and Bq (i.e. $A, B \leq_* q$) guarantee. It can be proved that addition of fuzzy reals is commutative and associative, $\overline{0}$ is the neutral element, and it extends addition of crisp reals.

A similarly straightforward application of Zadeh's principle to multiplication on Q (which is not monotonous w.r.t. $\leq$) would yield a counter-intuitive results. Like in classical Dedekind reals, one must restrict Zadeh's extension to subdomains of rationals where multiplication is monotonous (i.e., positive and negative rationals) and take the union of Zadeh's extensions on these pieces (I omit the details here for space reasons).

A task yet to be done is to define further operations on reals (subtraction, division, exponentiation, etc.) with suitable properties. Preliminary results (to be presented in a subsequent paper) suggest that these tasks are viable.

6 Fuzzy intervals

The formal theory presented in the previous sections can be extended to a theory of fuzzy intervals (often called just 'fuzzy numbers'), of which we give a brief sketch here.

Observe that since no special property of Q has been used, the results of the previous sections hold for the fuzzy Dedekind completion of any crisp poset (in particular, it always yields a fuzzy complete lattice). From the applicational point of view, probably the most useful are fuzzy intervals over crisp reals; further on we shall therefore assume that the crisp numbers (denoted by lowercase variables) are crisp reals instead of rationals (the results, however, again hold for any crisp ordered domain).

By (1), an upper Dedekind cut A is in fact an upper interval $\{q \mid A \leq q\}$. Obviously, the results for upper Dedekind cuts can be dualized for lower cuts as well; thus in the same way, a lower cut B is a lower interval $\{q \mid B \geq q\}$. A fuzzy interval

$$[A, B]_* =_{\mathrm{df}} \{q \mid A \leq q \,\&_*\, q \leq B\}$$

is therefore an intersection of an upper cut A and a lower cut B. In other words, the upper cut A represents the *left endpoint* of an upper interval $[A, +\infty)$; similarly B represents the *right endpoint* of $(-\infty, B]$, and $[A, B]_* = [A, +\infty) \cap_* (-\infty, B]$.

The operations of Section 5 have been motivated by (1); thus they are subject to this 'interval interpretation'. We thus get an algebra of intervals with natural operations induced by the cut operations on the endpoints, e.g.

$$[A, B]_* +_* [C, D]_* =_{\mathrm{df}} [A +_* C, B +_* D]_*$$

The crisp points where the kernel of an interval ends play an important role. In virtue of the lattice completeness of the system of cuts we can define them within the theory:

Definition 6.1. Let A be an upper cut and B a lower cut. Then we define the upper cut $A^{\leftarrow}$ and the lower cut $B^{\rightarrow}$ as follows:

$$\begin{aligned} A^{\leftarrow} &=_{\mathrm{df}} \inf \{q \mid \Delta A q\} \\ B^{\rightarrow} &=_{\mathrm{df}} \sup \{q \mid \Delta B q\} \end{aligned}$$

(If the system of underlying crisp numbers is a complete lattice, as in the case of crisp reals, the cuts $A^{\leftarrow}$ and $B^{\rightarrow}$ can be identified with the corresponding crisp numbers.)

Observe that in virtue of axiom 2 for Dedekind cuts (Section 4), $A^{\leftarrow}$ is in fact a *minimum* of the kernel of the cut (and dually). These crisp endpoints are preserved by arithmetical operations on cuts (since kernels behave classically in good definitions); thus, e.g., $(X +_* Y)^{\leftarrow} = X^{\leftarrow} + Y^{\leftarrow}$. (On the other hand, one can easily find counterexamples to $X \leq_* Y \rightarrow X^{\leftarrow} \leq_* Y^{\leftarrow}$ or the converse; only $\Delta(X \leq Y) \rightarrow X^{\leftarrow} \leq Y^{\leftarrow}$ holds.)

It can be observed that a fuzzy interval is normal iff $A^{\leftarrow} \leq B^{\rightarrow}$. In such a case the membership function of $[A, B]$ is that of A on $(-\infty, A^{\leftarrow}]$, that of B on $[B^{\rightarrow}, +\infty)$, and 1 on $[A^{\leftarrow}, B^{\rightarrow}]$.

If $A^{\leftarrow} = B^{\rightarrow}$, then there is exactly one element in the kernel of $[A, B]$. We will call such degenerate intervals *fuzzy points*. Due to the axioms for Dedekind cuts, fuzzy points satisfy the most usual requirements on 'fuzzy real numbers' (singleton kernel, convexity of cuts, monotony of membership function towards the central point). Conforming to the tradition of fuzzy mathematics, we can therefore (ambiguously, but intelligibly) denote representatives of the (crisp) equivalence class $\{[A, B] \mid A^{\leftarrow} = B^{\rightarrow} = r\}$ by $\widetilde{r}$.

The set of all fuzzy points is closed under usual arithmetical operations (since, as stated above, they preserve the crisp endpoints of cuts). Furthermore, their arithmetics (sketched above) extends the arithmetics of crisp numbers (thus, e.g., $\widetilde{1}+\widetilde{1} = \widetilde{2}$). However, the arithmetics of fuzzy points differs somewhat from the traditional arithmetics of fuzzy intervals, as our operations are defined separately for the upper and lower endpoints of fuzzy intervals. It is beyond the scope of this short paper to argue why this is well-motivated; it will be elaborated in more details in a separate paper. At present we only propose this new formal theory of fuzzy intervals and fuzzy points (or, "fuzzy numbers") for further study and for trying it in applications.

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References

[1] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[2] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[3] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[4] P. Cintula. The ŁΠ and ŁΠ$\frac{1}{2}$ propositional and predicate logics. *Fuzzy Sets and Systems*, 124(3):289–302, 2001.

[5] D. Dubois and H. Prade. Fuzzy intervals versus fuzzy numbers: Is there a missing concept in fuzzy set theory? In S. Gottwald, P. Hájek, U. Höhle, and E. P. Klement, editors, *Fuzzy Logics and Related Structures: Abstracts of 26th Linz Seminar on Fuzzy Set Theory*, pages 45–46, 2005.

[6] F. Esteva, L. Godo, and F. Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[7] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[8] G. Takeuti and S. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *Journal of Symbolic Logic*, 49(3):851–866, 1984.

Fuzzification of Groenendijk–Stokhof propositional erotetic logic

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1 Introduction

Fuzzy logic is a branch of many-valued logics aimed at capturing comparative degrees of truth and reasoning under vagueness. For a long time, fuzzy sets and fuzzy logic were rather an engineering tool than a well-developed mathematical theory. The advances in metamathematics of fuzzy logic achieved during past few years (esp. [5]), however, set the theory on a firm ground and made it possible to develop fuzzy generalizations of various branches of classical mathematics in the axiomatic way.

One of the fields in which many-valued logics can fruitfully be applied is the logic of questions. The importance of a many-valued approach to questions follows, i.a., from the fact that many questionnaires employ scaled answers rather than simple yes–no ones. In many cases, the scale of answers directly corresponds to comparative degrees of truth, which is the domain of fuzzy logic.[1] Furthermore, many questions in natural language ask for information about predicates which are not 'black and white' (i.e., 'crisp', in fuzzy terminology), but show a natural scale of truth.[2]

This paper develops a fuzzy generalization FGS of Groenendijk-Stokhof's system of erotetic logic (as described in [3] and [4], further referred to as GS). Since Groenendijk-Stokhof's system (also known as the *partition semantics of questions*) is based on intensional semantics of classical logic, fuzzy intensional semantics is developed first, within the framework of fuzzy class theory [1]. Our attention is restricted to propositional FGS, i.e. fuzzy yes–no questions.[3]

2 Classical Groenendijk–Stokhof semantics

In this section we repeat the basic definitions of intensional semantics for classical propositional logic and classical propositional Groenendijk-Stokhof system. For details, see [3] and [4].

Definition 2.1 (Intensional semantics)**.** Let W be a non-empty set. By a *valuation in* W we mean a function $\|\cdot\|$ taking formulae to subsets of W, such that $\|\neg\varphi\| = W - \|\varphi\|$,

[1] E.g., the scale 'yes, rather yes, rather no, no'. However, fuzzy logic is not applicable if the set of answers contains options like 'I don't know', since these are not *truth* degrees.

[2] For instance, if John is middle-sized, then the answer to the question 'Is John tall?' should be neither 'yes' nor 'no', but something in-between.

[3] While classical propositional GS is trivial, its fuzzified version is less so.

$\|\varphi \,\&\, \psi\| = \|\varphi\| \cap \|\psi\|$, $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$, $\|\varphi \to \psi\| = (W - \|\varphi\|) \cup \|\psi\|$. The pair $\mathcal{W} = \langle W, \|\cdot\|\rangle$ is called a *logical space,* the elements of W *indices* or *possible worlds,* the subsets of W *propositions.*

The proposition $\|\varphi\|$ is called the *intension of* φ (in $\mathcal{W}$). The *extension of* φ *in* $w \in W$ is the truth value of the statement that $w \in \|\varphi\|$; it will be denoted by $\|\varphi\|_w$.[4]

A formula φ *holds* in a logical space $\mathcal{W} = \langle W, \|\cdot\|\rangle$ (written $\mathcal{W} \models \varphi$) iff $\|\varphi\| = W$. A formula φ is a *tautology* (written $\models \varphi$) iff it holds in any logical space. A formula φ *entails* a formula ψ *in* $\langle W, \|\cdot\|\rangle$ iff $\|\varphi\| \subseteq \|\psi\|$. A formula φ *entails* a formula ψ (written $\varphi \models \psi$) iff φ entails ψ in any logical space.

Intensional semantics is adequate w.r.t. classical propositional calculus; i.e., a formula is provable in classical propositional calculus iff it is a tautology of intensional semantics. GS extends this semantics to interrogative formulae $?\varphi$ (read *whether* φ), where φ is any propositional formula.

Definition 2.2 (Semantics of interrogative formulae)**.** Let $\mathcal{W} = \langle W, \|\cdot\|\rangle$ be a logical space. The *extension* $\|?\varphi\|_w$ of $?\varphi$ in $w \in W$ is the proposition $\{w' \in W \mid \|\varphi\|_{w'} = \|\varphi\|_w\}$.

The *intension* $\|?\varphi\|$ of $?\varphi$ in $\mathcal{W}$ is the equivalence relation $\{\langle w, w'\rangle \in W^2 \mid \|\varphi\|_w = \|\varphi\|_{w'}\}$. The partition of W induced by this equivalence relation will be denoted by $W/\|?\varphi\|$.

Definition 2.3 (Answerhood and entailment of interrogatives)**.** Let $\langle W, \|\cdot\|\rangle$ be a logical space.

We say that ψ is a *direct answer* to $?\varphi$ in $\mathcal{W}$ iff $\|\psi\| \in W/\|?\varphi\|$. We say that ψ is an *answer* to $?\varphi$ in $\mathcal{W}$ (written $\psi \models^{\mathcal{W}} ?\varphi$) iff ψ entails a direct answer to $?\varphi$ in $\mathcal{W}$.

We say that $?\psi$ *entails* $?\varphi$ in $\mathcal{W}$ (written $?\psi \models^{\mathcal{W}} ?\varphi$) iff every answer to $?\psi$ is an answer to $?\varphi$ in $\mathcal{W}$. We say that $?\psi$ and $?\varphi$ are *equivalent* in $\mathcal{W}$ (written $?\psi \equiv^{\mathcal{W}} ?\varphi$) iff $?\psi$ entails $?\varphi$ in $\mathcal{W}$ and vice versa.

We say that these relations hold generally iff they hold in any logical space.

It is easy to prove that $?\psi \models^{\langle W, \|\cdot\|\rangle} ?\varphi$ iff the partition $W/\|?\psi\|$ refines the partition $W/\|?\varphi\|$, and that equivalence of interrogatives corresponds to the identity of partitions.

3 T-norm based fuzzy logic

In this section, the main ideas of t-norm based fuzzy logic are outlined and basic definitions are given. For details see [5].

T-norm based fuzzy logic is founded upon a few natural assumptions regarding the semantics of fuzzy conjunction: truth-functionality, associativity, commutativity, monotonicity, continuity, and classical values on $\{0, 1\}$. Such binary functions on $[0, 1]$ had already been studied in probability theory under the name *continuous triangular norms* (or *continuous t-norms*). Given a continuous t-norm $*$, the semantics of other propositional connectives can be defined in a natural way (e.g., the semantics of implication is the maximal function such that the internalization of modus ponens is valid). Generalizing Tarski's definitions in the obvious way, for each $[0, 1]$-valuation v of propositional variables and any formula φ we get a unique semantic value $\|\varphi\|_v \in [0, 1]$. A formula is a *tautology*

[4]Thus if $w \in \|\varphi\|$, we say that the extension of φ in w is 1 (the truth value 'true'); if $w \notin \|\varphi\|$, we say that it is 0 (the truth value 'false'). The intension of φ can be identified with the function that assigns to each possible world $w \in W$ the extension of φ in w.

w.r.t. a continuous t-norm $*$ iff it gets the value 1 under each valuation v. The set of all tautologies w.r.t. a continuous t-norm $*$ is called the *logic of* $*$ and denoted by PC($*$).

It turns out that some formulae are tautologies w.r.t. any continuous t-norm; we call them t-tautologies. It can be proved that the set of all t-tautologies is finitely axiomatizable. This gives rise to Basic Fuzzy Logic BL:

Definition 3.1 (BL)**.** Propositional logic BL is determined by the following axiom schemata and the deduction rule of modus ponens (the primitive connectives are $\to$, &, and $\bot$).

$$
\begin{array}{rl}
\text{(BL1)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(BL2)} & (\varphi \mathbin{\&} \psi) \to \varphi \\
\text{(BL3)} & (\varphi \mathbin{\&} \psi) \to (\psi \mathbin{\&} \varphi) \\
\text{(BL4)} & (\varphi \mathbin{\&} (\varphi \to \psi)) \to (\psi \mathbin{\&} (\psi \to \varphi)) \\
\text{(BL5a)} & (\varphi \to (\psi \to \chi)) \to ((\varphi \mathbin{\&} \psi) \to \chi) \\
\text{(BL5b)} & ((\varphi \mathbin{\&} \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \\
\text{(BL6)} & ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \\
\text{(BL7)} & \bot \to \varphi
\end{array}
$$

Further connectives are defined as follows:

$$
\begin{array}{rcl}
\varphi \wedge \psi & \equiv_{\text{df}} & \varphi \mathbin{\&} (\varphi \to \psi) \\
\varphi \vee \psi & \equiv_{\text{df}} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\
\varphi \leftrightarrow \psi & \equiv_{\text{df}} & (\varphi \to \psi) \mathbin{\&} (\psi \to \varphi) \\
\neg\varphi & \equiv_{\text{df}} & \varphi \to \bot \\
\top & \equiv_{\text{df}} & \neg\bot
\end{array}
$$

There are three salient continuous t-norms:[5] the minimum, also known as the Gödel t-norm $x * y = \min(x, y)$, the product $x * y = x \cdot y$, and the Łukasiewicz t-norm $x * y = \max(0, x+y-1)$. The sets of all tautologies w.r.t. these t-norms are called Gödel, product, and Łukasiewicz fuzzy logic, denoted G, Π and Ł, respectively.[6] They are axiomatizable by the following respective schematic extensions of BL:

$$
\begin{array}{rl}
\text{(G)} & \varphi \to (\varphi \mathbin{\&} \varphi) \\
\text{(Ł)} & \neg\neg\varphi \to \varphi \\
(\Pi) & (\neg(\varphi \mathbin{\&} \varphi) \to \neg\varphi) \mathbin{\&} (\neg\neg\varphi \to (((\psi \mathbin{\&} \varphi) \to (\chi \mathbin{\&} \varphi)) \to (\psi \to \chi)))
\end{array}
$$

The $[0, 1]$-semantics of $\wedge$, $\vee$, $\bot$ and $\top$ in any logic PC($*$) is that of minimum, maximum, 0, and 1, respectively. Furthermore, in any PC($*$), $\|\varphi \to \psi\| = \max\{z \mid z * \|\varphi\| \leq \|\psi\|\}$; in particular, $\|\varphi \to \psi\|_v = 1$ iff $\|\varphi\|_v \leq \|\psi\|_v$. Consequently $\|\varphi \leftrightarrow \psi\|_v = 1$ iff $\|\varphi\|_v = \|\psi\|_v$, and $\|\neg\varphi\|_v = 1$ iff $\|\varphi\|_v = 0$.

Except for G, all PC($*$) lack contraction (i.e., $\varphi \mathbin{\&} \varphi$ is generally stronger than φ), which justifies the presence of min-conjunction $\wedge$. If we add the law of excluded middle (i.e., the schema $\varphi \vee \neg\varphi$) to BL, we get classical logic.

A further unary propositional connective Δ (Baaz's delta) with the $[0, 1]$-semantics $\|\Delta\varphi\|_v = 1$ iff $\|\varphi\|_v = 1$, otherwise $\|\Delta\varphi\|_v = 0$, is often introduced. The resulting logics

[5] Not only are they most often used in applications, but it is proved that any continuous t-norm is a special kind of ordinal sum of these three t-norms (Mostert–Shields' characterization theorem).

[6] Ł and G coincide respectively with Łukasiewicz and Gödel infinite-valued logics. G extends intuitionistic logic with Dummett's prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$.

BLΔ, GΔ, ŁΔ, and ΠΔ are axiomatized by the axioms of the respective fuzzy logic plus the following axioms for Δ:

(Δ1)	$\Delta\varphi \vee \neg\Delta\varphi$
(Δ2)	$\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(Δ3)	$\Delta\varphi \to \varphi$
(Δ4)	$\Delta\varphi \to \Delta\Delta\varphi$
(Δ5)	$\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

The deduction rules for logics with Δ are modus ponens and Δ-necessitation (from φ infer $\Delta\varphi$).

In order to develop fuzzy mathematics, fuzzy predicate calculus is necessary. The syntax of first-order fuzzy logic is classical (except for the differences in propositional connectives, i.e. the presence of two conjunctions and possibly Δ). The quantifiers ∀ and ∃ are governed by the following axiom schemata (which assume that the term t is substitutable for x in φ and that x is not free in χ):

(∀1)	$(\forall x)\varphi(x) \to \varphi(t)$
(∃1)	$\varphi(t) \to (\exists x)\varphi(x)$
(∀2)	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$
(∃2)	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$
(∀3)	$(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$

The deduction rules are those of propositional logic plus generalization (from φ infer $(\forall x)\varphi$). Equality can be regarded as a logical symbol governed by the axioms of reflexivity $x = x$ and universal intersubstitutivity $x = y \to \Delta(\varphi(x) \leftrightarrow \varphi(y))$.

