Attachments

Paper I

INERTIAL EVOLUTION OF NON-LINEAR VISCOELASTIC SOLIDS IN THE FACE OF (SELF-)COLLISION

ANTONÍN ČEŠÍK[†], GIOVANNI GRAVINA[‡,*], AND MALTE KAMPSCHULTE[†]

Abstract. We study the time evolution of non-linear viscoelastic solids in the presence of inertia and (self-)contact. For this problem we prove the existence of weak solutions for arbitrary times and initial data, thereby solving an open problem in the field. Our construction directly includes the physically correct, measure-valued contact forces and thus obeys conservation of momentum and an energy balance. In particular, we prove an independently useful compactness result for contact forces.

1. Introduction

The study of contact and dynamic collisions between (elastic) bodies has a long history, starting from classical, 18th-century, physical considerations about conservation of energy and momentum and ranging into modern continuum mechanics. However in this, and in particular in the mathematical treatment of the latter, until now, there has always been a divide between more phenomenological and *ab initio* approaches.

This divide is a direct consequence of the difficulty and irreducibility of the full problem. If one is not able to fully treat dynamic contact between deformable elastic bodies, then one has to simplify the problem in one of several directions.

The first is to remove the ability of the bodies to deform, treating them as rigid bodies. However, when this is done, the problem immediately becomes ill-posed and needs to be supplemented by a phenomenological contact law. The second is to soften the contact itself, replacing the hard dichotomy of "in contact" vs. "not in contact" with a soft repulsion potential. Yet this also introduces an indirect contact law. The third possibility is to remove the dynamic aspects and focus on the static or quasistatic situation instead.

Additionally, even in this last case, there is a difference in difficulty between the collision of an elastic solid with a static obstacle and the collision of two elastic solids with each other or even an elastic solid deforming so far as to collide with itself. In this work for the first time, we prove the existence of weak solutions to this general case involving inertia and large deformations.

Specifically, consider one or more elastic solids, given in Lagrangian coordinates by a reference configuration $Q \subset \mathbb{R}^n$ (with multiple solids represented by multiple connected components of Q), as well as a deformation $\eta\colon Q \to \Omega \subset \mathbb{R}^n$ to describe the current configuration. To each deformation we attach an elastic energy $E(\eta)$ and to each change in configuration a dissipation potential $R(\eta, \partial_t\eta)$, which for physical reasons has to also depend on the deformation itself [2].

If the solid has density ρ in the reference configuration, then by Newton's second law we expect the solid to evolve according to

$$\rho\partial_{tt}\eta + DE(\eta) + D_2R(\eta, \partial_t\eta) = f,$$

where DE and D_2R denote the formal Fréchet derivatives with respect to η and $\partial_t\eta$ respectively, and f is an external force.

Of course this only works in the absence of collisions. As alluded to before, to deal with collisions, there needs to be another modeling assumption, which in turn needs to be justified. We choose here to adopt the absolute *minimal assumption*, which is also the only assumption that we can be sure holds universally, namely *non-interpenetration of matter*. Translated into the mathematical framework this

[†]Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

[‡]School of Mathematics and Statistical Sciences, Arizona State University, Tempe, AZ 85281

E-mail addresses: cesik@karlin.mff.cuni.cz, ggravina@asu.edu, kampschulte@karlin.mff.cuni.cz.

Date: December 15, 2023.

2020 *Mathematics Subject Classification.* 35Q74, 49J40, 74A30, 74B20, 74M15.

Key words and phrases. Non-interpenetration of matter, variational inequalities, non-simple materials.

Author ORCIDs. Antonín Češík: https://orcid.org/0000-0003-0207-6044, Giovanni Gravina: https://orcid.org/0000-0001-8985-7964, Malte Kampschulte: https://orcid.org/0000-0002-9150-4460.

means that the only additional information we assume in modeling is that η is injective, except possibly for a set of measure zero.[1]

Such a restriction of the set of admissible deformations by its very nature results in a Lagrange-multiplier, which we can readily identify as the contact force σ. This force has to be supported on the contact set and because of considerations involving conservation of energy and momentum, it has to be of equal magnitude and opposite direction at each pair of points where solids touch. Additionally, if we do not assume any friction, it has to point in the same direction as the respective interior normal of the physical configuration.

With this, we can summarize the system under study as

$$\begin{cases} \rho\partial_{tt}\eta + DE(\eta) + D_2R(\eta,\partial_t\eta) = \sigma + f, \\ \sigma = |\sigma| n_\eta, \ \operatorname{supp}(\sigma) \subset C_\eta, \end{cases} \tag{1.1}$$

where C_η is the set of (self-)contact points and n_η denotes the interior normal of the deformed configuration. Additionally, we require an "equal and opposite" assumption on the contact force, which is easier to write in weak form (see Definition 3.6 below). We can then give an abridged version of out main result as

Theorem (Theorem 2.5 (abridged))**.** *Under some (physical) assumptions on E and R, for any sufficiently reasonable initial data (injective a.e. and of finite energy) and any time $T > 0$, the system (1.1) has a weak solution in the interval $[0,T]$. This solution obeys conservation of momentum and the physical energy inequality.*

We mention here that in the full result we also treat the case in which the evolution happens in the presence of rigid, immovable obstacles.

The proof of the theorem relies mainly on variational methods. On one hand, this is not surprising, as more classical PDE methods (e.g., fixed-points or Galerkin approaches) are often inadequate to handle the difficulties that arise from the fact that the non-interpenetration constraint results in a non-convex, non-linear state space. On the other hand, so far, variational methods have generally been restricted to static and quasistatic systems. Indeed, for the problem at hand, we build quite explicitly on the work [15] by Palmer and Healey, where the authors study self-contact for the static case. Their results have been recently extended to the quasistatic case by Krömer and Roubíček [13].[2]

The crucial ingredient in our proofs is the method of using two time-scales developed in [3], which shows a way to lift almost any quasistatic weak existence result to a corresponding weak existence result for the associated inertial problem.

For the details of this approach, we also refer to [3, Sec. 3], where the corresponding result without collisions is shown (see also the introductory sections of [4], where an attempt is made to elaborate on some of the more general underpinnings of this method). However, we aim to keep the use of this method in the current paper self-contained.

Finally we note that throughout the paper we will restrict our attention to generalized second order materials (compare in particular with [10] and the references therein), i.e., we assume that the elastic energy will depend on the second derivative $\nabla^2\eta$ of the deformation. While at first glance this might seem like a departure from the classical theory of elasticity, we note that for the study of (self-)contact such a restriction is necessary. Indeed, not only is it needed to avoid issues arising from points where the Jacobian $\det \nabla\eta$ vanishes, but also because without the resulting C^1-regularity that the theory implies, microscopic oscillations of the exterior normal can lead to boundary microstructure and produce artificial friction (see Remark 3.11), all of which is outside of the scope of the current paper.

1.1. **Structure of the paper.** In Section 2, we will give a precise definition of the assumptions we require in terms of energy and the dissipation, as well as a precise statement of the main theorem and some possible extensions and corollaries. Next, in Section 3, we will go on and derive some of the properties related to contact forces as well as their convergence behavior, which might be of independent interest. The bulk of the paper will then be devoted to the proof of the main theorem, first by showing an auxiliary quasistatic result in Section 4 and then by using this to generate an approximating sequence of time-delayed solutions to the actual equation in Section 5.

[1] In the absence of rigid bodies and point-masses this turns out to also be a sufficient assumption. For more details see the discussion in Section 6.

[2] As a necessary step towards the proof, we also improve their result to give a quantization of the contact force as a measure, which in [13] was only characterized as part of a distribution. We thus in fact solve both of their open problems (See Remark 5.1). In particular, we believe that the more detailed treatment of convergence of contact forces used for this might be of independent interest.

Finally, in Section 6 we then relate this result to the actual physics we are aiming to describe by giving an example energy-dissipation pair that satisfies the assumptions and by discussing how the result is connected to momentum conservation. We also show by an example that the condition of non-interpenetration of matter that we use is indeed sufficient and there are no phenomenological contact laws needed to arrive at the correct behavior.

1.2. **Notation.** Throughout this paper we use the following notation, unless stated otherwise.

Deformations will be denoted by $\eta\colon Q \to \Omega$ or $\eta\colon [0,T]\times Q \to \Omega$ in the case of time-dependence, where $Q \subset \mathbb{R}^n$ is a reference configuration, and $\Omega \subset \mathbb{R}^n$ a possibly unbounded containing domain.

Points in Q are notated as x. Functions which depend on space and time are always notated time first e.g. $\eta(t,x)$. If we want to consider deformations for a fixed time, we will also use $\eta(t) := \eta(t,\cdot)$ to ease notation.

For any $x \in \partial Q$, the vector $n_Q(x)$ denotes the interior unit normal to ∂Q at x. Given additionally a sufficiently regular deformation $\eta\colon Q \to \Omega$, we will use

$$n_\eta(x) := \frac{\operatorname{cof} \nabla\eta(x) n_Q(x)}{|\operatorname{cof} \nabla\eta(x) n_Q(x)|}$$

to denote the interior unit normal of $\eta(Q)$ at $\eta(x)$. In case η is also time-dependent we use $n_\eta(t,x)$ in place of $n_{\eta(t)}(x)$. Finally for $y \in \partial\Omega$, the vector $n_\Omega(y)$ denotes the *exterior*[3] unit normal at y.

We use the usual notations $W^{k,p}(Q;\mathbb{R}^n)$ for Sobolev spaces and $L^q((0,T);W^{k,p}(Q;\mathbb{R}^n))$ for the respective Bochner spaces. A subscript Γ is used to denote spaces of functions whose trace vanishes on that set, e.g. $W^{k,p}_\Gamma(Q) := \{u \in W^{k,p}(Q) : u|_\Gamma = 0\}$. Additionally, we denote by $M(K;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued Radon measures and by $M^+(K)$ the set of non-negative Radon measures on a compact set K. For a measure $\sigma \in M(K;\mathbb{R}^n)$ and $\varphi \in C(K;\mathbb{R}^n)$ we denote

$$\langle \sigma, \varphi\rangle := \int_K \varphi \cdot d\sigma.$$

Spaces are written out in full (e.g., $L^2((0,T);W^{1,2}(Q;\mathbb{R}^n)))$, but when writing norms, we usually omit both the domain and the image if there is no chance of confusion (e.g., we write $\|f\|_{L^2}$ instead of $\|f\|_{L^2(Q;\mathbb{R}^n)}$). Additionally, when dealing with linear operators on Sobolev spaces, we write the linear argument in angled brackets, e.g.

$$A\langle u\rangle := \langle A, u\rangle_{(W^{k,p})^*\times W^{k,p}}$$

for $A\colon W^{k,p}(Q) \to \mathbb{R}$ and $u \in W^{k,p}(Q)$.

1.3. **Acknowledgements.** The authors acknowledge the support of the the Primus research programme of Charles University under grant No. PRIMUS/19/SCI/01. The research of A.Č. and M.K. was partly funded by the ERC-CZ grant LL2105. A.Č. further acknowledges the support of Charles University, project GA UK No. 393421. The work of G.G. and M.K. was partially supported by the Charles University research program No. UNCE/SCI/023 and by the Czech Science Foundation (GAČR) under grant No. GJ19-11707Y.

1.4. **Research data policy and data availability statements.** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

2. Modelling of viscoelastic materials and statement of the main results

2.1. **Viscoelastic solids.** The time evolution of a viscoelastic solid body in $\mathbb{R}^n$ can be described in Lagrangian coordinates by a (time dependent) deformation of a reference configuration $Q \subset \mathbb{R}^n$, which in the following we typically denote by $\eta\colon [0,T]\times Q \to \mathbb{R}^n$. The set $Q \subset \mathbb{R}^n$ will be a $C^{1,\alpha}$-smooth, bounded domain, or alternatively a disjoint union of finitely many of such domains in order to describe multiple bodies. We assume that the movement of the solid is confined to the set $\Omega \subset \mathbb{R}^n$ which is a $C^{1,\alpha}$-smooth domain, but possibly unbounded (e.g. $\Omega = \mathbb{R}^n$, half-space, etc.). Furthermore, in order to rule out non-physical phenomena such as self-interpenetration, we restrict our attention to deformations that are almost everywhere globally injective and orientation preserving. These assumptions are encoded in the class of admissible deformations, which we define using the Ciarlet–Nečas condition [5] as

$$\mathcal{E} := \left\{ \eta \in W^{2,p}(Q;\mathbb{R}^n) : \eta(Q) \subset \Omega, \eta|_\Gamma = \eta_0, \det\nabla\eta > 0, \text{ and } \mathcal{L}^n(\eta(Q)) = \int_Q \det\nabla\eta(x)\,dx \right\}. \tag{2.1}$$

[3]To reduce the distinction between cases, it is best to not think of Ω as the domain, but of $\mathbb{R}^n \setminus \Omega$ as a fixed, rigid obstacle. Thus n_Ω is the interior normal of that obstacle, in the same way n_η is the interior normal of the movable solids.

Here we use η_0 to denote a given admissible (initial) deformation and let Γ be a (fixed) measurable subset of ∂Q. Note, however, that for the main result of this paper we do not assume that $\mathcal{H}^{n-1}(\Gamma) > 0$ and refer the reader to Remark 2.2 for more information. Here and in the following we assume that $p > n$. In particular, this implies that every $\eta \in \mathcal{E}$ admits a representative of class $C^{1,1-\frac{n}{p}}$. Throughout the rest of the paper we identify η with this regular representative without further notice. Additionally, we assume a constant Lagrangian density $\rho \in (0, \infty)$, but remark that all our arguments also work for variable densities, as long as these are bounded from above and away from zero.

Next, we specify the assumptions on the energy-dissipation pair (E, R). To be precise, we assume that the elastic energy $E\colon W^{2,p}(Q;\mathbb{R}^n) \to (-\infty, \infty]$ has the following properties:

(E.1) There exists $E_{\min} > -\infty$ such that $E(\eta) \geq E_{\min}$ for all $\eta \in W^{2,p}(Q;\mathbb{R}^n)$. Moreover, $E(\eta) < \infty$ for every $\eta \in W^{2,p}(Q;\mathbb{R}^n)$ with $\inf_Q \det \nabla\eta > 0$.

(E.2) For every $E_0 \geq E_{\min}$ there exists $\varepsilon_0 > 0$ such that $\det \nabla\eta \geq \varepsilon_0$ for all η with $E(\eta) \leq E_0$.

(E.3) For every $E_0 \geq E_{\min}$ there exists a constant C such that

$$\|\nabla^2 \eta\|_{L^p} \leq C$$

for all η with $E(\eta) < E_0$.

(E.4) E is weakly lower semicontinuous, that is,

$$E(\eta) \leq \liminf_{k\to\infty} E(\eta_k)$$

whenever $\eta_k \rightharpoonup \eta$ in $W^{2,p}(Q;\mathbb{R}^n)$. Moreover, E is continuous with respect to strong convergence in $W^{2,p}(Q;\mathbb{R}^n)$.

(E.5) E is differentiable in its effective domain with derivative $DE(\eta) \in (W^{2,p}(Q;\mathbb{R}^n))^*$ given by

$$DE(\eta)\langle\varphi\rangle = \frac{d}{d\varepsilon} E(\eta + \varepsilon\varphi))\bigg|_{\varepsilon=0}.$$

Furthermore, DE is bounded on any sub-level set of E and $DE(\eta_k)\langle\varphi\rangle \to DE(\eta)\langle\varphi\rangle$ whenever $\eta_k \to \eta$ in $W^{2,p}(K;\mathbb{R}^n)$ for all K compactly contained in $\overline{Q}$ with $\operatorname{dist}(K,\Gamma) > 0$ and $\varphi \in W^{2,p}_\Gamma(Q;\mathbb{R}^n)$.

(E.6) DE satisfies

$$\liminf_{k\to\infty} (DE(\eta_k) - DE(\eta))\langle(\eta_k - \eta)\psi\rangle \geq 0$$

for all $\psi \in C^\infty_\Gamma(Q;[0,1])$ and all sequences $\eta_k \rightharpoonup \eta$ in $W^{2,p}(Q;\mathbb{R}^n)$. In addition, DE satisfies the following Minty-type property: If

$$\limsup_{k\to\infty} (DE(\eta_k) - DE(\eta))\langle(\eta_k - \eta)\psi\rangle \leq 0$$

for all $\psi \in C^\infty_\Gamma(Q;[0,1])$, then $\eta_k \to \eta$ in $W^{2,p}(K;\mathbb{R}^n)$ for all K compactly contained in $\overline{Q}$ with $\operatorname{dist}(K,\Gamma) > 0$.

Additionally, we assume that the dissipation potential $R\colon W^{2,p}(Q;\mathbb{R}^n) \times W^{1,2}(Q;\mathbb{R}^n) \to [0,\infty)$ satisfies the following properties:

(R.1) R is weakly lower semicontinuous in its second argument, that is, for all $\eta \in W^{2,p}(Q;\mathbb{R}^n)$ and every $b_k \rightharpoonup b$ in $W^{1,2}(Q;\mathbb{R}^n)$ we have that

$$R(\eta, b) \leq \liminf_{k\to\infty} R(\eta, b_k)$$

(R.2) R is homogeneous of degree 2 with respect to its second argument, that is,

$$R(\eta, \lambda b) = \lambda^2 R(\eta, b)$$

for all $\lambda \in \mathbb{R}$.

(R.3) R admits the following Korn-type inequality: For any $\varepsilon_0 > 0$, there exists K_R such that

$$K_R \|b\|^2_{W^{1,2}} \leq \|b\|^2_{L^2} + R(\eta, b)$$

for all $\eta \in \mathcal{E}$ with $\det \nabla\eta > \varepsilon_0$ and all $b \in W^{1,2}(Q;\mathbb{R}^n)$.

(R.4) R is differentiable in its second argument, with derivative $D_2R(\eta, b) \in (W^{1,2}(Q;\mathbb{R}^n))^*$ given by

$$D_2R(\eta, b)\langle\varphi\rangle := \frac{d}{d\varepsilon} R(\eta, b + \varepsilon\varphi)\bigg|_{\varepsilon=0}.$$

Furthermore, the map $(\eta, b) \mapsto D_2R(\eta, b)$ is bounded and weakly continuous with respect to both arguments, that is,

$$\lim_{k\to\infty} D_2R(\eta_k, b_k)\langle\varphi\rangle = D_2R(\eta, b)\langle\varphi\rangle$$

holds for all $\varphi \in W^{1,2}(Q;\mathbb{R}^n)$ and all sequences $\eta_k \rightharpoonup \eta$ in $W^{2,p}(Q;\mathbb{R}^n)$ and $b_k \rightharpoonup b$ in $W^{1,2}(Q;\mathbb{R}^n)$.

We also introduce a variant of (R.3) that will be used for the quasistatic case in the form of

(R.3_q) R admits the following Korn-type inequality: For any $\varepsilon_0 > 0$, there exists K_R such that

$$K_R\|b\|^2_{W^{1,2}} \leq R(\eta, b)$$

for all $\eta \in \mathcal{E}$ with $\det \nabla\eta > \varepsilon_0$ and all $b \in W^{1,2}_\Gamma(Q;\mathbb{R}^n)$

We mention here that the assumptions on the energy-dissipation pair are standard within the framework of second-order viscoelastic materials (see in particular [10], [13], and the references therein). For the convenience of the reader, explicit examples of E and R that satisfy the assumptions above are given in Section 6.

Remark 2.1. *Note that in particular* (R.2) *and* (R.4) *allow us to derive some additional growth conditions on the dissipation and its derivative. First of all, by taking the derivative of the identity in* (R.2) *with respect to b and dividing by λ, we get*

$$D_2R(\eta, \lambda b) = \lambda D_2 R(\eta, b) \tag{2.2}$$

i.e., D_2R is homogeneous of degree 1 with respect to its second argument.

Furthermore we can prove that for any $E_0 > E_{\min}$ there exists a constant C such that

$$\|D_2R(\eta, b)\|_{(W^{1,2})^*} \leq C\,\|b\|_{W^{1,2}}\,, \tag{2.3}$$

$$2R(\eta, b) \leq C\,\|b\|^2_{W^{1,2}} \tag{2.4}$$

for all $b \in W^{1,2}(Q;\mathbb{R}^n)$ and all $\eta \in W^{2,p}(Q;\mathbb{R}^n)$ with $E(\eta) \leq E_0$. To see this, assume that (2.3) *is not true. Then there exist sequences $\{\eta_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ with $\|D_2R(\eta_k, b_k)\|_{(W^{1,2})^*} > k\,\|b_k\|_{W^{1,2}}$ and $E(\eta_k) \leq E_0$. Additionally, due to the 1-homogeneity of D_2R (see* (2.2) *above), we can assume without loss that $\|b_k\|_{W^{1,2}} = 1$. This allows us to use*(E.3) *to pick weakly converging subsequences (which we do not relabel) and respective limits η and b. But then on one hand, by* (R.4)*, $D_2R(\eta_k, b_k)$ converges and thus $\|D_2R(\eta_k, b_k)\|_{(W^{1,2})^*}$ needs to stay bounded, and on the other hand, by our assumption it is larger than k, which is a contradiction. This proves* (2.3)*. Finally* (2.4) *follows from* (2.3) *by noting that due to* (R.2) *we have*

$$2R(\eta, b) = D_2R(\eta, b)\langle b\rangle \leq \|D_2R(\eta, b)\|_{(W^{1,2})^*}\,\|b\|_{W^{1,2}} \leq C\,\|b\|^2_{W^{1,2}}\,.$$

Remark 2.2. *The difference between* (R.3) *and* (R.3_q) *is subtle but central to the difference between quasistatic and inertial evolutions. The reasons for this do not only become evident in the proof, but also have a physical explanation. Indeed, in contrast to the full inertial problem, in the quasistatic regime there is no automatic conservation of linear or rotational momentum. As a result, when considering physical dissipations such as $R(\eta, b) = \int_Q |\nabla\eta^T\nabla b + \nabla b^T\nabla\eta|^2\,dx$, which are invariant under Galilean transformations, we need to include additional restrictions to the admissible deformations in $\mathcal{E}$, such as (partial) Dirichlet boundary data or a fixed center of mass and rotation around it.*

Remark 2.3 (On Dirichlet boundary data and contact)**.** *As it is a common occurrence in various applications, we incorporated the potential for Dirichlet boundary data into our formulation. The handling of these boundary conditions during the evolution is mostly standard; however, some subtleties arise when it comes to contact. When a freely moving part of the body comes into contact with a section of the solid where the Dirichlet boundary condition is specified, the latter behaves like a fixed obstacle. Since we are able to deal with fixed obstacles, this situation does not present any issues.*

What can potentially be more problematic, however, is the transition between the fixed and the free part of the boundary. As we require to cut off test functions in proximity of the fixed part of the boundary, we inevitably lose control over the resulting contact force. Notice that as long as this only happens to one of the sides that comes into contact, this is not an issue. Indeed, there is a corresponding opposite and equal force on the other side, which carries the same information that was lost due to the cut-off. In particular, we only run into issues if contact happens on both sides at points or regions where the fixed portion of the boundary transitions into the freely moving boundary.

To avoid this situation and keep the mathematical details manageable, we require $\eta_0|_\Gamma$ to be injective and that $\eta_0(\Gamma) \cap \partial\Omega = \emptyset$. We note however that more general situations can be handled with some additional care as well.

2.2. **Statement of the main results.** The precise definition of (weak) solution to the initial value problem considered in this paper can be formulated as follows.

Definition 2.4. *Let $T > 0$, $\eta_0 \in \mathcal{E}$, $\eta^* \in L^2(Q;\mathbb{R}^n)$, and $f \in L^2((0,T);L^2(Q;\mathbb{R}^n))$ be given. We say that*

$$\eta \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^n))$$

with $E(\eta) \in L^\infty((0,T))$ is a weak solution to (1.1) in $(0,T)$ if $\eta(0) = \eta_0$ and the variational inequality

$$\int_0^T DE(\eta(t))\langle\varphi(t)\rangle + D_2R(\eta(t),\partial_t\eta(t))\langle\varphi(t)\rangle\,dt$$
$$- \rho\langle\eta^*,\varphi(0)\rangle_{L^2} - \int_0^T \rho\langle\partial_t\eta(t),\partial_t\varphi(t)\rangle_{L^2}\,dt \geq \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt \quad (2.5)$$

holds for all $\varphi \in C([0,T];T_\eta(\mathcal{E})) \cap C^1_c([0,T);L^2(Q;\mathbb{R}^n))$. Here the set $T_\eta(\mathcal{E})$ denotes the class of admissible perturbations for the deformation η; its precise definition is given below in Definition 3.24.

Furthermore, we say that this η is a weak solution with a contact force if additionally it satisfies

$$\int_0^T DE(\eta(t))\langle\varphi(t)\rangle + D_2R(\eta(t),\partial_t\eta(t))\langle\varphi(t)\rangle\,dt$$
$$- \rho\langle\eta^*,\varphi(0)\rangle_{L^2} - \int_0^T \rho\langle\partial_t\eta(t),\partial_t\varphi(t)\rangle_{L^2}\,dt = \int_{[0,T]\times\partial Q} \varphi(t,x)\cdot d\sigma(t,x) + \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt$$

for all $\varphi \in C([0,T];W^{2,p}_\Gamma(Q;\mathbb{R}^n)) \cap C^1_c([0,T);L^2(Q;\mathbb{R}^n))$, where $\sigma \in M([0,T]\times\partial Q;\mathbb{R}^n)$ is a contact force satisfying the action-reaction principle in the sense of Definition 3.6.

Observe that in view of its regularity, η belongs to the space $C_w([0,T];W^{2,p}(Q;\mathbb{R}^n))$. Therefore, we have $\eta(t) \in W^{2,p}(Q;\mathbb{R}^n)$ for all $t \in [0,T]$, and in particular the initial condition $\eta(0) = \eta_0$ holds in the classical sense.

With this in hand, we can state the main result of this paper.

Theorem 2.5. *Let E and R be as in (E.1)–(E.6) and (R.1)–(R.4), respectively, and let $T > 0$, $\eta_0 \in \mathcal{E}$, $\eta^* \in L^2(Q;\mathbb{R}^n)$, and $f \in L^2((0,T);L^2(Q;\mathbb{R}^n))$ be given. Then (1.1) admits a weak solution with a contact force in $(0,T)$ in the sense of Definition 2.4, where the resulting contact force σ has no concentrations in time. Additionally this solution satisfies the energy inequality*

$$E(\eta(t)) + \frac{\rho}{2}\|\partial_t\eta(t)\|^2_{L^2} + \int_0^t 2R(\eta(s),\partial_t\eta(s))\,ds \leq E(\eta_0) + \frac{\rho}{2}\|\eta^*\|^2_{L^2} + \int_0^t \langle f(s),\partial_t\eta(s)\rangle_{L^2}\,ds$$

for almost all $t \in [0,T]$.

Let us now give a brief description of the method used in the proof. Following the approach developed in [3], our goal will be to approximate solutions to the original problem with solutions to suitably defined initial value problems for equations of first order, gradient flow type. Thus, we begin by considering a version of (1.1) where the inertial term is replaced by a time discretization. To be precise, for a fixed $h > 0$, we consider the problem

$$\rho\frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h} + DE(\eta^{(h)}(t)) + D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t)) = f(t) + \sigma^{(h)}(t), \quad (2.6)$$

complemented by an initial condition in the form of $\eta^{(h)}(0) = \eta_0$. We begin by finding a solution to (2.6) in the interval $[0,h]$, under the assumption that $\partial_t\eta^{(h)}$ is known for $t < 0$ (to be precise, one can assume that $\partial_t\eta^{(h)} = \eta^*$ on $[-h,0]$). This way, the term $\partial_t\eta^{(h)}(t-h)$ on the left-hand side of (2.6) can be regarded as a forcing term, and the standard machinery of minimizing movements (see [8]) yields a solution $\eta^{(h)}$ defined on $[0,h]$. We then consider (2.6) on the interval $[h,2h]$, again with the understanding that the time-shifted time derivative $\partial_t\eta^{(h)}(t-h)$ should not be regarded as part of the solution, but as a known term, in this case given by the solution found in the previous step. Iterating this procedure leads to a piecewise-defined function, still denoted by $\eta^{(h)}$, defined on the whole time interval $[0,T]$. A solution to the original problem can then be obtained by passing to the limit with $h \to 0^+$. This delicate limiting process is explained in detail in Section 5.

It is worth noting that this method allows us to derive a corresponding existence result (including contact forces) for the quasistatic case (see Corollary 5.2). In particular, in this paper we solve the two open problems formulated in Remarks 3.2 and 3.3 in [13].

2.3. **Outlook and future research directions.** The flexibility of the methods suggests that the results obtained can be generalized to different situations. The list below includes some of the directions that we plan to investigate in the future.

- **Irregular domains:** To simplify proofs and discussion, we have focused on domains that are somewhat regular, i.e. per construction the normals vary continuously and there are no edges or corners. In order to deal with more complex geometries, more care needs to be taken to study admissible directions for the contact force. In the static case (see [16] and [18]), this difficulty has been dealt with via the use of the Clarke-subdifferential [6]; it would be interesting to extend these results to the fully dynamic case as well.
- **Friction:** Similarly, we have simplified the situation by ignoring the effects of friction. In particular, dynamic friction could be of interest here, as it is purely dissipative as well as dependent on the contact force itself. It would thus perfectly fit into the framework presented.
- **Homogenization:** A common explanation of friction is via microscopic irregularities in the surface. As our result allows us to consider the effect of contact between macroscopic irregularities, it should be worth studying the resulting limit regimes when the scale of these irregularities is sent to zero.
- **Fracture mechanics:** A longstanding, active topic in solid mechanics is the study of fractures. While these represent the pulling apart of material and thus the opposite of collisions, they naturally result in situations where disconnected parts of the solid are close to each other. In particular in shear fractures immediate contact is to be expected. Extending our methods to this case thus seems like a natural target for future research. We mention here an important contribution of Dal Maso and Larsen [7], where the authors introduce a minimizing movements scheme for the study of the wave equation on domains with (evolving) cracks.
- **Fluid structure interactions:** Collisions between elastic bodies rarely occur in a vacuum. Instead, in numerous physical applications the volume that separates the solids is typically filled with a fluid (for example, air). Depending on the fluid, the boundary conditions, and the regularity of all surfaces involved, the presence of a fluid can result in large changes in behavior, in some situations even entirely preventing collisions (we refer to [11] for more information and to [9] for a study of rebound dynamics). When contact is theoretically possible, existence results for this kind of systems generally break at the time of first collision. Thus, it is natural to ask whether the methods presented in this paper can be used to extend solutions past that point.

3. Contact forces, admissible test functions, and their convergence

Before we begin with the proof of the main theorem, we first need to complete its statement with a more precise discussion of two related concepts: contact forces and the set of admissible test functions. These and their properties will not only be crucial in what follows, but some of the considerations here should be of independent interest for proving related results. Throughout the section we let $I \subset \mathbb{R}$ be a closed time interval. For consistency with the strategy outlined at the end of Subsection 2.2, I will play the role of $[0,h]$ in our study of the quasistatic problem (see Section 4) and $[0,T]$ when considering the full problem (see Section 5).

3.1. **Contact set and forces.** Let us begin by giving the definitions of contact set and contact force for a given deformation η. For the convenience of the reader, we recall some well-known properties that will be used throughout the rest of the section.

Definition 3.1 (Contact set)**.** *(i) Let $\eta \in \mathcal{E}$. The* (Lagrangian) contact set *of η is defined via*

$$C_\eta := \{x \in \overline{Q} : \eta(x) \in \partial\Omega \text{ or } \eta^{-1}(\eta(x)) \neq \{x\}\}.$$

Note that C_η consists of points of self-contact as well as points of contact with the boundary of the fixed domain Ω.

(ii) Let $\eta\colon I \times \overline{Q} \to \mathbb{R}^n$ be such that $\eta(t) \in \mathcal{E}$ for all $t \in I$, then we define its (Lagrangian) contact set *as*

$$C_\eta := \{(t,x) \in I \times \overline{Q} : x \in C_{\eta(t)}\},$$

where $C_{\eta(t)}$ denotes the contact set for the deformation $\eta(t,\cdot)$, defined as in (i).

The following result contains well-known structural properties of the contact set. In particular, by the regularity of η and the Ciarlet-Nečas condition (see (2.1)), one can show that there are no contact points in the interior.

Lemma 3.2. *For $\eta \in \mathcal{E}$, let C_η be given as in Definition 3.1. Then $C_\eta \subset \partial Q$. Furthermore, for $x \in C_\eta$ we have that*

(i) *if $\eta(x) \in \partial\Omega$, then $\eta^{-1}(\eta(x)) = \{x\}$ and $n_\eta(x)$ coincides with the interior unit normal vector to $\partial\Omega$ at $\eta(x)$;*
(ii) *if $\eta^{-1}(\eta(x)) \neq \{x\}$, then $\eta^{-1}(\eta(x)) = \{x, y\}$ for some $y \in \partial Q$ and $n_\eta(x) + n_\eta(y) = 0$.*

For a proof of Lemma 3.2 we refer to Theorem 2 in [5] (see also Lemma 2 in [15]). The time-dependent version stated below follows from the same argument with straightforward changes. Below we use η^{-1} to denote the inverse with respect to only the space variable, that is, $\eta^{-1}(t, \eta(t, x)) := \{z \in \overline{Q} : \eta(t, z) = \eta(t, x)\}$.

Lemma 3.3. *For $\eta\colon I \times \overline{Q} \to \mathbb{R}^n$ with $\eta(t) \in \mathcal{E}$ for all $t \in I$, let C_η be given as in Definition 3.1. Then $C_\eta \subset I \times \partial Q$. Furthermore, for $(t, x) \in C_\eta$ we have that*

(i) *if $\eta(t, x) \in \partial\Omega$, then $\eta^{-1}(t, \eta(t, x)) = \{x\}$ and $n_\eta(t, x)$ coincides with the interior unit normal vector to $\partial\Omega$ at $\eta(t, x)$;*
(ii) *if $\eta^{-1}(t, \eta(t, x)) \neq \{x\}$, then $\eta^{-1}(t, \eta(t, x)) = \{x, y\}$ and $n_\eta(t, x) + n_\eta(t, y) = 0$.*

Remark 3.4. *Note that (E.2) guarantees that the normal field n_η inherits the continuity of $\nabla\eta$. Indeed, if $\eta \in \mathcal{E}$ is such that $E(\eta) \leq E_0$, then by (E.2) and (E.3) we have that*

$$|(\nabla\eta)^{-1}| \leq \frac{|\nabla\eta|^{n-1}}{\det \nabla\eta} \leq C$$

for some constant C that depends only on E_0. In particular, this implies that $|(\nabla\eta)^{-T} n_Q| \geq \varepsilon_0$ for some ε_0 depending on E_0. In turn, we have that

$$n_\eta = \frac{\operatorname{cof} \nabla\eta\, n_Q}{|\operatorname{cof} \nabla\eta\, n_Q|} = \frac{(\nabla\eta)^{-T} n_Q}{|(\nabla\eta)^{-T} n_Q|}$$

belongs to $C^{0,\alpha}(\partial Q; \mathbb{R}^n)$, with Hölder seminorm bounded by a constant that only depends on E_0.

Additionally, for time dependent deformations, we note that if $\eta\colon I \times Q \to \Omega$ is such that $E(\eta(t)) < E_0$, then by an application of the chain rule, we see that $n_{\eta(t)}$ inherits some of the regularity of $\partial_t \nabla\eta$, e.g.

$$\left\|\partial_t n_{\eta(t)}\right\|_{L^2} \leq C \left\|\partial_t \nabla\eta(t)\right\|_{L^2}$$

for a constant C depending only on Q and E_0.

From this we can derive a well-known result about up-to the boundary local injectivity.

Lemma 3.5. *Let $\eta \in \mathcal{E}$ with $E(\eta) < \infty$. Then there exists a positive radius r depending on $E(\eta)$ such that the restriction of η to $B_r(x) \cap \overline{Q}$ is injective for all $x \in \overline{Q}$.*

Proof. Suppose that we have $x, y \in \overline{Q}$ with $\eta(x) = \eta(y)$ and $x \neq y$. Then by Lemma 3.2 we have that $x, y \in \partial Q$ and $n_\eta(x) = -n_\eta(y)$, so in particular $|n_\eta(x) - n_\eta(y)| = 2$. As noted in Remark 3.4, the seminorm $|n_\eta|_{C^{0,\alpha}}$ can be bounded in terms of only $E(\eta)$. Therefore $|x - y| > 2r$ with r depending only on $E(\eta)$. This implies that η must be injective on every ball of radius r. □

With this in hand, we can define contact forces as follows.

Definition 3.6 (Contact force).
(i) *Let $\eta \in \mathcal{E}$. Then a* contact force *for η is a vector valued measure $\sigma \in M(\partial Q; \mathbb{R}^n)$ with $\operatorname{supp} \sigma \subset C_\eta$ and with the property that on its support σ points in the direction of n_η in the sense that there exists a non-negative measure $|\sigma| \in M^+(\partial Q)$ such that $d\sigma = n_\eta d|\sigma|$, that is,*

$$\int_{\partial Q} \varphi \cdot d\sigma = \int_{\partial Q} \varphi \cdot n_\eta \, d|\sigma|$$

for all $\varphi \in C(\partial Q; \mathbb{R}^n)$.

(ii) *Let $\eta\colon I \times \partial Q \to \mathbb{R}^n$ be such that $\eta(t) \in \mathcal{E}$ for all $t \in I$ and assume that η is Borel measurable when considered as a mapping from I into $W^{2,p}(Q; \mathbb{R}^n)$. Then a* contact force *for η is a vector valued measure $\sigma \in M(I \times \partial Q; \mathbb{R}^n)$ with $\operatorname{supp} \sigma \subset C_\eta$ and with the property that on its support σ points in the direction of n_η in the sense that there exists a non-negative measure $|\sigma| \in M^+(I \times \partial Q)$ such that $d\sigma = n_\eta d|\sigma|$, that is,*

$$\int_{I \times \partial Q} \varphi \cdot d\sigma = \int_{I \times \partial Q} \varphi \cdot n_\eta \, d|\sigma|$$

for all $\varphi \in C(I \times \partial Q; \mathbb{R}^n)$.

(*iii*) *We say that a contact force* σ satisfies the action-reaction principle at self-contact *if*

$$\int_{\partial Q} (\varphi \circ \eta) \cdot d\sigma = 0$$

for all $\varphi \in C_c(\Omega; \mathbb{R}^n)$. *Similarly, for time-dependent deformations, we say that* σ satisfies the action-reaction principle at self-contact *if*

$$\int_{I \times \partial Q} (\varphi \circ \eta) \cdot d\sigma = 0$$

for all $\varphi \in C_c(I \times \Omega; \mathbb{R}^n)$.

Remark 3.7. *By the polar decomposition of measures (see, for example, Corollary 1.29 in* [1]*), every measure* $\mu \in M(I \times \partial Q; \mathbb{R}^n)$ *can be decomposed as* $d\mu = g\, d|\mu|$*, where* $|\mu| \in M^+(I \times \partial Q)$ *is the total variation of* μ *and* $g: I \times \partial Q \to \mathbb{R}^n$ *is Borel measurable with* $|g| \equiv 1$ *everywhere on* $I \times \partial Q$*. Thus, if* $\operatorname{supp} \sigma \subset C_\eta$*, the definition above says that* μ *is a contact force whenever this decomposition holds with* $g = n_\eta$ *on* $I \times \partial Q$*.*

3.2. **Compactness-Closure theorems for contact forces.** In this section we investigate compactness and closure properties of contact forces. These will enable us to conclude that contact forces associated to approximate solutions will converge to the contact force of the limiting solution. We present time-independent and time-dependent versions of these results, as both will be needed throughout the rest of the paper. However, we omit the proofs for the former case since these follow from analogous (but simpler) arguments.

Theorem 3.8 (Compactness-Closure for contact forces)**.** *Let* $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathcal{E}$ *be given and assume that there exist* $\eta \in \mathcal{E}$ *and a constant* E_0*, independent of* k*, such that* $\eta_k \to \eta$ *in* $C^1(Q; \mathbb{R}^n)$*,* $E(\eta_k) \le E_0$ *for all* k*, and* $E(\eta) \le E_0$*. For every* k*, let* $\sigma_k \in M(\partial Q; \mathbb{R}^n)$ *be a contact force for* η_k *and assume that*

$$\sup_k \|\sigma_k\|_{M(\partial Q;\mathbb{R}^n)} \le C.$$

Then there exist a subsequence (not relabelled) and a limit measure σ *such that* $\sigma_k \overset{*}{\rightharpoonup} \sigma$ *in* $M(\partial Q; \mathbb{R}^n)$*. Moreover,* σ *is a contact force for* η*, and if* σ_k *satisfies the action-reaction principle at self-contact for all* k*, then so does* σ*.*

Theorem 3.9 (Compactness-Closure for contact forces, time-dependent)**.** *Let* $\eta_k, \eta: I \times Q \to \mathbb{R}^n$ *be such that* $\eta_k(t), \eta(t) \in \mathcal{E}$ *for all* t *and all* $k \in \mathbb{N}$ *and assume that* $E(\eta_k(t)) \le E_0$ *for all* k*,* $E(\eta(t)) \le E_0$ *for some* E_0 *independent of* k*. Furthermore, assume that* $\eta_k(t) \to \eta(t)$ *in* $C^1(Q; \mathbb{R}^n)$ *uniformly in* t *for* $t \in I$*, that* η_k *is Borel measurable in time for all* k*, and that* $n_\eta \in C(I \times \partial Q; \mathbb{R}^n)$*. For every* k*, let* $\sigma_k \in M(I \times \partial Q; \mathbb{R}^n)$ *be a contact force for* η_k *and assume that*

$$\sup_k \|\sigma_k\|_{M(I\times\partial Q;\mathbb{R}^n)} \le C.$$

Then there exist a subsequence (not relabelled) and a limit measure σ *such that* $\sigma_k \overset{*}{\rightharpoonup} \sigma$ *in* $M(I \times \partial Q; \mathbb{R}^n)$*. Moreover,* σ *is a contact force for* η *and if* σ_k *satisfies the action-reaction principle at self-contact for all* k*, then so does* σ*.*

Proof. By Definition 3.6, we have that $d\sigma_k = n_{\eta_k} d|\sigma_k|$ with $|\sigma_k| \in M^+(I \times \partial Q)$ and $\operatorname{supp} \sigma_k \subset C_{\eta_k}$. Since the sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ is bounded in $M(I \times \partial Q; \mathbb{R}^n)$, we must have that $\{|\sigma_k|\}_{k\in\mathbb{N}}$ is bounded in $M^+(I \times \partial Q)$. Thus, eventually extracting a subsequence (which we do not relabel), we have that $|\sigma_k| \overset{*}{\rightharpoonup} |\sigma|$ for some $|\sigma| \in M^+(I \times \partial Q)$. Let $\sigma \in M(I \times \partial Q; \mathbb{R}^n)$ be defined by setting $d\sigma = n_\eta d|\sigma|$. We claim that $\sigma_k \overset{*}{\rightharpoonup} \sigma$ in $M(I \times \partial Q; \mathbb{R}^n)$. To prove the claim, let $g \in C(I \times \partial Q; \mathbb{R}^n)$ and observe that

$$\int_{I\times\partial Q} g \cdot n_{\eta_k}\, d|\sigma_k| - \int_{I\times\partial Q} g \cdot n_\eta\, d|\sigma| = \int_{I\times\partial Q} g \cdot (n_{\eta_k} - n_\eta)\, d|\sigma_k| + \int_{I\times\partial Q} g \cdot n_\eta\, d(|\sigma_k| - |\sigma|). \tag{3.1}$$

It is worth noting that the first term on the right-hand side of (3.1) is well defined since n_{η_k} is Borel measurable by assumption and $|\sigma_k|$ is a non-negative Radon measure. Moreover, observe that

$$\left| \int_{I\times\partial Q} g \cdot (n_{\eta_k} - n_\eta) d|\sigma_k| \right| \le \|n_{\eta_k} - n_\eta\|_{L^\infty} \||\sigma_k|\|_{M^+} \le C \|n_{\eta_k} - n_\eta\|_{L^\infty} \to 0$$

as $k \to \infty$ since $n_{\eta_k} \to n_\eta$ uniformly on $I \times \partial Q$. Finally, the continuity of $g \cdot n_\eta$ and the fact that $|\sigma_k| \overset{*}{\rightharpoonup} |\sigma|$ imply that, as $k \to \infty$, the last term on the right-hand side of (3.1) vanishes as well. This proves the claim.

Next, we prove that $\operatorname{supp}\sigma \subset C_\eta$. To this end, fix $(t,x) \in \operatorname{supp}\sigma$. By the weak convergence of measures, we can find $(t_k,x_k) \in \operatorname{supp}\sigma_k \subset C_{\eta_k}$ such that $(t_k,x_k) \to (t,x)$. There are now two possibilities: either there is a further subsequence (not relabelled) such that $\eta_k(t_k,x_k) \in \partial\Omega$, or there exist points $y_k \in \partial Q$ with $y_k \neq x_k$ such that $\eta_k(t_k,x_k) = \eta_k(t_k,y_k)$. In the first case, using the estimate

$$|\eta(t,x) - \eta_k(t_k,x_k)| \leq |\eta(t,x) - \eta(t_k,x_k)| + |\eta(t_k,x_k) - \eta_k(t_k,x_k)|,$$

the uniform convergence of η_k to η and the uniform continuity of η, we get that $\eta_k(t_k,x_k) \to \eta(t,x)$. Since $\eta_k(t_k,x_k) \in \partial\Omega$ for all k, we must also have that $\eta(t,x) \in \partial\Omega$, and therefore $(t,x) \in C_\eta$. In the second case, in view of the fact that $E(\eta_k(t_k)) \leq E_0$, Lemma 3.5 gives the existence of a minimal distance r (which only depends on E_0) with the property that $|x_k - y_k| \geq r$. By the compactness of ∂Q, eventually extracting a further subsequence, we can assume that $y_k \to y \in \partial Q$. Reasoning as above, by the uniform convergence we see that $\eta(t,x) = \eta(t,y)$. Since necessarily $x \neq y$, we conclude that $(t,x) \in C_\eta$ also in this case.

Finally, we note that for any $\varphi \in C_c(\Omega;\mathbb{R}^n)$ we have $\varphi \circ \eta_k \to \varphi \circ \eta$ uniformly. This, together with the convergence of $\sigma_k \overset{*}{\rightharpoonup} \sigma$, implies that the action-reaction principle continues to hold in the limit. □

Remark 3.10. *An added difficulty in the proof is that we do not have continuity in time, but only Borel measurability. This makes justifying that $\sigma_k \overset{*}{\rightharpoonup} \sigma$ not an immediate consequence of the convergence $|\sigma_k| \overset{*}{\rightharpoonup} |\sigma|$. However, as shown above, continuity of the limit and uniform convergence can be used to justify this convergence. While these assumptions may not seem natural at first glance, we mention here that they arise from the construction of solutions to the quasistatic problem. To be precise, we will consider approximations that are piecewise constant and uniformly bounded in time and prove that these converge uniformly to a limiting deformation that is continuous in time.*

Remark 3.11. *Theorem 3.8 and Theorem 3.9 are designed for higher order materials and the associated sense of convergence. In fact, the statements are in general not true if the convergence is not strong enough. Specifically, if there is no pointwise convergence of $\nabla\eta_k$, then the limit measure might not lie in the contact set. Additionally, it is necessary to have a strong sense of convergence for the normal vector to guarantee that the limit measure is still pointing in the normal direction. Indeed, take $Q = [0,1]^2$ and consider a sequence of deformations such that*

$$\eta_k(0,x_2) = \eta_k(1,1-x_2) = \left(\frac{1}{k}\sin(kx_2), x_2\right)$$

for all $x_2 \in [0,1]$. This can be done in such a way that η_k converges to some η with

$$\eta(0,x_2) = \eta(1,1-x_2) = (0,x_2)$$

in a sense that does not imply pointwise convergence of $\nabla\eta_k$, e.g., weakly in $W^{2,2}(Q;\mathbb{R}^2)$.

Now, observe that every contact set C_{η_k} contains points x for which $n_{\eta_k}(x)$ points in direction $(1,1)$. In particular, we can construct contact forces σ_k which only point in this direction and have unit mass. But then there exist a subsequence and a limit measure σ with the same property. However, all normals associated to C_η are of the form $(0,\pm 1)$.

Yet, we also note that if we are not studying self contact, but contact between a deformable solid and an immovable obstacle, the approach presented above can be adapted to work also for lower order materials, as the obstacle's normal is fixed.

Finally we note that if the measures σ_k are additionally uniformly L^2 in time, as will be the case for the corresponding quasistatic result, then the same holds for the limit. Before stating this result, we discuss the notion of a measure being "L^2 in time", namely the space $L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$.

Remark 3.12 (Definition of $L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$)**.** *We use $L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$ to denote the space of all $\sigma\colon I \to M(\partial Q;\mathbb{R}^n)$ which are weak* measurable, satisfy*

$$\|\sigma\|^2_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))} := \int_I \|\sigma(t)\|^2_{M(\partial Q;\mathbb{R}^n)} dt < \infty,$$

and any such two $\sigma, \tilde\sigma$ are considered to be equivalent if $\sigma(t) = \tilde\sigma(t)$ for $\mathcal{L}^1$-a.e. $t \in I$. Here we recall that σ is said to be weakly measurable if*

$$t \mapsto \int_{\partial Q} \varphi \cdot d\sigma(t)$$

is measurable for all $\varphi \in C(\partial Q;\mathbb{R}^n)$.
Observe that any $\sigma \in L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$ *can be regarded as an element of* $M(I\times\partial Q;\mathbb{R}^n)$ *by setting*

$$\int_{I\times\partial Q}\varphi(t,x)\cdot d\sigma(t,x) := \int_I\int_{\partial Q}\varphi(t,x)\cdot d\sigma(t)\,dt, \quad \varphi\in C(I\times\partial Q;\mathbb{R}^n).$$

Conversely, if $\sigma\in M(I\times\partial Q;\mathbb{R}^n)$ *can be represented as*

$$\int_{I\times\partial Q}\varphi(t,x)\cdot d\sigma(t,x) = \int_I\int_{\partial Q}\varphi(t,x)\cdot d\nu_t\,\phi(t)\,dt, \quad \varphi\in C(I\times\partial Q;\mathbb{R}^n)$$

with $\phi\in L^2(I;\mathbb{R})$ *and* $\{\nu_t\}_{t\in I}\subset M(\partial Q;\mathbb{R}^n)$ *a bounded weakly* measurable family of measures, then one can regard* σ *as belonging to* $\sigma\in L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$ *by setting*

$$\sigma(t) := \phi(t)\nu_t,$$

for $\mathcal{L}^1$*-a.e.* $t\in I$.

Lemma 3.13. *Let* $\sigma_k \in L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$ *with* $\|\sigma_k\|_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))}\le C$ *be such that* $\sigma_k \overset{*}{\rightharpoonup} \sigma$ *in* $M(I\times\partial Q;\mathbb{R}^n)$. *Then* $\sigma\in L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$ *and* $\|\sigma\|_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))}\le C$.

Proof. By the disintegration theorem (see e.g. [1, Theorem 2.28]) if we define $\mu\in M^+(I)$ by $\mu(A) = |\sigma|(A\times\partial Q)$ for any Borel set $A\subset I$, we obtain a weakly* measurable family of measures $\{\nu_t\}_{t\in I}\subset M(\partial Q;\mathbb{R}^n)$ with $\|\nu_t\|_{M(\partial Q;\mathbb{R}^n)} = 1$ such that $\sigma = \mu(dt)\otimes\nu_t$. More precisely, we have that

$$\int_{I\times\partial Q}\varphi\,d\sigma = \int_I\int_{\partial Q}\varphi(t,x)\,d\nu_t(x)\,d\mu(t), \quad \varphi\in C(I\times\partial Q).$$

We now show that μ has L^2 density with respect to the Lebesgue measure on I. To see this, let $g\in C(I)$ with $\|g\|_{L^2}\le 1$. Then for all k we have that

$$C\ge\|\sigma_k\|_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))}\ge\int_I\|\sigma_k(t)\|_{M(\partial Q;\mathbb{R}^n)}g(t)\,dt = \int_{I\times\partial Q}\varphi\,d|\sigma_k| \tag{3.2}$$

where $\varphi(t,x) := g(t)$. Since $|\sigma_k| \overset{*}{\rightharpoonup} |\sigma|$ in $M(I\times\partial Q)$, (3.2) implies that

$$C\ge\int_{I\times\partial Q}\varphi\,d|\sigma| = \int_I\int_{\partial Q}\varphi(t,x)\,d\nu_t(x)\,d\mu(t) = \int_I g(t)\,d\mu(t). \tag{3.3}$$

Since this is true for all g, (3.3) shows that μ defines a linear functional on $C(I)$ which is bounded with respect to the L^2-norm. By the Hahn–Banach theorem, this functional admits an extension to $L^2(I)$ and can be represented by $\phi\in L^2(I)$, in the sense that

$$\int_I g\,d\mu = \int_I g\phi\,dt, \quad g\in C(I).$$

Thus, as argued at the end of Remark 3.12, we see that $\sigma\in L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))$. Moreover, we have $\|\sigma\|_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))} = \|\phi\|_{L^2(I)}\le C$. This completes the proof. □

Remark 3.14. *Even the stronger assumption of* $\sigma_k\in L^2(I;M(\partial Q))$*, i.e. strong measurability, (which is satisfied for our approximation in Theorem 4.6) does not help, as the strong measurability may be lost in the weak* limit. This is closely related to the fact that* $L^2(I;M(\partial Q))$ *is not the dual space to* $L^2(I;C(\partial Q))$*, as* $M(\partial Q)$ *does not satisfy the Radon-Nikodým property (see e.g.* [12, Section 1.3]*).*

3.3. Normals and almost normals.

In the core sections of the paper, it would prove convenient to use the normal field n_η as a way of perturbing an admissible deformation. This will however not be allowed as normal vectors lack the needed regularity. We overcome this difficulty by introducing an "almost" normal field that is sufficiently regular for our purposes.

Definition 3.15 (Almost normals)**.** (i) *Let* $\eta\in\mathcal{E}$. *Then* $\tilde n_\eta\colon\overline{Q}\to\mathbb{R}^n$ *is an* almost normal *to* η*, if*

$$\tilde n_\eta(x)\cdot n_\eta(x) > \frac12\ \forall x\in\partial Q \qquad \text{and} \qquad |\tilde n_\eta(x)|\le 1\ \forall x\in\overline{Q}.$$

(ii) *Let* $\eta\in L^\infty(I;W^{2,p}(Q;\mathbb{R}^n))\cap W^{1,2}(I;W^{1,2}(Q;\mathbb{R}^n))$ *be such that* $E(\eta)\in L^\infty(I)$. *Then a vector field* $\tilde n_\eta\colon I\times\overline{Q}\to\mathbb{R}^n$ *is called* almost normal *to* η*, if*

$$\tilde n_\eta(t,x)\cdot n_\eta(t,x) > \frac12\ \forall(t,x)\in I\times\partial Q \qquad \text{and} \qquad |\tilde n_\eta(t,x)|\le 1\ \forall(t,x)\in I\times\overline{Q}.$$

Notice in particular that almost normals are defined also in the interior of Q. While the existence of a sufficiently regular (with respect to the space variables) almost normal $\tilde{n}_\eta$ is well known to specialists in the field, the precise statement used in the following and its proof are included here for the convenience of the reader. As before, we present both the time-independent and time-dependent versions, but only prove the slightly more complicated time-dependent version.

Proposition 3.16. *Let $\eta \in \mathcal{E}$ with $E(\eta) < \infty$. Then there exists an almost normal $\tilde{n}_\eta$ to η with $\tilde{n}_\eta \in C^{k_0}(\overline{Q};\mathbb{R}^n)$ for all $k_0 \in \mathbb{N}$ satisfying $\|\tilde{n}_\eta\|_{C^{k_0}(Q;\mathbb{R}^n)} \leq C_{k_0}$, where C_{k_0} depends only on $E(\eta)$ and k_0.*

Proposition 3.17. *Let $\eta \in L^\infty(I;\mathcal{E}) \cap W^{1,2}(I;W^{1,2}(Q;\mathbb{R}^n))$ be such that $E(\eta(t)) \leq E_0$ for all $t \in I$. Then there exists an almost normal $\tilde{n}_\eta$ to η with $\tilde{n}_\eta \in L^\infty(I;C^{k_0}(\overline{Q};\mathbb{R}^n))$ for all $k_0 \in \mathbb{N}$, satisfying $\|\tilde{n}_\eta\|_{L^\infty(I;C^{k_0}(Q;\mathbb{R}^n))} \leq C_{k_0}$, where C_{k_0} depends only on E_0 and k_0. Moreover, we can have that $\tilde{n}_\eta \in W^{1,2}(I;L^2(Q;\mathbb{R}^n))$ with $\|\partial_t \tilde{n}_\eta\|_{L^2(I\times Q;\mathbb{R}^n)} \leq C\|\partial_t \nabla\eta\|_{L^2(I\times Q;\mathbb{R}^n)}$, where C depends only on E_0.*

Proof. Reasoning as in Remark 3.4, due to the uniform bound on $E(\eta(t))$ and the fact that $\eta \in C(I;C^{1,\alpha}(Q;\mathbb{R}^n))$, we obtain that $n_\eta \in C(I;C^{0,\alpha}(\partial Q;\mathbb{R}^n))$. Notice that we can find $\delta > 0$ such that for all $t \in I$ and all $x, \tilde{x} \in \partial Q$ it holds that

$$n_\eta(t,x) \cdot n_\eta(t,\tilde{x}) > 3/4$$

whenever $|x - \tilde{x}| < \delta$. We then consider an extension of n_Q (still denoted by n_Q) to the δ-neighborhood of Q, namely Q_δ, such that $n_Q \in C^{0,\alpha}(Q_\delta;\mathbb{R}^n)$ with

$$\|n_Q\|_{C^{0,\alpha}(Q_\delta;\mathbb{R}^n)} \leq C(\delta)\|n_Q\|_{C^{0,\alpha}(\partial Q;\mathbb{R}^n)}$$

and with the property that $|n_Q| \equiv 1$ on P_δ, where P_δ denotes the δ-neighborhood of ∂Q. We then consider the extension of the deformed normal n_η to $I \times Q_\delta$ (again, still denoted by n_η) by

$$n_\eta(t,x) = \frac{\operatorname{cof} \nabla\eta(t,x)\, n_Q(x)}{|\operatorname{cof} \nabla\eta(t,x)\, n_Q(x)|}|n_Q(x)|, \quad (x,t) \in I \times Q_\delta \text{ if } |n_Q(x)| \neq 0$$

and by zero otherwise, where we use an extension of η to $I \times Q_\delta$ by a standard extension operator from the fixed domain $I \times Q$, which in particular can be chosen linear and so that it preserves the norms in $L^\infty(I;W^{2,p}(Q_\delta;\mathbb{R}^n))$ and $W^{1,2}(I;W^{1,2}(Q_\delta;\mathbb{R}^n))$ up to a constant. Note that this in particular implies sufficient regularity on the extended domain, so that the previous expression is well defined as $\operatorname{cof} \nabla\eta(t,x)\, n_Q(x)$ is uniformly continuous and bounded away from zero on ∂Q.

Arguing as in Remark 3.4 we see that $n_\eta \in C(I;C^{0,\alpha}(Q_\delta;\mathbb{R}^n))$ with Hölder seminorm dependent only on E_0. Moreover, by an application of the chain rule, we see that

$$\|\partial_t n_\eta(t)\|_{L^2(Q)} \leq C \|\partial_t \nabla\eta(t)\|_{L^2(Q)}$$

for a constant C depending only on Q and E_0.

Choose then $\tilde{\delta} > 0$ (possibly smaller that δ, dependent only on E_0) so that for all $t \in I$, all $x \in \partial Q$, and all $\tilde{x} \in Q_\delta$ we have that

$$n_\eta(t,x) \cdot n_\eta(t,\tilde{x}) > 1/2$$

whenever $|x - \tilde{x}| < \tilde{\delta}$. Finally, we mollify the (extended) normal field in space, that is, we set $\tilde{n}_\eta := n_\eta * \xi_{\tilde{\delta}}$ in $I \times \overline{Q}$, where $*$ is convolution with respect to x and $\xi_{\tilde{\delta}}$ is the standard mollification kernel with parameter $\tilde{\delta}$. Then, for each time t, we have that

$$\|\tilde{n}_\eta(t)\|_{C^{k_0}(\overline{Q};\mathbb{R}^n)} \leq C_{k_0}\|n_\eta(t)\|_{C(\partial Q;\mathbb{R}^n)},$$

where C_{k_0} depends only on E_0 and k_0, and

$$\|\partial_t \tilde{n}_\eta(t)\|_{L^2(Q)} \leq \|\partial_t n_\eta(t)\|_{L^2(Q)} \leq C \|\partial_t \nabla\eta(t)\|_{L^2(Q)}.$$

As one can readily check, $\tilde{n}_\eta$ is an almost normal to η in the sense of Definition 3.15. This concludes the proof. □

3.4. **Test functions for problems involving self-contact.** We can now introduce the cone of admissible test functions for our variational inequality and provide practical characterizations that are used throughout the rest of the paper. For our purposes, we need to develop this theory for both the static and the dynamical case. We believe that these characterizations are of independent interest and thus provide the precise statements for both cases.

In the following we use X to denote a Banach space that embeds compactly into $C^{1,\alpha}(Q;\mathbb{R}^n)$. Throughout the paper we apply the results of this section for $X = W^{k_0,2}(Q;\mathbb{R}^n)$ and $X = W^{2,p}(Q;\mathbb{R}^n)$ (with suitable conditions on k_0 and p).

3.4.1. *Test functions for static problems.* We begin by giving a definition of the admissible test functions.

Definition 3.18. *Let $\eta \in \mathcal{E}$ be a given deformation with $E(\eta) < \infty$. Then we define the corresponding cone of admissible test functions (i.e., the tangent cone to $\mathcal{E}$) as*

$$T_\eta(\mathcal{E} \cap X) \coloneqq \left\{ \varphi \in W^{2,p}(Q;\mathbb{R}^n) \cap X : \forall x \in C_\eta \sum_{z \in \eta^{-1}(\eta(x))} \varphi(z) \cdot n_\eta(z) \geq 0, \varphi|_\Gamma = 0 \right\}. \tag{3.4}$$

In case $X = W^{2,p}(Q;\mathbb{R}^n)$, we omit it in the notation.

We remark here for clarity that in view of Lemma 3.2, the sum in (3.4) consists of either one (in the case of contact with $\partial\Omega$) or two (in the case of self-contact) elements. Intuitively, the inequality in the definition of $T_\eta(\mathcal{E} \cap X)$ signifies that admissible test functions, i.e. admissible perturbations of the deformation η, cannot point outside of Ω whenever η already lies on $\partial\Omega$, and cannot further displace the deformed configuration in a way that would cause interpenetration of matter.

Lemma 3.19. *Let $U, V \subset \mathbb{R}^n$ be open, disjoint sets with $\partial U \cap \partial V \neq \emptyset$. Let $p \in \partial U \cap \partial V$ be such that ∂U, ∂V are C^1 near p and let vectors $u, v \in \mathbb{R}^n$ be such that*

$$u \cdot n_U(p) + v \cdot n_V(p) \leq -\delta < 0.$$

Then there exist $\varepsilon_0 > 0$ and $\rho > 0$ depending on δ, $|u - v|$, and the C^1 moduli of continuity of ∂U and ∂V (in a neighborhood of p), such that $B_{\rho\varepsilon}(p + \varepsilon u) \subset V + \varepsilon v$ for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. As the proof follows from elementary consideration, we only give a sketch of the general idea. By assumption, the vector $w = u - v$ at the point p points inside V. Since the boundary of V is of class C^1, there exists a non-empty open cone in V with vertex at p in the direction w. The height of the cone gives ε_0, and the aperture of the cone gives ρ. □

Proposition 3.20. *Let $\eta \in \mathcal{E}$ and $\varphi \in W^{2,p}_\Gamma(Q;\mathbb{R}^n)$. If there exists $x \in C_\eta$ with*

$$-\delta \coloneqq \sum_{z \in \eta^{-1}(\eta(x))} \varphi(z) \cdot n_\eta(z) < 0,$$

then there is $\varepsilon_0 > 0$ depending on δ, φ, and $E(\eta)$ such that $\eta + \varepsilon\varphi \notin \mathcal{E}$ for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let x and $\delta > 0$ be as in the statement. If $\eta(x) \in \partial\Omega$ we set $U = \eta(Q)$, $V = \mathbb{R}^n \setminus \overline{\Omega}$, $p = \eta(x)$, $u = \varphi(x)$, and $v = 0$. Recalling that the C^1-modulus of continuity of $\eta(\partial Q)$ depends on $E(\eta)$ (see Remark 3.4), the previous lemma gives $\varepsilon_0 > 0$ such that $\eta(x) + \varepsilon\varphi(x) \notin \overline{\Omega}$ for all $\varepsilon \in (0, \varepsilon_0)$, which implies $(\eta + \varepsilon\varphi)(Q) \not\subset \Omega$, and therefore that $\eta + \varepsilon\varphi \notin \mathcal{E}$.

If this is not the case then by Lemma 3.2 we have that $\eta^{-1}(\eta(x)) = \{x, y\}$ for some $y \neq x$. Then by Lemma 3.5 we can find a radius $r > 0$ (which depends on $E(\eta)$) such that $U = \eta(B_r(x) \cap Q)$ and $V = \eta(B_r(y) \cap Q)$ are disjoint. As before, denote $p = \eta(x)$, $u = n_\eta(x)$, $v = n_\eta(y)$ and apply the previous lemma. We thus have $\varepsilon_0 > 0$ and $\rho > 0$ such that

$$B_{\rho\varepsilon}(\eta(x)) + \varepsilon\varphi(x) \subset \eta(B_r(y) \cap Q) + \varepsilon\varphi(y)$$

for all $\varepsilon \in (0, \varepsilon_0)$. Eventually replacing r with a smaller number so that $|\varphi(w) - \varphi(y)| \leq \rho$ for all $w \in B_r(y)$, we get that

$$\operatorname{dist}(\eta(B_r(y) \cap Q) + \varepsilon\varphi(y), (\eta + \varepsilon\varphi)(B_r(y) \cap Q)) \leq \rho\varepsilon.$$

Consequently, we have that

$$\eta(x) + \varepsilon\varphi(x) \in (\eta + \varepsilon\varphi)(B_r(y) \cap Q).$$

for all $\varepsilon \in (0, \varepsilon_0)$. Since by Lemma 3.2 self-contact cannot happen at an interior point, this implies that $\eta + \varepsilon\varphi \notin \mathcal{E}$. This concludes the proof. □

Equipped with this, we can derive some useful characterizations for the cone of admissible test functions.

Proposition 3.21 (Characterizations of $T_\eta(\mathcal{E} \cap X)$)**.** *Let $\eta \in \mathcal{E} \cap X$ and $\varphi \in X$. The following are equivalent:*

(i) *$\varphi \in T_\eta(\mathcal{E} \cap X)$.*

(ii) *There exists a sequence $\{\varphi_k\}_k$ such that $\varphi_k \to \varphi$ in X, $\varphi_k = 0$ on Γ, and*

$$\sum_{z \in \eta^{-1}(\eta(x))} \varphi_k(z) \cdot n_\eta(z) > 0 \tag{3.5}$$

for all $k \in \mathbb{N}$ and all $x \in C_\eta$.

(*iii*) *There exist a sequence of positive real numbers* $\{\varepsilon_k\}_k$ *and a sequence of test functions* $\{\varphi_k\}_k$ *with* $\varphi_k \to \varphi$ *in* X *such that* $\eta + \varepsilon\varphi_k \in \mathcal{E}$ *for all* $\varepsilon \in [0, \varepsilon_k)$.

