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Report on the doctoral thesis "Variational strategies in material sciences: Analysis & Numerics" by Antonín Češík

Dear Professor Rokyta.

I am delighted by having the possibility of reviewing Antonín Češík's doctoral thesis, which I consider to be a very interesting piece of work.

Overview on the thesis. Antonín Češík's thesis focuses on the discussion of different analytical and numerical aspects of the dynamics of viscoelastic materials at large deformations. One is interested in proving the existence of weak solutions to hyperbolic viscoelastic systems in presence of self-contact or contact with an incompressible viscous fluid under slip boundary conditions.

The key technical point used to tackle these challenging, highly nonlinear problems is a two-scale approximation procedure, originally introduced in B. Benesová, M. Kampschulte, and S. Schwarzacher, A variational approach to hyperbolic evolutions and fluid-structure interactions, *J. Eur. Math. Soc. (JEMS)*, 26 (2024), no. 12, 4615–4697. This consists in an approximation of the hyperbolic problem by a sequence of parabolic problems with delay, combined with a second-level time discretization. This two-scale approximation technique offers an alternative to more classical *local* PDE methods, allowing to take *global* constraints into the picture.

This potential is fully exploited in Antonín Češík's thesis by considering contact and fluid-structure interaction questions. Moreover, the thesis provides some theoretical numerical counterpart for this approximation procedure, proving stability and, in some circumstances, optimal a-priori error estimates.

Structure of the work. The thesis consists in a collection of four papers, of which one is already published on *Calculus of Variations and Partial Differential Equations*, and three are currently submitted.

Paper 1 is centered on the existence for all times for the dynamic problem of hyperelasticity, past possible self-contact events. The material is assumed to be viscous and of so-called second-grade type. Contact naturally calls for augmenting the state of the system by additionally taking the reaction force at contact points into account. Such a strategy, already followed by A. Z. Palmer and T. Healey in the static case and by S. Krömer and T. Roubíček in the quasistatic case, is here eventually brought to the hyperbolic case.

Paper 2 is a follow-up of Paper 1 targeting the relevant case of solids with merely Lipschitz continuous boundaries. This extension requires a delicate discussion on normal and tangent material directions and the corresponding allowed perturbations, eventually restricting the set of admissible tests. The existence result is in the same direction of the one from Paper 1, asking for considering the additional unknown reaction force which is supported at the contact boundary.

Paper 3 investigates the stability and convergence of the two-scale time discretization of the dynamic problem of hyperelasticity in presence of both viscosity and second-grade regularization terms. A specific technical issue here is the nonconvexity of the energy, which is nonetheless dominated by the second-order nature of the elastic response. In the general case, the time discretization is proved to be unconditionally stable. In case the highest-order spatial term is linear, an optimal linear convergence rate is proved to hold.

Paper 4 considers a fluid-structure interaction problem. The two-scale approximation method has already been used in the setting of fluid-structure interaction the paper by B. Benesová, M. Kampschulte, and S. Schwarzacher mentioned above. In particular, they prove global existence for Navier-Stokes incompressible fluids and dynamic, hyperelastic, viscous, second-grade solids under the no-slip boundary conditions at the contact. This is here extended to slip boundary conditions, by just requiring the coincidence of normal components and fluid and solid velocities. This extension is physically very relevant and mathematically quite challenging. In particular, it calls for dealing with some additional class of test functions, specifically probing the fluid-dynamic subproblem. The main result is the existence of a weak solution, up to the first collision time. I expect that the treatment of contact will be included in some further work, allowing to prove the existence of a global weak solution.

I find the thematic focus of the thesis to be extremely coherent. On the one hand, all papers deal with the same material model, which is fairly general, albeit necessarily regularized by viscous and second-grade effects. On the other hand, the basic two-scale approximation technique is underpinning all results. Still, the thesis offers a fresh view on different and very relevant issues.

Scholarly merit. I believe the results of this thesis to be original, interesting, and timely, definitely a contribution to knowledge. The reported results are not routine applications of known tools but required original developments, which are surely to attract attention and developments in the future.

One has to mention that the thesis offers a first solution to the long-standing open issue of self-contact in the hyperbolic case. The thesis proposes an original take on this problem, allowing for the first time to consider post-collision, global-in-time dynamics.

At the same time, existence for slip conditions in dynamic large-deformation theories was unprecedented. In this direction, the thesis might open the way to consider friction conditions, with potentially even wider application.

All results of the thesis ultimately contribute to foundational issues in solid mechanics and fluid-structure interactions. As such, the results may well be expect to potentially have impact on more applied areas besides mathematics.

Referencing to the relevant literature is rather complete and the presented results are well-placed into context. Proofs are presented in good detail and are, to the best of my understanding, correct. In addition to the four papers, Antonín Češík provides a short introduction to the thesis, which I find very well organized and informative.

As it is customary in research in mathematics, it is impossible to distinguish the contribution of each single author in a collaboration. I nonetheless had the opportunity of extensively discussing the results contained in the thesis with Antonín Češík and I have no doubts about his ability of individually performing creative scientific work.

Concluding statement (quotable). I believe Antonín Češík's doctoral work to be of very high quality. The results obtained are remarkable, contributing first answers to long-standing questions in the field. The developed techniques are delicate and bound to attract further attention in the near future. Antonín Češík's results are of a foundational flavor in solid and fluid mechanics. As such, they have relevance also beyond the purely mathematical realm.

I have no hesitation in recommending the thesis for acceptance.

Kind regards,

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