The *standard semantics* for fuzzy predicate calculi is a straightforward generalization of Tarski's semantics to $[0, 1]$. The interpretation of predicates and functors of arity n in a model with the universe M are functions from M^n to $[0, 1]$ (for predicates) or to M (for functors); equality is interpreted as the identity on M. The semantics of ∀ and ∃ is that of infimum and supremum, respectively. The first-order logics G and GΔ are complete w.r.t. the standard $[0, 1]$-semantics; first-order BL, Ł and Π (with or without Δ), however, are not.[7]

4 Fuzzy class theory

Within fuzzy predicate calculus, axiomatic theory of fuzzy sets can be developed. For most purposes, however, one does not need a full-fledged set theory over fuzzy logic, since it is usually not necessary to consider the membership of *sets* in sets. The theory of membership of (atomic) *individuals* in fuzzy sets—i.e., fuzzy *class* theory—is much simpler; it has been elaborated in [1] over a richer fuzzy logic ŁΠ, which contains all the connectives of GΔ, ŁΔ, and ΠΔ. An easy inspection of proofs in [1] shows that the theorems of [1] that do not mix connectives of different logics remain valid in its

[7]They are complete w.r.t. special classes of distributive residuated lattices. Since our main motivation is the interval $[0, 1]$, we shall not discuss this general semantics (it can be found in [5]). The results of [1] reduce the relevant part of fuzzy class theory (see Section 4) to fuzzy propositional calculus, for which the completeness w.r.t. $[0, 1]$ holds.

fragments GΔ, ŁΔ, and ΠΔ. The adaptation of fuzzy class theory FCT developed in [1] for an extension $\mathcal{F}$ of BLΔ will be denoted by $\mathcal{F}$CT.

The language of $\mathcal{F}$CT has two sorts of variables: object variables $x, y, \dots$ and class variables $X, Y, \dots$ (there are no universal variables). The only primitive predicate is the membership predicate $\in$ between objects and classes. $\mathcal{F}$CT enjoys full class comprehension, i.e., for any formula $\varphi(x)$ there is a function symbol[8] $\{x \mid \varphi(x)\}$ and the comprehension axiom $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$. The classes are understood extensionally, therefore $\mathcal{F}$CT adopts the axiom of extensionality $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \to X = Y$.

The intended models consist of a universe U, which is the range of object variables, and the set $U^{[0,1]}$ of all functions from U to $[0,1]$, which is the range of class variables. The truth value of the formula $x \in X$ in a model $\mathbf{M}$ under an evaluation e of class and object variables is defined as the value of the function $e(X)$ on $e(x)$. The semantic value of the comprehension term $\{x \mid \varphi(x)\}$ in $\mathbf{M}$ under e is the function $f \colon U \to [0,1]$ such that for any $a \in U$, $f(a)$ is the truth value of $\varphi(x)$ in $\mathbf{M}$ under the evaluation $e_{x:a}$, where $e_{x:a}$ coincides with e except that $e_{x:a}(x) = a$. It is easy to prove all comprehension axioms as well as extensionality in such models; for details see [1].

We repeat here several definitions and theorems of [1] that will be needed later on.

Definition 4.1 (Fuzzy class operations and relations)**.**

$\emptyset$	$=_{\text{df}}$	$\{x \mid \bot\}$	empty class
V	$=_{\text{df}}$	$\{x \mid \top\}$	universal class
$-X$	$=_{\text{df}}$	$\{x \mid \neg(x \in X)\}$	complement
$X \cup Y$	$=_{\text{df}}$	$\{x \mid x \in X \vee x \in Y\}$	union
$X \cap Y$	$=_{\text{df}}$	$\{x \mid x \in X \mathbin{\&} x \in Y\}$	strong intersection
$X \subseteq Y$	$\equiv_{\text{df}}$	$(\forall x)(x \in X \to x \in Y)$	inclusion

Convention 4.2. In what follows, let the notation $\varphi(p_1, \dots, p_n)$ imply that the formula φ contains no propositional variables other than $p_1, \dots, p_n$. The formula $\varphi \mathbin{\&} \dots \mathbin{\&} \varphi$ (n times) is abbreviated by φ^n. Furthermore, we abbreviate $(\forall x)(x \in X \to \varphi)$ as $(\forall x \in X)\varphi$, $(\exists x)(x \in X \mathbin{\&} \varphi)$ as $(\exists x \in X)\varphi$, and $\{x \mid x \in X \mathbin{\&} \varphi\}$ as $\{x \in X \mid \varphi\}$. If $\varphi(p_1, \dots, p_n)$ is a propositional formula and $\psi_1, \dots, \psi_n$ are any formulae, then $\varphi(\psi_1, \dots, \psi_n)$ denotes the formula φ in which all occurrences of p_i are replaced by ψ_i (for all $i \leq n$).

Definition 4.3 (n-ary class operation)**.** Let φ be a propositional formula. We define the n-ary class operation induced by φ as

$$\mathrm{Op}_\varphi(X_1, \dots, X_n) =_{\text{df}} \{x \mid \varphi(x \in X_1, \dots, x \in X_n)\}.$$

The following lemmata are corollaries of more general theorems of [1]; their direct proofs are given in Appendix A.

Lemma 4.4. *Let $\varphi(p_1, \dots, p_n)$ and $\psi(p_1, \dots, p_n)$ be propositional formulae. Then $\mathcal{F} \vdash \varphi \to \psi$ iff $\mathcal{F}\mathrm{CT} \vdash \mathrm{Op}_\varphi(X_1, \dots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \dots, X_n)$.*

Lemma 4.5. $\mathcal{F}\mathrm{CT} \vdash (X \subseteq Y \mathbin{\&} Y \subseteq Z) \to X \subseteq Z$.

[8]For function symbols in fuzzy logics see [6].

5 Fuzzy intensional semantics

We want to generalize classical intensional semantics to fuzzy intensional semantics, i.e., to allow propositions to be fuzzy sets. Since we have a formal theory of fuzzy sets, viz the theory of fuzzy classes $\mathcal{F}$CT, we want to define the semantical notions in this theory (thus we shall be able to prove results on entailment within its framework). First we shall give an intuitive motivation for our definitions.

Let us work in $\mathcal{F}$CT. Given a (possibly fuzzy) class W (to be informally interpreted as a logical space), certain class operations of $\mathcal{F}$CT (union, intersection, etc.) on (possibly fuzzy) subclasses of W correspond directly to propositional connectives (disjunction, conjunction, etc., respectively). Subclasses $A \subseteq W$ can therefore aptly be called propositions and taken for the range of intensions of propositional formulae of fuzzy logic $\mathcal{F}$. The extension of a proposition A in $w \in W$ is expressed by the formula '$w \in A$'.[9]

It is natural to say that the proposition A entails B iff for all $w \in W$, the extension of A in w implies that of B in w.[10] This condition can be expressed as $(\forall w \in W)(w \in A \rightarrow w \in B)$, i.e., according to the definitions of $\mathcal{F}$CT, $W \cap A \subseteq B$. Similarly we can say that a proposition A holds in W iff it holds in all indices $w \in W$, formally $(\forall w \in W)(w \in A)$, i.e. $W \subseteq A$.

In these considerations, propositions $A \subseteq W$ represent intensions of propositional formulae of a fuzzy logic $\mathcal{F}$. The assignment $\|\cdot\|$ of propositions $A \subseteq W$ to formulae obeying the rules of correspondence between propositional connectives and class operations (e.g., $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$) can therefore be construed as an intensional semantics for propositional formulae in the logical space $\langle W, \|\cdot\|\rangle$.

We of course intend tautologicity to be defined as validity in all logical spaces, i.e., for all couples $\langle W, \|\cdot\|\rangle$. However, the assignment $\|\cdot\|$ is not an object of our theory;[11] thus we cannot quantify over it, and another formal solution is required.

It can be observed that in classical intensional semantics, the function $\|\cdot\|$ is in fact a translation of propositional formulae to the language of a theory of subsets of some basic set. Similarly, we can define fuzzy intensional semantics by giving a *translation* of propositional formulae to the language of a theory of *fuzzy* subsets of some basic set (favourably, a part of fuzzy class theory $\mathcal{F}$CT).[12] Interpreting propositional variables as class *variables,* propositional connectives as the corresponding class operations, and choosing a class *variable* W, we get the generality we need. The translation is adequate in the sense that a propositional formula is provable in fuzzy logic $\mathcal{F}$ iff the general validity of its translation is provable in $\mathcal{F}$CT (and so holds in every model of $\mathcal{F}$CT).

Let us elaborate this idea formally:[13]

---

[9]These definitions look the same as in the classical case, but notice that now the extensions can have truth values between 0 and 1 and the propositions can be fuzzy classes.

[10]This definition (of *local* entailment) allows the inference from A to B in w if A entails B (by detachment). Note that the entailment itself is a fuzzy notion.

[11]It could become an object of the theory after some strenghtening of $\mathcal{F}$CT, which would allow us to encode propositional formulae and classes of classes, but we shall not pursue this line here.

[12]We are thus giving an *interpretation* (a direct syntactic model) of fuzzy propositional calculus in $\mathcal{F}$CT. By means of this interpretation, any model of $\mathcal{F}$CT together with a valuation of free variables yields a fuzzy intensional model for the original propositional formulae.

[13]Since W can be construed as only a part of a larger logical space W' (whose subclass W is the class of those worlds to which we currently restrict our attention), we shall not further require that propositions be subclasses of W. The relativization of quantifiers in the definitions guarantees that only the worlds in W are taken into account when evaluating entailment of propositions.

Definition 5.1 (Fuzzy intensional semantics)**.** The translation $\|\cdot\|$ of the formulae of propositional fuzzy logic $\mathcal{F}$ to $\mathcal{F}$CT is defined as follows:

The translation $\|p_i\|$ of an atomic formula p_i is a class variable A_i. The translation of a complex formula $\varphi(p_1, \ldots, p_n)$ is

$$\|\varphi(p_1, \ldots, p_n)\| =_{\mathrm{df}} \mathrm{Op}_\varphi(\|p_1\|, \ldots, \|p_n\|).$$ [14]

Theorem 5.2 (Adequacy of fuzzy intensional semantics)**.**

$$\mathcal{F} \vdash \varphi \quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash W \subseteq \|\varphi\|$$

The proof is given in Appendix A. Similarly it is shown that $\mathcal{F} \vdash \varphi \to \psi$ iff $\mathcal{F}\mathrm{CT} \vdash W \cap \|\varphi\| \subseteq \|\psi\|$. This correspondence justifies writing $\models \varphi$ instead of $W \subseteq \|\varphi\|$, and $\varphi \models \psi$ instead of $W \cap \|\varphi\| \subseteq \|\psi\|$.[15] The notation can conveniently be generalized to any class terms of $\mathcal{F}$CT, defining $(\models A) \equiv_{\mathrm{df}} (W \subseteq A)$ and $(A \models B) \equiv_{\mathrm{df}} (W \cap A \subseteq B)$.[16] We further define logical equivalence of propositions as their mutual entailment: $(A \equiv B) \equiv_{\mathrm{df}} (A \models B) \,\&\, (B \models A)$.

Theorem 5.3 (Properties of fuzzy entailment)**.** *It is provable in* $\mathcal{F}$CT *that* $\Delta(W \subseteq W \cap W)$ *implies*[17]

$$[(A \models B) \;\&\; (B \models C)] \;\to\; (A \models C) \tag{1}$$
$$[(A \equiv B) \;\&\; (B \equiv C)] \;\to\; (A \equiv C) \tag{2}$$
$$[(A \equiv A') \;\&\; (B \equiv B')] \;\to\; [(A \models B) \leftrightarrow (A' \models B')] \tag{3}$$
$$(\varphi \models \psi) \;\to\; (\neg\psi \models \neg\varphi) \tag{4}$$

Proof: See Appendix A. QED

It should be stressed that the semantic notions defined here are graded, and can have truth values between 0 and 1. Thus even though the theorems on fuzzy answerhood derived here have syntactically the same form as their classical counterparts, in $\mathcal{F}$CT they express more general statements, namely that the truth value of the consequent is not less than that of the antecedent. Thus, e.g., the formula (4) should be interpreted as '$\neg\psi$ entails $\neg\varphi$ *at least in the degree* in which φ entails ψ', rather than a crisp statement that '$\neg\varphi$ entails $\neg\psi$ if φ entails ψ'. The same is true about the notions of answerhood and entailment of questions defined in the next Section.

---

[14] It can be observed that the definition works naturally, i.e., $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$, etc.

[15] Formulae of $\mathcal{F}$ are translated by $\|\cdot\|$ to class terms of $\mathcal{F}$CT (thus their semantical values in models of $\mathcal{F}$CT are fuzzy propositions). The semantical notions of tautologicity and entailment are expressed as certain formulae of $\mathcal{F}$CT. They can combine to complex semantical statements like $(\varphi \models \chi) \,\&\, (\psi \models \chi) \to (\varphi \,\&\, \psi \models \chi)$, which again are formulae of $\mathcal{F}$CT (so in models they may have truth values between 0 and 1). If they are provable in $\mathcal{F}$CT, we take them for valid semantical laws (they are 1-true in all models.)

[16] Thus we can also write $A \models \|\varphi\|$, or shortly $A \models \varphi$, etc.

[17] This condition is automatically satisfied in G, or if W is crisp. For each of the statements it can be somewhat weakened: e.g., the condition $W \subseteq W \cap W \cap W$ is sufficient for (1).

6 Fuzzy semantics for questions

Having defined intensional semantics in $\mathcal{F}$CT for propositional formulae, we want to extend this semantics to interrogative formulae $?\varphi$. There are two (classically equivalent) options as to how to understand the question $?\varphi$:

a. What is the truth value of φ?

b. Is it the case that φ?

We shall discuss both cases separately. We first interpret $?\varphi$ as the question about the truth value of φ.

Let us fix some *crisp* logical space W.[18] Then ψ answers such a question (which fact we shall symbolize $\psi \models_{\mathrm{t}} ?\varphi$) iff the truth value of ψ determines the truth value of φ. This amounts to the condition that for any indices $w, w' \in W$, if $\|\psi\|_w = \|\psi\|_{w'}$, then $\|\varphi\|_w = \|\varphi\|_{w'}$. Since the identity of truth values is expressed by the equivalence connective defuzzified by Δ (see Section 3), and the extension of φ in w is expressed by $w \in \|\varphi\|$ (see Section 5), the defining condition for $\psi \models_{\mathrm{t}} ?\varphi$ in $\mathcal{F}$CT reads

$$(\forall w, w' \in W)[\Delta(w \in \|\psi\| \leftrightarrow w' \in \|\psi\|) \rightarrow \Delta(w \in \|\varphi\| \leftrightarrow w' \in \|\varphi\|)]. \tag{5}$$

Again we can extend the notation and write $A \models_{\mathrm{t}} ?\varphi$, $\psi \models_{\mathrm{t}} ?B$, and $A \models_{\mathrm{t}} ?B$ for arbitrary class terms A and B, not restricting our definition to propositions definable by propositional formulae.

If we define the truth-equivalence relation R_X induced by (a proposition) X as[19]

$$R_X =_{\mathrm{df}} \{\langle u, v\rangle \mid \Delta(u \in X \leftrightarrow v \in X)\}$$

then the answerhood condition can be rewritten as

$$A \models_{\mathrm{t}} ?B \ \equiv_{\mathrm{df}} \ W^2 \cap R_A \subseteq R_B.$$

Following GS, we can identify the intension of $?\varphi$ and the relation $R_{\|\varphi\|}$. The proposition $\{w' \in W \mid \langle w, w'\rangle \in R_{\|\varphi\|}\}$ can be understood as the direct true answer to $?\varphi$ in w, i.e., the extension of $?\varphi$ in w. Truth-value based entailment and equivalence of $?B$ and $?C$ can be defined standardly as

$$?B \models_{\mathrm{t}} ?C \quad \equiv_{\mathrm{df}} \quad (\forall A)[(A \models_{\mathrm{t}} ?B) \rightarrow (A \models_{\mathrm{t}} ?C)] \tag{6}$$
$$?B \equiv_{\mathrm{t}} ?C \quad \equiv_{\mathrm{df}} \quad (?B \models_{\mathrm{t}} ?C) \ \&\ (?C \models_{\mathrm{t}} ?B) \tag{7}$$

It can be observed that these notions of answerhood and entailment are crisp. In fact, they correspond to answerhood and entailment for questions $?\alpha(\|\varphi\|_w = \alpha)$ of classical predicative GS in the intended models of $\mathcal{F}$CT.[20] As such, they bring little new to the topic; there is, however, a natural fuzzification of these semi-classical notions, obtained by omitting one or both of the Δ's in (5):

[18] Fuzzy W is also meaningful, but the definitions would need much more careful discussion.

[19] We need to extend the language of $\mathcal{F}$CT by tuples of objects $\langle x_1, \ldots, x_n\rangle$ here. This can be done by adding functors for forming tuples and accessing their components, and axiom schemata saying that tuples equal iff their respective components equal. We then define $W^2 =_{\mathrm{df}} \{\langle u, v\rangle \mid u \in W \ \&\ v \in W\}$. For details see [1].

[20] See Section 4 and [4].

Definition 6.1 (Fuzzy truth-value based answerhood).

$$A \models_{\text{ft}} ?B \ \equiv_{\text{df}}\ (\forall w, w' \in W)[\Delta(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)]$$
$$A \models_{\text{fft}} ?B \ \equiv_{\text{df}}\ (\forall w, w' \in W)[(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)]$$

The corresponding notions of entailment and equivalence of questions are defined as in (6) and (7), respectively.

It can be observed that the third option, viz discarding only the first Δ in (5), would lead to a counter-intuitive notion of answerhood, since it would admit cases when φ itself does not answer $?\varphi$ (this follows from the fact that $\chi \to \Delta\chi$ is not a theorem of BLΔ).

All $A \models_{\text{t}} ?B$, $A \models_{\text{ft}} ?B$, and $A \models_{\text{fft}} ?B$ are 1-true in a model if the partition of W by the truth-levels of A refines the partition by the truth-levels of B.[21] Unlike crisp $\models_{\text{t}}$, which otherwise is absolutely false, its graded variants $\models_{\text{ft}}$ and $\models_{\text{fft}}$ partially tolerate the flaws in the match of truth-levels. The truth value of $A \models_{\text{ft}} ?B$ is high iff the truth value of B does not change *too much* within the truth-levels of A.[22] In other words, a proposition *more-or-less* answers $?B$ if its truth value *more-or-less* determines the truth value of B. Different t-norm logics provide different measures of tolerance for imperfection in satisfying the answerhood condition.