Additionally, the condition

(*iv*) *there exists a curve* $\Phi \in C([0,\varepsilon_0); \mathcal{E} \cap X) \cap C^1([0,\varepsilon_0); X)$ *such that* $\Phi(0) = \eta$ *and* $\Phi'(0^+) = \varphi$

implies any of (*i*)-(*iii*) *and conversely if* φ *satisfies the condition imposed on* φ_k *in* (3.5)*, then this implies* (*iv*).[4]

Proof. (*i*) $\implies$ (*ii*)**:** It is enough to consider $\varphi_k := \varphi + \frac{1}{k}\xi_\Gamma \tilde{n}_\eta$ where $\tilde{n}_\eta$ is the smooth almost normal to η given by Proposition 3.16 and $\xi_\Gamma \colon \overline{Q} \to \mathbb{R}$ is a smooth function that vanishes on Γ and is otherwise positive. Indeed, then for $x \in C_\eta$ we have that

$$\sum_{z\in\eta^{-1}(\eta(x))} \varphi_k(z) \cdot n_\eta(z) \geq \sum_{z\in\eta^{-1}(\eta(x))} \frac{1}{k}\xi_\Gamma(z)\tilde{n}_\eta(z) \cdot n_\eta(z) \geq \frac{1}{2k} \sum_{z\in\eta^{-1}(\eta(x))} \xi_\Gamma(z) > 0.$$

(*ii*) $\implies$ (*iii*)**:** This is proved in Proposition 3 in [15] for the case of self-contact and $X = W^{2,p}$, but the argument easily extends to general X. The case of contact with the boundary is the same, as it can be treated like a part of the solid on which $\varphi = 0$.

(*iii*) $\implies$ (*i*)**:** Let $\{\varphi_k\}_k$ be as in (*iii*). We claim that $\varphi_k \in T_\eta(\mathcal{E} \cap X)$. In fact, in view of Proposition 3.20 $\varphi_k \notin T_\eta(\mathcal{E} \cap X)$ implies $\eta + \varepsilon\varphi_k \notin \mathcal{E}$ for ε arbitrarily small, which is a contradiction with (*iii*). Now it is apparent that $T_\eta(\mathcal{E} \cap X)$ is closed with respect to uniform convergence, therefore also with respect to convergence in X. Thus $\varphi \in T_\eta(\mathcal{E} \cap X)$. This proves that (*i*)–(*iii*) are equivalent.

(*iv*) $\implies$ (*i*)**:** To see this, let $x \in C_\eta$ and set

$$\tilde{\varphi}(\varepsilon, x) := \frac{\Phi(\varepsilon, x) - \eta(x)}{\varepsilon}.$$

We claim that we must have

$$\lim_{\varepsilon\to 0^+} \sum_{z\in\eta^{-1}(\eta(x))} \tilde{\varphi}(\varepsilon, z) \cdot n_\eta(z) \geq 0. \tag{3.6}$$

Notice that the limit in (3.6) exists since $\Phi \in C^1([0,\varepsilon_0); X)$ and that furthermore it equals $\sum_{z\in\eta^{-1}(\eta(x))} \varphi(z) \cdot n_\eta(z)$. To prove the claim, arguing by contradiction, assume that there exists a number $\delta > 0$ with

$$\sum_{z\in\eta^{-1}(\eta(x))} \tilde{\varphi}(\varepsilon, z) \cdot n_\eta(z) \leq -\delta < 0$$

for all ε small enough. Then, by Proposition 3.20 we have that $\Phi(\varepsilon, \cdot) = \eta(\cdot) + \varepsilon\tilde{\varphi}(\varepsilon, \cdot) \notin \mathcal{E}$ for all $\varepsilon > 0$ sufficiently small and we have thus arrived at a contradiction.

(*i*)+(3.5) $\implies$ (*iv*)**:** Set $\Phi(\varepsilon, x) := \eta(x) + \varepsilon\varphi(x)$. Then clearly $\Phi(0) = \eta$ and $\Phi'(0^+) = \varphi$. Additionally Φ has the required regularity. It is only left to check that $\Phi(\varepsilon, \cdot) \in \mathcal{E}$. For this, we note that per definition $\Phi(0) \in \mathcal{E}$ and there is $\varepsilon_0 > 0$ such that $\Phi(\varepsilon, \cdot)$ is globally injective (and maps to the interior of Ω) for any $\varepsilon \in (0, \varepsilon)$.

In fact it is enough to check for injectivity at the boundary. For any points $x, y \in \partial Q$ such that $\eta(x)$ and $\eta(y)$ (resp. $\eta(x)$ and $\partial\Omega$) have a fixed minimum distance, this follows from continuity. Similarly for any points $x, y \in Q$ that are close to each other, the same follows from local injectivity (see Lemma 3.5). This allows us to restrict our attention to pairs (x, y) in an arbitrary neighborhood of the compact set $\{(\tilde{x}, \tilde{y}) \in \partial Q \times \partial Q : \eta(\tilde{x}) = \eta(\tilde{y})\}$ (resp. points x in a neighborhood of $\{\tilde{x} \in \partial Q : \eta(\tilde{x}) \in \partial\Omega\}$).

But then we can choose this neighborhood small enough so that the condition in (3.5) extends to those pairs as well. Together with the definition of the normal vector, for any such pair (x, y), this allows us to pick a unit vector n such that $n \cdot (\eta(x) - \eta(y)) = 0$ and $(\varphi(x) - \varphi(y)) \cdot n > 0$. This then implies that $\Phi(\varepsilon, x)$ and $\Phi(\varepsilon, y)$ cannot coincide. A similar result holds for $z \in \partial\Omega$ in place of $\eta(y)$. □

Remark 3.22. *Note that the approximating sequences in (ii) and (iii) are necessary. For a general* φ *satisfying one of the conditions, it is not true that* $\eta + \varepsilon\varphi \in \mathcal{E}$ *for any* $\varepsilon > 0$*. Even if one excludes tangential movement, a condition like* $\varphi \cdot n \geq 0$ *only guarantees that the points of* C_η *themselves are not displaced in a way that would violate interpenetration of matter or move outside of* Ω*, but it cannot be used to conclude the same about any neighborhood.*

---

[4] Note that this implies that (3.5) characterizes the interior of $T_\eta(\mathcal{E} \cap X)$. Additionally, for a sufficiently regular η, it is not hard to prove that (*iv*) is in fact equivalent to the other three conditions. However, this proof involves splitting φ into its precise normal and tangential components and is thus not easily transferred to the general situation.

To see this in a simple example, consider the following situation. Take Ω to be the upper half-plane, Q a subset of the upper half-plane such that ∂Q includes $(-1,1)\times\{0\}$ and $\eta(x_1,x_2) := (x_1, x_2 + x_1^2)$ as well as $\varphi(x_1,x_2) := (0, 2x_1)$. Then there is contact only at $(0,0)$, where $\varphi\cdot n = 0$, but for any $\varepsilon > 0$ we have $(\eta(s,0)+\varepsilon\varphi_k(s,0))_2 < 0$ for some s small enough and thus a non-superficial intersection with the boundary.

However, φ is still an admissible test function in the sense of (i) and one can even construct the curve

$$\Phi(t,x) := (x_1, x_2 + (x_1+t)^2)$$

which satisfies $\Phi(0,\cdot) = \eta$ and $\frac{d}{dt}|_{t=0}\Phi = \varphi$ and as such is admissible in the sense of (iv).

The next result shows that the set of admissible test functions is well-behaved with respect to sequences of approximating deformations.

Proposition 3.23. *Let $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathcal{E}\cap X$ be given with $E(\eta_k)$ uniformly bounded and assume that there exists $\eta \in \mathcal{E}\cap X$ such that $\eta_k \to \eta$ in X. Then, for every $\varphi \in T_\eta(\mathcal{E}\cap X)$ there exists a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ such that $\varphi_k \to \varphi$ in X and with the property that $\varphi_k \in T_{\eta_k}(\mathcal{E}\cap X)$ for all $k\in\mathbb{N}$.*

Proof. By Proposition 3.21 (*ii*), we can approximate φ by a sequence of $\{\varphi_l\}_{l\in\mathbb{N}}$ for which (3.5) holds. Additionally, as we have seen in the proof of Theorem 3.9, any converging sequence of contact points $\{x_k\}_k$ with $x_k \in C_{\eta_k}$ has to converge to a contact point $x \in C_\eta$. But then by the fact that C_η is compact, n_{η_k} converges uniformly, and since each element of $\{\varphi_l\}_{l\in\mathbb{N}}$ is continuous, we have that for fixed l, (3.5) holds for all η_k with k large enough. In particular, we can use this to pick a non-decreasing sequence l_k such that $\varphi_{l_k} \in T_{\eta_k}(\mathcal{E}\cap X)$. □

3.4.2. *Test functions for evolutionary problems.* Throughout this section, let

$$\eta \in L^\infty(I;\mathcal{E}\cap X)\cap W^{1,2}(I^\circ; W^{1,2}(Q;\mathbb{R}^n)) \quad \text{with} \quad E(\eta(t)) \le E_0 \text{ for all } t\in I. \tag{3.7}$$

Note that this matches exactly the regularity that we ask for solutions to our time-dependent problem (see Definition 2.4). Recall that by the Aubin-Lions lemma η admits a representative in $C(I; C^{1,\alpha}(Q;\mathbb{R}^n))$. In the following, we always work with this representative without further notice.

Our aim here is to present the time-dependent versions of the results in the previous section. They follow along the same lines, therefore we will be brief and only indicate the differences to the static versions.

Definition 3.24. *Let η be given as in (3.7). By a slight abuse of notation, we denote*

$$C(I;T_\eta(\mathcal{E}\cap X)) := \{\varphi \in C(I; W^{2,p}(Q;\mathbb{R}^n)\cap X) : \varphi(t) \in T_{\eta(t)}(\mathcal{E}\cap X), t\in I\},$$

where $T_{\eta(t)}(\mathcal{E}\cap X)$ is understood as in the time-independent Definition 3.18.

Lemma 3.25. *Let η be given as in (3.7) and let $\varphi \in C(I;X)$ be such that $\varphi(t,\cdot) = 0$ on Γ. If there exists $(t,x)\in C_\eta$ with*[5]

$$-\delta := \sum_{z\in\eta^{-1}(t,\eta(t,x))} \varphi(t,z)\cdot n_\eta(t,z) < 0,$$

then there is $\varepsilon_0 > 0$ depending on δ, φ and E_0 such that $\eta(t)+\varepsilon\varphi(t) \notin \mathcal{E}$ for all $\varepsilon \in (0,\varepsilon_0)$.

This is a straightforward consequence of Proposition 3.20. Therefore, we can formulate the time-dependent version of the various characterizations of our test functions.

Proposition 3.26 (Characterizations of $C(I;T_\eta(\mathcal{E}\cap X))$)**.** *The following are equivalent:*

- (*i*) *$\varphi \in C(I;T_\eta(\mathcal{E}\cap X))$.*
- (*ii*) *There exists a sequence $\{\varphi_k\}_k$ with $\varphi_k \to \varphi$ in $C(I;X)$ such that $\varphi(t,\cdot) = 0$ on Γ, and*
$$\sum_{z\in\eta^{-1}(t,\eta(t,x))} \varphi_k(t,z)\cdot n_\eta(t,z) > 0 \tag{3.8}$$
for all $(t,x)\in C_\eta$ and all $k\in\mathbb{N}$.
- (*iii*) *There exist a sequence of positive real numbers $\{\varepsilon_k\}_k$ and a sequence of test functions $\{\varphi_k\}_k$ with $\varphi_k \to \varphi$ in $C(I;X)$ such that $\eta(t)+\varepsilon\varphi_k(t)\in\mathcal{E}$ for $\mathcal{L}^1$-a.e. $t\in I$ and all $\varepsilon\in[0,\varepsilon_k)$;*

Additionally the condition

- (*iv*) *there exists $\Phi \in C([0,\varepsilon_0); L^\infty(I;\mathcal{E}\cap X))\cap C^1([0,\varepsilon_0); L^\infty(I;X))$ such that $\Phi(0) = \eta$ and $\Phi'(0^+) = \varphi$*

[5]As before, η^{-1} denotes the preimage of $\eta(t,\cdot)$.

implies any of (i)-(iii) and conversely if φ satisfies the condition imposed on φ_k in (3.8), then this implies (iv).

The proof follows analogously as the proof of Proposition 3.21 with the following changes: The smooth almost normal $\tilde{n}_\eta$ is now the time-dependent version from Proposition 3.17; instead of Proposition 3.20 one has to invoke the time-dependent version Lemma 3.25.

Proposition 3.27. *Let $\{\eta_k\}_k \subset L^\infty(I;\mathcal{E}\cap X)\cap W^{1,2}(I^\circ;W^{1,2}(Q;\mathbb{R}^n))$ with $E(\eta_k(t)) \le E_0$ be given and assume that there exists η satisfying (3.7) such that $\eta_k(t)\to\eta(t)$ in X, uniformly in t for $t\in I$. Then, for every $\varphi \in C(I;T_\eta(\mathcal{E}\cap X))$ there exists a sequence $\{\varphi_k\}_k$ such that $\varphi_k \to \varphi$ in $C(I;X)$ and with the property that $\varphi_k \in C(I;T_{\eta_k}(\mathcal{E}\cap X))$ for all $k\in\mathbb{N}$. Additionally, if $J\subset I$ and $\varphi \in C(I;T_\eta(\mathcal{E}\cap X))\cap C^1_c(J;L^2(Q;\mathbb{R}^n))$, then we can find a sequence $\{\varphi_k\}_k$ as above but with the property that $\varphi_k \in C(I;T_{\eta_k}(\mathcal{E}\cap X))\cap C^1_c(J;L^2(Q;\mathbb{R}^n))$ for all $k\in\mathbb{N}$.*

Again, the proof is the same as in Proposition 3.23, using the time-dependent almost normal and uniform convergence in time.

4. Existence results for the quasistatic regime

This section is concerned with the study of the quasistatic counterpart to (1.1). To be precise, for a given energy-dissipation pair (E,R), we prove existence of solutions to

$$DE(\eta) + D_2R(\eta,\partial_t\eta) = f \tag{4.1}$$

in a framework compatible with the standard assumptions of second-order nonlinear elasticity. As a particular case of our analysis, we obtain an existence result for a parabolic variant of (1.1), that is,

$$\rho\frac{\partial_t\eta}{h} + DE(\eta) + D_2R(\eta,\partial_t\eta) = f + \rho\frac{\zeta}{h}, \tag{4.2}$$

where $\zeta(t)$ will later correspond to $\partial_t\eta(t-h)$ and the two terms thus form the difference quotient

$$\rho\frac{\partial_t\eta(t)-\partial_t\eta(t-h)}{h}.$$

While (4.2) has no direct physical relevance, solutions to this problem will play a fundamental role when inertial effects are taken into account (see Section 5).

Similarly to [15], where the authors study equilibrium configurations for a second-order nonlinear solid, our presentation is divided into two parts. First, we prove the existence of solutions to a variational inequality via minimizing movements. Then, in the second part of the section, we recover the governing equations by reintroducing contact forces. These forces can be interpreted as a Lagrange multiplier associated to the global injectivity constraint.

4.1. Existence of weak solutions via minimizing movements.

Throughout the section we consider an energy-dissipation pair (E,R), which can be thought of as a regularized version of the energy-dissipation pair introduced in Section 2 where, roughly speaking, the underlying space $W^{2,p}$ is replaced by $W^{k_0,2}$, with $k_0-\frac{n}{2} > 2-\frac{n}{p}$, so that $W^{k_0,2}(Q;\mathbb{R}^n)$ compactly embeds in $W^{2,p}(Q;\mathbb{R}^n)$. Note that the leading term in E is now quadratic (see Section 5), which results in a corresponding linear term in the equation. The assumptions (E.1)–(E.5) and (R.1)–(R.4) are thus replaced by their regularized versions as follows.

We assume that $E\colon W^{k_0,2}(Q;\mathbb{R}^n)\to(-\infty,\infty]$ satisfies the following properties:

- (E′.1) There exists $E_{\min} > -\infty$ such that $E(\eta)\ge E_{\min}$ for all $\eta\in W^{k_0,2}(Q;\mathbb{R}^n)$. Moreover, $E(\eta)<\infty$ for every $\eta\in W^{k_0,2}(Q;\mathbb{R}^n)$ with $\inf_Q \det\nabla\eta > 0$.
- (E′.2) For every $E_0\ge E_{\min}$ there exists $\varepsilon_0>0$ such that $\det\nabla\eta\ge\varepsilon_0$ for all η with $E(\eta)\le E_0$.
- (E′.3) For every $E_0\ge E_{\min}$ there exists a constant C such that

  $$\|\nabla^{k_0}\eta\|_{L^2}\le C$$

  for all η with $E(\eta)\le E_0$.
- (E′.4) E is weakly lower semicontinuous, that is,

  $$E(\eta)\le\liminf_{k\to\infty}E(\eta_k)$$

  whenever $\eta_k \rightharpoonup \eta$ in $W^{k_0,2}(Q;\mathbb{R}^n)$.

(E′.5) E is differentiable in its effective domain with derivative $DE(\eta) \in (W^{k_0,2}(Q;\mathbb{R}^n))^*$ given by

$$DE(\eta)\langle\varphi\rangle = \frac{d}{d\varepsilon}E(\eta+\varepsilon\varphi))\Big|_{\varepsilon=0}.$$

Furthermore, DE is bounded and continuous with respect to weak convergence in $W^{k_0,2}(Q;\mathbb{R}^n)$ on any sub-level set of E.

Furthermore, we assume that the dissipation potential $R\colon W^{k_0,2}(Q;\mathbb{R}^n) \times W^{k_0,2}(Q;\mathbb{R}^n) \to [0,\infty)$ satisfies the following properties:

(R′.1) R is weakly lower semicontinuous in its second argument, that is, for all $\eta \in W^{k_0,2}(Q;\mathbb{R}^n)$ and every $b_k \rightharpoonup b$ in $W^{k_0,2}(Q;\mathbb{R}^n)$ we have that

$$R(\eta,b) \le \liminf_{k\to\infty} R(\eta,b_k).$$

(R′.2) R is homogeneous of degree 2 with respect to its second argument, that is,

$$R(\eta,\lambda b) = \lambda^2 R(\eta,b)$$

for all $\lambda \in \mathbb{R}$.

(R′.3) R admits the following Korn-type inequality: For all $\varepsilon_0 > 0$, there exists a constant K_R such that

$$K_R\|b\|^2_{W^{k_0,2}} \le \|b\|_{L^2} + R(\eta,b)$$

for all $\eta \in \mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n)$ with $\det\nabla\eta > \varepsilon_0$ and $b \in T_\eta(\mathcal{E}\cap W^{k_0,2})$.

(R′.4) R is differentiable in its second argument, with derivative $D_2R(\eta,b) \in (W^{k_0,2}(Q;\mathbb{R}^n))^*$ given by

$$D_2R(\eta,b)\langle\varphi\rangle := \frac{d}{d\varepsilon}R(\eta,b+\varepsilon\varphi)\Big|_{\varepsilon=0}.$$

Furthermore, the map $(\eta,b) \mapsto D_2R(\eta,b)$ is bounded and weakly continuous with respect to both arguments, that is,

$$\lim_{k\to\infty} D_2R(\eta_k,b_k)\langle\varphi\rangle = D_2R(\eta,b)\langle\varphi\rangle$$

holds for all $\varphi \in W^{k_0,2}(Q;\mathbb{R}^n)$ and all sequences $\eta_k \rightharpoonup \eta$ and $b_k \rightharpoonup b$ in $W^{k_0,2}(Q;\mathbb{R}^n)$.

The quasistatic equivalent of (R′.3) is formulated as follows:

(R′.3$_\mathrm{q}$) R admits the following Korn-type inequality: For any $\varepsilon_0 > 0$, there exists K_R such that

$$K_R\|b\|^2_{W^{k_0,2}} \le R(\eta,b)$$

for all $\eta \in \mathcal{E}$ with $\det\nabla\eta > \varepsilon_0$ and all $b \in W^{k_0,2}_\Gamma(Q;\mathbb{R}^n)$.

Next, we give the precise definition of a solution to (4.1).

Definition 4.1. *Let $h > 0$, $\eta_0 \in \mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n)$, and $f \in L^2((0,h);L^2(Q;\mathbb{R}^n))$ be given. We say that*

$$\eta \in W^{1,2}((0,h);W^{k_0,2}(Q;\mathbb{R}^n)) \cap L^\infty((0,h);\mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n)) \quad \text{with} \quad E(\eta) \in L^\infty((0,h))$$

is a solution to (4.1) in $(0,h)$ if $\eta(0) = \eta_0$ and

$$\int_0^h DE(\eta(t))\langle\varphi(t)\rangle + D_2R(\eta(t),\partial_t\eta(t))\langle\varphi(t)\rangle\,dt \ge \int_0^h \langle f(t),\varphi(t)\rangle_{L^2}\,dt \tag{4.3}$$

holds for all $\varphi \in C([0,h];T_\eta(\mathcal{E}\cap W^{k_0,2}))$.

Remark 4.2. *Let us comment here on Definition 4.1.*

(*i*) *It is worth noting that the initial condition $\eta(0) = \eta_0$ in Definition 4.1 is satisfied in the classical sense since every element η of $W^{1,2}((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$ admits a representative in the space $C_w([0,h];W^{k_0,2}(Q;\mathbb{R}^n))$.*

(*ii*) *We also remark that the variational inequality (4.3) is preserved along sequences of test functions $\{\varphi_k\}_k$ such that $\varphi_k \rightharpoonup \varphi$ in $L^2((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$.*

(*iii*) *The inequality in (4.3) is a consequence of the injectivity constraint imposed on the class of admissible deformations. In particular, it is evident from the fact that $C_c^\infty(Q;\mathbb{R}^n) \subset T_\eta(\mathcal{E}\cap X)$ that if $\varphi(t,\cdot) \in C_c^\infty(Q;\mathbb{R}^n)$ then (4.3) holds as an equality. Similarly, equality holds if collisions can be excluded a priori.*

The existence of solutions to (4.1) is established in the following theorem.

Theorem 4.3. *Let E satisfy (E′.1)–(E′.5), R satisfy (R′.1), (R′.2), (R′.3$_q$) as well as (R′.4) and let h, η_0, and f be given as above. Then there exists a solution η to (4.1) in the sense of Definition 4.1. Furthermore, for every $t \in [0,h]$, η satisfies the energy inequality*

$$E(\eta(t)) + 2\int_0^t R(\eta(s), \partial_t\eta(s))\,ds \le E(\eta_0) + \int_0^t \langle f(s), \partial_t\eta(s)\rangle_{L^2}\,ds. \tag{4.4}$$

Proof. For the convenience of the reader, we divide the proof into several steps.
Step 1: Given $M \in \mathbb{N}$, we set $\tau := h/M$ and decompose $[0,h]$ into subintervals $[k\tau,(k+1)\tau]$ of length τ. Moreover, for every $1 \le k \le M$ we let

$$f_k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(t)\,dt \in L^2(Q;\mathbb{R}^n).$$

We then define recursively $\eta_k \in \mathcal{E} \cap W^{k_0,2}(Q;\mathbb{R}^n)$ to be a solution of the minimization problem for

$$\mathcal{J}_k(\eta) := E(\eta) + \tau R\left(\eta_{k-1}, \tfrac{\eta-\eta_{k-1}}{\tau}\right) - \tau\left\langle f_k, \tfrac{\eta-\eta_{k-1}}{\tau}\right\rangle_{L^2}, \tag{4.5}$$

that is,

$$\eta_k \in \arg\min\left\{\mathcal{J}_k(\eta) : \eta \in \mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n)\right\}. \tag{4.6}$$

We remark that the existence of η_k is a consequence of the direct method in the calculus of variations. Indeed (E′.3) and (R′.3$_q$) imply that $\mathcal{J}_k$ is coercive and the rest follows from (E′.1), (E′.2), (E′.4) and (R′.1). Therefore, in view of (4.6) we have that $\mathcal{J}_k(\eta_k) \le \mathcal{J}_k(\eta_{k-1})$; moreover, since $R(\eta,0) = 0$ for all η (see (R′.2)), the inequality can be rewritten as

$$E(\eta_k) + \tau R\left(\eta_{k-1}, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right) \le E(\eta_{k-1}) + \tau\left\langle f_k, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right\rangle_{L^2}. \tag{4.7}$$

Additionally, (R′.3$_q$) implies that

$$\frac{K_R}{\tau}\|\eta_k - \eta_{k-1}\|_{L^2}^2 \le \tau R\left(\eta_{k-1}, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right). \tag{4.8}$$

Notice that by Young's inequality we have that

$$\tau\left\langle f_k, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right\rangle_{L^2} \le \frac{\tau}{2c}\|f_k\|_{L^2}^2 + \frac{c}{2\tau}\|\eta_k - \eta_{k-1}\|_{L^2}^2. \tag{4.9}$$

Thus, combining (4.8) and (4.9) with (4.7) yields

$$E(\eta_k) + \frac{\tau}{2}R\left(\eta_{k-1}, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right) \le E(\eta_{k-1}) + \frac{\tau}{2c}\|f_k\|_{L^2}^2. \tag{4.10}$$

Summing over $k = 1,\dots,m$ in (4.10), where $m \le M$, we arrive at

$$E(\eta_m) + \sum_{k=1}^m \frac{\tau}{2}R\left(\eta_{k-1}, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right) \le E(\eta_0) + \frac{1}{2c}\sum_{k=1}^m \tau\|f_k\|_{L^2}^2. \tag{4.11}$$

Notice that an application of Jensen's inequality yields

$$\sum_{k=1}^m \tau\|f_k\|_{L^2}^2 \le \sum_{k=1}^m \int_{(k-1)\tau}^{k\tau} \|f(t)\|_{L^2}^2\,dt \le \|f\|_{L^2(L^2)}^2. \tag{4.12}$$

Combining (4.11) and (4.12) we can find a constant C_0 which depends only on Q, K_R, $E(\eta_0)$ and $\|f\|_{L^2(L^2)}$ such that the following estimates hold uniformly in τ:

$$\sup_{k\le M} E(\eta_k) \le C_0, \tag{4.13}$$

$$\sum_{k=1}^M \tau\left\|\frac{\eta_k - \eta_{k-1}}{\tau}\right\|_{L^2}^2 \le C_0, \tag{4.14}$$

$$\sum_{k=1}^M \tau R\left(\eta_{k-1}, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right) \le C_0. \tag{4.15}$$

Next, for $t \in [(k-1)\tau, k\tau)$ we let

$$\underline{\eta}_\tau(t) := \eta_{k-1}, \qquad \overline{\eta}_\tau(t) := \eta_k, \qquad \text{and} \qquad \eta_\tau(t) := \frac{k\tau - t}{\tau}\eta_{k-1} + \frac{t-(k-1)\tau}{\tau}\eta_k.$$

Notice that due to the coercivity of E, the families of functions $\{\underline{\eta}_\tau\}_\tau$, $\{\overline{\eta}_\tau\}_\tau$, and $\{\eta_\tau\}_\tau$ are uniformly bounded in $L^\infty((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$. Moreover, from (4.14) and (4.15) we also obtain that

$$\int_0^h \|\partial_t \eta_\tau(t)\|_{L^2}^2\,dt \le C_0,$$

$$K_R \int_0^h \|\nabla^{k_0}\partial_t \eta_\tau(t)\|_{L^2}^2\,dt \le C_0,$$

where K_R is the constant in the Korn's inequality for R (see ($R'3_q$)), which is bounded thanks to (4.13). Reasoning as above we have that $\partial_t\eta_\tau$ is bounded in $L^2((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$. Consequently, $\{\eta_\tau\}_\tau$ is bounded in $W^{1,2}((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$, and eventually extracting a subsequence (which we do not relabel) we find $\underline{\eta}$, $\overline{\eta} \in L^\infty((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$ and $\eta \in W^{1,2}((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$ such that

$$\underline{\eta}_\tau \overset{*}{\rightharpoonup} \underline{\eta} \ \text{ in } L^\infty((0,h);W^{k_0,2}(Q;\mathbb{R}^n)),$$
$$\overline{\eta}_\tau \overset{*}{\rightharpoonup} \overline{\eta} \ \text{ in } L^\infty((0,h);W^{k_0,2}(Q;\mathbb{R}^n)),$$

and

$$\eta_\tau \rightharpoonup \eta \quad \text{in } W^{1,2}((0,h);W^{k_0,2}(Q;\mathbb{R}^n)). \tag{4.16}$$

Next, we claim that $\underline{\eta} = \overline{\eta} = \eta$. Indeed, by (4.8) and (4.15) we see that

$$\|\eta_k - \eta_{k-1}\|_{W^{k_0,2}}^2 \le \frac{C_0\tau}{c}. \tag{4.17}$$

In turn, we have that for $t \in [(k-1)\tau, k\tau)$

$$\|\eta_\tau(t) - \overline{\eta}_\tau(t)\|_{W^{k_0,2}}^2 = \left\|\frac{k\tau - t}{\tau}(\eta_{k-1} - \eta_k)\right\|_{W^{k_0,2}}^2 \le \frac{C_0\tau}{c}. \tag{4.18}$$

To prove the claim, it is enough to notice that in view of (4.18),

$$\left|\int_0^h \langle \varphi(t), \eta_\tau(t) - \overline{\eta}_\tau(t)\rangle\,dt\right| \le \lim_{\tau\to 0}\sqrt{\frac{C_0\tau}{c}}\int_0^h \|\varphi(t)\|_{(W^{k_0,2})^*}\,dt = 0$$

holds for all $\varphi \in L^1((0,h);W^{k_0,2}(Q;\mathbb{R}^n)^*)$.

We conclude this step by proving that $\eta \in L^\infty((0,h);\mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n))$. The result, in turn, is obtained by showing that the sequence $\{\overline{\eta}_\tau\}_\tau$ converges to η uniformly; to be precise, we will show that (up to the extraction of a subsequence)

$$\overline{\eta}_\tau \to \eta \quad \text{in } L^\infty((0,h);W^{2,p}(Q;\mathbb{R}^n)). \tag{4.19}$$

Notice that this readily implies the claim since $\eta \in L^\infty((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$ and furthermore since $\mathcal{E}$ is closed with respect to weak convergence in $W^{2,p}(Q;\mathbb{R}^n)$. To prove (4.19), we argue as follows: By (4.16), together with an application of the Aubin–Lions lemma we see that $\eta_\tau \to \eta$ in $C([0,h];W^{2,p}(Q;\mathbb{R}^n))$. Therefore, (4.18) implies that

$$\begin{aligned}\|\eta(t) - \overline{\eta}_\tau(t)\|_{W^{2,p}} &\le \|\eta(t) - \eta_\tau(t)\|_{W^{2,p}} + \|\eta_\tau(t) - \overline{\eta}_\tau(t)\|_{W^{2,p}}\\ &\le \|\eta - \eta_\tau\|_{L^\infty(W^{2,p})} + C\|\eta_\tau(t) - \overline{\eta}_\tau(t)\|_{W^{k_0,2}}\\ &\le \|\eta - \eta_\tau\|_{L^\infty(W^{2,p})} + C\sqrt{\frac{C_0\tau}{c}}\end{aligned}$$

holds for $\mathcal{L}^1$-a.e. $t \in [0,h]$. Letting $\tau \to 0$ concludes the proof of (4.19).

Step 2: Next, we show that the function η obtained in the previous step is a solution to (4.1) in the sense of Definition 4.1. To this end, consider $\varphi \in W^{k_0,2}(Q;\mathbb{R}^n)$ such that there is a curve $\Phi \in C^0([0,\varepsilon_0);\mathcal{E}\cap W^{k_0,2}(Q;\mathbb{R}^n)) \cap C^1([0,\varepsilon_0);W^{k_0,2}(Q;\mathbb{R}^n))$ with $\Phi'(0^+) = \varphi$, as in Proposition 3.21 (*iv*). By the minimality of η_k we obtain that

$$0 \le \frac{d}{d\varepsilon}\mathcal{J}_k(\Phi(\varepsilon))\bigg|_{\varepsilon=0} = \left[DE(\eta_k) + D_2R\left(\eta_{k-1}, \tfrac{\eta_k - \eta_{k-1}}{\tau}\right)\right]\langle\varphi\rangle - \langle f_k, \varphi\rangle_{L^2}. \tag{4.20}$$

By Proposition 3.21 (*ii*), the same is then also true for all $\varphi \in T_\eta(\mathcal{E}\cap W^{k_0,2})$.

Let $\{\tau_j\}_{j\in\mathbb{N}} \subset (0,1)$ with $\tau_j \to 0^+$ be a decreasing subsequence for which (4.16) and (4.19) hold, and consider $\varphi \in C([0,h];T_\eta(\mathcal{E}\cap W^{k_0,2}))$. By Proposition 3.27, there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ such that $\varphi_j \to \varphi$ in $C([0,h];W^{k_0,2})$ and $\varphi_j \in C([0,h];T_{\overline{\eta}_j}(\mathcal{E}\cap W^{k_0,2}))$. In particular, since this implies that $\varphi_j(t) \in T_{\eta_k}(\mathcal{E}\cap W^{k_0,2})$ for $t \in [(k-1)\tau_j, k\tau_j)$, for $\mathcal{L}^1$-a.e. such t we can rewrite (4.20) as

$$\left[DE(\overline{\eta}_{\tau_j}(t)) + D_2R(\underline{\eta}_{\tau_j}(t), \partial_t\eta_{\tau_j}(t))\right]\langle\varphi_j(t)\rangle - \langle f_{\tau_j}(t), \varphi_j(t)\rangle_{L^2} \geq 0, \tag{4.21}$$

where $f_{\tau_j}(t) := f_k$ for all $t \in [(k-1)\tau_j, k\tau_j)$. Integrating (4.21) over $[(k-1)\tau_j, k\tau_j)$ and summing up the resulting inequalities we get

$$\int_0^h \left[DE(\overline{\eta}_{\tau_j}(t)) + D_2R(\underline{\eta}_{\tau_j}(t), \partial_t\eta_{\tau_j}(t))\right]\langle\varphi_j(t)\rangle\,dt \geq \int_0^h \langle f_{\tau_j}(t), \varphi_j(t)\rangle_{L^2}\,dt. \tag{4.22}$$

Notice that $f_{\tau_j}(t) \to f(t)$ for $\mathcal{L}^1$-a.e. $t \in [0,h]$ (to be precise, for every t which is a Lebesgue point). Then by Lebesgue's dominated convergence theorem we also have that $f_{\tau_j} \to f$ in $L^2((0,h);L^2(Q;\mathbb{R}^n))$. In turn, we can pass to the limit pass to the limit as $\tau_j \to 0^+$ on the right-hand side of (4.22). Convergence for the terms on the left-hand side is a consequence of (E′.5) and (R′.4). Thus, we have shown that η satisfies (4.3) for all $\varphi \in C([0,h];T_\eta(\mathcal{E}\cap W^{k_0,2}))$.

Step 3: As it remains to verify that η satisfies the energy inequality (4.4), the goal of this step is to show that $-\partial_t\eta$ is an admissible test function. For $t \in (\varepsilon, h-\varepsilon)$, let

$$\partial_t^\varepsilon\eta(t) := \frac{\eta(t-\varepsilon)-\eta(t)}{\varepsilon}.$$

Let $\{\chi_\varepsilon\}_\varepsilon$ be a family of smooth cut-off functions with the property that $\chi_\varepsilon \in C_c((\varepsilon,h-\varepsilon);[0,1])$ and $\chi_\varepsilon \to 1$ in $L^2((0,h))$. Then, $\chi_\varepsilon\partial_t^\varepsilon\eta \to -\partial_t\eta$ in $L^2((0,h);W^{k_0,2}(Q;\mathbb{R}^n))$. Fix $(t,x) \in C_\eta$. We claim that

$$\lim_{\varepsilon\to 0^+} \sum_{z\in\eta^{-1}(t,\eta(t,x))} \partial_t^\varepsilon\eta(t,z)\cdot n_\eta(t,x) \geq 0.$$

Indeed, if this is not the case then we can find a sequence $\varepsilon_j \to 0^+$ and a number $\delta > 0$ such that $\sum_{z\in\eta^{-1}(t,\eta(t,x))} \partial_t^{\varepsilon_j}\eta(t,z)\cdot n_\eta(t,z) \leq -\delta$ for all j. In turn, by Proposition 3.20, we must have that

$$\eta(t-\varepsilon_j,\cdot) = \eta(t,\cdot) + \varepsilon_j\partial_t^{\varepsilon_j}\eta(t,\cdot) \notin \mathcal{E}$$

for all j sufficiently large. We thus reached a contradiction and the claim is proved. Now let

$$\varphi_{\varepsilon,\delta} := \chi_\varepsilon\partial_t^\varepsilon\eta + \delta\xi_\Gamma\tilde{n}_\eta,$$

where $\xi_\Gamma\colon \overline{Q} \to \mathbb{R}$ is a smooth function that vanishes on Γ and is otherwise positive. Then, for all small values of $\delta > 0$, in view of Proposition 3.26 we have that $\varphi_{\varepsilon,\delta} \in C([0,h];T_\eta(\mathcal{E}\cap W^{k_0,2}))$ for all ε sufficiently small. The desired result follows by testing with $\varphi_{\varepsilon,\delta}$ in (4.3), letting $\varepsilon \to 0^+$ first, and then $\delta \to 0^+$. A similar argument shows that $-\chi_{[0,t]}\partial_t\eta$ is an admissible test function.

Step 4: In view of the previous step, we can substitute $\varphi = -\chi_{[0,t]}\partial_t\eta$ in the variational inequality (4.3) and obtain that

$$\int_0^t [DE(\eta(s)) + D_2R(\eta(s),\partial_t\eta(s))]\langle\partial_t\eta(s)\rangle\,ds \leq \int_0^t \langle f, \partial_t\eta(s)\rangle_{L^2}\,ds.$$

Using the fundamental theorem of calculus for the term involving the elastic energy and (R′.2), the previous inequality can be rewritten as

$$E(\eta(t)) - E(\eta(0)) + 2\int_0^t R(\eta(s),\partial_t\eta(s))\,ds \leq \int_0^t \langle f(s), \partial_t\eta(s)\rangle_{L^2}\,ds.$$

This concludes the proof. □

As a corollary, we show the existence of solutions to (4.2).

Corollary 4.4. *Let E and η_0 be given as above and assume that for some $h > 0$*

$$\tilde{R}(\eta,b) := R(\eta,b) + \frac{\rho}{2h}\|b\|_{L^2}^2, \tag{4.23}$$

where R now satisfies (R′.1)–(R′.4). Define additionally $\tilde{f} := f + \rho\frac{\zeta}{h}$ for some $\zeta \in L^2((0,h);L^2(Q;\mathbb{R}^n))$. Then the conclusions of Theorem 4.3 continue to hold. In particular, this yields the existence of a weak solution to (4.2) (in the sense of Definition 4.1), where (4.3) can be rewritten as

$$\int_0^h [DE(\eta(t)) + D_2\tilde{R}(\eta(t),\partial_t\eta(t))]\langle\varphi(t)\rangle\,dt + \int_0^h \left\langle \rho\frac{\partial_t\eta(t)-\zeta}{h}, \varphi(t)\right\rangle_{L^2} dt \leq \int_0^h \langle f(t), \varphi(t)\rangle_{L^2}\,dt. \tag{4.24}$$

Furthermore, for every $t \in [0,h]$, η satisfies the energy inequality

$$E(\eta(t)) + 2\int_0^t \tilde{R}(\eta(s), \partial_t\eta(s))\, ds + \frac{\rho}{2h}\int_0^t \|\partial_t \eta(s)\|_{L^2}^2\, ds \\ \leq E(\eta_0) + \int_0^t \langle f(s), \partial_t\eta(s)\rangle_{L^2}^2\, ds + \frac{\rho}{2h}\int_0^t \|\zeta\|_{L^2}^2\, ds. \quad (4.25)$$

Proof. We note that since R satisfies (R′.3) then (R′.3$_q$) holds for $\tilde{R}$. Similarly, since (R′.1), (R′.2) and (R′.4) hold for R, they also hold for $\tilde{R}$. As a result, we can apply Theorem 4.3.

What is left is the energy inequality. For this we note that rewriting the energy inequality for $\tilde{R}$ and $\tilde{f}$ in terms of R and f and using Young's inequality yields

$$E(\eta(t)) - E(\eta(0)) + 2\int_0^t R(\eta(s), \partial_t\eta(s)) + \frac{\rho}{h}\|\partial_t\eta(s)\|_{L^2}\, ds \leq \int_0^t \left\langle f(s) + \rho\tfrac{\zeta}{h}, \partial_t\eta(s)\right\rangle_{L^2} ds$$
$$\leq \int_0^t \left[\langle f(s), \partial_t\eta(s)\rangle_{L^2} + \frac{\rho}{2h}\|\zeta(s)\|_{L^2}^2 + \frac{\rho}{2h}\|\partial_t\eta(s)\|_{L^2}^2\right] ds.$$

Rearranging the terms on both sides then results in the claim. □

4.2. **Existence of the contact force.** We now refine the results of the previous section to show that the inequality (4.3) can be characterized by the existence of a contact force.

Definition 4.5 (Solution with a contact force)**.** *Let $\eta_0 \in \mathcal{E} \cap W^{k_0,2}(Q;\mathbb{R}^n)$ and $f \in L^2((0,h); L^2(Q;\mathbb{R}^n))$ be given. We say that*

$$\eta \in W^{1,2}((0,h); W^{k_0,2}(Q;\mathbb{R}^n)) \cap L^\infty((0,h); \mathcal{E} \cap W^{k_0,2}(Q;\mathbb{R}^n)), \quad \sigma \in L^2_{w^*}((0,h); M(\partial Q;\mathbb{R}^n)),$$

where σ is a contact force for η (see Definition 3.6), is a solution with a contact force *to* (4.1) *if $\eta(0) = \eta_0$ and*

$$\int_0^h [DE(\eta(t)) + D_2R(\eta(t), \partial_t\eta(t))]\langle\varphi(t)\rangle\, dt = \int_0^h \langle\sigma(t), \varphi(t)\rangle\, dt + \int_0^h \langle f(t), \varphi(t)\rangle_{L^2}\, dt \quad (4.26)$$

for all $\varphi \in C([0,h]; W^{k_0,2}_\Gamma(Q;\mathbb{R}^n))$.

Theorem 4.6. *Let E satisfy* (E′.1)–(E′.5)*, R satisfy* (R′.1)*,* (R′.2)*,* (R′.3$_q$) *as well as* (R′.4)*. Given $\eta_0 \in \mathcal{E} \cap W^{k_0,2}(Q;\mathbb{R}^n)$ and $f \in L^2((0,h); L^2(Q;\mathbb{R}^n))$, there exists a solution with a contact force to* (4.1)*. Moreover, η satisfies the energy inequality* (4.4) *for all $t \in [0,h]$, and the contact force satisfies $\sigma \in L^2_{w^*}((0,h); M(\partial Q;\mathbb{R}^n))$ with estimate*

$$\int_0^h \|\sigma(t)\|_{M(\partial Q;\mathbb{R}^n)}^2 dt \leq C_{E_0}\left(h + \|\partial_t\eta\|_{L^2(W^{1,2})}^2 + \|f\|_{L^2(L^2)}^2\right). \quad (4.27)$$

Remark 4.7. *If we applied the Lagrange multiplier theorem to our constrained minimization problem, we would get the contact "force" only in the form of a distribution, namely as an element of $(W^{k_0,2}(Q;\mathbb{R}^n))^*$. An estimate in the same space for the time-continuous, parabolic solution is directly available by using the equation, cf.* [13]*. However* [15] *proved that in the steady case, one in fact obtains a measure–valued contact force in the sense our definition. We will use their argument here, but in addition to that we employ a quantitative estimate on the norm of the measure to retain the desired regularity when passing to the limit as $\tau \to 0$.*

Proof. We use the approximation η_τ resp. $\overline{\eta}_\tau$ and all other relevant notation from the proof of Theorem 4.3. Fix $k \in \{1,\dots,M\}$ and recall that we have that

$$D\mathcal{J}_k(\eta_k) = DE(\eta_k) + D_2R\left(\eta_{k-1}, \tfrac{\eta_k - \eta_{k-1}}{\tau}\right) - f_k \in (W^{k_0,2}(Q;\mathbb{R}^n))^*.$$

Step 1: By the assumptions on $\eta_0|_\Gamma$ (see Remark 2.3) and the local injectivity, we can pick a compact subset $K \subset \partial Q$ such that $\eta(t)|_{\partial Q\setminus K}$ is always injective. Then by the action-reaction principle, estimating $\|\sigma_k\|_{M(K;\mathbb{R}^n)}$ is enough to estimate $\|\sigma_k\|_{M(\partial Q;\mathbb{R}^n)}$. Without loss of generality, we can choose $\xi_\Gamma \in C^\infty_\Gamma(Q;[0,1])$ such that $\xi_\Gamma(x) = 1$ for all $x \in K$. We will now follow[6] [15] to get the existence of a

[6]To avoid confusion, we note that since we use interior normals, some signs and inequalities have to be turned around.

contact force for the auxiliary problem. Let us denote

$$M^+_{\eta_k} := \Big\{ (\ell, w) \in \mathbb{R} \times C(\partial Q) : \exists \varphi \in W^{k_0,2}_\Gamma(Q, \mathbb{R}^n) \text{ s.t. } D\mathcal{J}_k(\eta_k)\langle\varphi\rangle \le \ell$$

$$\text{and } \forall x \in C_{\eta_k} \text{ we have } \sum_{z\in\eta^{-1}(\eta(x))} \varphi(z)\cdot n_{\eta_k}(z) \ge \sum_{z\in\eta^{-1}(\eta(x))} w(z) \Big\},$$

$$M^- := \{(\ell, w) \in \mathbb{R} \times C(\partial Q) : \ell \le 0, w \ge 0\}.$$

Now, if $\ell < 0$ and $w > 0$, then any $\varphi \in W^{k_0,2}_\Gamma(Q;\mathbb{R}^n)$ satisfying

$$\sum_{z\in\eta^{-1}(\eta(z)))} \varphi(z)\cdot n_{\eta_k}(z) \ge \sum_{z\in\eta^{-1}(\eta(x))} w(z) > 0$$

for all $x \in C_\eta$ is an admissible test function by Proposition 3.21. Hence φ satisfies (by the previous proof) the variational inequality

$$D\mathcal{J}_k(\eta_k)\langle\varphi\rangle \ge 0 > \ell,$$

and thus $(\ell, w) \notin M^+_{\eta_k}$. This means that $M^+_{\eta_k} \cap \operatorname{int} M^- = \emptyset$. Since both sets are convex, we can find a separating hyperplane. That is, there exists $(\lambda_0, \sigma_0) \in (\mathbb{R}\times C(\partial Q))^* = \mathbb{R}\times M(\partial Q)$ with $(\lambda_0, \sigma_0) \ne (0, 0)$ such that

$$\begin{aligned} \lambda_0\ell + \langle\sigma_0, w\rangle &\le 0, \quad (\ell, w) \in M^+_{\eta_k}, \\ \lambda_0\ell + \langle\sigma_0, w\rangle &\ge 0, \quad (\ell, w) \in M^-. \end{aligned} \tag{4.28}$$

The latter inequality shows that for every $w \ge 0$, since $(0, w) \in M^-$, we have that $\langle\sigma_0, w\rangle \ge 0$. Hence the measure $\sigma_0 \in M^+(\partial Q)$ is non-negative. Furthermore, for any $w \in C(\partial Q)$ with $w \ge 0$ and $\operatorname{supp} w \cap C_{\eta_k} = \emptyset$ we see immediately that $(0, w) \in M^+_{\eta_k} \cap M^-$ and thus $\langle\sigma_0, w\rangle = 0$. This proves that $\operatorname{supp}\sigma_0 \subset C_{\eta_k}$.

Now, given $\varphi \in W^{k_0,2}_\Gamma(Q;\mathbb{R}^n)$ we have that $(D\mathcal{J}_k(\eta_k)\langle\varphi\rangle, n_{\eta_k}\cdot\varphi) \in M^+_{\eta_k}$, and so

$$\lambda_0 D\mathcal{J}_k(\eta_k)\langle\varphi\rangle + \langle\sigma_0, n_{\eta_k}\cdot\varphi\rangle \le 0.$$

Repeating the argument with $-\varphi$ in place of φ we obtain that

$$\lambda_0 D\mathcal{J}_k(\eta_k)\langle\varphi\rangle + \langle\sigma_0, n_{\eta_k}\cdot\varphi\rangle = 0. \tag{4.29}$$

Fix $\delta > 0$ and denote $\varphi_\delta = \delta\xi_\Gamma\tilde{n}_{\eta_k}$, where $\tilde{n}_{\eta_k}$ is the smooth almost normal from Corollary 3.16. Then $\varphi_\delta \in C^2_\Gamma(\overline{Q};\mathbb{R}^n)$ and moreover $\|\varphi_\delta\|_{W^{k_0,2}(Q;\mathbb{R}^n)} \le \tilde{C}\delta$, where by the estimate from Proposition 3.17 and the energy estimate (4.13), $\tilde{C}$ depends only on $C_0 = E(\eta_0) + \|f\|_{L^2(L^2)}$. Then, for every $x \in C_{\eta_k} \cap K$, we have that

$$\sum_{z\in\eta^{-1}(\eta(x))} \varphi_\delta(z)\cdot n_{\eta_k}(z) = \delta \sum_{z\in\eta^{-1}(\eta(x))} \tilde{n}_{\eta_k}(z)\cdot n_{\eta_k}(z) \ge \frac{\delta}{2}.$$

This shows that $(D\mathcal{J}_k(\eta_k)\langle\varphi_\delta\rangle, \delta/2) \in M^+_{\eta_k}$, where $\delta/2$ is treated as a constant function. Hence by (4.28) we obtain that

$$\lambda_0 D\mathcal{J}_k(\eta_k)\langle\varphi_\delta\rangle + \langle\sigma_0, \delta/2\rangle \le 0. \tag{4.30}$$

From this we can see that $\lambda_0 < 0$. Indeed, if $\lambda_0 = 0$, then $\sigma_0 \ne 0$ and $\langle\sigma_0, \delta/2\rangle > 0$, and we thus reach a contradiction; if $\lambda_0 > 0$, take $\ell \ge D\mathcal{J}_k(\eta_k)\langle\varphi_\delta\rangle$ with $\ell > 0$. Then $(\ell, \delta/2) \in M^+_{\eta_k}$ and $\lambda_0\ell + \langle\sigma_0, \delta/2\rangle > 0$, yielding a contradiction in this case as well.

Therefore, we can define $\sigma_k \in M(\partial Q;\mathbb{R}^n)$ via $d\sigma_k = -\frac{1}{\lambda_0} n_{\eta_k}\, d\sigma_0$, which is a contact force for η_k since we have verified that $\lambda_0 < 0$, $\sigma_0 \in M^+(\partial Q)$ and $\operatorname{supp}\sigma_0 \subset C_{\eta_k}$.

Step 2: Now we estimate

$$\|\sigma_k\|_{M(\partial Q;\mathbb{R}^n)} \le 2\|\sigma_k\|_{M(K;\mathbb{R}^n)} \le -\frac{2}{\lambda_0}\langle\sigma_0, \xi_\Gamma\rangle \le \frac{4}{\delta} D\mathcal{J}_k(\eta_k)\langle\varphi_\delta\rangle.$$

Setting $\sigma_\tau(t) := \sigma_k$ on $t \in [(k-1)\tau, k\tau)$, we multiply this last inequality by $\psi \in C([0, h];\mathbb{R}^+)$ and integrate over $[0, h]$ to obtain

$$\begin{aligned} \int_0^h \|\sigma_\tau(t)\|_{M(\partial Q;\mathbb{R}^n)}\, \psi(t)\, dt &\le 4\int_0^h D\mathcal{J}_{\lfloor t/\tau\rfloor}(\overline{\eta}_\tau(t))\langle\xi_\Gamma\tilde{n}_{\overline{\eta}_\tau}(t)\rangle\psi(t)\, dt \\ &\le 4\int_0^h \Big([DE(\overline{\eta}_\tau(t)) + D_2R(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t))]\langle\xi_\Gamma\tilde{n}_{\overline{\eta}_\tau}(t)\rangle - \langle f(t), \xi_\Gamma\tilde{n}_{\underline{\eta}_\tau}(t)\rangle_{L^2} \Big)\psi(t)\, dt. \end{aligned}$$

Since by (E′.5) DE is bounded on sublevel sets of E, in view of Proposition 3.17 we obtain that

$$\int_0^h DE(\overline{\eta}_\tau(t))\langle \xi_\Gamma \tilde{n}_{\overline{\eta}_\tau}(t)\rangle \psi(t)\,dt \le C_{k_0} \max_k \|DE(\eta_k)\|_{(W^{k_0,2})^*} \int_0^h \psi(t)\,dt.$$

Furthermore, by (2.3) we arrive at

$$\int_0^h D_2R(\underline{\eta}_\tau(t), \partial_t \eta_\tau(t))\langle \xi_\Gamma \tilde{n}_{\overline{\eta}_\tau}(t)\rangle \psi(t)\,dt \le C_{E_0} \int_0^h \|\partial_t \eta_\tau\|_{W^{1,2}} \psi(t)\,dt.$$

Finally, since the remaining term satisfies the estimate

$$\int_0^h \langle f(t), \xi_\Gamma \tilde{n}_{\overline{\eta}_\tau}(t)\rangle \psi(t)\,dt \le C_{E_0} \int_0^h \|f(t)\|_{L^2} \psi(t)\,dt,$$

we conclude that

$$\int_0^h \|\sigma_\tau(t)\|^2_{M(\partial Q;\mathbb{R}^n)} dt \le C_{E_0} \left(h + \|\partial_t \eta_\tau\|^2_{L^2(W^{1,2})} + \|f\|^2_{L^2(L^2)}\right). \tag{4.31}$$

In view of the energy estimate, the right-hand side in (4.31) is bounded by a constant (independent of τ). This means that we have estimated σ_τ in $L^2_{w^*}((0,h); M(\partial Q;\mathbb{R}^n))$, so in particular also in $M([0,h]\times \partial Q;\mathbb{R}^n)$.

Step 3: Upon dividing by $-\lambda_0$ and summing over k, (4.29) can be rewritten as

$$\int_0^h D\mathcal{J}_{\lfloor t/\tau \rfloor}(\overline{\eta}_\tau(t))\langle \varphi\rangle - \langle \sigma_\tau(t), \varphi\rangle\,dt = 0.$$

By compactness of contact forces (see Theorem 3.9) we can find a converging subsequence (which we do not relabel) such that

$$\sigma_\tau \overset{*}{\rightharpoonup} \sigma \quad \text{in } M([0,h]\times\partial Q;\mathbb{R}^n),$$

where σ is a contact force for η. By this convergence, upon sending $\tau \to 0$, the equation (4.26) holds for all $\varphi \in W^{k_0,2}_\Gamma(Q;\mathbb{R}^n)$, and by a density argument for all $\varphi \in C([0,h], W^{k_0,2}_\Gamma(Q;\mathbb{R}^n))$. Moreover, thanks to (4.31), we are in a position to apply Lemma 3.13 to conclude that $\sigma \in L^2_{w^*}((0,h); M(\partial Q;\mathbb{R}^n))$ and that it satisfies (4.27). This concludes the proof. ☐

We now state a version of Corollary 4.4 that includes contact forces.

Corollary 4.8. *Let E and η_0 be given as above and assume that for some $h > 0$*

$$\tilde{R}(\eta, b) := R(\eta, b) + \frac{\rho}{2h}\|b\|^2_{L^2}, \tag{4.32}$$

where R satisfies (R′.1)–(R′.4). Define additionally $\tilde{f} := f + \rho\frac{\zeta}{h}$ for some $\zeta \in L^2((0,h); L^2(Q;\mathbb{R}^n))$. Then the conclusions of Theorem 4.6 continue to hold. In particular, this yields the existence of a weak solution with a contact force to (4.2) (in the sense of Definition 4.5), where (4.26) can be rewritten as

$$\int_0^h \left[DE(\eta(t)) + D_2R(\eta(t), \partial_t \eta(t))\right]\langle \varphi(t)\rangle\,dt + \int_0^h \rho\left\langle \frac{\partial_t \eta(t) - \zeta(t)}{h}, \varphi(t)\right\rangle_{L^2} dt = \int_0^h \langle \sigma(t), \varphi(t)\rangle\,dt + \int_0^h \langle f(t), \varphi(t)\rangle_{L^2}\,dt. \tag{4.33}$$

Furthermore, for every $t \in [0,h]$, η satisfies the energy inequality in (4.25) and the contact force satisfies $\sigma \in L^2_{w^*}((0,h); M(\partial Q;\mathbb{R}^n))$ with

$$\int_0^h \|\sigma(t)\|^2_{M(\partial Q;\mathbb{R}^n)} dt \le C_{E_0}\left(h + \frac{\|\zeta\|^2_{L^2(W^{1,2})}}{h} + \|f\|^2_{L^2(L^2)}\right). \tag{4.34}$$

Proof. The result is an immediate consequence of Theorem 4.6 and Corollary 4.4, thus we omit the details. ☐

5. Proof of the main theorem

In this section we present the proof of Theorem 2.5. Additionally we conclude the section by returning to the corresponding quasistatic problem. In particular, we present an improvement of the existence result of a solution with contact force that was previously obtained by Krömer and Roubíček in [13].

Proof of Theorem 2.5. We divide the proof into several steps.
Step 1: Given $L \in \mathbb{N}$, we set $h := T/L$ and decompose $[0,T]$ into subintervals $[(j-1)h, jh]$, with $j \geq 1$, of length h. Let $E^{(h)} \colon W^{k_0,2}(Q;\mathbb{R}^n) \to (-\infty,\infty]$ be defined via

$$E^{(h)}(\eta) := E(\eta) + h^{a_0}\|\nabla^{k_0}\eta\|_{L^2}^2, \tag{5.1}$$

where $k_0 - \frac{n}{2} > 2 - \frac{n}{p}$ and $a_0 > 0$ is to be determined. Let η_0 be given as in the statement. In this step we prove that there exists a sequence $\{\eta_0^{(h)}\}_h \subset W^{k_0,2}(Q;\mathbb{R}^n)$ such that $\eta_0^{(h)} \to \eta_0$ in $W^{2,p}(Q;\mathbb{R}^n)$ as $h \to 0$, and for all but finitely many values of h we have that $\eta_0^{(h)}(Q) \subset \Omega$ and $\eta_0^{(h)}$ is a.e. globally injective on $\overline{Q}$. Furthermore, we show that this sequence can be chosen with the property that

$$E^{(h)}(\eta_0^{(h)}) \to E(\eta_0). \tag{5.2}$$

We present the proof for the case $\eta_0 \in \partial\mathcal{E}$. Indeed, if $\eta_0 \in \operatorname{int}\mathcal{E}$ the proof follows from a similar but simpler argument. We begin by considering perturbations of the initial condition of the form $\eta_0 + h^{b_0}\tilde{n}_{\eta_0}$, with $b_0 > 0$, where $\tilde{n}_{\eta_0}$ is given as in Proposition 3.16. Reasoning as in Proposition 3.21 (see also Proposition 3 in [15]), if h is sufficiently small we have that $(\eta_0 + h^{b_0}\tilde{n}_{\eta_0})(Q) \subset \Omega$ and furthermore $\eta_0 + h^{b_0}\tilde{n}_{\eta_0}$ is a.e. globally injective on $\overline{Q}$. Recall that $\eta_0 + h^{b_0}\tilde{n}_{\eta_0}$ is injective in $B_r(x) \cap \overline{Q}$ for all x provided that r is given as in Lemma 3.5. Moreover, if h is sufficiently small, for x, y with $|x-y| \geq r$ we have that

$$|\eta_0(x) + h^{b_0}\tilde{n}_{\eta_0}(x) - \eta_0(y) - h^{b_0}\tilde{n}_{\eta_0}(y)| \geq ch^{b_0}$$

for some $c > 0$. Similarly, $\operatorname{dist}(\eta_0(x) + h^{b_0}\tilde{n}_{\eta_0}(x), \partial\Omega) \geq ch^{b_0}$ for all $x \in \overline{Q}$. This implies that there exist a smooth open set Q' such that $Q \subset Q'$ and $\operatorname{dist}(Q, \partial Q') \geq ch^{b_0}$, and an extension of $\eta_0 + h^{b_0}\tilde{n}_{\eta_0}$ to Q' (not relabeled) with the property that $(\eta_0 + h^{b_0}\tilde{n}_{\eta_0})(Q') \subset \Omega$ and such that $\eta_0 + h^{b_0}\tilde{n}_{\eta_0}$ is a.e. globally injective on $\overline{Q'}$. Fix $b_1 > b_0$ and consider

$$\xi_h(x) := \frac{1}{h^{nb_1}}\xi\left(\frac{x}{h^{b_1}}\right), \tag{5.3}$$

where ξ is the standard mollifier. We then define

$$\eta_0^{(h)}(x) := [(\eta_0 + h^{b_0}\tilde{n}_{\eta_0}) * \xi_h](x) \tag{5.4}$$

and claim that $\eta_0^{(h)}$ has the desired properties for all h sufficiently small. Since $\eta_0^{(h)} \to \eta_0$ in $W^{2,p}(Q;\mathbb{R}^n)$, (E.4) implies that $E(\eta_0^{(h)}) \to E(\eta_0)$. Moreover, as one can readily check, we have that

$$\|\nabla^{k_0}\eta_0^{(h)}\|_{L^2} \leq Ch^{b_1(2-k_0)}(\|\eta_0\|_{W^{2,p}} + h^{b_0}\|\tilde{n}_{\eta_0}\|_{C^2}). \tag{5.5}$$

Consequently, if $a_0 > b_1(k_0 - 2)$, by (5.5) we obtain that

$$h^{a_0}\|\nabla^{k_0}\eta_0^{(h)}\|_{L^2} \to 0$$

as $h \to 0$, thus proving (5.2).
Step 2: Next, we consider the time-delayed equation

$$\rho\frac{\partial_t\eta(t) - \partial_t\eta(t-h)}{h} + DE^{(h)}(\eta(t)) + D_2R^{(h)}(\eta(t), \partial_t\eta(t)) = f(t), \tag{5.6}$$

on the interval $[0,h]$, where $E^{(h)}$ is defined as in (5.1) and

$$R^{(h)}(\eta, b) := R(\eta, b) + h\|\nabla^{k_0}b\|_{L^2}^2.$$

Problem (5.6) is complemented by the initial condition $\eta(0) = \eta_0^{(h)}$, with $\eta_0^{(h)}$ given as in the previous step. Moreover, we define $\partial_t\eta(t) := \eta^*$ for all $t < 0$. Notice that (5.6) can be recast as

$$D\tilde{E}(\eta) + D_2\tilde{R}(\eta, \partial_t\eta) = \tilde{f}, \tag{5.7}$$

where

$$\tilde{E}(\eta) := E^{(h)}(\eta), \quad \tilde{R}(\eta, b) := R^{(h)}(\eta, b) + \frac{1}{2h}\|b\|_{L^2}^2, \quad \tilde{f}(t) := f(t) + \frac{\rho}{h}\partial_t\eta(t-h). \tag{5.8}$$

As one can readily check, the energy-dissipation pair $(\tilde{E}, \tilde{R})$ satisfies the assumptions of Section 4. Thus, an application of Corollary 4.4 yields the existence of a solution to (5.7) in $[0,h]$, denoted $\eta_1^{(h)}$. For $j \geq 2$, we can then iteratively construct solutions to (5.7) in each of the intervals $[(j-1)h, jh]$, with the difference, however, that the initial condition for η and the time-shifted derivative $\partial_t\eta(t-h)$ are given by the solution in the previous time interval. To be precise, $\eta_j^{(h)}$ is a solution to (5.7) on the interval $[(j-1)h, jh]$ with initial condition $\eta_j^{(h)}((j-1)h) = \eta_{j-1}^{(h)}((j-1)h)$ and forcing term (see (5.8)) given by

$$\tilde{f}(t) := f(t) + \frac{\rho}{h}\partial_t\eta_{j-1}^{(h)}(t-h).$$

Note that the result of each such step yields well posed data for the next step by the energy estimate (4.25).

We then define

$$\eta^{(h)}(t) := \eta_j^{(h)}(t) \tag{5.9}$$

for all $t \in [(j-1)h, jh]$.

Step 3: Recall that each of the $\eta_j^{(h)}$ satisfies the energy inequality (4.25) with the respective initial time $(j-1)h$ and corresponding initial data. Fix $t \in [(j-1)h, jh]$. Adding together the energy estimate for $\eta_j^{(h)}$ as well as all those for the previous steps with respect to the final time of the corresponding interval, we end up with

$$\begin{aligned} E^{(h)}(\eta^{(h)}(t)) &+ \int_0^t 2R^{(h)}(\eta^{(h)}(s), \partial_t\eta^{(h)}(s))\,ds + \frac{\rho}{2h}\int_{t-(j-1)h}^t \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds \\ &\le E^{(h)}(\eta_0^{(h)}) + \frac{\rho}{2}\|\eta^*\|_{L^2}^2 + \int_0^t \langle f(s), \partial_t\eta^{(h)}(s)\rangle_{L^2}\,ds \\ &\le E^{(h)}(\eta_0^{(h)}) + \frac{\rho}{2}\|\eta^*\|_{L^2}^2 + \frac{1}{2\delta}\|f\|_{L^2(L^2)}^2 + \frac{\delta}{2}\int_0^t \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds \\ &\le C + \frac{\delta}{2}\int_0^T \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds, \end{aligned} \tag{5.10}$$

where C is a constant that depends only on η_0, η^*, ρ, T, and f. In view of (E.1) and recalling that $R^{(h)}$ is nonnegative, the previous inequality implies that

$$\frac{\rho}{2h}\int_{t-(j-1)h}^t \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds \le C + \frac{\delta}{2}\int_0^T \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds. \tag{5.11}$$

If we now let $t = T$, by multiplying both side of (5.11) by h and summing over j we obtain

$$\frac{\rho}{2}\int_0^T \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds \le Lh\left(C + \frac{\delta}{2}\int_0^T \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds\right).$$

Recalling that $Lh = T$ and setting $\delta := T/2$ gives the bound

$$\int_0^T \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds \le C, \tag{5.12}$$

where C is a constant independent of h. Furthermore, as an immediate consequence of (5.10) and (5.12), we obtain

$$\begin{aligned} \sup_{t\in[0,T]} \left\{E^{(h)}(\eta^{(h)}(t)) + \frac{\rho}{2h}\int_{t-h}^t \|\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds\right\} &\le C, \\ \int_0^T R^{(h)}(\eta^{(h)}(s), \partial_t\eta^{(h)}(s))\,ds &\le C. \end{aligned} \tag{5.13}$$

In view of (5.8), (5.13) implies that

$$\int_0^T \|\nabla\partial_t\eta^{(h)}(s)\|_{L^2}^2 + h\|\nabla^{k_0}\partial_t\eta^{(h)}(s)\|_{L^2}^2\,ds \le C. \tag{5.14}$$

From (5.13) we see that $\{\eta^{(h)}\}_h$ is bounded in $L^\infty((0,T); W^{2,p}(Q;\mathbb{R}^n))$. Furthermore, (5.13) and (5.14) imply that $\partial_t\eta^{(h)}$ is uniformly bounded in $L^2((0,T); W^{1,2}(Q;\mathbb{R}^n))$, from which it follows that $\eta^{(h)}$ is bounded in $W^{1,2}((0,T); W^{1,2}(Q;\mathbb{R}^n))$. In particular, we obtain that (eventually extracting a subsequence, which we do not relabel)

$$\begin{aligned} \eta^{(h)} &\overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty((0,T); W^{2,p}(Q;\mathbb{R}^n)), \\ \eta^{(h)} &\rightharpoonup \eta \quad \text{in } W^{1,2}((0,T); W^{1,2}(Q;\mathbb{R}^n)), \end{aligned} \tag{5.15}$$

to some function $\eta \in L^\infty((0,T); W^{2,p}(Q;\mathbb{R}^n)) \cap W^{1,2}((0,T); W^{1,2}(Q;\mathbb{R}^n))$. An application of the classical Aubin–Lions lemma can be used to show that, up to the extraction of a further subsequence,

$$\eta^{(h)} \to \eta \text{ in } C^0([0,T]; C^{1,\alpha}(Q;\mathbb{R}^n)), \tag{5.16}$$

where α is any number in $(0, 1-n/p)$. In particular, we have that $\eta \in L^\infty((0,T);\mathcal{E})$. Note also that with these convergences and the lower-semicontinuity of its left hand side, the first inequality in (5.10) converges to the desired energy inequality.