The answerhood notion $\models_{\text{fft}}$ strengthens the condition and requires further that the closeness of the truth values of the answer imply the closeness of those being asked for. In Ł, $A \models_{\text{fft}} ?B$ is 1-true iff for any $w, w' \in W$, the difference of the truth values of A in w and w' does not exceed the difference of the truth values of B in w and w'. For Π and G, replace the word 'difference' in the previous sentence respectively by 'ratio' and 'smaller' (where 'the smaller of the truth values' means 1 if they are equal).

Obviously $\models_{\text{ft}}$ is the weakest of the three notions:

Theorem 6.2. *$\mathcal{F}$CT proves*

$$(A \models_{\text{fft}} ?B) \ \to\ (A \models_{\text{ft}} ?B) \tag{8}$$
$$(A \models_{\text{t}} ?B) \ \to\ (A \models_{\text{ft}} ?B) \tag{9}$$

For the proof see Appendix A; counter-examples to the remaining implications are easy to find.[23]

Theorem 6.3. *Let* $\circ$ *be* ft *or* fft*. Then* $\mathcal{F}$CT *proves*

$$(A \equiv B) \ \to\ (A \models_{\circ} ?B) \tag{10}$$
$$(A \equiv B) \ \to\ (?A \equiv_{\circ} ?B) \tag{11}$$
$$(A \equiv B) \ \to\ [(A \models_{\text{fft}} ?C) \leftrightarrow (B \models_{\text{fft}} ?C)] \tag{12}$$
$$(?A \models_{\circ} ?B) \ \to\ [(?B \models_{\circ} ?C) \to (?A \models_{\circ} ?C)] \tag{13}$$
$$(?A \equiv_{\circ} ?B) \ \to\ [(?B \equiv_{\circ} ?C) \to (?A \equiv_{\circ} ?C)] \tag{14}$$

[21] We slightly abuse the language here for brevity's sake. It would be more accurate to speak about the partition of the *evaluation of* W and the truth-levels of the *evaluations* of A and B in the model. We use a similar license in the following paragraphs.

[22] The exact meaning of 'does not change too much' is given by the semantics of the equivalence connective, which in t-norm logics expresses the closeness of truth values. In particular, in Ł and Π it respectively expresses the difference and ratio of truth values, while in G it yields the smaller of the truth values (unless they are equal, in which case it is 1-true).

[23] E.g., to disprove $(A \models_{\text{ft}} ?B) \to (A \models_{\text{fft}} ?B)$, use a two-element intended model with the universe $\{a, b\}$ and assign the function $\{\langle a, 0.5\rangle, \langle b, 0.6\rangle\}$ to A, $\{\langle a, 0.5\rangle, \langle b, 0.7\rangle\}$ to B, and $\{\langle a, 1\rangle, \langle b, 1\rangle\}$ to W. Then the truth value of $A \models_{\text{ft}} ?B$ in Ł is 1, while $A \models_{\text{fft}} ?B$ evaluates only to 0.9.

For the proofs see Appendix A. Two-element counter-examples show that (10) and (11) are not valid for t in place of $\circ$, and that fft in (12) cannot be replaced by t or ft.

Because of their motivation, the notions of answerhood defined above are sensitive with respect to operations that can change the (exact or approximate) match of truth values. Therefore, answerhood is not preserved by usual logical operations (except for equivalence).[24] An example of preservation properties that can be proved is the following theorem:

Theorem 6.4. *Let* $\diamond$ *be a (primitive or defined) connective congruent w.r.t.* $\leftrightarrow$*, i.e., such that* $\mathcal{F} \vdash [(\varphi \leftrightarrow \psi) \mathbin{\&} (\varphi' \leftrightarrow \psi')] \to [(\varphi \diamond \varphi') \leftrightarrow (\psi \diamond \psi')]$*. Then* $\mathcal{F}$CT *proves*

$$[(\varphi \models_{\mathrm{ft}} ?\psi) \mathbin{\&} (\varphi \models_{\mathrm{ft}} ?\psi')] \to [\varphi \models_{\mathrm{ft}} ?(\psi \diamond \psi')]$$

For the proof, see Appendix A. In particular, the statement holds for &, $\wedge$, $\vee$, or $\leftrightarrow$ substituted for $\diamond$, and can easily be generalized to any arity of $\diamond$. A further discussion of the truth-value based notion of answerhood, entailment, and equivalence of questions is given in Section 7.

Let us now investigate the other interpretation of $?\varphi$. We again work in $\mathcal{F}$CT and now allow W to be fuzzy. The yes–no question 'Is it the case that φ?' is answered by a proposition A iff A either entails φ (then it is an affirmative answer) or entails $\neg\varphi$ (a negative answer):

Definition 6.5. A proposition A is an *affirmative* answer to $?\varphi$ iff $A \models \varphi$. It is a *negative* answer to $?\varphi$ iff $A \models \neg\varphi$. It is a *yes–no* answer (in symbols $A \models ?\varphi$) iff it is an affirmative answer or a negative answer:

$$A \models ?\varphi \;\equiv_{\mathrm{df}}\; (A \models \varphi) \vee (A \models \neg\varphi)$$

Since entailment of fuzzy propositions is generally a fuzzy notion, so is yes–no answerhood: answers can be, not only fully affirmative or negative, but also *partially* affirmative or *partially* negative (or neither).

Theorem 6.6. $\mathcal{F}$CT *proves that* $\Delta(W \subseteq W \cap W)$ *implies*[25]

$$(A \models B) \to [(B \models ?\varphi) \to (A \models ?\varphi)] \tag{15}$$
$$(A \equiv B) \to [(B \models ?\varphi) \leftrightarrow (A \models ?\varphi)] \tag{16}$$
$$(\varphi \equiv \psi) \to [(A \models ?\varphi) \to (A \models ?\psi)] \tag{17}$$

Proof: See Appendix A. QED

Theorem 6.7. $\mathcal{F}$CT *proves that if* $\Delta(W \subseteq W \cap W)$*, then affirmative and negative answers exclude each other, i.e.,*

$$[(\psi^+ \models \varphi) \mathbin{\&} (\psi^- \models \neg\varphi)] \to (\models \neg(\psi^+ \mathbin{\&} \psi^-))$$

Proof: See Appendix A. QED

[24] Again, the two-element counter-examples to $[(\varphi \models_\circ ?\chi) \mathbin{\&} (\psi \models_\circ ?\chi)] \to (\varphi \diamond \psi \models_\circ ?\chi)$ for $\diamond$ replaced by &, $\wedge$, or $\vee$ are easy to find.

[25] See footnote 17 on page 217.

It can be noticed that the consequent in Theorem 6.7 cannot be strengthened to $\models \neg(\psi^+ \wedge \psi^-)$. In Ł, e.g., an answer ψ can be *both* partially affirmative and partially negative (only $\psi \mathbin{\&} \neg\psi$ must be false).[26]

Following GS, we can define yes–no entailment and equivalence of questions in the standard way:

Definition 6.8 (yes–no entailment and equivalence of questions)**.**

$$
\begin{aligned}
?\varphi \models ?\psi \quad &\equiv_{\mathrm{df}} \quad (\forall A)[(A \models ?\varphi) \rightarrow (A \models ?\psi)] \\
?\varphi \equiv ?\psi \quad &\equiv_{\mathrm{df}} \quad (?\varphi \models ?\psi) \mathbin{\&} (?\psi \models ?\varphi)
\end{aligned}
$$

Theorem 6.9. *It is provable in $\mathcal{F}$CT that $\Delta(W \subseteq W \cap W)$ implies*

$$(?\varphi \models ?\psi) \quad \rightarrow \quad [(?\psi \models ?\chi) \rightarrow (?\varphi \models ?\chi)] \tag{18}$$
$$(?\varphi \equiv ?\psi) \quad \rightarrow \quad [(?\psi \equiv ?\chi) \rightarrow (?\varphi \equiv ?\chi)] \tag{19}$$
$$(\varphi \equiv \varphi') \quad \rightarrow \quad [(?\varphi \models ?\psi) \rightarrow (?\varphi' \models ?\psi)] \tag{20}$$
$$(\psi \equiv \psi') \quad \rightarrow \quad [(?\varphi \models ?\psi) \rightarrow (?\varphi \models ?\psi')] \tag{21}$$
$$(\varphi \equiv \psi) \quad \rightarrow \quad (?\varphi \models ?\psi) \tag{22}$$
$$(?\varphi \models ?\psi) \quad \rightarrow \quad (?\varphi \models ?\neg\psi) \tag{23}$$

Proof: See Appendix A. QED

Since obviously $\varphi \models \varphi$, from (22) and (23) it follows that $?\varphi \models ?\varphi$ and $?\varphi \models ?\neg\varphi$. The converse, $?\neg\varphi \models ?\varphi$, does not hold generally, since $\neg\neg\varphi \rightarrow \varphi$ is not a theorem of BL. There are examples from natural language that this result does not contradict intuition: if negation behaves in some context as the bivalent negation of G or Π (there are such contexts—e.g., *not guilty* can be regarded as bivalent, even though there are degrees of *guilt*), then a negative answer to $?\neg\varphi$ need not be affirmative enough to $?\varphi$. The equivalence of $?\varphi$ and $?\neg\varphi$ does, however, hold in Ł or for crisp φ.

7 Conclusions

We have seen that the two interpretations of the question $?\varphi$ in fuzzy logic give rise to two different kinds of fuzzy answerhood notions. Although these notions coincide in classical logic, their properties in fuzzy logic are considerably different. It appears that the number of interesting theorems that can be derived in $\mathcal{F}$CT is larger with yes–no answerhood than with truth-value based answerhood. The following observations can shed some light upon this fact.

The definition of yes–no answerhood $\models$ conforms better to the methodology of [2], according to which the truth-value semantics of fuzzy logic is only secondary to the rules of inference that hold for fuzzy propositions. Since there is little sense in asserting that some fuzzy proposition (e.g., 'John loves Mary') is true exactly in the degree (say) 0.7845, fuzzy truth values must only be regarded as a *model* underlying the rules of inference valid for fuzzy propositions (even though these rules may originally have been descried by means

[26] An example from natural language for such a situation is, e.g., an answer to the question 'Is he old?' giving some middle age, which both partially affirms and partially denies seniority. (Yet, since *old* and *not old* are mutually exclusive, the truth degrees of affirmation and denial must be low enough for their strong conjunction to be false.)

of this model). The doctrine of not speaking explicitly about the truth degrees, but rather hiding them in the semantical meta-level of a formal theory, is one of the design principles of $\mathcal{F}$CT, which has been used here as the framework for fuzzy intensional semantics and erotetic logic.

Although formulated formally in $\mathcal{F}$CT (ergo, without an explicit reference to truth values), the definitions of $\models_{\mathrm{t}}$, $\models_{\mathrm{ft}}$, and $\models_{\mathrm{fft}}$ capture in fact the answerhood conditions for the question about the *truth value* of φ, rather than about the fuzzy proposition φ itself.[27] Therefore these notions, though useful when working with particular models, are not particularly well-suited for investigation in $\mathcal{F}$CT, which only captures general laws valid in all models, rather than particular truth values. Nevertheless, the theorems of Section 6 show that at least some properties of truth-value answerhood are universally valid and can be proved in $\mathcal{F}$CT.

Fuzzy intensional semantics developed here for the purposes of fuzzy erotetic logic is general enough to serve as the basis for a similar fuzzification of other kinds of modal (epistemic, deontic, etc.) logic. Since our semantic notions of entailment and answerhood are defined as certain formulae of $\mathcal{F}$CT, they are compatible with the formalism proposed in [2] and [1] as a unified framework for a large part of fuzzy mathematics, and directly applicable in other formal theories within the framework.

A Formal proofs

In this Appendix, we give the formal proofs of the theorems of the preceding sections. In the proofs we shall freely use the transitivity of implication, (i.e., the axiom (BL1) plus twice modus ponens), (BL3), (BL5a), and (BL5b) without explicit notices. All statements of the form $\mathrm{BL} \vdash \varphi$ or $\mathrm{BL}\Delta \vdash \varphi$ refer to [5] where they are proved.

Lemma A.1. *The following formulae are theorems of* BLΔ*:*

$$(\forall w)(\varphi \to \psi) \to [(\forall w)\varphi \to (\forall w)\psi] \tag{24}$$
$$(\varphi \to \psi) \to [(\chi \to \varphi) \to (\chi \to \psi)] \tag{25}$$
$$(\varphi \to \psi) \to [\nu \to (\varphi \to \psi)] \tag{26}$$
$$[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \mathbin{\&} \varphi') \to (\psi \mathbin{\&} \psi')] \tag{27}$$
$$[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \vee \varphi') \to (\psi \vee \psi')] \tag{28}$$
$$[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \wedge \varphi') \to (\psi \wedge \psi')] \tag{29}$$
$$[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi^2 \to \chi)] \tag{30}$$
$$(\varphi \to \psi) \to (\Delta\varphi \to \psi) \tag{31}$$
$$(\Delta\varphi \to \Delta\psi) \to (\Delta\varphi \to \psi) \tag{32}$$
$$[(\nu \to \nu^3) \mathbin{\&} ((\nu \mathbin{\&} \varphi) \to \psi) \mathbin{\&} ((\nu \mathbin{\&} \psi) \to \chi)] \to ((\nu \mathbin{\&} \varphi) \to \chi) \tag{33}$$
$$[(\nu \to \nu^2) \mathbin{\&} ((\nu \mathbin{\&} \varphi) \to \psi)] \to ((\nu \mathbin{\&} \neg\psi) \to \neg\varphi) \tag{34}$$
$$\Delta(\nu \to \nu^2) \to (\nu \to \nu^3) \tag{35}$$

Proof: (24) and (27) are proved in [5].

(26) is an instance of $\mathrm{BL} \vdash \varphi \to (\psi \to \varphi)$.

[27] This can be seen from the fact that propositions stronger than A need not answer $?\varphi$, even if A itself does. This would be counter-intuitive for answerhood of the question about φ, but is quite natural for querying about truth values, since the truth values of stronger propositions may be much different from those of A and the distinctions may become lost.

(25) follows from (BL1) by (BL5a), (BL3), and (BL5b).

(28) From $(\varphi \to \psi) \to [(\psi \to (\psi \vee \psi')) \to (\varphi \to (\psi \vee \psi'))]$, which is an instance of (BL1), and BL $\vdash \psi \to (\psi \vee \psi')$ we get $(\varphi \to \psi) \to (\varphi \to (\psi \vee \psi'))$. Similarly, using in addition BL $\vdash (\psi' \vee \psi) \to (\psi \vee \psi')$, we get $(\varphi' \to \psi') \to (\varphi' \to (\psi \vee \psi'))$. Thus by (27) we get $[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \to (\psi \vee \psi')) \mathbin{\&} (\varphi' \to (\psi \vee \psi'))]$, whence (28) follows from BL $\vdash [(\varphi \to \chi) \mathbin{\&} (\varphi' \to \chi)] \to [(\varphi \vee \varphi') \to \chi]$.

(29) is proved similarly as (28), only using $\wedge$ instead of $\vee$ and antecedents instead of consequents of implications.

(30) follows from the instance $[(\varphi \to (\psi \to \chi)) \mathbin{\&} (\varphi \to \psi)] \to [(\varphi \mathbin{\&} \varphi) \to (\psi \mathbin{\&} (\psi \to \chi))]$ of (27) and BL $\vdash [\psi \mathbin{\&} (\psi \to \chi)] \to \chi$.

(31) follows from (Δ3) and the instance $(\Delta\varphi \to \varphi) \to [(\varphi \to \psi) \to (\Delta\varphi \to \psi)]$ of (BL1).

(32) follows from (Δ3) and the instance $(\Delta\psi \to \psi) \to [(\Delta\varphi \to \Delta\psi) \to (\Delta\varphi \to \psi)]$ of (25).

(33) Take the instance of (BL1)
$((\nu \mathbin{\&} \varphi) \to \psi) \to [(\psi \to \chi) \to ((\nu \mathbin{\&} \varphi) \to \chi)]$; thence by (26) we get
$((\nu \mathbin{\&} \varphi) \to \psi) \to [\nu \to [(\psi \to \chi) \to ((\nu \mathbin{\&} \varphi) \to \chi)]]$; applying (30) we get
$((\nu \mathbin{\&} \varphi) \to \psi) \to [[\nu \to (\psi \to \chi)] \to [((\nu^3 \mathbin{\&} \varphi) \to \chi)]]$,
whence (33) readily follows.

(34) From BL $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ by (26) we get
$\nu \to [(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)]$; then by (30) we have
$[(\nu \mathbin{\&} \varphi) \to \psi] \to [(\nu^2 \mathbin{\&} \neg\psi) \to \neg\varphi]$, whence (34).

(35) The instance $[\Delta(\nu \to \nu^2) \mathbin{\&} \nu] \to \nu^2$ of (Δ3) used twice in (27) and BL$\Delta \vdash (\Delta\varphi \mathbin{\&} \Delta\varphi) \leftrightarrow \Delta\varphi$ yield $[\Delta(\nu \to \nu^2) \mathbin{\&} \nu^2] \to \nu^4$, i.e., $\Delta(\nu \to \nu^2) \to (\nu^2 \to \nu^4)$. Since $\Delta(\nu \to \nu^2) \to (\nu \to \nu^2)$ by (Δ3), we get also $\Delta(\nu \to \nu^2) \to (\nu \to \nu^4)$, whence by (BL2) we obtain (35). QED

Proof of Lemma 4.4 The substitution of $x \in X_i$ for p_i (for all $i \leq n$) everywhere in the proof of $\varphi \to \psi$ in $\mathcal{F}$ transforms it into the proof of

$$\varphi(x \in X_1, \ldots, x \in X_n) \to \psi(x \in X_1, \ldots, x \in X_n)$$

in first-order $\mathcal{F}$. Generalization on x then yields

$$(\forall x)(\varphi(x \in X_1, \ldots, x \in X_n) \to \psi(x \in X_1, \ldots, x \in X_n))$$

which is exactly $\mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$ by the definitions and axioms of $\mathcal{F}$CT.