Step 4: Recall that each of the $\eta_j^{(h)}$ solves a variational inequality on the interval $[(j-1)h, jh]$. Rewriting these inequalities in terms of $\eta^{(h)}$ (see (5.9)) and summing over j yields that

$$\int_0^T \left[DE^{(h)}(\eta^{(h)}(t)) + D_2R^{(h)}(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\right]\langle\varphi(t)\rangle\,dt \\ + \int_0^T \frac{\rho}{h}\langle\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), \varphi(t)\rangle_{L^2}\,dt \geq \int_0^T \langle f(t), \varphi(t)\rangle_{L^2}\,dt \tag{5.17}$$

holds for all $\varphi \in C([0,T]; T_{\eta^{(h)}}(\mathcal{E}))$, as well as

$$\int_0^T \left[DE^{(h)}(\eta^{(h)}(t)) + D_2R^{(h)}(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\right]\langle\varphi(t)\rangle\,dt \\ + \int_0^T \frac{\rho}{h}\langle\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), \varphi(t)\rangle_{L^2}\,dt = \int_0^T \langle f(t), \varphi(t)\rangle_{L^2} + \langle\sigma^{(h)}(t), \varphi(t)\rangle\,dt \tag{5.18}$$

for all $\varphi \in C([0,T], W_\Gamma^{k_0,2}(Q;\mathbb{R}^n))$. The goal of this step is to obtain the technical intermediate results needed to pass to the limit as $h \to 0^+$ (or, alternatively, as $L \to \infty$) in (5.17) and (5.18). As observed in [3], the main difficulty is in passing to the limit with $DE(\eta^{(h)})$. Indeed, by assumption DE is only continuous with respect to the strong topology of $W^{2,p}$. To improve the convergence of $\eta^{(h)}$ from what is provided by (5.15) and (5.16), we rely on the Minty property of DE (that is, (E.6)). A key step in this direction is to obtain a uniform estimate in $L^2((0,T); L^2(Q;\mathbb{R}^n))$ for the difference quotients that approximate the inertial term. To this end, we let

$$b^{(h)} := \fint_t^{t+h} \partial_t\eta^{(h)}(s)\,ds = \frac{\eta^{(h)}(t+h) - \eta^{(h)}(t)}{h}. \tag{5.19}$$

Then, (5.13) and (5.14) imply that $b^{(h)}$ is bounded in $L^2((0,T); W^{1,2}(Q;\mathbb{R}^n))$. Recall that if $\varphi \in C_c^\infty((0,T)\times Q;\mathbb{R}^n)$ then (5.17) holds as an equality. Then, reasoning as in Lemma 3.7 in [3], we obtain that $\partial_t b^{(h)}$ is bounded in $L^2((0,T); W^{-k_0,2}(Q;\mathbb{R}^n))$. Indeed, we have that

$$\int_0^T \frac{\rho}{h}\langle\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), \varphi(t)\rangle_{L^2}\,dt \leq \int_0^T \Big(\|DE(\eta^{(h)}(t))\|_{(W^{2,p})^*} + h^{a_0}\|\nabla^{k_0}\eta^{(h)}(t)\|_{L^2} \\ + \|D_2R(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\|_{(W^{1,2})^*} \\ + h\|\nabla^{k_0}\partial_t\eta^{\eta}(t)\|_{L^2} + \|f(t)\|_{L^\infty}\Big)\|\varphi(t)\|_{W^{k_0,2}}\,dt. \tag{5.20}$$

In view of (5.13), (5.14), (E.5), and (R.4), it follows from (5.20) that

$$\int_0^T \rho\left\|\frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h}\right\|^2_{W^{-k_0,2}} dt \leq C, \tag{5.21}$$

for some constant C independent of h. Consequently, an application of the Aubin–Lions lemma shows that $b^{(h)}$ is precompact in $L^2((0,T); L^2(Q;\mathbb{R}^n))$. As one can readily check (see in particular Lemma 3.8 in [3]), up to the extraction of a subsequence (which we do not relabel), we have that $b^{(h)} \to \partial_t\eta$ in $L^2((0,T); L^2(Q;\mathbb{R}^n))$.

Regarding the contact force, we can apply Corollary 4.8 with $E^{(h)}$, $R^{(h)}$, f and $\zeta = \partial_t\eta^{(h)}(\cdot - h)$, and use $\xi_\Gamma\tilde{n}_{\eta^{(h)}}$ as a test function in (4.33), where $\xi_\Gamma \in C_\Gamma^\infty(Q;[0,1])$ is given as in the proof of Theorem 4.6. Note that this is allowed, as it follows from Proposition 3.17 that $\tilde{n}_{\eta^{(h)}} \in L^\infty((0,T); C^{k_0}(\overline{Q};\mathbb{R}^n))$. Using this together with the fact that $\|\sigma^{(h)}(t)\|_{M(\partial Q;\mathbb{R}^n)} \leq 4\langle\sigma^{(h)}(t), \xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle$, we obtain that for $\mathcal{L}^1$-a.e.

$t \in (0,T)$

$$\frac{1}{4}\|\sigma^{(h)}(t)\|_{M(\partial Q;\mathbb{R}^n)} \leq \left[DE^{(h)}(\eta^{(h)}(t)) + D_2R^{(h)}(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\right]\langle\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle + \rho\left\langle\frac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\right\rangle_{L^2} - \langle f(t),\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle_{L^2}.$$

Reasoning as in the proof of Theorem 4.6, we multiply the previous inequality by $\psi \in L^\infty((0,T);\mathbb{R}^+)$ and integrate the resulting expression over $(0,T)$ to obtain

$$\begin{aligned}\frac{1}{4}\int_0^T \|\sigma^{(h)}(t)\|_{M(\partial Q;\mathbb{R}^n)}\psi(t)dt \leq \int_0^T \Big\{ &\left[DE(\eta^{(h)}(t)) + D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\right]\langle\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle \\ &+ 2h^{a_0}\left\langle\nabla^{k_0}\eta^{(h)}(t),\nabla^{k_0}(\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t))\right\rangle_{L^2} + 2h\left\langle\nabla^{k_0}\partial_t\eta^{(h)}(t),\nabla^{k_0}(\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t))\right\rangle_{L^2} \\ &+ \rho\left\langle\frac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\right\rangle_{L^2} - \langle f(t),\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle_{L^2}\Big\}\psi(t)\,dt. \end{aligned} \tag{5.22}$$

We now estimate all the terms on the right hand side. Recall that $\left\|\xi_\Gamma\tilde{n}_{\eta^{(h)}}\right\|_{L^\infty(C^{k_0})} \leq C$ by Proposition 3.17. By (5.13) and (E.5) we see that

$$\begin{aligned}\int_0^T DE(\eta^{(h)}(t))\langle\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle\psi(t)\,dt \\ \leq \left\|DE(\eta^{(h)})\right\|_{L^\infty((W^{2,p})^*)}\left\|\xi_\Gamma\tilde{n}_{\eta^{(h)}}\right\|_{L^2(W^{2,p})}\|\psi\|_{L^2} \leq C\|\psi\|_{L^2};\end{aligned} \tag{5.23}$$

similarly, by (5.14) and (2.3) it follows that

$$\begin{aligned}\int_0^T D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\langle\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle\psi(t)\,dt \\ \leq \left\|D_2R(\eta^{(h)},\partial_t\eta^{(h)})\right\|_{L^2((W^{1,2})^*)}\left\|\xi_\Gamma\tilde{n}_{\eta^{(h)}}\right\|_{L^\infty(W^{1,2})}\|\psi\|_{L^2} \leq C\|\psi\|_{L^2}.\end{aligned} \tag{5.24}$$

Notice also that by (5.13) we have that

$$\begin{aligned}\int_0^T 2h^{a_0}\left\langle\nabla^{k_0}\eta^{(h)}(t),\nabla^{k_0}(\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t))\right\rangle_{L^2}\psi(t)\,dt \\ \leq 2h^{a_0}\left\|\nabla^{k_0}\eta^{(h)}(t)\right\|_{L^\infty(L^2)}\left\|\xi_\Gamma\tilde{n}_{\eta^{(h)}}\right\|_{L^2(W^{k_0,2})}\|\psi\|_{L^2} \leq h^{\frac{a_0}{2}}C\|\psi\|_{L^2},\end{aligned} \tag{5.25}$$

that an application of (5.14) yields

$$\begin{aligned}\int_0^T 2h\left\langle\nabla^{k_0}\partial_t\eta^{(h)}(t),\nabla^{k_0}(\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t))\right\rangle_{L^2}\psi(t)\,dt \\ \leq 2h\left\|\nabla^{k_0}\partial_t\eta\right\|_{L^2(L^2)}\left\|\xi_\Gamma\tilde{n}_{\eta^{(h)}}\right\|_{L^2(W^{k_0,2})}\|\psi\|_{L^2} \leq \sqrt{h}C\|\psi\|_{L^2},\end{aligned} \tag{5.26}$$

and that the estimate

$$\int_0^T -\langle f(t),\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\rangle_{L^2}\psi(t)\,dt \leq \|f\|_{L^2(L^2)}\left\|\xi_\Gamma\tilde{n}_{\eta^{(h)}}\right\|_{L^\infty(L^2)}\|\psi\|_{L^2} \leq C\|\psi\|_{L^2} \tag{5.27}$$

follows as simple consequence of Hölder's inequality.

Finally, for the inertial term, we first consider the case where $\psi \equiv 1$. Then we have by a discrete partial integration (i.e., by a change of variables)

$$\begin{aligned}&\int_0^T \rho\left\langle\frac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\right\rangle_{L^2}\psi(t)\,dt \\ &= \int_0^{T-h}\rho\left\langle\partial_t\eta^{(h)}(t),\xi_\Gamma\frac{\tilde{n}_{\eta^{(h)}}(t)-\tilde{n}_{\eta^{(h)}}(t+h)}{h}\right\rangle_{L^2}dt \\ &+ \int_{T-h}^T\frac{\rho}{h}\left\langle\partial_t\eta^{(h)}(t),\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\right\rangle_{L^2}dt - \int_0^h\frac{\rho}{h}\left\langle\partial_t\eta^{(h)}(t-h),\xi_\Gamma\tilde{n}_{\eta^{(h)}}(t)\right\rangle_{L^2}dt.\end{aligned}$$

For the second to last term on the right-hand side of the identity above, we estimate

$$\begin{aligned}\int_{T-h}^{T} \frac{\rho}{h} \left\langle \partial_t \eta^{(h)}(t), \xi_\Gamma \tilde{n}_{\eta^{(h)}}(t) \right\rangle_{L^2} dt &\leq \frac{\rho}{h} \left(\int_{T-h}^{T} \left\| \partial_t \eta^{(h)}(t) \right\|_{L^2}^2 dt \right)^{1/2} \left(\int_{T-h}^{T} \left\| \xi_\Gamma \tilde{n}_{\eta^{(h)}}(t) \right\|_{L^2}^2 dt \right)^{1/2} \\ &\leq \rho \left(\fint_{T-h}^{T} \left\| \partial_t \eta^{(h)}(t) \right\|_{L^2}^2 dt \right)^{1/2} \left\| \xi_\Gamma \tilde{n}_{\eta^{(h)}} \right\|_{L^\infty(L^2)} \leq C\end{aligned}$$

and a similar, easier estimate holds for the first, as $\partial_t \eta^{(h)}(t-h)$ for $t < h$ is in fact already given by the initial data. This leaves us with

$$\int_0^{T-h} \rho \left\langle \partial_t \eta^{(h)}(t), \xi_\Gamma \frac{\tilde{n}_{\eta^{(h)}}(t) - \tilde{n}_{\eta^{(h)}}(t+h)}{h} \right\rangle_{L^2} dt \leq C\rho \left\| \partial_t \eta^{(h)} \right\|_{L^2(L^2)} \left\| \partial_t \tilde{n}_{\eta^{(h)}} \right\|_{L^2(L^2)} \leq C.$$

Together with (5.22)–(5.27) this immediately shows that

$$\int_0^T \|\sigma^{(h)}(t)\|_{M(\partial Q;\mathbb{R}^n)} dt \leq C,$$

thus proving that $\sigma^{(h)}$ is uniformly bounded in $L^1([0,T]; M(\partial Q;\mathbb{R}^n))$.

Finally we want to prove the absence of concentrations. For this, fix $t_0 \in [0,T)$, $\delta > 0$ and repeat the previous estimate with ψ as the characteristic function of the small time interval $[t_0, t_0+\delta]$. As the normal direction cannot change too abruptly, if δ is small enough, we can choose $\tilde{n}_{\eta^{(h)}}$ to be piecewise constant in time. With this, as parts of the interval now cancels, we can estimate the inertial term using

$$\begin{aligned}&\int_{t_0}^{t_0+\delta} \rho \left\langle \frac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h}, \xi_\Gamma \tilde{n}_\eta^{(h)}(t) \right\rangle_{L^2} dt \\ &= \fint_{t_0+\delta-h}^{t_0+\delta} \rho \left\langle \partial_t \eta^{(h)}(t), \xi_\Gamma \tilde{n}_\eta^{(h)}(t) \right\rangle_{L^2} dt - \fint_{t_0-h}^{t_0} \rho \left\langle \partial_t \eta^{(h)}(t), \xi_\Gamma \tilde{n}_\eta^{(h)}(t+h) \right\rangle_{L^2} dt \\ &\leq \fint_{t_0+\delta-h}^{t_0+\delta} \rho \left\| \partial_t \eta^{(h)}(t) \right\|_{L^2} \left\| \xi_\Gamma \tilde{n}_\eta^{(h)}(t) \right\|_{L^2} dt + \fint_{t_0-h}^{t_0} \rho \left\| \partial_t \eta^{(h)}(t) \right\|_{L^2} \left\| \xi_\Gamma \tilde{n}_\eta^{(h)}(t+h) \right\|_{L^2} dt \\ &\leq C \left(\sqrt{\fint_{t_0+\delta-h}^{t_0+\delta} \|\partial_t \eta^{(h)}(t)\|_{L^2}^2 dt} + \sqrt{\fint_{t_0-h}^{t_0} \|\partial_t \eta^{(h)}(t)\|_{L^2}^2 dt} \right) \sup_{t\in[0,T]} \sqrt{|\operatorname{supp} \tilde{n}_{\eta^{(h)}}(t)|} \\ &\leq C \sup_{t\in[0,T]} \sqrt{|\operatorname{supp} \tilde{n}_{\eta^{(h)}}(t)|},\end{aligned} \tag{5.28}$$

where in the last step we have used (5.13).

From the construction of $\tilde{n}_{\eta^{(h)}}$ (see Proposition 3.17), it is clear that the support can be chosen arbitrarily small, albeit at the cost of increasing the norms of its derivatives. For any $\varepsilon > 0$, we can thus choose $\tilde{n}_{\eta^{(h)}}$ in such a way that the last term in (5.28) is bounded by $\varepsilon/2$. As all the other terms are estimated in terms of $\|\psi\|_{L^2}$ (see (5.23)–(5.27)) with constant depending only on energy bounds and the choice of support of $\tilde{n}_{\eta^{(h)}}$, we can now choose δ small enough so that the contributions from all these remaining terms also sum up to at most $\varepsilon/2$. Then

$$\frac{1}{4} \int_{t_0}^{t_0+\delta} \|\sigma^{(h)}(t)\|_{M(\partial Q;\mathbb{R}^n)} \psi(t) dt < \varepsilon \tag{5.29}$$

independently of t_0 and h, which proves that $\sigma^{(h)} \in L^1((0,T); M(\partial Q;\mathbb{R}^n))$ in fact cannot develop concentrations in time.

Step 5: With this at hand, we proceed to prove that $\eta^{(h)}(t) \to \eta(t)$ in $W^{2,p}(K;\mathbb{R}^n)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, where $K \subset \overline{Q}$ is such that $\operatorname{dist}(K,\Gamma) > 0$. Let $\tilde{\eta}^{(h)}$ be the map obtained by considering the convolution of an extension of η (given by a standard extension operator) with a standard mollifier with parameter h^{b_1} (defined in space-time similarly to (5.3)). Let $\psi \in C^\infty_\Gamma(Q;[0,1])$ be given. Straightforward computations show that

$$\begin{aligned}\|\tilde{\eta}^{(h)}\psi\|_{W^{k_0,2}} &\leq Ch^{b_1(2-k_0)} \|\eta\|_{W^{2,p}}, \\ \|\partial_t \tilde{\eta}^{(h)}\psi\|_{W^{k_0,2}} &\leq Ch^{b_1(1-k_0)} \|\partial_t \eta\|_{W^{1,2}},\end{aligned}$$

where C is a constant that does not depend on h. Let

$$\varphi^{(h)} := (\eta^{(h)} - \tilde{\eta}^{(h)})\psi.$$

Then, by (E.6) we have that

$$\begin{aligned}0 &\le \limsup_{h\to 0} \int_0^T \Big[DE(\eta^{(h)}(t)) - DE(\eta(t))\Big]\langle(\eta^{(h)}(t)-\eta(t))\psi(t)\rangle\,dt\\ &= \limsup_{h\to 0} \int_0^T DE(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt\\ &= \limsup_{h\to 0} \int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt - 2h^{a_0}\int_0^T \langle\nabla^{k_0}\eta^{(h)}(t), \nabla^{k_0}\varphi^{(h)}(t)\rangle_{L^2}\,dt, \qquad (5.30)\end{aligned}$$

where the first equality is obtained by using that $\tilde{\eta}^{(h)} \to \eta$ in $L^2((0,T); W^{2,p}(Q;\mathbb{R}^n))$ together with the fact that $DE(\eta^{(h)})$ is uniformly bounded in $L^2((0,T); (W^{2,p}(Q;\mathbb{R}^n))^*)$ as a consequence of (E.5). Observe that

$$\begin{aligned}\langle\nabla^{k_0}\eta^{(h)}(t), \nabla^{k_0}(\eta^{(h)}(t)\psi(t))\rangle_{L^2} &= \langle\nabla^{k_0}\eta^{(h)}(t), (\nabla^{k_0}\eta^{(h)}(t))\psi(t) + L^{(h)}(t)\rangle_{L^2}\\ &\ge \langle\nabla^{k_0}\eta^{(h)}(t), L^{(h)}(t)\rangle_{L^2},\end{aligned}$$

where $L^{(h)}$ only involves derivatives of $\eta^{(h)}$ of order less than k_0. Using the fact that $\eta^{(h)}$ is bounded in $L^\infty((0,T); W^{2,p}(Q;\mathbb{R}^n))$ (see (5.13)) and that $h^{a_0}\nabla^{k_0}\eta^{(h)}$ is bounded in $L^\infty((0,T); L^2(Q;\mathbb{R}^n))$, a standard interpolation inequality shows that $h^{a_0}\|L^{(h)}\|_{L^\infty L^2} \to 0$. Combining this with (5.30) we obtain that

$$\begin{aligned}0 &\le \limsup_{h\to 0} \int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt + Ch^{a_0}\int_0^T \|\nabla^{k_0}\eta^{(h)}(t)\|_{L^2}\|\tilde{\eta}^{(h)}(t)\psi(t)\|_{W^{k_0,2}}\,dt\\ &\le \limsup_{h\to 0} \int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt + Ch^{\frac{a_0}{2}-b_1(2-k_0)}\\ &= \limsup_{h\to 0} \int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt, \qquad (5.31)\end{aligned}$$

where the last step follows by choosing b_1 sufficiently small. Observe that this condition is strictly stronger than what is required on b_1 from the first step. We can then use (5.18) to rewrite the last term in (5.31) as

$$\begin{aligned}\int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt = &-\int_0^T D_2R^{(h)}(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt\\ &-\int_0^T \frac{\rho}{h}\langle\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), \varphi^{(h)}(t)\rangle_{L^2}\,dt\\ &+\int_0^T \langle f(t), \varphi^{(h)}(t)\rangle_{L^2}\,dt + \int_0^T \langle\sigma^{(h)}, \varphi^{(h)}(t)\rangle dt.\end{aligned}$$

Notice that

$$\begin{aligned}D_2R^{(h)}(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle &= D_2R(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\\ &\quad + 2h\langle\nabla^{k_0}\partial_t\eta^{(h)}(t), \nabla^{k_0}\varphi^{(h)}(t)\rangle_{L^2}. \qquad (5.32)\end{aligned}$$

In view of (R.4) and (5.15), we have that $D_2R(\eta^{(h)}(t), \partial_t\eta^{(h)}) \to D_2R(\eta(t), \partial_t\eta(t))$ in $W^{1,2}(Q;\mathbb{R}^n)^*$ for a.e. $t \in (0,T)$. Since $\varphi^{(h)}(t) \to 0$ in $W^{1,2}(Q;\mathbb{R}^n)$ for a.e. $t \in (0,T)$ we have get that

$$D_2R(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle \to 0$$

as $h \to 0$. Moreover, observe that

$$h|\langle\nabla^{k_0}\partial_t\eta^{(h)}(t), \nabla^{k_0}\varphi^{(h)}(t)\rangle_{L^2}| \le h^{\frac{1}{2}-\frac{a_0}{2}}\|h^{\frac{1}{2}}\nabla^{k_0}\partial_t\eta^{(h)}(t)\|_{L^2}\|h^{\frac{a_0}{2}}\nabla^{k_0}\varphi^{(h)}(t)\|_{L^2}. \qquad (5.33)$$

Using (5.13) and (5.14), we obtain that the right-hand side of (5.33) vanishes, provided that $a_0 < 1$ (and that b_1 is chosen accordingly). In turn, Lebesgue's dominated convergence theorem implies that

$$\int_0^T D_2R^{(h)}(\eta^{(h)}(t), \partial_t\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt \to 0$$

as $h \to 0$. Next, we claim that

$$\lim_{h\to 0}\int_0^T \frac{1}{h}\langle\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h), \varphi^{(h)}(t)\rangle_{L^2}\,dt = 0.$$

To see this, we begin by rewriting the previous integral as

$$\int_0^T \frac{1}{h}\langle \partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h),\varphi^{(h)}(t)\rangle_{L^2}\,dt = \int_0^T \left\langle \partial_t\eta^{(h)}(t), (\tilde\eta^{(h)}(t)-\eta^{(h)}(t))\frac{\psi(t+h)-\psi(t)}{h}\right\rangle_{L^2} dt$$
$$+\int_0^T \left\langle \partial_t\eta^{(h)}(t), \left(\frac{\tilde\eta^{(h)}(t+h)-\tilde\eta^{(h)}(t)}{h} - \frac{\eta^{(h)}(t+h)-\eta^{(h)}(t)}{h}\right)\psi(t+h)\right\rangle_{L^2} dt. \quad (5.34)$$

Here, we notice that the first term on the right hand side of (5.34) converges to zero by recalling that $\partial_t\eta^{(h)} \rightharpoonup \partial_t\eta$ in $L^2((0,T);L^2(Q;\mathbb{R}^n))$ (see (5.15)) and by also noticing that $\eta^{(h)}-\tilde\eta^{(h)} \to 0$ in $L^2((0,T);L^2(Q;\mathbb{R}^n))$ by the properties of the mollification. Similarly, the second term vanishes as $h\to 0$ since we have previously shown that $b^{(h)}$ (defined in (5.19)) admits a converging subsequence in $L^2((0,T);L^2(Q;\mathbb{R}^n))$ and the analogous convergence continues to hold when $\eta^{(h)}$ is replaced with its mollification $\tilde\eta^{(h)}$. For the term involving the contact force, we note that Step 4 and (5.16) together imply

$$\lim_{h\to 0}\left|\int_0^T \langle\sigma^{(h)}(t),\varphi^{(h)}(t)\rangle_{L^2}\,dt\right| \le \lim_{h\to 0}\int_0^T \left\|\sigma^{(h)}(t)\right\|_{M(\partial Q;\mathbb{R}^n)}\left\|\varphi^{(h)}(t)\right\|_{C^0(\partial Q;\mathbb{R}^n)} dt = 0.$$

Since clearly we also have that

$$\lim_{h\to 0}\int_0^T \langle f(t),\varphi^{(h)}(t)\rangle_{L^2}\,dt = 0,$$

we conclude that

$$\limsup_{h\to 0}\int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi^{(h)}(t)\rangle\,dt = 0.$$

In view of (5.31) and (E.6), this implies the desired result.

Step 6: Let $\varphi\in C([0,T];T_\eta(\mathcal{E}))\cap C_c^1([0,T);L^2(Q;\mathbb{R}^n))$ be given and let $\{\varphi_h\}_h$ be given as in Proposition 3.27. Then (5.17) holds for φ_h and we can pass to the limit as $h\to 0$. To be precise, we have that

$$\int_0^T DE^{(h)}(\eta^{(h)}(t))\langle\varphi_h(t)\rangle\,dt \to \int_0^T DE(\eta(t))\langle\varphi(t)\rangle\,dt,$$
$$\int_0^T D_2R^{(h)}(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\langle\varphi_h(t)\rangle\,dt \to \int_0^T D_2R(\eta(t),\partial_t\eta(t))\langle\varphi(t)\rangle\,dt,$$

and clearly also

$$\int_0^T \langle f(t),\varphi_h(t)\rangle_{L^2}\,dt \to \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt.$$

Finally, recalling that $\partial_t\eta^{(h)}$ is to be understood as η^* for $t<0$ and assuming without loss of generality that φ_h is extended constantly for negative times and that it vanishes for $t\ge T-h$, we observe that

$$\begin{aligned}
&\int_0^T \frac{\rho}{h}\langle\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h),\varphi_h(t)\rangle_{L^2}\,dt\\
&= -\int_0^{T-h}\frac{\rho}{h}\langle\partial_t\eta^{(h)}(t),\varphi_h(t+h)-\varphi_h(t)\rangle_{L^2}\,dt - \int_{-h}^0\frac{\rho}{h}\langle\eta^*,\varphi_h(t+h)\rangle_{L^2}\,dt\\
&= -\int_0^{T-h}\frac{\rho}{h}\langle\partial_t\eta^{(h)}(t),\int_t^{t+h}\partial_t\varphi_h(s)\,ds\rangle_{L^2}\,dt - \int_{-h}^0\frac{\rho}{h}\langle\eta^*,\varphi_h(t+h)\rangle_{L^2}\,dt\\
&= -\int_0^T\frac{\rho}{h}\int_{t-h}^t\langle\partial_t\eta^{(h)}(s),\partial_t\varphi_h(t)\,ds\rangle_{L^2}\,dt - \int_{-h}^0\frac{\rho}{h}\langle\eta^*,\varphi_h(t+h)\rangle_{L^2}\,dt\\
&= -\int_0^T\rho\langle b^{(h)}(t),\partial_t\varphi_h(t)\rangle_{L^2}\,dt - \int_{-h}^0\frac{\rho}{h}\langle\eta^*,\varphi_h(t+h)\rangle_{L^2}\,dt\\
&\to -\int_0^T\rho\langle\partial_t\eta(t),\partial_t\varphi(t)\rangle_{L^2}\,dt - \rho\langle\eta^*,\varphi(0)\rangle_{L^2}.
\end{aligned}$$

This shows that η is a solution to the variational inequality (2.5).

The last missing piece of the main result of this paper is to show that the previous procedure in fact yields existence of a solution to the full problem with a contact force. For this it is enough to show the convergence of contact forces. By the estimate on $\sigma^{(h)}$ in Step 4 and the convergence of $\eta^{(h)}$, Theorem 3.9 guarantees the existence of a subsequence (which we do not relabel) such that $\sigma^{(h)}\overset{*}{\rightharpoonup}\sigma$ in $M([0,T]\times\partial Q;\mathbb{R}^n)$, where $\sigma\in M([0,T]\times\partial Q;\mathbb{R}^n)$ is a contact force for η and satisfies the action-reaction principle (see Definition 3.6). Finally, in view of (5.29) we also obtain that σ has no concentrations in time. Since the weak* convergence of measures is enough to pass to the limit in the term with $\sigma^{(h)}$ and since all the other terms in the weak formulation have been shown to convergence, this concludes the proof. □

Remark 5.1 (On the relation to [13])**.** *In* [13]*, the authors studied the existence of solutions to the quasistatic problem* (4.1) *via an implicit time discretization and arrived at results comparable with those obtained in Theorem 4.6. We comment here on the two main differences between the results.*

- (i) *The more important difference is that the methodology used in* [13] *does not allow one to recover the contact force as a measure. In turn, devising techniques that allow to avoid this loss of regularity for the contact force was posed an open problem in Remark 3.2 in* [13]*. To be precise, in their proof the authors derive all bounds on the contact force directly from the weak equation. As a result, the best regularity that one can achieve is that of the worst other term in the equation, and only allows to conclude that compactness holds in a negative Sobolev space. In contrast, we use a more detailed analysis of the discrete setting to provide explicit bounds in the space of measures, which in turn allows us to conclude existence of a proper limit measure.*
- (ii) *The other difference is that throughout Section 4 we consider energies and dissipation potentials for which the term of highest order is quadratic. In fact, this is only done in order to more easily obtain the precise energy estimate we need for proof of Theorem 2.5. However, as shown in the proof of Theorem 2.5, as long as the remaining highest order term still guarantees the compact embedding into C^1 and quasimonotonicity, we can use* (E.6) *together with a standard Minty-type argument to get rid of the regularizing term. In fact, this provides another contrast to* [13]*, where this leads to an additional error term, as it is not possible to use the equation to both deal with the contact force and the estimate needed for the Minty-type argument at the same time.*

The existence of a solution with contact force for the quasistatic case is shown in the corollary below. We can thus claim to have solved the open problem formulated in Remark 3.2 in [13].

Corollary 5.2. *Let E and R be as in* (E.1)–(E.6) *and* (R.1), (R.2), (R.3$_\mathrm{q}$), *and* (R.4)*, respectively, $T>0$, and let $\eta_0 \in \mathcal{E}$ and $f \in L^2((0,T);L^2(Q;\mathbb{R}^n))$ be given. Then the quasistatic problem* (4.1) *admits a weak solution with a contact force, that is, there exist*

$$\eta \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^n))$$

with $E(\eta) \in L^\infty(0,T)$ and $\eta(0)=\eta_0$ and a contact force $\sigma \in L^2_{w^*}((0,T);M(\partial Q;\mathbb{R}^n))$ (see Definition 3.6) such that

$$\int_0^T [DE(\eta(t)) + D_2R(\eta(t),\partial_t\eta(t))]\,\langle\varphi(t)\rangle\,dt = \int_0^T \langle\sigma(t),\varphi(t)\rangle\,dt + \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt$$

holds for all $\varphi \in C([0,T];W^{2,p}_\Gamma(Q;\mathbb{R}^n))$.

Proof. The proof is analogous to that of Theorem 2.5, but simpler. To be precise, let $E^{(h)}, R^{(h)}$ be the regularized energy-dissipation pair considered in the proof of Theorem 2.5, and let $\{\eta_0^{(h)}\}_h$ be given as in (5.4). Reasoning as above, an application of Theorem 4.3 yields the existence of a family of deformations $\{\eta^{(h)}\}_h$ that solves the variational inequality for the regularized problem, and such that

$$E^{(h)}(\eta^{(h)}(t)) + 2\int_0^t R^{(h)}(\eta^{(h)}(s),\partial_t\eta^{(h)}(s))\,ds \le E^{(h)}(\eta_0^{(h)}) + \int_0^t \langle f(s),\partial_t\eta^{(h)}(s)\rangle_{L^2}\,ds. \tag{5.35}$$

We remark here that the parameter h is only introduced in order to apply the theory developed in Section 4 and that in the absence of the inertial term we no longer require to consider a time-delayed problem in subintervals of length h, but rather we directly study the problem on the entire interval $[0,T]$. Since

$$K_R\|b\|^2_{W^{1,2}} \le R(\eta,b) \le R^{(h)}(\eta,b),$$

by Young's inequality we obtain that

$$E^{(h)}(\eta^{(h)}(t)) + \int_0^t R^{(h)}(\eta^{(h)}(s),\partial_t\eta^{(h)}(s))\,ds \le C,$$

where C only depends on $E(\eta_0)$, f, and K_R. The rest of the proof follows the same strategy as that of Theorem 2.5. However, the argument presented in Steps 4 through 6 can be simplified by formally substituting $\rho = 0$, which in particular allows to ignore any difficulties involving the inertia-term and use L^2 in time bounds there. Thus, we omit the details. □

Remark 5.3. *In the preprint version of this article we claimed that also in the inertial case the contact force is square-integrable in time, i.e.* $\sigma \in L^2_{w^*}([0,T]; M(\partial Q))$*. However, while revising for publication, we realized that there was an incorrect estimate in the respective proof. While this does not affect our conclusions for the quasistatic regime (see Corollary 5.2), for the inertial problem we can only guarantee that* σ *is a measure without any concentration in time. Nevertheless we still conjecture that our original bounds will end up being correct.*

Looking at the proof of Theorem 2.5, in particular at Step 4, we have good L^2*-in-time bounds on almost all of the terms involved. This, together with the weak equation itself, means that the only way for* σ *to have a non square-integrable part is if it is cancelled by a similar (but opposite) contribution coming from the inertial term. Additionally, this contribution will have to happen at the boundary and in normal direction, as this is where the contact force is supported and pointed.*

It now stands to reason that such a concentration in $\partial_{tt}\eta$ *can only happen precisely at the instant of collision, since once two parts of the solid are in touch their behavior in normal direction will be similar to that of a single solid. But since locally those instances are naturally isolated in time, the proven absence of concentrations implies that these should not give any meaningful contribution. We were however unable to prove this rigorously.*

6. Physical considerations

The aim of this section is to display the physical content of the purely mathematical discussion in the rest of this paper. The way to this is threefold. First we will illustrate by an example that our assumptions on energy and dissipation are reasonable. Then we will consider how our result is intricately related to momentum conservation, both globally as well as in a localized version. Finally we show that in contrast to e.g. rigid body or thin film dynamics, no additional contact law is required in our bulk setting, as the contact set (see Definition 3.1), and thus the region of possibly instantaneous momentum change, is of lower dimension and thus has no influence on the total momentum.

6.1. **An example energy-dissipation pair.** We have already remarked on the need for a second order material in order to have sufficient regularity of the normal vectors. Additionally, it is an indirect requirement to allow us a control on the determinant in general, which in turn is related to local injectivity as well as the existence of a Korn-type inequality. Nevertheless, while we only ever use the relatively generic assumptions detailed in Section 2, we have an example energy-dissipation pair in mind which fulfils all of those and at the same time can be considered physical. For a full discussion, we refer in particular to [3, Sec. 2.3].

A good choice of energy has to be frame invariant, i.e. it should not change under rotations in the image. At the same time, it needs to penalize compression, but in such a way that there still is a well defined Fréchet-derivative at any deformation of finite energy. Finally, it should still be related to the simpler models studied in engineering. Perhaps the most simple candidate that unifies these requirements is given by

$$E(\eta) := \int_Q \mathcal{C}(\nabla\eta^T\nabla\eta - I) : (\nabla\eta^T\nabla\eta - I) + \frac{c_1}{(\det \nabla\eta)^a} + c_2 \left|\nabla^2\eta\right|^p dx,$$

where $\mathcal{C}$ is a fourth order tensor, c_1, c_2 are positive constants, and $a > \frac{pn}{p-n}$ is needed to guarantee injectivity (see [10]). Here the first term corresponds to classical elasticity theory, while the others can be seen as small, frame invariant perturbations needed in order to deal with large and irregular deformations.

Similar to the energy, the example dissipation we have in mind needs to be frame invariant as well; as a result, it is actually better to consider the dissipation potential as a function depending on $\partial_t(\nabla\eta^T\nabla\eta)$ instead (see also [2] for a detailed discussion). This leads directly to the most simple example of such a dissipation satisfying our assumptions:

$$R(\eta, \partial_t\eta) := \int_Q \left|\partial_t\nabla\eta^T\nabla\eta + \nabla\eta^T\partial_t\nabla\eta\right|^2 dx = \int_Q \left|\partial_t(\nabla\eta^T\nabla\eta)\right|^2 dx.$$

Korn-type inequalities (see (R.3)) for this choice of the dissipation potential have been show by Neff [14] and Pompe [17].

6.2. **Momentum conservation and collision forces.** The total momentum in the Lagrangian representation is given by the integral of the momentum density $\int_Q \rho\partial_t\eta\, dx$. This quantity is conserved in the dynamical case. Globally we can note that we can test each of the three equations we obtain in the course of the proof of Theorem 2.5 (Euler–Lagrange, time-delayed and hyperbolic) with $\varphi := e_i$ to obtain the respective versions:

$$\begin{aligned} \frac{\rho}{h}\int_Q \frac{\eta_k^i - \eta_{k-1}^i}{\tau}\,dx &= \int_{\partial Q} d\sigma_k^i + \frac{\rho}{h}\int_Q \zeta_k^i\,dx \\ \frac{\rho}{h}\int_Q \partial_t\eta^i(t)\,dx &= \int_{\partial Q} d\sigma^i + \frac{\rho}{h}\int_Q \partial_t\eta^i(t-h)\,dx \\ \frac{d}{dt}\rho\int_Q \partial_t\eta^i\,dx &= \int_{\partial Q} d\sigma^i \end{aligned}$$

where σ is the respective contact force and we use that physically reasonable energy and dissipation functionals only act on the spatial derivatives, so since $\nabla e_i = 0$ we have $DE(\eta)\langle e_i\rangle = 0$ and so on.

For the second equation, we also note that it implies

$$\rho\partial_t \fint_{t-h}^t \int_Q \partial_t\eta^i\,dxds = \frac{\rho}{h}\int_Q \partial_t\eta^i(t)\,dx - \frac{\rho}{h}\int_Q \partial_t\eta^i(t-h)\,dx = \int_{\partial Q} d\sigma^i,$$

which again shows that time averages are the suitable quantities to study when considering convergence of the time delayed equation.

Additionally, since the contact force we obtain satisfies the action-reaction principle, it is not hard to show that all its contributions from self-contacts have to cancel in the end, which leaves us with

$$\frac{d}{dt}\rho\int_Q \partial_t\eta\,dx = \int_{\eta^{-1}(\partial\Omega)} d\sigma.$$

We can thus immediately conclude that the total momentum can only change if there is contact with the exterior boundary $\partial\Omega$ and it can only do so in normal direction.

Similarly we have a momentum density as conserved quantity: There is a symmetric matrix valued stress tensor, in the form of a distribution $A \in (W^{1,p}(Q;\mathbb{R}^{n\times n}))^*$ fulfilling

$$\nabla\cdot A = DE + DR.$$

Now, testing the final equation (the same can also be done in case of the approximations, with similar results) with φe_i for some $\varphi \in C^\infty([0,T]\times Q)$ yields

$$\int_Q \varphi(T)\rho\partial_t\eta^i(T)\,dx - \int_0^T\int_Q \rho\partial_t\eta^i\partial_t\varphi\,dxdt + \int_0^T \langle\nabla\cdot A_i,\varphi\rangle = \int_0^T\int_{\partial Q}\varphi d\sigma^i\,dt + \int_Q \varphi(0)\rho\partial_t\eta^i(0)\,dx,$$

which is of course a weak formulation of

$$\partial_t(\rho\partial_t\eta^i) = \nabla\cdot A_i + \sigma^i,$$

that is, the physical conservation of momentum in continuum mechanics.

6.3. **An example of "true bouncing".** It should be noted that we only show existence of *a* solution, and that at no point we claim that solutions must be unique. While this is primarily due to the non-linear nature of the elastic energy, the potential lack of uniqueness can be seen to have important implications in a context where contact is allowed, especially for the resulting rebound dynamics.

When considering the reduced example of a point particle or a rigid body colliding with a fixed obstacle, one typically specifies an additional contact law, usually in the form of a reflection of (a fraction of) the velocity across the contact plane. This, however, is not the approach that we followed in this paper. Instead, we only prohibit entering the obstacle, which results in an obvious source of non-uniqueness. In particular, if we were to apply our method to this case, the expected result would be the rigid body getting stuck at or sliding along the obstacle, as these correspond to the solutions with the least possible change in velocity.

We claim that this unphysical behavior is not possible with our approach, since the elastic solids that we consider have full dimension. The main reason for this is that, while there has to be an instantaneous change of velocity at the point of contact, which is not specified by the equations, this change of velocity happens instantaneously only at the point (or possibly at the surface) of contact and only at the time of contact. As this is a set of lower dimension, its unspecified influence on the total momentum and on the kinetic energy is negligible. Only for times after contact, these change continuously via the contact force and conversion into other energy types respectively, all of which are accounted for by the equations.

We illustrate this with an example.

Example 6.1 (Bouncing ball). *Let $\Omega = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ be the half plane and consider a ball of radius r, with uniform density ρ as the elastic solid, i.e. $Q = B_r(0) \subset \mathbb{R}^n$ and an elastic energy E for which this is the only rest configuration, i.e. the critical points of E are precisely the Euclidean transformations, which all minimize E with zero energy. We assume that initially, the solid is in such a configuration away from the wall, i.e., $\eta_0(x) = x + le_1$ with $l > r$ and it has uniform initial velocity $v_0 = -e_1$ pointing directly towards the wall, so that there will be a collision.*

Now consider the solid's center of mass at time t, which we will denote by $y(t) := \fint_Q \eta(t)\,dx$. Per definition we know that it evolves along with the total momentum, i.e.,

$$m\dot{y}(t) = p(t) := \fint_Q \rho\partial_t\eta(t)\,dx.$$

Of this we have derived in the previous subsection that it can only change through contact forces arising from collisions with the boundary $\partial\Omega$, which for our half space Ω means that $\dot{p}_1 \geq 0$ and $\dot{p}_i = 0$ for $i > 1$, as those forces only have one normal direction to act in. As the geometry implies that $y_1(t) > 0$ for all times and since the initial conditions imply $\dot{y}_1(0) = -1$, we know that contact will happen. In turn, the main question is to understand how the center of mass will evolve afterwards. In particular, our goal is to show that there is a time T such that $\dot{y}_1(T) > 0$, which means that the ball will perpetually move away from the wall, i.e. we have "true bouncing".

To see this, we analyze the situation further. First we observe that, before contact, the ball cannot deform. Indeed, this is guaranteed by the equation, but also for energy reasons, as uniform velocity is already the minimizer of kinetic energy for given momentum, i.e. there is no other way to obtain nonzero elastic energy in the energy balance. So at the time t_0 of first contact we have $y_1(t_0) = r$. Now, as the change of total momentum cannot have concentrations in time, and $\dot{y}_1(t_0) = -1$, there is a time $t_1 > t_0$ for which $y_1(t_1) < r$. If we assume that there is no rebound, i.e. $\dot{y}_1 \leq 0$, then this needs to hold true for all future times as well.

But this is at odds with the tendency of the solid to evolve towards a relaxed configuration in the absence of forces. Specifically assume that there is no bouncing, i.e. $y(t)$ stays bounded. As this implies $\dot{y}_1 \leq 0$, then there exists a limit $y_1(t) \searrow y_\infty < r$, and obviously $p_1(t) \nearrow 0$ as well. Combining the latter with a consequence of the energy estimate, that

$$\frac{\rho}{2}\|\partial_t\eta(t)\|_{L^2}^2 + \int_0^t c\|\nabla\partial_t\eta\|_{L^2}^2\,dt \leq \frac{\rho}{2}\|\partial_t\eta(t)\|_{L^2}^2 + \int_0^t R(\eta,\partial_t\eta)^2 dt$$

is bounded for some $c > 0$, we get that $\partial_t\eta(t) \to 0$ in L^2 exponentially by a Gronwall argument. This in turn implies the existence of a pointwise limit $\eta(t) \to \eta_\infty$ almost everywhere, and we can choose a sequence of times $\{t_k\}_k$ with $t_k \to \infty$ such that $\nabla\partial_t\eta(t_k) \to 0$ in L^2 since the its integral in time is bounded. Similarly, we know that the integral of the total contact force is equal to the change in momentum and thus remains bounded. Without loss of generality, we can then choose t_k in such a way that this quantity vanishes as well. Finally, compactness gives us that $\eta(t_k) \rightharpoonup \eta_\infty$ in $W^{2,p}(Q;\mathbb{R}^n)$.

But then, we see from the equation that all terms vanish, except possibly for $DE(\eta_\infty)$, which in turn implies that also $DE(\eta_\infty) = 0$. Hence, we conclude that η_∞ is a translation of the identity that maps into Ω. Thus, we have shown that $\fint_Q \eta_\infty\,dx \geq r > y_\infty$, which is a contradiction.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] S. S. Antman, *Physically unacceptable viscous stresses*, Zeitschrift für angewandte Mathematik und Physik, 49 (1998), pp. 980–988.

[3] B. Benešová, M. Kampschulte, and S. Schwarzacher, *A variational approach to hyperbolic evolutions and fluid-structure interactions*, J. Eur. Math. Soc. (online first), (2023).

[4] B. Benešová, M. Kampschulte, and S. Schwarzacher, *Variational methods for fluid–structure interaction and porous media*, Nonlinear Analysis: Real World Applications, 71 (2023), p. 103819.

[5] P. G. Ciarlet and J. Nečas, *Injectivity and self-contact in nonlinear elasticity*, Arch. Rational Mech. Anal., 97 (1987), pp. 171–188.

[6] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Society for Industrial and Applied Mathematics, 1990.

[7] G. Dal Maso and C. J. Larsen, *Existence for wave equations on domains with arbitrary growing cracks*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 22 (2011), pp. 387–408.

[8] E. De Giorgi, *New problems on minimizing movements*, in Boundary value problems for partial differential equations and applications, vol. 29 of RMA Res. Notes Appl. Math., Masson, Paris, 1993, pp. 81–98.

[9] G. Gravina, S. Schwarzacher, O. Souček, and K. Tůma, *Contactless rebound of elastic bodies in a viscous incompressible fluid*, Journal of Fluid Mechanics, 942 (2022), p. A34.

[10] T. J. Healey and S. Krömer, *Injective weak solutions in second-gradient nonlinear elasticity*, ESAIM Control Optim. Calc. Var., 15 (2009), pp. 863–871.

[11] M. Hillairet, *Lack of collision between solid bodies in a 2d incompressible viscous flow*, Communications in Partial Differential Equations, 32 (2007), pp. 1345–1371.

[12] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, *Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer Science, 2016.

[13] S. Krömer and T. Roubíček, *Quasistatic viscoelasticity with self-contact at large strains*, J. Elasticity, 142 (2020), pp. 433–445.

[14] P. Neff, *On Korn's first inequality with non-constant coefficients*, Proc. Roy. Soc. Edinburgh Sect. A, 132 (2002), pp. 221–243.

[15] A. Palmer and T. Healey, *Injectivity and self-contact in second-gradient nonlinear elasticity*, Calculus of Variations and Partial Differential Equations, 56 (2017).

[16] A. Z. Palmer, *Variations of deformations with self-contact on Lipschitz domains*, Set-Valued Var. Anal., 27 (2019), pp. 807–818.

[17] W. Pompe, *Korn's first inequality with variable coefficients and its generalization*, Comment. Math. Univ. Carolin., 44 (2003), pp. 57–70.

[18] F. Schuricht, *Variational approach to contact problems in nonlinear elasticity*, Calculus of Variations and Partial Differential Equations, 15 (2002), pp. 433–449.

Paper II

INERTIAL (SELF-)COLLISIONS OF VISCOELASTIC SOLIDS WITH LIPSCHITZ BOUNDARIES

ANTONÍN ČEŠÍK[†], GIOVANNI GRAVINA[‡], AND MALTE KAMPSCHULTE[†,¶,*]

Abstract. We continue our study, started in [ČGK24], of (self-)collisions of viscoelastic solids in an inertial regime. We show existence of weak solutions with a corresponding contact force measure in the case of solids with only Lipschitz-regular boundaries. This necessitates a careful study of different concepts of tangent and normal cones and the role these play both in the proofs and in the formulation of the problem itself. Consistent with our previous approach, we study contact without resorting to penalization, i.e. by only relying on a strict non-interpenetration condition. Additionally, we improve the strategies of our previous proof, eliminating the need for regularization terms across all levels of approximation.

1. Introduction

As in our previous paper [ČGK24], we consider the (self-)collision of viscoelastic solids under the influence of inertia. That is, we are interested in showing the existence of weak solutions to

$$\rho\partial_{tt}\eta + DE(\eta) + D_2R(\eta,\partial_t\eta) = f + \sigma \tag{1.1}$$

where $\eta\colon Q \to \mathbb{R}^n$ is a deformation of a reference configuration, ρ is the mass density, E and R denote a 2nd order elastic energy and dissipation potential, respectively (see the following section for the precise assumptions), f is a given force, and σ is a solution-dependent contact force pointing in a generalized surface normal direction. This force purely derives from a non-interpenetration condition, i.e. from the assumption that all deformations are restricted to some container Ω and are injective almost everywhere.

While also in our previous work we did not prescribe the topology or the general shape of the solids, we restricted ourselves to the case of smooth (i.e. $C^{1,\alpha}$) boundaries. This was a natural class, compatible with the regularity imposed on deformations by the type of second order elastic energies considered there.

However, the main reason for this restriction was that it not only greatly simplifies the type of contact that can occur, but also gave us a unique, well-defined normal vector at every point of the boundary. Furthermore, this normal vector was not only continuous along the boundary, but also stable with respect to convergence of deformations, two properties we relied on heavily in our proofs.

It is worth mentioning that the study of the static minimization case has a long history, and for this it has long been known how to properly generalize obstacle problems to encompass less regular, i.e. Lipschitz, boundaries (see, for example, [Pal18]; see also [Sch02]).

A way to deal with non-smooth boundaries is to work with normals and tangents in a generalized sense. Specifically, variational analysis offers different notions of generalized tangents and normals to any set in $\mathbb{R}^n$, cf. [RW98]. This is the approach that we take and which will be discussed in Section 3. An alternative would be to use the Clarke subdifferential for Lipschitz functions (see [Cla87]), and through it define generalized tangents and normals for Lipschitz sets.

The aim of this paper is now to apply those considerations to the dynamic case using the time-delayed approximations developed in [BKS23]. This generalization comes with non-trivial analytical challenges. Indeed, not only does it require a more nuanced discussion of contact forces compared to [ČGK24], but in addition, a crucial step of the proof (the time-delayed energy inequality), which relied on testing with

[†]Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

[‡]School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona, USA

[†]Institute of Mathematics, Faculty of Mathematics and Computer Science, University of Leipzig, Germany

* Corresponding author

E-mail addresses: cesik@karlin.mff.cuni.cz, ggravina@asu.edu, kampschulte@karlin.mff.cuni.cz.

2020 *Mathematics Subject Classification.* 35Q74, 49J40, 74A30, 74B20, 74M15.

Key words and phrases. Non-interpenetration of matter, variational inequalities, non-simple materials.

Author ORCIDs. Antonín Češík: https://orcid.org/0000-0003-0207-6044, Giovanni Gravina: https://orcid.org/0000-0001-8985-7964, Malte Kampschulte: https://orcid.org/0000-0002-9150-4460.

the time-derivative, can now no longer be obtained in the same way (see Remark 6.4). We instead have to employ a completely different strategy to obtain a similar estimate.

With this, the outline of this paper is as follows: In the next section, we introduce our assumptions and state the main results. Sections 3-5 will then be concerned with introducing notation and properties of tangent and normal cones, contact forces, and admissible test functions, respectively. In Section 6 we will show existence of weak solutions for the quasistatic problem as an intermediate step. Finally, in Section 7 we will then use the results from the previous section as a building block for proving the main theorem, thus establishing the existence of weak solutions for the full problem with inertia.

2. Assumptions and main results

2.1. **Viscoelastic solids.** The assumptions we impose on the solid mirror those found in [ČGK24], with the key difference that in order to deal with corners and related phenomena, in this paper we expanded our scope to encompass general Lipschitz boundaries, instead of requiring them to be of class $C^{1,\alpha}$. While we will reintroduce the definitions for clarity, we refer the reader to [ČGK24] for a more detailed discussion of the various aspects that are not directly tied to the lowered regularity of ∂Q.

The solid body will be described in Lagrangian coordinates by a (time dependent) deformation of a reference configuration $Q \subset \mathbb{R}^n$, denoted by $\eta\colon [0,T] \times Q \to \Omega$. The set $Q \subset \mathbb{R}^n$ will be a Lipschitz, bounded domain, or alternatively a disjoint union of finitely many of such domains. The confining set $\Omega \subset \mathbb{R}^n$ will similarly be a (possibly unbounded) Lipschitz domain. The class of admissible deformations consists of maps that satisfy the Ciarlet–Nečas condition [CN87] and are thus injective almost everywhere. To be precise, we set

$$\mathcal{E} := \left\{ \eta \in W^{2,p}(Q;\mathbb{R}^n) : \eta(Q) \subset \Omega, \eta|_\Gamma = \eta_0, \det \nabla\eta > 0, \text{ and } \mathcal{L}^n(\eta(Q)) = \int_Q \det \nabla \eta(x)\, dx \right\}. \tag{2.1}$$

Here we use η_0 to denote a given admissible (initial) deformation, $\mathcal{L}^n$ is the Lebesgue measure and Γ is a (fixed) measurable subset of ∂Q, not necessarily with positive measure, i.e. possibly empty. Furthermore, we require $\eta_0|_\Gamma$ to be injective and $\eta_0(\Gamma) \cap \partial\Omega = \emptyset$ (see [ČGK24, Remark 2.3] for more information). Here and in the following we assume that $p > n$. Thus we can assume that $\eta \in C^{1,1-\frac{n}{p}}(\overline{Q};\mathbb{R}^n)$ for all $\eta \in \mathcal{E}$ and always work with this precise representative.

In order to simplify the notation, in the following we write $A\langle u\rangle$ as a shorthand for the duality pairing $\langle A, u\rangle_{(W^{k,p})^*\times W^{k,p}}$, whenever $A\colon W^{k,p}(Q;\mathbb{R}^n) \to \mathbb{R}$ is linear and $u \in W^{k,p}(Q;\mathbb{R}^n)$. Moreover, we use the subscript Γ to denote spaces of functions whose trace vanishes on that set, e.g. $W^{k,p}_\Gamma(Q) := \{u \in W^{k,p}(Q) : u|_\Gamma = 0\}$.

Next, we specify the assumptions on the energy-dissipation pair (E,R). The canonical examples we have in mind are

$$E(\eta) := \int_Q \mathcal{C}(\nabla\eta^T\nabla\eta - I) : (\nabla\eta^T\nabla\eta - I) + \frac{c_1}{(\det\nabla\eta)^a} + c_2 \left|\nabla^2\eta\right|^p\, dx$$
$$R(\eta,\partial_t\eta) := \int_Q \nabla\eta^T\nabla\partial_t\eta + \nabla\partial_t\eta^T\nabla\eta dx,$$

where $\mathcal{C}$ is a positive definite fourth order tensor, c_1 and c_2 are positive constants, and $a > \frac{pn}{p-n}$ to guarantee injectivity (see [HK09]). However in the proofs, we will not use this explicit form. Instead we only assume that the elastic energy $E\colon W^{2,p}(Q;\mathbb{R}^n) \to (-\infty,\infty]$ has the following properties:

(E.1) There exists $E_{\min} > -\infty$ such that $E(\eta) \geq E_{\min}$ for all $\eta \in W^{2,p}(Q;\mathbb{R}^n)$. Moreover, $E(\eta) < \infty$ for every $\eta \in W^{2,p}(Q;\mathbb{R}^n)$ with $\inf_Q \det\nabla\eta > 0$.

(E.2) For every $E_0 \geq E_{\min}$ there exists $\varepsilon_0 > 0$ such that $\det\nabla\eta \geq \varepsilon_0$ on Q for all η with $E(\eta) \leq E_0$.

(E.3) For every $E_0 \geq E_{\min}$ there exists a constant C such that

$$\|\nabla^2\eta\|_{L^p} \leq C$$

for all η with $E(\eta) < E_0$.

(E.4) E is weakly lower semicontinuous, that is,

$$E(\eta) \leq \liminf_{k\to\infty} E(\eta_k)$$

whenever $\eta_k \rightharpoonup \eta$ in $W^{2,p}(Q;\mathbb{R}^n)$. Moreover, E is continuous with respect to strong convergence in $W^{2,p}(Q;\mathbb{R}^n)$.

(E.5) E is differentiable in its effective domain with derivative $DE(\eta) \in (W^{2,p}(Q;\mathbb{R}^n))^*$ given by

$$DE(\eta)\langle\varphi\rangle = \frac{d}{d\varepsilon}E(\eta+\varepsilon\varphi)\Big|_{\varepsilon=0}.$$

Furthermore, DE is bounded on any sub-level set of E and $DE(\eta_k)\langle\varphi\rangle \to DE(\eta)\langle\varphi\rangle$ whenever $\eta_k \to \eta$ in $W^{2,p}(K;\mathbb{R}^n)$ for all K compactly contained in $\overline{Q}$ with $\operatorname{dist}(K,\Gamma) > 0$ and $\varphi \in W^{2,p}_\Gamma(Q;\mathbb{R}^n)$.

(E.6) DE satisfies

$$\liminf_{k\to\infty}(DE(\eta_k) - DE(\eta))\langle(\eta_k - \eta)\psi\rangle \geq 0$$

for all $\psi \in C^\infty_\Gamma(Q;[0,1])$ and all sequences $\eta_k \rightharpoonup \eta$ in $W^{2,p}(Q;\mathbb{R}^n)$. In addition, DE satisfies the following Minty-type property: If

$$\liminf_{k\to\infty}(DE(\eta_k) - DE(\eta))\langle(\eta_k - \eta)\psi\rangle \leq 0$$

for all $\psi \in C^\infty_\Gamma(Q;[0,1])$, then $\eta_k \to \eta$ in $W^{2,p}(K;\mathbb{R}^n)$ for all K compactly contained in $\overline{Q}$ with $\operatorname{dist}(K,\Gamma) > 0$.

Similarly, we assume that the dissipation potential $R\colon W^{2,p}(Q;\mathbb{R}^n) \times W^{1,2}(Q;\mathbb{R}^n) \to [0,\infty)$ is a function satisfying the following properties:

(R.1) R is weakly lower semicontinuous in its second argument, that is, for all $\eta \in W^{2,p}(Q;\mathbb{R}^n)$ and every $b_k \rightharpoonup b$ in $W^{1,2}(Q;\mathbb{R}^n)$ we have that

$$R(\eta,b) \leq \liminf_{k\to\infty} R(\eta,b_k).$$

(R.2) R is convex and homogeneous of degree 2 with respect to its second argument, that is,

$$R(\eta,\lambda b) = \lambda^2 R(\eta,b)$$

for all $\lambda \in \mathbb{R}$.

(R.3) R admits the following Korn-type inequality: For any $\varepsilon_0 > 0$, there exists K_R such that

$$K_R\|b\|^2_{W^{1,2}} \leq \|b\|^2_{L^2} + R(\eta,b)$$

for all $\eta \in \mathcal{E}$ with $\det\nabla\eta \geq \varepsilon_0$ and all $b \in W^{1,2}(Q;\mathbb{R}^n)$.

(R.4) R is differentiable in its second argument, with derivative $D_2R(\eta,b) \in (W^{1,2}(Q;\mathbb{R}^n))^*$ given by

$$D_2R(\eta,b)\langle\varphi\rangle := \frac{d}{d\varepsilon}R(\eta,b+\varepsilon\varphi)\Big|_{\varepsilon=0}.$$

Furthermore, the map $(\eta,b) \mapsto D_2R(\eta,b)$ is bounded and weakly continuous with respect to both arguments, that is,

$$\lim_{k\to\infty} D_2R(\eta_k,b_k)\langle\varphi\rangle = D_2R(\eta,b)\langle\varphi\rangle$$

holds for all $\varphi \in W^{1,2}(Q;\mathbb{R}^n)$ and all sequences $\eta_k \rightharpoonup \eta$ in $W^{2,p}(Q;\mathbb{R}^n)$ and $b_k \rightharpoonup b$ in $W^{1,2}(Q;\mathbb{R}^n)$.

We also introduce a variant of (R.3) that will be used for the quasistatic case (see Theorem 6.2) in the form of

(R.3$_\text{q}$) R admits the following Korn-type inequality: For any $\varepsilon_0 > 0$, there exists K_R such that

$$K_R\|b\|^2_{W^{1,2}} \leq R(\eta,b)$$

for all $\eta \in \mathcal{E}$ with $\det\nabla\eta \geq \varepsilon_0$ and all $b \in W^{1,2}_\Gamma(Q;\mathbb{R}^n)$.

We mention here that the assumptions on the energy-dissipation pair are standard within the framework of second-order viscoelastic materials (see in particular [HK09], [KR20], and the references therein).

Finally, as in [ČGK24, Remark 2.1] we note that these assumptions imply the following:

$$D_2R(\eta,\lambda b) = \lambda D_2R(\eta,b) \tag{2.2}$$

$$\|D_2R(\eta,b)\|_{(W^{1,2})^*} \leq C\,\|b\|_{W^{1,2}}\,, \tag{2.3}$$

$$2R(\eta,b) \leq C\,\|b\|^2_{W^{1,2}} \tag{2.4}$$

for all $b \in W^{1,2}(Q;\mathbb{R}^n)$ and all $\eta \in W^{2,p}(Q;\mathbb{R}^n)$ with $E(\eta) \leq E_0$ and $C = C(E_0)$.

2.2. **Statement of the main result.** The precise definition of (weak) solution to the initial value problem considered in this paper can be formulated as follows.

Definition 2.1. *Let $T > 0$, $\eta_0 \in \mathcal{E}$, $\eta^* \in L^2(Q;\mathbb{R}^n)$, and $f \in L^2((0,T);L^2(Q;\mathbb{R}^n))$ be given. We say that*

$$\eta \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^n))$$

with $E(\eta) \in L^\infty((0,T))$ is a weak solution *to* (1.1) *in $(0,T)$ with initial deformation η_0 and initial velocity η^* if $\eta(0) = \eta_0$ and the variational inequality*

$$\int_0^T DE(\eta(t))\langle\varphi(t)\rangle + D_2R(\eta(t),\partial_t\eta(t))\langle\varphi(t)\rangle\,dt \\ - \rho\langle\eta^*,\varphi(0)\rangle_{L^2} - \int_0^T \rho\langle\partial_t\eta(t),\partial_t\varphi(t)\rangle_{L^2}\,dt \geq \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt \quad (2.5)$$

holds for all $\varphi \in C([0,T];T^0_\eta(\mathcal{E})) \cap C^1_c([0,T);L^2(Q;\mathbb{R}^n))$. Here the set $T^0_\eta(\mathcal{E})$ denotes the class of reduced admissible perturbations for the deformation η; its precise definition is given below in Lemma 5.1.

Furthermore, we say that such a function η is a weak solution with a contact force σ *if additionally it satisfies*

$$\int_0^T DE(\eta(t))\langle\varphi(t)\rangle + D_2R(\eta(t),\partial_t\eta(t))\langle\varphi(t)\rangle\,dt \\ - \rho\langle\eta^*,\varphi(0)\rangle_{L^2} - \int_0^T \rho\langle\partial_t\eta(t),\partial_t\varphi(t)\rangle_{L^2}\,dt = \int_{[0,T]\times\partial Q} \varphi(t,x)\cdot d\sigma(t,x) + \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt$$

for all $\varphi \in C([0,T];W^{2,p}_\Gamma(Q;\mathbb{R}^n)) \cap C^1_c([0,T);L^2(Q;\mathbb{R}^n))$, where $\sigma \in M([0,T]\times\partial Q;\mathbb{R}^n)$ is a contact force satisfying the action-reaction principle in the sense of Definition 4.4.

Observe that in view of its regularity, η belongs to the space $C_w([0,T];W^{2,p}(Q;\mathbb{R}^n))$. Therefore, we have that $\eta(t) \in W^{2,p}(Q;\mathbb{R}^n)$ for all $t \in [0,T]$, and in particular the initial condition $\eta(0) = \eta_0$ holds in the classical sense. Moreover, it is worth mentioning that if it is known *a priori* that the contact force $\sigma = 0$ and that η is sufficiently regular, then for some choices of E and R (see, e.g., the example pair given in Section 2.1) we can conclude that the equation holds in a pointwise sense.

With this in hand, we can state the main result of this paper.

Theorem 2.2. *Let E and R be as in* (E.1)–(E.6) *and* (R.1)–(R.4)*, respectively, and let $T > 0$, $\eta_0 \in \mathcal{E}$, $\eta^* \in L^2(Q;\mathbb{R}^n)$, and $f \in L^2((0,T);L^2(Q;\mathbb{R}^n))$ be given. Then* (1.1) *admits a weak solution with a contact force in $(0,T)$ in the sense of Definition 2.1, where the resulting contact force σ has no concentrations in time (in the sense that all sets $\{t\}\times\partial Q$ are of measure zero). Additionally, this solution satisfies the energy inequality*

$$E(\eta(t)) + \frac{\rho}{2}\|\partial_t\eta(t)\|^2_{L^2} + \int_0^t 2R(\eta(s),\partial_t\eta(s))\,ds \leq E(\eta_0) + \frac{\rho}{2}\|\eta^*\|^2_{L^2} + \int_0^t \langle f(s),\partial_t\eta(s)\rangle_{L^2}\,ds$$

for $\mathcal{L}^1$-a.e. $t \in [0,T]$.

As in our previous paper, a similar quasistatic result is also available in the form of Theorem 6.2.

Like mentioned above, the precise definitions of admissible perturbations and of contact forces will be given later. For now, let us note that we want to allow those perturbations that do not result in overlap and thus expect forces that are, in some sense, normal to the surface. For the case of Lipschitz boundaries we consider here, this is not trivial. First of all, a given direction of perturbation at a point might seem be admissible at that specific location, but due to the regularity required of test functions, it might result in overlap somewhere nearby. This greatly complicates finding any local condition on the admissibility of test functions. Secondly, as the solutions will be constructed using an approximation and classical normals can vary greatly for a Lipschitz boundary, we require a compatible notion of closure that allows for some convergence and compactness of contact forces. All of this will be the subject of the next few sections.

In light of all of this, it is particularly worth noting that, in contrast to [ČGK24], we do not provide a characterization of weak solutions in terms of an inequality for test functions in the whole abstract tangent space to the set of configurations. Instead we only consider a somewhat restricted subset. This is not an oversight, but a crucial feature of the problem. Indeed, in order to work with a set of solutions that is closed under convergence, we have to contend with situations where forces may appear in such a way that prevents the solids from moving or deforming in a direction that otherwise seems admissible.

We defer a more detailed discussion of this issue to Remark 5.6, as it requires several definitions that will be given in the intervening sections.

3. Tangent and normal cones

We begin by recalling basic definitions and some properties for cones of generalized tangents and normals. For a more detailed treatment we refer to the monograph [RW98], whose notation we also try to follow. However as notation tends to vary between authors, we aim for this section to be mostly self-contained.

Recall that $Q \subset \mathbb{R}^n$ is open, bounded, and (strongly) Lipschitz, that is, ∂Q is given locally by the graph of a Lipschitz function.

Definition 3.1 (Cone). *A set $C \subset \mathbb{R}^n$ is called a* cone *if $w \in C$ implies $\lambda w \in C$ for all $\lambda \geq 0$. For any cone C, the* polar cone *of C is defined as*

$$C^* := \{w \in \mathbb{R}^n : w \cdot v \leq 0 \text{ for all } v \in C\}.$$

Note that a cone needs to be neither closed nor convex, whereas the polar cone is by definition both closed and convex. In particular C^{**} will be the closed convex hull of C.