Conversely, let e be an evaluation that refutes $\varphi \to \psi$ (we use the Completeness Theorem for propositional fuzzy logics here, see [5]). We construct a model $\mathbf{M}$ of $\mathcal{F}$CT that refutes $\mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$ as follows. Let the universe of $\mathbf{M}$ contain a single element a, and let the class variables X_i be represented by the functions that assign $e(p_i)$ to a. It is trivial to check that $\mathbf{M}$ models $\mathcal{F}$CT and refutes $\mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$. By the Soundness Theorem of the first-order logic $\mathcal{F}$ (see [5]) the proof is done. QED

Proof of Lemma 4.5: From the instance $(x \in X \to x \in Y) \to [(x \in Y \to x \in Z) \to (x \in X \to x \in Z)]$ of (BL1), generalization on x and distribution of the quantifier by (24) yields the required formula $[(\forall x)(x \in X \to x \in Y) \mathbin{\&} (\forall x)(x \in Y \to x \in Z)] \to (\forall x)(x \in X \to x \in Z)$. QED

Proof of Theorem 5.2 In BL it is provable that φ is equivalent to $\top \to \varphi$. Since further $\mathrm{Op}_\top = \mathrm{V}$, we get from Lemma 4.4:

$$\begin{aligned} &\mathcal{F} \vdash \varphi(p_1, \ldots, p_n) \quad \text{iff} \\ &\quad \text{iff} \quad \mathcal{F} \vdash \top \to \varphi(p_1, \ldots, p_n) \\ &\quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash \mathrm{Op}_\top(\|p_1\|, \ldots, \|p_n\|) \subseteq \mathrm{Op}_\varphi(\|p_1\|, \ldots, \|p_n\|) \\ &\quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash \mathrm{V} \subseteq \|\varphi(p_1, \ldots, p_n)\| \\ &\quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash W \subseteq \|\varphi(p_1, \ldots, p_n)\| \end{aligned}$$

The last equivalence follows in one direction from the monotonicity of $\subseteq$ (Lemma 4.5); the other direction is obtained by generalization on W and specification to V. QED

Proof of Theorem 5.3: (1) follows from (33) and a general theorem of [1], but it is not difficult to derive it from (33) directly. Substitute $w \in W$, $w \in A$, $w \in B$, and $w \in C$ into (33) for ν, φ, ψ, and χ, respectively. Then generalizing on w and distributing the quantifier by (24) (using (BL5)), we get (1) expanded according to the definitions of $\mathcal{F}$CT. (Use (35) to get the stated precondition of the theorem.)

(2) follows from (27) and the instances of (1):
$[(A \models B) \mathbin{\&} (B \models C)] \to (A \models C)$ and $[(C \models B) \mathbin{\&} (B \models A)] \to (C \models A)$.

(3) It follows from (1) that
$[(A' \models A) \mathbin{\&} (B \models B')] \to [(A \models B) \to (A' \models B')]$ and
$[(A \models A') \mathbin{\&} (B' \models B)] \to [(A' \models B') \to (A \models B)]$. Now use (27).

(4) follows from (34) in the same way as (1) from (33). (Note that the converse of (4) does not hold.) QED

Proof of Theorem 6.2: (8) is proved by generalization on $w \in W$ and $w' \in W$ of the instance $[(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)] \to [\Delta(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)]$ of (31), and distribution of both quantifiers over the principal implication by (24). (The rule of bounded generalization follows from (26); the analogue of (24) for quantifiers relativized to a *crisp* domain follows easily from (30).)

(9) is proved in the same way as (8) from the instance $[\Delta(w \in A \leftrightarrow w' \in A) \to \Delta(w \in B \leftrightarrow w' \in B)] \to [\Delta(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)]$ of (32). QED

Proof of Theorem 6.3 In the proof, the restriction of all quantifiers to W is omitted for simplicity's sake. It is an easy, but tedious exercise to verify that the proof works in the same way with all quantifiers restricted to crisp W. We shall use Xww' as shorthand for $w \in X \leftrightarrow w' \in X$.

(10) $A \equiv B$ amounts to $A \subseteq B \mathbin{\&} B \subseteq A$ here, whence by specification we get $(A \equiv B) \to [(w' \in A \to w' \in B) \mathbin{\&} (w \in B \to w \in A)]$. The transitivity of implication entails $[(w' \in A \to w' \in B) \mathbin{\&} (w \in B \to w \in A) \mathbin{\&} (w \in A \leftrightarrow w' \in A)] \to (w \in B \to w' \in B)$; thus we get $[(A \equiv B) \mathbin{\&} (w \in A \leftrightarrow w' \in A)] \to (w \in B \to w' \in B)$. Similarly $[(A \equiv B) \mathbin{\&} (w' \in A \leftrightarrow w \in A)] \to (w' \in B \to w \in B)$. Since $\mathrm{BL} \vdash [(\chi \to \varphi) \mathbin{\&} (\chi \to \psi)] \to [(\chi \to (\varphi \wedge \psi))]$ and $\mathrm{BL} \vdash [(\varphi \to \psi) \wedge (\psi \to \varphi)] \to (\varphi \leftrightarrow \psi)$, we get $(A \equiv B) \to [(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)]$. Generalization on w and w' plus the axiom ($\forall$2) conclude the proof. (10) for $\models_{\mathrm{ft}}$ follows a fortiori (see Theorem 6.2).

(11) In the proof of (10) we have proved $(A \equiv B) \to (Aww' \to Bww')$. By (25) with χ instantiated to Cww' we get $(A \equiv B) \to [(Cww' \to Aww') \to (Cww' \to Bww')]$. Generalization on w, w' plus ($\forall$2) and (24), and generalization on C plus ($\forall$2) conclude the proof. (11) for $\models_{\mathrm{ft}}$ is proved in the same way, only using $\Delta Cww'$ when instantiating χ in (25).

(12) As in the proof of (10) we prove $(A \equiv B) \to (Bww' \to Aww')$. Since further BL $\vdash$ $[Aww' \,\&\, (Aww' \to Cww')] \to Cww'$, we get $[(A \equiv B) \,\&\, Bww' \,\&\, (Aww' \to Cww')] \to Cww'$, i.e., $(A \equiv B) \to [(Aww' \to Cww') \to (Bww' \to Cww')]$. Generalization on w and w' plus the axiom ($\forall$2) conclude the proof.

(13) From the instance $((D \models ?A) \to (D \models ?B)) \to [((D \models ?B) \to (D \models ?C)) \to ((D \models ?A) \to (D \models ?C))]$ of (BL1), by generalization on D and distribution of the quantifier by (24) we get (13).

(14) follows from (13) by (27). QED

Proof of Theorem 6.4 Let us denote $\|\varphi\|$, $\|\psi\|$, $\|\psi'\|$, and $\|\psi \diamond \psi'\|$ by A, B, B', and C respectively, and adopt the conventions of the Proof of Theorem 6.3.

The precondition of the present theorem gives $(Bww' \,\&\, B'ww') \to Cww'$, whence $[\Delta Aww' \to (Bww' \,\&\, B'ww')] \to (\Delta Aww' \to Cww')$ by (25). Thence by (27) and BLΔ $\vdash$ $\Delta\chi \to (\Delta\chi \,\&\, \Delta\chi)$ we get $[(\Delta Aww' \to Bww') \,\&\, (\Delta Aww' \to B'ww')] \to (\Delta Aww' \to Cww')$. By generalization on w and w' and distribution of the quantifiers by (24) we get the required formula.

That &, $\wedge$, $\vee$, and $\leftrightarrow$ substituted for $\diamond$ satisfy the precondition of the theorem follows from (27), (29), (28), and transitivity of $\leftrightarrow$, respectively. QED

Proof of Theorem 6.6: (15) From (1) we have
$[(A \models B) \,\&\, (B \models \varphi)] \to (A \models \varphi)$ and $[(A \models B) \,\&\, (B \models \neg\varphi)] \to (A \models \neg\varphi)$.
Thence by (28) it follows that
$[((A \models B) \,\&\, (B \models \varphi)) \vee ((A \models B) \,\&\, (B \models \neg\varphi))] \to ((A \models \varphi) \vee (A \models \neg\varphi))$;
now by BL $\vdash [(\chi \,\&\, \psi) \vee (\chi \,\&\, \psi')] \leftrightarrow [\chi \,\&\, (\psi \vee \psi')]$ we get
$[((A \models B) \,\&\, ((B \models \varphi) \vee (B \models \neg\varphi)))] \to ((A \models \varphi) \vee (A \models \neg\varphi))$, which is (15).

(16) From (15) we get
$(A \models B) \to [(B \models ?\varphi) \to (A \models ?\varphi)]$ and $(B \models A) \to [(A \models ?\varphi) \to (B \models ?\varphi)]$,
whence by (27) we get (16).

(17) From (1) it follows that
$(\varphi \models \psi) \to [(A \models \varphi) \to (A \models \psi)]$ and, using (4),
$(\psi \models \varphi) \to [(A \models \neg\varphi) \to (A \models \neg\psi)]$. Then
$[(\varphi \equiv \psi) \,\&\, ((A \models \varphi) \vee (A \models \neg\varphi))] \to ((A \models \psi) \vee (A \models \neg\psi))$ as in (15). QED

Proof of Theorem 6.7: By (27) we get
$[(\psi^+ \to \varphi) \,\&\, (\psi^- \to \neg\varphi)] \to [(\psi^+ \,\&\, \psi^-) \to (\varphi \,\&\, \neg\varphi)]$.
Since BL $\vdash (\chi \to (\varphi \,\&\, \neg\varphi)) \to \neg\chi$, we have
$[(\psi^+ \to \varphi) \,\&\, (\psi^- \to \neg\varphi)] \to \neg(\psi^+ \,\&\, \psi^-)$.
Then proceed as in the proof of (34) and (1). QED

Proof of Theorem 6.9: (18) and (19) are proved exactly as (13) and (14).

(20) From (17) we have $(\varphi \equiv \varphi') \to [(A \models ?\varphi') \to (A \models ?\varphi)]$. Thus from
$((A \models ?\varphi') \to (A \models ?\varphi)) \to [((A \models ?\varphi) \to (A \models ?\psi)) \to ((A \models ?\varphi') \to (A \models ?\psi))]$,
which is an instance of (BL1), we get
$(\varphi \equiv \varphi') \to [((A \models ?\varphi) \to (A \models ?\psi)) \to ((A \models ?\varphi') \to (A \models ?\psi))]$.
Then generalize on A and distribute the quantifier according to ($\forall$2) and (24).

(21) From (17) we have $(\psi \equiv \psi') \to [(A \models ?\psi) \to (A \models ?\psi')]$.
As in the proof of (20) we derive
$(\psi \equiv \psi') \to [((A \models ?\varphi) \to (A \models ?\psi)) \to ((A \models ?\varphi) \to (A \models ?\psi'))]$
and proceed as in the previous case.

(22) From (17) we have $(\varphi \equiv \psi) \to [(A \models ?\varphi) \to (A \models ?\psi)]$.
Then generalize on A and use ($\forall$2).

(23) From BL $\vdash \psi \to \neg\neg\psi$ we can infer $\psi \models \neg\neg\psi$ and by (1) get
$(A \models \psi) \to (A \models \neg\neg\psi)$, whence
$[(A \models ?\varphi) \to (A \models ?\psi)] \to [(A \models ?\varphi) \to (A \models ?\neg\psi)]$.
To finish the proof we generalize on A and apply (24). QED

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References

[1] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[2] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[3] J. Groenendijk and M. Stokhof. Partitioning logical space. 2nd ESSLLI Annotated Handout, 1990.

[4] J. Groenendijk and M. Stokhof. Questions. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 1055–1124. Elsevier and MIT Press, 1997.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[6] P. Hájek. Function symbols in fuzzy logic. In *Proceedings of the East–West Fuzzy Colloquium*, pages 2–8, Zittau/Görlitz, 2000. IPM.

Topology in Fuzzy Class Theory: Basic notions

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Abstract: In the formal and fully graded setting of Fuzzy Class Theory (or higher-order fuzzy logic) we make an initial investigation into basic notions of fuzzy topology. In particular we study graded notions of fuzzy topology regarded as a fuzzy system of open or closed fuzzy sets and as a fuzzy system of fuzzy neighborhoods. We show their basic graded properties and mutual relationships provable in Fuzzy Class Theory and give some links to the traditional notions of fuzzy topology.

1 Introduction

Fuzzy topology is among the fundamental disciplines of fuzzy mathematics whose development was stimulated from the very beginning of the invention of fuzzy sets [10]. Following the role of topology in classical mathematics, fuzzy topology should capture the notions of openness, neighborhood, closure, etc., within the setting of fuzzy set theory. The paper [15] by Höhle and Šostak, which is contained in the special issue of Fuzzy Sets and Systems (1995) on fuzzy topology, mentions and classifies a number of conceptual frameworks (lattice-, model-, and category-theoretical) that have arisen during past decades. A detailed and up-to-date exposition of many-valued and fuzzy topologies, mostly based on a categorical viewpoint, is contained in the monograph [14] by Höhle.

This paper follows the footsteps of Ying's attempt [17] to establish fuzzy topology as a non-elementary theory over many-valued logic. We make initial steps towards understanding fuzzy topology as an axiomatic higher-order theory over Hájek-style [12] formal fuzzy logic, following the methodology for formal fuzzy mathematics described in [7]. According to the classification proposed in [15], the models of our theory are closest to "L-fuzzy topologies as characteristic morphisms". However, the apparatus of Fuzzy Class Theory, employed in this paper, makes our notions and the way in which they can be studied quite distinct from (and in some aspects more general than) other approaches to fuzzy topology.

The paper is organized as follows: Section 2 gives a brief exposition of Fuzzy Class Theory and the definitions needed in the paper. Section 3 studies the graded notion of fuzzy topology regarded as a fuzzy system of open (or closed) fuzzy sets. Section 4 then studies graded fuzzy topologies regarded as fuzzy systems of fuzzy neighborhoods.

2 Preliminaries

Fuzzy Class Theory FCT, introduced in [5], is an axiomatization of Zadeh's notion of fuzzy set in formal fuzzy logic. Here we use its variant defined over IMTL$_\triangle$ [11], the logic of all left-continuous t-norms whose residual negation is involutive (we shall call them *IMTL t-norms;* the most important example is the Łukasiewicz t-norm $x * y =_{\mathrm{df}} \max(0, x + y - 1)$).

Remark 2.1. We have the following reasons for choosing IMTL$_\triangle$ for the ground logic: the logic MTL$_\triangle$ [11] of all left-continuous t-norms is arguably [3] the weakest fuzzy logic with good inferential properties for fully graded fuzzy mathematics in the framework of formal fuzzy logic [7]. IMTL$_\triangle$ extends it with the law of double negation, which is in fuzzy topology needed for the correspondence between open and closed fuzzy sets. A generalization of fuzzy topology to the logic MTL$_\triangle$ (with independent systems of open and closed fuzzy sets) will be the subject of some future paper.

We assume the reader's familiarity with IMTL$_\triangle$; for details on this logic see [11]. Here we only recapitulate its standard $[0, 1]$ semantics:

$\&$	...	a left-continuous t-norm $*$ with involutive residual negation
$\to$	...	the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$
$\wedge$, $\vee$	...	min, max
$\neg$	...	$x \Rightarrow 0$; in IMTL$_\triangle$ it is involutive, due to the axiom $\neg\neg\varphi \to \varphi$
$\veebar$	...	the t-conorm dual to $*$ (since $\varphi \veebar \psi$ is defined as $\neg(\neg\varphi \,\&\, \neg\psi)$)
$\leftrightarrow$	...	the bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
$\triangle$	...	$\triangle x = 1 - \mathrm{sgn}(1 - x)$
$\forall$, $\exists$	...	inf, sup; by involutiveness, $(\exists x)\neg\varphi \leftrightarrow \neg(\forall x)\varphi$

Definition 2.1. *Fuzzy Class Theory* FCT is a formal theory over multi-sorted first-order fuzzy logic (in this paper, IMTL$_\triangle$), with the sorts of variables for

- atomic objects (lowercase letters $x, y, \dots$)
- fuzzy classes of atomic objects (uppercase letters $A, B, \dots$)
- fuzzy classes of fuzzy classes of atomic objects (Greek letters $\tau, \sigma, \dots$)
- fuzzy classes of the third order (calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$)
- etc., in general for fuzzy classes of the n-th order ($X^{(n)}, Y^{(n)}, \dots$)

Besides the crisp identity predicate $=$, the language of FCT contains:

- the membership predicate $\in$ between objects of successive sorts
- class terms $\{x \mid \varphi\}$, for any formula φ and any variable x of any order
- symbols $\langle x_1, \dots, x_k\rangle$ for k-tuples of individuals $x_1, \dots, x_k$ of any order

FCT has the following axioms (for all formulae φ and variables of all orders):

- the logical axioms of multi-sorted first-order logic IMTL$_\triangle$
- the axioms of crisp identity: (i) $x = x$, (ii) $x = y \to (\varphi(x) \to \varphi(y))$, (iii) $\langle x_1, \dots, x_k\rangle = \langle y_1, \dots, y_k\rangle \to x_1 = y_1 \,\&\, \dots \,\&\, x_k = y_k$

Table 1: Abbreviations used in the formulae of FCT

Ax	$\equiv_{\mathrm{df}}$	$x \in A$
$x_1 \ldots x_k$	$=_{\mathrm{df}}$	$\langle x_1, \ldots, x_k \rangle$
$x \notin A$	$\equiv_{\mathrm{df}}$	$\neg(x \in A)$, and similarly for other predicates
$(\forall x \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\forall x)(x \in A \to \varphi)$
$(\exists x \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\exists x)(x \in A \mathbin{\&} \varphi)$
$(\forall x, y \in A)\varphi$	$\equiv_{\mathrm{df}}$	$(\forall x \in A)(\forall y \in A)\varphi$, similarly for $\exists$
$\{x \in A \mid \varphi\}$	$=_{\mathrm{df}}$	$\{x \mid x \in A \mathbin{\&} \varphi\}$
$\{t(x_1, \ldots, x_k) \mid \varphi\}$	$=_{\mathrm{df}}$	$\{z \mid z = t(x_1, \ldots, x_k) \mathbin{\&} \varphi\}$
φ^n	$\equiv_{\mathrm{df}}$	$\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (n times)

- the comprehension axioms: $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$
- the extensionality axioms: $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$

Remark 2.2. Notice that in FCT, fuzzy sets are rendered as a *primitive notion* rather than modeled by membership functions. In order to capture this distinction, fuzzy sets are in FCT called *fuzzy classes;* the name *fuzzy set* is reserved for membership functions in the models of the theory.

The models of FCT are systems of fuzzy sets of all orders over a fixed crisp universe of discourse, with truth degrees taking values in an IMTL$_\triangle$-chain (e.g., the interval $[0,1]$ equipped with an IMTL t-norm). Thus all theorems on fuzzy classes provable in FCT are true statements about **L**-valued fuzzy sets, for any IMTL$_\triangle$-chain **L**. Notice however that the theorems of FCT have to be derived from its axioms by the rules of the fuzzy logic IMTL$_\triangle$ rather than classical Boolean logic. For details on proving theorems of FCT see [8] or [6].

Convention 2.1. In formulae of FCT, we employ usual abbreviations known from classical mathematics, including those listed in Table 1. Usual rules of precedence apply to the connectives of IMTL$_\triangle$. Furthermore we define standard defined notions of FCT, summarized in Table 2, for all orders of fuzzy classes.

Remark 2.3. Notice that in FCT, not only the membership predicate $\in$, but all defined notions are in general fuzzy (unless they are defined as provably crisp). FCT thus presents a fully graded approach to fuzzy mathematics. The importance of full gradedness in fuzzy mathematics is explained in [8, 4, 2]: its main merit lies in that it allows inferring relevant information even when a property of fuzzy sets is not fully satisfied. Fuzzy topology has a long tradition of attempting full gradedness, cf. graded definitions and theorems e.g. in [14, 17].

Remark 2.4. It should be noted that fully graded theories have some peculiar features in which they differ from both classical mathematics and traditional fuzzy mathematics. A detailed account of the unusual features of fully graded theories is given in [9]; some of them can also be found in [8] (available online). Here we only briefly stress the main features of graded mathematics:

- Since $\varphi \to \varphi \& \varphi$ is not a generally valid law of fuzzy logic, premises may occur several times in theorems. A typical graded theorem has the form $\varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n} \to \psi$,

Table 2: Defined notions of FCT

$\emptyset$	$=_{\mathrm{df}}$	$\{x \mid 0\}$	empty class
V	$=_{\mathrm{df}}$	$\{x \mid 1\}$	universal class
$\operatorname{Ker} A$	$=_{\mathrm{df}}$	$\{x \mid \triangle Ax\}$	kernel
αA	$=_{\mathrm{df}}$	$\{x \mid \alpha \mathbin{\&} Ax\}$	α-resize
$-A$	$=_{\mathrm{df}}$	$\{x \mid \neg Ax\}$	complement
$A \cap B$	$=_{\mathrm{df}}$	$\{x \mid Ax \mathbin{\&} Bx\}$	(strong) intersection
$A \cup B$	$=_{\mathrm{df}}$	$\{x \mid Ax \veebar Bx\}$	(strong) union
$A \times B$	$=_{\mathrm{df}}$	$\{xy \mid Ax \mathbin{\&} By\}$	Cartesian product
$\bigcup \tau$	$=_{\mathrm{df}}$	$\{x \mid (\exists A \in \tau)(x \in A)\}$	class union
$\bigcap \tau$	$=_{\mathrm{df}}$	$\{x \mid (\forall A \in \tau)(x \in A)\}$	class intersection
$\operatorname{Pow}(A)$	$=_{\mathrm{df}}$	$\{X \mid X \subseteq A\}$	power class
$\operatorname{Crisp}(A)$	$\equiv_{\mathrm{df}}$	$(\forall x)\triangle(Ax \vee \neg Ax)$	crispness
$\operatorname{Ext}_E A$	$\equiv_{\mathrm{df}}$	$(\forall x, y)(Exy \mathbin{\&} Ax \to Ay)$	E-extensionality
$A \subseteq B$	$\equiv_{\mathrm{df}}$	$(\forall x)(Ax \to Bx)$	inclusion
$A \approxeq B$	$\equiv_{\mathrm{df}}$	$(A \subseteq B) \mathbin{\&} (B \subseteq A)$	(strong) bi-inclusion

where φ^k abbreviates $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (k times, where φ^0 is 1). The multiplicity k_i of the premise φ_i shows how strongly it influences (the lower bound for) the truth of ψ (when only partially true), and depends on how many times the premise is used in the derivation of ψ from $\varphi_1, \ldots, \varphi_k$. The exponent k in φ^k can also take the conventional value "$\triangle$", where $\varphi^{\triangle}$ is understood as $\triangle\varphi$ (recall that $\varphi^{\triangle} \to \varphi^n$ for all n).

- If a complex notion Φ is defined as a conjunction $\varphi_1 \mathbin{\&} \ldots \mathbin{\&} \varphi_n$, then the conjuncts φ_i will get different multiplicities in different theorems. It is therefore appropriate to parameterize Φ by the multiplicities of the components φ_i and define it as $\Phi^{k_1,\ldots,k_n} \equiv_{\mathrm{df}} \varphi_1^{k_1} \mathbin{\&} \ldots \mathbin{\&} \varphi_n^{k_n}$. (All graded topological notions in the following sections will be defined in this way.) We can write just Φ^k instead of $\Phi^{k_1,\ldots,k_n}$ if $k_i = k$ for all i, and just Φ if $k_i = 1$ for all i.

The following defined predicates will be employed in the next sections.

Definition 2.2. We define the following (graded) unary predicates:

$\cup$-closedness:	$\operatorname{uc}(\tau)$	$\equiv_{\mathrm{df}}$	$(\forall A, B \in \tau)(A \cup B \in \tau)$
$\cap$-closedness:	$\operatorname{ic}(\tau)$	$\equiv_{\mathrm{df}}$	$(\forall A, B \in \tau)(A \cap B \in \tau)$
$\bigcup$-closedness:	$\operatorname{Uc}(\tau)$	$\equiv_{\mathrm{df}}$	$(\forall \nu \subseteq \tau)\big(\bigcup \nu \in \tau\big)$
$\bigcap$-closedness:	$\operatorname{Ic}(\tau)$	$\equiv_{\mathrm{df}}$	$(\forall \nu \subseteq \tau)\big(\bigcap \nu \in \tau\big)$
$\subseteq$-upperness:	$\operatorname{Upper}(\tau)$	$\equiv_{\mathrm{df}}$	$(\forall A, B)(A \subseteq B \mathbin{\&} A \in \tau \to B \in \tau)$
being a filter:	$\operatorname{Filter}^{v,e,u,i}(\tau)$	$\equiv_{\mathrm{df}}$	$(\mathrm{V} \in \tau)^v \mathbin{\&} (\emptyset \notin \tau)^e \mathbin{\&} \operatorname{Upper}^u(\tau) \mathbin{\&} \operatorname{ic}^i(\tau)$

3 Topology as a system of open (closed) fuzzy classes

In classical mathematics, topology can be introduced in several equivalent ways—by open sets, closed sets, neighborhoods, closure, etc. In FCT, however, these approaches yield different concepts. In this paper, we make an initial investigation into two of them, namely the system of open (or closed) classes (in this section) and the system of neighborhoods (in Sect. 4). Due to the limited size of this paper we present only some of the initial results and have to omit all proofs.

The fuzzification of the concept of open (closed) fuzzy topology presented in Def. 3.1 follows the methodology sketched in [13, §5] and formally elaborated in [5, §7], i.e., reinterpreting the formulae of the classical definition in fuzzy logic.[1]

Definition 3.1. We define an *(open)* (e, v, i, u)*–fuzzy topology* and a *closed* (e, v, u, i)*–fuzzy topology* respectively by the predicates

$$\begin{aligned} \mathsf{OTop}^{e,v,i,u}(\tau) &\equiv_{\mathrm{df}} (\emptyset \in \tau)^e \ \&\ (\mathrm{V} \in \tau)^v \ \&\ \mathrm{ic}^i(\tau) \ \&\ \mathrm{Uc}^u(\tau) \\ \mathsf{CTop}^{e,v,u,i}(\sigma) &\equiv_{\mathrm{df}} (\emptyset \in \sigma)^e \ \&\ (\mathrm{V} \in \sigma)^v \ \&\ \mathrm{uc}^u(\sigma) \ \&\ \mathrm{Ic}^i(\sigma) \end{aligned}$$

(see Remark 2.4 for the meaning of the parameters e, v, u, i).

Note that this concept of topology is graded, i.e., the predicate $\mathsf{OTop}^{e,v,i,u}$ determines the *degree* to which τ is an open (e, v, i, u)–fuzzy topology.

Example 3.1. Let $*$ be an IMTL t-norm and $\Rightarrow$ its residuum. The $*$-based Zadeh models of open $(1, 1, \triangle, \triangle)$–fuzzy topology, i.e., of the predicate

$$\mathsf{OTop}^{1,1,\triangle,\triangle}(\tau) \equiv \emptyset \in \tau \ \&\ \mathrm{V} \in \tau \ \&\ \triangle\, \mathrm{ic}(\tau) \ \&\ \triangle\, \mathrm{Uc}(\tau)$$

are functions $\tau\colon [0,1]^{\mathrm{V}} \to [0,1]$ satisfying the following conditions:

(i) $\tau(A) * \tau(B) \leq \tau(A \cap B)$ for every $A, B \in [0,1]^{\mathrm{V}}$

(ii) $\bigwedge_{A \in [0,1]^{\mathrm{V}}} (\nu(A) \Rightarrow \tau(A)) \leq \tau\left(\bigcup \nu\right)$ for every $\nu\colon [0,1]^{\mathrm{V}} \to [0,1]$

where $(A \cap B)(x) = A(x) * B(x)$ and $\left(\bigcup \nu\right)(x) = \bigvee_{A \in [0,1]^{\mathrm{V}}} (\nu(A) * A(x))$. Since both (i) and (ii) are crisp, the degree to which τ is a $(1, 1, \triangle, \triangle)$-fuzzy topology equals $\tau(\emptyset) * \tau(\mathrm{V})$. These models cover fuzzy topologies studied under the name "L-fuzzy topologies of Höhle type" [15].

In $\mathrm{IMTL}_\triangle$, open and closed topologies are interdefinable:

Definition 3.2. Let $\tau_c =_{\mathrm{df}} \{A \mid -A \in \tau\}$.

Theorem 3.1. FCT *proves:* $\mathsf{OTop}(\tau) \leftrightarrow \mathsf{CTop}(\tau_c)$*,* $\mathsf{CTop}(\sigma) \leftrightarrow \mathsf{OTop}(\sigma_c)$*.*

Definition 3.3. Given a class of classes τ, we define the *interior* and *closure* in τ as follows:

$$\begin{aligned} \mathrm{Int}_\tau(A) &=_{\mathrm{df}} \bigcup \{B \in \tau \mid B \subseteq A\} \\ \mathrm{Cl}_\tau(A) &=_{\mathrm{df}} \bigcap \{B \in \tau_c \mid A \subseteq B\} \end{aligned}$$

[1]The requirement that both $\emptyset$ and the ground set be open can meaningfully be reinterpreted in fuzzy logic in several ways; here we restrict ourselves to the weakest one, requiring openness just for the two classes $\emptyset$ and V. Stronger notions of topology (e.g., *stratified* topology [14] with the condition $\alpha\mathrm{V} \in \tau$ for all truth degrees α) will be studied in subsequent papers.

Theorem 3.2. *It is provable in* FCT*:*

(i) $\mathrm{Int}_\tau(A) \subseteq A$

(ii) $A \subseteq B \to \mathrm{Int}_\tau(A) \subseteq \mathrm{Int}_\tau(B)$

(iii) $A \in \tau \to \mathrm{Int}_\tau(A) \approxeq A$

(iv) $\mathrm{Int}_\tau(A \cap B) \cap \mathrm{Int}_\tau(A \cap B) \subseteq \mathrm{Int}_\tau(A) \cap \mathrm{Int}_\tau(B)$

Theorem 3.3 (OTop and the interior operator)**.** *It is provable in* FCT*:*

(i) $\mathsf{OTop}^{0,0,0,1}(\tau) \to \mathrm{Int}_\tau(A) \in \tau$

(ii) $\mathsf{OTop}^{0,0,0,1}(\tau) \to \mathrm{Int}_\tau(\mathrm{Int}_\tau(A)) \approxeq \mathrm{Int}_\tau(A)$

(iii) $\mathsf{OTop}^{0,0,1,0}(\tau) \to \mathrm{Int}_\tau(A) \cap \mathrm{Int}_\tau(B) \subseteq \mathrm{Int}_\tau(A \cap B)$

(iv) $\mathsf{OTop}^{0,1,0,0}(\tau) \to \mathrm{Int}_\tau(\mathrm{V}) \approxeq \mathrm{V}$

Since $\mathrm{Cl}_\tau(A) = -\mathrm{Int}_\tau(-A)$ is provable in FCT, the next two theorems are just dual counterparts of Th. 3.2 and 3.3.

Theorem 3.4. *It is provable in* FCT*:*

(i) $A \subseteq \mathrm{Cl}_\tau(A)$

(ii) $A \subseteq B \to \mathrm{Cl}_\tau(A) \subseteq \mathrm{Cl}_\tau(B)$

(iii) $A \in \tau_c \to \mathrm{Cl}_\tau(A) \approxeq A$

(iv) $\mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B) \subseteq \mathrm{Cl}_\tau(A \cup B) \cup \mathrm{Cl}_\tau(A \cup B)$

Theorem 3.5 (OTop and the closure operator)**.** *It is provable in* FCT*:*

(i) $\mathsf{OTop}^{0,0,0,1}(\tau) \to \mathrm{Cl}_\tau(A) \in \tau_c$

(ii) $\mathsf{OTop}^{0,0,0,1}(\tau) \to \mathrm{Cl}_\tau(\mathrm{Cl}_\tau(A)) \approxeq \mathrm{Cl}_\tau(A)$

(iii) $\mathsf{OTop}^{0,0,1,0}(\tau) \to \mathrm{Cl}_\tau(A \cup B) \subseteq \mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B)$

(iv) $\mathsf{OTop}^{0,1,0,0}(\tau) \to \mathrm{Cl}_\tau(\emptyset) \approxeq \emptyset$

Definition 3.4. A predicate expressing that A is a *neighborhood* of x in τ is defined as

$$\mathrm{Nb}_\tau(x, A) \equiv_{\mathrm{df}} (\exists B \in \tau)(B \subseteq A \;\&\; x \in B)$$

The system of all neighborhoods of x will be denoted by $\nu_x =_{\mathrm{df}} \{A \mid \mathrm{Nb}_\tau(x, A)\}$.

Theorem 3.6 (OTop and neighborhoods)**.** *It is provable in* FCT*:*

(i) $x \in \bigcap \nu_x$

(ii) $\mathrm{Nb}_\tau(x, A) \leftrightarrow x \in \mathrm{Int}_\tau(A)$

(iii) $\mathsf{OTop}(\tau) \to \mathrm{Filter}(\nu_x) \;\&\; (\forall A \in \nu_x)(\exists B \in \nu_x)(B \subseteq A \;\&\; (\forall y \in B)\, \mathrm{Nb}_\tau(y, B))$

In general, the system of all open fuzzy topologies is not closed under arbitrary intersections. Nevertheless, the system of all open $\triangle$–fuzzy topologies is at least closed under crisp intersections, which allows introducing the notion of open fuzzy topology generated by a subbase of fuzzy classes:

Theorem 3.7. *Let $\mathcal{X}$ be a fuzzy class of the third order. Then* FCT *proves:*

$$\mathrm{Crisp}(\mathcal{X}) \mathbin{\&} (\forall \tau \in \mathcal{X})\big(\triangle\, \mathsf{OTop}(\tau)\big) \rightarrow \triangle\, \mathsf{OTop}\big(\bigcap \mathcal{X}\big)$$

Definition 3.5. Let σ be a fuzzy class of fuzzy classes. Then we define

$$\tau_\sigma =_{\mathrm{df}} \bigcap \{\tau' \mid \triangle(\mathsf{OTop}(\tau') \mathbin{\&} \sigma \subseteq \tau')\}$$

By Th. 3.7, FCT proves that $\triangle\,\mathsf{OTop}(\tau_\sigma)$, and obviously also that τ_σ is the least open $\triangle$–fuzzy topology containing σ.

Example 3.2. *Interval open fuzzy topology.* Let $\leq$ be a crisp dense ordering (e.g., of real or rational numbers). The fuzzy properties of being an upper resp. lower class in $\leq$ are defined by the predicates

$$\begin{aligned} \mathrm{Upper}_{\leq}(A) &\equiv_{\mathrm{df}} (\forall p, q)(p \leq q \mathbin{\&} Ap \rightarrow Aq) \\ \mathrm{Lower}_{\leq}(A) &\equiv_{\mathrm{df}} (\forall p, q)(p \geq q \mathbin{\&} Bp \rightarrow Bq) \end{aligned}$$

Fuzzy intervals $[A, B]$ in $\leq$ can be defined [1] as intersections $A \cap B$ of two fuzzy classes A, B, where $\triangle\,\mathrm{Upper}_{\leq}(A) \mathbin{\&} \triangle\,\mathrm{Lower}_{\leq}(B)$. An *open* fuzzy interval can be defined by the following fuzzy predicate:[2]

$$\begin{aligned} \mathrm{Op}([A, B]) \equiv_{\mathrm{df}} \; & \triangle(\mathrm{Upper}_{\leq}(A)) \mathbin{\&} (\forall p)(Ap \rightarrow (\exists q < p)Aq) \mathbin{\&} \\ & \triangle(\mathrm{Lower}_{\leq}(B)) \mathbin{\&} (\forall p)(Bp \rightarrow (\exists q > p)Bq) \end{aligned}$$

By Th. 3.7, the fuzzy system $\sigma = \{[A, B] \mid \mathrm{Op}([A, B])\}$ of open fuzzy intervals generates an open fuzzy topology τ_σ—the *interval open fuzzy topology* of $\leq$. It can be shown that σ itself is $\cap$-closed; since furthermore $\cap$ distributes over $\bigcup$, FCT proves that $\tau_\sigma = \{\bigcup \nu \mid \nu \subseteq \sigma\}$ (just like in the crisp interval topology).

4 Topology given by a neighborhood relation

The following definition of fuzzy topology is an internalization in fuzzy logic of the conditions required from the system of neighborhoods.[3]

Definition 4.1. We define a *neighborhood* (i, j, k, l)*–fuzzy topology* by the predicate

$$\begin{aligned} \mathsf{NTop}^{i,j,k,l}(\mathrm{Nb}) \equiv_{\mathrm{df}} \; & \triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker}\,\mathrm{Pow}(\mathrm{V})) \mathbin{\&} \\ & ((\forall x, A)(\mathrm{Nb}(x, A) \rightarrow x \in A))^i \mathbin{\&} \\ & ((\forall x, A, B)(\mathrm{Nb}(x, A) \mathbin{\&} A \subseteq B \rightarrow \mathrm{Nb}(x, B)))^j \mathbin{\&} \\ & ((\forall x, A, B)(\mathrm{Nb}(x, A) \mathbin{\&} \mathrm{Nb}(x, B) \rightarrow \mathrm{Nb}(x, A \cap B)))^k \mathbin{\&} \\ & ((\forall x, A)(\mathrm{Nb}(x, A) \rightarrow (\exists B \subseteq A)(\mathrm{Nb}(x, B) \mathbin{\&} (\forall y \in B)\,\mathrm{Nb}(y, B)))^l \end{aligned}$$

[2]Observe that it generalizes the notion of crisp open interval, by the requirement of the appropriate left- or right- continuity of the characteristic function of the interval.