Definition 3.2 (Tangent and normal cones). *For Q as above and $x \in \overline{Q}$, we define the following cones:*

(i) Tangent cone[1]

$$T_Q(x) := \left\{ \lim_{i\to\infty} \frac{x_i - x}{\tau_i} : x_i \to x, x_i \in \overline{Q}, \tau_i \to 0^+ \right\} = \limsup_{\tau\to 0^+} \frac{Q - x}{\tau};$$

(ii) Regular tangent cone

$$\widehat{T}_Q(x) := \left\{ v \in \mathbb{R}^n : \forall x_i \to x, x_i \in \overline{Q}, \exists v_i \in T_Q(x_i), v_i \to v \right\} = \liminf_{y\to x, y\in\overline{Q}} T_Q(y);$$

(iii) Convexified normal cone

$$\overline{N}_Q(x) := \widehat{T}_Q(x)^*.$$

We also define the respective cones in the deformed configuration via

$$T_\eta(x) := T_{\eta(Q\cap B_r(x))}(\eta(x))$$

for any $\eta \in \mathcal{E}$ with $E(\eta) < \infty$ and $r > 0$ such that η is injective on $\overline{Q} \cap B_r(x)$.[2] *Similar notations are used for the other cones introduced above. We also use the shorthand $T_\eta(t, x) := T_{\eta(t)}(x)$ when dealing with time-dependent deformations.*

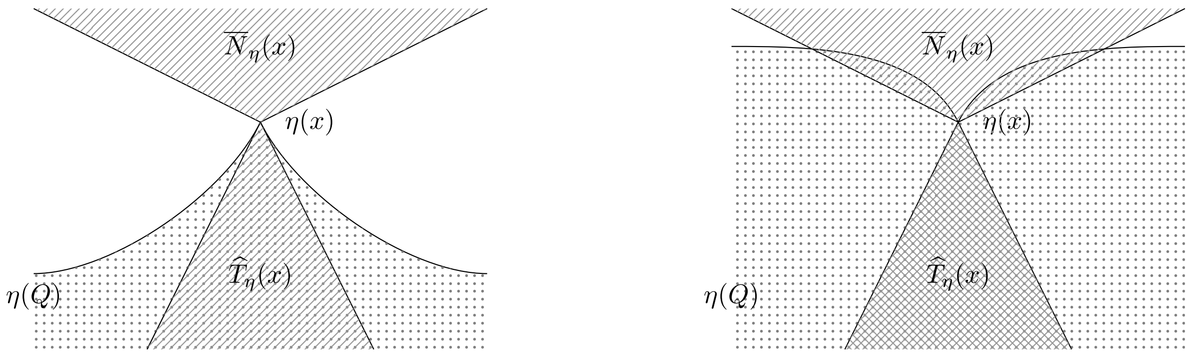


Figure 1. Interior regular tangent and exterior convexified normal cone for corners with acute and obtuse angle respectively.

Here we recall that while $T_Q(x)$ describes the usual set of tangent vectors at x, the regular tangent cone $\widehat{T}_Q(x)$ describes the smaller set of vectors which are (up to a vanishing error) also tangent vectors to any point in a neighborhood of x. This not only gives us a pointwise condition on directions which are "safe to move in" in a whole neighborhood, but it also forms a key ingredient to the approximation of test functions. See Figure 1 for an illustration.

[1] Here lim sup and lim inf denote the set-theoretic versions in a metric space (cf. [RW98, Chapter 4]).

[2] Notice that we cannot simply use $T_{\eta(Q)}(\eta(x))$. While this would work in the absence of contact, when collisions do occur, we need to distinguish the different, physically unconnected parts of the solid, which this definition fails to do. Also note that this is well defined, as any η of finite energy is locally injective by Lemma 4.2 (iv).

Remark 3.3. *The definition of $\overline{N}_Q(x)$ we use is not the classic definition. Instead, one usually proceeds via some intermediate steps. First, one constructs the regular normal cone as the polar of $T_Q(x)$. Then, similarly to $\widehat{T}_Q(x)$, one takes the normal cone to be the corresponding* lim sup. *Finally, since the resulting set can be nonconvex, one takes the convex hull. However, it follows from* [RW98, Theorem 6.28] *that these two approaches are equivalent. Thus, since we will only ever use $\overline{N}_Q(x)$ as a polar cone, we have opted to take this directly as a definition.*

Notice that the definition of $\widehat{T}_Q(x)$ readily implies that if for $w \in \mathbb{R}^n$ there are $\varepsilon_0 > 0$ and $r > 0$ such that $y + \varepsilon w \in \overline{Q}$ for all $\varepsilon \in (0, \varepsilon_0]$ and all $y \in B_r(x)$, then $w \in \widehat{T}_Q(x)$. More interestingly, in the interior of $\widehat{T}_Q(x)$ the converse holds as well, provided that the condition above holds for a neighborhood of w. To be more precise, one can show the following:

Proposition 3.4 ([RW98, Theorem 6.36])**.** *For $x \in \partial Q$, the following are equivalent:*

(i) $w \in \operatorname{int} \widehat{T}_Q(x)$;

(ii) There exist $\varepsilon_0 > 0$, $r > 0$, $\delta > 0$ such that $y + \varepsilon v \in \overline{Q}$ for all $\varepsilon \in [0, \varepsilon_0]$, all $y \in \overline{Q} \cap B_r(x)$, and all $v \in B_\delta(w)$.

As we deal with a Lagrangian representation of the solid, it is useful to remember how tangential and normal cones in the image relate to their respective counterparts in the reference configuration. For this we note their transformation behavior.

Lemma 3.5 (Transformation behavior of cones)**.** *For any $\eta \in \mathcal{E}$ and any $x \in \partial Q$ we have*

$$\widehat{T}_\eta(x) = [\nabla \eta(x)] \widehat{T}_Q(x) \qquad \text{and} \qquad \overline{N}_\eta(x) = [\operatorname{cof} \nabla \eta(x)] \overline{N}_Q(x),$$

where cof *denotes the cofactor matrix.*

Proof. For any sequence $\{x_i\}_i \subset \overline{Q}$ with $(x_i - x)/\tau_i \to v$ where $\tau_i \to 0^+$, we have that

$$\frac{\eta(x_i) - \eta(x)}{\tau_i} \to [\nabla \eta(x)] v.$$

As η is locally invertible, the analogous formula holds for η^{-1}. This implies the first statement. The second statement follows from the definition of $\overline{N}_Q(x)$ and the well-known formula $\nabla \eta(x) \operatorname{cof} \nabla \eta(x)^T = (\det \nabla \eta)\mathrm{I}$. ☐

Finally, throughout the remaining sections we will frequently use the fact that a Lipschitz boundary implies that the tangent cone cannot be too degenerate. This is made precise in the following lemma.

Lemma 3.6. *Denote by $L_{\theta,v}$ the cone in the direction $v \in \mathbb{R}^n \setminus \{0\}$, with opening angle $\theta \in (0, \pi/2)$, i.e.*

$$L_{\theta,v} := \{w \in \mathbb{R}^n : w \cdot v \geq \cos\theta |w||v|\}. \tag{3.1}$$

Then the following hold:

(i) There exist $\theta \in (0, \pi/2)$ and $0 \neq v_1, \dots, v_k \in \mathbb{R}^n$ with $|v_i| = 1$ such that for every $x \in \partial Q$ we have that $L_{\theta,v_i} \subset \widehat{T}_Q(x)$ for some i. In particular, $\operatorname{int} \widehat{T}_Q(x) \neq \emptyset$. Note that θ is independent of x, and depends only on the Lipschitz continuity of ∂Q.

(ii) Given $E_0 \geq E_{\min}$, there exist $\vartheta \in (0, \pi/2)$, vectors $0 \neq v_1, \dots, v_k \in \mathbb{R}^n$, a covering $\{G_1, \dots, G_k\}$ of ∂Q, and points $x_i \in G_i$ such that: For every $\eta \in \mathcal{E}$ with $E(\eta) \leq E_0$ we have that for every $x \in G_i$ it holds $L_{\vartheta,w_i} \subset \widehat{T}_\eta(x)$, where $w_i := [\nabla \eta(x_i)] v_i$. Note that ϑ, k, and the vectors v_i depend only on E_0 and Q. In particular, these can be chosen in such a way that they do not depend on η.

Proof. We divide the proof into two steps.
Step 1: Recall that by our assumptions on Q there exists a finite covering of ∂Q, namely $\{G_1, \dots, G_k\}$, such that for every $i = 1, \dots k$ we have that $\Gamma_i := \partial Q \cap G_i$ is the graph of a Lipschitz function in the direction of $v_i \in \mathbb{R}^n$, $|v_i| = 1$, and furthermore that $Q \cap G_i$ is the region above the graph. Then for every i we must have that

$$y + L_{\theta,v_i} \cap B_{r_y}(y) \subset \overline{Q} \cap G_i$$

for some $r_y > 0$ which depends on the distance of y to the relative boundary of Γ_i in ∂Q. In particular, this implies that $L_{\theta,v_i} \subset T_Q(y)$ for all $y \in \Gamma_i$, which in turn shows that $L_{\theta,v_i} \subset \widehat{T}_Q(x)$ for all $x \in \Gamma_i$. This concludes the proof of the first statement.
Step 2: The second result follows from similar considerations. To see this, consider the cone $L_{\theta,v}$. Then, using the fact that $\nabla \eta$ is uniformly continuous (with modulus of continuity that depends only on E_0) and that $\det \nabla \eta > 0$ in $\overline{Q}$ (see (E.2)), we must have that the transformed cone $[\nabla \eta(x)] L_{\theta,v}$ contains a

cone of the form $L_{\vartheta,w}$, where ϑ depends only on θ and E_0, and $w := [\nabla\eta(x)]v$. For $i = 1, \dots, k$, let G_i, x_i, and v_i be as in the previous step and set $w_i := [\nabla\eta(x)]v_i$. Then, for every $x \in G_i$ we have that

$$L_{\vartheta,w_i} \subset [\nabla\eta(x)]L_{\theta,v_i} \subset [\nabla\eta(x)]\widehat{T}_Q(x).$$

The desired result follows from an application of Lemma 3.5. □

We conclude with a technical lemma.

Lemma 3.7. *Let $C \subset \mathbb{R}^n$ be a cone with nonempty interior and let $v_0 \in \operatorname{int} C \setminus \{0\}$. Then there exists $\beta > 0$ such that the cone*

$$K_\beta := \{w \in \mathbb{R}^n : w \cdot v \leq \sin\beta |w||v| \text{ for all } v \in C\}$$

(i.e., K_β is the polar cone of C but with opening angle enlarged by β) has the property that $v_0 \cdot w < 0$ for all $w \in K_\beta \setminus \{0\}$.

Proof. Since v_0 is in the interior of C, there exists $\beta_0 > 0$ such that $L_{\beta_0,v_0} \subset C$ (see the definition of L_{β_0,v_0} in (3.1)). We claim that any $0 < \beta < \beta_0$ has the desired property. To prove the claim, let $w \in K_\beta \setminus \{0\}$. Notice that if $w = cv_0$ for $c \in \mathbb{R}$, then necessarily $c < 0$ and there is nothing else to do. Otherwise, we can find $0 \neq v \in \partial C$ such that v_0, v, and w are coplanar and v lies between v_0 and w. Such a vector $v \in \partial C$ can be found by letting $v := av_0 + bw$ for some choice of $a, b > 0$. Then, since $L_{\beta_0,v_0} \subset C$, we have that the angle between v_0 and v must be at least β_0. Furthermore, using the definition of K_β we see that the angle between v and w must be at least $\pi/2 - \beta$. This shows that the angle between v_0 and w is strictly larger than $\pi/2$, and therefore concludes the proof. □

4. Description of the geometry and forces at collision

In this section we introduce the tools required in our analysis of both quasistatic and dynamic collisions. In the following we let $I \subset \mathbb{R}$ be a closed and bounded time interval.

Definition 4.1 (Contact set). *For every $\eta \in \mathcal{E}$, the* contact set *of η is defined via*

$$C_\eta := \{x \in \overline{Q} : \eta(x) \in \partial\Omega \text{ or } \eta^{-1}(\eta(x)) \neq \{x\}\}. \tag{4.1}$$

For time-dependent deformations, that is, if $\eta\colon I \times \overline{Q} \to \mathbb{R}^n$ is such that $\eta(t) \in \mathcal{E}$ for all $t \in I$, we define the contact set *as*

$$C_\eta := \{(t, x) \in I \times \overline{Q} : x \in C_{\eta(t)}\},$$

where $C_{\eta(t)}$ is the contact set of $\eta(t, \cdot)$ as defined in (4.1).

The following lemma collects useful properties of the contact set.

Lemma 4.2. *Let $\eta \in \mathcal{E}$ be given and let C_η be as in Definition 4.1. Then the following hold:*

- *(i) C_η is a closed subset of ∂Q;*
- *(ii) There exists a number $M \in \mathbb{N}$ that depends only on Q and $E(\eta)$ such that $\eta^{-1}(\eta(x))$ consists of at most M points;*
- *(iii) If $\eta(x) = \eta(y)$ for some $x \neq y$, then $\operatorname{int}\widehat{T}_\eta(x) \cap \operatorname{int}\widehat{T}_\eta(y) = \emptyset$;*
- *(iv) If $\eta(x) = \eta(y)$ for some $x \neq y$, then $|x - y| > r$ for some $r > 0$ only depending on $E(\eta)$ but not η itself.*

Proof. For a proof of the statements in (i) and (iii) we refer the reader to [Pal18, Lemma 2.1] (note the slightly different notation there); (ii) is a direct consequence of Lemma 3.6. Finally, (iii) implies that x and y cannot lie in the same set G_i of Lemma 3.6. Since these sets form a finite open cover that is independent of η, this implies the existence of a minimal distance r as in the statement of (iv). □

Remark 4.3. *We mention here that the uniform local injectivity in Lemma 4.2 (iv) actually only depends on the regularity and the lower bound on the Jacobian. Since this will be used later in the paper (see the proof of Lemma 5.1), we recall this argument for the convenience of the reader.*

Assume that $\|\nabla\eta\|_{C^0} < \infty$ and $\inf_Q \det \nabla\eta > 0$. Then, as one can readily check, there exists a constant c, depending only on those two quantities, such that $|\nabla\eta(x)v| \geq c|v|$ for all $x \in \overline{Q}$ and all $v \in \mathbb{R}^n$. Consequently, for every $x, y \in \overline{Q}$ we have that

$$|\eta(x) - \eta(y)| \geq |\nabla\eta(x)(x - y)| - o(|x - y|) \geq c|x - y| - o(|x - y|). \tag{4.2}$$

To conclude, observe that the right-hand side of (4.2) is bounded away from zero whenever x and y are sufficiently close to each other.

Recall that every $\sigma \in M(X;\mathbb{R}^n)$ (that is, every Radon measure on the compact space X with values in $\mathbb{R}^n$) admits a polar decomposition of the form $d\sigma = g\,d|\sigma|$ in the sense that

$$\int_X \varphi \cdot d\sigma = \int_X \varphi \cdot g\,d|\sigma|$$

for all $\varphi \in C(X;\mathbb{R}^n)$, where $|\sigma| \in M^+(X)$ is the total variation of σ and $g \in L^1(X, |\sigma|;\mathbb{R}^n)$ is such that $|g| \leq 1$. With this at hand, we proceed to define contact forces as follows.

Definition 4.4 (Contact force). *(i) Let $\eta \in \mathcal{E}$. A* contact force *for η is a vector-valued measure $\sigma \in M(\partial Q;\mathbb{R}^n)$ with $\operatorname{supp}\sigma \subset C_\eta$ and the property that it points in the interior normal direction in the sense that the function $g\colon \partial Q \to \mathbb{R}^n$ in the polar decomposition $d\sigma = g\,d|\sigma|$ satisfies $-g(x) \in \overline{N}_\eta(x)$ for σ-a.e. $x \in C_\eta$.*

(ii) Let $\eta\colon I \times \overline{Q} \to \mathbb{R}^n$ be such that $\eta(t) \in \mathcal{E}$ for every $t \in I$ and assume that η is Borel measurable when considered as a mapping from I to $W^{2,p}(Q;\mathbb{R}^n)$. Then a contact force *for η is a vector-valued measure $\sigma \in M(I\times\partial Q;\mathbb{R}^n)$ with $\operatorname{supp}\sigma \subset C_\eta$ and the property that it points in the normal direction in the sense that the function $g\colon I \times \partial Q \to \mathbb{R}^n$ in the polar decomposition $d\sigma = g\,d|\sigma|$ satisfies $-g(t,x) \in \overline{N}_\eta(t,x)$ for σ-a.e. $(t,x) \in C_\eta$.*

(iii) Additionally, we say that a contact force σ satisfies the action-reaction principle at self-contact *if*

$$\int_{\partial Q} (\varphi \circ \eta) \cdot d\sigma = 0$$

for all $\varphi \in C_c(\Omega;\mathbb{R}^n)$. Similarly, for time-dependent deformations, we say that σ satisfies the action-reaction principle at self-contact *if*

$$\int_{I\times\partial Q} (\varphi \circ \eta) \cdot d\sigma = 0$$

for all $\varphi \in C_c(I \times \Omega;\mathbb{R}^n)$, where, to simplify notation we write $(\varphi\circ\eta)(t,x) := \varphi(t, \eta(t,x))$.

Note that, as is consistent with physics, contact forces always point in interior normal direction. However in contrast to [ČGK24] where we used interior normals, $\overline{N}_\eta(x)$ is always pointing in exterior direction, to remain consistent with standard notations in [RW98]. Thus, compared to our previous work, the signs in some of the equations are flipped. Note also that the requirement that η is Borel measurable in time (see Definition 4.4 (ii)) is necessary to ensure the compatibility with the Borel measure σ, without requiring η to be continuous with respect to the variable t.

Next we make use of the fact that locally $\overline{N}_\eta(x)$ is defined in duality with $\widehat{T}_\eta(x)$, something that can be turned into a similar duality between continuous functions and measures.

Lemma 4.5 (Characterization of contact forces)**.** *The following hold.*

(i) For $\eta \in \mathcal{E}$, let $\sigma \in M(\partial Q;\mathbb{R}^n)$ be such that $\operatorname{supp}\sigma \subset C_\eta$. Then σ is a contact force for η if and only if

$$\int_{\partial Q} \varphi \cdot d\sigma \geq 0$$

for all $\varphi \in C(\partial Q;\mathbb{R}^n)$ with $\varphi(x) \in \widehat{T}_\eta(x)$ for $x \in \partial Q$.

(ii) Let $\eta\colon I \times \overline{Q} \to \mathbb{R}^n$ be such that $\eta(t) \in \mathcal{E}$ for every $t \in I$ and assume that η is Borel measurable when considered as a mapping from I to $W^{2,p}(Q;\mathbb{R}^n)$. Let $\sigma \in M(I \times \partial Q;\mathbb{R}^n)$ be such that $\operatorname{supp}\sigma \subset C_\eta$. Then σ is a contact force for η if and only if

$$\int_{I\times\partial Q} \varphi \cdot d\sigma \geq 0 \tag{4.3}$$

for all $\varphi\colon I \times \partial Q$ satisfying $\varphi(t,\cdot) \in C(\partial Q;\mathbb{R}^n)$, $t \in I$, Borel measurable in t and bounded in $I \times \partial Q$, with $\varphi(t,x) \in \widehat{T}_\eta(t,x)$ for $(t,x) \in I \times \partial Q$.

Proof. We only present the proof of the time-dependent characterization in statement (ii). Indeed, the proof of (i) follows from analogous (but simpler) arguments.

Step 1: Let σ be a contact force for η. Then, by the definition (see Definition 4.4 (ii)) we have that $d\sigma = g\,d|\sigma|$, where $-g(t,x) \in \overline{N}_\eta(t,x)$ for σ-a.e. $(t,x) \in C_\eta$. Consequently, by the polarity relation between $\widehat{T}_\eta(t,x)$ and $\overline{N}_\eta(t,x)$ (i.e., by the definition of $\overline{N}_\eta(t,x)$ given in Definition 3.2), we have that $g(t,x)\cdot\varphi(t,x) \geq 0$ σ-a.e. holds for every φ as in the statement. Therefore,

$$\int_{I\times\partial Q} \varphi \cdot d\sigma = \int_{I\times\partial Q} \varphi \cdot g\,d|\sigma| \geq 0,$$

thus proving that (4.3) holds as desired.
Step 2: Next, we show that any measure measure σ with $\operatorname{supp}\sigma \subset C_\eta$ that satisfies (4.3) is a contact force for η. To this end, arguing by contradiction, assume that σ is not a contact force for η. Then we can find a measurable set $S \subset I \times \partial Q$ with $|\sigma|(S) > 0$ with the property that $-g(t,x) \notin \overline{N}_\eta(t,x)$ for $(t,x) \in S$.

Let $h\colon S \to \mathbb{R}^n$ be Borel measurable with $h(t,x) \in \operatorname{int}\widehat{T}_\eta(t,x)$ and such that $h(t,x)\cdot g(t,x) < 0$ for σ-a.e. $(t,x) \in S$. Then, an application of Lusin's theorem yields the existence of a compact set $K \subset S$ with $|\sigma|(K) > 0$ and a function $\tilde{h} \in C(I \times \partial Q; \mathbb{R}^n)$ such that $h(t,x) = \tilde{h}(t,x)$ for all $(t,x) \in K$. For $t \in I$, we let $K_t := \{x \in \partial Q : (t,x) \in K\}$. Moreover, using the fact that $\tilde{h}$ is uniformly continuous and by Proposition 3.4, for every t we can find a set $G_t \supset K_t$ that is open with respect to the subspace topology of ∂Q and with the property that $\tilde{h}(t,x) \in \widehat{T}_\eta(t,x)$ for all $x \in G_t$.

Next, consider a sequence of cutoff functions $\psi_i\colon I \times \partial Q \to [0,1]$ that are Borel measurable in time and satisfy $\psi_i(t) \in C(\partial Q;[0,1])$ and $\chi_K \leq \psi_i \leq \chi_G$, and such that $\psi_i \to \chi_K$ in $L^1(I \times \partial Q, \sigma)$ as $i \to \infty$. We can take, for example, $\psi_i(t,x) := \max\{1 - i\operatorname{dist}(x,K_t), 0\}$ for $1/i \leq \operatorname{dist}(K_t, \partial Q \setminus G_t)$. Then, denoting $\varphi_i := \psi_i\tilde{h}$ and recalling that $h(t,x)\cdot g(t,x) < 0$ on K, we obtain that

$$\int_{I\times\partial Q} \varphi_i \cdot d\sigma = \int_{I\times\partial Q} \psi_i\tilde{h}\cdot g\, d|\sigma| \to \int_K h\cdot g\, d|\sigma| < 0.$$

In particular, the same inequality holds for $\varphi := \varphi_{n_0}$, for some $i_0 \in \mathbb{N}$ that is sufficiently large. Notice that φ is Borel measurable in time, $\varphi(t) \in C(\partial Q;\mathbb{R}^n)$, $\varphi(t,x) \in \widehat{T}_\eta(t,x)$ for all $(t,x) \in \partial Q$ and that moreover it satisfies

$$\int_{\partial Q} \varphi \cdot d\sigma < 0.$$

Thus, we have reached a contradiction to (4.3). This concludes the proof. □

Compactness and closure properties of contact forces will play a fundamental role in Section 6 and Section 7, where existence results are obtained via a limiting process involving approximate solutions.

Theorem 4.6 (Compactness-closure of contact forces)**.** *The following hold.*

(*i*) *Let $\{\eta_k\}_k \subset \mathcal{E}$ be a sequence of deformation with $E(\eta_k) \leq E_0$ for some $E_0 > E_{\min}$ and assume that there exists $\eta \in \mathcal{E}$ with $E(\eta) \leq E_0$ such that $\eta_k \to \eta$ in $C^1(Q;\mathbb{R}^n)$. For every k, let σ_k be a contact force for η_k with*

$$\sup_k \|\sigma_k\|_{M(\partial Q;\mathbb{R}^n)} < \infty.$$

Then there exist a subsequence (which we do not relabel) and a limit measure σ such that $\sigma_k \overset{*}{\rightharpoonup} \sigma$ in $M(\partial Q;\mathbb{R}^n)$. Moreover, σ is a contact force for η and if each σ_k satisfies the action-reaction principle at self-contact, then so does σ.

(*ii*) *Let $\{\eta_k\}_k$ be a sequence of time-dependent deformations, that is, for every k we have that $\eta_k\colon I \times \overline{Q} \to \mathbb{R}^n$ is Borel measurable in time and such that $\eta_k(t) \in \mathcal{E}$ and $E(\eta_k(t)) \leq E_0$ hold for all $t \in I$. Furthermore, assume that $\eta_k(t) \to \eta(t)$ in $C^1(\overline{Q};\mathbb{R}^n)$ uniformly in t, where $\eta \in C(I; C^{1,\alpha}(\overline{Q};\mathbb{R}^n))$ is such that $\eta(t) \in \mathcal{E}$ and $E(\eta(t)) \leq E_0$ for all $t \in I$. For every k, let $\sigma_k \in M(I \times \partial Q;\mathbb{R}^n)$ be a contact force for η_k with*

$$\sup_k \|\sigma_k\|_{M(I\times\partial Q;\mathbb{R}^n)} < \infty.$$

Then there exist a subsequence (which we do not relabel) and a limit measure σ such that $\sigma_k \overset{*}{\rightharpoonup} \sigma$ in $M(I\times\partial Q;\mathbb{R}^n)$. Moreover, σ is a contact force for η and if each σ_k satisfies the action-reaction principle at self-contact, then so does σ.

Proof. We present the proof of (*ii*). The proof of the time-independent statement in (*i*) is analogous but simpler, and therefore we omit it.

By the weak compactness of measures (eventually extracting a subsequence) we have that $\sigma_k \overset{*}{\rightharpoonup} \sigma$. Thus, it remains to verify that σ is a contact force for η.

We begin by showing that $\operatorname{supp}\sigma \subset C_\eta$. We mention here that the proof of this fact follows from the same argument used in [ČGK24, Theorem 3.9]. Indeed, for $(t,x) \in \operatorname{supp}\sigma$, using the fact that $\sigma_k \rightharpoonup \sigma$, we have that there are points $(t_k,x_k) \in \operatorname{supp}\sigma_k$ such that $(t_k,x_k) \to (t,x)$. Since $\operatorname{supp}\sigma_k \subset C_{\eta_k}$ by assumption, there are now two cases: Either there is a subsequence such that $\eta_k(t_k,x_k) \in \partial\Omega$, or there exist points $y_k \neq x_k \in \partial Q$ with $\eta_k(t_k,x_k) = \eta_k(t_k,y_k)$. In the first case, the uniform convergence of η_k and the uniform continuity of η imply that $\eta_k(t_k,x_k) \to \eta(t,x) \in \partial\Omega$ and thus $(t,x) \in C_\eta$. In the second case, recall that by Lemma 4.2 (*iv*) there exists a minimal distance $r > 0$ such that $|x_k - y_k| \geq r$ holds for every k. Eventually extracting a further subsequence, we have $y_k \to y \in \partial Q$. As above, the uniform convergence of η_k and the uniform continuity of η imply that $\eta(t,x) = \eta(t,y)$ with $x \neq y$. Hence, also in this case we have shown that $(t,x) \in C_\eta$. We note here that even though (in contrast to [ČGK24]) there can be more than two points touching at the same location, this complication is readily circumvented by taking a subsequence of y_k.

With this at hand, we can now show that σ is indeed a contact force for η by using the characterization in Lemma 4.5 (ii). To this end, fix $\varphi\colon I \times \partial Q \to \mathbb{R}^n$ as in Lemma 4.5 (ii) and let φ_k be defined via

$$\varphi_k(t,x) := \nabla\eta_k(t,x)(\nabla\eta(t,x))^{-1}\varphi(t,x).$$

Then, in view of Lemma 3.5 we have that $\varphi_k(t,x) \in \widehat{T}_{\eta_k}(t,x)$ and furthermore, by the uniform convergence of $\nabla\eta_k \to \nabla\eta$ in $I \times \partial Q$, we obtain that $\varphi_k \to \varphi$ uniformly in $I \times \partial Q$. Since σ_k is a contact force for η_k, by Lemma 4.5 (ii) we have that

$$\int_{I\times\partial Q} \varphi_k(t,x) \cdot d\sigma_k(t,x) \geq 0.$$

Next, observe that

$$\int_{I\times\partial Q} \varphi \cdot d\sigma = \int_{I\times\partial Q} \varphi \cdot d(\sigma - \sigma_k) + \int_{I\times\partial Q} (\varphi - \varphi_k) \cdot d\sigma_k + \int_{I\times\partial Q} \varphi_k \cdot d\sigma_k,$$

and that, passing to the limit as $k \to \infty$, the first two terms on the right-hand side go to zero. This shows that

$$\int_{I\times\partial Q} \varphi \cdot d\sigma = \lim_{k\to\infty} \int_{I\times\partial Q} \varphi_k \cdot d\sigma_k \geq 0,$$

and in turn, once again by Lemma 4.5 (ii), we obtain that σ is a contact force for η.

Finally, for any $\varphi \in C_c(I \times \Omega; \mathbb{R}^n)$ we have that $\varphi \circ \eta_k \to \varphi \circ \eta$ uniformly in $I \times \partial Q$. This, together with the convergence $\sigma_k \overset{*}{\rightharpoonup} \sigma$, shows that if σ_k satisfies the action-reaction principle for every k, then the action-reaction principle continues to hold in the limit. □

We conclude this section by noting that L^2-in-time estimates remain true in the weak* limit.

Lemma 4.7. *Let $\sigma_k \in L^2_{w^*}(I; M(\partial Q; \mathbb{R}^n))$ with $\|\sigma_k\|_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))} \leq C$ be such that $\sigma_k \overset{*}{\rightharpoonup} \sigma$ in $M(I \times \partial Q; \mathbb{R}^n)$. Then $\sigma \in L^2_{w^*}(I; M(\partial Q; \mathbb{R}^n))$ and $\|\sigma\|_{L^2_{w^*}(I;M(\partial Q;\mathbb{R}^n))} \leq C$.*

For a proof and a discussion of the weak* measurability involved, we refer to [ČGK24, Remark 3.12, Lemma 3.13].

5. Admissible directions and test functions

In this section we study the space of admissible directions. These are, roughly speaking, infinitesimal perturbations of the deformed configuration that do not cause self-interpenetration and for which $\eta(\cdot, \overline{Q}) \subset \overline{\Omega}$. To be precise, we let

$$T_\eta(\mathcal{E}) := \{\varphi \in W^{2,p}(Q; \mathbb{R}^n) : \exists \varepsilon_1 > 0 \text{ and } \Phi \in C([0,\varepsilon_1); \mathcal{E}) \cap C^1([0,\varepsilon_1); W^{2,p}(Q;\mathbb{R}^n)) \text{ with } \Phi(0) = \eta \text{ and } \Phi'(0^+) = \varphi\}. \tag{5.1}$$

The following result is adapted from [Pal18, Proposition 3.1] and provides a useful characterization of a large subset of $T_\eta(\mathcal{E})$.

Lemma 5.1 (Strictly interior directions are admissible)**.** *For $\eta \in \mathcal{E}$, let*

$$T^0_\eta(\mathcal{E}) := \{\varphi \in W^{2,p}(Q; \mathbb{R}^n) : \varphi|_\Gamma = 0, \varphi(x) \in \operatorname{int} \widehat{T}_\eta(x) \text{ for all } x \text{ with } \eta(x) \in \partial\Omega, \text{ and } \varphi(x) - \varphi(y) \in \operatorname{int} \widehat{T}_\eta(x) - \operatorname{int} \widehat{T}_\eta(y) \text{ for all } x \neq y \text{ with } \eta(x) = \eta(y)\}. \tag{5.2}$$

Then $T^0_\eta(\mathcal{E}) \subset T_\eta(\mathcal{E})$.

Proof. Let $\varphi \in T^0_\eta(\mathcal{E})$ be given and set $\eta_\varepsilon := \eta + \varepsilon\varphi$. In the following we will show that the function $\Phi(\varepsilon) := \eta_\varepsilon$ then satisfies the conditions of (5.1).

First of all, observe that $\eta_\varepsilon = \eta_0$ when restricted to Γ. Next, recall that by (E.2) there exists $\varepsilon_0 > 0$ that only depends on $E(\eta)$ such that $\det \nabla\eta \geq \varepsilon_0$ in $\overline{Q}$. Since $\varphi \in C^{1,\alpha}(Q)$, we have that

$$\det \nabla\eta_\varepsilon \geq \frac{\varepsilon_0}{2}, \tag{5.3}$$

provided that ε is sufficiently small. Reasoning as in Remark 4.3, we also obtain that, for all ε such that (5.3) holds, η_ε is injective when restricted to balls of radius r, where r only depends on $E(\eta)$.

We now claim that η_ε is (globally) injective on $\overline{Q}$ for all $\varepsilon > 0$ small enough. Indeed, arguing by contradiction, assume that there are a monotone decreasing sequence $\varepsilon_i \to 0^+$ and points $x_i \neq y_i \in \overline{Q}$ such that $\eta_{\varepsilon_i}(x_i) = \eta_{\varepsilon_i}(y_i)$. By the compactness of $\overline{Q}$ (up to the extraction of a subsequence, which we do not relabel) we can assume that $x_i \to x$ and $y_i \to y$. We now consider three cases for $\eta(x_i) - \eta(y_i)$ and see that none of them can happen for a subsequence, resulting in a contradiction.

Case 1:

$$\eta(y_i) - \eta(x_i) \in \operatorname{int} \widehat{T}_\eta(x) - \operatorname{int} \widehat{T}_\eta(y). \tag{5.4}$$

Since the local injectivity of η_{ε_i} for i large enough implies that $|x_i - y_i| \geq r$, then necessarily we must have that $|x - y| \geq r$. Using the fact that $\eta_{\varepsilon_i} \to \eta$ uniformly we obtain that $\eta(x) = \eta(y)$, and by Lemma 4.2 (i) we conclude that $x, y \in \partial Q$. Note that if $\eta(y_i) - \eta(x_i) \in \operatorname{int} \widehat{T}_\eta(x) - \operatorname{int} \widehat{T}_\eta(y)$ holds for infinitely many values of i, then by Proposition 3.4 there exists a subsequence such that $\eta(y_i) - \eta(x_i) \in \operatorname{int} \widehat{T}_\eta(x_i) - \operatorname{int} \widehat{T}_\eta(y_i)$ for all i large enough. This means, by Proposition 3.4, that $\eta(B_{r/2}(x_i) \cap Q)$ and $\eta(B_{r/2}(y_i) \cap Q)$ intersect, in contradiction with the interior injectivity of η, Lemma 4.2 (i).

Case 2:

$$\eta(x_i) = \eta(y_i). \tag{5.5}$$

Observe that if (5.5) holds, then we have that $\varphi(x_i) - \varphi(y_i) \in \operatorname{int} \widehat{T}_\eta(x_i) - \operatorname{int} \widehat{T}_\eta(y_i)$ by (5.2), and therefore $\eta_{\varepsilon_i}(x_i) \neq \eta_{\varepsilon_i}(y_i)$, as guaranteed by Lemma 4.2 (iii).

Case 3:

$$\eta(y_i) - \eta(x_i) \notin (\operatorname{int} \widehat{T}_\eta(x) - \operatorname{int} \widehat{T}_\eta(y)) \cup \{0\}. \tag{5.6}$$

Let K_β be the enlarged polar cone given by Lemma 3.7 with $C = \widehat{T}_\eta(x) - \widehat{T}_\eta(y)$ and $v_0 = \varphi(x) - \varphi(y) \in \operatorname{int} \widehat{T}_\eta(x) - \operatorname{int} \widehat{T}_\eta(y)$. Then, for all $w \in K_\beta$ we have that

$$(\varphi(x) - \varphi(y)) \cdot w < 0. \tag{5.7}$$

Let p_i be the projection of $\eta(y_i) - \eta(x_i)$ onto K_β and observe $p_i \neq 0$ by (5.6). Then, using the fact that $p_i \cdot (\eta(y_i) - \eta(x_i)) \geq 0$, we obtain that

$$\frac{p_i}{|p_i|} \cdot (\eta_{\varepsilon_i}(y_i) - \eta_{\varepsilon_i}(x_i)) \geq \varepsilon_i \frac{p_i}{|p_i|} \cdot (\varphi(y_i) - \varphi(x_i)). \tag{5.8}$$

Notice that (eventually extracting a subsequence) $p_i/|p_i| \to p \in K_\beta \setminus \{0\}$, and therefore, by (5.7) we have that

$$\frac{p_i}{|p_i|} \cdot (\varphi(y_i) - \varphi(x_i)) \to p \cdot (\varphi(y) - \varphi(x)) > 0. \tag{5.9}$$

In particular, combining (5.8) and (5.9), we obtain that

$$\frac{p_i}{|p_i|} \cdot (\eta_{\varepsilon_i}(y_i) - \eta_{\varepsilon_i}(x_i)) > 0 \tag{5.10}$$

holds for all i sufficiently large. Finally, (5.10) implies that $\eta_{\varepsilon_i}(x_i) \neq \eta_{\varepsilon_i}(y_i)$.

Therefore we have reached the desired contradiction and thus shown that η_ε is injective on $\overline{Q}$ for all ε sufficiently small.

A similar but simpler argument can be used to show that $\eta_\varepsilon(\overline{Q}) \subset \Omega$. Indeed, the obstacle $\mathbb{R}^n \setminus \Omega$ can be treated as a non-moving part of the solid. This concludes the proof. □

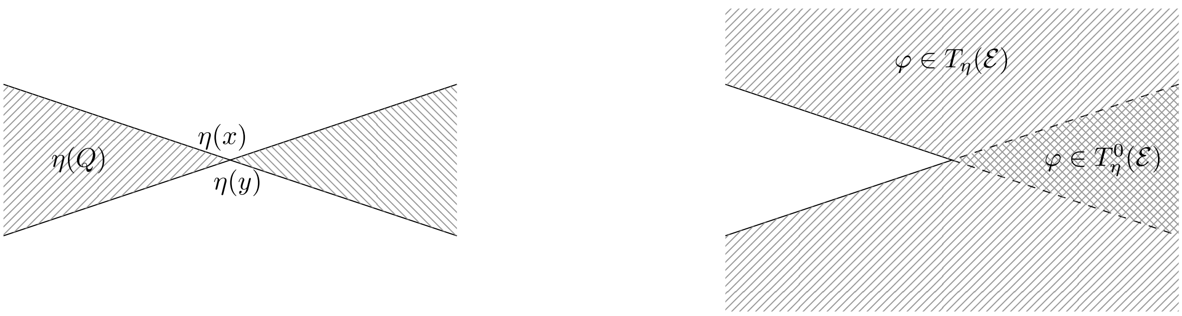


FIGURE 2. Comparison of $T^0_\eta(\mathcal{E})$ and $T_\eta(\mathcal{E})$ as described in Remark 5.2. The first image shows two parts of the solid touching at their corners x, y with $\eta(x) = \eta(y)$. For y as preimage of the right side, the second image illustrates the possible values of $\varphi(y)$.

Remark 5.2. *Note that, in general, the inclusion in Lemma 5.1 is strict. To see this, consider the case of a head-on collision where the contact happens between two outside corners of otherwise smooth, well-separated surfaces. Then the cones $\widehat{T}_\eta(x)$ and $\widehat{T}_\eta(y)$ have the same opening angles and the resulting set $T^0_\eta(\mathcal{E})$ is relatively small when compared to $T_\eta(\mathcal{E})$, that is, the set of all the directions it is possible to move in.*

In contrast, if we replace one of the these angles with the complement of the other one, creating an inside corner, then at the corner the regular tangent cones will in fact not change, but now the closure of $T^0_\eta(\mathcal{E})$ is precisely the set of directions it is possible to move in.

In fact, this shows that it is not possible to give a pointwise characterization of $T_\eta(\mathcal{E})$ in terms of the regular tangent cone. It is reasonable to ask if there is a better characterization in terms of e.g. a different definition of tangent cones but to the best of our knowledge there seems to be no way to do so, as the set of admissible directions ultimately depends on the complete behavior (e.g. oscillations and the directions they occur in) of the boundary in a neighborhood of the contact point.

For our purposes however, neither is such characterization actually required, as the preceding lemma gives us a sufficient amount of admissible directions for all required estimates, nor would having such a characterization be particularly helpful. Indeed, while it would allow for a stronger control on the directions of the Lagrange multipliers in the initial minimization, these are not the contact forces occurring in the final equation, which arise only as limits, whose behavior is better captured using regular normal and tangent cones.

Additionally, since it is not convex, characterizing the set of admissible directions actually does not characterize the set of admissible test functions for the equation. Consider for example the above case of two angles meeting. These can "slide" along each other in two directions, so both of these directions provide admissible test functions. But then so does their average, even though moving in that average direction immediately results in an overlap.

All of this is in contrast to the case of a smooth boundary (cmp. [ČGK24]), where there is no need to distinguish different cones, as all different types of tangential cones for a given point will be the same half-space and there is only ever a single normal direction.

To allow us to better quantify contact forces in the proof we now require a specific type of test function.

Definition 5.3. *Given $\eta \in \mathcal{E}$, we say that $\tilde{t}_\eta \colon \overline{Q} \to \mathbb{R}^n$ is a* uniformly interior vector field *for η, if there is an angle $\vartheta \in (0, \pi/2)$ and a constant $0 < c \leq 1$ such that*

$$\tilde{t}_\eta(x) \cdot n \leq -\sin\vartheta |\tilde{t}_\eta(x)||n| \quad \text{and} \quad c \leq |\tilde{t}_\eta(x)| = 1 \qquad \text{for all} \quad x \in \partial Q \text{ and } n \in \overline{N}_\eta(x),$$

and $|\tilde{t}|$ is bounded on $\overline{Q}$. Similarly, given $\eta \in L^\infty(I;\mathcal{E}) \cap W^{1,2}(I; W^{1,2}(Q;\mathbb{R}^n))$ with $E(\eta) \in L^\infty(I)$, we say that $\tilde{t}_\eta \colon I \times \overline{Q} \to \mathbb{R}^n$ is a uniformly interior vector field *for η, if there is an angle $\vartheta \in (0, \pi/2)$ such that for all $t \in I$*

$$\tilde{t}_\eta(t,x) \cdot n \leq -\sin\vartheta |\tilde{t}_\eta(t,x)||n| \quad \text{and} \quad |\tilde{t}_\eta(t,x)| = 1 \qquad \text{for all} \quad x \in \partial Q \text{ and } n \in \overline{N}_\eta(x),$$

and $|\tilde{t}(t,\cdot)|$ is bounded on $\overline{Q}$.

Note that the polarity of $\widehat{T}_\eta(x)$ and $\overline{N}_\eta(x)$ and the strict positivity of $\sin\vartheta$ imply that $\tilde{t}_\eta(x) \in \operatorname{int} \widehat{T}_\eta(x)$. To be precise, the definition implies that the cone $L_{\vartheta, \tilde{t}_\eta(x)} \subset \widehat{T}_\eta(x)$ (see (3.1)). We show below that for any deformation η there exists a *smooth* uniformly interior vector field.

Proposition 5.4. *The following hold.*

(*i*) *For every $\eta \in \mathcal{E}$ there exists a uniformly interior vector field $\tilde{t}_\eta$ in the sense of Definition 5.3 with $\tilde{t}_\eta \in C^{k_0}(\overline{Q};\mathbb{R}^n)$ and such that $\|\tilde{t}_\eta\|_{C^{k_0}(\overline{Q};\mathbb{R}^n)} \leq C$ for all $k_0 \in \mathbb{N}$, where C is a constant that depends only on $E(\eta)$ and k_0.*

(*ii*) *For every $\eta \in L^\infty(I;\mathcal{E}) \cap W^{1,2}(I;W^{1,2}(Q;\mathbb{R}^n))$ with $\|E(\eta)\|_{L^\infty(I)} \leq E_0$ there exists a uniformly interior vector field $\tilde{t}_\eta$ in the sense of Definition 5.3 with $\tilde{t}_\eta \in C(I;C^{k_0}(\overline{Q};\mathbb{R}^n))$ and such that $\|\tilde{t}_\eta\|_{C(I;C^{k_0}(\overline{Q};\mathbb{R}^n))} \leq C_{k_0}$ for all $k_0 \in \mathbb{N}$, where C_{k_0} is a constant that depends only on E_0 and k_0. Moreover it can be chosen so that $\tilde{t}_\eta \in W^{1,2}(I;L^2(Q;\mathbb{R}^n))$ with the estimate $\|\partial_t \tilde{t}_\eta\|_{L^2(I\times Q;\mathbb{R}^n)} \leq C\|\partial_t \nabla\eta\|_{L^2(I\times Q;\mathbb{R}^n)}$ with C depending on E_0.*

Proof. We only present the proof of (*ii*). The proof of the time-independent statement in (*i*) follows from a similar argument (see also [Pal18, Proposition 3.1]). Let $\{G_1, \dots, G_k\}$ be a covering of ∂Q with corresponding points $x_i \in G_i$ and directions $v_i \in S^{n-1}$ and ϑ for E_0 given as in the proof of Lemma 3.6. Choose $\delta > 0$ so small that $\{G_1, \dots, G_k\}$ covers also P_δ, the δ-neighborhood of ∂Q. Note that this δ still depends only on Q and E_0. Now let $\{\psi_1, \dots, \psi_k\}$ be a partition of unity on P_δ subordinated to the covering $\{G_1, \dots, G_k\}$. To be precise, $\psi_i \in C_c^\infty(G_i, [0,1])$ for every $i = 1, \dots, k$ and $\sum_{i=1}^k \psi_i(x) = 1$ for all $x \in P_\delta$.

Next, we construct $\tilde{t}_Q \in C^\infty(Q_\delta; \mathbb{R}^n)$, where Q_δ is the δ-neighborhood of Q, by setting (we consider each ψ_i to be extended outside G_i by 0)

$$\tilde{t}_Q(x) := \sum_{i=1}^k \psi_i(x) v_i, \quad x \in Q_\delta.$$

Then $\tilde{t}_Q$ is a smooth uniformly interior vector field to Q with angle ϑ, moreover satisfying $c \le |\tilde{t}_Q| \le 1$ on P_δ, where $0 < c \le 1$ depends only on Q.

We consider an extension of η to $I \times Q_\delta$ preserving all the norms, in particular the $L^\infty(I; W^{2,p}(Q_\delta; \mathbb{R}^n))$ and $W^{1,2}(I; W^{1,2}(Q_\delta; \mathbb{R}^n))$ norms, up to a constant. With this define

$$t_\eta(t,x) = \frac{\nabla\eta(t,x)\,\tilde{t}_Q(x)}{|\nabla\eta(t,x)\,\tilde{t}_Q(x)|}|\tilde{t}_Q(x)|, \quad t \in I, \quad x \in Q_\delta.$$

Because of the uniform lower bound on the Jacobian which can be extended to Q_δ for δ small enough, we see that $t_\eta \in C(I; C^{1,\alpha}(Q_\delta; \mathbb{R}^n))$ with Hölder seminorm dependent only on E_0. Further, we see by an application of the chain rule, that

$$\|\partial_t t_\eta(t)\|_{L^2(Q_\delta;\mathbb{R}^n)} \le C \|\partial_t \nabla\eta\|_{L^2(Q;\mathbb{R}^n)}$$

with C depending on E_0.

Now choose $\tilde{\delta} > 0$, possibly smaller than δ but only dependent on E_0, such that it holds for all $t \in I$ and all $x \in \partial Q$ and $\tilde{x} \in Q_\delta$ with $|x - \tilde{x}| \le \tilde{\delta}$ that

$$t_\eta(t,x) \cdot t_\eta(t,\tilde{x}) \ge \cos(\vartheta/2).$$

We mollify in space, that is put $\tilde{t}_\eta := t_\eta * \xi_\delta$, where $*$ is the convolution in space and ξ_δ the smooth mollification kernel with radius δ. Because of the above choice of $\tilde{\delta}$, one can readily check that $\tilde{t}_\eta$ is a uniformly interior field for η in the sense of Definition 5.3, with the angle $\vartheta/2$. Finally, regarding the regularity of $\tilde{t}_\eta$, we see that for each $t \in I$ we have

$$\|\tilde{t}_\eta(t)\|_{C^{k_0}(\overline{Q};\mathbb{R}^n)} \le C_{k_0} \|t_\eta(t)\|_{C(\partial Q;\mathbb{R}^n)}$$

with C_{k_0} depending on E_0 and k_0, and

$$\|\partial_t \tilde{t}_\eta(t)\|_{L^2(Q;\mathbb{R}^n)} \le C \|\partial_t t_\eta(t)\|_{L^2(Q;\mathbb{R}^n)}$$

with C depending on E_0, which combining with the above inequalities finishes the proof. □

Next, we show that the set of strictly interior directions, namely $T^0_\eta(\mathcal{E})$, is well-behaved with respect to sequences of approximating deformations.

Proposition 5.5. *The following hold.*

- *(i) Let $\{\eta_k\}_k \subset \mathcal{E}$ be given and assume that there exists $\eta \in \mathcal{E}$ such that $\eta_k \to \eta$ in $C^1(\overline{Q}; \mathbb{R}^n)$. Then, for every $\varphi \in T^0_\eta(\mathcal{E})$ there exists a sequence $\{\varphi_k\}_k$ such that $\varphi_k \to \varphi$ in $W^{2,p}(Q; \mathbb{R}^n)$ and with the property that $\varphi_k \in T^0_{\eta_k}(\mathcal{E})$ for all k sufficiently large.*
- *(ii) Let $\{\eta_k\}_k \subset L^\infty(I; \mathcal{E})$ be given and assume that there exists $\eta \in W^{1,2}(I; W^{2,p}(Q; \mathbb{R}^n))$ with $\|E(\eta)\|_{L^\infty(I)} \le E_0$ such that $\eta_k \to \eta$ and $\nabla\eta_k \to \nabla\eta$ uniformly on $I \times \overline{Q}$. Then, for every $\varphi \in C(I; T^0_\eta(\mathcal{E}))$ there exists a sequence $\{\varphi_k\}_k$ such that $\varphi_k \to \varphi$ in $C(I; W^{2,p}(Q; \mathbb{R}^n))$ and with the property that $\varphi_k \in C(I; T^0_{\eta_k}(\mathcal{E}))$ for all k sufficiently large. Moreover, let $J \subset I$ and assume that $\varphi \in C(I; T^0_\eta(\mathcal{E})) \cap C^1_c(J; L^2(Q; \mathbb{R}^n))$. Then there exists a sequence $\{\varphi_k\}_k$ as above but with the additional property that $\varphi_k \in C(I; T^0_{\eta_k}(\mathcal{E})) \cap C^1_c(J; L^2(Q; \mathbb{R}^n))$ for all k sufficiently large.*

Proof. We only present the proof of the slightly more complicated time-dependent statement *(ii)*.

Let $\xi_\Gamma : \overline{Q} \to \mathbb{R}$ be a smooth function that satisfies $\xi_\Gamma(x) = 0$ for $x \in \Gamma$ and $\xi_\Gamma(x) > 0$ for $x \in \overline{Q} \setminus \overline{\Gamma}$. Let φ be as in *(ii)* and $\tilde{t}_\eta$ be as in Proposition 5.4 *(ii)* (with $k_0 \ge 2$). For $m \in \mathbb{N}$, we then define

$$\varphi_m := \varphi + \frac{1}{m}\tilde{t}_\eta \xi_\Gamma. \tag{5.11}$$

As one can readily check (see (5.2)), we have that $\varphi_m \in C(I; T^0_\eta(\mathcal{E}))$ for all m; furthermore, it is evident that $\varphi_m \to \varphi$ in $C(I; W^{2,p}(Q; \mathbb{R}^n))$ as $m \to \infty$.

Fix $m > 0$. We claim that $\varphi_m \in C(I; T^0_{\eta_k}(\mathcal{E}))$ for all k sufficiently large. Indeed, if this is not the case then we can find a subsequence of $\{\eta_k\}_k$ (which we do not relabel) and contact points $\{(t_k, x_k)\}_k$ with $(t_k, x_k) \in C_{\eta_k}$ such that either there exists $\{y_k\}_k$ with $x_k \ne y_k$ and $\eta_k(t_k, x_k) = \eta_k(t_k, y_k)$

and $\varphi_m(t_k,x_k) - \varphi_m(t_k,y_k) \notin \operatorname{int}\widehat{T}_{\eta_k}(t_k,x_k) - \operatorname{int}\widehat{T}_{\eta_k}(t_k,y_k)$ or $\eta_k(t_k,x_k) \in \partial\Omega$ and $\varphi_m(t_k,x_k) \notin \operatorname{int}\widehat{T}_{\eta_k}(t_k,x_k)$.

Since that the approach for handling the latter case is comparable, we consider only the first case. Eventually extracting a further subsequence (which again we do not relabel), we can find $t \in I$ and $x, y \in \partial Q$ such that $t_k \to t$, $x_k \to y$, $y_k \to y$, $x \neq y$, and $\eta(t,x) = \eta(t,y)$.

Recalling that $\varphi_m \in C(I; T^0_\eta(\mathcal{E}))$, by (5.2) and Proposition 5.4 *(ii)*, we can find two sets K_1 and K_2 such that

$$\varphi_m(t,x) - \varphi_m(t,y) \in K_1 \subset\subset K_2 \subset\subset \operatorname{int}\widehat{T}_\eta(t,x) - \operatorname{int}\widehat{T}_\eta(t,y).$$

In particular, Proposition 3.4, the regularity of η, and Lemma 3.5 imply that

$$\varphi_m(t',x') - \varphi_m(t',y') \in K_2 \subset\subset \operatorname{int}\widehat{T}_\eta(t',x') - \operatorname{int}\widehat{T}_\eta(t',y') \tag{5.12}$$

for every t' near t, every x' near x and every y' near y. Since by assumption we have that $\nabla\eta_k \to \nabla\eta$ uniformly on $I \times \overline{Q}$, another application of Lemma 3.5 yields that

$$K_2 \subset \operatorname{int}\widehat{T}_{\eta_k}(t_k,x_k) - \operatorname{int}\widehat{T}_{\eta_k}(t_k,y_k)$$

holds for all k sufficiently large, where K_2 is given as in (5.12).

In turn, this implies that $\varphi_m(t_k,x_k) - \varphi_m(t_k,y_k) \in \operatorname{int}\widehat{T}_{\eta_k}(t_k,x_k) - \operatorname{int}\widehat{T}_{\eta_k}(t_k,y_k)$, and we have therefore reached a contradiction.

To prove the second part of the statement, let

$$\varphi_m := \varphi + \frac{1}{m}\tilde{t}_\eta \xi_\Gamma \psi_J,$$

where ψ_J is an opportunely defined smooth cut-off function to ensure that $\varphi_m \in C^1_c(J; L^2(Q;\mathbb{R}^n))$. The argument above can then be repeated with only straightforward changes. This concludes the proof. $\square$

FIGURE 3. Approximate configurations and construction of a contact force without a fixed sign for all potential movement directions, as described in Remark 5.6. Specifically, moving the right hand solid in direction v is part of $T_\eta(\mathcal{E})$ but not $T^0_\eta(\mathcal{E})$.

Remark 5.6. *Proposition 5.5 already gives us a hint as to why we need to consider the set $T^0_\eta(\mathcal{E})$ instead of $T_\eta(\mathcal{E})$ in the weak inequality (2.1). As we construct solutions through approximations, we require some notion of stability under convergence. To be precise, as the set of admissible test functions necessarily depends on the deformation itself, when studying the convergence of approximate solutions, we can only consider test functions that can be approximated by other test functions which are admissible along the sequence of converging deformations.*

The sets $T_\eta(\mathcal{E})$ do not have this property. Or spoken in a more abstract way, $\{(\eta, b) : \eta \in \mathcal{E}, b \in T_\eta(\mathcal{E})\}$ is not closed in any reasonable topology. To see this, consider a limit configuration consisting of two corners touching at their tips. This can arise as the limit of a sequence of configurations where the two parts of the solid touch along their sides (see Figure 3). Even restricting to rigid motions of one of the sides, it is clear that in the approximation there is a whole half-space of directions that cause overlap and are thus not admissible. In contrast, for the limit configuration, the only rigid motions in $T_\eta(\mathcal{E})$ that are excluded are those that directly cause the tip to enter into the other corner.[3]

The same issue also directly translates into a consideration about contact forces: As the approximations have well-defined normals, we can have a clearly defined contact force for each of them. Assuming the right scaling, this force can persist in the limit. Indeed, this is precisely the reason why we can only assume that the final contact force has a direction in $\overline{N}_Q(x)$ instead of the smaller set $T_Q(x)^*$.

We note here that this does not mean that the set of test functions, nor the set of possible directions for contact forces we choose are optimal. However, this proved to be good enough to obtain a satisfactory

[3]Note also that the resulting set $T_\eta(\mathcal{E})$ is non-convex. As the weak inequality is linear in its test function, this here this would allow to enlarge the set of test functions even further.

existence theory and is likely the best that can be done using local characterizations. For example, for this type of corners there are only two possible directions of contact forces, which are in fact determined by the side of the overlap the approximation is coming from. Even for isolated corners, this argument is difficult to formalize, and still much harder to generalize to an arbitrary setting of Lipschitz boundaries. In contrast, the approach we use is mostly based on a well-established theory and relies mainly on the abstract minimization structure instead of a precise characterization of contact forces or admissible testing directions. It is thus also much easier to generalize to other settings.

Remark 5.7. *Formally, $T_\eta(\mathcal{E})$ is the tangent cone to $\mathcal{E}$ in the space $W^{2,p}(Q;\mathbb{R}^n)$. One can then ask what the corresponding regular tangent cone is, i.e. $\widehat{T}_\eta(\mathcal{E}) := \liminf_{\tilde\eta\to\eta,\tilde\eta\in\mathcal{E}} T_{\tilde\eta}(\mathcal{E})$. The space $T^0_\eta(\mathcal{E})$ appears to be a reasonable candidate, as Proposition 5.5 implies that $T^0_\eta(\mathcal{E}) \subset \widehat{T}_\eta(\mathcal{E})$ and shows that $T^0_\eta(\mathcal{E})$ behaves similarly with respect to convergence. Conveniently, this would also formally identify the set of contact forces with the corresponding convexified normal cone.*

We conjecture that this is indeed a characterization, i.e. $T^0_\eta(\mathcal{E}) = \widehat{T}_\eta(\mathcal{E})$. It is not hard to convince oneself that it is true in standard cases such as a smooth boundary, where both equal $T_\eta(\mathcal{E})$, or for known geometries by a direct contradiction: If without loss of generality $\varphi \in T_\eta(\mathcal{E})$ with $\varphi(x) \notin \widehat{T}_\eta(x)$ at contact and the other side is not moving, then there is an approximation $\{x_k\}_k$ of x for which $\varphi(x) \notin T_\eta(x_k)$. Constructing an approximation $(\eta_k)_k$ of η with contact at x_k that results in collision when moving in the direction $\varphi(x_k) \approx \varphi(x)$ is then a simple exercise in the case of e.g. isolated corners.

For general Lipschitz boundaries, this proof strategy runs into technical issues. Nevertheless we were not able to find a counterexample and thus leave this characterization as an open problem. In fact our technique is compatible with using $\widehat{T}_\eta(\mathcal{E})$ in place of $T^0_\eta(\mathcal{E})$, as we mainly rely its behavior under convergence. However we believe that the explicit characterization that $T^0_\eta(\mathcal{E})$ offers to be more valuable in practice.

Remark 5.8. *Figure 3 also illustrates why, due to the collisions, we cannot expect stability or even uniqueness of the evolution. In the case of two corners colliding almost head on, with an arbitrary small offset, they will slide along each other on either side, resulting in two distant solutions with arbitrarily close initial data. It the symmetric case of a head on collision on the other hand, no side is prefered, but either seems energetically preferable to the symmetric solution, so this symmetry needs to be broken.*

While we do not claim this as a rigorous proof, we use it to highlight that this question is non-trivial. Internally, from the terms of the solid and their (pseudo-)monotone nature, one would expect them to behave nicely. However as the above example shows, it is not possible to use this in a standard Gronwall-argument. Indeed the contact force is stable under convergence of solutions, however as seen above it is not stable under perturbations of the solution. A proper study of this issue, proving e.g. uniqueness of solutions for all initial values except a small degenerate set like the head on collision above, would thus be novel in itself and far exceed the scope of the current article.

6. Quasistatic evolution of viscoelastic solids with Lipschitz boundaries

In this section, we study the quasistatic counterpart of our problem. Besides being of independent interest, this will also serve as a building block for the main result of this paper. We mainly follow the strategy presented in [ČGK24, Section 4], in combination with the results of the previous sections, in order to treat the case of Lipschitz boundaries. Thus, we will only sketch the parts of the proof that are identical to those in [ČGK24] and focus more on the differences. In particular, a crucial step in our analysis involves establishing an energy inequality. For this, we follow a different approach than the one in [ČGK24], which allows us to work without an additional regularization (see Remark 6.4 below). This has the advantage that, in contrast to [ČGK24, Section 4], we do not need an additional set of assumptions for the regularized problem.

The quasistatic problem in question is

$$DE(\eta) + D_2R(\eta, \partial_t\eta) = f. \tag{6.1}$$

As a particular case, we obtain an existence theory for the parabolic equation

$$\rho\frac{\partial_t\eta}{h} + DE(\eta) + D_2R(\eta, \partial_t\eta) = f + \rho\frac{\zeta}{h}, \tag{6.2}$$

where $\rho\frac{\partial_t\eta}{h}$ can be thought of as derivative of an additional dissipation and ζ is a given function. Later in Section 7 we will make the choice $\zeta(t) = \partial_t\eta(t-h)$. Observe that combining these two newly introduced terms yields the difference quotient

$$\rho\frac{\partial_t\eta(t) - \partial_t\eta(t-h)}{h}.$$

Weak solutions to (6.1) are defined as follows.

Definition 6.1. *Let $h > 0$, $\eta_0 \in \mathcal{E}$, and $f \in L^2((0,h); L^2(Q;\mathbb{R}^n))$ be given. We say that*

$$\eta \in W^{1,2}((0,h); W^{1,2}(Q;\mathbb{R}^n)) \cap L^\infty((0,h);\mathcal{E}) \quad \textit{with} \quad E(\eta) \in L^\infty((0,h))$$

is a solution to (6.1) in $(0,h)$ if $\eta(0) = \eta_0$ and

$$\int_0^h DE(\eta(t))\langle\varphi(t)\rangle + D_2R(\eta(t), \partial_t\eta(t))\langle\varphi(t)\rangle\, dt \geq \int_0^h \langle f(t), \varphi(t)\rangle_{L^2}\, dt \tag{6.3}$$

holds for all $\varphi \in C([0,h]; T^0_\eta(\mathcal{E}))$,[4] where $T^0_\eta(\mathcal{E})$ is the cone defined in (5.2).

The existence of solutions to (6.1) is established in the following theorem.

Theorem 6.2. *Let E satisfy (E.1)–(E.6), R satisfy (R.1), (R.2), (R.3$_\mathrm{q}$) as well as (R.4), and let h, η_0, and f be given as above. Then there exists a solution η to (6.1) in the sense of Definition 6.1.*

Proof. We adopt the strategy employed in the initial two steps of the proof of [ČGK24, Theorem 4.3]. To facilitate the comparison and highlight the differences, we retain the same structure.

Step 1: Given $M \in \mathbb{N}$, we set $\tau := h/M$ and decompose $[0,h]$ into subintervals $[k\tau, (k+1)\tau]$ of length τ. Moreover, for every $1 \leq k \leq M$ we let

$$f_k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(t)\, dt \in L^2(Q;\mathbb{R}^n). \tag{6.4}$$

We then define $\eta_k \in \mathcal{E}$ recursively via

$$\eta_k \in \arg\min\left\{\mathcal{J}_k(\eta) : \eta \in \mathcal{E}\right\}, \tag{6.5}$$

where

$$\mathcal{J}_k(\eta) := E(\eta) + \tau R\left(\eta_{k-1}, \tfrac{\eta-\eta_{k-1}}{\tau}\right) - \tau\left\langle f_k, \tfrac{\eta-\eta_{k-1}}{\tau}\right\rangle_{L^2}. \tag{6.6}$$

The existence of η_k as in (6.5) follows by a standard application of the direct method in the calculus of variations. Observe that since $\mathcal{J}_k(\eta_k) \leq \mathcal{J}_k(\eta_{k-1})$, expanding these inequalities and summing over $k = 1, \ldots, m$, $m \leq M$, yields a uniform estimate in the form of

$$E(\eta_m) + \sum_{k=1}^m \tau R\left(\eta_{k-1}, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right) \leq E(\eta_0) + \tau\sum_{k=1}^m \left\langle f_k, \tfrac{\eta_k-\eta_{k-1}}{\tau}\right\rangle_{L^2}. \tag{6.7}$$

Next, for $t \in [(k-1)\tau, k\tau)$, we let

$$\underline{\eta}_\tau(t) := \eta_{k-1}, \qquad \overline{\eta}_\tau(t) := \eta_k, \qquad \text{and} \qquad \eta_\tau(t) := \frac{k\tau - t}{\tau}\eta_{k-1} + \frac{t-(k-1)\tau}{\tau}\eta_k. \tag{6.8}$$

Reasoning as in [ČGK24, Theorem 4.3], (6.7) and the assumptions on E and R yield the existence of a subsequence (which we do not relabel) and a limiting deformation $\eta \in W^{1,2}((0,h); W^{1,2}(Q;\mathbb{R}^n)) \cap L^\infty((0,h);\mathcal{E})$ such that

$$\underline{\eta}_\tau \overset{*}{\rightharpoonup} \eta \ \text{ in } L^\infty((0,h); W^{2,p}(Q;\mathbb{R}^n)),$$
$$\overline{\eta}_\tau \overset{*}{\rightharpoonup} \eta \ \text{ in } L^\infty((0,h); W^{2,p}(Q;\mathbb{R}^n)),$$

and

$$\eta_\tau \rightharpoonup \eta \ \text{ in } W^{1,2}((0,h); W^{1,2}(Q;\mathbb{R}^n)). \tag{6.9}$$

Step 2: A standard argument in the calculus of variations guarantees that each function η_k (see (6.5)) satisfies the Euler-Lagrange inequality

$$0 \leq D\mathcal{J}_k(\eta_k)\langle\varphi\rangle \tag{6.10}$$

for all $\varphi \in T^0_{\eta_k}(\mathcal{E})$. Summing up these inequalities and replacing all terms with their time-dependent counterparts (see (6.8)), we obtain that

$$0 \leq \int_0^h DE(\overline{\eta}_\tau(t))\langle\varphi(t)\rangle + D_2R(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t))\langle\varphi(t)\rangle - \langle f_\tau(t), \varphi(t)\rangle\, dt \tag{6.11}$$

for all $\varphi \in L^\infty((0,h); W^{2,p}(Q;\mathbb{R}^n))$ with $\varphi(t) \in T^0_{\overline{\eta}_\tau(t)}(\mathcal{E})$ for $\mathcal{L}^1$-a.e. $t \in (0,h)$. Therefore, in order to conclude the proof, we are left to show that all terms in (6.11) converge to their formal limit. For the last two, this is essentially a direct consequence of the weak convergence, together with (R.4) and

[4] Here we use $L^\infty((0,h);\mathcal{E})$ as a shorthand for the space of functions in $L^\infty((0,h); W^{2,p}(Q;\mathbb{R}^n))$ that belong to $\mathcal{E}$ for a.e. $t \in (0,h)$. Similarly, $C([0,h]; T^0_\eta(\mathcal{E}))$ denotes the subset of $C([0,h]; W^{2,p}(Q;\mathbb{R}^n))$ consisting of all functions with the property that $\varphi(t) \in T^0_{\eta(t)}(\mathcal{E})$ for all $t \in [0,h]$.

Proposition 5.5 to ensure a strong approximation of test functions. We note here that in order to apply Proposition 5.5, we need to show that $\nabla\overline{\eta}_\tau \to \nabla\eta$ uniformly in $I \times Q$. This, in turn, can be obtained by observing that, in view of the definition, $\overline{\eta}_\tau(t) = \overline{\eta}_\tau(\lceil \frac{t}{\tau} \rceil \tau) = \eta_\tau(\lceil \frac{t}{\tau} \rceil \tau)$, and therefore

$$\|\overline{\eta}_\tau(t) - \eta(t)\|_{C^{1,\alpha}} \leq \left\|\eta_\tau(\lceil \tfrac{t}{\tau} \rceil \tau) - \eta_\tau(t)\right\|_{C^{1,\alpha}} + \|\eta_\tau(t) - \eta(t)\|_{C^{1,\alpha}}.$$

Now, we have that the former term converges to 0 by uniform continuity of the η_τ, while the latter term converges to 0 uniformly by the convergence in $C([0,h]; C^{1,\alpha}(Q;\mathbb{R}^n))$ (which, up to the extraction of a further subsequence, follows by an application of the Aubin-Lions lemma).

For the potential energy however, due to the lack of regularization, we have to deviate from [ČGK24] and use a Minty-type argument that in [ČGK24] was only needed at a later stage. To this end, let $\psi \in C^\infty([0,h]; C^\infty_\Gamma(Q;[0,1]))$ be given. Then, by (E.6) and Lebesgue's dominated convergence theorem we see that

$$\begin{aligned} 0 &\leq \limsup_{\tau\to 0} \int_0^h [DE(\overline{\eta}_\tau(t)) - DE(\eta(t))]\langle(\overline{\eta}_\tau(t) - \eta(t))\psi(t)\rangle \, dt \\ &= \limsup_{\tau\to 0} \int_0^h DE(\overline{\eta}_\tau(t))\langle(\overline{\eta}_\tau(t) - \eta(t))\psi(t)\rangle \, dt. \\ &= \limsup_{\tau\to 0} \int_0^h DE(\overline{\eta}_\tau(t))\langle\varphi_\tau(t) + \delta_\tau \tilde{t}_{\overline{\eta}_\tau}(t)\psi(t)\rangle \, dt, \end{aligned} \tag{6.12}$$

where

$$\varphi_\tau := (\overline{\eta}_\tau - \eta - \delta_\tau \tilde{t}_{\overline{\eta}_\tau})\psi.$$

We recall here that $\tilde{t}_{\overline{\eta}_\tau}$ is the uniformly interior vector field given by Proposition 5.4. Notice that $\delta_\tau \in \mathbb{R}$ can be chosen in such a way that $-\varphi_\tau \in C([0,h]; T^0_\eta(\mathcal{E}))$ and $\delta_\tau \to 0$ as $\tau \to 0$. Observe that the (approximate) Euler–Lagrange inequality (6.11) for $-\varphi_\tau$ can be rewritten as

$$\int_0^h DE(\overline{\eta}_\tau(t))\langle\varphi_\tau(t)\rangle \, dt \leq \int_0^h -D_2R(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t))\langle\varphi_\tau(t)\rangle + \langle f_\tau(t), \varphi_\tau(t)\rangle \, dt. \tag{6.13}$$

Eventually extracting a subsequence (which we do not relabel), we have that

$$\varphi_\tau(t) \to 0 \qquad \text{in } W^{1,2}(Q;\mathbb{R}^n) \tag{6.14}$$

for $\mathcal{L}^1$-a.e. t. Moreover, since $f_\tau \to f$ in $L^2((0,h); L^2(Q;\mathbb{R}^n))$, an application of Lebesgue's dominated convergence theorem yields that

$$\int_0^h \langle f_\tau(t), \varphi_\tau(t)\rangle \, dt \to 0 \tag{6.15}$$

as $\tau \to 0$. Since $\{D_2R(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t))\}_\tau$ is bounded in $(W^{1,2})^*$ uniformly in t (see (R.4)), by (6.14) we have that

$$\int_0^h D_2R(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t))\langle\varphi_\tau(t)\rangle \, dt \to 0 \tag{6.16}$$

as $\tau \to 0$. Combining (6.12) with (6.13), (6.15), and (6.16) we conclude that

$$\begin{aligned} \limsup_{\tau\to 0} \int_0^h &[DE(\overline{\eta}_\tau(t)) - DE(\eta(t))]\langle(\overline{\eta}_\tau(t) - \eta(t))\psi(t)\rangle \, dt \\ &\leq \limsup_{\tau\to 0} \left| \int_0^h DE(\overline{\eta}_\tau(t))\langle\delta_\tau \tilde{t}_{\overline{\eta}_\tau}(t)\psi(t)\rangle \, dt \right| \\ &\leq \limsup_{\tau\to 0} \int_0^h \delta_\tau \|DE(\overline{\eta}_\tau(t))\|_{(W^{2,p})^*} \|\tilde{t}_{\overline{\eta}_\tau}(t)\psi(t)\|_{W^{2,p}} \, dt = 0. \end{aligned}$$

Since ψ is arbitrary, we must have that

$$\limsup_{\tau\to 0} [DE(\overline{\eta}_\tau(t)) - DE(\eta(t))]\langle(\overline{\eta}_\tau(t) - \eta(t))\psi(t)\rangle \leq 0$$

for a.e. t. Consequently, (E.6) implies that $\overline{\eta}_\tau(t) \to \eta(t)$ in $W^{2,p}(K;\mathbb{R}^n)$ for all K compactly contained in $\overline{Q}$ with $\operatorname{dist}(K,\Gamma) > 0$ and for $\mathcal{L}^1$-a.e. t. Thus, we have shown that, by (E.5), $DE(\overline{\eta}_\tau(t)) \to DE(\eta(t))$ in $(W^{2,p}(Q;\mathbb{R}^n))^*$. Since $\{DE(\overline{\eta}_\tau(t))\}_\tau$ is bounded in $(W^{2,p}(Q;\mathbb{R}^n))^*$ uniformly in t, we claim that this is enough to prove the existence of a solution. Indeed, let φ be as in Definition 6.1, then using Proposition 5.5 we can find $\varphi_\tau \to \varphi$ that is admissible for the approximate Euler–Lagrange variational inequality (see (6.11)) and we can then let $\tau \to 0$. □

6.1. **Energy inequality.** We will derive the energy inequality using the previous construction and the so called Moreau-Yosida approximation. This approach, which is commonly employed in minimizing movements and can be traced back to De Giorgi, effectively circumvents any regularity issues associated with test functions by relying solely on the metric properties of energy and dissipation. Given that we will primarily utilize it with the time-delayed approximation, we will prove the energy inequality with the respective terms from the inertia. Importantly, this proof is applicable even when $\rho = 0$, that is, in the quasistatic case. Our approach draws inspiration from [AGS05, Sec. 3.1], although some modifications are necessary to accommodate forces and delve deeper into the discussion of inertial terms. On the other hand, the specific nature of our problem allows for certain simplifications, facilitating the analysis.

Theorem 6.3 (Energy inequality)**.** *The solutions constructed for equation (6.1) satisfy the energy inequality*

$$E(\eta(s)) + \int_0^s 2R(\eta(t), \partial_t\eta(t))\,dt \le E(\eta_0) + \int_0^s \langle f(t), \partial_t \eta(t)\rangle_{L^2}\,dt \tag{6.17}$$

for all $s \in [0,h]$. Similarly, for the time-delayed equation (6.2), we have that

$$E(\eta(s)) + \int_0^s 2R(\eta(t), \partial_t\eta(t))\,dt + \frac{\rho}{2h}\int_0^s \|\partial_t\eta(t)\|_{L^2}^2\,dt$$
$$\le E(\eta_0) + \frac{\rho}{2h}\int_0^s \|\zeta(t)\|_{L^2}^2\,dt + \int_0^s \langle f(t), \partial_t\eta(t)\rangle_{L^2}\,dt \tag{6.18}$$

holds for all $s \in [0,h]$.

Proof. Since the estimate in (6.17) is a special case of (6.18) with $\rho = 0$, it suffices to prove the validity of the latter inequality.

Fix τ and $k \in \{1,\dots,M\}$. We define the Moreau-Yosida approximation by considering the family of functionals

$$E_\theta(\eta;\eta_{k-1}) := E(\eta) + \theta R\left(\eta_{k-1}, \tfrac{\eta-\eta_{k-1}}{\theta}\right) + \frac{\theta\rho}{2h}\left\|\tfrac{\eta-\eta_{k-1}}{\theta} - \zeta_k\right\|_{L^2}^2 - \theta\left\langle f_k, \tfrac{\eta-\eta_{k-1}}{\theta}\right\rangle_{L^2}$$

for $\theta \in (0,\tau]$ as well as their minimizers:

$$\eta_\theta \in \operatorname*{arg\,min}_{\eta\in\mathcal{E}} E_\theta(\eta;\eta_{k-1}).$$

Here ζ_k is used to denote the time average of ζ over the interval $[(k-1)\tau, k\tau)$ (see (6.4)). Reasoning as in the proof of Theorem 6.2, the existence of η_θ follows by an application of the direct method in the calculus of variations.

What we now want to do, is to vary θ between 0 and τ. In particular, we want to consider the (at the moment formal) equation

$$E_\tau(\eta_\tau;\eta_{k-1}) - \lim_{\theta\searrow 0} E_\theta(\eta_\theta;\eta_{k-1}) = \int_0^\tau \frac{d}{d\theta}E_\theta(\eta_\theta;\eta_{k-1})\,d\theta. \tag{6.19}$$

The reason for this is that, for $\theta = \tau$, without loss of generality we can identify η_τ with η_k and get the energy and part of the other terms we need, while on the other hand $E_\theta(\eta_\theta;\eta_{k-1})$ can be estimated by $E(\eta_{k-1})$ for $\theta \searrow 0$. Finally, the integral on the right hand side will give the missing terms.

Let us begin with the limit $\theta \searrow 0$. Comparing the minimizer η_θ with the admissible competitor η_{k-1}, we obtain that $E_\theta(\eta_{k-1};\eta_{k-1}) \ge E_\theta(\eta_\theta;\eta_{k-1})$ and thus

$$E(\eta_{k-1}) + \frac{\theta\rho}{2h}\|\zeta_k\|_{L^2}^2 \ge E_\theta(\eta_\theta;\eta_{k-1}).$$

This in turn immediately implies that

$$\limsup_{\theta\searrow 0} E_\theta(\eta_\theta;\eta_{k-1}) \le E(\eta_{k-1}). \tag{6.20}$$

Expanding the right hand side further, we also have

$$E(\eta_{k-1}) + \frac{\theta\rho}{2h}\|\zeta_k\|_{L^2}^2 \ge E(\eta_\theta) + \theta R\left(\eta_{k-1}, \tfrac{\eta_\theta-\eta_{k-1}}{\theta}\right) + \frac{\theta\rho}{2h}\left\|\tfrac{\eta_\theta-\eta_{k-1}}{\theta} - \zeta_k\right\|_{L^2}^2 - \theta\left\langle f_k, \tfrac{\eta_\theta-\eta_{k-1}}{\theta}\right\rangle_{L^2}.$$

Multiplying the previous inequality by θ, separating the dissipation-like terms of lower order, using Korn's inequality (see (R.3)) and the 2-homogenity of R, we get that

$$K_R\|\eta_\theta - \eta_{k-1}\|_{W^{1,2}}^2 \le R(\eta_k, \eta_\theta - \eta_{k-1}) + \frac{\rho}{2h}\|\eta_\theta - \eta_{k-1}\|_{L^2}^2$$
$$\le \theta\left(E(\eta_{k-1}) - E(\eta_\theta)\right) + \frac{\theta\rho}{h}\langle \eta_\theta - \eta_{k-1}, \zeta_k\rangle_{L^2} + \theta\langle f_k, \eta_\theta - \eta_{k-1}\rangle_{L^2} - \frac{\theta^2\rho}{h}\|\zeta_k\|_{L^2}^2.$$

Now, the scalar products can be estimated using Young's inequality and, for $\theta < \tau$ small enough, their parts involving η_θ can be absorbed on the left hand side. Furthermore, recall that E is bounded from below, that $E(\eta_k)$ is uniformly bounded (see (6.7)), and notice that we can drop the last term because of its sign. This then yields the uniform estimate

$$c\,\|\eta_\theta - \eta_{k-1}\|^2_{W^{1,2}(Q)} \le \theta\left(E(\eta_{k-1}) - E_{\min} + \frac{\rho}{2h}\|f_k\|^2_{L^2} + \frac{\rho}{2h}\|\zeta_k\|^2_{L^2}\right).$$

In particular, from this we infer that $\eta_\theta \to \eta_{k-1}$ for $\theta \searrow 0$.

Next, we compute the derivative $\frac{d}{d\theta}E_\theta(\eta_\theta;\eta_{k-1})$. For this, let $\theta_1, \theta_2 \in (0,\tau]$ be given. Then a direct calculation yields that

$$\begin{aligned}E_{\theta_1}(\eta_{\theta_1};\eta_{k-1}) - E_{\theta_2}(\eta_{\theta_2};\eta_{k-1}) &\le E_{\theta_1}(\eta_{\theta_2};\eta_{k-1}) - E_{\theta_2}(\eta_{\theta_2};\eta_{k-1})\\ &= (\theta_1-\theta_2)\left[-\frac{1}{\theta_1\theta_2}\left(R(\eta_{k-1},\eta_{\theta_2}-\eta_{k-1}) + \frac{\rho}{2h}\|\eta_{\theta_2}-\eta_{k-1}\|^2_{L^2}\right) + \frac{\rho}{2h}\|\zeta_k\|^2_{L^2}\right].\end{aligned}$$

Similarly, we also get

$$\begin{aligned}E_{\theta_1}(\eta_{\theta_1};\eta_{k-1}) - E_{\theta_2}(\eta_{\theta_2};\eta_{k-1}) &\ge E_{\theta_1}(\eta_{\theta_1};\eta_{k-1}) - E_{\theta_2}(\eta_{\theta_1};\eta_{k-1})\\ &= (\theta_1-\theta_2)\left[-\frac{1}{\theta_1\theta_2}\left(R(\eta_{k-1},\eta_{\theta_1}-\eta_{k-1}) + \frac{\rho}{2h}\|\eta_{\theta_1}-\eta_{k-1}\|^2_{L^2}\right) + \frac{\rho}{2h}\|\zeta_k\|^2_{L^2}\right].\end{aligned}$$

Dividing by $\theta_1 - \theta_2$ then gives us an upper and a lower bound on the Lipschitz constant of $\theta \mapsto E_\theta(\eta_\theta;\eta_{k-1})$, which is uniform whenever $\theta_1,\theta_2 > \theta_0$ for a fixed $\theta_0 > 0$, thus proving that this map is locally Lipschitz. Furthermore, sending $\theta_1 \to \theta_2$ we get that

$$\frac{d}{d\theta}E_\theta(\eta_\theta;\eta_{k-1}) = -\frac{1}{\theta^2}R(\eta_{k-1},\eta_\theta-\eta_{k-1}) - \frac{1}{\theta^2}\frac{\rho}{2h}\|\eta_\theta-\eta_{k-1}\|^2_{L^2} + \frac{\rho}{2h}\|\zeta_k\|^2_{L^2} \tag{6.21}$$

holds for almost all $\theta \in (0,\tau]$.

Next, combining (6.19), (6.20), and (6.21) yields

$$\begin{aligned}E(\eta_k) + \tau R\left(\eta_{k-1},\tfrac{\eta_k-\eta_{k-1}}{\tau}\right) + \frac{\tau\rho}{2h}\left\|\tfrac{\eta_k-\eta_{k-1}}{\tau} - \zeta_k\right\|^2_{L^2} - \tau\left\langle f_k,\tfrac{\eta_k-\eta_{k-1}}{\tau}\right\rangle_{L^2} - E(\eta_{k-1})\\ \le -\int_0^\tau R\left(\eta_{k-1},\tfrac{\eta_\theta-\eta_{k-1}}{\theta}\right) + \frac{\rho}{2h}\left\|\tfrac{\eta_\theta-\eta_{k-1}}{\theta}\right\|^2_{L^2} d\theta + \tau\frac{\rho}{2h}\|\zeta_k\|^2_{L^2}.\end{aligned}$$

After reordering, we end up with

$$\begin{aligned}E(\eta_k) + \int_0^\tau R\left(\eta_{k-1},\tfrac{\eta_k-\eta_{k-1}}{\tau}\right) + R\left(\eta_{k-1},\tfrac{\eta_\theta-\eta_{k-1}}{\theta}\right)d\theta\\ + \frac{\rho}{2h}\int_0^\tau \left\|\tfrac{\eta_k-\eta_{k-1}}{\tau}-\zeta_k\right\|^2_{L^2} + \left\|\tfrac{\eta_\theta-\eta_{k-1}}{\theta}\right\|^2_{L^2} d\theta \le E(\eta_{k-1}) + \tau\frac{\rho}{2h}\|\zeta_k\|^2_{L^2} + \tau\left\langle f_k,\tfrac{\eta_k-\eta_{k-1}}{\tau}\right\rangle_{L^2}.\end{aligned}$$

This telescopes into an estimate over all of $[0,s]$ (where without loss of generality we can assume s to be a multiple of τ), which, after dropping the term including ζ_k on the left hand side, can be rewritten as

$$\begin{aligned}E(\overline{\eta}_\tau(s)) + \int_0^s R(\underline{\eta}_\tau(t),\partial_t\eta_\tau(t)) + R(\underline{\eta}_\tau(t),\beta_\tau(t))\,dt + \frac{\rho}{2h}\int_0^s\|\beta_\tau(t)\|^2_{L^2}\,dt\\ \le E(\eta_0) + \frac{\rho}{2h}\int_0^s\|\zeta(t)\|^2_{L^2}\,dt + \int_0^s\langle f(t),\partial_t\eta_\tau(t)\rangle_{L^2}\,dt\end{aligned}\tag{6.22}$$

where $\overline{\eta}_\tau, \underline{\eta}_\tau$, and η_τ are defined as in (6.8) and β_τ is the so called De Giorgi-interpolation, defined by

$$\beta_\tau(t) := \frac{\eta_\theta - \eta_{k-1}}{\theta} \qquad \text{where } \theta \in (0,\tau], t = \tau k + \theta \text{ and } \eta_\theta \text{ is defined as above.}$$

Now we send $\tau \to 0$ again. Then, as shown in the proof of Theorem 6.2, eventually extracting a subsequence we have that the linear and affine approximations, as well as the time-derivative of the affine approximation converge to η and $\partial_t\eta$, respectively.

Next, we notice that (6.22) yields a uniform bound on β_τ in $L^2((0,h);W^{1,2}(Q;\mathbb{R}^n))$. Therefore, eventually extracting a further subsequence (which we do not relabel), we can assume that β_τ converges weakly to a limit β. The next step will be identifying the limit β with $\partial_t\eta$.

For this we note that η_θ fulfills an Euler-Lagrange inequality similar to that for η_{k+1}. Using the previous definition this inequality reads as

$$DE(\overline{\eta}_\tau)\langle\varphi\rangle + D_2R(\underline{\eta}_\tau,\beta_\tau)\langle\varphi\rangle + \frac{\rho}{h}\langle\beta_\tau-\zeta_\tau,\varphi\rangle_{L^2} \ge \rho\langle f,\varphi\rangle_{L^2}$$

for almost all $t \in [0,h]$ and admissible test functions φ. We now integrate the previous inequality and send $\tau \to 0$. The only non-trivial convergence here is that of the first term, which we have already dealt with before. Then finally we end up with

$$\int_0^s DE(\eta)\langle\varphi\rangle + D_2R\,(\eta,\beta)\,\langle\varphi\rangle + \frac{\rho}{h}\,\langle\beta-\zeta,\varphi\rangle_{L^2}\,dt \geq \int_0^s \rho\,\langle f,\varphi\rangle_{L^2}\,dt$$

for all admissible test functions. In particular this includes all compactly supported test functions. Thus, for these test functions, subtracting from this the equation for the time-delayed solution (6.2) we get

$$\int_0^s D_2R\,(\eta,\beta)\,\langle\varphi\rangle - D_2R\,(\eta,\partial_t\eta)\,\langle\varphi\rangle + \frac{\rho}{h}\,\langle\beta-\partial_t\eta,\varphi\rangle_{L^2}\,dt = 0$$

which implies that $\beta = \partial_t\eta$, since $D_2R(\eta,\cdot)$ is the derivative of a convex function and thus a monotone operator.

Finally, with this at hand and in view of the lower-semicontinuity of the terms on the left hand side of (6.22), we obtain the desired energy inequality. □

Remark 6.4 (Energy estimates in the presence of corners)**.** *The previous proof significantly differs from its "regular boundary" counterpart in [ČGK24]. In that context, we derived a nearly identical estimate by simply testing with $-\partial_t\eta$. This process relied on satisfying two distinct admissibility criteria.*

The first requirement was having sufficient regularity of $\partial_t\eta$, given that $DE(\eta)$ is a distribution of order -2 and that the bounds obtained from the dissipation $R(\eta,\partial_t\eta)$ only give control on the first derivative of $\partial_t\eta$. We achieved this by introducing an extra regularization term in the energy. The same strategy could be applied in this scenario.

The other, more crucial requirement was that of $\partial_t\eta$ pointing in an admissible direction. This way, we were able to test the variational inequality, yielding the correct sign for the energy inequality. An equivalent approach would have been to test the equation involving the contact force with opportunely defined test functions and showing in this way that the additional forcing term has the right sign. Indeed, since the force always points in the interior normal direction and movement can never cause an overlap, this was precisely the case in [ČGK24].

However, for a less smooth boundary this approach no longer works that seamlessly, as evidenced by the example given in Remark 5.6. There, we arrived at a contact force pointing in a direction not blocked by any other solid, i.e. there can be evolutions of the solid where the two corners simply glide past each other in either direction. Consequently, testing the contact measure with $\partial_t\eta$ can potentially produce terms of either sign. This again is the same reason why we can only recover an inequality for $T^0_\eta(\mathcal{E})$ and not for the larger set $T_\eta(\mathcal{E})$, as the above example would contradict the latter. Of course, it is natural to conjecture that having such a passing solution would be incompatible with generating such a force in the approximation. However, formalizing this argument is non-trivial and as we have shown, this complication can be circumvented by a more abstract approach.

6.2. **Existence of the contact force.** Our goal now is to refine the inequality in (6.3) by recasting it as an equation involving a contact force. To achieve this, we introduce the following definition.

Definition 6.5 (Solution with a contact force)**.** *Let $\eta_0 \in \mathcal{E}$ and $f \in L^2((0,h);L^2(Q;\mathbb{R}^n))$ be given. We say that the pair*

$$\eta \in W^{1,2}((0,h);W^{1,2}(Q;\mathbb{R}^n)) \cap L^\infty((0,h);\mathcal{E}), \qquad \sigma \in L^2_{w^*}((0,h);M(\partial Q;\mathbb{R}^n)),$$

where σ is a contact force for η (see Definition 4.4), is a solution with a contact force *to (6.1) if $\eta(0)=\eta_0$ and*

$$\int_0^h \left[DE(\eta(t)) + D_2R(\eta(t),\partial_t\eta(t))\right]\langle\varphi(t)\rangle\,dt = \int_0^h \langle\sigma(t),\varphi(t)\rangle\,dt + \int_0^h \langle f(t),\varphi(t)\rangle_{L^2}\,dt \tag{6.23}$$

for all $\varphi \in C([0,h];W^{2,p}_\Gamma(Q;\mathbb{R}^n))$.

The existence of solutions with a contact force for problem (6.1) is established in the following theorem.

Theorem 6.6. *Under the assumptions of Theorem 6.2, there exists a solution with a contact force to (6.1). Moreover, the solution satisfies the energy inequality*

$$E(\eta(t)) + 2\int_0^t R(\eta(s),\partial_t\eta(s))\,ds \leq E(\eta(0)) + \int_0^t \langle f(s),\partial_t\eta(s)\rangle\,ds$$

for $t \in [0,h]$ and the estimate

$$\int_0^h \|\sigma(t)\|^2_{M(\partial Q;\mathbb{R}^n)} dt \le C_0 \left(h + \|\partial_t \eta\|^2_{L^2((0,h);W^{1,2}(Q))} + \|f\|^2_{L^2((0,h)\times Q)} \right), \tag{6.24}$$

where C_0 is a constant that depends only on $E(\eta_0)$.

Proof. Here we continue to use the notation introduced in the proof of Theorem 6.2. In particular, we recall that η_k (see (6.5)) is a minimizer for $\mathcal{J}_k$ (see (6.6)). We divide the proof into a several steps.
Step 1 (Lagrange multiplier is a measure): In this first step we use the argument described in [Pal18, Theorem 3.1] to show that there is a contact force for η_k, namely $\sigma_k \in M(\partial Q;\mathbb{R}^n)$, such that

$$D\mathcal{J}_k(\eta_k)\langle\varphi\rangle - \langle\sigma_k, \varphi\rangle = 0 \tag{6.25}$$

for all $\varphi \in W^{2,p}_\Gamma(Q;\mathbb{R}^n)$. We begin by letting

$$M^-_{\eta_k} := \{(\ell, w) \in \mathbb{R} \times C(\partial Q;\mathbb{R}^n) : \ell \le 0 \text{ and } w(x) \in \operatorname{int} \widehat{T}_{\eta_k}(x)\ \forall x \in C_{\eta_k}\},$$
$$M^+_{\eta_k} := \{(\ell, w) \in \mathbb{R} \times C(\partial Q;\mathbb{R}^n) : \exists \varphi \in \mathcal{A}^w_{\eta_k} \text{ s.t. } D\mathcal{J}_k(\eta_k)\langle\varphi\rangle \le \ell\},$$

where for $w \in C(\partial Q;\mathbb{R}^n)$

$$\mathcal{A}^w_{\eta_k} := \{\varphi \in W^{2,p}_\Gamma(Q;\mathbb{R}^n) : \varphi(x) - \varphi(y) - w(x) + w(y) \in \operatorname{int} \widehat{T}_{\eta_k}(x) - \operatorname{int} \widehat{T}_{\eta_k}(y)$$
$$\text{if } \eta_k(x) = \eta_k(y) \text{ with } x \ne y, \text{ and } \varphi(x) - w(x) \in \operatorname{int} \widehat{T}_{\eta_k}(x) \text{ if } \eta_k(x) \in \partial\Omega\}.$$

As one can readily check, the sets $M^+_{\eta_k}$ and $M^-_{\eta_k}$ are convex as a consequence of the convexity of the regular tangent cones. Moreover, $M^-_{\eta_k} \ne \emptyset$ by Proposition 5.4 (i) and has nonempty interior in view of Proposition 3.4.

Next, we claim that $M^+_{\eta_k} \cap \operatorname{int} M^-_{\eta_k} = \emptyset$. Indeed, notice that if $(\ell, w) \in \operatorname{int} M^-_{\eta_k}$ then $\mathcal{A}^w_{\eta_k} \subset T^0_\eta(\mathcal{E})$, and therefore (see (6.10)) we have that

$$D\mathcal{J}_k(\eta_k)\langle\varphi\rangle \ge 0 > \ell.$$

This shows that $(\ell, w) \notin M^+_{\eta_k}$, thus proving the claim. Therefore, by the Hahn-Banach theorem there exists a separating hyperplane, that is, a pair $(0,0) \ne (\lambda_0, \sigma_0) \in \mathbb{R} \times M(\partial Q;\mathbb{R}^n)$ such that

$$\lambda_0 \ell + \langle\sigma_0, w\rangle \ge 0, \qquad \text{for all } (\ell, w) \in M^-_{\eta_k},$$
$$\lambda_0 \ell + \langle\sigma_0, w\rangle \le 0, \qquad \text{for all } (\ell, w) \in M^+_{\eta_k}.$$

Notice that if $w \in C(\partial Q;\mathbb{R}^n)$ is such that $C_{\eta_k} \cap \operatorname{supp} w = \emptyset$, then $(0, w) \in M^+_{\eta_k} \cap M^-_{\eta_k}$ and therefore $\langle\sigma_0, w\rangle = 0$. This shows $\operatorname{supp} \sigma_0 \subset C_{\eta_k}$. Thus, by Lemma 4.5 we have that σ_0 is a contact force for η_k.

Let ξ_Γ be given as in Proposition 5.5 and notice that $(2D\mathcal{J}_k(\eta_k)\langle\tilde{t}_{\eta_k}\xi_\Gamma\rangle, \tilde{t}_{\eta_k}\xi_\Gamma) \in M^+_{\eta_k}$ by the definition of $M^+_{\eta_k}$. Therefore

$$\lambda_0 2D\mathcal{J}_k(\eta_k)\langle\tilde{t}_{\eta_k}\xi_\Gamma\rangle + \langle\sigma_0, \tilde{t}_{\eta_k}\xi_\Gamma\rangle \le 0. \tag{6.26}$$

From this we see that $\lambda_0 < 0$. Indeed, since $\lambda_0 = 0$ implies $\sigma_0 \ne 0$ (by the Hahn-Banach theorem) and since $\langle\sigma_0, \tilde{t}_{\eta_k}\xi_\Gamma\rangle > 0$ in view of the fact that $\tilde{t}_{\eta_k}$ is a uniformly interior field, then necessarily we must have that $\lambda_0 \ne 0$. On the other hand, if $\lambda_0 > 0$ then by taking $\ell > \max(0, 2D\mathcal{J}_k(\eta_k)\langle\tilde{t}_{\eta_k}\xi_\Gamma\rangle)$ we get that $(\ell, \tilde{t}_{\eta_k}\xi_\Gamma) \in M^+_{\eta_k}$. However, in this case we readily reach a contradiction by noticing that $\lambda_0 \ell + \langle\sigma_0, \tilde{t}_{\eta_k}\xi_\Gamma\rangle > 0$.

Finally, set $\sigma_k := -\sigma_0/\lambda_0$. Then σ_k is a contact force for η_k and a simple computation shows that equation (6.25) is satisfied.
Step 2 (bound on the measure): For σ_k as above, let σ_τ be defined via

$$\sigma_\tau(t) := \sigma_k, \qquad t \in [\tau(k-1), \tau k). \tag{6.27}$$

By the assumptions on $\eta_0|_\Gamma$ and the local injectivity of η, we can pick a compact subset $K \subset \partial Q$ such that $\eta(t)|_{\partial Q \setminus K}$ is always injective. Then by the action-reaction principle, estimating $\|\sigma_k\|_{M(K;\mathbb{R}^n)}$ is enough to estimate $\|\sigma_k\|_{M(\partial Q;\mathbb{R}^n)}$. Wlog. we can choose ξ_Γ from before such that $\xi_\Gamma(x) = 1$ for all $x \in K$. Let us use $\xi_\Gamma \tilde{t}_{\eta_k}$ with $\tilde{t}_{\eta_k} \in C^2(\overline{Q};\mathbb{R}^n)$ from Proposition 5.4 (i) as a test function. This gives us (as $d\sigma_k = g_k\, d|\sigma_k|$, see Definition 4.4), denoting $\alpha := \sin\vartheta$ from Definition 5.3,

$$-D\mathcal{J}_k(\eta_k)\langle\xi_\Gamma \tilde{t}_{\eta_k}\rangle = \langle\sigma_k, \xi_\Gamma \tilde{t}_{\eta_k}\rangle = \int_{\partial Q} \xi_\Gamma \tilde{t}_{\eta_k} \cdot g_k d|\sigma_k| \ge \alpha \int_K |\tilde{t}_{\eta_k}||g_k| d|\sigma_k| = \alpha\|\sigma_k\|_{M(K;\mathbb{R}^n)}. \tag{6.28}$$

Thus

$$\|\sigma_k\|_{M(\partial Q;\mathbb{R}^n)} \le -\frac{2}{\alpha}\left(DE(\eta_k)\langle\tilde{t}_{\eta_k}\rangle + DR\left(\eta_{k-1}, \frac{\eta_k - \eta_{k-1}}{\tau}\right)\langle\tilde{t}_{\eta_k}\rangle - \langle f_k, \tilde{t}_{\eta_k}\rangle \right) \tag{6.29}$$

Multiplying this by $\psi \in C([0,h];\mathbb{R}^+)$ and integrating over $t \in (0,h)$

$$\int_0^h \|\sigma_\tau(t)\|_{M(\partial Q;\mathbb{R}^n)}\psi(t)dt \le -\frac{2}{\alpha}\int_0^h \left(DE(\overline{\eta}_\tau(t))\langle \tilde{t}_{\overline{\eta}_\tau}(t)\rangle + DR\left(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t)\right)\langle \tilde{t}_{\overline{\eta}_\tau}(t)\rangle - \langle \overline{f}_\tau(t), \tilde{t}_{\overline{\eta}_\tau}(t)\rangle\right)\psi(t)\,dt.$$

Let us estimate all terms on the right hand side. First,

$$\left|\int_0^h DE(\overline{\eta}_\tau(t))\langle \tilde{t}_{\overline{\eta}_\tau}(t)\rangle\psi(t)\,dt\right| \le \|DE(\overline{\eta}_\tau)\|_{L^\infty((W^{2,p})^*)}\|\tilde{t}_{\overline{\eta}_\tau}\|_{L^\infty(W^{2,p})}\sqrt{h}\|\psi\|_{L^2((0,h))} \tag{6.30}$$

and by the energy inequality Theorem 6.3 and by (E.5), we have $\|DE(\overline{\eta}_\tau)\|_{L^\infty((W^{2,p})^*)}$ bounded by a constant C_{E_0} depending only on E_0. Next, using (2.3) we get

$$\left|\int_0^h DR\left(\underline{\eta}_\tau(t), \partial_t\eta_\tau(t)\right)\langle \tilde{t}_{\overline{\eta}_\tau}(t)\rangle\psi(t)\,dt\right| \le \int_0^h C_{E_0}\|\partial_t\eta_\tau(t)\|_{W^{1,2}}\|\tilde{t}_{\overline{\eta}_\tau}(t)\|_{W^{1,2}}\psi(t)\,dt \tag{6.31}$$

$$\le C_{E_0}\|\partial_t\eta_\tau\|_{L^2(W^{1,2})}\|\tilde{t}_{\overline{\eta}_\tau}\|_{L^\infty(W^{1,2})}\|\psi\|_{L^2((0,h))} \tag{6.32}$$

and finally

$$\int_0^h -\langle \overline{f}_\tau(t), \tilde{t}_{\overline{\eta}_\tau}(t)\rangle\psi(t)\,dt \le \|f\|_{L^2(L^2)}\|\tilde{t}_{\overline{\eta}_\tau}\|_{L^\infty(L^2)}\|\psi\|_{L^2((0,h))}. \tag{6.33}$$

So altogether, given that by Proposition 5.4 (i) $\tilde{t}_{\overline{\eta}_\tau}$ is bounded in all the norms above by a constant, that

$$\int_0^h \|\sigma_\tau(t)\|_{M(\partial Q;\mathbb{R}^n)}\psi(t)dt \le \frac{1}{\alpha}C_{E_0}(\sqrt{h} + \|\partial_t\eta_\tau\|_{L^2(W^{1,2})} + \|f\|_{L^2(L^2)})\|\psi\|_{L^2((0,h))} \tag{6.34}$$

Taking a supremum over ψ such that $\|\psi\|_{L^2((0,h))} \le 1$ yields

$$\|\sigma_\tau\|_{L^2((0,h);M(\partial Q;\mathbb{R}^n))} \le \frac{1}{\alpha}C_{E_0}(\sqrt{h} + \|\partial_t\eta_\tau\|_{L^2((0,h);W^{1,2}(Q))} + \|f\|_{L^2((0,h)\times Q)}). \tag{6.35}$$

Hence, we have proved that $\sigma_\tau \in L^2((0,h);M(\partial Q;\mathbb{R}^n))$ and moreover the bound is independent of τ, since by Theorem 6.3 $\|\partial_t\eta_\tau\|_{L^2((0,h);W^{1,2}(Q))}$ is bounded independently of τ.

Step 3 (passing to the limit): We have from Step 1 that

$$\int_0^h DE(\overline{\eta}_\tau)\langle\varphi\rangle + DR(\underline{\eta}_\tau, \partial_t\eta_\tau)\langle\varphi\rangle - \langle\sigma_\tau,\varphi\rangle\,dt = \int_0^h \langle f_\tau,\varphi\rangle\,dt, \quad \varphi \in W^{2,p}_\Gamma(Q;\mathbb{R}^n).$$

By the estimate (6.35), using the compactness of contact forces from Theorem 4.6 we have that for a subsequence of $\tau \to 0$ (which we do not relabel) we have

$$\sigma_\tau \overset{*}{\rightharpoonup} \sigma \text{ in } M([0,h]\times\partial Q),$$

where σ is a contact force for η. Moreover it satisfies the equation (passage to the limit in all of the other terms has already been established in the proof of Theorem 6.2)

$$\int_0^h DE(\eta)\langle\varphi\rangle + DR(\eta,\partial_t\eta)\langle\varphi\rangle - \langle\sigma,\varphi\rangle\,dt = \int_0^h \langle f,\varphi\rangle\,dt$$

by a density argument for all $\varphi \in C([0,h];W^{2,p}_\Gamma(Q;\mathbb{R}^n))$. Moreover by (6.35) and Lemma 4.7 we have also $\sigma \in L^2_{w^*}((0,h);M(\partial Q;\mathbb{R}^n))$ with the same estimate as (6.35), that is

$$\|\sigma\|_{L^2_{w^*}((0,h);M(\partial Q;\mathbb{R}^n))} \le \frac{1}{\alpha}C_{E_0}\left(\sqrt{h} + \|\partial_t\eta\|_{L^2((0,h);W^{1,2}(Q))} + \|f\|_{L^2((0,h)\times Q)}\right),$$

finishing the proof. □

7. Proof of the main result

In this section, we adapt the several steps in the proof of Theorem 2.4 in [ČGK24, Section 5] to our more general situation. Thus, we will again only sketch the arguments that are identical in the case of regular boundaries and mainly focus on the differences. To make the comparison easier, we keep the same structure.

Proof of Theorem 2.2. Step 1: Recall that by using a different approach for the energy inequality, we were able to obtain the results of the previous section without relying on additional regularization terms. Therefore, in the current framework, there is no need to regularize the initial data. We can thus skip this step.

Step 2: For fixed h, on each interval $[lh,(l+1)h]$, for $l \in \mathbb{N}$, we now iteratively apply Theorem 6.2 with

$$\tilde{R}^{(h)}(\eta,b) := R(\eta,b) + \frac{\rho}{2h}\|b\|_{L^2}^2 \qquad \text{and} \qquad \tilde{f}(t) := f(t) + \frac{\rho}{h}\partial_t\eta(t-h) \tag{7.1}$$

in place of R and f. Here we implicitly set $\partial_t\eta(t) := \eta^*$ for $t \leq 0$.

Note that on each interval, the energy inequality for the previous interval gives us a control on $\partial_t\eta$ which guarantees that $\tilde{f}$ is well defined in the next interval. It is immediate to see that if R is given as in (R.3), then $\tilde{R}^{(h)}$ satisfies (R.3_{q}); in particular, this implies that all the assumptions of Theorem 6.2 are satisfied for all l.

Next, we observe that the piecewise constant function defined on the entire interval $[0,T]$ obtained by gluing the solutions on the sub-intervals $[lh,(l+1)h]$, namely $\eta^{(h)}$, is a weak solution to

$$\rho\frac{\partial_t\eta^{(h)}(t) - \partial_t\eta^{(h)}(t-h)}{h} + DE(\eta^{(h)}(t)) + D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t)) = f(t), \tag{7.2}$$

for each $h > 0$.

Step 3: A closer look at the energy inequalities (see Theorem 6.3) for each interval $[lh,(l+1)h]$ reveals that the terms for kinetic and potential energy on each side of the equation cancel when summed up. In turn, we find that

$$\begin{aligned} E(\eta^{(h)}(s)) + \fint_{s-h}^{s} \frac{\rho}{2}\left\|\partial_t\eta^{(h)}(t)\right\|_{L^2}^2 dt + \int_0^s 2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\,dt& \\ \leq E(\eta_0) + \frac{\rho}{2}\|\eta^*\|_{L^2}^2 + \int_0^s \left\langle f(t),\partial_t\eta^{(h)}(t)\right\rangle_{L^2} dt& \end{aligned} \tag{7.3}$$

holds for all $s \in [0,T]$. From this, but also using the properties of E and R, one easily derives uniform bounds (with respect to h) for

$$\eta^{(h)} \in L^\infty((0,T);W^{2,p}(Q;\mathbb{R}^n)) \qquad \text{and} \qquad \partial_t\eta^{(h)} \in L^2((0,T);W^{1,2}(Q;\mathbb{R}^n)). \tag{7.4}$$

In particular, this implies the existence of a limit deformation η and a subsequence such that

$$\begin{aligned} &\eta^{(h)} \overset{*}{\rightharpoonup} \eta \ \text{ in } L^\infty((0,T);W^{2,p}(Q;\mathbb{R}^n)),\\ &\eta^{(h)} \rightharpoonup \eta \ \text{ in } W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^n)). \end{aligned} \tag{7.5}$$

Step 4: This step is dedicated to improving the convergences in (7.5). This is necessary in order to pass to the limit as $h \to 0$ in the term that contains $DE(\eta^{(h)})$ in the weak formulation for (7.2). To this end, we first consider the auxiliary function

$$b^{(h)}(t) := \fint_t^{t+h} \partial_t\eta^{(h)}(s)\,ds.$$

From the bounds on $\partial_t\eta^{(h)}$ we quickly derive that $b^{(h)} \in L^2((0,T);W^{1,2}(Q;\mathbb{R}^n))$. Additionally, noting that using (7.2) we have that

$$\partial_t b^{(h)}(\cdot) = \frac{\partial_t\eta^{(h)}(\cdot+h) - \partial_t\eta^{(h)}(\cdot)}{h} \in \Big(L^2((0,T);W_0^{2,p}(Q;\mathbb{R}^n))\Big)^* \tag{7.6}$$

admits a uniform bound, we can use the Aubin-Lions theorem to prove that $b^{(h)}$ converges strongly (eventually extracting a further subsequence) in $L^2((0,T)\times Q;\mathbb{R}^n)$ and we can further identify its limit with $\partial_t\eta$.

Now note that using our definitions for $\tilde{R}^{(h)}$ and $\tilde{f}$, the solution $\eta^{(h)}$ satisfies

$$\begin{aligned} &\int_0^T DE(\eta^{(h)}(t))\langle\varphi(t)\rangle + D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\langle\varphi(t)\rangle\,dt - \rho\langle\eta^*,\varphi(0)\rangle_{L^2} \\ &\quad + \int_0^T \rho\left\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h},\varphi(t)\right\rangle_{L^2} dt = \int_{[0,T]\times\partial Q}\varphi(t,x)\cdot d\sigma(t,x) + \int_0^T \langle f(t),\varphi(t)\rangle_{L^2}\,dt \end{aligned} \tag{7.7}$$

for all $\varphi \in C([0,T];W_\Gamma^{2,p}(Q;\mathbb{R}^n))$.

Now let $\tilde{t}_{\eta^{(h)}}$ be given as in Proposition 5.4, $\xi_\Gamma \in C^\infty(Q;[0,1])$ be a cutoff that satisfies $\xi_\Gamma(x) = 0$ for $x \in \Gamma$ and $\xi_\Gamma(x) > 0$ for $x \in \overline{Q}\setminus\overline{\Gamma}$, and let $\psi \in L^\infty((0,T))$. We then use $\xi_\Gamma\tilde{t}_{\eta^{(h)}}\psi$ as a test function in

(7.7). The rest of this step then proceeds precisely as in [ČGK24] and we will thus only highlight some of the details.

Note that $\tilde{t}_{\eta^{(h)}} \in L^\infty((0,T);C^2(\overline{Q};\mathbb{R}^n))$. From this we obtain

$$\int_0^T \|\sigma^{(h)}(t)\|_{M(\partial Q;\mathbb{R}^n)}\psi(t)dt \le \frac{1}{\alpha}\int_0^T \Big\{ \Big[DE(\eta^{(h)}(t)) + D_2R(\eta^{(h)}(t),\partial_t\eta^{(h)}(t))\Big] \langle \xi_\Gamma \tilde{t}_{\eta^{(h)}}(t)\rangle \\ + \rho\left\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h}, \xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\right\rangle_{L^2} - \langle f(t),\xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\rangle_{L^2}\Big\}\psi(t)\,dt. \quad (7.8)$$

Taking $\psi = 1$, we can then proceed exactly as in the corresponding step of Theorem 2.4 in [ČGK24] and estimate all the terms on the right-hand side of (7.8). We recall here that the difference quotient used to approximate the inertial term can be rewritten as

$$\int_0^T \rho\left\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h}, \xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\right\rangle_{L^2} dt = \int_0^{T-h} \rho\left\langle \partial_t\eta^{(h)}(t), \xi_\Gamma \tfrac{\tilde{t}_{\eta^{(h)}}(t)-\tilde{t}_{\eta^{(h)}}(t+h)}{h}\right\rangle_{L^2} dt \\ + \int_{T-h}^T \frac{\rho}{h}\langle \partial_t\eta^{(h)}(t),\xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\rangle_{L^2}\,dt - \int_0^h \frac{\rho}{h}\langle \partial_t\eta^{(h)}(t-h),\xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\rangle_{L^2}\,dt, \quad (7.9)$$

and that the first term on the right-hand is controlled by the bounds on $\partial_t\tilde{t}_{\eta^{(h)}}$ obtained in Proposition 5.4. Together with the other estimates, this implies that $\sigma^{(h)} \in L^1_{w^*}((0,T);M(\partial Q;\mathbb{R}^n))$. Consequently, Theorem 4.6 guarantees the existence of a subsequence (which we do not relabel) such that $\sigma^{(h)} \overset{*}{\rightharpoonup} \sigma$ in $M([0,T]\times\partial Q;\mathbb{R}^n)$, where $\sigma \in M([0,T]\times\partial Q;\mathbb{R}^n)$ is a contact force for η and satisfies the action-reaction principle.

Next we consider the case where ψ is the characteristic function of a small interval $[t_0,t_0+\delta]$. Observe that $\tilde{t}_{\eta^{(h)}}$ can be constructed in such a way that it is piecewise constant in time and that with this choice, reasoning as in (7.9), we have that

$$\int_{t_0}^{t_0+\delta} \rho\left\langle \tfrac{\partial_t\eta^{(h)}(t)-\partial_t\eta^{(h)}(t-h)}{h}, \xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\right\rangle_{L^2} dt \\ = \fint_{t_0+\delta-h}^{t_0+\delta} \rho\left\langle \partial_t\eta^{(h)}(t),\xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\right\rangle_{L^2} dt - \fint_{t_0-h}^{t_0} \rho\left\langle \partial_t\eta^{(h)}(t),\xi_\Gamma\tilde{t}_{\eta^{(h)}}(t)\right\rangle_{L^2} dt$$

which can easily be bounded uniformly by the energy-inequality and in fact arbitrarily small, by limiting the support of $\tilde{t}_{\eta^{(h)}}$ to a smaller neighborhood of ∂Q. Together with the fact that the estimates of the other terms depend on $\|\psi\|_{L^2}$ and thus vanish when $\delta \to 0$, this implies the absence of any concentrations in time. Again, we refer the reader to [ČGK24] for more details, as the proof of these estimates is unchanged.

Step 5: With this in hand, using (E.6), we can conclude that

$$0 \le \liminf_{h\to 0}\int_0^T [DE(\eta^{(h)}) - DE(\eta)]\langle(\eta^{(h)}-\eta)\psi\rangle\,dt$$

for all $\psi \in C^\infty_\Gamma(Q;[0,1])$. Again, as in Theorem 6.2, we do not need to deal with the regularization, making this argument more straightforward when compared to our proof in [ČGK24]. The only additional term we get after integrating by parts is

$$-\liminf_{h\to 0}\int_0^T \left\langle \tfrac{\partial_t\eta^{(h)}(\cdot+h)-\partial_t\eta^{(h)}(\cdot)}{h}, \psi(\eta^{(h)}-\eta)\right\rangle_{L^2} dt = \liminf_{h\to 0}\int_0^T \left\langle b^{(h)},\psi\partial_t(\eta^{(h)}-\eta)\right\rangle_{L^2} dt,$$

which converges to zero. Moreover, we have that $\partial_t\eta^{(h)} \rightharpoonup \partial_t\eta$ and $b^{(h)} \to \partial_t\eta$, respectively, in $L^2((0,T)\times Q;\mathbb{R}^n)$. Combining these facts with (E.6) we then conclude that $\eta^{(h)}(t) \to \eta(t)$ in $W^{2,p}(K;\mathbb{R}^n)$ for all K compactly contained in $\overline{Q}$ with $\operatorname{dist}(K,\Gamma) > 0$ and for almost all $t \in [0,T]$.

Step 6: The previously shown convergences now allow us to pass to the limit with all terms in the equation, including the one involving the contact force. This readily implies the existence of a solution to the full problem with a contact force and therefore completes the proof. □

Declarations. A.Č. and M.K. acknowledge the support of the the ERC-CZ grant LL2105, and the Czech Science Foundation (GAČR) under grant No. 23-04766S. A.Č. further acknowledges the support of Charles University, project GA UK No. 393421, and Charles University Research Centre program No. UNCE/24/SCI/005. The authors have no other conflicts of interest regarding this publication.

References

[AGS05] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2005.

[BKS23] Barbora Benešová, Malte Kampschulte, and Sebastian Schwarzacher. A variational approach to hyperbolic evolutions and fluid-structure interactions. *J. Eur. Math. Soc. (online first)*, 2023.

[ČGK24] Antonín Češík, Giovanni Gravina, and Malte Kampschulte. Inertial evolution of non-linear viscoelastic solids in the face of (self-)collision. *Calc. Var. & PDE*, 63(2):55, 2024.

[Cla87] Frank H. Clarke. *Optimization and Nonsmooth Analysis (Classics in Applied Mathematics)*. Society for Industrial Mathematics, 1987.

[CN87] Philippe G. Ciarlet and Jindřich Nečas. Injectivity and self-contact in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 97(3):171–188, September 1987.

[HK09] Timothy J. Healey and Stefan Krömer. Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(4):863–871, July 2009.

[KR20] Stefan Krömer and Tomáš Roubíček. Quasistatic viscoelasticity with self-contact at large strains. *J. Elasticity*, 142(2):433–445, 2020.

[Pal18] Aaron Zeff Palmer. Variations of deformations with self-contact on lipschitz domains. *Set-Valued and Variational Analysis*, 27(3):807–818, June 2018.

[RW98] R. Tyrrell Rockafellar and Roger J. B. Wets. *Variational Analysis*. Springer Berlin Heidelberg, 1998.

[Sch02] Friedemann Schuricht. Variational approach to contact problems in nonlinear elasticity. *Calculus of Variations and Partial Differential Equations*, 15(4):433–449, December 2002.

Paper III

STABILITY AND CONVERGENCE OF IN TIME APPROXIMATIONS OF HYPERBOLIC ELASTODYNAMICS VIA STEPWISE MINIMIZATION

ANTONÍN ČEŠÍK AND SEBASTIAN SCHWARZACHER

Abstract. We study step-wise time approximations of non-linear hyperbolic initial value problems. The technique used here is a generalization of the minimizing movements method, using two time-scales: one for velocity, the other (potentially much larger) for acceleration. The main applications are from elastodynamics namely so-called generalized solids, undergoing large deformations. The evolution follows an underlying variational structure exploited by step-wise minimisation. We show for a large family of (elastic) energies that the introduced scheme is stable; allowing for non-linearities of highest order. If the highest order can assumed to be linear, we show that the limit solutions are regular and that the minimizing movements scheme converges with optimal linear rate. Thus this work extends numerical time-step minimization methods to the realm of hyperbolic problems.

1. Introduction

We study step-wise time approximations of hyperbolic non-linear initial value problems. For this we consider $Q \subset \mathbb{R}^n$ a bounded Lipschitz domain and a time interval $[0, T]$. The partial differential equations considered here are of the form

$$\begin{aligned} &\partial_{tt}\eta(t,x) + DE(\eta(t,x)) = f(t,x) \text{ for } (t,x) \in [0,T] \times Q \\ &\eta(0,x) = \eta_0(x), \quad \partial_t\eta(0,x) = \eta_*(x) \text{ for } x \in Q \end{aligned} \tag{1.1}$$

where E is some energy functional, DE its (Fréchet) derivative, η_0, η_* given initial conditions and f a given right hand side. We supplement the problem with prescribed boundary values. Our motivational example are the dynamics of *largely deforming* elastic solids. Therefore one critical challenge of this paper is to allow for elastic energies that include negative powers of the Jacobian (see (1.2)). This means the energies can be non-convex and be defined over a *non-convex state space* (as for example the space of vector fields with positive Jacobians a.e.).

In the case of *a convex state space*, implicit Euler or variational schemes have been studied exhaustively, see for instance [FD97, CD04, DSET01, Pro08, HP10]. Further in situations where additionally the energy is assumed to be convex, more approaches are applicable, see for instance the classical works [Kač86, Pul84]. Note that none of the above literature is applicable for largely deforming solid evolutions. In particular, none treats non-convex state spaces. Actually, it seems that the case of a hyperbolic PDE describing largely deforming solid evolutions has not been treated before. There are so far only numeric results for the quasi-static and visco-elastic case [BGN10, RT21].

We follow the scheme developed in [BKS23b] where via step-wise minimization a second order in time evolution was approximated. The heart of the method was to use two different time scales – the *velocity scale* τ and the (potentially much larger) *acceleration scale* h. Accordingly, $\partial_{tt}\eta$ is approximated as

$$\partial_{tt}\eta(t) \approx \frac{\frac{\eta(t)-\eta(t-\tau)}{\tau} - \frac{\eta(t-h)-\eta(t-h-\tau)}{\tau}}{h}.$$

In [BKS23b] the fact was exploited that for a fixed acceleration scale a gradient flow structure can be used naturally on the length scale of h. This opened the door to use variational strategies for hyperbolic evolutions.

In this paper we wish to investigate the potential of the method for numerical discrete-in-time approximations.

We consider this question worthy since step-wise minimisation is a rather well established approximation strategy for gradient flows ever since the seminal works of DeGiorgi [DG93]. It

Key words and phrases. Minimizing movements, Hyperbolic evolutions, Time-discretisation, Elastodynamics, Solids with large deformations, MSC2020 classes: 65M12, 74H15, 74H80, 74B20, 74H55, 74H30.

has been used widely both for analysis and numerics. See [Kru98, May00, BGN10, BK11, GO13, LO16, RSS17, MR20, RT21] and the references therein for some examples of applications, but there are many more.

In this work we analyse the potential of step-minimisation for numerics of hyperbolic problems, by providing respective stability and convergence results.

The main motivating application are hyperbolic evolutions for *elastic solids that may deform largely* [BKS23b, Chapter 3]. A typical example for the elastic energy (see [Bal76, Dac07]) is

$$E_1(\eta) := \begin{cases} \int_Q \frac{1}{8}\big(\mathcal{C}(\nabla\eta^T\nabla\eta - I)\big)\cdot\big(\nabla\eta^T\nabla\eta - I\big) + \frac{1}{(\det\nabla\eta)^a}dx, & \text{for } \eta, \det\nabla\eta > 0 \text{ a.e. in } Q \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Here $\mathcal{C}$ is a positive definite tensor of elastic constants and a a given exponent. In E_1 the first term corresponds to the Saint Venant-Kirchhoff energy, the second models the resistance of the solids to infinite compression.

It is the non-convexity of the state space (not of the functional) that precludes using other approaches such as fixed point methods; but a minimizer can still be found [Bal76, Dac07]. Observe that this non-convexity stems from the physical requirement that compressing a bulk solid into zero volume requires infinite energy.[1] Hence, this example is a strong motivation for the *utility and potential necessity of variational schemes.*

Although largely deforming elastic solids are our main motivation, the scheme has the potential to be used in other possibly more complicated scenarios. These include fluid-structure interaction scenarios [BKS23b], [BKS23a], [KSS23], (elasto-)plastic motions [MR06, MR16, MRS17, RSS17], also handling damage [MR06] or temperature [MR20, BFK23]. For that reason we additionally provide abstract assumptions for stability and convergence.

The setting also includes the case of ordinary differential equations in $\mathbb{R}^n$. In this setting the results are illustrated with some numerical experiments highlighting the effects of the two time-scales on the approximation and the sharpness of our respective estimates; see Section 4.

The current state of the art in the analysis for *evolutionary large of deformation solids* conventionally assumes the elastic energy to also depend on higher order derivatives. Such solids are known as *hyperelastic solids* [Ngu00, HK09, Dog00, KR19] and conventionally add to E_1 a second functional $E_2\colon W^{2,q}(Q) \to [0,\infty]$ or $E_2\colon W^{k_0,p}(Q) \to [0,\infty]$ which is typically of the form

$$E_2(\eta) = \int_Q (1 + |\nabla^2\eta|)^{q-2}\left|\nabla^2\eta\right|^2 dx \text{ or} \tag{1.3}$$

$$E_2(\eta) = \int_Q \left|\nabla^{k_0}\eta\right|^2 dx. \tag{1.4}$$

For the latter one DE_2 is a linear operator.

In this paper we provide stability and convergence results for the variationally constructed two-scale time-discrete solutions.

All results demonstrated here are valid for arbitrary choices of τ and h, provided they are sufficiently small in relation to the non-convexity. This might seem surprising with regard to the fact that in the analytic results [BKS23b] using the hyperbolic variational scheme the independent limit passage of $\tau \to 0$ and subsequently $h \to 0$ is essential. This is one of the reasons we believe the estimates demonstrated here to be relevant for pure analytic applications independently from its value for numerics, see also Section 1.

Note also that all the presented results presume the absence of (self-)collisions. Collisions in (visco-)elastodynamics have been analytically treated in [ČGK24], where it is demonstrated that the collision produces a Lagrange multiplier as a surface measure. We consider treating collisions in the case of fully time-discrete scheme to be of interest, but leave it to a future work.

We include here the main results for the model case presented above and $\tau = h$. The general results for arbitrary τ and h and for general assumptions on $DE(\eta)$ can be found in Theorem 2.27, Theorem 3.8 and Theorem 3.5.

[1] To see this, consider the following two deformations: identity and rotation by 180 degrees. They both have finite energy, but their convex combination is a singular matrix, and thus of infinite energy. Therefore the state space is necessarily non-convex.

Our main stability theorem deals with the approximation (2.17), which includes a dissipation term (vanishing in the limit $\tau \to 0$). The dissipation term stabilizes the scheme so that there is no increase in energy with time. One may see this term as an artificial viscosity of order τ.

Theorem 1.1 (Stability for the scheme with artificial viscosity). *Let $E(\eta) = E_1(\eta) + E_2(\eta)$, where E_1 is given by* (1.2) *and E_2 by* (1.3) *or* (1.4) *satisfying $q > n$ or $k_0 > n/2 + 1$. Let η_k be the variational approximation obtained by step-wise minimization*

$$\eta_k = \arg\min_{\eta \in \mathcal{E}} \frac{\tau^2}{2} \left\| \frac{\frac{\eta - \eta_{k-1}}{\tau} - \frac{\eta_{k-1} - \eta_{k-2}}{\tau}}{\tau} \right\|_{L^2}^2 + E(\eta) + \frac{c\tau^2}{2} \left\| \nabla \frac{\eta_k - \eta_{k-1}}{\tau} \right\|_{L^2}^2 - \langle f_k, \eta \rangle_{L^2}, \quad (1.5)$$

where $\mathcal{E}$ is defined in (2.4) *and $c > 0$. Then there exists a $\tau_0 > 0$ and a $c_0 > 0$ depending on the assumptions on E, the initial data and the right hand side, such that for all $0 < \tau \leq \tau_0$ and for $c \geq c_0$, the stability estimate*

$$E(\eta_k) + \frac{1}{2} \left\| \frac{\eta_k - \eta_{k-1}}{\tau} \right\|_{L^2}^2 \leq \left(E(\eta_0) + \frac{1}{2} \|\eta_*\|_{L^2}^2 + \frac{1}{2} \|f\|_{L^2((0,T);L^2)}^2 \right),$$

is satisfied whenever the right hand side is finite and $(\eta_k)_{k=1,\dots,\lfloor T/\tau \rfloor}$ does not reach a collision.

Next we show that (given uniform convexity of the leading term, which is the case for both (1.3) and (1.4)) one obtains an energy estimate also for the direct approach, meaning without the extra dissipation term. In this case energy can increase in time, even in the case $f = 0$, however note that the increase is controlled by $(1 + 4C\tau^2 k) \leq (1 + 4C\tau T) \to 1$ with $\tau \to 0$. Though this stability estimate is weaker than the previous one, we include it as it is of possible interest due to the easier implementation of this scheme.

Theorem 1.2 (Stability of the direct approach). *Let E be as above and moreover E_2 be uniformly convex. Let η_k be the variational approximation obtained by step-wise minimization*

$$\eta_k = \arg\min_{\eta \in \mathcal{E}} \frac{\tau^2}{2} \left\| \frac{\frac{\eta - \eta_{k-1}}{\tau} - \frac{\eta_{k-1} - \eta_{k-2}}{\tau}}{\tau} \right\|_{L^2}^2 + E(\eta) - \langle f_k, \eta \rangle_{L^2}, \quad (1.6)$$

where $\mathcal{E}$ is defined in (2.4)*. Then there exists a $\tau_0 > 0$ depending on the initial data and the right hand side, such that for all $0 < \tau \leq \tau_0$, the stability estimate*

$$E(\eta_k) + \frac{1}{2} \left\| \frac{\eta_k - \eta_{k-1}}{\tau} \right\|_{L^2}^2 \leq \left(E(\eta_0) + \frac{1}{2} \|\eta_*\|_{L^2}^2 + \frac{1}{2} \|f\|_{L^2((0,T);L^2)}^2 \right) (1 + 4C\tau^2 k),$$

is satisfied whenever the right hand side is finite and $(\eta_k)_{k=1,\dots,\lfloor T/\tau \rfloor}$ does not reach a collision. The constant C depends on the given data only.

In the case of a linear higher order term (i.e. corresponding to the case (1.4)) we prove that solutions are unique and hence the scheme converges (for arbitrary initial data). If the initial data is smooth we deduce that the scheme converges with a linear rate (note that in the statement below, both sides of the inequality are squared). This result is valid for both schemes, the direct approach as well as the approach with artificial viscosity.

Theorem 1.3 (Convergence rate). *Let $E(\eta) = E_1(\eta) + E_2(\eta)$, where E_1 is given by* (1.2) *and E_2 by* (1.4) *satisfying $q > n$ or $k_0 > n/2 + 1$. Let further η be the solution of* (1.1) *with boundary values* (2.10), (2.11)*. We denote the error $e_k = \eta_k - \eta(\tau k)$, where η_k is the variational approximation obtained by step-wise minimization defined by* (1.5) *or* (1.6)*. There exist constants C_1, C and $\tau_0 > 0$ depending on the given data, such that for all $0 < \tau \leq \tau_0$ the following convergence rate estimate holds*

$$\frac{1}{2} \left\| \nabla^{k_0} e_k \right\|_{L^2}^2 + \frac{1}{2} \left\| \frac{e_k - e_{k-1}}{\tau} \right\|_{L^2}^2 \leq \tau^2 C T e^{C_1 k \tau},$$

whenever the data has the following higher regularity

$$\eta_0, \eta_* \in W^{3k_0,2}(Q), \quad f \in W^{2,2}((0,T); L^2(Q)).$$

Please observe that the rate shown here is the same as is known to be optimal in the case of parabolic evolutions with convex energies. Nevertheless we include some simple numerical experiment in Section 4 that shows the optimality of the rates shown here. In particular this simple example shows already that choosing $\tau = h$ seems to be the optimal choice in general, which is in accordance with the estimates in Theorem 3.8.

To close the estimate we show higher regularity for the $W^{k_0,2}$-case (1.4), see Theorem 3.5 for the full statement.

Theorem 1.4 (Regularity)**.** *Under precisely the same assumptions as in Theorem 1.3, we find*

$$\partial_t \eta \in L^\infty((0,T); W^{2k_0,2}_{loc}(Q)), \quad \partial_{tt}\eta \in L^\infty((0,T); W^{k_0,2}(Q)), \quad \partial_{ttt}\eta \in L^\infty((0,T); L^2(Q))$$

and $\Delta^{k_0}\partial_t\eta \in L^\infty((0,T); L^2(Q))$*, with natural bounds.*

We note here that the difference of the hyperbolic to the parabolic case is that more regularity for the initial state and external forcing has to be assumed, as no smoothing effect over time is available.

In conclusion, the variational time-discrete approximation possesses very strong properties regarding the question of stability and convergence. Indeed, for a family of non-convex and non-linear settings, which are in some sense of lower order, the respective hyperbolic evolution has the same stability and/or convergence properties as in the convex and/or linear case, provided the term of leading order is convex and/or linear.

Remark (Evolutions with dissipation potential)**.** The stability and the convergence estimate are both valid in the presence of dissipation under rather general assumptions, which can be checked easily using the methods introduced in this paper. We decided to put our focus on pure hyperbolic motions here, since even so hyperbolic motions are physically very relevant, much less analysis is available in that regime. Indeed, surveying the state of the art in elastodynamics including large deformation shows that in most analytic results a dissipation potential for the elastic deformation is assumed. This was in particular the case in the recent existence result allowing for inertia [BKS23b, Section 3].

Remark (Relevance for existence theory)**.** The stability and regularity result seem to have the potential for future use in analysis. Already in the existence theory, obtaining a *hyperbolic* energy estimate that is given here in form of a stability estimate *on the τ-level* was previously not known and allows to circumvent the testing of $\partial_t\eta$ on the h-level as was done in the previous literature of the method [BKS23a, BKS23b].

Reversively the regularity theory allows to show that $\partial_t\eta$ is also a test-function a-posteriori, which allows for precise uniqueness and/or the quantification of distances between solutions.

1.1. Structure of the paper.

The paper consists of two main parts, Sections 2 and 3, and is supplemented with a numeric experiment in Section 4.

Section 2 is about the stability question. Here we first clarify the abstract assumptions necessary for the stability in 2.1, and the particular assumptions for elastodynamics 2.2. For these we show that they are satisfied for the leading elastic example in 2.4. Second, we provide some general Gronwall inequality for discrete schemes with two scales in 2.5, which will then be used to show the stability. This is one of the technical highlights of this part, another is the critical non-convexity estimate for elastic energies in 2.3. The variational time-stepping scheme is introduced in 2.6 and the stability estimate is then proved in 2.8–2.10.

In Section 3, we focus on the case of elasto-dynamics for which we explicitly prove regularity and convergence rate. We also provide abstract assumptions at the beginning of the section. In 3.1 we show in-time regularity for hyperbolic elastodynamics. Theorem 3.8 is then proved at the end of the section.

We conclude the paper with some numerical experiments in Section 4 that imply that the rates are optimal, that the appearance of τ_0 in the stability and convergence results is necessary, and the loss of convergence in case h and τ differ. The experiment is merely to show that our results are in coherence with expectations from the ODE theory. We leave the implementation for largely deforming solids to a future work.

2. Stability

2.1. General setting for stability.

We formulate our problem of interest in a rather general case. Let X be a reflexive Banach space, which densely embeds into some Hilbert space H.

Further, let $\mathcal{E} \subset X$ be a weakly closed subset. Let the energy $E\colon X \to (-\infty, \infty]$ satisfy Assumptions 2.2 below. We consider the problem

$$\begin{aligned} \partial_{tt}\eta + DE(\eta) = f \quad &\text{in } X^*, \qquad \text{a.e. on } (0,T) \\ \eta(t) \in \mathcal{E} &\text{ for a.e. } t \in (0,T) \\ \eta(0) &= \eta_0 \\ \partial_t \eta(0) &= \eta_*, \end{aligned} \tag{2.1}$$

where DE denotes the (Fréchet) derivative of E. We assume that the initial conditions and the right hand side have

$$\eta_0 \in X, \quad \eta_* \in H, \quad f \in L^2((0,T);H).$$

For convenience we denote

$$L^\infty((0,T);\mathcal{E}) = \{\eta \in L^\infty((0,T);X) : \eta(t) \in \mathcal{E} \text{ for a.e. } t \in (0,T)\}. \tag{2.2}$$

Now we define the notion of weak solution to the problem above. By $\langle\,,\,\rangle_H$ resp. $\langle\,,\,\rangle_X$ we denote the scalar product on H resp. the duality pairing between X^* and X.

Definition 2.1 (Weak solution in general setting)**.** We say that

$$\eta \in L^\infty((0,T);\mathcal{E}) \quad \text{with} \quad \|E(\eta)\|_{L^\infty((0,T))} < \infty \quad \text{and} \quad \partial_t\eta \in L^\infty((0,T);H)$$

is a weak solution to (2.1) if[2] $\eta(0) = \eta_0$ and

$$\langle \eta_*, \varphi(0)\rangle_H + \int_0^T -\langle \partial_t\eta, \partial_t\varphi\rangle_H + \langle DE(\eta), \varphi\rangle_X \,\mathrm{d}t = \int_0^T \langle f, \varphi\rangle_H \,\mathrm{d}t$$

for all $\varphi \in L^2((0,T);X) \cap W^{1,2}((0,T);H)$ with $\varphi(T) = 0$.