[3]The first condition only determines the type of the neighborhood predicate (i.e., that it is a relation between points and classes), therefore its full validity is required.

Definition 4.2. Let $\triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker\,Pow(V)})$. Then we define (as usual) the system of Nb-*open* classes:

$$\tau_{\mathrm{Nb}} =_{\mathrm{df}} \{A \mid (\forall x \in A)\, \mathrm{Nb}(x, A)\}$$

It can be shown that even if Nb is a neighborhood fuzzy topology to degree one, τ_{Nb} still need not be an open fuzzy topology (in particular, it need not be closed under arbitrary unions). Only the following holds:

Theorem 4.1. FCT *proves:* $\triangle\,\mathsf{NTop}(\mathrm{Nb}) \rightarrow (\forall \sigma \subseteq \tau_{\mathrm{Nb}})\,(\bigcup(\sigma \cap \sigma) \in \tau_{\mathrm{Nb}})$.

This motivates a modified notion of open fuzzy topology:

Definition 4.3. We define the following predicates:

$$\begin{aligned} \mathrm{U_2c}(\tau) &\equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\bigcup(\sigma \cap \sigma) \in \tau\big) \\ \mathsf{O_2Top}^{e,v,i,u}(\tau) &\equiv_{\mathrm{df}} (\emptyset \in \tau)^e \;\&\; (\mathrm{V} \in \tau)^v \;\&\; \mathrm{ic}^i(\tau) \;\&\; \mathrm{U_2c}^u(\tau) \end{aligned}$$

Theorem 4.2. FCT *proves:*

$$(\exists x, A)\,\mathrm{Nb}(x, A) \;\&\; \mathsf{NTop}^{1,3,1,1}(\mathrm{Nb}) \rightarrow \mathsf{O_2Top}(\tau_{\mathrm{Nb}}) \;\&\; (\mathrm{Nb}(x, A) \leftrightarrow \mathrm{Nb}_{\tau_{\mathrm{Nb}}}(x, A))$$

Thus, a "sufficiently monotone" non-empty neighborhood topology determines a corresponding open "topology" which is closed under the operation $\bigcup(\sigma \cap \sigma)$ rather than under usual unions $\bigcup \sigma$. Such systems are met quite often in fully graded fuzzy topology:

Example 4.1. It is well-known from traditional fuzzy mathematics that the system of fuzzy sets fully extensional w.r.t. a fuzzy relation R is closed under unions of arbitrary crisp systems of fuzzy sets and under min-intersections of crisp pairs of fuzzy sets (i.e., it forms a fuzzy topology in the traditional, non-graded sense of [16]). In the *graded* framework of FCT it can be proved that the *fuzzy* system of R-extensional classes $\{A \mid \mathrm{Ext}_R\, A\}$ is closed under $\bigcup(\sigma \cap \sigma)$ (but not under arbitrary fuzzy unions), and provided $R \subseteq R \cap R$ (which holds e.g. if R is crisp), it satisfies $\mathsf{O_2Top}$.

Both OTop and $\mathsf{O_2Top}$ topologies are closed under crisp unions, which leads to a further generalization of the notion of open fuzzy topology:

Definition 4.4. We define the following predicates:

$$\begin{aligned} \mathrm{U_{\triangle}c}(\tau) &\equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\mathrm{Crisp}(\sigma) \rightarrow \bigcup \sigma \in \tau\big) \\ \mathsf{O_{\triangle}Top}^{e,v,i,u}(\tau) &\equiv_{\mathrm{df}} (\emptyset \in \tau)^e \;\&\; (\mathrm{V} \in \tau)^v \;\&\; \mathrm{ic}^i(\tau) \;\&\; \mathrm{U_{\triangle}c}^u(\tau) \end{aligned}$$

The models of $\mathsf{O_{\triangle}Top}$ are among frequently studied fuzzy topological structures called "L-fuzzy topologies of Šostak-type" according to the classification proposed in [15].

The definition of the interior operator needs to be modified to have good properties in neighborhood fuzzy topologies:

$$\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) =_{\mathrm{df}} \bigcup\{B \mid \triangle(B \in \tau_{\mathrm{Nb}} \;\&\; B \subseteq A)\}$$

Theorem 4.3. *It is provable in* FCT*:*

(i) $\mathsf{NTop}^{0,1,0,0}(\mathrm{Nb}) \rightarrow \triangle(\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \in \tau_{\mathrm{Nb}})$

(ii) $A \subseteq B \rightarrow \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \subseteq \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(B)$

(iii) $\triangle(A \in \tau_{\mathrm{Nb}}) \rightarrow \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) = A$

Theorem 4.4 (NTop and interior operator)**.** *It is provable in* FCT*:*

(i) $\triangle(\mathrm{V} \in \tau_{\mathrm{Nb}}) \rightarrow \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(\mathrm{V}) = \mathrm{V}$

(ii) $\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \subseteq A$

(iii) $\mathsf{NTop}^{0,1,0,0}(\mathrm{Nb}) \rightarrow \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A)) = \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A)$

(iv) $\mathsf{NTop}^{0,0,1,0}(\mathrm{Nb}) \rightarrow \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \cap \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(B) \subseteq \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A \cap B)$

(v) $\mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A \cap B) \cap \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A \cap B) \subseteq \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(A) \cap \mathrm{Int}'_{\tau_{\mathrm{Nb}}}(B)$

The following theorem guarantees that neighborhoods defined from a (sufficiently union-closed) open fuzzy topology are exactly the neighborhoods in the sense of predicate NTop.

Theorem 4.5 (OTop and NTop)**.** *It is provable in* FCT*:*

$$\mathsf{OTop}^{1,1,1,2}(\tau) \rightarrow \mathsf{NTop}(\mathrm{Nb}_\tau) \mathbin{\&} (A \in \tau \leftrightarrow (\forall x \in A)\, \mathrm{Nb}_\tau(x, A))$$

Example 4.2. *Interval neighborhood fuzzy topology.* The (fuzzy) system of open fuzzy intervals of Example 3.2 allows introducing the *interval neighborhood fuzzy topology* w.r.t. a crisp dense ordering $\leq$, by taking

$$\mathrm{Nb}(x, X) \equiv_{\mathrm{df}} (\exists A, B) \triangle (\mathrm{Op}([A, B]) \mathbin{\&} [A, B] \subseteq X \mathbin{\&} x \in [A, B])$$

Then it can be shown that FCT proves $\triangle\, \mathsf{NTop}(\mathrm{Nb})$, and in virtue of Th. 4.2, $\triangle\, \mathsf{O_2Top}(\tau_{\mathrm{Nb}})$ and $\mathrm{Nb} = \mathrm{Nb}_{\tau_{\mathrm{Nb}}}$. Notice, however, that the interval open topology of Example 3.2 differs from the interval neighborhood topology introduced here, since in the latter all classes open to degree 1 are crisp.

5 Conclusions

We have introduced two notions of fuzzy topology in the graded framework of Fuzzy Class Theory and investigated their basic properties; where appropriate, we gave links to similar notions of fuzzy topology studied previously in traditional fuzzy mathematics. Most of our notions generalize usual concepts of fuzzy topology by allowing full gradedness of all defined predicates and functions. Proofs of the graded theorems, though omitted here due to the limited space, are rather simple and show the strength of the apparatus of higher-order fuzzy logic in fuzzy topology. The results open a wide area of fully graded topological theory and show the possibility of the investigation of more advanced graded topological notions by means of Fuzzy Class Theory.

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References

[1] L. Běhounek. Towards a formal theory of fuzzy Dedekind reals. In E. Montseny and P. Sobrevilla, editors, *Proceedings of the Joint 4th Conference of EUSFLAT and the 11th LFA*, pages 946–954, Barcelona, 2005.

[2] L. Běhounek. Extensionality in graded properties of fuzzy relations. In *Proceedings of 11th IPMU Conference*, pages 1604–1611, Paris, 2006. Edition EDK.

[3] L. Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159:1153–1164, 2008.

[4] L. Běhounek, U. Bodenhofer, and P. Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159:1729–1772, 2008.

[5] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[6] L. Běhounek and P. Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[7] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[8] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[9] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, 2007.

[10] C. L. Chang. Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, 24:182–190, 1968.

[11] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[12] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[13] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[14] U. Höhle. *Many Valued Topology and Its Applications*. Kluwer, Boston, MA, 2001.

[15] U. Höhle and A. Šostak. A general theory of fuzzy topological spaces. *Fuzzy Sets and Systems*, 73:131–149, 1995.

[16] R. Lowen. Fuzzy topological spaces and fuzzy compactness. *Journal of Mathematical Analysis and Applications*, 56:623–633, 1976.

[17] M. Ying. Fuzzy topology based on residuated lattice-valued logic. *Acta Mathematica Sinica (English Series)*, 17:89–102, 2001.

Interior-based topology in Fuzzy Class Theory

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Abstract: Fuzzy topology based on interior operators is studied in the fully graded framework of Fuzzy Class Theory. Its relation to graded notions of fuzzy topology given by open sets and neighborhoods is shown.

Keywords: Fuzzy topology, Fuzzy Class Theory, interior operator, neighborhood.

1 Introduction

Fuzzy topology is a discipline of fuzzy mathematics developed since the beginning of the theory of fuzzy sets [13, 16, 21, 20, 22, 19]. Besides established approaches to fuzzy topology (categorial, lattice-valued, etc.), recent advances in metamathematics of fuzzy logic have enabled an approach to fuzzy topology based on formal fuzzy logic. The framework of higher-order fuzzy logic, also known as Fuzzy Class Theory [4], is especially suitable for fuzzy topology, as it easily accommodates fuzzy sets of fuzzy sets (of arbitrary orders), which are constantly encountered in fuzzy topology.

In classical mathematics, topology can be defined in several equivalent ways: by a system of open (closed) sets, by a system of neighborhoods, or by an interior (closure) operator. These definitions, however, are no longer equivalent in fuzzy logic. Notions of fuzzy topology given by open sets and neighborhoods have been investigated in the framework of Fuzzy Class Theory in [9]. In the present paper we focus on fuzzy topology given by interior operators. Unlike the authors of previous studies of interior and closure operators (e.g., [15, 10, 11]), we work in the fully graded and formal framework of Fuzzy Class Theory, following the methodology of [6]. This approach yields a specific kind of results [8], incomparable to those obtained in traditional fuzzy mathematics: they are on the one hand more general (namely fully graded, i.e., admitting partially valid assumptions), while on the other hand limited to the scope of applicability of deductive fuzzy logic [2].

2 Preliminaries

Fuzzy Class Theory FCT, introduced in [4], is an axiomatization of Zadeh's notion of fuzzy set in formal fuzzy logic. We use its variant defined over $\mathrm{MTL}_\triangle$ [14], the logic of all left-continuous t-norms, which is arguably [2] the weakest fuzzy logic with good inferential properties for fully graded fuzzy mathematics in the framework of formal fuzzy logic [6].

We assume the reader's familiarity with MTL$_\triangle$; for details on this logic see [14]. Here we only recapitulate its standard real-valued semantics:

$\&$	...	a left-continuous t-norm $*$
$\to$	...	the residuum $\Rightarrow$ of $*$, defined as $x \Rightarrow y =_{\mathrm{df}} \sup\{z \mid z * x \leq y\}$
$\wedge, \vee$	...	min, max
$\neg$	...	$x \Rightarrow 0$
$\leftrightarrow$	...	the bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
$\triangle$	...	$\triangle x = 1 - \mathrm{sgn}(1 - x)$
$\forall, \exists$	...	inf, sup

Definition 2.1. *Fuzzy Class Theory* FCT is a formal theory over multi-sorted first-order fuzzy logic (in this paper, MTL$_\triangle$), with the sorts of variables for

- Atomic objects (lowercase letters $x, y, \dots$)
- Fuzzy classes of atomic objects (uppercase letters $A, B, \dots$)
- Fuzzy classes of fuzzy classes of atomic objects (Greek letters $\tau, \sigma, \dots$)
- Fuzzy classes of the 3rd order (in this paper denoted by sans serif letters $\mathsf{A}, \mathsf{B}, \mathsf{a}, \mathsf{b}, \dots$)
- Etc., in general for fuzzy classes of the n-th order $(X^{(n)}, Y^{(n)}, \dots)$

Besides the crisp identity predicate $=$, the language of FCT contains:

- The membership predicate $\in$ between objects of successive sorts
- Class terms $\{x \mid \varphi\}$ of order $n + 1$, for any variable x of any order n and any formula φ
- Symbols $\langle x_1, \dots, x_k \rangle$ for k-tuples of individuals $x_1, \dots, x_k$ of any order

FCT has the following axioms (for all formulae φ and variables of all orders):

- The logical axioms of multi-sorted first-order logic MTL$_\triangle$
- The axioms of crisp identity:

$$
\begin{aligned}
&x = x \\
&x = y \to (\varphi(x) \to \varphi(y)) \\
&\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle \to x_i = y_i
\end{aligned}
$$

- The comprehension axioms:

$$y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$$

- The extensionality axioms:

$$(\forall x) \triangle (x \in A \leftrightarrow x \in B) \to A = B$$

Table 1: Abbreviations used in the formulae of FCT

Ax	$\equiv_{\rm df}$	$x \in A$
$x_1 \dots x_k$	$=_{\rm df}$	$\langle x_1, \dots, x_k \rangle$
$x \notin A$	$\equiv_{\rm df}$	$\neg(x \in A)$
$(\forall x \in A)\varphi$	$\equiv_{\rm df}$	$(\forall x)(x \in A \to \varphi)$
$(\exists x \in A)\varphi$	$\equiv_{\rm df}$	$(\exists x)(x \in A \mathbin{\&} \varphi)$
$(\forall x_1, \dots, x_k \in A)\varphi$	$\equiv_{\rm df}$	$(\forall x_1 \in A) \dots (\forall x_k \in A)\varphi$
$(\exists x_1, \dots, x_k \in A)\varphi$	$\equiv_{\rm df}$	$(\exists x_1 \in A) \dots (\exists x_k \in A)\varphi$
$\{x \in A \mid \varphi\}$	$=_{\rm df}$	$\{x \mid x \in A \mathbin{\&} \varphi\}$
$\{t(x_1, \dots, x_k) \mid \varphi\}$	$=_{\rm df}$	$\{z \mid z = t(x_1, \dots, x_k) \mathbin{\&} \varphi\}$
$y = F(x)$	$\equiv_{\rm df}$	Fxy, if $\triangle \operatorname{Fnc} F$ (see Tab. 2) is proved or assumed
φ^n	$\equiv_{\rm df}$	$\varphi \mathbin{\&} \dots \mathbin{\&} \varphi$ (n times)
$\varphi^\triangle$	$\equiv_{\rm df}$	$\triangle\varphi$

Note that in FCT, fuzzy sets are rendered as a *primitive notion* rather than modeled by membership functions. In order to capture this distinction, fuzzy sets are in FCT called *fuzzy classes;* the name *fuzzy set* is reserved for membership functions in the models of the theory.

The models of FCT are systems of fuzzy sets (and fuzzy relations) of all orders over a crisp universe of discourse, with truth degrees taking values in an $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$ (e.g., the interval $[0, 1]$ equipped with a left-continuous t-norm); thus all theorems on fuzzy classes provable in FCT are true statements about $\mathbf{L}$-valued fuzzy sets. Notice however that the theorems of FCT have to be derived from its axioms by the rules of the fuzzy logic $\mathrm{MTL}_\triangle$ rather than classical Boolean logic. For details on proving theorems of FCT see [7] or [5].

In formulae of FCT we employ usual abbreviations known from classical mathematics or traditional fuzzy mathematics, including those listed in Table 1. Usual rules of precedence apply to the connectives of $\mathrm{MTL}_\triangle$. Furthermore we define standard derived notions of FCT, summarized in Table 2, for all orders of fuzzy classes.

Fuzzy counterparts of classical mathematical notions are in the present paper defined following the methodology sketched in [18, §5] and further elaborated in [4, §7], namely by choosing a suitable formula that expresses the classical definitions and re-interpreting it in fuzzy logic.

A distinguishing feature of FCT is that not only the membership predicate $\in$, but all defined notions are in general fuzzy (unless they are defined as provably crisp). FCT thus presents a fully graded approach to fuzzy mathematics. The importance of full gradedness in fuzzy mathematics is explained in [7, 3, 1]: its main merit lies in that it allows inferring relevant information even when a property of fuzzy sets is not fully satisfied. Fuzzy topology has a long tradition of attempting full gradedness, cf. graded definitions and theorems in [19, 22].

3 Open fuzzy topology

In classical mathematics, topology introduced by means of open sets is given by a crisp system τ of crisp subsets of a ground set V, where τ is required to satisfy certain conditions (closedness under $\bigcup, \cap, \emptyset, \mathrm{V}$, and possibly further properties, e.g., separation axioms).