Assumptions 2.2. The energy $E\colon X \to (-\infty,\infty]$ satisfies the following assumptions:

(A.1) $E(\eta) < \infty$ for $\eta \in \mathcal{E}$.
(A.2) There is $E_{\min} > -\infty$ such that $E(\eta) \geq E_{\min}$ for all $\eta \in X$.
(A.3) E is Fréchet differentiable at each $\eta \in \operatorname{int}\mathcal{E}$, and $DE\colon \operatorname{int}\mathcal{E} \to X^*$ is strongly continuous.
(A.4) E is coercive in the following sense: for every $E_{\min} \leq K < \infty$ the sublevel set $\{\eta \in X : E(\eta) \leq K\}$ is bounded in X.
(A.5) There is a Banach space Z, such that $X \subset Z$, and a linear operator $L\colon Z \to Z^*$ which is bounded, symmetric and elliptic (meaning that for some $\lambda > 0$ it holds $\langle Lz, z\rangle \geq \lambda\|z\|_Z^2$ for all $z \in Z$), such that E satisfies the *non-convexity estimate in* Z: For every $E_{\min} \leq K < \infty$ there exists C depending on K such that

$$\langle DE(\eta_1), \eta_1 - \eta_0\rangle \geq E(\eta_1) - E(\eta_0) - C\,\|\eta_1 - \eta_0\|_Z^2$$

for all $\eta_1, \eta_0 \in \mathcal{E}$ with $E(\eta_1), E(\eta_0) \leq K$.

The non-convexity estimate is exactly what will enable us to prove the stability of our approximation. Important is its relation to $\mathcal{E}$, which can here be any weakly closed set.

The operator L will be used to produce an artificial stabilization in the approximation (2.23). In this setting it will be enough to have the non-convexity estimate in Z. In case we do not include an extra stabilization term, we can obtain a stability estimate (see 2.27) under the stronger assumption

(A.5') E satisfies the following *non-convexity estimate in* H: For every $E_{\min} \leq K < \infty$ there exists C depending on K such that

$$\langle DE(\eta_1), \eta_1 - \eta_0\rangle \geq E(\eta_1) - E(\eta_0) - C\,\|\eta_1 - \eta_0\|_H^2$$

for all $\eta_1, \eta_0 \in \mathcal{E}$ with $E(\eta_1), E(\eta_0) \leq K$.

[2] Note that since $\eta \in W^{1,\infty}((0,T);H)$, the value $\eta(0) \in H$ is well-defined.

2.2. **Setting for elastodynamics.** As the most prominent application we present the case of dynamic evolution of an elastic solid. The solid is described in Lagrangian coordinates. This means that there is a bounded Lipschitz reference domain $Q \subset \mathbb{R}^n$, and at each time t the solid is described by a deformation $\eta(t) \colon Q \to \mathbb{R}^n$. Then we seek $\eta \colon (0,T) \times Q \to \mathbb{R}^n$ a solution to the equation

$$\begin{aligned} \partial_{tt}\eta + DE(\eta) &= f \quad \text{in } L^2(Q), \qquad \text{a.e. on } (0,T), \\ \eta(t) &\in \mathcal{E} \quad \text{for a.e. } t \in (0,T) \\ \eta(0) &= \eta_0, \\ \partial_t \eta(0) &= \eta_*, \\ \eta(t,x) &= x, \quad x \in \Gamma_D, \\ \partial_\nu \eta(t,x) &= \nu, \quad x \in \Gamma_N \end{aligned} \tag{2.3}$$

for given initial conditions and the right hand side

$$\eta_0 \in \mathcal{E}, \quad \eta_* \in L^2(Q), \quad f \in L^2((0,T); L^2(Q)),$$

where $\Gamma_D \cup \Gamma_N = \partial Q$ and $|\Gamma_D|_{d-1} > 0$. Here we denote by ν the outer normal to ∂Q resp. by ∂_ν the corresponding normal derivative. Further boundary conditions for higher order derivatives naturally appear depending on E_2. For more details see the remark on boundary conditions at the end of this subsection.

In any case the set of admissible deformations is

$$\mathcal{E} = \left\{ \eta \in X : |\eta(Q)| = \int_Q \det \nabla \eta \, \mathrm{d}x, \quad \eta|_{\Gamma_D}(x) = x \text{ for } x \in \Gamma_D \right\}, \tag{2.4}$$

where the function space X is one of the following two cases:

$$X = W^{2,q}(Q) \quad \text{or} \quad X = W^{k_0,2}(Q). \tag{2.5}$$

We will henceforth refer to the former as *the* $W^{2,q}$*-case* and to the latter as *the* $W^{k_0,2}$*-case*. Regarding the exponents, we assume either

$$q > n \quad \text{or} \quad k_0 > n/2 + 1. \tag{2.6}$$

Note that in particular if $n = 2$ or $n = 3$, in the $W^{k_0,2}$-case it is enough to choose $k_0 \geq 3$. In both cases we have the compact embedding $X \subset\subset C^{1,\alpha}(Q)$ for $0 < \alpha < \min(1, \alpha_0)$, where either

$$\alpha_0 = 1 - n/q \quad \text{or} \quad \alpha_0 = k_0 - 1 - n/2. \tag{2.7}$$

The condition $|\eta(Q)| = \int_Q \det \nabla \eta \, \mathrm{d}x$ is called the Ciarlet–Nečas condition and it entails a.e. injectivity of η, cf. [CN87]. We can readily see that $\mathcal{E}$ is weakly closed, as the condition is stable under weak convergence in X.

Remark. It may be worthwhile to comment on the exclusion of self-contact, which is presumed for all of our results. Firstly, by (E.3) it follows that deformations of bounded energy are locally injective. More precisely, in dependence of the energy E_0 there exists a radius $\delta_0 > 0$ such that all deformations $\eta \in \mathcal{E}$ with $E(\eta) \leq E_0$ are injective on all subsets of radius at most δ_0. Secondly, every globally injective deformation has an L^2-neighborhood of radius γ_0 (depending on E_0) such that all deformations in this neighborhood are globally injective. This means that, at least for short times, collisions are excluded. For a more detailed discussion including proofs, see [BKS23b, Section 2.1].

Recall the notation (2.2) and also denote

$$W^{2,2}_D(Q) = \{u \in W^{2,2}(Q) : u|_{\Gamma_D} = 0\} \tag{2.8}$$

Now we define the weak solutions for elastodynamics, consistently with the general Definition 2.1.

Definition 2.3 (Weak solution for elastodynamics)**.** We say that η with

$$\eta \in L^\infty((0,T); \mathcal{E}) \quad \text{with} \quad \|E(\eta)\|_{L^\infty((0,T))} < \infty \quad \text{and} \quad \partial_t \eta \in L^\infty((0,T); L^2(Q)). \tag{2.9}$$

is a weak solution to (2.3) if

$$\int_0^T -\langle \partial_t \eta, \partial_t \phi \rangle + \langle DE(\eta), \phi \rangle \, \mathrm{d}t + \langle \eta_*, \phi(0) \rangle = \int_0^T \langle f, \phi \rangle \, \mathrm{d}t$$

for all $\phi \in C^\infty([0,T]; C^\infty(Q;\mathbb{R}^n))$ with $\phi|_{(0,T)\times\Gamma_D} = 0$ and $\phi(T) = 0$.

Now let us specify the assumptions of the elastic energy.

Assumptions 2.4. We assume the *energy* $E\colon X \to (-\infty,\infty]$ can be written as the sum

$$E(\eta) = E_1(\eta) + E_2(\eta)$$

and E_1, E_2 satisfy the following assumptions. There exists a density e for E_1, that is $e\colon \mathbb{R}^{n\times n} \to (-\infty,\infty]$, such that E_1 is of the form

$$E_1(\eta) = \int_Q e(\nabla\eta)\,\mathrm{d}x, \quad \eta \in X,$$

and moreover it holds:

(E.1) $e \in C^2(\mathbb{R}^{n\times n}_{\det>0})$, where $\mathbb{R}^{n\times n}_{\det>0} = \{M \in \mathbb{R}^{n\times n} : \det M > 0\}$.
(E.2) There is $e_{\min} > -\infty$ such that $e(\xi) \geq e_{\min}$ for all $\xi \in \mathbb{R}^{n\times n}$.
(E.3) For all $K < \infty$ there exists $\varepsilon_0 > 0$ such that for each $\eta \in X$, $E(\eta) \leq K$ implies $\det \nabla\eta \geq \varepsilon_0$ in Q.
(E.4) For (a sequence of) $\xi \in \mathbb{R}^{n\times n}_{\det>0}$ with $\det\xi \to 0$ it holds $e(\xi) \to \infty$. Further, for $\xi \in \mathbb{R}^{n\times n}$ with $\det\xi \leq 0$ it holds $e(\xi) = \infty$.
(E.5) E_2 is convex, coercive (in the sense that sublevel sets of E_2 are bounded subsets of X) and Fréchet differentiable on X.

Analogously to the general case (i.e. (A.5) and (A.5')), we include separately the stronger convexity assumption that allows to use the approximation without extra stabilization term.

(E.5') E_2 is uniformly convex on $W^{2,2}(Q)$. This means there exists $c > 0$ so that it holds for all $\eta \in X$ and $w \in X$

$$\langle D^2E_2(\eta), w\otimes w\rangle \geq c\left\|\nabla^2 w\right\|^2_{L^2(Q)}.$$

Note that in contrast to the abstract setting, we do not assume the non-convexity estimate (A.5). In fact, it will be proven in Lemma 2.6 that this estimate follows from the other properties and (E.5).

Remark (Notation)**.** To avoid confusion with derivatives, we denote the gradient of $\eta\colon Q \to \mathbb{R}^n$ with respect to $x \in Q$ by ∇ (or ∇_x), whereas the gradient of $e\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ will be denoted by ∇_ξ. Similarly for higher derivatives. Moreover, the derivative of E (resp. E_1 or E_2) will be denoted by D, to emphasize that it is a derivative of a functional on the infinite-dimensional space X.

Remark (More general boundary conditions)**.** For simplicity we take throughout the paper the assumption that we have boundary conditions

$$g(x) = x \tag{2.10}$$

as otherwise the estimates do not change significantly but are much harder to follow. In the case of general boundary function g that is assumed to be extended to Q by G in an appropriate sense (here it means in some Sobolev space $W^{k,p}(Q)$ and with strictly positive Jacobian) we can define

$$\eta(t,x) = g(x) \text{ for } (t,x) \in [0,T]\times\Gamma_D \text{ and } \partial_\nu(\eta(t,x) - G(x)) = 0 \text{ on } [0,T]\times\Gamma_N. \tag{2.11}$$

The related testing space to these boundary conditions is

$$W = \{\phi \in W^{1,1}(Q) \,:\, \phi(x) = 0 \text{ for } x \in \Gamma_D\}.$$

For the higher order derivatives the respective boundary conditions of Navier type are defined via $E_2 : W^{k,p}(Q) \to [0,\infty]$, such that

$$DE_2(\eta - G) \in (W^{k,p}(Q)\cap W)^*$$

In the example of the k-Laplacian $E_2(\eta) = \frac{1}{2}\int_Q |\nabla^k\eta|^2\,\mathrm{d}x$ this becomes

$$\int_Q \nabla^k(\eta - G)\cdot\nabla^k\phi\,dx = \int_Q (-\Delta)^k(\eta - G)\cdot\phi\,dx,$$

which means additionally to (2.11) that

$$\partial_\nu \Delta^{k-1}(\eta - G) = 0 \text{ on } \Gamma_N,$$
$$\partial_\nu \nabla^{k-l-1}\Delta^l(\eta - G) = 0 \text{ on } \partial Q \text{ for } l \in \{0, ..., k-2\}.$$

2.3. **Non-convexity estimate for elastic solids.** As was indicated already, the non-convexity estimate (A.5) is essential for the stability estimates. Previous versions of this estimate in the literature do not allow the estimate for general distances, see [MR20, Proposition 3.2].

As a first step, we show that e and its derivatives are uniformly bounded, with bound depending on the energy only.

Lemma 2.5. *Let $K \in \mathbb{R}$. Then there exists a $C_K \in \mathbb{R}$ such that every η belonging to the weak solution class* (2.9) *with $\|E(\eta)\|_{L^\infty((0,T))} \le K$ satisfies*

$$e(\nabla\eta), |\nabla_\xi e(\nabla\eta)|, |\nabla_\xi^2 e(\nabla\eta)| \le C_K, \quad \textit{in } (0,T) \times Q.$$

Proof. Using the assumption (E.3) we see that

$$\det \nabla\eta \ge \varepsilon_0 \quad \text{in } (0,T) \times Q$$

with ε_0 depending only on K. Moreover, since X is embedded into $W^{1,\infty}$, we see that, by boundedness of E_2, $|\nabla\eta(t,x)| \le C$, where C depends on K and the embedding $X \subset W^{1,\infty}$. This means that

$$\nabla\eta(t,x) \in \mathcal{K} := \{M \in \mathbb{R}^{n\times n} : \det M \ge \varepsilon_0, |M| \le C\}.$$

Since $\mathcal{K}$ is a compact set contained in $\mathbb{R}^{n\times n}_{\det>0}$ where e is C^2 by (E.1), we know that $e, \nabla_\xi e, \nabla_\xi^2 e$ are bounded on $\mathcal{K}$. Since $\mathcal{K}$ depends only on K, the proof is finished. □

Now we will estimate the non-convexity of E_1 in terms of the distance of gradients.

Lemma 2.6 (Non-convexity estimate (I))**.** *Suppose that we have $\eta_0, \eta_1 \in \mathcal{E}$ with*

$$E(\eta_0), E(\eta_1) \le K < \infty.$$

Then there exists a constant C_1 depending only on K such that

$$\langle DE_1(\eta_1), \eta_1 - \eta_0\rangle \ge E_1(\eta_1) - E_1(\eta_0) - C_1 \|\nabla\eta_1 - \nabla\eta_0\|_{L^2}^2.$$

Proof. Throughout the proof, any constant named C_i with any index i depends only on K. By coercivity of E following from (E.5), we have that $\|\eta_0\|_X, \|\eta_1\|_X \le C_K$ and thus $\|\nabla\eta_0\|_{L^\infty}, \|\nabla\eta_1\|_{L^\infty} \le C_\infty$. Let ε_0 be the lower bound on the determinant from (E.3), corresponding to K. Now the set

$$\mathcal{A} = \{M \in \mathbb{R}^{n\times n} : \det M \ge \varepsilon_0, |M| \le C_\infty\}$$

is compact in $\mathbb{R}^{n\times n}_{\det>0}$, and

$$\mathcal{B} = \{M \in \mathbb{R}^{n\times n} : \det M \ge \varepsilon_0/2, |M| \le 2C_\infty\}$$

is likewise compact in $\mathbb{R}^{n\times n}_{\det>0}$, with $\mathcal{A} \subset \operatorname{int}\mathcal{B}$. So there exists $r > 0$ such that $B_r(\mathcal{A}) \subset \mathcal{B}$, in other words,

$$\text{for all } A \in \mathcal{A} \text{ and for all } B \in \mathbb{R}^{n\times n}, |B - A| < r \text{ implies } B \in \mathcal{B}.$$

Now let us split our expression in two parts:

$$\langle DE_1(\eta_1), \eta_1 - \eta_0\rangle = \int_{\{|\nabla\eta_1 - \nabla\eta_0| \le r\}} \nabla_\xi e(\nabla\eta_1) : \nabla(\eta_1 - \eta_0)\,\mathrm{d}x + \int_{\{|\nabla\eta_1 - \nabla\eta_0| > r\}} \nabla_\xi e(\nabla\eta_1) : \nabla(\eta_1 - \eta_0)\,\mathrm{d}x.$$

For the second part, we recall from Lemma 2.5 that $|\nabla_\xi e(\nabla\eta_1)| \le C_K$ and we get

$$\int_{\{|\nabla\eta_1 - \nabla\eta_0| > r\}} \nabla_\xi e(\nabla\eta_1) : \nabla(\eta_1 - \eta_0)\,\mathrm{d}x \ge -\frac{C_K}{r} \|\nabla\eta_1 - \nabla\eta_0\|_{L^2}^2.$$

Let us now for $\theta \in [0,1]$ denote $\eta_\theta = \theta\eta_1 + (1-\theta)\eta_0$. Then for the first term, we apply pointwisely (i.e. for each x) the Taylor theorem of the second order (with respect to θ). So we obtain that there exists $\theta\colon Q \to [0,1]$ such that

$$\int_{\{|\nabla\eta_1 - \nabla\eta_0| \le r\}} \nabla_\xi e(\nabla\eta_1) : \nabla(\eta_1 - \eta_0)\,\mathrm{d}x = \int_{\{|\nabla\eta_1 - \nabla\eta_0| \le r\}} e(\nabla\eta_1) - e(\nabla\eta_0) + \nabla_\xi^2 e(\nabla\eta_\theta)(\nabla\eta_1 - \nabla\eta_0, \nabla\eta_1 - \nabla\eta_0)\,\mathrm{d}x$$

$$= E_1(\eta_1) - E_1(\eta_0) - \int_{\{|\nabla\eta_1 - \nabla\eta_0| > r\}} e(\nabla\eta_1) - e(\nabla\eta_0)\,\mathrm{d}x + \int_{\{|\nabla\eta_1 - \nabla\eta_0| \le r\}} \nabla_\xi^2 e(\nabla\eta_\theta)(\nabla\eta_1 - \nabla\eta_0, \nabla\eta_1 - \nabla\eta_0)\,\mathrm{d}x$$

The middle term we estimate similarly as before

$$-\int_{\{|\nabla\eta_1 - \nabla\eta_0| > r\}} e(\nabla\eta_1) - e(\nabla\eta_0)\,\mathrm{d}x \ge -2\frac{C_K}{r^2}\,\|\nabla\eta_1 - \nabla\eta_0\|^2 .$$

It thus remains to estimate the last quadratic term, namely it suffices to show that

$$|\nabla_\xi^2 e(\nabla\eta_\theta)| \le C_2 \quad \text{on} \quad \{|\nabla\eta_1 - \nabla\eta_0| \le r\}.$$

Due to our assumption, we have $\nabla\eta_0(x), \nabla\eta_1(x) \in \mathcal{A}$ for all $x \in Q$. Therefore if $|\nabla\eta_1(x) - \nabla\eta_0(x)| \le r$, then $\nabla\eta_{\theta(x)}(x) \in \mathcal{B}$. So the inequality holds with $C_2 = \max_{\mathcal{B}} |\nabla_\xi^2 e|$, which is finite due to $e \in C^2(\mathbb{R}^{n\times n}_{\det>0})$ and $\mathcal{B}$ being compact in $\mathbb{R}^{n\times n}_{\det>0}$. Putting together all the inequalities, we see that we proved our claim with $C_1 = \frac{C_K}{r} + 2\frac{C_K}{r^2} + C_2$. □

Lemma 2.7. *Assumptions 2.4,* (E.1)–(E.5) *imply Assumptions 2.2,* (A.1)–(A.5)*.*

Proof. It is readily seen that (E.1) implies (A.1) and (E.2) implies (A.2). The Fréchet differentiability of E_1 on $\operatorname{int}\mathcal{E}$ follows from (E.1) combined with (E.3), as then we can see that the derivative at $\eta \in \operatorname{int}\mathcal{E}$ in the direction $\gamma \in X$ is then

$$\langle DE_1(\eta), \gamma\rangle = \int_Q \nabla_\xi e(\nabla\eta) : \nabla\gamma\,\mathrm{d}x,$$

differentiability of E_2 is already assumed in (E.5). This shows (A.3). Coercivity (A.4) is thanks to (E.5) and the fixed boundary values on Γ_D. The non-convexity estimate (A.5) follows from Lemma 2.6 and (E.5). □

In case no stabilizer is considered the argument needs to be refined, which is done by Lemma 2.8: By interpolation and using (E.5'), the $W^{2,2}$-uniform convexity of E_2, we can estimate the non-convexity of E in terms of L^2 distance only. This is shown in the following lemma.

Lemma 2.8 (Non-convexity estimate (II))**.** *Let E_2 satisfy additionally* (E.5') *and suppose that we have $\eta_0, \eta_1 \in \mathcal{E}$ with*

$$E(\eta_0), E(\eta_1) \le K < \infty.$$

Then there exists a constant C depending only on K such that

$$\langle DE(\eta_1), \eta_1 - \eta_0\rangle \ge E(\eta_1) - E(\eta_0) - C\,\|\eta_1 - \eta_0\|_{L^2}^2 .$$

Proof. Using Taylor theorem of the second order, we get for some $\xi \in [0,1]$, $\eta_\xi = \xi\eta_1 + (1-\xi)\eta_0$

$$\begin{aligned}\langle DE_2(\eta_1), \eta_1 - \eta_0\rangle &= E_2(\eta_1) - E_2(\eta_0) + \frac{1}{2}\langle D^2E_2(\eta_\xi), (\eta_1 - \eta_0)\otimes(\eta_1 - \eta_0)\rangle \\ &\ge E_2(\eta_1) - E_2(\eta_0) + c\left\|\nabla^2(\eta_1 - \eta_0)\right\|_{L^2}^2 .\end{aligned}$$

Interpolate (thanks to fixed boundary values on Γ_D)

$$C_1\,\|\nabla\eta_1 - \nabla\eta_0\|_{L^2}^2 \le c\left\|\nabla^2\eta_1 - \nabla^2\eta_0\right\|_{L^2}^2 + C\,\|\eta_1 - \eta_0\|_{L^2}^2 .$$

Combining these two inequalities and plugging this into the result of Lemma 2.6 gives the desired result. □

Lemma 2.9. *Assumptions 2.4* (E.1)-(E.4), (E.5') *imply Assumptions 2.2* (A.1)-(A.4), (A.5')*.*

Proof. The non-convexity estimate (A.5') follows from Lemma 2.6. The validity of (A.1)-(A.4) has already been shown in Lemma 2.7. □

Let us now in the equation (2.3) rewrite the term $DE_1(\eta)$ in terms of e. Assume η lies in the spaces (2.9) and compute the Gateaux derivative at $\eta(t) \in \mathcal{E}$ in a direction $\gamma \in X$. Recall that $X \subset W^{2,2}(Q)$ so we can use the chain rule to obtain at time t

$$\langle DE_1(\eta(t)), \gamma\rangle = \int \nabla_\xi e(\nabla_x\eta) : \nabla_x\gamma = \int_Q \sum_{i,j=1}^n \partial_{\xi_j^i} e(\nabla\eta)\partial_{x_i}\gamma = -\int_Q \sum_{i,j=1}^n \partial_{x_i}(\partial_{\xi_j^i} e(\nabla\eta))\gamma_j$$

$$= -\int_Q \sum_{i,j,k,l=1}^{n} \partial^2_{\xi^i_j \xi^k_l} e(\nabla\eta)\partial^2_{x_i x_l}\eta_k \gamma_j = -\int_Q \nabla^2_\xi e(\nabla_x \eta) : \nabla^2_x \eta \cdot \gamma.$$

then by the previous Lemma 2.5 we see $\nabla^2_\xi e(\nabla\eta) \in L^\infty((0,T)\times Q)$ and $\nabla^2\eta \in L^2((0,T)\times Q)$, so then $DE_1(\eta) = -\nabla^2_\xi e(\nabla_x\eta) : \nabla^2_x\eta \in L^2((0,T)\times Q)$ in the usual sense.

So then our equation (2.3) can be written as

$$\partial_{tt}\eta + DE_2(\eta) - \nabla^2_\xi e(\nabla_x\eta) : \nabla^2_x \eta = 0.$$

2.4. **Prototypical energy.** As a prototype of the highest-order convex regularizing part of the energy we can put

$$\text{if } X = W^{k_0,2}(Q): \quad E_2(\eta) = \frac{1}{2}\left\|\nabla^{k_0}\eta\right\|^2_{L^2(Q)} = \frac{1}{2}\int_Q \left|\nabla^{k_0}\eta\right|^2 \,\mathrm{d}x, \tag{2.12}$$

or

$$\text{if } X = W^{2,q}(Q): \quad E_2(\eta) = \frac{1}{q}\int_Q (1+|\nabla^2\eta|)^{q-2}|\nabla^2\eta|^2 \,\mathrm{d}x. \tag{2.13}$$

They are both uniformly convex thus satisfy (E.5'), the latter thanks to $t \mapsto (1+|t|)^{q-2}|t|^q$ being uniformly convex. Further, the q-biLaplacian

$$E_2(\eta) = \frac{1}{q}\left\|\nabla^2\eta\right\|^q_{L^q(Q)} = \frac{1}{q}\int_Q |\nabla^2\eta|^q \,\mathrm{d}x \tag{2.14}$$

satisfies the convexity (E.5), but not the uniform convexity (E.5').

As there is no difference in the analysis we simplify the physical energy (1.2) to its determinant part and use as prototype of the energy density for E_1

$$e(\xi) = \begin{cases} \dfrac{1}{(\det\xi)^a}, & \det\xi > 0 \\ \infty, & \det\xi \le 0 \end{cases} \tag{2.15}$$

with $a > n/\alpha_0$ (recall that α_0 is defined by (2.7)). It can readily be checked that the next theory also holds for E_1 being in the form of (1.2).

Theorem 2.10. *The prototypical energy E_1 defined in* (2.15) *satisfies Assumptions 2.4* (E.1)-(E.4).

Proof. We only need to check the lower bound on the determinant (E.3), since all other properties are clear. This is essentially proven in [HK09], but for completeness we give a concise proof here. Let $\eta \in \mathcal{E}$ with $E(\eta) \le K$ for some given $K < \infty$. Thus we have by the coercivity of E on X a bound on η in X, thus also a bound on $\|\nabla\eta\|_{C^{0,\alpha}}$ and therefore $\|\det\nabla\eta\|_{C^{0,\alpha}} \le c_\alpha$, where c_α depends on K.

Because the boundary of Q is Lipschitz continuous, there exists a constant $c_L > 0$ and a $\delta_0 > 0$, such that for all $\delta \in (0,\delta_0]$ we have for all $x \in \overline{Q}$

$$|B_\delta(x) \cap Q| \ge c_L \delta^n.$$

Let $x_0 \in \overline{Q}$ be such that $\det\nabla\eta(x_0) = \min_{x\in\overline{Q}} \det\nabla\eta(x) > 0$. Put $\varepsilon_0 = \min(\det\nabla\eta(x_0), \delta_0^\alpha)$ and take $0 < \delta \le \delta_0$ arbitrary. Therefore, for $x_0 \in Q$ such that $\det\nabla\eta(x_0) \ge \varepsilon_0$ we have

$$\begin{aligned} K \ge E(\eta) &\ge \int_Q \frac{1}{(\det\nabla\eta)^a}\,\mathrm{d}x \ge \int_{B_\delta(x_0)\cap Q} \frac{1}{\det\nabla\eta(x)^a}\,\mathrm{d}x \\ &\ge \int_{B_\delta(x_0)\cap Q} \frac{1}{(\det\nabla\eta(x_0) - |\det\nabla\eta(x) - \det\nabla\eta(x_0)|)^a}\,\mathrm{d}x \\ &\ge \int_{B_\delta(x_0)\cap Q} \frac{1}{(\varepsilon_0 + c_\alpha\delta^\alpha)^a}\,\mathrm{d}x \ge c_L \frac{\delta^n}{(\varepsilon_0 + c_\alpha\delta^\alpha)^a}. \end{aligned}$$

As δ was arbitrary number smaller than δ_0, we can choose $\delta = \varepsilon_0^{1/\alpha}$ and obtain

$$K \ge c_L \frac{\varepsilon_0^{n/\alpha}}{(\varepsilon_0 + c_\alpha\varepsilon_0)^a} = \varepsilon_0^{n/\alpha - a}\frac{c_L}{(1+c_\alpha)^a}$$

and since $n/\alpha - a < 0$, we obtain

$$\det \nabla\eta(x_0) \geq \varepsilon_0 \geq \left(\frac{K(1+c_\alpha)^a}{c_L}\right)^{\frac{1}{n/\alpha - a}}.$$

Since the right hand side depends only on K, this gives the lower bound on $\det \nabla\eta$ and proves (E.3). □

Remark. For the prototypical energy density we can calculate more explicitly the bounds from Lemma 2.5 as follows. Directly from (2.15) see that

$$e(\nabla\eta) \leq \varepsilon_0^{-1/a}.$$

Next, we compute the first and second gradient of e. We have, as $\nabla_\xi \det \xi = \operatorname{cof} \xi$,

$$\nabla_\xi e(\xi) = -\frac{a}{(\det \xi)^{a+1}} \operatorname{cof} \xi,$$

so the lower bound on the determinant (E.3), along with the bound on $\|\nabla\eta\|_{L^\infty}$ suffices to bound $\nabla_\xi e(\nabla\eta)$.

For the second gradient we compute

$$\nabla_\xi^2(\det \xi) = (\partial_{\xi_l^k}(-1)^{i+j} \det \xi_{\hat{j}}^{\hat{i}})_{i,j,k,l} = \left(\begin{cases} 0, & i = k \text{ or } j = l, \\ (-1)^{i+j+k+l} \det \xi_{\widehat{jl}}^{\widehat{ik}}, & \text{else} \end{cases}\right)_{i,j,k,l}$$

where $\xi_{\hat{j}}^{\hat{i}}$ is the matrix ξ with row i and column j deleted, likewise for $\xi_{\widehat{jl}}^{\widehat{ik}}$ there are rows i, k and columns j, l deleted[3]. Therefore for our prototype we have for any $\xi \in \mathbb{R}^{n\times n}_{\det>0}$

$$\nabla_\xi^2 e(\xi) = \frac{a(a+1)}{(\det \xi)^{a+2}} \operatorname{cof} \xi \otimes \operatorname{cof} \xi + \frac{-a}{(\det \xi)^{a+1}} \nabla_\xi(\operatorname{cof} \xi)$$

or in components after plugging in $\xi := \nabla\eta$ this reads as

$$\nabla_\xi^2 e(\nabla_x \eta) = \left(\begin{cases} \frac{a(a+1)}{(\det \nabla\eta)^{a+2}} \det(\nabla\eta)_{\hat{j}}^{\hat{i}} \det(\nabla\eta)_{\hat{l}}^{\hat{k}}, & i = k \text{ or } j = l, \\ \frac{a(a+1)}{(\det \nabla\eta)^{a+2}} (-1)^{i+j+k+l} \det(\nabla\eta)_{\hat{j}}^{\hat{i}} \det(\nabla\eta)_{\hat{l}}^{\hat{k}} + (-1)^{i+j+k+l} \frac{-a}{(\det \nabla\eta)^{a+1}} \det(\nabla\eta)_{\widehat{jl}}^{\widehat{ik}}, & \text{else} \end{cases}\right)_{i,j,k,l}$$

so again, the lower bound on $\det \nabla\eta$ and the bound on $\|\nabla\eta\|_{L^\infty}$ suffices to calculate an explicit bound on $|\nabla_\xi^2 e(\nabla_x \eta)|$.

2.5. **Gronwall–type inequalities.** To show the stability and convergence rate of our scheme (introduced in the next section), we will make use of some inequalities of discrete Gronwall type. For completeness, we start with the classical version of the Gronwall inequality, including a short proof.

Lemma 2.11 (Discrete Gronwall inequality)**.** *Let $a_0, \ldots, a_n \geq 0$ and $c_0, \ldots, c_{n-1} \geq 0$ satisfy*

$$a_k \leq a_0 + \sum_{i=0}^{k-1} c_i a_i, \quad k = 1, \ldots, n.$$

Then

$$a_k \leq a_0 \prod_{i=0}^{k-1}(1 + c_i) \leq a_0 \exp\left(\sum_{i=0}^{k-1} c_i\right), \quad k = 0, \ldots, n.$$

Proof. By induction, we prove the stronger inequality

$$a_0 + \sum_{i=0}^{k-1} c_i a_i \leq a_0 \prod_{i=0}^{k-1}(1 + c_i).$$

For $k = 1$ both sides are equal to $a_0(1 + c_0)$. Then for $k > 1$ proceed by induction

$$a_0 + \sum_{i=0}^{k-1} c_i a_i \leq a_0 + \sum_{i=0}^{k-2} c_i a_i + c_{k-1} a_{k-1} \leq a_0 \prod_{i=0}^{k-2}(1+c_i) + c_{k-1} a_0 \prod_{i=0}^{k-2}(1+c_i) = a_0(1+c_{k-1}) \prod_{i=0}^{k-2}(1+c_i)$$

[3]If after deleting we would have 0×0 matrix, this determinant is defined as 1.

which concludes the proof, using $1+c_i \le e^{c_i}$ to obtain the second inequality. $\square$

In fact, it will be more useful for us to shift the indices by 1.

Lemma 2.12 (Discrete Gronwall inequality, shifted k)**.** *Let* $a_0,\dots,a_n \ge 0$ *satisfy with* $0 \le c_1,\dots,c_n < 1$

$$a_k \le a_0 + \sum_{i=1}^{k} c_i a_i, \quad k = 1,\dots,n.$$

Then it holds

$$a_k \le a_0 \prod_{i=1}^{k} (1-c_i)^{-1}, \quad k = 1,\dots,n.$$

In particular, if also $c_1,\dots,c_n \le 1/2$ *then*

$$a_k \le a_0 \exp\left(2\sum_{i=1}^{k} c_i\right), \quad k = 1,\dots,n.$$

Proof. For $k=1$ we have $a_1 \le a_0 + c_1 a_1$, so it is enough to subtract $c_1 a_1$ and divide by $1-c_1$.

For $k \ge 2$ proceed by induction. Write

$$a_k \le a_0 + \sum_{i=1}^{k} c_i a_i \le a_0 + \sum_{i=1}^{k-1} c_i a_0 \prod_{j=1}^{i} (1-c_j)^{-1} + a_k c_k = a_0 + \sum_{i=1}^{k-1} \frac{c_i}{1-c_i} a_0 \prod_{j=1}^{i-1} (1-c_j)^{-1} + a_k c_k$$

subtract $c_k a_k$ and use $\frac{c_i}{1-c_i} = (1-c_i)^{-1} - 1$ so that we get a telescoping sum

$$a_k(1-c_k) \le a_0 + \sum_{i=1}^{k-1} (1-c_i)^{-1} a_0 \prod_{j=1}^{i-1} (1-c_j)^{-1} - \sum_{i=1}^{k-1} a_0 \prod_{j=1}^{i-1} (1-c_j)^{-1} = a_0 \prod_{i=k}^{k-1} (1-c_i)^{-1}$$

which is, after division by $1-c_k$, what we wanted.

In the case that $c_i \le 1/2$ we can use $(1-c_i)^{-1} \le 1 + 2c_i \le e^{2c_i}$ to obtain the second inequality. $\square$

Remark. Notice that the requirement $c_i < 1$ is natural here, since for $c_i \ge 1$ the inequality in the assumption does not pose any restriction on a_i.

The following version of Gronwall inequality, including a square-root term, will enable us to get the natural estimate with the forcing term f being present in the equation.

Lemma 2.13 (Discrete Gronwall inequality with square root)**.** *Let* $a_0,\dots,a_n \ge 0$ *satisfy with* $c_0,\dots c_{n-1} \ge 0$ *and* $d_0,\dots d_{n-1} \ge 0$ *the inequality*

$$a_k \le a_{k-1} + c_{k-1} a_{k-1} + d_{k-1}\sqrt{a_{k-1}}, \quad k = 1,\dots,n.$$

Then it holds

$$a_k \le \left(\sqrt{a_0} + \frac{1}{2}\sum_{i=0}^{k-1} d_i\right)^2 \prod_{i=0}^{k-1} (1+c_i) \le \left(\sqrt{a_0} + \frac{1}{2}\sum_{i=0}^{k-1} d_i\right)^2 e^{\sum_{i=0}^{k-1} c_i}, \quad k = 1,\dots,n.$$

Proof. We show the first inequality by induction on k. For $k=1$:

$$a_1 \le a_0 + c_0 a_0 + d_0\sqrt{a_0} \le \left(\sqrt{a_0} + \frac{d_0}{2}\right)^2 + c_0 a_0 \le \left(\sqrt{a_0} + \frac{d_0}{2}\right)^2 (1+c_0).$$

Now if the inequality holds for $k-1$, we get

$$a_k \le a_{k-1} + c_{k-1} a_{k-1} + d_{k-1}\sqrt{a_{k-1}} \le \left(\sqrt{a_{k-1}} + \frac{d_{k-1}}{2}\right)^2 (1+c_{k-1})$$

$$\le \left(\left(\sqrt{a_0} + \frac{1}{2}\sum_{i=0}^{k-2} d_i\right) \prod_{i=0}^{k-2} \sqrt{1+c_i} + \frac{1}{2} d_{k-1}\right)^2 (1+c_{k-1}) \le \left(\sqrt{a_0} + \frac{1}{2}\sum_{i=0}^{k-1} d_i\right)^2 \prod_{i=0}^{k-1} (1+c_i),$$

so the first inequality is proven. To show the second one we conclude with $1+c_i \le e^{c_i}$. $\square$

In fact we will need a version with index on the right (that is, at the a_k and d_k coefficients) shifted by 1, which is needed due to using an implicit scheme in our approximation that we shall introduce.

Lemma 2.14 (Discrete Gronwall inequality with square root, shifted k)**.** *Let $a_0,\dots,a_n \geq 0$ satisfy with $0 \leq c_1,\dots,c_n < 1$ and $d_1,\dots d_n \geq 0$ the inequality*

$$a_k \leq a_{k-1} + c_k a_k + d_k\sqrt{a_k}, \quad k = 1,\dots,n.$$

Then it holds

$$a_k \leq \left(\sqrt{a_0} + \sum_{i=1}^{k} \frac{d_i}{\sqrt{1-c_i}}\right)^2 \prod_{i=1}^{k}(1-c_i)^{-1}, \quad k = 1,\dots,n.$$

In particular, for $0 < c_i \leq 1/2$ we have also

$$a_k \leq \left(\sqrt{a_0} + \sum_{i=1}^{k} \frac{d_i}{\sqrt{1-c_i}}\right)^2 \exp\left(2\sum_{i=1}^{k} c_i\right), \quad k = 1,\dots,n.$$

Proof. Proceed to show the first inequality by induction and assume it holds for $k-1$. For $k=0$ the inequality is trivially true. Rewrite the assumed inequality to

$$(1-c_k)a_k - d_k\sqrt{a_k} \leq a_{k-1},$$

so that by completing the square on the left we get

$$\left(\sqrt{1-c_k}\sqrt{a_k} - \frac{d_k}{2\sqrt{1-c_k}}\right)^2 \leq a_{k-1} + \frac{d_k^2}{4(1-c_k)}.$$

From this, we express a_k, and use $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$ to obtain

$$a_k \leq \left(\sqrt{a_{k-1} + \frac{d_k^2}{4(1-c_k)}} + \frac{d_k}{2\sqrt{1-c_k}}\right)^2 \frac{1}{1-c_k} \leq \left(\sqrt{a_{k-1}} + \frac{d_k}{\sqrt{1-c_k}}\right)^2 \frac{1}{1-c_k}.$$

Thus by induction

$$a_k \leq \left(\left(\sqrt{a_0} + \sum_{i=1}^{k-1} \frac{d_i}{\sqrt{1-c_i}}\right) \prod_{i=1}^{k-1} \sqrt{(1-c_i)^{-1}} + \frac{d_k}{\sqrt{1-c_k}}\right)^2 \frac{1}{1-c_k}$$
$$\leq \left(\sqrt{a_0} + \sum_{i=1}^{k} \frac{d_i}{\sqrt{1-c_i}}\right)^2 \prod_{i=1}^{k}(1-c_i)^{-1}$$

which proves the desired inequality. Finally, note that for $0 < c_i \leq 1/2$ it holds $(1-c_i)^{-1} \leq 1 + 2c_i \leq e^{2c_i}$ proving the second inequality. $\square$

Two-scale Gronwall inequalities. Here we state the two-scale analogues of Lemma 2.12 and Lemma 2.14, respectively. The particular form of the inequality is suitable to estimates of solutions arising from the minimization scheme (2.17).

Theorem 2.15 (Two–scale Gronwall inequality I)**.** *Let $M, N \in \mathbb{N}$ and let us have the sequences $a_k^\ell, b_k^\ell, d_k^\ell \geq 0$, $k = 0,\dots,N$, $\ell = 0,\dots,M-1$ satisfying $a_0^\ell = a_N^{\ell-1}$, $b_0^\ell = b_N^{\ell-1}$, $\ell = 1,\dots,M$. Assume we have for some $0 \leq c < 1/2N$ the estimate*

$$a_k^\ell + \frac{1}{N}b_k^\ell \leq a_{k-1}^\ell + \frac{1}{N}b_k^{\ell-1} + ca_k^\ell + cb_k^\ell + d_k^\ell, \quad k = 1,\dots,N,\ \ell = 0,\dots,M-1,$$

where we put $b_k^{-1} := b_0^0$, $k = 1,\dots,N$. Then it holds

$$\max_{k=1,\dots,N}\left(a_k^\ell + \frac{1}{N}\sum_{i=1}^{k} b_i^\ell\right) \leq \left(a_0^0 + b_0^0 + \sum_{l=1}^{\ell}\sum_{k=1}^{N} d_k^\ell\right)(1-cN)^{-\ell}, \quad \ell = 1,\dots,M-1.$$

Proof. Let $k_\ell := \operatorname{argmax}_{k=1,\dots,N} a_k^\ell + \frac{1}{N}\sum_{i=1}^{k} b_i^\ell$. Then we have after summing $1,\dots,k_\ell$

$$a_{k_\ell}^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^\ell \leq a_0^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^{\ell-1} + c\sum_{k=1}^{k_\ell} a_k^\ell + c\sum_{k=1}^{k_\ell} b_k^\ell + \sum_{k=1}^{k_\ell} d_k^\ell$$

Now first remember that $a_0^\ell = a_N^{\ell-1}$, denote $\alpha_\ell := a_{k_\ell}^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^\ell$ (so that $\alpha_\ell = \max_{k=1,\dots,N} a_k^\ell + \frac{1}{N}\sum_{i=1}^{k} b_i^\ell$). Use the inequalities

$$a_N^{\ell-1} + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^{\ell-1} \le a_N^{\ell-1} + \frac{1}{N}\sum_{k=1}^{N} b_k^{\ell-1} \le a_{k_{\ell-1}}^{\ell-1} + \frac{1}{N}\sum_{k=1}^{k_{\ell-1}} b_k^{\ell-1} = \alpha_{\ell-1}$$

$$c\sum_{k=1}^{k_\ell} b_k^\ell \le cN\alpha_\ell$$

$$c\sum_{k=1}^{k_\ell} a_k^\ell \le ck_\ell\alpha_\ell \le cN\alpha_\ell$$

$$\sum_{k=1}^{k_\ell} d_k^\ell \le \sum_{k=1}^{N} d_k^\ell$$

so it becomes

$$\alpha_\ell \le \alpha_{\ell-1} + 2cN\alpha_\ell + \sum_{k=1}^{N} d_k^\ell.$$

Summing this over ℓ we get

$$\alpha_\ell \le \alpha_0 + 2cN\sum_{l=1}^{\ell}\alpha_l + \sum_{l=1}^{\ell}\sum_{k=1}^{N} d_k^l$$

so applying Lemma 2.12 (using in this lemma a_0 as $\alpha_0 + \sum_{l=1}^{\ell}\sum_{k=1}^{N} d_k^l$) we see that

$$\alpha_\ell \le \left(\alpha_0 + \sum_{l=1}^{\ell}\sum_{k=1}^{N} d_k^l\right)(1-cN)^{-\ell},$$

which finishes the proof, since $\alpha_0 = a_0^0 + b_0^0$. □

Theorem 2.16 (Two-scale Gronwall inequality II)**.** *Let $M, N \in \mathbb{N}$ and let us have the sequences $a_k^\ell, b_k^\ell, d_k^\ell \ge 0$, $k = 0,\dots,N$, $\ell = 0,\dots,M-1$ satisfying $a_0^\ell = a_N^{\ell-1}$, $b_0^\ell = b_N^{\ell-1}$, $\ell = 1,\dots,M$. Assume we have for some $0 \le c < 1/N$ the estimate*

$$a_k^\ell + \frac{1}{N}b_k^\ell \le a_{k-1}^\ell + \frac{1}{N}b_k^{\ell-1} + cb_k^\ell + d_k^\ell\sqrt{b_k^\ell}, \quad k = 1,\dots,N,\ \ell = 0,\dots,M-1,$$

where we put $b_k^{-1} := b_0^0$, $k = 1,\dots,N$. Then it holds

$$\max_{k=1,\dots,N}\left(a_k^\ell + \frac{1}{N}\sum_{i=1}^{k} b_i^\ell\right) \le \left(\sqrt{a_0^0 + b_0^0} + \frac{1}{\sqrt{1-cN}}\sum_{l=1}^{\ell}\sqrt{N\sum_{k=1}^{N}(d_k^\ell)^2}\right)^2 (1-cN)^{-\ell}, \ell = 1,\dots,M-1.$$

Proof. Let $k_\ell := \operatorname{argmax}_{k=1,\dots,N} a_k^\ell + \frac{1}{N}\sum_{i=1}^{k} b_l^\ell$ for $\ell = 1,\dots,N$. Then we have after summing over $1,\dots,k_\ell$

$$a_{k_\ell}^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^\ell \le a_0^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^{\ell-1} + c\sum_{k=1}^{k_\ell} b_k^\ell + \sum_{k=1}^{k_\ell} d_k^\ell\sqrt{b_k^\ell}.$$

Now denote $\alpha_\ell := a_{k_\ell}^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^\ell$ (so that $\alpha_\ell = \max_{k=1,\dots,N} a_k^\ell + \frac{1}{N}\sum_{i=1}^{k} b_i^\ell$), remember that $a_0^\ell = a_N^{\ell-1}$ and we see

$$a_0^\ell + \frac{1}{N}\sum_{k=1}^{k_\ell} b_k^{\ell-1} \le a_N^{\ell-1} + \frac{1}{N}\sum_{k=1}^{N} b_k^{\ell-1} \le \alpha_{\ell-1}$$

$$c\sum_{k=1}^{k_\ell} b_k^\ell \le cN\alpha_\ell$$

$$\sum_{k=1}^{k_\ell} d_k^\ell\sqrt{b_k^\ell} \le \sqrt{N\sum_{k=1}^{k_\ell}(d_k^\ell)^2}\sqrt{\frac{1}{N}\sum_{k=1}^{k_\ell} b_k^\ell} \le \sqrt{N\sum_{k=1}^{N}(d_k^\ell)^2}\sqrt{\alpha_\ell},$$

so that in total the inequality reads

$$\alpha_\ell \leq \alpha_{\ell-1} + cN\alpha_\ell + \sqrt{N\sum_{k=1}^{N}(d_k^\ell)^2}\sqrt{\alpha_\ell}$$

Now we use the discrete square root Gronwall Lemma 2.14 with shifted index for α_ℓ. This yields

$$\alpha_\ell \leq \left(\sqrt{\alpha_0} + \frac{1}{\sqrt{1-cN}}\sum_{l=1}^{\ell}\sqrt{N\sum_{k=1}^{N}(d_k^l)^2}\right)^2 (1-cN)^{-\ell}$$

which is the desired inequality, since as before $\alpha_0 = a_0^0 + b_0^0$. □

2.6. **The numerical scheme and the definition of stability.** Let us now define an appropriate notion of stability for a scheme approximating the solution of (2.3). For this we will perform some heuristical formal a-priori estimates.

Assume formally that η is a solution and that $\partial_t\eta$ is an admissible test function. For the purpose of our formal estimates, assume that it holds

$$\langle DE(\eta), \partial_t\eta\rangle = \partial_t E(\eta). \tag{2.16}$$

This is a formal application of the chain rule. Then using a test function $\partial_t\eta$ gives

$$\frac{1}{2}\partial_t \|\partial_t\eta\|^2 + \langle DE(\eta), \partial_t\eta\rangle = \langle f, \partial_t\eta\rangle \leq \|f\|_{L^2}\|\partial_t\eta\|_{L^2}.$$

Using this, it follows using a square-root Gronwall type argument that

$$\frac{1}{2}\|\partial_t\eta(t)\|_{L^2}^2 + E(\eta(t)) \leq \left(\sqrt{\frac{1}{2}\|\eta_*\|_{L^2}^2 + E(\eta_0)} + \frac{1}{2}\int_0^t \|f\|_{L^2}\,\mathrm{d}t\right)^2$$

Accordingly we call an approximation stable if it satisfies an appropriate substitute of the above estimate. This is defined below.

Definition 2.17. Let $\tilde\eta$ be an approximation of the solution.[4] We say that the approximation is *stable* if it satisfies with some $C \geq 0$ an estimate

$$\frac{1}{2}\|\partial_t\tilde\eta(t)\|_{L^2}^2 + E(\tilde\eta(t)) \leq \left(\sqrt{\frac{1}{2}\|\eta_*\|_{L^2}^2 + E(\eta_0)} + C\int_0^t \|f\|_{L^2}\,\mathrm{d}t\right)^2.$$

Let us now introduce the **Minimizing movement scheme**, that approximates (2.3) with the appropriate stability. The minimizing movement scheme without dissipation is very similar and can be found in (2.26).

Consider the following time-stepping scheme, for the two time scales $0 < \tau \leq h$. We follow the scheme of [BKS23b] with the distinction that we keep our scheme discrete in h. For simplicity and ease of notation[5] assume $h = N\tau$ and $T = Mh$ with $N, M \in \mathbb{N}$. Define the discrete times $t_k^\ell := \ell h + k\tau$, notice in particular that $t_0^\ell = t_N^{\ell-1}$. For $k = 1, \ldots, N$ and $\ell = 0, \ldots, M$ define the approximation via the step-wise minimization of

$$\eta_k^\ell = \arg\min_{\eta\in\mathcal{E}} \frac{\tau h}{2}\left\|\frac{\frac{\eta-\eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}\right\|_{L^2}^2 + E(\eta) + \frac{c\tau^2}{2}\left\|\nabla\frac{\eta-\eta_{k-1}^\ell}{\tau}\right\|_{L^2}^2 - \langle f_k^\ell, \eta\rangle_{L^2}, \tag{2.17}$$

where we start from the initial conditions $\eta_0^0 := \eta_0$ and for $\ell = 0$, the fraction $\frac{\eta_k^{-1}-\eta_{k-1}^{-1}}{\tau}$ is replaced by η_*. Moreover we take $\eta_0^{\ell+1} := \eta_N^\ell$, since $t_0^{\ell+1} = t_N^\ell$. The constant c is a factor in front of the regularizer that is chosen large enough to compensate the non-convexity of the energy. It will be specified during the proof of Theorem 2.22. The term f_k^ℓ, a discretization of the right hand side, is defined as

$$f_k^\ell := \fint_0^\tau \fint_0^h f(t_{k-1}^{\ell-1} + s + \sigma)\,\mathrm{d}s\,\mathrm{d}\sigma. \tag{2.18}$$

[4]At this point we do not specify in which sense it is an approximation, apart from saying that $\tilde\eta$ lies in the correct space, that is (2.9).

[5]It is possible to include the cases that h resp. T is not an integer multiple of τ resp. h, and the resulting complications of this are essentially notational.

The reason for this particular choice of discretization of f will be apparent later in Section 3. The Euler-Lagrange equation of the minimizer indeed approximates the hyperbolic evolution, as can be seen from Lemma 2.19. In particular a stabilization term of the form $-c\tau\Delta\partial_t\hat{\eta}^{(h)}_{(\tau)}$ appears.

2.7. **A priori bounds on energy.**

Lemma 2.18. *The minimizer $\eta^\ell_k \in \mathcal{E}$ of (2.17) exists. If $\eta^\ell_k \in \partial\mathcal{E}$, then a (self-)collision occurred.*

Proof. Existence follows from the direct method. From the lower bound on the determinant (E.3) we see that we are away from the part of $\partial\mathcal{E}$ corresponding to the case when $\det\nabla\eta$ vanishes somewhere. In particular, since the minimizer satisfies $\det\nabla\eta^\ell_k > 0$, it is possible to take variations in all directions if and only if η^ℓ_k is injective on $\partial\Omega$ (this is well-known [Cia88], see also the discussion in [BKS23b]). In other words, it holds $\eta^\ell_k \in \operatorname{int}\mathcal{E}$ if and only if there is no (self-)collision. □

Now denote the piecewise constant approximation

$$\overline{\eta}^{(h)}_{(\tau)}(t) = \eta^\ell_k, \quad t \in [t^\ell_{k-1}, t^\ell_k)$$

and

$$\overline{f}^{(h)}_{(\tau)}(t) = f^\ell_k, \quad t \in [t^\ell_{k-1}, t^\ell_k).$$

Denote the piecewise affine approximation

$$\hat{\eta}^{(h)}_{(\tau)}(t) = \frac{t - t^\ell_{k-1}}{\tau}\eta^\ell_{k-1} + \frac{t^\ell_k - t}{\tau}\eta^\ell_k \quad \text{for } t \in [t^\ell_{k-1}, t^\ell_k),$$

so that $\hat{\eta}^{(h)}_{(\tau)}(t^\ell_k) = \eta^\ell_k$ and $\hat{\eta}^{(h)}_{(\tau)}$ is affine on each of the intervals $[t^\ell_{k-1}, t^\ell_k]$.

We now can see that this is a time-discrete solution of our problem (2.3), in the following sense

Lemma 2.19. *Assume that no collision happened, that is $\eta^\ell_k \in \operatorname{int}\mathcal{E}$ for all k and ℓ. Then it holds for a.a. times $t \in (0,T)$ that*

$$\frac{\partial_t\hat{\eta}^{(h)}_{(\tau)}(t) - \partial_t\hat{\eta}^{(h)}_{(\tau)}(t-h)}{h} + DE\left(\overline{\eta}^{(h)}_{(\tau)}(t)\right) - c\tau\Delta\partial_t\hat{\eta}^{(h)}_{(\tau)}(t) = f(t).$$

Proof. Since η^ℓ_k is a minimizer of (2.17) and it is an interior point, by (A.3) E is differentiable at η^ℓ_k, and we have the Euler-Lagrange equation

$$\frac{\frac{\eta^\ell_k - \eta^\ell_{k-1}}{\tau} - \frac{\eta^{\ell-1}_k - \eta^{\ell-1}_{k-1}}{\tau}}{h} + DE(\eta^\ell_k) - c\tau\frac{\eta^\ell_k - \eta^\ell_{k-1}}{\tau} = f^\ell_k.$$

Using the notation for the piecewise constant and piecewise affine interpolations above, observe that it holds

$$\partial_t\hat{\eta}^{(h)}_{(\tau)}(t) = \frac{\eta^\ell_k - \eta^\ell_{k-1}}{\tau}, \quad t \in (t^\ell_{k-1}, t^\ell_k).$$

From this we see that the stated equation is satisfied for $t \in (0,T) \setminus \{t^\ell_k : k = 1,\dots,N, \ell = 0,\dots M\}$, i.e. all times except finitely many. Thus we have proven the claim. □

Approximation of the right hand side. Now we verify that the discretization of the right hand side is well-behaved. We state two lemmas, first with only one scale, second with two scales that we use in our approximation.

Lemma 2.20. *Let $f \in L^p((0,T);X)$ with X a Banach space and $1 \le p < \infty$. Define*

$$\overline{f}^{(\tau)}(t) = f_k, t \in [t_{k-1}, t_k), \quad \text{where } f_k = ⨍_{t_{k-1}}^{t_k} f \,\mathrm{d}t.$$

Then $\left\|\overline{f}^{(\tau)}\right\|_{L^p((0,T);X)} \le \|f\|_{L^p((0,T);X)}$ and moreover $\overline{f}^{(\tau)} \to f$ in $L^p((0,T);X)$ as $\tau \to 0$.

Proof. For the first part, use the Jensen inequality

$$\left\|\overline{f}^{(\tau)}\right\|^p_{L^p((0,T);X)} = \sum_{k=1}^N \tau \left\| ⨍_{t_{k-1}}^{t_k} f(t)\,\mathrm{d}t \right\|^p_X \leq \sum_{k=1}^N \tau ⨍_{t_{k-1}}^{t_k} \|f(t)\|^p_X \,\mathrm{d}t = \|f\|^p_{L^p((0,T);X)}.$$

To prove the convergence, fix $\varepsilon > 0$ and find $g \in C([0,T];X)$ with $\|f-g\|_{L^p((0,T);X)} \leq \varepsilon$. By uniform continuity of g we find $\tau > 0$ such that $|t-s| < \tau$ implies $\|g(t)-g(s)\|_X \leq \varepsilon$. Then by Jensen inequality and the uniform continuity

$$\begin{aligned}\left\|g - \overline{g}^{(\tau)}\right\|^p_{L^p([0,T];X)} &= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left\| ⨍_{t_{k-1}}^{t_k} g(t) - g(s)\,\mathrm{d}s \right\|^p_X \mathrm{d}t \\ &\leq \sum_{k=1}^N \int_{t_{k-1}}^{t_k} ⨍_{t_{k-1}}^{t_k} \|g(t)-g(s)\|^p_X \,\mathrm{d}s\,\mathrm{d}t \leq T\varepsilon^p.\end{aligned}$$

Moreover by linearity $\overline{f}^{(\tau)} - \overline{g}^{(\tau)} = \overline{(f-g)}^{(\tau)}$, so $\left\|\overline{f}^{(\tau)} - \overline{g}^{(\tau)}\right\|^p_{L^p((0,T);X)} \leq \|f-g\|^p_{L^p((0,T);X)}$ by the first part. We conclude the proof by the triangle inequality

$$\begin{aligned}\left\|f - \overline{f}^{(\tau)}\right\|_{L^p((0,T);X)} &\leq \|f-g\|_{L^p((0,T);X)} + \left\|g - \overline{g}^{(\tau)}\right\|_{L^p((0,T);X)} + \left\|\overline{g}^{(\tau)} - \overline{f}^{(\tau)}\right\|_{L^p((0,T);X)} \\ &\leq \varepsilon + \sqrt[p]{T}\varepsilon + \varepsilon.\end{aligned}$$

□

Lemma 2.21. *Let $f \in L^p((0,T);X)$ with X a Banach space and $1 \leq p < \infty$. Extend f by 0 outside $(0,T)$ and define*

$$f_k^\ell := ⨍_0^\tau ⨍_0^h f(t_{k-1}^{\ell-1} + s + \sigma)\,\mathrm{d}s\,\mathrm{d}\sigma, \quad \overline{f}^{(h)}_{(\tau)}(t) := f_k^\ell, \quad t \in [t_{k-1}^\ell, t_k^\ell).$$

Then $\left\|\overline{f}^{(h)}_{(\tau)}\right\|_{L^p((0,T);X)} \leq \|f\|_{L^p((0,T);X)}$ and moreover $\overline{f}^{(h)}_{(\tau)} \to f$ in $L^p((0,T);X)$ as $h, \tau \to 0$.

Proof. For the first claim, use twice Jensen inequality as follows

$$\begin{aligned}\left\|\overline{f}^{(h)}_{(\tau)}\right\|^p_{L^p((0,T);X)} &= \sum_{\ell=1}^M \sum_{k=1}^N \tau \left\|f_k^\ell\right\|^p_X = \sum_{\ell=1}^M \sum_{k=1}^N \tau \left\| ⨍_0^\tau ⨍_0^h f(t_{k-1}^{\ell-1} + s + \sigma)\,\mathrm{d}s\,\mathrm{d}\sigma \right\|^p_X \\ &\leq \sum_{\ell=1}^M \sum_{k=1}^N \int_0^\tau \left\| ⨍_0^h f(t_{k-1}^{\ell-1} + s + \sigma)\,\mathrm{d}s \right\|^p_X \mathrm{d}\sigma = \sum_{\ell=1}^M \sum_{k=1}^N \int_{t_{k-1}^{\ell-1}}^{t_k^{\ell-1}} \left\| ⨍_0^h f(t+s)\,\mathrm{d}s \right\|^p_X \mathrm{d}t \\ &= \int_0^T \left\| ⨍_0^h f(t+s)\,\mathrm{d}s \right\|^p_X \mathrm{d}t \leq \int_0^T ⨍_0^h \|f(t+s)\|^p_X \,\mathrm{d}s\,\mathrm{d}t = ⨍_0^h \int_0^T \|f(t+s)\|^p_X \,\mathrm{d}t\,\mathrm{d}s \\ &\leq ⨍_0^h \|f\|^p_{L^p((0,T);X)} \,\mathrm{d}s = \|f\|^p_{L^p((0,T);X)}.\end{aligned}$$

Fix $\varepsilon > 0$. Find $g \in C([0,T];X)$ with $\|f-g\|_{L^p((0,T);X)} \leq \varepsilon$. Then, by uniform continuity of g, find $h_0 > 0$ such that for all $|t-s| \leq h_0$ it holds $\|g(t)-g(s)\|_X \leq \varepsilon$. Then, using this and Jensen inequality, for $\tau \leq h \leq h_0$

$$\begin{aligned}\left\|g - \overline{g}^{(h)}_{(\tau)}\right\|^p_{L^p((0,T);X)} &= \sum_{\ell=1}^M \sum_{k=1}^N \int_{t_{k-1}^\ell}^{t_k^\ell} \left\| ⨍_0^\tau ⨍_0^h g(t) - g(t_{k-1}^{\ell-1} + s + \sigma)\,\mathrm{d}s\,\mathrm{d}\sigma \right\|^p_X \mathrm{d}t \\ &\leq \sum_{\ell=1}^M \sum_{k=1}^N \int_{t_{k-1}^\ell}^{t_k^\ell} ⨍_0^\tau ⨍_0^h \left\|g(t) - g(t_{k-1}^{\ell-1} + s + \sigma)\right\|^p_X \mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \leq T\varepsilon^p.\end{aligned}$$

Further, by linearity $\overline{f}^{(h)}_{(\tau)} - \overline{g}^{(h)}_{(\tau)} = \overline{(f-g)}^{(h)}_{(\tau)}$ so by the first part $\left\|\overline{f}^{(h)}_{(\tau)} - \overline{g}^{(h)}_{(\tau)}\right\|^p_{L^p((0,T);X)} \leq \|f-g\|^p_{L^p((0,T);X)}$. We then conclude the proof by the triangle inequality

$$\left\|\overline{f}^{(h)}_{(\tau)} - f\right\|_{L^p((0,T);X)} \leq \left\|\overline{f}^{(h)}_{(\tau)} - \overline{g}^{(h)}_{(\tau)}\right\|_{L^p((0,T);X)} + \left\|\overline{g}^{(h)}_{(\tau)} - g\right\|_{L^p((0,T);X)} + \|g-f\|_{L^p((0,T);X)}$$

$$\leq \varepsilon + \sqrt[p]{T}\varepsilon + \varepsilon.$$

□

2.8. **Stability – elastic solid.** We can now state our main stability result for the approximation (2.17). Note that we have an energy estimate with bound given by the initial condition and the right hand side, and in particular the energy can not grow in time in case no right hand side is considered.

Theorem 2.22 (Stability with dissipation)**.** *There exists an $h_0 > 0$ and $c > 0$ depending on $E(\eta_0)$, $\|\eta_*\|_{L^2(Q)}$, $\|f\|_{L^2((0,T)\times Q)}$, the assumptions on E and T, such that for all $N\tau = h \leq h_0$ if the corresponding approximation η_k^ℓ does not reach a collision, i.e. satisfies $\eta_k^\ell \in \operatorname{int}\mathcal{E}$ for all k and ℓ, then the following stability estimate holds for all ℓ*

$$\max_{k=1,\dots,N}\left(E(\eta_k^\ell) + \frac{1}{N}\sum_{i=1}^{k}\left\|\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau}\right\|_{L^2}^2\right) \leq \left(\sqrt{E(\eta_0) + \frac{1}{2}\|\eta_*\|_{L^2}^2} + \|f\|_{L^2((0,h\ell)\times Q)}\right)^2. \tag{2.19}$$

Proof. To ease the notation, the norm $\|\cdot\|$ without any index is the $L^2(Q)$ norm, and $\langle\cdot,\cdot\rangle$ is the $L^2(Q)$ scalar product (or dual pairing of X and X^* in the DE terms).

We proceed by induction on ℓ. Thus assume that the inequality (2.19) holds for $\ell - 1$ (and every k), and we want to prove it for ℓ. We first show the following auxiliary estimate for $k = 1, \dots, N$:

$$E(\eta_k^\ell) \leq K := \left(\sqrt{E(\eta_0) + \frac{1}{2}\|\eta_*\|_{L^2}^2} + \|f\|_{L^2((0,T)\times Q)}\right)^2 + h_0\|f\|_{L^2((0,T)\times Q)}^2. \tag{2.20}$$

For this, take η_{k-1}^ℓ as competitor in (2.17), which implies

$$\frac{\tau h}{2}\left\|\frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}\right\|_{L^2}^2 + E(\eta_k^\ell) + \frac{c\tau^2}{2}\left\|\nabla\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau}\right\|_{L^2}^2$$
$$\leq \frac{\tau h}{2}\left\|\frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau h}\right\|^2 + E(\eta_{k-1}^\ell) + \langle f_k^\ell, (\eta_k^\ell - \eta_{k-1}^\ell)\rangle_{L^2}.$$

We then find that

$$\frac{\tau}{2h}\left\|\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\|_{L^2}^2 + E(\eta_k^\ell) + \frac{c\tau^2}{2}\left\|\nabla\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau}\right\|_{L^2}^2$$
$$\leq \frac{\tau}{2h}\left\|\frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\|^2 + E(\eta_{k-1}^\ell) + \langle f_k^\ell, (\eta_k - \eta_{k-1})\rangle_{L^2}$$
$$\leq \frac{\tau}{2h}\left\|\frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\|^2 + E(\eta_{k-1}^\ell) + \left\|f_k^\ell\right\|\tau\left\|\frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\| + \left\|f_k^\ell\right\|\tau\left\|\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\|$$
$$\leq \frac{\tau}{h}\left\|\frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\|^2 + E(\eta_{k-1}^\ell) + h\tau\left\|f_k^\ell\right\|^2 + \frac{\tau}{2h}\left\|\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}\right\|^2.$$

Subtracting $\frac{\tau}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}\right\|^2$ on both sides and summing over k, we find that the $E(\eta_k^\ell)$ term telescopes, so that

$$E(\eta_k^\ell) + \sum_{i=1}^{k}\frac{c\tau^2}{2}\left\|\nabla\frac{\eta_i^\ell - \eta_{i-1}^\ell}{\tau}\right\|^2 \leq E(\eta_0^\ell) + \sum_{i=1}^{k}\frac{\tau}{h}\left\|\frac{\eta_i^{\ell-1} - \eta_{i-1}^{\ell-1}}{\tau}\right\|^2 + h\sum_{i=1}^{k}\tau\left\|f_i^\ell\right\|^2.$$

Now remember that we have $\eta_0^\ell = \eta_N^{\ell-1}$ and (2.19) with $\ell - 1$ holds, so after a Jensen inequality $h\sum_{i=1}^k \tau\left\|f_i^\ell\right\|^2 \leq \|f\|_{L^2((\ell-1)h,\ell h)\times Q)}^2$ and $0 < h \leq h_0$ we show (2.20).

Equipped with this, we come to the main estimate. Since we assume that $\eta_k^\ell \in \operatorname{int}\mathcal{E}$, E is differentiable at η_k^ℓ and we can test the minimizer with $\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}$, which gives

$$\left\langle \frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle + \left\langle DE(\eta_k^\ell), \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle + c\tau\left\|\nabla\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 = \left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle.$$

Using $a(a-b)=\frac{a^2}{2}-\frac{b^2}{2}+\frac{(a-b)^2}{2}$, we obtain for the first term

$$\begin{aligned}&\left\langle \frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle \\ &\qquad = \frac{1}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 + \frac{1}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}\right\|^2 - \frac{1}{2h}\left\|\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}\right\|^2.\end{aligned}$$

Now multiply by τ, omit the middle term, use Lemma 2.6 (note carefully that by (2.20) the resulting constant C_1 is independent of ℓ and k), and obtain

$$\begin{aligned}&\frac{\tau}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 + E(\eta_k^\ell) + c\tau^2\left\|\nabla\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 \\ &\qquad \le \frac{\tau}{2h}\left\|\frac{\eta_k^{\ell-1}-\eta_k^{\ell-1}}{\tau}\right\|^2 + E(\eta_{k-1}^\ell) + \tau\left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle + C_1\tau^2\left\|\nabla\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2,\end{aligned}$$

where we precisely here choose $c>2C_1$. Hence

$$\begin{aligned}&\frac{\tau}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 + E(\eta_k^\ell) + C_1\tau^2\left\|\nabla\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 \\ &\qquad \le \frac{\tau}{2h}\left\|\frac{\eta_k^{\ell-1}-\eta_k^{\ell-1}}{\tau}\right\|^2 + E(\eta_{k-1}^\ell) + \tau\left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle.\end{aligned} \tag{2.21}$$

Using the inequality

$$\tau\left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle \le \tau\left\|f_k^\ell\right\|\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|$$

we get, denoting $a_k^\ell = E(\eta_k^\ell)$, $b_k^\ell=\frac{1}{2}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2$, $d_k^\ell=\tau\left\|f_k^\ell\right\|$ and $N=h/\tau$ the inequality

$$a_k^\ell + \frac{1}{N}b_k^\ell \le a_{k-1}^\ell + \frac{1}{N}b_k^{\ell-1} + d_k^\ell\sqrt{b_k^\ell}.$$

Now we are in a position to use the Two-scale Gronwall inequality with square root, Theorem 2.16. Thus, we obtain

$$\max_{k=1,\dots,N}\left(a_k^\ell + \frac{1}{N}\sum_{i=1}^k b_i^\ell\right) \le \left(\sqrt{a_0^0+b_0^0} + \sum_{l=1}^\ell\sqrt{N\sum_{k=1}^N (d_k^\ell)^2}\right)^2, \quad \ell=1,\dots,M-1.$$

So this reads

$$\max_{k=1,\dots,N}\left(E(\eta_k^\ell) + \frac{1}{N}\sum_{i=1}^k\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2\right) \le \left(\sqrt{E(\eta_0)+\frac{1}{2}\|\eta_*\|^2} + \sum_{l=1}^\ell h\sqrt{\frac{1}{N}\sum_{k=1}^N\|f_k^\ell\|^2}\right)^2.$$

Finally we use Lemma 2.21, from which we get by Jensen inequality

$$\sum_{l=1}^\ell h\sqrt{\frac{1}{N}\sum_{k=1}^N\|f_k^\ell\|^2} \le \|f\|_{L^2((0,h\ell)\times Q)},$$

so that the induction on ℓ is finished. □

We also note in the following lemma that we can read an estimate on the additional dissipation term.

Lemma 2.23 (Dissipation estimate)**.** *Further, the approximation satisfies*

$$c\tau \left\| \partial_t \nabla \hat{\eta}^{(h)}_{(\tau)} \right\|^2_{L^2((0,T)\times Q)} = c\tau \sum_{\ell=1}^{M} \sum_{k=1}^{N} \tau \left\| \nabla \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|^2 \leq C \tag{2.22}$$

with C depending on η_0, η_*, f.

Proof. After summing (2.21) over k and ℓ, we obtain

$$C_1\tau \sum_{\ell=1}^{M} \sum_{k=1}^{N} \tau \left\| \nabla \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|^2 \leq \frac{1}{2}\|\eta_*\|^2 + E(0) - E_{\min} + \sum_{\ell=1}^{M} \sum_{k=1}^{N} \tau \left\langle f_k^\ell, \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\rangle.$$

The last term is estimated by

$$\sum_{\ell=1}^{M} \sum_{k=1}^{N} \tau \left\langle f_k^\ell, \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\rangle \leq \sum_{\ell=1}^{M} h \sqrt{\frac{1}{N}\sum_{k=1}^{N} \|f_k^\ell\|^2} \sqrt{\frac{1}{N}\sum_{k=1}^{N} \left\| \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|^2}.$$

Since by the stability estimate 2.22 the last term is bounded by a constant, we are finished. □

2.9. **Stability – general.** For the general case, namely assuming Assumptions 2.2, the minimizing movement approximation (2.17) becomes

$$\eta_k^\ell = \arg\min_{\eta\in\mathcal{E}} \frac{\tau h}{2} \left\| \frac{\frac{\eta-\eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h} \right\|_H^2 + E(\eta) + \frac{c\tau^2}{2} \left\langle L\frac{\eta-\eta_k^{\ell-1}}{\tau}, \frac{\eta-\eta_k^{\ell-1}}{\tau} \right\rangle_Z - \langle f_k^\ell, \eta\rangle_H, \tag{2.23}$$

It is readily seen that the argument above for the elastic solids goes through in the general case. We thus have the following generalized version of Theorem 2.22, including Lemma 2.23.

Theorem 2.24 (Stability with dissipation, general case)**.** *There exists a $h_0 > 0$ and $c_0 > 0$ depending on $E(\eta_0)$, $\|\eta_*\|_H$, $\|f\|_{L^2((0,T);H)}$, the assumptions on E and T such that for all $N\tau = h \leq h_0$ if $c > c_0$ in* (2.23) *and the corresponding approximation η_k^ℓ satisfies $\eta_k^\ell \in \operatorname{int}\mathcal{E}$ we have the stability estimate*

$$\max_{k=1,\dots,N} \left(E(\eta_k^\ell) + \frac{1}{2N}\sum_{i=1}^{k} \left\| \frac{\eta_i^\ell - \eta_{i-1}^\ell}{\tau} \right\|_H^2 \right) \leq \left(\sqrt{E(\eta_0) + \frac{1}{2}\|\eta_*\|_H^2} + \|f\|_{L^2((0,\ell h);H)} \right)^2,$$

further the approximation satisfies

$$c\lambda\tau \sum_{\ell=1}^{M} \sum_{k=1}^{N} \tau \left\| \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|_Z^2 \leq C \tag{2.24}$$

with C depending on η_0, η_*, f.

2.10. **Stability – ODE.** It is worth mentioning that second-order ordinary differential equations fit into our framework, and therefore our approximation is also stable here.

Namely, consider the equation with unknown $x\colon (0,T) \to \mathbb{R}^n$

$$\begin{aligned} x'' + \nabla E(x) &= f, \\ x(t) &\in \mathcal{E}, \\ x(0) &= x_0, \\ x'(0) &= x_*, \end{aligned} \tag{2.25}$$

where $f \in L^2((0,T);\mathbb{R}^n)$, $x_0, x_* \in \mathbb{R}^n$, $E\colon \mathbb{R}^n \to (-\infty,\infty]$ and $\mathcal{E} \subset \mathbb{R}^n$ closed with nonempty interior (and $X = H = \mathbb{R}^n$).

Moreover let $E \in C(\mathbb{R}^n, (-\infty,\infty]) \cap C^2(\mathcal{E})$ and $\mathcal{E} \subset \{z \in \mathbb{R}^n : E(z) < \infty\}$ and E be coercive in the sense $\lim_{|x|\to\infty} E(x) = \infty$. We only need to show the non-convexity estimate (A.5), the rest of the properties (A.1)–(A.4) are clear.

Lemma 2.25. *The function E satisfies the non-convexity estimate: For all $x, y \in \mathcal{E}$ we find*

$$\nabla E(y) \cdot (y - x) \geq E(y) - E(x) - C|x - y|^2,$$

where C depends on $K = \max(E(x), E(y))$.

Proof. Let $x, y \in \mathcal{E}$. Given K from the statement, let us introduce a cutoff $\tilde{E}$ of E such that $\tilde{E} \colon \mathbb{R}^n \to \mathbb{R}$, $\tilde{E} \in C^2(\mathbb{R}^n)$, $\tilde{E}(z) = E(z)$ for all $z \in \mathcal{E}$ with $E(z) \leq K$, and such that the $\|\tilde{E}\|_{C^2(\mathbb{R}^n)}$ norm depends only on K. This we can certainly achieve by smoothly cutting off E near the compact set $\{z \in \mathcal{E} : E(z) \leq K\}$, which depends only on K.

Then we find that for some z on the line segment between x and y

$$\tilde{E}(x) = \tilde{E}(y) + \nabla \tilde{E}(y) \cdot (x - y) + \frac{1}{2} \nabla^2 \tilde{E}(z) : (y - x) \otimes (y - x)$$

Since $E = \tilde{E}$ on a neighbourhood of x and y, we find that

$$\nabla E(y) \cdot (y - x) \geq E(y) - E(x) - \frac{1}{2} \|\tilde{E}\|_{C^2(\mathbb{R}^n)} |y - x|^2.$$

Since we have chosen $\tilde{E}$ such that $\|\tilde{E}\|_{C^2(\mathbb{R}^n)}$ depends only on K, this finishes the proof. □

2.11. **Stability estimate without stabilizing term–elastic solids.** In the approximation (2.17) we have used a stabilizing dissipation term. In this section, we show that if we assume the stronger convexity assumption (E.5'), we obtain a stability estimate without introducing the stabilizing dissipation term. Although this estimate is weaker than the previous one, we consider it possibly of independent interest due to easier implementation of this scheme.
Therefore, throughout this section, we assume (E.5') instead of (E.5), respectively (A.5') instead of (A.5).

The time-steping scheme (2.17) is now replaced by the following minimization:

$$\eta_k^\ell = \arg\min_{\eta \in \mathcal{E}} \frac{\tau h}{2} \left\| \frac{\frac{\eta - \eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}}{h} \right\|_{L^2}^2 + E(\eta) - \langle f_k^\ell, \eta \rangle, \tag{2.26}$$

where as before $\eta_0^0 := \eta_0$ and for $\ell = 0$, the fraction $\frac{\eta_k^{-1} - \eta_{k-1}^{-1}}{\tau}$ is replaced by η_*. Moreover we take $\eta_0^{\ell+1} := \eta_N^\ell$, since $t_0^{\ell+1} = t_N^\ell$. The term f_k^ℓ is as before defined by (2.18).

Analogously as before we find (denoting the piecewise constant/affine interpolations as before) that an approximation can be constructed.

Lemma 2.26. *The minimizer $\eta_k^\ell \in \mathcal{E}$ exists. If $\eta_k^\ell \in \partial\mathcal{E}$, then a (self-)collision occurred. Assume that no collision happened, that is $\eta_k^\ell \in \operatorname{int}\mathcal{E}$ for all k and ℓ. Then it holds for a.a. times $t \in (0, T)$ that*

$$\frac{\partial_t \hat{\eta}_{(\tau)}^{(h)}(t) - \partial_t \hat{\eta}_{(\tau)}^{(h)}(t - h)}{h} + DE\left(\overline{\eta}_{(\tau)}^{(h)}(t)\right) = f(t).$$

This is analogous to Lemma 2.19, only the stabilizing term $-c\tau\Delta\partial_t \eta_{(\tau)}^{(h)}(t)$ is now missing. Now we present the proof of the stability in this case. Notice in particular on the right hand side, the term $(1 + 4C\tau h\ell)$ which linearly depends on τ and approaches 1 with $\tau \to 0$.

Theorem 2.27 (Stability for elastic solid)**.** *There exists a $h_0 > 0$ and $C > 0$ depending on $E(\eta_0)$, $\|\eta_*\|_{L^2(Q)}$, $\|f\|_{L^2((0,T)\times Q)}$, the assumptions on E and T, such that for all $N\tau = h \leq h_0$ with $h\ell \leq T$ the following holds: If the corresponding approximation η_k^ℓ does not reach a collision, i.e. it satisfies $\eta_k^\ell \in \operatorname{int}\mathcal{E}$ for all k and ℓ, then the following stability estimate holds*

$$\max_{k=1,\dots,N} \left(E(\eta_k^\ell) + \frac{1}{N} \sum_{i=1}^k \left\| \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|_{L^2}^2 \right) \leq \left(\sqrt{E(\eta_0) + \frac{1}{2} \|\eta_*\|_{L^2}^2} + \|f\|_{L^2((0,h\ell)\times Q)} \right)^2 (1 + 4C\tau h\ell). \tag{2.27}$$

Proof. As before to ease the notation, the norm $\|\cdot\|$ without any index is the $L^2(Q)$ norm, and $\langle \cdot, \cdot \rangle$ is the $L^2(Q)$ scalar product (or dual pairing of X and X^* in the DE terms).

We proceed by induction on ℓ. Thus assume that the inequality (2.27) holds for $\ell-1$ (and every k), and we want to prove it for ℓ. Analogous to the damped case we need the following auxiliary estimate for $k=1,\dots,N$:

$$E(\eta_k^\ell) \le K := (1+4C\tau T)\left(\sqrt{E(\eta_0)+\frac{1}{2}\|\eta_*\|_{L^2}^2} + \|f\|_{L^2((0,T)\times Q)}\right)^2 + h_0\|f\|^2_{L^2((0,T)\times Q)}. \quad (2.28)$$

But this estimate does however follow line by line by the argument in the proof of Theorem 2.22. Indeed, the damping part is not used for any of the absorbed terms.

Hence we assume that $\eta_k^\ell \in \operatorname{int}\mathcal{E}$, with E is differentiable at η_k^ℓ and we can test the minimizer with a uniform bound. Now taking $\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}$ as a test function gives

$$\left\langle \frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle + \left\langle DE(\eta_k^\ell), \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle = \left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle.$$

Using $a(a-b)=\frac{a^2}{2}-\frac{b^2}{2}+\frac{(a-b)^2}{2}$, we obtain for the first term

$$\left\langle \frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle$$
$$= \frac{1}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 + \frac{1}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}\right\|^2 - \frac{1}{2h}\left\|\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}\right\|^2.$$

Now multiply by τ, omit the middle term, use the non convexity estimate from Lemma 2.8 (again note that C is independent of k, ℓ by (2.28)) and obtain

$$\frac{\tau}{2h}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2 + E(\eta_k^\ell) \le \frac{\tau}{2h}\left\|\frac{\eta_k^{\ell-1}-\eta_k^{\ell-1}}{\tau}\right\|^2 + E(\eta_{k-1}^\ell) + \tau\left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle + C\tau^2\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2$$

where the constant C depends only on K.

Using the inequality

$$\tau\left\langle f_k^\ell, \frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\rangle \le \tau\left\|f_k^\ell\right\|\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|$$

we get, denoting $a_k^\ell = E(\eta_k^\ell)$, $b_k^\ell = \frac{1}{2}\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2$, $c = 2C\tau^2$, $d_k^\ell = \tau\left\|f_k^\ell\right\|$ and $N = h/\tau$ the inequality

$$a_k^\ell + \frac{1}{N}b_k^\ell \le a_{k-1}^\ell + \frac{1}{N}b_k^{\ell-1} + cb_k^\ell + d_k^\ell\sqrt{b_k^\ell}.$$

Now we are in a position to use the Two-scale Gronwall inequality with square root, Theorem 2.16. Thus, provided $c < \frac{1}{N}$ (i.e. $\tau h < \frac{1}{2C}$, which is guaranteed by $h < h_0 := \sqrt{2C}$), we obtain

$$\max_{k=1,\dots,N}\left(a_k^\ell + \frac{1}{N}\sum_{i=1}^k b_i^\ell\right) \le \left(\sqrt{a_0^0+b_0^0} + \frac{1}{\sqrt{1-cN}}\sum_{l=1}^\ell\sqrt{N\sum_{k=1}^N (d_k^\ell)^2}\right)^2 (1-cN)^{-\ell}, \quad \ell = 1,\dots,M-1.$$

Further, for $c \le \frac{1}{2N}$ (i.e. $\tau h \le \frac{1}{4C}$, which is guaranteed by $h \le h_0 := \frac{1}{2\sqrt{C}}$) we have $(1-cN)^{-\ell} \le 1+2cN\ell$, so that this implies

$$\max_{k=1,\dots,N}\left(E(\eta_k^\ell) + \frac{1}{N}\sum_{i=1}^k\left\|\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}\right\|^2\right) \le \left(\sqrt{E(\eta_0)+\frac{1}{2}\|\eta_*\|^2} + \sum_{l=1}^\ell h\sqrt{\frac{1}{N}\sum_{k=1}^N\|f_k^\ell\|^2}\right)^2 (1+4C\tau h\ell).$$

Finally using Lemma 2.21, from which we know, using a Jensen inequality

$$\sum_{l=1}^\ell h\sqrt{\frac{1}{N}\sum_{k=1}^N\|f_k^\ell\|^2} \le \sqrt{\sum_{l=1}^\ell\sum_{k=1}^N\tau\|f_k^\ell\|^2} \le \|f\|_{L^2((0,h\ell)\times Q)}$$

the proof is finished.

□

2.12. **Stability estimate without stabilizing term–general.** The conditions in Assumption 2.2, replacing $(A.5)$ by $(A.5')$ are chosen precisely in such a way that we have the same stability result as in the previous section. Hence the proof is the same as the proof for elastic solids after making the obvious changes, therefore we omit it. And we reach with the following stability estimate:

Theorem 2.28. *There exists a $h_0 > 0$ and $C > 0$ depending on $E(\eta_0)$, $\|\eta_*\|_H$, $\|f\|_{L^2((0,T);H)}$, the assumptions on E and T such that for all $N\tau = h \le h_0$ if the corresponding approximation η_k^ℓ satisfies $\eta_k^\ell \in \operatorname{int}\mathcal{E}$ we have the stability estimate*

$$\max_{k=1,\dots,N}\left(E(\eta_k^\ell) + \frac{1}{2N}\sum_{i=1}^{k}\left\|\frac{\eta_i^\ell - \eta_{i-1}^\ell}{\tau}\right\|_H^2\right) \le \left(\sqrt{E(\eta_0) + \frac{1}{2}\|\eta_*\|_H^2} + \frac{1}{2}\|f\|_{L^2((0,T);H)}\right)^2 (1{+}4C\tau h\ell),$$

for all ℓ, with $\ell h \le T$.

Remark. Another possibility, in case (E.5') holds, we can achieve stabilization by using $c\tau\partial_t\hat{\eta}^{(h)}_{(\tau)}$, that is the minimization

$$\eta_k^\ell = \arg\min_{\eta\in\mathcal{E}} \frac{\tau h}{2}\left\|\frac{\frac{\eta-\eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}\right\|_{L^2}^2 + E(\eta) + \frac{c\tau^2}{2}\left\|\frac{\eta-\eta_k^{\ell-1}}{\tau}\right\|^2 - \langle f_k^\ell, \eta\rangle.$$

The proof of the stability goes through the same way as in Theorem 2.22, utilizing the estimate of Lemma 2.8 instead of Lemma 2.6, and gives same estimate as in Theorem 2.22.

3. Convergence rate

Now our focus will be on quantifying the convergence results, namely showing that the under some regularity conditions, our scheme *converges to the solution with a linear rate.* Here we focus on the model case of elastic energies. This means we stick to Assumptions 2.4. Further, throughout the entire section we assume the $W^{k_0,2}$-case, that is

$$X = W^{k_0,2}(Q), \quad \text{and} \quad E_2(\eta) = \frac{1}{2}\left\|\nabla^{k_0}\eta\right\|_{L^2(Q)}^2. \tag{3.1}$$

The convergence analysis here can also be applied to more abstract settings. For that one needs to assume besides Assumption 2.2 that the estimates in Lemma 3.6 and Lemma 3.7 have to be satisfied. In particular this follows when additionally to Assumption 2.2 the following two assumptions are made:

(A.6) The limit solution η satisfies

$$\left\|\fint_0^a DE(\eta(t+s)) - DE(\eta(t))\,\mathrm{d}s\right\|_{X^*} \le Ca$$

for all $t \in [0, T-a]$ and $a \in [0, \tau_0]$.

(A.7) For all $\alpha, \beta \in \mathcal{E}$, $\gamma \in X$

$$\langle DE(\alpha) - DE(\beta)), \gamma\rangle \le C\|\alpha-\beta\|_X \|\gamma\|_H$$

are satisfied.

Please note that these assumptions are true for a large class of problems including the ODE examples in the numeric section.

3.1. **Time-regularity.** In order to indicate the validity of the convergence analysis, we first include here some higher order a-priori estimates for smooth solutions. Our technique is to introduce the dissipation term of the form $\varepsilon(-\Delta)^{k_0}\partial_t\eta_\varepsilon$ and then remove the term after having obtained uniform estimates.

We have, for each $\varepsilon > 0$, given

$$\eta_0 \in \mathcal{E}, \quad \eta_* \in L^2(Q), \quad f \in L^2((0,T);L^2(Q)), \tag{3.2}$$

a solution η_ε of

$$\begin{aligned}\partial_{tt}\eta_\varepsilon + (-\Delta)^{k_0}\eta_\varepsilon + \varepsilon(-\Delta)^{k_0}\partial_t\eta_\varepsilon - \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)) &= f\\ \eta_\varepsilon(t) &\in \mathcal{E}, \quad t\in(0,T)\\ \eta_\varepsilon(0) &= \eta_0\\ \partial_t\eta_\varepsilon(0) &= \eta_*\\ \eta_\varepsilon(t,x) &= x, \quad x\in\Gamma_D\\ \partial_\nu\eta_\varepsilon(t,x) &= \nu(x), \quad x\in\Gamma_N.\end{aligned}\tag{3.3}$$

provided no collision happens in the time interval $(0,T)$.

By the previous, we have the existence of η_ε with

$$\eta_\varepsilon\in L^\infty((0,T);W^{k_0,2}(Q)),\quad \partial_t\eta_\varepsilon\in L^\infty((0,T);L^2(Q)),\quad \partial_t\eta_\varepsilon\in L^2((0,T);W^{k_0,2}(Q))$$

satisfying the estimates

$$\|\eta_\varepsilon\|_{L^\infty((0,T);W^{k_0,2}(Q))}, \|\partial_t\eta_\varepsilon\|_{L^\infty((0,T);L^2(Q))}, \sqrt{\varepsilon}\,\|\partial_t\eta_\varepsilon\|_{L^2((0,T);W^{k_0,2}(Q))} \le C(\eta_0,\eta_*,f),\tag{3.4}$$

here and further $C(\dots)$ is a constant depending on the parameters in parenthesis. Indeed, one can use the approximation

$$\eta_k^\ell = \operatorname*{arg\,min}_{\eta\in\mathcal{E}} \frac{\tau h}{2}\left\|\frac{\frac{\eta-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}\right\|_{L^2}^2 + \varepsilon\frac{\tau}{2}\left\|\nabla^{k_0}\frac{\eta-\eta_{k-1}^\ell}{\tau}\right\|_{L^2}^2 + E(\eta) - \langle f_k^\ell,\eta\rangle,$$

with the same initial conditions as used in (2.17). Following step by step the argument in Theorem 2.27, uniform stability estimates are available with the dissipation additionally appearing on the left hand side. Hence by the usual weak compactness results and compact embeddings a solution to (3.3) can be established by letting $\tau,h\to 0$.

We now investigate higher time regularity, given regularity of initial conditions and the right hand side. In particular we note which estimate do or do not depend on ε. For ease of notation let us denote

$$W_D^{k_0,2}(Q) := \{\eta\in W^{k_0,2}(Q) : \eta(x)=x \text{ for } x\in\Gamma_D\}.$$

We begin with the following lemma.

Lemma 3.1. *The following problem for β*

$$\begin{aligned}\partial_{tt}\beta + (-\Delta)^{k_0}\beta + \varepsilon(-\Delta)^{k_0}\partial_t\beta &= g, \quad \text{in } (0,T)\\ \beta(0) &= \beta_0\\ \partial_t\beta(0) &= \beta_*\end{aligned}$$

has, for given data

$$g\in L^2((0,T);W^{-1,2}(Q)),\quad \beta_0\in W_D^{k_0,2}(Q),\quad \beta_*\in L^2(Q)$$

a unique solution β with

$$\beta\in L^\infty((0,T);W_D^{k_0,2}(Q)),\quad \partial_t\beta\in L^\infty((0,T);L^2(Q)),\quad \partial_t\beta\in L^2((0,T);W^{k_0,2}(Q))$$

satisfying the estimate

$$\|\beta\|_{L^\infty((0,T);W^{k_0,2}(Q))}, \|\partial_t\beta\|_{L^\infty((0,T);L^2(Q))}, \sqrt{\varepsilon}\,\|\partial_t\beta\|_{L^2((0,T);W^{k_0,2}(Q))} \le C = C(\varepsilon,\beta_0,\beta_*,g).\tag{3.5}$$

Proof. We show the statement by Galerkin approximation. Let $\{w_k\}_{k\in\mathbb{N}}\subset W^{k_0,2}(Q)$ be an orthogonal basis, moreover orthonormal in $L^2(Q)$. Then for $n\in\mathbb{N}$ we solve the following system of ODE

$$\begin{aligned}\alpha_k''(t) + \alpha_k(t)\left\|\nabla^{k_0}w_k\right\|_{L^2}^2 + \alpha_k'(t)\varepsilon\left\|\nabla^{k_0}w_k\right\|_{L^2}^2 &= \langle g,w_k\rangle, \quad k=1,\dots,n.\\ \alpha_k(0) &= \langle\beta_0,w_k\rangle_{L^2},\\ \alpha_k'(0) &= \langle\beta_*,w_k\rangle_{L^2}.\end{aligned}$$

The existence of absolutely continuous solutions $\alpha_k \colon (0,T) \to \mathbb{R}$ is standard theory of ODE. Then $\beta_n(t) = \sum_{k=1}^n \alpha_k(t) w_k$ solves the equation

$$\begin{aligned} \langle \partial_{tt}\beta_n, w_k\rangle_{L^2} + \langle \nabla^{k_0}\beta_n, \nabla^{k_0} w_k\rangle_{L^2} + \varepsilon\langle \nabla^{k_0}\partial_t\beta_n, \nabla^{k_0} w_k\rangle_{L^2} &= \langle g, w_k\rangle, \quad k = 1,\dots,n,\\ \langle \beta_n(0), w_k\rangle_{L^2} &= \langle \beta_0, w_k\rangle_{L^2},\\ \langle \partial_t\beta_n(0), w_k\rangle_{L^2} &= \langle \beta_*, w_k\rangle_{L^2}. \end{aligned}$$

Now we multiply by α_k' and sum for $k = 1,\dots,n$ (i.e. use $\partial_t\beta_n$ as a test function), so we obtain

$$\frac{1}{2}\partial_t \left\|\partial_t\beta_n\right\|_{L^2}^2 + \frac{1}{2}\partial_t \left\|\nabla^{k_0}\beta_n\right\|_{L^2}^2 + \varepsilon \left\|\nabla^{k_0}\partial_t\beta_n\right\|_{L^2}^2 = \langle g, \partial_t\beta_n\rangle_{L^2} \le \frac{1}{2\varepsilon}\left\|g\right\|_{W^{-1,2}}^2 + \frac{\varepsilon}{2}\left\|\nabla\partial_t\beta_n\right\|_{L^2}^2$$

Absorbing the last term and using the Gronwall inequality gives

$$\frac{1}{2}\left\|\partial_t\beta_n\right\|_{L^\infty((0,T);L^2)}^2 + \frac{1}{2}\left\|\nabla^{k_0}\beta_n\right\|_{L^\infty((0,T);L^2)}^2 + \frac{\varepsilon}{2}\left\|\nabla^{k_0}\partial_t\beta_n\right\|_{L^2((0,T);L^2)}^2 \le C(\varepsilon, \beta_0, \beta_*, g).$$

Passing $n \to \infty$ gives the result. Finally, uniqueness of the solution can be readily seen from linearity. □

The lemma will now be used to show a better regularity of η_ε.

Lemma 3.2. *Let it further be satisfied that*

$$\eta_0 \in W^{2k_0,2}(Q), \quad \eta_* \in W^{2k_0,2}(Q), \quad \partial_t f \in W^{1,2}((0,T);L^2(Q)). \tag{3.6}$$

Let η_ε be a solution of

$$\begin{aligned} \partial_{tt}\eta_\varepsilon + (-\Delta)^{k_0}\eta_\varepsilon + \varepsilon(-\Delta)^{k_0}\partial_t\eta_\varepsilon - \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)) &= f,\\ \eta_\varepsilon(0) &= \eta_0,\\ \partial_t\eta_\varepsilon(0) &= \eta_*. \end{aligned}$$

Then it holds

$$\partial_t\eta_\varepsilon \in L^\infty((0,T);W^{k_0,2}(Q)), \quad \partial_{tt}\eta_\varepsilon \in L^\infty((0,T);L^2(Q)), \quad \partial_{tt}\eta_\varepsilon \in L^2((0,T);W^{k_0,2}(Q))$$

with the estimate

$$\left\|\partial_t\eta_\varepsilon\right\|_{L^\infty((0,T);W^{k_0,2}(Q))}, \left\|\partial_{tt}\eta_\varepsilon\right\|_{L^\infty((0,T);L^2(Q))}, \sqrt{\varepsilon}\left\|\partial_{tt}\eta_\varepsilon\right\|_{L^2((0,T);W^{k_0,2}(Q))} \le C(\varepsilon, \eta_0, \eta_*, f).$$

Proof. Consider the problem for β

$$\begin{aligned} \partial_{tt}\beta + (-\Delta)^{k_0}\beta + \varepsilon(-\Delta)^{k_0}\partial_t\beta &= \partial_t f + \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon)\\ \beta(0) &= \eta_*\\ \partial_t\beta(0) &= f(0) - (-\Delta)^{k_0}\eta_0 - \varepsilon(-\Delta)^{k_0}\eta_* + \operatorname{div}(\nabla_\xi e(\nabla\eta_0)). \end{aligned} \tag{3.7}$$

We are in a position to use Lemma 3.1 with

$$g = \partial_t f + \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon), \quad \beta_0 = \eta_*, \quad \beta_* = f(0) - (-\Delta)^{k_0}\eta_0 - \varepsilon(-\Delta)^{k_0}\eta_* + \operatorname{div}(\nabla_\xi e(\nabla\eta_0)).$$

Note that g, β_0, β_* lie in the correct spaces to apply Lemma 3.1, as $\nabla_\xi e(\nabla\eta_\varepsilon) \in L^\infty((0,T)\times Q)$ and $\nabla\partial_t\eta_\varepsilon \in L^2((0,T)\times Q)$ implies $g \in L^2((0,T);W^{-1,2}(Q))$, and also assumptions (3.6) imply $\beta_* \in L^2(Q)$. Thus Lemma 3.1 gives existence of β with the respective estimates (3.5).

Now we need to check that $\beta = \partial_t\eta_\varepsilon$. For this we define

$$\tilde{\eta}_\varepsilon(t) := \eta_0 + \int_0^t \beta \,\mathrm{d}t.$$

Clearly $\partial_t\tilde{\eta}_\varepsilon = \beta$ and now we need to check that $\tilde{\eta}_\varepsilon = \eta_\varepsilon$. We will show this by arguing that $\tilde{\eta}_\varepsilon$ solves the same linear equation as η_ε.

For this we integrate the equation (3.7) over $(0,t)$ to obtain

$$\begin{aligned} &\partial_t\beta(t) - \partial_t\beta(0) + (-\Delta)^{k_0}\tilde{\eta}_\varepsilon(t) - (-\Delta)^{k_0}\eta_0 + \varepsilon(-\Delta^{k_0})\beta(t) - \varepsilon(-\Delta^{k_0})\beta_0\\ &\qquad = f(t) - f(0) + \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon(t)) - \operatorname{div}(\nabla_\xi e(\nabla\eta_0)). \end{aligned}$$

Using the initial conditions on β we thus conclude that $\nabla\eta_\varepsilon$ solves the linear equation with initial conditions

$$\begin{aligned} \partial_{tt}\tilde{\eta}_\varepsilon + (-\Delta)^{k_0}\tilde{\eta}_\varepsilon + \varepsilon(-\Delta^{k_0})\partial_t\tilde{\eta}_\varepsilon &= f + \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon))\\ \tilde{\eta}_\varepsilon(0) &= \eta_0 \end{aligned}$$

$$\partial_t \tilde{\eta}_\varepsilon(0) = \eta_*.$$

Since η_ε solves the same linear equation and its solutions are unique, it follows that $\eta_\varepsilon = \tilde{\eta}_\varepsilon$. Thus $\beta = \partial_t \eta_\varepsilon$ and the proof is finished. ☐

We now can prove the time regularity estimate for η_ε, which is *independent* of ε.

Theorem 3.3 (Time regularity)**.** *Let it further be satisfied*

$$\eta_0 \in W^{2k_0,2}(Q), \quad \eta_* \in W^{2k_0,2}(Q), \quad \partial_t f \in L^2((0,T);L^2(Q)).$$

Then the solution η_ε of (3.3) *satisfies*

$$\partial_{tt}\eta_\varepsilon \in L^\infty((0,T);L^2(Q)), \quad \partial_t \eta_\varepsilon \in L^\infty((0,T);W^{k_0,2}(Q))$$

with the ε-independent estimate

$$\|\partial_{tt}\eta_\varepsilon\|_{L^\infty((0,T);L^2(Q))} + \|\partial_t\eta_\varepsilon\|_{L^\infty((0,T);W^{k_0,2}(Q))} \le C = C(\eta_0, \eta_*, f).$$

Proof. As shown during the proof of Lemma 3.2, η_ε satisfies the equation

$$\begin{aligned}
\partial_{ttt}\eta_\varepsilon + (-\Delta)^{k_0}\partial_t\eta_\varepsilon + \varepsilon(-\Delta)^{k_0}\partial_{tt}\eta_\varepsilon &= \partial_t f + \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon)\\
\eta_\varepsilon(0) &= \eta_0\\
\partial_t\eta_\varepsilon(0) &= \eta_*\\
\partial_{tt}\eta_\varepsilon(0) &= f(0) - (-\Delta)^{k_0}\eta_0 - \varepsilon(-\Delta)^{k_0}\eta_* + \operatorname{div}(\nabla_\xi e(\nabla\eta_0))
\end{aligned}$$

By the same lemma we have $\partial_{tt}\eta_\varepsilon \in L^\infty((0,T);L^2(Q)) \cap L^2((0,T);W^{k_0,2}(Q))$ so we can use $\partial_{tt}\eta_\varepsilon$ as a test function. This yields

$$\begin{aligned}
\frac{1}{2}\partial_t \|\partial_{tt}\eta_\varepsilon\|_{L^2}^2 + \frac{1}{2}\partial_t \left\|\nabla^{k_0}\partial_t\eta_\varepsilon\right\|_{L^2}^2 + \varepsilon\left\|\nabla^{k_0}\partial_{tt}\eta_\varepsilon\right\|_{L^2}^2 &= \langle\partial_t f, \partial_{tt}\eta_\varepsilon\rangle + \langle\operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon), \partial_{tt}\eta_\varepsilon\rangle\\
&\le \|\partial_t f + \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon)\|_{L^2}\|\partial_{tt}\eta_\varepsilon\|_{L^2}.
\end{aligned}$$

We now apply the chain rule (for which we have enough regularity by above)

$$\operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon) = \nabla_\xi^2 e(\nabla\eta_\varepsilon)\nabla^2\eta_\varepsilon\nabla\partial_t\eta_\varepsilon + \nabla_\xi e(\nabla\eta_\varepsilon)\nabla^2\partial_t\eta_\varepsilon.$$

Now estimate by Hölder inequality and the Sobolev inequality for the continuous embedding $W^{k_0,2}(Q) \subset W^{1,\infty}(Q)$ and the Poincaré inequality (here c_s resp. c_p is the Sobolev resp. Poincaré constant)

$$\begin{aligned}
\|\operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)\nabla\partial_t\eta_\varepsilon)\|_{L^2} &\le \left\|\nabla_\xi^2 e(\nabla\eta_\varepsilon)\right\|_{L^\infty}\left\|\nabla^2\eta_\varepsilon\right\|_{L^2}\|\nabla\partial_t\eta_\varepsilon\|_{L^\infty} + \|\nabla_\xi e(\nabla\eta_\varepsilon)\|_{L^\infty}\left\|\nabla^2\partial_t\eta_\varepsilon\right\|_{L^2}\\
&\le \left(c_s\left\|\nabla_\xi^2 e(\nabla\eta_\varepsilon)\right\|_{L^\infty}\left\|\nabla^2\eta_\varepsilon\right\|_{L^2} + c_p\|\nabla_\xi e(\eta_\varepsilon)\|_{L^\infty}\right)\left\|\nabla^{k_0}\partial_t\eta_\varepsilon\right\|_{L^2}
\end{aligned}$$

Note that by the energy estimates (3.4) and Lemma 2.5 we have

$$\left\|\nabla^2\eta_\varepsilon\right\|_{L^2}, \left\|\nabla^2 e(\nabla\eta_\varepsilon)\right\|_{L^\infty} \le C = C(E(\eta_0), \|\eta_*\|_{L^2}, f).$$

Therefore we have

$$\frac{1}{2}\left(\partial_t\|\partial_{tt}\eta_\varepsilon\|_{L^2}^2 + \partial_t\left\|\nabla^{k_0}\partial_t\eta_\varepsilon\right\|_{L^2}^2\right) + \varepsilon\underbrace{\left\|\nabla^{k_0}\partial_{tt}\eta_\varepsilon\right\|_{L^2}^2}_{\ge 0} \le C\left(1 + \|\partial_{tt}\eta_\varepsilon\|_{L^2}^2 + \left\|\nabla^{k_0}\partial_t\eta_\varepsilon\right\|_{L^2}^2\right)$$

where C is independent of ε. Applying the Gronwall inequality to $\|\partial_{tt}\eta_\varepsilon\|_{L^2}^2 + \left\|\nabla^{k_0}\partial_t\eta_\varepsilon\right\|_{L^2}^2$ we obtain the desired estimate

$$\|\partial_{tt}\eta_\varepsilon\|_{L^\infty((0,T);L^2(Q))}^2 + \left\|\nabla^{k_0}\partial_t\eta_\varepsilon\right\|_{L^\infty((0,T);L^2(Q))}^2 \le e^T\left(\|\eta_*\|_{L^2}^2 + \left\|\nabla^{k_0}\eta_0\right\|_{L^2}^2 + CT\right),$$

where the right hand side depends on η_0, η_*, f. ☐

Theorem 3.4 (Higher time regularity)**.** *Assume it holds*

$$\eta_0, \eta_* \in W^{3k_0,2}(Q), \quad f \in W^{2,2}((0,T);L^2(Q)).$$

and further assume that $\left\|\Delta^{2k_0}\eta_*\right\|_{L^2(Q)} \le \frac{1}{\varepsilon}$, then

$$\partial_{tt}\eta_\varepsilon \in L^\infty((0,T);W^{k_0,2}(Q)), \quad \partial_{ttt}\eta_\varepsilon \in L^\infty((0,T);L^2(Q)), \quad \partial_{ttt}\eta_\varepsilon \in L^2((0,T);W^{k_0,2}(Q)).$$

with the ε-independent estimate

$$\|\partial_{tt}\eta_\varepsilon\|_{L^\infty((0,T);W^{k_0,2}(Q))} + \|\partial_{ttt}\eta_\varepsilon\|_{L^\infty((0,T);L^2(Q))} + \sqrt{\varepsilon}\,\|\partial_{ttt}\eta_\varepsilon\|_{L^2((0,T);W^{k_0,2}(Q))} \le C(\eta_0,\eta_*,f).$$

Proof. This follows by an iteration of the previous proof. Use

$$\begin{aligned} g &= \partial_{tt} f + \partial_{tt}\operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)),\\ \beta_0 &= f(0) - (-\Delta)^{k_0}\eta_0 - \varepsilon(-\Delta)^{k_0}\eta_* + \operatorname{div}(\nabla_\xi e(\nabla\eta_0)),\\ \beta_* &= \partial_t f(0) - (-\Delta)^{k_0}\eta_* - \varepsilon(-\Delta)^{k_0}\beta_0 + \operatorname{div}(\nabla_\xi^2 e(\nabla\eta_0)\nabla\eta_*), \end{aligned}$$

we can readily verify the assumptions of Lemma 3.1, that is $g \in L^2((0,T);W^{-1,2}(Q))$, $\beta_0 \in W^{k_0,2}(Q)$, $\beta_* \in L^2(Q)$. Proceed as in the proof of Lemma 3.2 to see that $\beta = \partial_{tt}\eta_\varepsilon$ has the given regularity with given estimates. □

No dissipation limit. We will now see that when the dissipation vanishes, we obtain as the limit $\varepsilon \to 0$ a solution to the problem without dissipation.

Theorem 3.5. *For $\eta_0 \in W^{k_0,2}_D(Q)$, $\eta_* \in L^2(Q)$, $f \in L^2((0,T);L^2(Q))$. Then there exists a weak solution to*

$$\begin{aligned} \partial_{tt}\eta + (-\Delta)^{k_0}\eta - \operatorname{div}(\nabla_\xi e(\nabla\eta)) &= f\\ \eta(0) &= \eta_0\\ \partial_t\eta(0) &= \eta_*. \end{aligned} \tag{3.8}$$

If further

$$\eta_0 \in W^{2k_0,2}(Q), \quad \eta_* \in W^{2k_0,2}(Q), \quad \partial_t f \in L^2((0,T);L^2(Q)),$$

then

$$\partial_{tt}\eta \in L^\infty((0,T);L^2(Q)), \quad \partial_t\eta \in L^\infty((0,T);W^{k_0,2}(Q)), \quad \eta \in L^\infty((0,T);W^{2k_0,2}_{loc}(Q))$$

with

$$\|\partial_{tt}\eta\|_{L^\infty((0,T);L^2(Q))} + \|\partial_t\eta\|_{L^\infty((0,T);W^{k_0,2}(Q))} + \left\|\Delta^{k_0}\eta\right\|_{L^\infty((0,T);L^2(Q))} \le C(\eta_0,\eta_*,f).$$

If further,

$$\eta_0,\eta_* \in W^{3k_0,2}(Q), \quad f \in W^{2,2}((0,T);L^2(Q)),$$

then

$$\partial_{tt}\eta \in L^\infty((0,T);W^{k_0,2}(Q)), \quad \partial_{ttt}\eta \in L^\infty((0,T);L^2(Q)), \quad \partial_t\eta \in L^\infty((0,T);W^{2k_0,2}_{loc}(Q))$$

with

$$\|\partial_{ttt}\eta\|_{L^\infty((0,T);L^2(Q))} + \|\partial_{tt}\eta\|_{L^\infty((0,T);W^{k_0,2}(Q))} + \left\|\Delta^{k_0}\partial_t\eta\right\|_{L^\infty((0,T);L^2(Q))} \le C(\eta_0,\eta_*,f). \tag{3.9}$$

Proof. Let for $\varepsilon > 0$ be η_ε a solution of

$$\begin{aligned} \partial_{tt}\eta_\varepsilon + (-\Delta)^{k_0}\eta_\varepsilon + \varepsilon(-\Delta)^{k_0}\partial_t\eta_\varepsilon - \operatorname{div}(\nabla_\xi e(\nabla\eta_\varepsilon)) &= f\\ \eta_\varepsilon(0) &= \eta_0\\ \partial_t\eta_\varepsilon(0) &= \eta_*. \end{aligned}$$

From the estimates (3.4) we see that there is a subsequence of $\varepsilon \to 0$ (here and below not relabelled), so that

$$\eta_\varepsilon \overset{*}{\rightharpoonup} \eta \text{ in } L^\infty((0,T);W^{k_0,2}(Q)), \quad \partial_t\eta_\varepsilon \overset{*}{\rightharpoonup} \partial_t\eta \text{ in } L^\infty((0,T);L^2(Q)),$$

and moreover, by the Aubin-Lions lemma and the compact embedding of $W^{k_0,2}(Q)$ into $C^{1,\alpha}(Q)$, we see that (for a further subsequence)

$$\eta_\varepsilon \to \eta \text{ in } C([0,T];C^{1,\alpha}(Q)). \tag{3.10}$$

It remains to check that η solves the equation (3.8). Let us test the equation for η_ε with a test function $\varphi \in C^\infty_c([0,T) \times Q)$. Then we see

$$\langle\eta_*,\varphi(0)\rangle + \int_0^T -\langle\partial_t\eta_\varepsilon,\partial_t\varphi\rangle + \langle\nabla^{k_0}\eta_\varepsilon,\nabla^{k_0}\varphi\rangle + \varepsilon\langle\nabla^{k_0}\partial_t\eta_\varepsilon,\nabla^{k_0}\varphi\rangle + \langle\nabla_\xi e(\nabla\eta_\varepsilon),\nabla\varphi\rangle\,dt = \int_0^T \langle f,\varphi\rangle\,dt.$$

Limit passage $\varepsilon \to 0$ in the first two terms on the left is from the weak convergence of η_ε resp. $\partial_t \eta_\varepsilon$, and in the last term on the left from the strong convergence (3.10). Finally, regarding the dissipation term we estimate

$$\left| \int_0^T \varepsilon \langle \nabla^{k_0} \partial_t \eta_\varepsilon, \nabla^{k_0} \varphi \rangle \,\mathrm{d}t \right| \le \sqrt{\varepsilon}\sqrt{\varepsilon} \left\| \nabla^{k_0} \partial_t \eta_\varepsilon \right\|_{L^2((0,T);L^2(Q))} \left\| \nabla^{k_0} \varphi \right\|_{L^2((0,T);L^2(Q))} \to 0,$$

since by the (3.4), $\sqrt{\varepsilon} \left\| \nabla^{k_0} \partial_t \eta_\varepsilon \right\|_{L^2((0,T);L^2(Q))}$ is bounded independent of ε. So passing to the limit $\varepsilon \to 0$ we have

$$\langle \eta_*, \varphi(0) \rangle + \int_0^T -\langle \partial_t \eta, \partial_t \varphi \rangle + \langle \nabla^{k_0} \eta, \nabla^{k_0} \varphi \rangle + \langle \nabla_\xi e(\nabla \eta), \nabla \varphi \rangle \,\mathrm{d}t = \int_0^T \langle f, \varphi \rangle \,\mathrm{d}t.$$

Further, the initial condition $\eta(0) = \eta_0$ is satisfied due to $\eta_\varepsilon(0) = \eta_0$ and the strong convergence (3.10).

The regularity estimates except for the $\Delta^{k_0}\eta$ and $\Delta^{k_0}\partial_t\eta$ follow by the uniform in ε estimates in Theorem 3.3 and Theorem 3.4. Further, they imply in the first case that $f(t) - \partial_{tt}\eta(t) + \operatorname{div}(\nabla_\xi e(\nabla\eta(t))) \in L^\infty((0,T); L^2(Q))$, which implies that

$$(-\Delta)^{k_0}\eta(t) = f(t) - \partial_{tt}\eta(t) + \operatorname{div}(\nabla_\xi e(\nabla \eta(t)))$$

almost everywhere and the estimate of $\Delta^{k_0}\eta$ follows.

Similarly, we find in the second case that $\partial_t f(t) - \partial_{ttt}\eta(t) + \operatorname{div}(\nabla_\xi^2 e(\nabla\eta(t))\nabla\partial_t\eta(t))$ is bounded in $L^\infty((0,T); L^2(Q))$. Hence

$$(-\Delta)^{k_0}\partial_t\eta(t) = \partial_t f(t) - \partial_{ttt}\eta(t) + \operatorname{div}(\nabla_\xi^2 e(\nabla\eta(t))\nabla\partial_t\eta(t)),$$

is satisfied almost everywhere and the estimate of $\Delta^{k_0}\partial_t\eta$ follows.

The regularity in $W^{2k_0,2}_{\mathrm{local}}(Q)$ follows by the local regularity theory for the (k_0)-Laplace equation (which follows by applying iteratively the local theory for the Poisson equation). □

Remark (Space regularity)**.** Please observe that in many situations the fact that $\Delta^{k_0}\eta \in L^2(Q)$ implies global higher spacial regularity up to $\eta \in W^{2k_0,2}(Q)$. This regularity however sensitively depends on the regularity and shape of the domain. For the sake of the generality of domains and boundary values we decided to not make this specific here. Certainly, local estimates are always available by the classical theory for the Poisson equation.

Remark (Improvement of regularity with dissipation)**.** In case that the equation includes a dissipation term $(-\Delta)^{k_0}\partial_t\eta$, we can observe a time regularizing effect, known from parabolic equations. Therefore in that case, it is possible to start with initial data of no higher regularity, namely only (3.2). By a suitable testing of the equation we get that the regularity improves in an arbitrarily short time interval, and therefore it is possible to take this new time as initial and perform the procedure above.

3.2. **Convergence rate – elastic solid.** Now we turn to the main question of this section, the rate of convergence of our scheme. In this section we again, to ease the notation, adopt the convention that $\|\cdot\|$ without any index is the $L^2(Q)$-norm, resp. $\langle\cdot,\cdot\rangle$ is the $L^2(Q)$-scalar product (or dual pairing in the DE_1 terms). Moreover we strengthen the assumption (E.1) so that e has one more derivative:

(E.1') $e \in C^3(\mathbb{R}^{n\times n}_{\det>0})$, where $\mathbb{R}^{n\times n}_{\det>0} = \{M \in \mathbb{R}^{n\times n} : \det M > 0\}$.

The solution η solves the equation (2.3) To compare η with our minimizing movements approximation $\{\eta_k^\ell\}_{k,\ell}$ defined by (2.17) or by (2.26), we will make a discrete version of η. For this let us integrate the equation in (2.3) over time $s \in (t-h, t)$ and then $t \in (t_{k-1}^\ell, t_k^\ell)$ and divide by τ and h, to get (recall the definition (2.18) of f_k^ℓ)

$$\frac{\frac{\eta(t_k^\ell)-\eta(t_{k-1}^\ell)}{\tau} - \frac{\eta(t_k^{\ell-1})-\eta(t_{k-1}^{\ell-1})}{\tau}}{h} + \fint_0^\tau \fint_0^h DE(\eta(t_{k-1}^\ell + s + \sigma))\,\mathrm{d}s\,\mathrm{d}\sigma = f_k^\ell \tag{3.11}$$

Please recall that we have introduced two schemes for construction of η_k^ℓ, one with stabilization term (2.17) and one without it (2.26). All calculations are made here for η_k^ℓ with the dissipation term, that is from the approximation (2.17). We remark here that the version without stabilization (2.26), the only difference is that the term $-c\tau\Delta\frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau}$ will not appear. Thus all the calculations below go through (slightly more simply) also in this case, and we get the same result.

Let us thus recall the Euler-Lagrange equation for $\eta_k^\ell \in \operatorname{int}\mathcal{E}$ (recall that we exclude collisions):

$$\frac{\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}-\frac{\eta_k^{\ell-1}-\eta_{k-1}^{\ell-1}}{\tau}}{h}+DE(\eta_k^\ell)-c\tau\Delta\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}=f_k^\ell. \tag{3.12}$$

Subtract (3.12) and (3.11), denote the error term by $e_k^\ell=\eta_k^\ell-\eta(t_k^\ell)$ and get

$$\frac{\frac{e_k^\ell-e_{k-1}^\ell}{\tau}-\frac{e_k^{\ell-1}-e_{k-1}^{\ell-1}}{\tau}}{h}-c\tau\Delta\frac{\eta_k^\ell-\eta_{k-1}^\ell}{\tau}+\fint_0^\tau\fint_0^h DE(\eta_k^\ell)-DE(\eta(t_{k-1}^\ell+s+\sigma))\,\mathrm{d}s\,\mathrm{d}\sigma=0$$

We add and subtract $DE(\eta(t_k^\ell))-c\tau\Delta\frac{\eta(t_k^\ell)-\eta(t_{k-1}^\ell)}{\tau}$, so that

$$\begin{aligned}&\frac{\frac{e_k^\ell-e_{k-1}^\ell}{\tau}-\frac{e_k^{\ell-1}-e_{k-1}^{\ell-1}}{\tau}}{h}+DE(\eta_k^\ell)-DE(\eta(t_k^\ell))-c\tau\Delta\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\\&\qquad=\fint_0^\tau\fint_0^h DE(\eta(t_{k-1}^\ell+s+\sigma))-DE(\eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma+c\tau\Delta\frac{\eta(t_k^\ell)-\eta(t_{k-1}^\ell)}{\tau}.\end{aligned}$$

Use as a test function $\frac{e_k^\ell-e_{k-1}^\ell}{\tau}$ to obtain

$$\begin{aligned}&\frac{1}{2h}\left\|\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\|^2+\frac{1}{2h}\left\|\frac{e_k^\ell-e_{k-1}^\ell}{\tau}-\frac{e_k^{\ell-1}-e_{k-1}^{\ell-1}}{\tau}\right\|^2+\left\langle\nabla^{k_0}e_k^\ell,\nabla^{k_0}\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle+c\tau\left\|\nabla\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\|^2\\&\qquad=\frac{1}{2h}\left\|\frac{e_k^{\ell-1}-e_{k-1}^{\ell-1}}{\tau}\right\|^2-\left\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)),\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle\\&+\left\langle\fint_0^\tau\fint_0^h DE(\eta(t_{k-1}^\ell+s+\sigma))-DE(\eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma,\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle-c\tau\left\langle\nabla\frac{\eta(t_k^\ell)-\eta(t_{k-1}^\ell)}{\tau},\nabla\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle\end{aligned}$$

so that

$$\begin{aligned}&\frac{1}{2h}\left\|\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\|^2+\frac{1}{2h}\left\|\frac{e_k^\ell-e_{k-1}^\ell}{\tau}-\frac{e_k^{\ell-1}-e_{k-1}^{\ell-1}}{\tau}\right\|^2+\frac{1}{2\tau}\left\|\nabla^{k_0}e_k^\ell\right\|^2+\frac{1}{2\tau}\left\|\nabla^{k_0}e_k^\ell-\nabla^{k_0}e_{k-1}^\ell\right\|^2\\&+c\tau\left\|\nabla\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\|^2=\frac{1}{2h}\left\|\frac{e_k^{\ell-1}-e_{k-1}^{\ell-1}}{\tau}\right\|^2+\frac{1}{2\tau}\left\|\nabla^{k_0}e_{k-1}^\ell\right\|^2-\left\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)),\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle\\&+\left\langle\fint_0^\tau\fint_0^h DE(\eta(t_{k-1}^\ell+s+\sigma))-DE(\eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma,\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle-c\tau\left\langle\nabla\frac{\eta(t_k^\ell)-\eta(t_{k-1}^\ell)}{\tau},\nabla\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\rangle.\end{aligned} \tag{3.13}$$

Therefore we need to estimate the two terms containing the difference of energies. These will be estimated in the following two lemmas.

Lemma 3.6. *There exist C_1, C_2 depending on the energy bound of Theorem 2.27 such that*

$$\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)),e_k^\ell-e_{k-1}^\ell\rangle\le\tau C_1\left\|\nabla^{k_0}e_k^\ell\right\|^2+\tau C_2\left\|\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\|^2$$

Proof. Write

$$\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)),e_k^\ell-e_{k-1}^\ell\rangle=\int_Q(\nabla_\xi e(\nabla\eta_k^\ell)-\nabla_\xi e(\nabla\eta(t_k^\ell)):\nabla(e_k^\ell-e_{k-1}^\ell)\,\mathrm{d}x \tag{3.14}$$

and denote $\eta_\theta=\theta\eta_k^\ell+(1-\theta)\eta(t_k^\ell)$ for $\theta\in[0,1]$. We modify the energy density e so that it remains bounded on $\nabla\eta_\theta$ over Q, in the following way.

Let us take $\tilde e\colon\mathbb{R}^{n\times n}\to\mathbb{R}$, $\tilde e\in C^3(\mathbb{R}^{n\times n})$, $\tilde e(\xi)=e(\xi)$ for $\xi\in\mathbb{R}^{n\times n}$, $\det\xi\ge\varepsilon_0$, $|\xi|\le C$, where ε_0 is the lower bound on the determinant (E.3) and C is from the energy estimate of

Theorem 2.27, such that $\|\tilde{e}\|_{C^3(\mathbb{R}^{n\times n})}$ depends only on these constants. Such $\tilde{e}$ can be constructed e.g. as

$$\tilde{e} = e\cdot(\chi_{\mathbb{R}^{n\times n}_{\det\geq\varepsilon_0}+B_\delta(0)} * \psi_\delta),\text{ where } 2\delta = \operatorname{dist}\big(\mathbb{R}^{n\times n}_{\det\leq 0}, \{\xi\in\mathbb{R}^{n\times n} : \det\xi\geq\varepsilon_0, |\xi|\leq C\}\big),$$

where χ_M is the characteristic function of M, ψ_δ is the standard mollifier and $*$ denotes convolution.

Then we have $e(\nabla\eta_k^\ell) = \tilde{e}(\nabla\eta_k^\ell)$ and $e(\nabla\eta(t_k^\ell)) = \tilde{e}(\nabla\eta(t_k^\ell))$. Thus, since $\tilde{e}$ is everywhere finite and C^3, we can write

$$\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)), e_k^\ell-e_{k-1}^\ell\rangle = \int_Q\int_0^1\nabla_\xi^2\tilde{e}(\nabla(\theta\eta_k^\ell+(1-\theta)\eta(t_k^\ell)))\,\mathrm{d}\theta : \nabla e_k^\ell : \nabla(e_k^\ell-e_{k-1}^\ell)\,\mathrm{d}\theta\,\mathrm{d}x$$

Integrating by parts, this gives

$$= -\int_Q\int_0^1\nabla_\xi^3\tilde{e}(\nabla\eta_\theta)\nabla^2\eta_\theta\,\mathrm{d}\theta : \nabla e_k^\ell : (e_k^\ell-e_{k-1}^\ell)\,\mathrm{d}x - \int_Q\int_0^1\nabla_\xi^2\tilde{e}(\nabla\eta_\theta)\,\mathrm{d}\theta : \nabla^2 e_k^\ell\cdot(e_k^\ell-e_{k-1}^\ell)\,\mathrm{d}x.$$

Hence we can use the Hölder inequality and obtain

$$\begin{aligned}&\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)), e_k^\ell-e_{k-1}^\ell\rangle\\ &\leq\int_0^1\left\|\nabla_\xi^3\tilde{e}(\nabla\eta_\theta)\right\|_{L^\infty}\left\|\nabla^2\eta_\theta\right\|\left\|\nabla e_k^\ell\right\|_{L^\infty}\left\|e_k^\ell-e_{k-1}^\ell\right\|+\left\|\nabla_\xi^2\tilde{e}(\nabla\eta_\theta))\right\|_{L^\infty}\left\|\nabla^2 e_k^\ell\right\|\left\|e_k^\ell-e_{k-1}^\ell\right\|\mathrm{d}\theta.\end{aligned}$$

We have, by our estimates, (C independent of time)

$$\left\|\nabla_\xi^3\tilde{e}(\nabla\eta_\theta)\right\|_{L^\infty}, \left\|\nabla^2\eta_\theta\right\|\leq C$$

and by the embedding $W^{1,\infty}(Q)\subset W^{k_0,2}(Q)$

$$\left\|\nabla e_k^\ell\right\|_{L^\infty}\leq C\left\|\nabla^{k_0}e_k^\ell\right\|$$

which in total, after Poincaré inequality $\left\|\nabla^2 e_k^\ell\right\|\leq C\left\|\nabla^{k_0}e_k^\ell\right\|$ gives

$$\langle DE_1(\eta_k^\ell)-DE_1(\eta(t_k^\ell)), e_k^\ell-e_{k-1}^\ell\rangle\leq C\left\|\nabla^{k_0}e_k^\ell\right\|\left\|e_k^\ell-e_{k-1}^\ell\right\|\leq C_1\tau\left\|\nabla^{k_0}e_k^\ell\right\|^2+C_2\tau\left\|\frac{e_k^\ell-e_{k-1}^\ell}{\tau}\right\|^2$$

with constants C_1, C_2 independent of τ. ☐

Remark. A more explicit construction of $\tilde{e}$ can be obtained in the case that e explicitly depends on $\det\nabla\eta$. Suppose that $e\in C^3(\mathbb{R}^{n\times n}_{\det>0})$ is of the form

$$e(\xi) = h(\xi,\det\xi)\qquad\text{with}\qquad h\in C^3(\mathbb{R}^{n\times n}\times(0,\infty)).$$

Then we can truncate

$$\tilde{h}(\xi,\det\xi) = h(\xi,\psi(\det\xi))),$$

where ψ is some smoothing of $\max(\varepsilon_0,\cdot)$, so e.g. $\psi\in C^3(\mathbb{R})$ with $\psi(t) = t$ for $t\geq\varepsilon_0$, $\psi(t) = \varepsilon_0/2$ for $t\leq\varepsilon_0/2$ and $|\psi''|\leq C/\varepsilon_0$. Then

$$\tilde{e}(\xi) = \tilde{h}(\xi,\det\xi)$$

fulfils the required properties.

Lemma 3.7. *We have for $C\equiv C(\eta_0,\eta_*,f)$ given by (3.9) that*

$$\left\langle⨍_0^\tau⨍_0^h DE(\eta(t_{k-1}^\ell+s+\sigma))-DE(\eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma, e_k^\ell-e_{k-1}^\ell\right\rangle\leq C(h+\tau)\left\|e_k^\ell-e_{k-1}^\ell\right\|.$$

Proof. We only provide here the estimate for E_2. The estimate for E_1 follows in the same way but needs much less regularity. Recall the definition of E_2 in (3.1) and compute, given the regularity from Theorem 3.5,

$$\begin{aligned}&\langle⨍_0^\tau⨍_0^h DE_2(\eta(t_{k-1}^\ell+s+\sigma))-DE_2(\eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma, e_k^\ell-e_{k-1}^\ell\rangle\\ &\qquad= ⨍_0^\tau⨍_0^h\int_Q\nabla^{k_0}(\eta(t_{k-1}^\ell+s+\sigma)-\eta(t_k^\ell)) : \nabla^{k_0}(e_k^\ell-e_{k-1}^\ell)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\sigma\end{aligned}$$

$$= \int_Q \fint_0^\tau \fint_0^h \nabla^{k_0}(\eta(t_{k-1}^\ell + s + \sigma) - \eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma : \nabla^{k_0}(e_k^\ell - e_{k-1}^\ell)\,\mathrm{d}x$$

$$= \int_Q \fint_0^\tau \fint_0^h (-\Delta)^{k_0}(\eta(t_{k-1}^\ell + s + \sigma) - \eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma : (e_k^\ell - e_{k-1}^\ell)\,\mathrm{d}x.$$

Now

$$\left\| \fint_0^\tau \fint_0^h (-\Delta)^{k_0}(\eta(t_{k-1}^\ell + s + \sigma) - \eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma \right\| \leq \fint_0^\tau \fint_0^h (s+\sigma) \left\| \partial_t(-\Delta)^{k_0}\eta \right\| \,\mathrm{d}s\,\mathrm{d}\sigma$$
$$= \left\| \partial_t(-\Delta)^{k_0}\eta \right\|_{L^\infty((0,T);L^2(Q))} \frac{\tau + h}{2},$$

which implies the result by Theorem 3.5. □

Finally we are in the position to show the convergence rate result.

Theorem 3.8 (Convergence rate for elastic solid)**.** *Let the initial data satisfy*

$$\eta_0 \in W^{2k_0,2}(Q), \quad \eta_* \in W^{2k_0,2}(Q), \quad \partial_t f \in L^2((0,T); L^2(Q)).$$

There exists a $h_0 > 0$ and constants C_1, C depending on $E(\eta_0)$, $\|\eta_*\|_{L^2}$, $\|f\|_{L^2((0,T);L^2(Q))}$ and T, such that for all $0 < \tau \leq h \leq h_0$ (recall that we have $N\tau = h$ and $Mh = T$, and the error term is defined as $e_k^\ell = \eta_k^\ell - \eta(t_k^\ell)$) the following convergence rate estimate holds

$$\max_{k=1,\dots,N} \left(\frac{1}{2} \left\| \nabla^{k_0} e_k^\ell \right\|^2 + \frac{1}{2N} \sum_{i=1}^k \left\| \frac{e_i^\ell - e_{i-1}^\ell}{\tau} \right\|^2 \right) \leq (\tau^2 + h^2) C T e^{C_1 h \ell}, \quad \ell = 1, \dots, M.$$

Proof. Throughout the proof C is a constant, depending on on $E(\eta_0)$, $\|\eta_*\|_{L^2}$, $\|f\|_{L^2((0,T);L^2(Q))}$ and T. From (3.13) we get, omitting the $\frac{1}{2h}\left\| \frac{e_k^\ell - e_{k-1}^\ell}{\tau} - \frac{e_k^{\ell-1} - e_{k-1}^{\ell-1}}{\tau} \right\|^2$ term and multiplying by τ,

$$\frac{\tau}{2h} \left\| \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\|^2 + \frac{1}{2} \left\| \nabla^{k_0} e_k^\ell \right\|^2 + \frac{1}{2} \left\| \nabla^{k_0} e_k^\ell - \nabla^{k_0} e_{k-1}^\ell \right\|^2 + c\tau^2 \left\| \nabla \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\|^2$$
$$\leq \frac{\tau}{2h} \left\| \frac{e_k^{\ell-1} - e_{k-1}^{\ell-1}}{\tau} \right\|^2 + \frac{1}{2} \left\| \nabla^{k_0} e_{k-1}^\ell \right\|^2 - \langle DE_1(\eta_k^\ell) - DE_1(\eta(t_k^\ell)), e_k^\ell - e_{k-1}^\ell \rangle$$
$$+ \left\langle \fint_0^\tau \fint_0^h DE(\eta(t_{k-1}^\ell + s + \sigma)) - DE(\eta(t_k^\ell))\,\mathrm{d}s\,\mathrm{d}\sigma, e_k^\ell - e_{k-1}^\ell \right\rangle$$
$$- c\tau^2 \left\langle \nabla \frac{\eta(t_k^\ell) - \eta(t_{k-1}^\ell)}{\tau}, \nabla \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\rangle.$$

Using the inequalities from Lemma 3.6 and Lemma 3.7

$$\frac{\tau}{2h} \left\| \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\|^2 + \frac{1}{2} \left\| \nabla^{k_0} e_k^\ell \right\|^2 + \frac{1}{2} \left\| \nabla^{k_0} e_k^\ell - \nabla^{k_0} e_{k-1}^\ell \right\|^2 + c\tau^2 \left\| \nabla \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\|^2$$
$$\leq \frac{\tau}{2h} \left\| \frac{e_k^{\ell-1} - e_{k-1}^{\ell-1}}{\tau} \right\|^2 + \frac{1}{2} \left\| \nabla^{k_0} e_{k-1}^\ell \right\|^2 + \tau C_1 \left\| \nabla^{k_0} e_k^\ell \right\|^2 + \tau C_2 \left\| \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\|^2 + C(h+\tau) \left\| e_k^\ell - e_{k-1}^\ell \right\|$$
$$- c\tau^2 \left\langle \nabla \frac{\eta(t_k^\ell) - \eta(t_{k-1}^\ell)}{\tau}, \nabla \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\rangle$$

For the last term, thanks to the regularity of Theorem 3.5, we can write $\Delta \frac{\eta(t_k^\ell) - \eta(t_{k-1}^\ell)}{\tau} = \frac{1}{\tau} \int_{t_{k-1}^\ell}^{t_k^\ell} \partial_t \Delta \eta \, dt$ to get the estimate

$$- c\tau^2 \left\langle \nabla \frac{\eta(t_k^\ell) - \eta(t_{k-1}^\ell)}{\tau}, \nabla \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\rangle = c\tau^2 \left\langle \Delta \frac{\eta(t_k^\ell) - \eta(t_{k-1}^\ell)}{\tau}, \frac{e_k^\ell - e_{k-1}^\ell}{\tau} \right\rangle$$

$$\leq c\tau^2\|\partial_t\Delta\eta\|_{L^\infty((0,T);L^2(Q))}\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|.$$

Finally use Young inequality and the estimate of Theorem 3.5

$$c\tau^2\|\partial_t\Delta\eta\|_{L^\infty((0,T);L^2(Q))}\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\| \leq C\tau^3 + C\tau\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2.$$

By a further Young inequality

$$C(\tau+h)\left\|e_k^\ell - e_{k-1}^\ell\right\| \leq \tau C(\tau^2+h^2) + \tau\frac{1}{2}\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2$$

we arrive at

$$\begin{aligned}
&\frac{\tau}{2h}\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2 + \frac{1}{2}\left\|\nabla^{k_0}e_k^\ell\right\|^2 + c\tau^2\left\|\nabla\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2\\
&\quad\leq \frac{\tau}{2h}\left\|\frac{e_k^{\ell-1} - e_{k-1}^{\ell-1}}{\tau}\right\|^2 + \frac{1}{2}\left\|\nabla^{k_0}e_{k-1}^\ell\right\|^2 + \tau C_1\left\|\nabla^{k_0}e_k^\ell\right\|^2 + \tau C_2\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2\\
&\qquad + \tau C(\tau^2+h^2) + \tau\frac{1}{2}\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2 + C\tau^3 + C\tau\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2.
\end{aligned}$$

Finally we get, denoting $a_k^\ell = \frac{1}{2}\left\|\nabla^{k_0}e_k^\ell\right\|^2$, $b_k^\ell = \frac{1}{2}\left\|\frac{e_k^\ell - e_{k-1}^\ell}{\tau}\right\|^2$ that

$$a_k^\ell + \frac{1}{N}b_k^\ell \leq a_{k-1}^\ell + \frac{1}{N}b_k^{\ell-1} + \tau(C_2 + 1/2 + C)b_k^\ell + \tau C(3\tau^2 + h^2).$$

Thus we can use the Two-scale Gronwall inequality, Theorem 2.15 and obtain, since $a_0^0 = 0$, $b_0^0 = 0$ (as η and η_k^ℓ satisfy the same initial conditions), and $MN\tau = T$, the desired estimate

$$\max_{k=1,\dots,N}\left(\frac{1}{2}\left\|\nabla^{k_0}e_k^\ell\right\|^2 + \frac{1}{2N}\sum_{i=1}^k\left\|\frac{e_i^\ell - e_{i-1}^\ell}{\tau}\right\|^2\right) \leq CT(\tau^2+h^2)e^{(C_2+1/2+C)h\ell}.$$

□

4. Numerical experiments

We discuss some numerical experiments on the case of ODE in one dimension. These experiments verify:

(1) The expected *optimality of the rates* demonstrated in this paper. See Figure 2 where it is seen that the rate is indeed linear.
(2) A characteristic danger in non-convex regimes, which is the possibility to “land in the wrong well”; summarized in the necessity of the appearance of τ_0 in the stability and convergence results. Indeed, if τ is to large one may get trapped in the wrong well; see Figure 1.
(3) The expected differences between the fully discrete, the time-delayed and the continuous solution. That is that the difference between the time-delayed solution and the time-discrete solution is of order τ, while the difference between the time-delayed solution and the limit solution is of order h. This also indicates that choosing $\tau = h$ is commonly optimal with regard to convergence; see Figure 1.