Table 2: Defined notions of FCT

$\emptyset$	$=_{\mathrm{df}}$	$\{x \mid 0\}$	empty class
V	$=_{\mathrm{df}}$	$\{x \mid 1\}$	universal class
$\operatorname{Ker} A$	$=_{\mathrm{df}}$	$\{x \mid \triangle Ax\}$	kernel
αA	$=_{\mathrm{df}}$	$\{x \mid \alpha \mathbin{\&} Ax\}$	α-resize
$-A$	$=_{\mathrm{df}}$	$\{x \mid \neg Ax\}$	complement
$A \cap B$	$=_{\mathrm{df}}$	$\{x \mid Ax \mathbin{\&} Bx\}$	(strong) intersection
$A \sqcap B$	$=_{\mathrm{df}}$	$\{x \mid Ax \wedge Bx\}$	min-intersection
$A \cup B$	$=_{\mathrm{df}}$	$\{x \mid Ax \veebar Bx\}$	(strong) union
$A \times B$	$=_{\mathrm{df}}$	$\{xy \mid Ax \mathbin{\&} By\}$	Cartesian product
$\operatorname{Rng} R$	$=_{\mathrm{df}}$	$\{y \mid (\exists x)Rxy\}$	range
$\bigcup \tau$	$=_{\mathrm{df}}$	$\{x \mid (\exists A \in \tau)(x \in A)\}$	class union
$\bigcap \tau$	$=_{\mathrm{df}}$	$\{x \mid (\forall A \in \tau)(x \in A)\}$	class intersection
$\operatorname{Pow} A$	$=_{\mathrm{df}}$	$\{X \mid X \subseteq A\}$	power class
$\operatorname{Hgt} A$	$\equiv_{\mathrm{df}}$	$(\exists x)Ax$	height
$\operatorname{Plt} A$	$\equiv_{\mathrm{df}}$	$(\forall x)Ax$	plinth
$\operatorname{Crisp} A$	$\equiv_{\mathrm{df}}$	$(\forall x)\triangle(Ax \vee \neg Ax)$	crispness
$\operatorname{Refl} R$	$\equiv_{\mathrm{df}}$	$(\forall x)Rxx$	reflexivity
$\operatorname{Trans} R$	$\equiv_{\mathrm{df}}$	$(\forall x, y, z)(Rxy \mathbin{\&} Ryz \to Rxz)$	transitivity
$\operatorname{Preord} R$	$\equiv_{\mathrm{df}}$	$\operatorname{Refl} R \mathbin{\&} \operatorname{Trans} R$	preorder
$\operatorname{Fnc} R$	$\equiv_{\mathrm{df}}$	$(\forall x, y, y')(Rxy \mathbin{\&} Rxy' \to y = y')$	functionality
$A \subseteq B$	$\equiv_{\mathrm{df}}$	$(\forall x)(Ax \to Bx)$	inclusion
$A \approx B$	$\equiv_{\mathrm{df}}$	$(A \subseteq B) \wedge (B \subseteq A)$	weak bi-inclusion
$A \approxeq B$	$\equiv_{\mathrm{df}}$	$(A \subseteq B) \mathbin{\&} (B \subseteq A)$	strong bi-inclusion

Generalization by admitting fuzzy subsets leads in FCT to regarding open fuzzy topology as a (possibly fuzzy) class of (possibly fuzzy) subclasses of the ground class V, i.e., a fuzzy class τ of the second order.[1]

When investigating open fuzzy topologies, we are interested in such τ that satisfy analogous (but fuzzified) closure conditions as in classical topology. These are given by the following predicates that express the (degree of) closedness of τ under $\bigcup$ and $\cap$:

$$\mathsf{ic}(\tau) \equiv_{\mathrm{df}} (\forall A, B \in \tau)(A \cap B \in \tau)$$
$$\mathsf{Uc}(\tau) \equiv_{\mathrm{df}} (\forall \sigma \subseteq \tau)\big(\bigcup \sigma \in \tau\big)$$

These conditions (plus $\emptyset \in \tau$ and $\mathrm{V} \in \tau$) can be regarded as characteristic of open fuzzy topology. However, when studying open fuzzy topologies, we do *not* in general require that these axioms be satisfied as in classical topology. This is because they are (like all formulae of FCT) interpreted in many-valued logic; thus they need not be simply true or false, but are always true *to some degree.* By restricting our attention just to the systems that fully satisfy the above axioms, we would completely disregard systems that satisfy them to a degree of, e.g., 0.9999, even though graded theorems of FCT can provide us with useful information about such systems. Therefore we study *all* systems τ, no matter to which degree they satisfy the above axioms. Similarly we proceed also in fuzzification of other definitions of fuzzy topology in the following sections.

[1]We keep the ground class crisp to avoid problems with quantification relativized to a fuzzy domain; generalization to fuzzy topological spaces with fuzzy universes is a topic for future work. Since in this paper we always work within a single topological space, we can identify the ground class with the universal class V.

It turns out [9] that besides the predicate Uc, also predicates of the following forms are often met in the study of fuzzy topology (for $m, n \geq 1$):

$$\mathsf{Uc}_{m,n}(\tau) \equiv_{\mathrm{df}} (\forall\sigma)\big(\sigma \subseteq^m \tau \to \bigcup(\sigma \cap \overset{n}{\ldots} \cap \sigma) \in \tau\big)$$

Note that because $\varphi \mathbin{\&} \varphi$ is not generally equivalent to φ in $\mathrm{MTL}_\triangle$ (nor in stronger fuzzy logics except for Gödel fuzzy logic or stronger), $\sigma \cap \overset{n}{\ldots} \cap \sigma$ does not generally equal σ (only $\sigma \cap \overset{n}{\ldots} \cap \sigma \subseteq \sigma$ holds for all σ). Similarly $(\sigma \subseteq \tau)^m$ is in general stronger than simple $\sigma \subseteq \tau$ if $m > 1$. Recall that the larger m, the stronger φ^m; informally φ^m can be understood as m-times stressed φ (consult, e.g., [7] for the role of multiple conjunction in formal proofs). Thus, like Uc, the predicate $\mathsf{Uc}_{m,n}$ expresses the closedness of τ under a certain operation similar to the union of subsystems, only the condition of what counts as a subsystem is strengthened by m and the union itself is strengthened by n.

By convention, we also admit the value "$\triangle$" for either m or n or both (cf. the last line of Table 1). Then, e.g., $\mathsf{Uc}_{\triangle,1}(\tau)$ expresses the closedness of τ under the unions of *crisp* subsystems of τ, while $\mathsf{Uc}_{1,\triangle}(\tau)$ expresses the closedness of τ under the unions of *kernels* of subsystems of τ (i.e., only full members of the subsystem enter the union).

For convenience, we define a predicate that puts the properties monitored in open fuzzy topologies together. Since each of the properties can appear with varying multiplicity in theorems, we have to add further indices that parameterize the multiplicity of each of the conditions:

Definition 3.1. We define the predicate indicating the degree to which τ is an (e, v, i, u, m, n)*–open fuzzy topology* as

$$\mathsf{OTop}^{e,v,i,u}_{m,n}(\tau) \equiv_{\mathrm{df}} (\emptyset \in \tau)^e \mathbin{\&} (\mathrm{V} \in \tau)^v \mathbin{\&} \mathsf{ic}^i(\tau) \mathbin{\&} \mathsf{Uc}^u_{m,n}(\tau)$$

For the sake of brevity, we drop the subscripts if both equal 1, and similarly for the superscripts.

The properties of open fuzzy topologies have been studied in [9]. Since in this paper we are mainly interested in the interior operator, we repeat here the definition of the interior operator induced by an open fuzzy topology and list its basic properties.

Definition 3.2. Given a class of classes τ, we define the *interior* of a class A in τ as

$$\mathrm{Int}_\tau(A) =_{\mathrm{df}} \bigcup\{B \in \tau \mid B \subseteq A\}$$

Proposition 3.3. *It is provable in* FCT*:*

(i) $\mathrm{Int}_\tau(A) \subseteq A$

(ii) $A \in \tau \to \mathrm{Int}_\tau(A) \approxeq A$

(iii) $A \subseteq B \to \mathrm{Int}_\tau(A) \subseteq \mathrm{Int}_\tau(B)$

(iv) $\mathrm{Int}_\tau(A \sqcap B) \subseteq \mathrm{Int}_\tau(A) \sqcap \mathrm{Int}_\tau(B)$

Proposition 3.4. *It is provable in* FCT*:*

(i) $\mathrm{V} \in \tau \to \mathrm{Int}_\tau(\mathrm{V}) \approxeq \mathrm{V}$

(ii) $\mathsf{Uc}(\tau) \to \mathrm{Int}_\tau(\mathrm{Int}_\tau(A)) \approxeq \mathrm{Int}_\tau(A)$

(iii) $\mathsf{ic}(\tau) \to \mathrm{Int}_\tau(A) \cap \mathrm{Int}_\tau(B) \subseteq \mathrm{Int}_\tau(A \cap B)$

Propositions 3.3 and 3.4 show that the interior operator generated by an open fuzzy topology τ satisfies properties expected from an interior operator—unconditionally in Proposition 3.3, and to a guaranteed degree (depending on the degree to which τ satisfies the conditions required from open fuzzy topologies) in Proposition 3.4.

If the antecedent conditions in Propositions 3.3 and 3.4 are fulfilled to the full degree, so are the conclusions. In particular, we have the following corollary:

Corollary 3.5. FCT *proves:*

(i) $\triangle(A \in \tau) \to \mathrm{Int}_\tau(A) = A$

(ii) $\triangle\,\mathsf{Uc}(\tau) \to \mathrm{Int}_\tau(\mathrm{Int}_\tau(A)) = \mathrm{Int}_\tau(A)$

In words, whenever a fuzzy class A is fully in τ, it equals its interior (no matter what conditions τ does or does not satisfy to which degree). Similarly, if τ is fully closed under fuzzy unions, interiors are stable in τ.

It will further be seen in Section 5 that an open fuzzy topology can vice versa be recovered from a primitive interior operator, under conditions similar to those above.

4 Neighborhood fuzzy topology

In classical mathematics, topology can also be introduced by assigning a system of neighborhoods to each point of a ground set V. Such a neighborhood system can be represented by a relation Nb between elements and subsets of V, where $\mathrm{Nb}(x, A)$ represents the fact that $A \subseteq \mathrm{V}$ is a neighborhood of $x \in \mathrm{V}$. The notion of neighborhood-based fuzzy topology, obtained by fuzzification of the classical notion in FCT, just allows the relation Nb and the class A in $\mathrm{Nb}(x, A)$ to be fuzzy.[2] Thus in FCT, neighborhood fuzzy topologies will be second-order relations between atomic objects and first-order classes, i.e., classes Nb such that $\triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker\,Pow\,V})$.

Neighborhood systems are in classical topology required to satisfy certain conditions. Fuzzified versions of these conditions will be of interest in neighborhood-based fuzzy topology, too:

Definition 4.1. Let Nb be a second-order class such that $\triangle(\mathrm{Nb} \subseteq \mathrm{V} \times \mathrm{Ker\,Pow}(\mathrm{V}))$. Then we define the following predicates:

$$\mathsf{N}_1(\mathrm{Nb}) \equiv_{\mathrm{df}} (\forall x)\,\mathrm{Nb}(x, \mathrm{V})$$
$$\mathsf{N}_2(\mathrm{Nb}) \equiv_{\mathrm{df}} (\forall x, A)(\mathrm{Nb}(x, A) \to x \in A)$$
$$\mathsf{N}_3(\mathrm{Nb}) \equiv_{\mathrm{df}} (\forall x, A, B)(\mathrm{Nb}(x, A) \mathbin{\&} A \subseteq B \to \mathrm{Nb}(x, B))$$
$$\mathsf{N}_4(\mathrm{Nb}) \equiv_{\mathrm{df}} (\forall x, A, B)(\mathrm{Nb}(x, A) \mathbin{\&} \mathrm{Nb}(x, B) \to \mathrm{Nb}(x, A \cap B))$$
$$\mathsf{N}_5(\mathrm{Nb}) \equiv_{\mathrm{df}} (\forall x, A)(\mathrm{Nb}(x, A) \to (\exists B)(B \subseteq A \mathbin{\&} \mathrm{Nb}(x, B) \mathbin{\&} (\forall y \in B)\,\mathrm{Nb}(y, B))$$

For convenience, we aggregate them in the following defined predicate:

[2] We again keep the ground set V crisp for simplicity and identify it with the universal class; see footnote 1.

Definition 4.2. We define the predicate indicating the degree to which Nb is a $(k_1, \ldots, k_5)$–*neighborhood fuzzy topology* as follows:

$$\mathsf{NTop}^{k_1,\ldots,k_5}(\mathrm{Nb}) \equiv_{\mathrm{df}} \mathrm{Nb} \subseteq^{\triangle} \mathrm{V} \times \mathrm{Ker\,Pow}(\mathrm{V}) \mathbin{\&} \mathop{\&}_{i=1}^{5} \mathsf{N}_i^{k_i}(\mathrm{Nb})$$

Basic properties of neighborhood fuzzy topologies and their relation to open fuzzy topologies have been summarized in [9]. Here we restrict our attention to their relationship to interior-based topologies. The following definition internalizes in FCT the classical definition of the interior of a class A:

Definition 4.3. Given a binary predicate Nb between elements and classes, we define

$$\mathrm{Int}_{\mathrm{Nb}}(A) =_{\mathrm{df}} \{x \mid \mathrm{Nb}(x, A)\}$$

The behavior of $\mathrm{Int}_{\mathrm{Nb}}$ w.r.t. Kuratowski's (fuzzified) axioms of interior operators is studied in the following section.

5 Interior fuzzy topology

In classical topology, an interior operator is a function Int that assigns to each subset A of a ground set V a set $\mathrm{Int}(A) \subseteq \mathrm{V}$. In FCT we allow both the argument A and the output $\mathrm{Int}(A)$ of the function to be fuzzy.[3] Fuzzy interior operators are thus construed as crisp second-order functions, i.e., classes Int such that $\mathrm{Int} \subseteq^{\triangle} \mathrm{Ker\,Pow}(\mathrm{V}) \times \mathrm{Ker\,Pow}(\mathrm{V}) \mathbin{\&} \triangle\,\mathrm{Fnc}(\mathrm{Int})$.

The degrees to which Int satisfies (fuzzy versions of) Kuratowski's axioms for interior operators are given by the following predicates:

Definition 5.1. For second-order classes Int such that $\mathrm{Int} \subseteq^{\triangle} \mathrm{Ker\,Pow}(\mathrm{V}) \times \mathrm{Ker\,Pow}(\mathrm{V}) \mathbin{\&} \triangle\,\mathrm{Fnc}(\mathrm{Int})$ we define the following predicates:

$$\begin{aligned}
&\mathsf{K}_1(\mathrm{Int}) \equiv_{\mathrm{df}} \mathrm{Int}(\mathrm{V}) \approxeq \mathrm{V}\\
&\mathsf{K}_2(\mathrm{Int}) \equiv_{\mathrm{df}} (\forall A)(\mathrm{Int}(A) \subseteq A)\\
&\mathsf{K}_3(\mathrm{Int}) \equiv_{\mathrm{df}} (\forall A)(\mathrm{Int}(\mathrm{Int}(A)) \approxeq \mathrm{Int}(A))\\
&\mathsf{K}_4(\mathrm{Int}) \equiv_{\mathrm{df}} (\forall A, B)(\mathrm{Int}(A) \cap \mathrm{Int}(B) \subseteq \mathrm{Int}(A \cap B))
\end{aligned}$$

Unlike in classical topology, in $\mathrm{MTL}_{\triangle}$ these conditions do not imply the monotonicity of Int. Therefore we define also the following predicates:

$$\begin{aligned}
\mathsf{Mon}(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A, B)(A \subseteq B \rightarrow \mathrm{Int}(A) \subseteq \mathrm{Int}(B))\\
\mathsf{K}_5(\mathrm{Int}) &\equiv_{\mathrm{df}} (\forall A, B)(\mathrm{Int}(A \sqcap B) \subseteq \mathrm{Int}(A) \sqcap \mathrm{Int}(B))
\end{aligned}$$

Although Mon and K_5 are not equivalent, the following relationships between them hold:

Proposition 5.2. *It is provable in* FCT*:*

1. $\mathsf{K}_5(\mathrm{Int}) \rightarrow \mathsf{Mon}(\mathrm{Int})$

[3] Again we keep V crisp and identify it with the universal class as in footnote 1. The function Int itself is conceived as crisp as well, to keep the correspondence to logical functions of [17]; if needed, it can be fuzzified by a similarity relation as in [1].

2. $\mathsf{Mon}^2(\mathrm{Int}) \to \mathsf{K}_5(\mathrm{Int})$

3. $\triangle\, \mathsf{K}_5(\mathrm{Int}) \leftrightarrow \triangle\, \mathsf{Mon}(\mathrm{Int})$

For convenience, we gather the conditions K_1–K_5 into one predicate ITop:[4]

Definition 5.3. We define the notion of $(k_1, \ldots, k_5)$*–interior fuzzy topology* by the predicate

$$\mathsf{ITop}^{k_1,\ldots,k_5}(\mathrm{Int}) \equiv_{\mathrm{df}} \mathrm{Int} \subseteq^{\triangle} \mathrm{Ker\,Pow(V)} \times \mathrm{Ker\,Pow(V)} \;\&\; \triangle\,\mathrm{Fnc}(\mathrm{Int}) \;\&\; \mathop{\&}_{i=1}^{5} \mathsf{K}_i^{k_i}(\mathrm{Int})$$

Open classes can be defined by means of the interior operator as usual:

$$\tau_{\mathrm{Int}} =_{\mathrm{df}} \{A \mid A \subseteq \mathrm{Int}(A)\}$$

The following graded theorem shows that if Int satisfies Kuratowski's axioms to a large degree, then τ_{Int} satisfies the properties of open fuzzy topologies to a large degree, and the interior operator generated by τ_{Int} equals Int to a large degree. Notice, however, that we have only got $\mathsf{OTop}_{2,1}(\tau_{\mathrm{Int}})$ rather than $\mathsf{OTop}(\tau_{\mathrm{Int}})$; in other words, we can only prove that the system of classes open w.r.t. a fuzzy Kuratowski interior operator is closed under unions of families "doubly included" in the system.

Theorem 5.4. FCT *proves:*

$$\mathsf{ITop}^{1,1,1,1,2}(\mathrm{Int}) \to \mathsf{OTop}_{2,1}(\tau_{\mathrm{Int}}) \;\&\; (\forall A)(\mathrm{Int}(A) \approxeq \mathrm{Int}_{\tau_{\mathrm{Int}}}(A))$$

Corollary 5.5. FCT *proves:*

$$\triangle\, \mathsf{ITop}(\mathrm{Int}) \to \triangle\, \mathsf{OTop}_{2,1}(\tau_{\mathrm{Int}}) \;\&\; \mathrm{Int} = \mathrm{Int}_{\tau_{\mathrm{Int}}}$$

Vice versa, interiors in well-behaved open fuzzy topologies are well-behaved fuzzy interior operators:

Theorem 5.6. FCT *proves:*

$$\mathsf{OTop}^{0,1,1,1}(\tau) \to \mathsf{ITop}(\mathrm{Int}_\tau) \;\&\; (\forall A)(A \in \tau \leftrightarrow A \subseteq \mathrm{Int}_\tau(A))$$

Corollary 5.7. FCT *proves:*

$$\triangle\, \mathsf{OTop}(\tau) \to \triangle\, \mathsf{ITop}(\mathrm{Int}_\tau) \;\&\; \tau = \tau_{\mathrm{Int}_\tau}$$

Neighborhoods can also be defined by means of the interior operator as usual:

$$\mathrm{Nb}_{\mathrm{Int}}(x, A) \equiv_{\mathrm{df}} x \in \mathrm{Int}(A)$$

It is immediate that $\mathrm{Nb}_{\mathrm{Int}}$ and $\mathrm{Int}_{\mathrm{Nb}}$ of Definition 4.3 are mutually inverse, i.e.,

$$\mathrm{Int} = \mathrm{Int}_{\mathrm{Nb}_{\mathrm{Int}}}$$
$$\mathrm{Nb} = \mathrm{Nb}_{\mathrm{Int}_{\mathrm{Nb}}}$$

Moreover we have the following correspondence between the predicates ITop and NTop:

[4]It is not much important whether we take K_5 or Mon in the definition of ITop, as Proposition 5.2 "translates" between the two variants.

Theorem 5.8. FCT *proves:*

1. $\mathsf{ITop}^{1,2,2,1,1}(\mathrm{Int}) \to \mathsf{NTop}(\mathrm{Nb}_{\mathrm{Int}})$

2. $\mathsf{NTop}^{1,3,3,2,1}(\mathrm{Nb}) \to \mathsf{ITop}(\mathrm{Int}_{\mathrm{Nb}})$

As a corollary, we get the perfect match between the conditions ITop and NTop when true to degree 1:

Corollary 5.9. FCT *proves:*

$$\triangle\,\mathsf{ITop}(\mathrm{Int}) \leftrightarrow \triangle\,\mathsf{NTop}(\mathrm{Nb}_{\mathrm{Int}}), \qquad \mathrm{Int} = \mathrm{Int}_{\mathrm{Nb}_{\mathrm{Int}}}$$
$$\triangle\,\mathsf{NTop}(\mathrm{Nb}) \leftrightarrow \triangle\,\mathsf{ITop}(\mathrm{Int}_{\mathrm{Nb}}), \qquad \mathrm{Nb} = \mathrm{Nb}_{\mathrm{Int}_{\mathrm{Nb}}}$$

We conclude by giving three examples of interior-based fuzzy topology.

Example 5.10. The operation sending a fuzzy class to its kernel is an interior operator that fully satisfies all of Kuratowski's axioms, as FCT proves

$$\begin{aligned} \mathrm{Ker}\,\mathrm{V} &= \mathrm{V} \\ \mathrm{Ker}\,A &\subseteq A \\ \mathrm{Ker}\,\mathrm{Ker}\,A &= \mathrm{Ker}\,A \\ \mathrm{Ker}\,A \cap \mathrm{Ker}\,B &= \mathrm{Ker}(A \cap B) \\ \mathrm{Ker}(A \sqcap B) &= \mathrm{Ker}\,A \sqcap \mathrm{Ker}\,B \end{aligned}$$

by [4, §3.4]. Thus $\triangle\,\mathsf{ITop}(\mathrm{Ker})$; we call it the *kernel fuzzy topology.*

In the kernel fuzzy topology, a class is fully open iff it is crisp: $\triangle(A \in \tau_{\mathrm{Ker}}) \leftrightarrow \mathrm{Crisp}\,A$. Partially open classes are those whose fuzzy elements only have low membership degrees. Since all crisp classes (including singletons) are open in the kernel fuzzy topology, it is a generalization of the notion of *discrete* crisp topology, with which it coincides in 2-valued models.

Example 5.11. Define the interior of A as $(\mathrm{Plt}\,A)\mathrm{V}$ (see Table 2 for the definitions of plinth and resize); i.e., $x \in \mathrm{Int}\,A \equiv_{\mathrm{df}} (\forall y)Ay$. In other words, the membership function of $\mathrm{Int}\,A$ is constant and all elements belong to $\mathrm{Int}\,A$ to the degree which is the infimum of the membership function of A. Then it is provable in FCT that $\triangle\,\mathsf{ITop}(\mathrm{Int})$; we call it the *plinth fuzzy topology.*

A class is fully open in the plinth topology iff it is a resize of the universal class. Thus, the plinth fuzzy topology is *stratified* (stratified topologies are defined as those in which all classes $\alpha\mathrm{V}$ are open [21, 19]). Partially open in the plinth topology are such classes whose membership functions have small amplitudes (i.e., the differences between their suprema and infima), as $\tau_{\mathrm{Int}} = \{A \mid \mathrm{Hgt}\,A \to \mathrm{Plt}\,A\}$. Since the only crisp open classes in the plinth topology are $\emptyset$ and V, it is a generalization of the notion of *anti-discrete* crisp topology (with which it coincides in 2-valued models).

Example 5.12. In [12], an operation of the *opening* of a fuzzy set under a fuzzy relation has been introduced. In [3] the definition has been generalized to the graded framework of FCT and its graded properties have been investigated. The definition can be rephrased as follows:

$$\mathrm{Int}_R(A) =_{\mathrm{df}} \{y \mid (\exists x)(Rxy \,\&\, (\forall z)(Rxz \to Az))\}$$

From results proved in [3] it follows that for any relation R, the operator fully satisfies the conditions K_2, K_3, and K_5. If R is a crisp preorder, then furthermore Int_R fully satisfies K_4. Since $\mathsf{K}_1(\mathrm{Int}_R)$ is equivalent to $\mathrm{V} \subseteq \mathrm{Rng}\, R$, we get

$$\triangle \, \mathrm{Preord}\, R \;\&\; \mathrm{Crisp}\, R \rightarrow \triangle \, \mathsf{ITop}(\mathrm{Int}_R)$$

This result can be generalized to a larger class of fuzzy relations: e.g., instead of crispness, $R = R \cap R$ is sufficient for $\triangle \, \mathsf{K}_4(\mathrm{Int}_R)$ if $\triangle \, \mathrm{Preord}\, R$; both conditions can further be relaxed if ITop is not required to degree 1. Furthermore it is shown in [3] that for any R we have $\mathrm{Int}_R = \mathrm{Int}_{\tau_{\mathrm{Int}_R}}$.

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References

[1] L. Běhounek. Extensionality in graded properties of fuzzy relations. In *Proceedings of 11th IPMU Conference*, pages 1604–1611, Paris, 2006. Edition EDK.

[2] L. Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159:1153–1164, 2008.

[3] L. Běhounek, U. Bodenhofer, and P. Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159:1729–1772, 2008.

[4] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[5] L. Běhounek and P. Cintula. Fuzzy class theory as foundations for fuzzy mathematics. In *Fuzzy Logic, Soft Computing and Computational Intelligence: 11th IFSA World Congress*, volume 2, pages 1233–1238, Beijing, 2005. Tsinghua University Press/Springer.

[6] L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[7] L. Běhounek and P. Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[8] L. Běhounek and P. Cintula. Features of mathematical theories in formal fuzzy logic. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 523–532. Springer, 2007.

[9] L. Běhounek and T. Kroupa. Topology in Fuzzy Class Theory: Basic notions. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, and W. Pedrycz, editors, *Foundations of Fuzzy Logic and Soft Computing (IFSA 2007)*, volume 4529 of *Lecture Notes in Artificial Intelligence*, pages 513–522. Springer, 2007.

[10] R. Bělohlávek and T. Funioková. Fuzzy interior operators. *International Journal of General Systems*, 33(4):415–430, 2004.

[11] R. Bělohlávek and T. Funioková and Vilém Vychodil. Fuzzy closure operators with truth stressers. *Logic Journal of the IGPL*, 13:503–513, 2005.

[12] U. Bodenhofer. A unified framework of opening and closure operators with respect to arbitrary fuzzy relations. *Soft Computing*, 7:220–227, 2003.

[13] C. L. Chang. Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, 24:182–190, 1968.

[14] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[15] W. Gähler. The general fuzzy filter approach to fuzzy topology I, II. *Fuzzy Sets and Systems*, 76:205–224, 225–246, 1995.

[16] J. A. Goguen. The fuzzy Tychonoff theorem. *Journal of Mathematical Analysis and Applications*, 43:734–742, 1973.

[17] P. Hájek. Function symbols in fuzzy logic. In *Proceedings of the East–West Fuzzy Colloquium*, pages 2–8, Zittau/Görlitz, 2000. IPM.

[18] U. Höhle. Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic. *Fuzzy Sets and Systems*, 24(3):263–278, 1987.

[19] U. Höhle. *Many Valued Topology and Its Applications*. Kluwer, Boston, MA, 2001.

[20] U. Höhle and A. Šostak. A general theory of fuzzy topological spaces. *Fuzzy Sets and Systems*, 73:131–149, 1995.

[21] R. Lowen. Fuzzy topological spaces and fuzzy compactness. *Journal of Mathematical Analysis and Applications*, 56:623–633, 1976.

[22] M. Ying. Fuzzy topology based on residuated lattice-valued logic. *Acta Mathematica Sinica (English Series)*, 17:89–102, 2001.

Part III

Mandatory annexes

Abstracts

English abstract

The dissertation consists of the author's published papers on logic-based fuzzy mathematics. It is accompanied with a cover study (Part I of the thesis), which introduces the area of logic-based fuzzy mathematics, argues for the significance of the area of research, presents the state of the art, indicates the author's contribution to the field, and comments on the papers comprising the thesis.

Fuzzy mathematics can be characterized as the study of *fuzzy structures,* i.e., mathematical structures in which the two values 0, 1 are at some points replaced by a richer system of degrees. Under the *logic-based* approach, fuzzy structures are formalized by means of *axiomatic theories* over suitable systems of *fuzzy logic,* whose rules replace the rules of classical logic in formal derivation of theorems. The main advantages of the logic-based approach are the general gradedness of defined notions, methodological clarity provided by the axiomatic method, and the applicability of a foundational architecture mimicking that of classical mathematics. Logic-based fuzzy mathematics is part of a broader area of non-classical mathematics (i.e., mathematical disciplines axiomatizable in non-classical logics), as well as a specific subfield of general fuzzy methods. Following earlier isolated developments in logic-based fuzzy set theory and arithmetic, a systematic logic-based study of fuzzy mathematics was made possible by recent advances of first-order fuzzy logic that opened the way for Henkin-style higher-order fuzzy logic (or simple fuzzy type theory), which is capable of serving as a foundational theory for logic-based fuzzy mathematics. The author's contribution to the development of logic-based fuzzy mathematics has been presented in the published papers that comprise the main body of the thesis.

The paper *On the difference between traditional and deductive fuzzy logic* clarifies methodological assumptions of formal fuzzy logic, contrasts them to those of traditional fuzzy mathematics, and indicates necessary conditions on systems of fuzzy logic suitable for logic-based fuzzy mathematics as developed in this thesis. The paper *From fuzzy logic to fuzzy mathematics: a methodological manifesto* (co-authored by P. Cintula) formulates methodological guidelines for logic-based fuzzy mathematics and proposes a foundational architecture analogous to that of classical mathematics, with three layers formed by first-order fuzzy logic, a foundational theory axiomatized in fuzzy logic, and particular mathematical disciplines developed within the foundational theory.

The paper *Fuzzy class theory* (co-authored by P. Cintula) introduces Henkin-style higher-order fuzzy logic (also called Fuzzy Class Theory or FCT) as an axiomatic approximation of Zadeh's notion of fuzzy set, and proposes it as a foundational theory for logic-based fuzzy mathematics. Metatheorems are proved for FCT that reduce a large part of elementary fuzzy set theory to propositional fuzzy logic, and the interpretability

of classical higher-order theories in FCT (by which classical mathematical structures are available within the theory) is shown in the paper.

The paper *Relations in Fuzzy Class Theory: initial steps* (co-authored by U. Bodenhofer and P. Cintula) develops the basic theory of fuzzy relations in FCT, which is a prerequisite for all other parts of formal fuzzy mathematics. The topics studied include basic graded properties of fuzzy relations, relational images and bounds, Valverde characterization theorems, and fuzzy partitions. The paper *Relational compositions in Fuzzy Class Theory* (co-authored by M. Daňková) reduces a large family of fuzzy relational and set-theoretical notions to fuzzy relational compositions, and presents methods for mass proofs of theorems on these notions. The paper *Extensionality in graded properties of fuzzy relations* introduces indistinguishability-relative graded properties of fuzzy relations and studies their relationship to the property of extensionality, to which they reduce in traditional fuzzy mathematics, but not in the logic-based setting.

The paper *Towards a formal theory of fuzzy Dedekind reals* constructs fuzzy real numbers as the lattice completion of the classical real line by fuzzy Dedekind cuts and gives some hints for logic-based fuzzy interval arithmetics. The paper *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* employs FCT as the formal semantics for a logic of fuzzy questions. Finally, the papers *Topology in Fuzzy Class Theory: basic notions* and *Interior-based topology in Fuzzy Class Theory* (both co-authored by T. Kroupa) introduce logic-based notions of fuzzy topology defined respectively by open or closed sets, neighborhoods, and interior operators, and study their mutual relationships.

Český abstrakt (Czech abstract)

Předložená disertační práce sestává z autorových publikovaných článků o logických základech fuzzy matematiky, doplněných shrnující studií (tvořící úvodní část disertace), ve které je představen na formálnělogický přístup k fuzzy matematice. Dále je v ní dokládána důležitost výzkumu v tomto oboru a charakerizován jeho současný stav, popsán autorův příspěvek k oboru a podány komentáře k jednotlivým článkům, z nichž se disertace skládá.

Fuzzy matematiku lze vymezit jako studium *fuzzy struktur,* tj. takových matematických struktur, v nichž je dvojice hodnot 0, 1 na některých místech nahrazena bohatším systémem stupňů. V přístupu založeném na formální logice jsou fuzzy struktury zachyceny prostřednictvím *axiomatických teorií* ve vhodných systémech *fuzzy logiky,* jejichž pravidla jsou použita pro formální odvozování teorémů namísto pravidel klasické logiky. Hlavními výhodami logického přístupu k fuzzy matematice jsou všeobecná gradualita definovaných pojmů, metodologická čistota daná aplikací axiomatické metody a použitelnost podobné základové architektury jako v klasické matematice. Na logice založená fuzzy matematika je součástí neklasické matematiky (tj. rodiny matematických teorií axiomatizovatelných v neklasických logikách), a zároveň tvoří specifickou část širšího oboru fuzzy metod. Systematické zkoumání fuzzy matematiky v přístupu založeném na logice, navazující na předchozí ojedinělé výzkumy podobného přístupu k teorii fuzzy množin a aritmetice, bylo umožněno nedávným pokrokem v oblasti prvořádové fuzzy logiky. Díky němu bylo možno vyvinout henkinovskou fuzzy logiku vyššího řádu (čili jednoduchou fuzzy teorii typů), jež může sloužit jako základová teorie pro formální fuzzy matematiku. Autorovy příspěvky k výzkumu logických základů fuzzy matematiky byly publikovány v článcích, které tvoří hlavní část disertace.

Článek *On the difference between traditional and deductive fuzzy logic* (K rozdílu mezi tradiční a deduktivní fuzzy logikou) vyjasňuje metodologické předpoklady formální fuzzy logiky ve srovnání s předpoklady tradiční fuzzy matematiky a stanovuje požadavky na systémy fuzzy logiky vyhovující takovému přístupu k fuzzy matematice, jaký je rozvíjen v této disertaci. V článku *From fuzzy logic to fuzzy mathematics: a methodological manifesto* (Od fuzzy logiky k fuzzy matematice – metodologický manifest, spoluautor P. Cintula) jsou formulovány metodologické zásady na logice založeného přístupu k fuzzy matematice a je navržena její základová architektura způsobem analogickým k základům klasické matematiky, se třemi vrstvami tvořenými prvořádovou fuzzy logikou, v ní axiomatizovanou základovou teorií a jednotlivými matematickými disciplínami vyvíjenými v rámci této základové teorie.

V článku *Fuzzy class theory* (Teorie fuzzy tříd, spoluautor P. Cintula) je zavedena henkinovská fuzzy logika vyššího řádu (zvaná též teorie fuzzy tříd, zkr. FCT z angl. Fuzzy Class Theory), jakožto axiomatická aproximace Zadehova pojmu fuzzy množiny. Tato teorie je zde navržena za základovou teorii pro formální fuzzy matematiku. V článku jsou dokázány metavěty FCT, které redukují značnou část elementární teorie fuzzy množin na výrokovou fuzzy logiku, a je ukázána interpretovatelnost klasických teorií vyššího řádu v FCT (díky níž jsou v FCT k dispozici klasické matematické struktury).

V článku *Relations in Fuzzy Class Theory: initial steps* (Relace v teorii fuzzy tříd – počáteční kroky, spoluautoři U. Bodenhofer a P. Cintula) jsou v rámci FCT vybudovány základy teorie fuzzy relací, jež tvoří nezbytný předpoklad zkoumání ostatních partií fuzzy matematiky. V článku se zkoumají zejména základní graduální vlastnosti fuzzy relací, obrazy, závory, valverdovské charakterizace a fuzzy rozklady. V článku *Relational compositions in Fuzzy Class Theory* (Skládání relací v teorii fuzzy tříd, spoluautorka M. Daňková) popisuje redukci rozsáhlé rodiny pojmů teorie fuzzy relací a fuzzy množin na pojem skládání fuzzy relací a ukazuje metodu hromadných důkazů vět o těchto pojmech. Článek *Extensionality in graded properties of fuzzy relations* (Extenzionalita u graduálních vlastností fuzzy relací) zavádí graduální vlastnosti fuzzy relací definované relativně vůči dané relaci nerozlišitelnosti a studuje jejich vztah k vlastnosti extenzionality, s níž v tradiční fuzzy matematice splývají, v přístupu založeném na logice se však od ní liší.

Článek *Towards a formal theory of fuzzy Dedekind reals* (Předběžné poznámky k formální teorii dedekindovských fuzzy reálných čísel) podává konstrukci fuzzy reálných čisel pomocí svazového zúplnění klasické reálné číselné osy fuzzy dedekindovskými řezy a uvádí některé výsledky potřebné k vybudování fuzzy intervalové aritmetiky. V článku *Fuzzification of Groenendijk–Stokhof propositional erotetic logic* (Fuzzifikace výrokové Groenendijkovy–Stokhofovy erótetické logiky) je aparát FCT použit jako formální sémantika pro logiku fuzzy otázek. V závěrečných článcích *Topology in Fuzzy Class Theory: basic notions* (Topologie v teorii fuzzy tříd – základní pojmy) a *Interior-based topology in Fuzzy Class Theory* (Topologie definovaná pomocí operátoru vnitřku v teorii fuzzy tříd, spoluautor obou článků T. Kroupa) jsou v rámci přístupu založeném na logice zavedeny pojmy fuzzy topologie definované pomocí otevřených či uzavřených množin, okolí bodů a operátoru vnitřku a prozkoumány jejich vzájemné vztahy.