We consider a double-well potential, with minima at 1 and −1

$$E(x) = (x^2-1)^2.$$

We will compare the solution of

$$\begin{aligned}
x''(t) + E'(x(t)) &= 0, \quad t \in (0,T)\\
x(0) &= x_0,\\
x'(0) &= x_*
\end{aligned}$$

with that of our minimizing movements approximation $\overline{x}_{(\tau)}^{(h)}$. We will consider here the case $\tau = h$, which is according to our theory a good choice. Recall that our minimizing movements approximation is then defined as

$$x_k = \arg\min_{x\in\mathbb{R}} \frac{\tau^2}{2}\left|\frac{\frac{x-x_{k-1}}{\tau} - \frac{x_{k-1}-x_{k-2}}{\tau}}{\tau}\right|^2 + E(x),$$

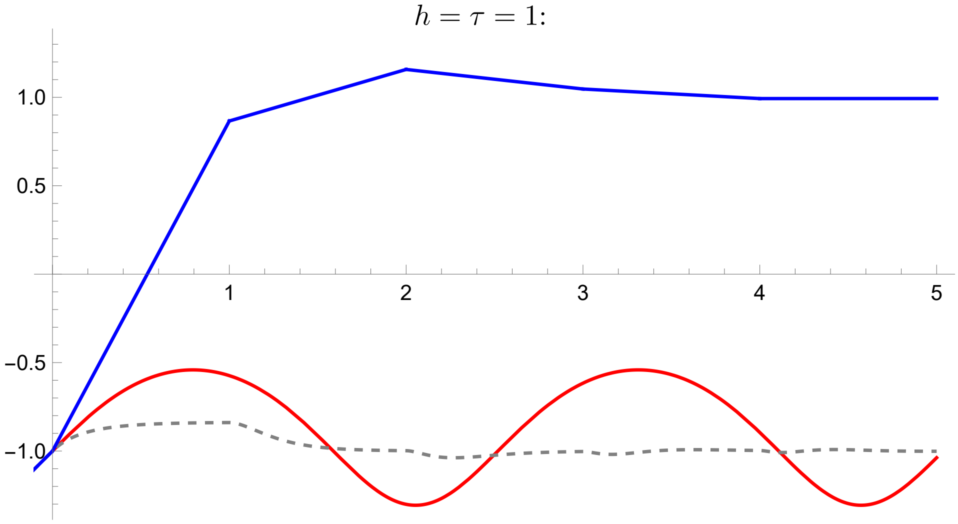
where for $k = 1$ we replace the fraction $\frac{x_0-x_{-1}}{\tau}$ by x_*, and the piecewise constant resp. piecewise affine interpolations $\overline{x}_{(\tau)}$ resp. $\hat{x}_{(\tau)}$ are defined as

$$\overline{x}_{(\tau)}(t) = x_k, \quad t \in ((k-1)\tau, k\tau],$$
$$\hat{x}_{(\tau)}(t) = \frac{t-(k-1)\tau}{\tau}x_k + \frac{k\tau - t}{\tau}x_{k-1}, \quad t \in [(k-1)\tau, k\tau].$$

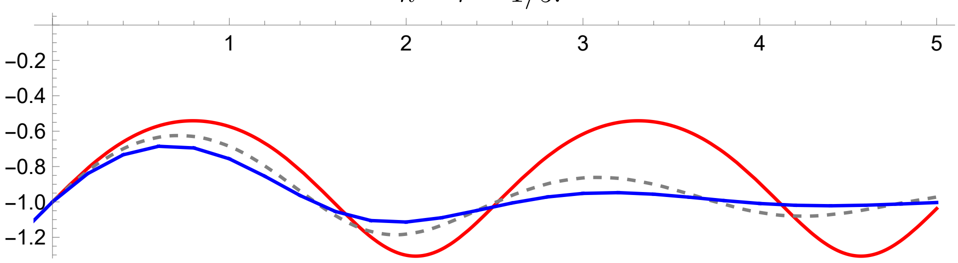
Further for comparison we include the solution of the *time-delayed equation*

$$\frac{x'(t)-x'(t-h)}{h} + E'(x(t)) = 0,$$
$$x(0) = x_0,$$
$$x'(s) = x_*, \; s \in (-h, 0].$$

$h = \tau = 1$:



$h = \tau = 1/5$:



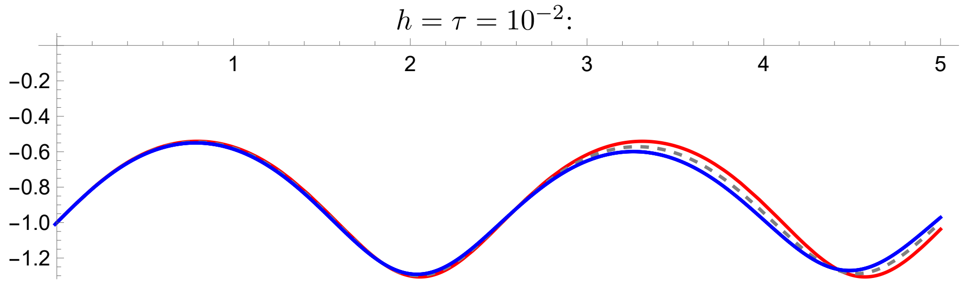


FIGURE 1. Comparison of the our approximation $\hat{x}_{(\tau)}$ (blue) with the time-delayed solution (dashed) and the limit solution (red). Note that with too large parameter $h = \tau = 1$ the solution overshoots and ends up in the other local minimum of E.

$\tau = h$	$\left\Vert x - \bar{x}_{(\tau)} \right\Vert_{L^\infty((0,T))}$
0.5	0.42947058
0.2	0.40930512
0.1	0.35324727
0.05	0.27452178
0.02	0.16104612
0.01	0.09451410
0.005	0.05162134
0.002	0.02183888
0.001	0.01113153
0.0005	0.00562018
0.0002	0.00226120
0.0001	0.00113302
0.00005	0.00056687
0.00002	0.00021291

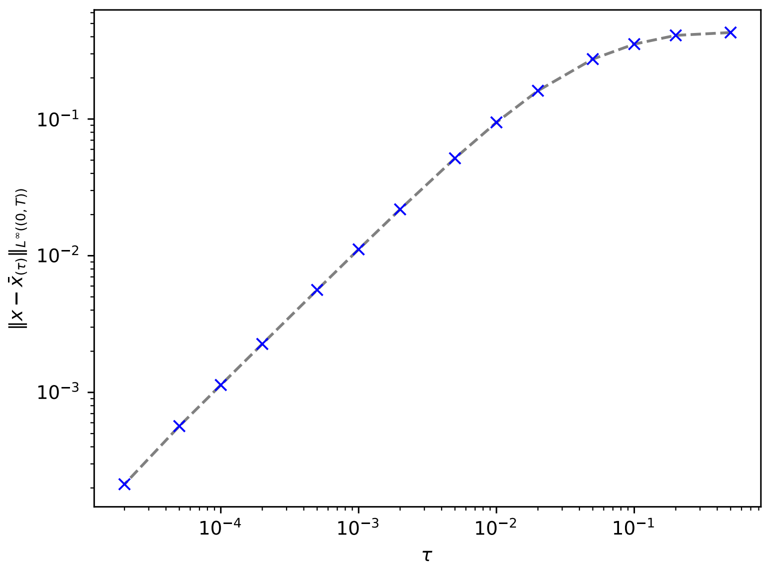


FIGURE 2. Table and graph (in log scale) showing the predicted error decay of the minimizing movements approximation, for different time steps.

Acknowledgements. The authors acknowledge the support of the ERC-CZ grant LL2105, and the support of Charles University, project GA UK No. 393421. S. S. further acknowledges the support of the VR-Grant 2022-03862 by the Swedish Research Council. A. Č. further acknowledges the support of the Czech Science Foundation (GAČR) grant No. 23-04766S and Charles University Research Centre program No. UNCE/24/SCI/005.

References

[Bal76] John M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 63(4):337–403, dec 1976.

[BFK23] Rufat Badal, Manuel Friedrich, and Martin Kružík. Nonlinear and linearized models in thermoviscoelasticity. *Archive for Rational Mechanics and Analysis*, 247(1):5, 2023.

[BGN10] John W. Barrett, Harald Garcke, and Robert Nürnberg. Numerical approximation of gradient flows for closed curves in $\mathbb{R}^d$. *IMA Journal of Numerical Analysis*, 30(1):4–60, January 2010.

[BK11] Sören Bartels and Martin Kružík. An efficient approach to the numerical solution of rate-independent problems with nonconvex energies. *Multiscale Modeling & Simulation*, 9(3):1276–1300, 2011.

[BKS23a] B. Benešová, M. Kampschulte, and S. Schwarzacher. Variational methods for fluid–structure interaction and porous media. *Nonlinear Analysis: Real World Applications*, 71:103819, jun 2023.

[BKS23b] Barbora Benešová, Malte Kampschulte, and Sebastian Schwarzacher. A variational approach to hyperbolic evolutions and fluid-structure interactions. *to appear in JEMS*, August 2023.

[CD04] Carsten Carstensen and Georg Dolzmann. Time-Space Discretization of the Nonlinear Hyperbolic System. *SIAM Journal on Numerical Analysis*, 42(1):75–89, January 2004.

[ČGK24] Antonín Češík, Giovanni Gravina, and Malte Kampschulte. Inertial evolution of non-linear viscoelastic solids in the face of (self-)collision. *Calculus of Variations and Partial Differential Equations*, 63(2):55, February 2024.

[Cia88] Philippe G. Ciarlet. *Mathematical Elasticity. Volume I, Three-dimensional Elasticity*. North-Holland ; Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co., Amsterdam, New York, 1988.

[CN87] Philippe G. Ciarlet and Jindřich Nečas. Injectivity and self-contact in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 97(3):171–188, sep 1987.

[Dac07] Bernard Dacorogna. *Direct Methods in the Calculus of Variations (Applied Mathematical Sciences)*. Springer, 2007.

[DG93] Ennio De Giorgi. New problems on minimizing movements. *Ennio de Giorgi: Selected Papers*, pages 699–713, 1993.

[Dog00] Issam Doghri. *Mechanics of Deformable Solids*. Springer Berlin Heidelberg, 2000.

[DSET01] Sophia Demoulini, David M. A. Stuart, and Athanasios E. Tzavaras. A Variational Approximation Scheme for Three-Dimensional Elastodynamics with Polyconvex Energy. *Archive for Rational Mechanics and Analysis*, 157(4):325–344, May 2001.

[FD97] G. Friesecke and G. Dolzmann. Implicit Time Discretization and Global Existence for a Quasi-Linear Evolution Equation with Nonconvex Energy. *SIAM Journal on Mathematical Analysis*, 28(2):363–380, March 1997.

[GO13] Nicola Gigli and Felix Otto. Entropic Burgers' equation via a minimizing movement scheme based on the Wasserstein metric. *Calculus of Variations and Partial Differential Equations*, 47(1-2):181–206, May 2013.

[HK09] Timothy J. Healey and Stefan Krömer. Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(4):863–871, jul 2009.

[HP10] Jonas Haehnle and Andreas Prohl. Approximation of nonlinear wave equations with nonstandard anisotropic growth conditions. *Mathematics of Computation*, 79(269):189–189, January 2010.

[Kač86] Jozef Kačur. Method of Rothe in evolution equations. *Equadiff 6*, pages 23–34, 1986.

[KR19] Martin Kružík and Tomáš Roubíček. *Mathematical Methods in Continuum Mechanics of Solids*. Springer International Publishing, 2019.

[Kru98] Martin Kružík. Numerical approach to double well problems. *SIAM journal on numerical analysis*, 35(5):1833–1849, 1998.

[KSS23] Malte Kampschulte, Sebastian Schwarzacher, and Gianmarco Sperone. Unrestricted deformations of thin elastic structures interacting with fluids. *Journal de Mathématiques Pures et Appliquées*, 173:96–148, may 2023.

[LO16] Tim Laux and Felix Otto. Convergence of the thresholding scheme for multi-phase mean-curvature flow. *Calculus of Variations and Partial Differential Equations*, 55(5):129, 2016.

[May00] Uwe F Mayer. A numerical scheme for moving boundary problems that are gradient flows for the area functional. *European Journal of Applied Mathematics*, 11(1):61–80, 2000.

[MR06] Alexander Mielke and Tomáš Roubíček. Rate-independent damage processes in nonlinear elasticity. *Mathematical Models and Methods in Applied Sciences*, 16(02):177–209, 2006.

[MR16] Alexander Mielke and Tomáš Roubíček. Rate-independent elastoplasticity at finite strains and its numerical approximation. *Mathematical Models and Methods in Applied Sciences*, 26(12):2203–2236, 2016.

[MR20] Alexander Mielke and Tomáš Roubíček. Thermoviscoelasticity in kelvin–voigt rheology at large strains. *Archive for Rational Mechanics and Analysis*, 238(1):1–45, October 2020.

[MRS17] Alexander Mielke, Riccarda Rossi, and Giuseppe Savaré. Global existence results for viscoplasticity at finite strain. *Archive for Rational Mechanics and Analysis*, 227(1):423–475, sep 2017.

[Ngu00] Quoc Son. Nguyen. *Stability and Nonlinear Solid Mechanics.* Wiley, 2000.

[Pro08] Andreas Prohl. Convergence of a Finite Element-Based Space-Time Discretization in Elastodynamics. *SIAM Journal on Numerical Analysis*, 46(5):2469–2483, January 2008.

[Pul84] Milan Pultar. Solutions of abstract hyperbolic equations by Rothe method. *Applications of Mathematics*, 29(1):23–39, 1984.

[RSS17] Filip Rindler, Sebastian Schwarzacher, and Endre Süli. Regularity and approximation of strong solutions to rate-independent systems. *Math. Models Methods Appl. Sci.*, 27:2511–2556, 2017.

[RT21] Tomáš Roubíček and Chrysoula Tsogka. Staggered explicit-implicit time-discretization for elastodynamics with dissipative internal variables. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55:S397–S416, 2021.

Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675, Prague, Czech Republic

Email address: cesik@karlin.mff.cuni.cz

Department of Mathematics, Uppsala University, Box 480 751 06 Uppsala & Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675, Prague, Czech Republic

Email address: schwarz@karlin.mff.cuni.cz

Paper IV

FLUID-STRUCTURE INTERACTIONS WITH SLIP

ANTONÍN ČEŠÍK, MALTE KAMPSCHULTE, AND SEBASTIAN SCHWARZACHER

Abstract. We construct weak solutions to the fluid-structure interaction problem of a Navier-Stokes fluid interacting with a nonlinear viscoelastic bulk solid. Our weak formulation consists of two types of test functions: continuous over the fluid-solid domain, and fluid-only test functions with nonzero tangential component at the boundary. We further show that this weak formulation is compatible with the strong formulation in the sense that regular weak solutions are also strong solutions.

Contents

1. Introduction

Recently there has been much interest in the construction of weak solutions for fluid-structure interaction problems. The former results have been in lower dimensional structures, such as plates or shells, interacting with fluids [BS18, SS22, MS22, MC13, MČ15, MMN$^+$22, MC13, KSS23, BS23, LR14]. More recently, a variational approach was devised [BKS23b] which allows for treatment of bulk solids interacting with fluids, giving rise to several new results in this setting [BKS24, BKS23a, KMT24]. In this paper we wish to contribute to this ongoing research, including the possibility of a *full slip* at the fluid-solid interface. We mention also the recent result [LMN24] treating slip in the case of an elastic shell.

We employ the physical setup of a viscoelastic bulk solid immersed in an incompressible Navier-Stokes fluid as in [BKS23b]. We fix a container $\Omega \subset \mathbb{R}^d$ and consider both the fluid and the solid to be confined to Ω. The solid is described with respect to a Lagrangian reference configuration $Q \subset \mathbb{R}^d$. The solid deformation is then $\eta\colon (0,T) \times Q \to \Omega$, so that $\eta(t) = \eta(t,\cdot)$ is the deformation of the solid at the time $t \in (0,T)$. The spatial dimension is $d \geq 2$ with

Date: July 31, 2024.

$d = 3$ and $d = 2$ being the most physically relevant ones. We assume that the fluid occupies the rest of the container not occupied by the solid. That is, the fluid at time t is defined on the domain $\Omega(t) = \Omega \setminus \eta(t, Q)$. and is determined by the velocity $v(t)\colon \Omega(t) \to \mathbb{R}^d$ and pressure $p\colon \Omega(t) \to \mathbb{R}$.

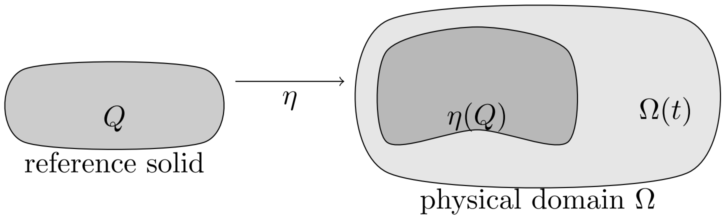


FIGURE 1. Geometrical depiction of the fluid-structure interaction problem.

Now let us describe the equations of motion. We assume, similarly as in previous works [BKS23b, ČGK24, BKS24], the for the solid the Piola-Kirchhoff stress tensor can be derived from the energy and dissipation potentials. Thus the momentum equation for the solid is

$$\rho_s \partial_{tt}\eta + DE(\eta) + D_2 R(\eta, \partial_t \eta) = \rho_s f \quad \text{in } Q$$

where ρ_s is the (reference) solid density and $f\colon (0,T) \times \Omega \to \mathbb{R}^d$ the external force in the Eulerian configuration. The fluid will be assumed to be satisfy an incompressible Navier-Stokes equation, so that

$$\rho_f(\partial_t v + v \cdot \nabla v) = \nu \Delta v - \nabla p + \rho_f f \quad \text{in } \Omega(t)$$

with the incompressibility condition

$$\operatorname{div} v = 0 \quad \text{in } \Omega(t).$$

The *kinematic coupling* to the fluid-solid will be through an *impermeability condition.* That is, it holds

$$v \cdot n = (\partial_t \eta \circ \eta^{-1}) \cdot n \quad \text{on } \partial\eta(t, Q)$$

where n is the normal vector to the fluid-solid interface $\partial\eta(t,Q)$ (i.e. also the normal vector to $\partial\Omega(t)$ on the interface part). On the outer boundary of the container, we also prescribe the impermeability condition

$$v \cdot n = 0 \quad \text{on } \partial\Omega.$$

Additionally to the kinematic coupling condition there will be the equalities of stress with the slipping law, and also of the solid hyperstress (which arises due to the presence of second gradients). All these boundary conditions are stated in the definition of strong solutions, namely Definition 3.3.

Further, the solid is presumed to satisfy the *no-interpenetration of matter* in the form of the Ciarlet-Nečas condition

$$|\eta(Q)| = \int_Q \det \nabla\eta(x)\, dx.$$

Finally, we complete the system by prescribing the initial conditions

$$\eta(0) = \eta_0, \quad \partial_t \eta(0) = \eta_*, \quad v(0) = v_0, \quad \Omega(0) = \Omega \setminus \eta_0(Q).$$

In contrast to no-slip boundary conditions in case of slip the tangential component of the fluid velocity is not fixed at the interface. Hence, as stated above, the kinematic coupling condition reduces to impermeability. In the framework of weak solution the test functions have to be adapted accordingly. Typically for fluid-structure interactions, the space of test function is non-linearly related to the solution. In contrast to the no-slip case, in the case of slip the *normal vector* of the geometry (that comes from the time-changing solid deformation) influences the test-function space. Hence the gradient of the solid deformation directly influences the weak-formulation. In order to construct a solution this higher order influence is reduced to the fluid equation.

This explains why our weak formulation consists of two types of test function – what we call the *coupled* and the *fluid-only* test functions. The coupled test functions are defined to be continuous over Ω, and the corresponding Lagrangian test function for the solid is pulled from $\eta(Q)$ back to Q by η. Then the fluid-only test functions are defined on $\Omega(t)$ and have the normal component zero on $\partial\eta(\cdot, Q)$. In essence only the tangential part of the fluid-only test function on $\partial\eta(\cdot, Q)$ is important, and this is what gives rise to the slip. (Compare to [BKS23b] where the only test functions are the continuous coupled ones, this results in no slip condition.) Please note that the splitting of the weak formulation has to be performed on all levels of approximation. This is necessary due to the regularity drop between solids (2nd order) and fluids (1st order). Related to this deviation it turns out suitable to derive the proper a-priori estimates already on the τ level (see below). Here we rely on our findings in [ČS23].

We then show the following existence result.

Theorem 1.1. *The fluid-structure interaction problem has a weak solution in the sense of Definition 3.1, until the time of the first solid-solid collision.*

The paper is organized as follows. In Section 2 we introduce some preliminary material on moving domains and the corresponding function spaces. In Section 3 we introduce the full weak formulation of the problem, as well as the strong formulation, and show their equivalence for regular solutions. In Section 4 we construct the solution by a variational time-stepping scheme by multiple levels of approximation.

1.1. **Assumptions on energy and dissipation.** We work in the context of *nonsimple* elastic materials [KR19], where the energy depends on higher gradient. It has been observed by [HK09] that under suitable growths of the energy, a well-known issue of obtaining lower bound on the Jacobian [Bal02] is circumvented, thus obtaining an Euler-Lagrange equation for the minimizer is possible. As in previous results, we stick to this setting here.

The set of admissible deformations is defined as

$$\mathcal{E} := \left\{ \eta \in W^{2,q}(Q;\mathbb{R}^d) : \eta(Q) \subset \Omega, \det \nabla\eta > 0, |\eta(Q)| = \int_Q \det \nabla\eta(x)\, dx \right\} \tag{1.1}$$

where $q > d$, so that we have the embedding $W^{2,q}(Q;\mathbb{R}^d) \rightharpoonup C^{1,1-d/q}(Q;\mathbb{R}^d)$. Here we specify the assumptions on the energy E and dissipation R. We assume that the elastic energy potential $E\colon W^{2,q}(Q;\mathbb{R}^d) \to (-\infty, \infty]$ has the following properties:

(E.1) There is $E_{\min} > -\infty$ such that $E(\eta) \geq E_{\min}$ for all $\eta \in W^{2,q}(Q;\mathbb{R}^d)$. Further, for $\eta \in W^{2,q}(Q;\mathbb{R}^d)$ with $\inf_Q \det \nabla\eta > 0$ it holds $E(\eta) < \infty$.

(E.2) For every $E_0 \geq E_{\min}$ there exists $\varepsilon_0 > 0$ such that $E(\eta) \leq E_0$ implies $\det \nabla\eta \geq \varepsilon_0$.

(E.3) For every $E_0 \geq E_{\min}$ there exists C such that $E(\eta) \leq E_0$ implies $\|\nabla^2\eta\|_{L^q} \leq C$.

(E.4) E is weakly lower semicontinuous. That is, for $\eta_k \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^d)$ it holds

$$E(\eta) \leq \liminf_{k\to\infty} E(\eta_k).$$

Further, E strongly continuous in $W^{2,q}(Q;\mathbb{R}^d)$.

(E.5) E is differentiable for $\eta \in \mathcal{E}$ with derivative $DE(\eta) \in (W^{2,q}(Q;\mathbb{R}^d))^*$ given by

$$DE(\eta)\langle\varphi\rangle = \frac{d}{ds} E(\eta + \varepsilon\varphi))\bigg|_{s=0}.$$

Furthermore, DE is bounded sublevel sets of E and continuous with respect to strong $W^{2,q}(Q;\mathbb{R}^d)$ convergence.

(E.6) The derivative DE satisfies

$$\liminf_{k\to\infty} (DE(\eta_k) - DE(\eta))\langle(\eta_k - \eta)\psi\rangle \geq 0$$

for all $\psi \in C_0^\infty(Q;[0,1])$ and all $\eta_k \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^d)$. Further, DE satisfies the following *Minty-type property*: If

$$\limsup_{k\to\infty} (DE(\eta_k) - DE(\eta))\langle(\eta_k - \eta)\psi\rangle \leq 0$$

for all $\psi \in C_0^\infty(Q;[0,1])$, then $\eta_k \to \eta$ in $W^{2,q}(Q;\mathbb{R}^d)$.

(E.7) E satisfies the following *non-convexity estimate*: For all $E_0 \geq E_{\min}$ there exists $C_1 \geq 0$ such that for all $\eta_1, \eta_0 \in \mathcal{E}$ with $E(\eta_1), E(\eta_0) \leq E_0$ it holds

$$DE(\eta_1)\langle \eta_1 - \eta_0 \rangle \geq E(\eta_1) - E(\eta_0) - C_1 \|\nabla \eta_1 - \nabla \eta_0\|_{L^2}^2.$$

For the the dissipation potential $R\colon W^{2,q}(Q;\mathbb{R}^d) \times W^{1,2}(Q;\mathbb{R}^d) \to [0,\infty)$ satisfies the following properties:

(R.1) R is weakly lower semicontinuous in its second argument. That is, for all $\eta \in W^{2,q}(Q;\mathbb{R}^d)$ and all $b_k \rightharpoonup b$ in $W^{1,2}(Q;\mathbb{R}^d)$ it holds

$$R(\eta, b) \leq \liminf_{k\to\infty} R(\eta, b_k)$$

(R.2) R is 2-homogeneous in its second argument, that is,

$$R(\eta, \lambda b) = \lambda^2 R(\eta, b), \quad \lambda \in \mathbb{R}.$$

(R.3) R admits the following *Korn-type inequality*: For every $\varepsilon_0 > 0$, there is K_R such that

$$K_R \|b\|_{W^{1,2}}^2 \leq \|b\|_{L^2}^2 + R(\eta, b)$$

for all $\eta \in \mathcal{E}$ with $\det \nabla \eta > \varepsilon_0$ and all $b \in W^{1,2}(Q;\mathbb{R}^d)$.

(R.4) R is differentiable in its second argument, with the derivative $D_2R(\eta, b) \in (W^{1,2}(Q;\mathbb{R}^d))^*$. Further, the map $(\eta, b) \mapsto D_2R(\eta, b)$ is bounded and weakly continuous with respect η and b. This means that for all $\varphi \in W^{1,2}(Q;\mathbb{R}^d)$ and all $\eta_k \rightharpoonup \eta$ in $W^{2,q}(Q;\mathbb{R}^d)$ and $b_k \rightharpoonup b$ in $W^{1,2}(Q;\mathbb{R}^d)$ it holds

$$\lim_{k\to\infty} D_2R(\eta_k, b_k)\langle \varphi \rangle = D_2R(\eta, b)\langle \varphi \rangle.$$

The model case which satisfies these assumptions is

$$R(\eta, \partial_t \eta) := \int_Q \left| (\nabla \partial_t \eta)^T \nabla \eta + (\nabla \eta)^T (\nabla \partial_t \eta) \right|^2 dx = \int_Q \left| \partial_t \left(\nabla \eta^T \nabla \eta \right) \right|^2 dx,$$

$$E(\eta) := \begin{cases} \int_Q \left[\frac{1}{8} \left| \nabla \eta^T \nabla \eta - I \right|_{\mathcal{C}} + \frac{1}{(\det \nabla \eta)^a} + \frac{1}{q} \left| \nabla^2 \eta \right|^q \right] dx & \text{if } \det \nabla \eta > 0 \text{ a.e. in } Q, \\ +\infty & \text{otherwise,} \end{cases}$$

denoting $\left|\nabla \eta^T \nabla \eta - I\right|_{\mathcal{C}} := \left(\mathcal{C}\left(\nabla \eta^T \nabla \eta - I\right)\right) \cdot \left(\nabla \eta^T \nabla \eta - I\right)$ with $\mathcal{C}$ being a positive definite tensor of elastic constants, and $a > \frac{qd}{q-d}$. For a more detailed discussion of this model see [BKS23b], and in particular for the discussion of the non-convexity estimate (E.7) see [ČS23].

1.2. **The approximation.** The construction of the solution will go through three levels of approximation that we shall briefly describe here.

Spatial regularization – κ level. At the κ-level we introduce in the problem a $W^{k_0+2,2}$-regularization of the solid energy and dissipation, as well as a $W^{k_0,2}$-regularization of the fluid dissipation. Here k_0 is chosen so large that $W^{k_0,2} \hookrightarrow W^{2,q}$. The equations with this regularization are then quadratic in the highest order, and thus using $(\partial_t \eta, v)$ as a test function is admissible as long as $\kappa > 0$. The two gradients more for the regularity of η are used in the approximation of test functions in Proposition 2.6, since there we use the spatial extension of the normal n (recall this is the normal to $\partial \eta(\cdot, Q)$) to the fluid domain.

Time-delayed equation – h level. The h-level corresponds to replacing the equations by a *time-delayed equation* where the second time derivative is discrete with scale h, and the first time derivative is continuous. So that then

$$\partial_{tt}\eta \approx \frac{\partial_t \eta - \partial_t \eta(t-h)}{h}, \quad \partial_t v + v \cdot \nabla v \approx \frac{v \circ \Phi_h - v(t-h)}{h}$$

where Φ_h is the flow map, defined by $\partial_t \Phi_t = v(t) \circ \Phi_t$ and $\Phi_0 = \mathrm{id}$. The flow map has to be constructed along with the solution.

1.2.1. *Minimizing movements – τ level.* The first level is the minimizing movement approximation of the time-delayed problem with τ-steps. Note that the time-delayed equation can, on an interval of length h, be seen as a gradient flow where $\partial_t\eta(\cdot - h)$ and $v(\cdot - h)$ can be seen as a given external force. Indeed, these quantities are already known from the previous h-interval. Note that to get the τ-independent estimate, we do this already on the discrete τ level based on [ČS23].

The approximation passes to the limits in the following order $\tau \to 0$, $h \to 0$, $\kappa \to 0$. This then results in the solution of the full problem.

For a more detailed discussion of the scheme we refer the reader to [BKS23b]. Keep in mind that in this original paper, the parameters κ and h are tied together and $h \to 0$, $\kappa \to 0$ is performed simultaneously. We chose to separate these two parameters for more clarity and also for future work which will include collisions, since these happen only as a result of the $\kappa \to 0$ limit (see Corollary 4.12).

2. Preliminaries

Here we state some notions for time-dependent domains and the respective function spaces. These will be later used for the Eulerian fluid domain.

2.1. **Moving domains.** Consider a domain variable in time, that is $\Omega(t) \subset \mathbb{R}^d$ be a Lipschitz domain for each $t \in (0,T)$, such that the time-space domain

$$\Omega_T := (0,T) \times \Omega(t) := \{(t,x) \in (0,T) \times \mathbb{R}^d : x \in \Omega(t)\} = \bigcup_{t\in(0,T)} \{t\} \times \Omega(t)$$

is open in $\mathbb{R}^{d+1}$. We say it is a moving domain, if Ω_T has moreover Lipschitz boundary in $\mathbb{R}^{d+1}$. We adopt the notation that by $\partial_x\Omega_T$ we mean only the lateral (spatial part) of the boundary, that is

$$\partial_x\Omega_T := \{(t,x) \in (0,T) \times \mathbb{R}^d : x \in \partial\Omega(t)\} = \bigcup_{t\in(0,T)} \{t\} \times \partial\Omega(t)$$

so that the true time-space boundary in $\mathbb{R}^{d+1}$ of Ω_T is

$$\partial\Omega_T = \partial_x\Omega_T \cup \{0\} \times \overline{\Omega}(0) \cup \{T\} \times \overline{\Omega}(T)$$

Denote by $n(t) = n(t,\cdot)\colon \partial\Omega(t) \to \mathbb{R}^d$ the normal to the Lipschitz boundary of $\Omega(t)$, defined $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega(t)$. Since we assume $\Omega_T \subset \mathbb{R}^{d+1}$ to be also a Lipschitz domain, it has $\mathcal{H}^d$-a.e. defined normal $\tilde{n}\colon \partial\Omega_T \to \mathbb{R}^{d+1}$ which necessarily is of the form

$$\tilde{n}(t,x) = \begin{cases} (-\tilde{n}_t(t,x), n(t,x)), & t \in (0,T), x \in \partial\Omega(t), \\ (-1,0), & t = 0, x \in \Omega(0), \\ (1,0), & t = T, x \in \Omega(T), \end{cases}$$

for some $\tilde{n}_t\colon \partial_x\Omega_T \to \mathbb{R}$ which we may call the *normal velocity* (note that in all three cases the value on the right is written as $(a,b) \in \mathbb{R} \times \mathbb{R}^d$). Note that $\tilde{n}$ is not a unit vector, rather it is chosen so that the spatial part n is unit. We denote by dS the surface measure on $\partial\Omega(t)$, and dn denotes the vector-valued measure $dn := n\, dS$. The domain $\Omega(t)$ said is transported by the vector field $w\colon \overline{\Omega_T} \to \mathbb{R}^d$, if $w \cdot n = \tilde{n}_t$.

2.2. **Transport theorem.** Now let us prove a variant of the Reynolds transport theorem for this moving domain. We do not explicitly refer to a transport field in the statement. However if such a field exists, i.e. $\Omega(t)$ is transported by w in the sense above, we recover the standard Reynolds transport theorem.

Theorem 2.1 (Transport theorem)**.** *Let $\Omega(t)$ be a moving domain as above and $u \in C^1(\overline{\Omega}_T)$. Then it holds for all $t \in (0,T)$*

$$\frac{d}{dt}\int_{\Omega(t)} u(t,x)\,dx = \int_{\Omega(t)} \partial_t u(t,x)\,dx + \int_{\partial\Omega(t)} u(t,x)\tilde{n}_t(t,x)\,d\mathcal{H}^{d-1}(x)$$

Proof. Denote $U = (u, 0)\colon \Omega_T \to \mathbb{R}\times\mathbb{R}^d$. Pick $0 \le t < s \le T$ and invoke the divergence theorem for U on the Lipschitz domain $\Omega_{s,t} := \Omega_T \cap (s,t) \times \mathbb{R}^d$ to obtain

$$\int_{\Omega(s)} u(s,x)\,dx - \int_{\Omega(t)} u(t,x)\,dx = \int_t^s \int_{\Omega(r)} \partial_t u(r,x)\,dx\,dr + \int_t^s \int_{\partial\Omega(r)} u(r,x)\tilde{n}_t(r,x)\,d\mathcal{H}^{d-1}(x)\,dr.$$

Let us now pick $t \in (0,T)$ a Lebesgue point of $t \mapsto \int_{\Omega(t)} u(t,x)\,dx$ and of $t \mapsto \tilde{n}_t$. Then we compute, by above,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} u(t,x)\,dx &= \lim_{s\to t} \frac{\int_{\Omega(s)} u(s,x)\,dx - \int_{\Omega(t)} u(t,x)\,dx}{s-t} \\ &= \int_{\Omega(t)} \partial_t u(t,x)\,dx + \int_{\partial\Omega(t)} u(t,x)\tilde{n}_t(t,x)\,d\mathcal{H}^{d-1}(x),\end{aligned}$$

which finishes the proof. □

Remark. It is enough that u differentiable in time and having a trace, so that all manipulations in the preceding proof go through. However we will use it only for C^1 strong solutions in Theorem 3.4.

2.3. **Spaces on a moving domain.** Assume now $\Omega(t)$ is a moving domain.

We now rigorously describe the spaces

$$\begin{gathered} L^2((0,T); W^{1,2}(\Omega(t);\mathbb{R}^d)), \\ L^2((0,T); W^{1,2}_n(\Omega(t);\mathbb{R}^d)), \\ L^2((0,T); W^{1,2}_{\mathrm{div},n}(\Omega(t);\mathbb{R}^d)). \end{gathered}$$

First, classically we can identify the space $L^2((0,T); L^2(\Omega(t);\mathbb{R}^d))$ with $L^2(\Omega_T;\mathbb{R}^d)$. That is, we consider measurable functions $u\colon \Omega_T \to \mathbb{R}^d$ which are square integrable:

$$\int_{\Omega_T} |u|^2\,dx\,dt < \infty$$

so that by Fubini theorem, we have for a.e. $t \in (0,T)$ that $u(t) \in L^2(\Omega(t);\mathbb{R}^d)$ and it holds

$$\|u\|_{L^2((0,T);L^2(\Omega(t);\mathbb{R}^d))} := \int_0^T \int_{\Omega(t)} |u|^2\,dx\,dt < \infty$$

Let us henceforth assume $\Omega(t) \subset \Omega$, $t \in (0,T)$, for some fixed Lipschitz domain Ω. Clearly we can consider the zero extension to $u_0\colon (0,T)\times\Omega \to \mathbb{R}^d$ by

$$u_0(t,x) = \begin{cases} u(t,x), & x \in \Omega(t), \\ 0, & x \in \Omega\setminus\Omega(t) \end{cases}$$

so that then $u_0 \in L^2((0,T); L^2(\Omega;\mathbb{R}^d))$, and in this sense we will understand the embedding

$$L^2((0,T); L^2(\Omega(t);\mathbb{R}^d)) \hookrightarrow L^2((0,T); L^2(\Omega;\mathbb{R}^d)).$$

Now we say that $u \in L^2((0,T); W^{1,2}(\Omega(t);\mathbb{R}^d))$ if it holds

- $u \in L^2((0,T); L^2(\Omega(t);\mathbb{R}^d))$
- for a.e. $t \in (0,T)$ we have $u(t) \in W^{1,2}(\Omega(t);\mathbb{R}^d)$
- $\nabla u \in L^2((0,T); L^2((\Omega(t);\mathbb{R}^{d\times d}))$, where the function $\nabla u\colon \Omega_T \to \mathbb{R}^{d\times d}$ is a.e. defined by the previous point.

The norm on this space is defined as

$$\|u\|_{L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))} := \|u\|_{L^2((0,T);L^2(\Omega(t);\mathbb{R}^d))} + \|\nabla u\|_{L^2((0,T);L^2(\Omega(t);\mathbb{R}^{d\times d}))}$$

We now can consider the mapping $u \mapsto (u_0, (\nabla u)_0)$ to be an embedding

$$L^2((0,T); W^{1,2}(\Omega(t);\mathbb{R}^d)) \hookrightarrow L^2((0,T); L^2(\Omega;\mathbb{R}^d)) \times L^2((0,T); L^2(\Omega;\mathbb{R}^{d\times d})).$$

We now define the weak convergence in $L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))$ by weak convergence in the space on the right. Explicitly, we say that

$$u_n \rightharpoonup u \quad \text{in } L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))$$

if it holds

$$(u_n)_0 \rightharpoonup u_0 \quad \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)),$$
$$(\nabla u_n)_0 \rightharpoonup (\nabla u)_0 \quad \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^{d\times d})).$$

It is easy to see that the Banach-Alaoglu theorem continues to hold: If

$$\|u_n\|_{L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))} \leq C$$

then we can choose subsequence (not relabeled) such that

$$(u_n)_0 \rightharpoonup w \quad \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d))$$
$$(\nabla u_n)_0 \rightharpoonup A \quad \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^{d\times d}))$$

Clearly $\operatorname{supp} w \subset \Omega_T$, so that $w = u_0$ for some $u \in L^2((0,T);L^2(\Omega(t);\mathbb{R}^d))$. By choosing $\Phi \in C_c^\infty(\Omega_T;\mathbb{R}^{d\times d})$ we find that, as $n \to \infty$

$$\int_0^T \int_{\Omega(t)} u_n \cdot \operatorname{div} \Phi \, dx\, dt \to \int_0^T \int_{\Omega(t)} u \cdot \operatorname{div} \Phi \, dx\, dt,$$

where by integration by parts on the left this is equal to

$$= -\int_0^T \int_{\Omega(t)} \nabla u_n : \Phi \, dx \to -\int_0^T \int_{\Omega(t)} A : \Phi \, dx\, dt$$

showing that $A = (\nabla u)_0$.

We define the subspace of functions that have zero normal trace as

$$\begin{aligned} L^2((0,T);&W_n^{1,2}(\Omega(t);\mathbb{R}^d)) \\ &:= \{u \in L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d)) : u(t)|_{\partial\Omega(t)} \cdot n(t) = 0 \text{ on } \partial\Omega(t) \text{ for a.e. } t \in (0,T)\} \end{aligned}$$

where $u(t)|_{\partial\Omega(t)} \in W^{1/2,2}(\partial\Omega(t);\mathbb{R}^d)$ is the trace of $u(t)$. It is clear that this subspace is closed under the weak convergence in $L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))$, by compactness of the trace operator.

Finally, we consider also the space of divergence-free functions:

$$L^2((0,T);W^{1,2}_{\operatorname{div}}(\Omega(t);\mathbb{R}^d)) := \{u \in L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d)) : \operatorname{div} u = 0\},$$
$$L^2((0,T);W^{1,2}_{\operatorname{div},n}(\Omega(t);\mathbb{R}^d)) := \{u \in L^2((0,T);W^{1,2}_n(\Omega(t);\mathbb{R}^d)) : \operatorname{div} u = 0\}.$$

We call the moving domain admissible if there exists $w \in L^2((0,T);W^{1,2}_{\operatorname{div}}(\Omega(t);\mathbb{R}^d))$ with $w \cdot n = \tilde{n}_t$.

To define spaces with time derivative

$$W^{1,2}((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d)),$$

we proceed as follows. We say that

$$u \in W^{1,2}((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d)),$$

if there exists $\partial_t u \in L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))$, which is a time derivative of u in the sense that for any $\varphi \in C_c^1(\Omega_T)$ it holds

$$\int_0^T \int_{\Omega(t)} \partial_t u \cdot \varphi \, dx\, dt = -\int_0^T \int_{\Omega(t)} u \cdot \partial_t \varphi \, dx\, dt.$$

Spaces with higher derivatives or different integrability shall be defined analogously.

2.3.1. *Convergence of domains.* Now consider a sequence of moving domains. That is, for $i \in \mathbb{N}$ each $\Omega_T^{(i)}$ is a time dependent domain, as well as Ω_T. We say that $\Omega_T^{(i)} \to \Omega_T$ if they converge in the Hausdorff metric in space, uniformly in time, that is

$$\sup_{t\in(0,T)} \mathcal{H}(\Omega_T^{(i)}, \Omega_T) \to 0, \quad \text{where } \mathcal{H}(\Omega_T^{(i)}, \Omega_T) = \max\left(\sup_{x\in\Omega_T^{(i)}} \operatorname{dist}(x, \Omega_T), \sup_{y\in\Omega_T} \operatorname{dist}(y, \Omega_T^{(i)}) \right)$$

as $i \to \infty$. It is easy to see that it implies the convergence of their characteristic functions in L^1:

$$\chi_{\Omega_T^{(i)}} \to \chi_{\Omega_T} \quad \text{in } L^1((0,T)\times\Omega)$$

and in fact for any any L^q, $1 \le q < \infty$, as can be seen by

$$\left\| \chi_{\Omega_T^{(i)}} - \chi_{\Omega_T} \right\|_{L^q((0,T)\times\Omega)} = \left| \chi_{\Omega_T} \vartriangle \chi_{\Omega_T^{(i)}} \right|^{1/q} = \left\| \chi_{\Omega_T^{(i)}} - \chi_{\Omega_T} \right\|^{1/q}_{L^1((0,T)\times\Omega)},$$

where $\vartriangle$ denotes the symmetric difference. Observe also that whenever $K \subset \Omega_T$ is a compact set, then $K \subset \Omega_T^{(i)}$ for all i large enough. This will later be in particular important for us when K is the support of our chosen test function.

2.3.2. *Convergence of functions on different domains.* Let $u^{(i)} \in L^2((0,T); W^{1,2}(\Omega^{(i)}(t); \mathbb{R}^d))$ and $u \in L^2((0,T); W^{1,2}(\Omega(t); \mathbb{R}^d))$ where we have a sequence of converging time dependent domains $\Omega_T^{(i)} \to \Omega_T$. The space normals to these domains are denoted by n the normal to $\Omega(t)$, $n^{(i)}$ the normal to $\Omega^{(i)}$.

In this context we can define the weak convergences by zero extension, that is we say that

$$u^{(i)} \overset{\eta}{\rightharpoonup} u \quad \text{in } L^2((0,T); W^{1,2}(\Omega(t); \mathbb{R}^d))$$

if it holds

$$\begin{aligned} u_0^{(i)} &\rightharpoonup u_0 \quad \text{in } L^2((0,T); L^2(\Omega; \mathbb{R}^d)), \\ (\nabla u^{(i)})_0 &\rightharpoonup (\nabla u)_0 \quad \text{in } L^2((0,T); L^2(\Omega; \mathbb{R}^{d\times d})). \end{aligned}$$

We can see that the Banach-Alaoglu theorem continues to hold: If

$$\|u^{(i)}\|_{L^2((0,T);W^{1,2}(\Omega^{(i)}(t);\mathbb{R}^d))} \le C$$

then we can choose subsequence (not relabeled) such that

$$\begin{aligned} (u^{(i)})_0 &\rightharpoonup w \quad \text{in } L^2((0,T); L^2(\Omega; \mathbb{R}^d)) \\ (\nabla u^{(i)})_0 &\rightharpoonup A \quad \text{in } L^2((0,T); L^2(\Omega; \mathbb{R}^{d\times d})) \end{aligned}$$

By the convergence $\Omega_T^{(i)} \to \Omega_T$ we can easily see $\operatorname{supp} w \subset \Omega_T$, so that $w = u_0$ for some $u \in L^2((0,T); L^2(\Omega(t); \mathbb{R}^d))$. By choosing $\Phi \in C_c^\infty(\Omega_T; \mathbb{R}^{d\times d})$ we find that for any i large enough enough it holds $\Phi \in C_c^\infty(\Omega_T^{(i)}; \mathbb{R}^{d\times d})$, so that

$$\int_0^T \int_{\Omega^{(i)}(t)} u^{(i)} \cdot \operatorname{div} \Phi \, dx \, dt \to \int_0^T \int_{\Omega(t)} u \cdot \operatorname{div} \Phi \, dx \, dt$$

is equal to

$$= -\int_0^T \int_{\Omega^{(i)}(t)} \nabla u^{(i)} : \Phi \, dx \to -\int_0^T \int_{\Omega(t)} A : \Phi \, dx \, dt$$

(remember that $\chi_{\Omega^{(i)}}(t)$ converges strongly) showing that $A = (\nabla u)_0$.

Proposition 2.2. *If $u^{(i)} \in L^2((0,T); W_n^{1,2}(\Omega^{(i)}(t); \mathbb{R}^d))$ and $u^{(i)} \overset{\eta}{\rightharpoonup} u$, then the limit satisfies $u \in L^2((0,T); W_n^{1,2}(\Omega(t); \mathbb{R}^d))$. In other words, zero normal trace is preserved under weak convergence.*

Proof. Let us take a test function $\xi \in C^1_c((0,T)\times\mathbb{R}^d)$. Then we compute for a.e. $t\in(0,T)$ by the divergence theorem

$$\begin{aligned} 0 = \int_0^T \int_{\partial\Omega^{(i)}(t)} \xi u^{(i)} \cdot dn^{(i)}\,dt &= \int_0^T \int_{\Omega^{(i)}(t)} \nabla\xi\cdot u^{(i)} + \xi \operatorname{div} u^{(i)}\,dx\,dt \\ &\to \int_0^T\int_{\Omega(t)} \nabla\xi\cdot u + \xi\operatorname{div} u\,dx\,dt = \int_0^T\int_{\partial\Omega(t)} \xi u\cdot dn\,dt \end{aligned}$$

and since ξ was arbitrary, this shows $u\cdot n = 0$ on $\partial_x\Omega_T$. The convergence for $i\to\infty$ goes through, as it holds

$$\begin{aligned} u_0^{(i)} &\rightharpoonup u_0 && \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)),\\ (\operatorname{div} u^{(i)})_0 &\rightharpoonup (\operatorname{div} u)_0 && \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)),\\ \chi_{\Omega_T^{(i)}} &\to \chi_{\Omega_T} && \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)). \end{aligned}$$

□

In fact we can also treat converging normal boundary values.

Lemma 2.3 (Convergence of normal trace)**.** *Let $\Omega^{(i)}\to\Omega(t)$, $u^{(i)} \overset{\eta}{\rightharpoonup} u$ in $L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d))$ and $u^{(i)}\cdot n^{(i)}|_{\partial\Omega^{(i)}} = \phi^{(i)}$ with $\phi^{(i)}\colon \partial\Omega^{(i)}\to\mathbb{R}$ given such that $\phi^{(i)}\rightharpoonup\phi$ in the sense that $\phi^{(i)}\mathcal{H}^{d-1}|_{\partial\Omega^{(i)}} \overset{*}{\rightharpoonup} \phi\mathcal{H}^{d-1}|_{\partial\Omega(t)}$ as measures $M([0,T]\times\mathbb{R}^d)$, ie. for all $\psi\in C([0,T]\times\mathbb{R}^d)$ it holds*

$$\int_0^T\int_{\partial\Omega^{(i)}} \psi\phi^{(i)}\,dS\,dt \to \int_0^T\int_{\partial\Omega(t)} \psi\phi\,dS\,dt.$$

Then it holds $u\cdot n|_{\partial\Omega(t)} = \phi$. In other words

$$u^{(i)} \overset{\eta}{\rightharpoonup} u \implies u^{(i)}\cdot n^{(i)}\mathcal{H}^{d-1}|_{\partial\Omega^{(i)}} \overset{*}{\rightharpoonup} u\cdot n\mathcal{H}^{d-1}|_{\partial\Omega(t)}$$

Proof. It is sufficient to show that for all $\psi\in C([0,T]\times\mathbb{R}^d)$ it holds that

$$\int_0^T\int_{\partial\Omega(t)} \psi u\cdot dn = \int_0^T\int_{\partial\Omega(t)} \psi\phi\,dS.$$

We know that

$$\int_0^T\int_{\partial\Omega^{(i)}(t)} \psi u^{(i)}\cdot dn^{(i)}\,dt = \int_0^T\int_{\partial\Omega^{(i)}(t)} \psi\phi^{(i)}\,dS\,dt \to \int_0^T\int_{\partial\Omega(t)} \psi\phi\,dS\,dt$$

Rewrite the left side by the divergence theorem

$$\begin{aligned} \int_0^T\int_{\partial\Omega^{(i)}(t)} \psi u^{(i)}\cdot dn^{(i)}\,dt &= \int_0^T\int_{\Omega^{(i)}(t)} \nabla\psi\cdot u^{(i)} + \psi\operatorname{div} u^{(i)}\,dx\,dt \\ &\to \int_0^T\int_{\Omega(t)} \nabla\psi\cdot u + \psi\operatorname{div} u\,dx\,dt = \int_0^T\int_{\partial\Omega(t)} \psi u\cdot dn\,dt \end{aligned}$$

and we are finished. □

2.4. **Approximation of test functions.**

2.4.1. *Eulerian to Lagrangian and back.* We will have the deformation η mapping the Lagrangian solid Q to the deformed configuration $\eta(Q)$. Here we shall see that under the $W^{2,q}$ regularity, one can switch between Lagrangian and Eulerian domains at no cost. This statement is made precise below.

Lemma 2.4. *Let $\eta\in W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d)\cap L^\infty((0,T);W^{2,q}(Q;\mathbb{R}^d))$ with $\det\nabla\eta\geq\varepsilon_0$ and $\eta(t,\cdot)$ injective for all t. The mapping $\xi\mapsto\xi\circ\eta$ is linear bounded operator between the spaces*

$$\begin{aligned} L^1((0,T);W^{2,q}(\eta(Q);\mathbb{R}^d)) &\cap W^{1,1}_0((0,T);L^2(\eta(Q);\mathbb{R}^d)) \\ &\to L^1((0,T);W^{2,q}(Q;\mathbb{R}^d))\cap W^{1,1}_0((0,T);L^2(Q;\mathbb{R}^d)) \end{aligned}$$

with the bound C_η depending only on ε_0, $\|\eta\|_{L^\infty((0,T);W^{2,q}(Q;\mathbb{R}^d))}$ and $\|\partial_t\eta\|_{L^\infty((0,T);L^2(Q;\mathbb{R}^d))}$.

Proof. Linearity is clear. As in [BKS23b, A.4], we calculate

$$\begin{aligned}\|\nabla^2(\xi\circ\eta)\|_{L^q(Q)} &= \|(\nabla^2\xi\circ\eta\cdot\nabla\eta)\cdot\nabla\eta+\nabla\xi\circ\eta\cdot\nabla^2\eta\|_{L^q(Q)}\\ &\le \|\nabla\eta\|^2_{L^\infty(Q)}\|\nabla^2\xi\circ\eta\|_{L^q(Q)}+\|\nabla\xi\|_{L^\infty(\eta(Q))}\|\nabla^2\eta\|_{L^q(Q)}\end{aligned}$$

and use

$$\|\nabla^2\xi\circ\eta\|^q_{L^q(Q)} \le \int_Q |\nabla^2\xi\circ\eta|^q\frac{\det\nabla\eta}{\varepsilon_0}\,dx = \frac{1}{\varepsilon_0}\|\nabla^2\xi\|^q_{L^q(\eta(Q))}$$

which then proves

$$\|\xi\circ\eta\|_{L^1((0,T);W^{2,q}(Q;\mathbb{R}^d))} \le C_\eta\|\xi\|_{L^1((0,T);W^{2,q}(\eta(t,Q);\mathbb{R}^d))}$$

Now for the time derivative, compute

$$\partial_t(\xi\circ\eta) = \partial_t\xi\circ\eta+(\nabla\xi\circ\eta)\partial_t\eta$$

In the first term, we compute using the, we can see that

$$\|\partial_t\xi\circ\eta\|^2_{L^2(Q;\mathbb{R}^d)} \le \int_Q |\partial_t\xi\circ\eta|^2\frac{\det\nabla\eta}{\varepsilon_0}\,dx = \frac{1}{\varepsilon_0}\|\partial_t\xi\|^2_{L^2(\eta(Q);\mathbb{R}^d)}.$$

In the second term, with C coming from the Morrey inequality

$$\begin{aligned}\|(\nabla\xi\circ\eta)\partial_t\eta\|_{L^2(Q;\mathbb{R}^d)} \le \|\nabla\xi\circ\eta\|_{L^\infty(Q;\mathbb{R}^d)}\|\partial_t\eta\|_{L^2(Q;\mathbb{R}^d)} &= \|\nabla\xi\|_{L^\infty(\eta(Q);\mathbb{R}^d)}\|\partial_t\eta\|_{L^2(Q;\mathbb{R}^d)}\\ &\le C\|\nabla^2\xi\|_{L^q(\eta(Q);\mathbb{R}^d)}\|\partial_t\eta\|_{L^2(Q;\mathbb{R}^d)}\end{aligned}$$

In total, this shows that

$$\|\partial_t(\xi\circ\eta)\|_{L^1((0,T);L^2(Q))} \le \frac{1}{\varepsilon_0}\|\partial_t\xi\|_{L^1((0,T);L^2(Q))}+C\|\partial_t\eta\|_{L^\infty((0,T);L^2(Q))}\|\xi\|_{L^1((0,T);W^{2,q}(\eta(Q)))}$$

which concludes the proof. ☐

Lemma 2.5. *Let $\eta\in W^{2,q}(Q;\mathbb{R}^d)\cap W^{k_0,2}(Q;\mathbb{R}^d)$ with $\det\nabla\eta\ge\varepsilon_0$ be globally injective. Then the mapping*

$$\xi\mapsto\xi\circ\eta$$

is an isomorphism $W^{k_0,2}(\eta(Q);\mathbb{R}^d)\to W^{k_0,2}(Q;\mathbb{R}^d)$ with norm depending only on $\|\eta\|_{W^{2,q}(Q;\mathbb{R}^d)}$ and ε_0.

Proof. Proof follows from application of the chain rule to $\nabla^{k_0}(\xi\circ\eta)$, the embedding $W^{2,q}\hookrightarrow W^{1,\infty}$ and interpolation in the product which results from this chain rule, see [BKS23b, Lemma A.3]. ☐

2.4.2. *Extending the divergence-free domain.* We recall here [BKS23b, Proposition 2.22], which will be useful for convergences in the coupled euqation.

Proposition 2.6 (Approximation of coupled test functions)**.** *Fix a function*

$$\eta\in L^\infty([0,T];\mathcal{E})\cap W^{1,2}([0,T];W^{1,2}(Q;\mathbb{R}^d)) \qquad \text{with } \sup_{t\in T}E(\eta(t))<\infty$$

such that $\eta(t)\notin\partial\mathcal{E}$ for all $t\in[0,T]$. As before, set $\Omega(t)=\Omega\setminus\eta(t,Q)$. Let $\mathcal{T}_\eta$ be the set of admissible test functions, defined as

$$\begin{aligned}\mathcal{T}_\eta := \Big\{&(\phi,\xi)\in W^{1,2}\Big([0,T];W^{1,2}\Big(Q;\mathbb{R}^d\Big)\Big)\times L^2\Big([0,T];W^{1,2}_0\Big(\Omega;\mathbb{R}^d\Big)\Big):\\ &\phi=\xi\circ\eta \text{ on } [0,T]\times Q \text{ and } \operatorname{div}\xi(t)=0 \text{ in } \Omega(t)\}.\end{aligned}$$

Then the set

$$\tilde{\mathcal{T}}_\eta := \Big\{(\phi,\xi)\in\mathcal{T}_\eta \mid \xi\in C^\infty\Big([0,T];C^\infty_0\Big(\Omega;\mathbb{R}^d\Big)\Big),\operatorname{div}\xi(t,y)=0\Big\}$$

for all $t\in[0,T]$ and all y with $\operatorname{dist}(y,\Omega(t))<\varepsilon$ for some $\varepsilon>0$ is dense in $\mathcal{T}_\eta$ in the following sense: For every ε sufficiently small there exists a linear map $(\phi,\xi)\mapsto(\phi_\varepsilon,\xi_\varepsilon)\in\tilde{\mathcal{T}}_\eta$ such that

$$\operatorname{div}(\xi_\varepsilon(t,y))=0 \qquad \text{for all } y\in\Omega \text{ with } \operatorname{dist}(y,\Omega(t))\le\varepsilon.$$

Moreover, if $\xi \in L^2((0,T); W^{k_0,2}(\Omega(t);\mathbb{R}^d)$ *then*

$$\xi_\varepsilon \to \xi \quad \text{in } L^2((0,T); W^{k_0,2}(\Omega(t);\mathbb{R}^d) \quad \text{as } \varepsilon \to 0$$

and if $\eta \in L^\infty((0,T); W^{k_0+2,2}(Q;\mathbb{R}^d))$*, then*

$$\phi_\varepsilon \to \phi \quad \text{in } L^\infty((0,T); W^{k_0+2,2}(Q;\mathbb{R}^d)) \cap W^{1,2}((0,T); W^{1,2}(Q;\mathbb{R}^d))$$

Moreover the following bounds hold

$$\begin{aligned}
\|\xi_\varepsilon(t)\|_{W^{1,2}(\Omega(t);\mathbb{R}^d)} &\le c\|\xi(t)\|_{W^{1,2}\left(\Omega(t);\mathbb{R}^d\right)} \\
\|\xi_\varepsilon(t) - \xi(t)\|_{L^2(\Omega(t))} &\le c\varepsilon^{\frac{2}{d+2}}\|\xi(t)\|_{W^{1,2}(\Omega(t))} \\
\|\xi_\varepsilon(t)\|_{W^{k,a}(\Omega(t))} &\le c(\varepsilon)\|\xi(t)\|_{L^2(\Omega(t);\mathbb{R}^n)} \\
\|\phi_\varepsilon(t)\|_{W^{k,a}(Q)} &\le c\|\xi(t)\|_{C^k(\Omega)}\|\eta(t)\|_{W^{k,a}(Q)} \le c(\varepsilon)\|\xi(t)\|_{L^2(\Omega)}\|\eta(t)\|_{W^{k,a}(Q)}
\end{aligned}$$

For a proof see [BKS23b, Proposition 2.22].

2.4.3. *Universal Bogovskii.* We state here a version of the universal Bogovski operator. It is universal in the sense that we have the very same operator for domains which are similar enough, in a suitable sense. Classically, similar enough means star shaped with respect to the same ball [GHH06], and graph domains in the same coordinates [KSS23]. We shall further extend this to similar Lipschitz domains, respectively to close deformations.

Theorem 2.7 (Universal Bogovskii)**.** *Let* Ω *be a bounded L-Lipschitz domain. Fix a finite covering* $\mathcal{G}$ *of its boundary by L-Lipschitz graphs; by this we mean that* $\mathcal{G}$ *consists of open rectangles such that for each* $G \in \mathcal{G}$, $G \cap \Omega$ *is a subgraph of an L-Lipschitz function (in some direction). Let* $b \in C_c^\infty((0,T)\times\Omega)$ *with* $\int_\Omega b(t) = 1$ *for all* t*. Then there exists a universal Bogovskii operator* $\mathcal{B}\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega;\mathbb{R}^d)$ *which satisfies*

$$\operatorname{div}\mathcal{B}f = f - b\int_\Omega f$$

and moreover for any L-Lipschitz domain $\tilde\Omega$ *such that* $\mathcal{G}$ *is also a covering of its boundary by L-Lipschitz graphs (in the same sense as above) it also satisfies* $\mathcal{B}\colon C_c^\infty(\tilde\Omega) \to C_c^\infty(\tilde\Omega;\mathbb{R}^d)$ *and extends to a bounded operator*

$$\mathcal{B}\colon W_0^{k,p}(\tilde\Omega) \to W_0^{k+1,p}(\tilde\Omega;\mathbb{R}^d)$$

with norm of this operator depending on $\mathcal{G}$ *and* Ω *but independent of* $\tilde\Omega$*.*

Proof idea. Restricted to one rectangle $G \in \mathcal{G}$, this is shown in [KSS23, Corollary 3.4]. In the proof therein, the rectangle is covered by slices such that two neighboring slices are star-shaped with respect with the same cube and through a subordinate partition of unity the operator is extended from one slice to the next. In our case, the slices at the edges of the neighboring rectangles overlap, and in this way the construction can be extended from one rectangle to the next. For more details of the construction see [KSS23], proof of Theorem 3.3 and Corollary 3.4. □

Theorem 2.8 (Bogovskii for close deformations)**.** *Let* $\eta \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T); W^{1,2}(Q;\mathbb{R}^d)$*. Then for* $\delta > 0$ *small enough the following holds: Fix* $b \in C_c^\infty((0,T)\times\Omega_\eta)$ *with* $\int_{\Omega_\eta} b = 1$ *and* $\operatorname{dist}(\operatorname{supp} b, \partial\Omega_\eta) \ge \delta$*. Then exists a universal Bogovskii operator* $\mathcal{B}$ *for domains* Ω_η*, that is*

$$\mathcal{B}\colon C_c^\infty((0,T)\times\Omega_\eta) \to C_c^\infty((0,T)\times\Omega_\eta;\mathbb{R}^d)$$

which satisfies

$$\operatorname{div}\mathcal{B}f = f - b\int_{\Omega_\eta} f$$

and for every $\tilde\eta \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T); W^{1,2}(Q;\mathbb{R}^d)$ *with*

$$\|\eta(t) - \tilde\eta(t)\|_{W^{2,q}(Q;\mathbb{R}^d)} \le \gamma(\delta) \tag{2.1}$$

the operator $\mathcal{B}$ maps $\mathcal{B}\colon C_c^\infty((0,T)\times\Omega_{\tilde\eta})\to C_c^\infty((0,T)\times\Omega_{\tilde\eta};\mathbb{R}^d)$ and is moreover extended to a bounded operator for $1\le r\le\infty$ and $1<p<\infty$

$$\mathcal{B}\colon L^r((0,T);W_0^{k,p}(\Omega_{\tilde\eta}))\to L^r((0,T);W_0^{k+1,p}(\Omega_{\tilde\eta};\mathbb{R}^d))$$

with norm independent of $\tilde\eta$.

Proof. By the $C([0,T];C^{1,\alpha}(Q;\mathbb{R}^d))$ regularity of η, we can find a partition of the time interval $0=t_1<t_2<\dots<t_M=T$, such that on each interval (t_i,t_{i+2}) the assumptions of Theorem 2.7 are satisfied. Namely that there is a covering $\mathcal{G}_i$ which covers the boundary of $\Omega_\eta(t)$, $t\in(t_{i-1},t_i)$ by L-Lipschitz graphs and for any given $\tilde\eta$ as in the statement, $\mathcal{G}_i$ also covers the boundary of $\Omega_{\tilde\eta}(t)$, $t\in(t_{i-1},t_i)$ by L-Lipschitz graphs (because $W^{2,q}$ embeds into Lipschitz functions and the closeness (2.1)) so that Theorem 2.7 above gives us the Bogovskii operator $\mathcal{B}_i$. Consider a partition of unity $\{\psi_i\}$ on $(0,T)$ subordinate to the covering (t_i,t_{i+2}) we define in total the sought Bogovskii operator by

$$\mathcal{B}(f)(t)=\sum_{i=1}^{M-2}\psi_i(t)\mathcal{B}_i(f(t)).$$

All the desired properties of $\mathcal{B}$ follow. □

2.4.4. *Approximating fluid-only test functions.* Here, analogously to above, we state an approximation result for the test functions, this time to be used in the fluid-only equation. This will go by extending the normals and tangents to the deformed domains, by means of extending the deformation.

Extensions of η, and the normal and tangent. Throughout this section, assume that we have the limit deformation

$$\eta\in L^\infty((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))\cap W^{1,2}((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))\text{ with }E(\eta(t))\le E_0$$

We now consider for a.e. t an extension of $\eta(t)$ to $\overline{\eta}(t)\colon\mathbb{R}^d\to\mathbb{R}^d$ such that

$$\|\overline{\eta}(t)\|_{W^{k_0+2,2}(\mathbb{R}^d)}\le C\|\eta\|_{W^{k_0+2,2}(Q)}$$

(it can be done by a bounded linear extension operator $\overline{\cdot}\colon W^{k_0+2,2}(Q;\mathbb{R}^d)\to W^{k_0+2,2}(\mathbb{R}^d;\mathbb{R}^d)$), so then we have

$$\|\overline{\eta}\|_{L^\infty((0,T);W^{k_0+2,2}(\mathbb{R}^d;\mathbb{R}^d))}\le C\|\eta\|_{L^\infty((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))}$$
$$\|\partial_t\overline{\eta}\|_{L^2((0,T);W^{k_0+2,2}(\mathbb{R}^d;\mathbb{R}^d))}\le C\|\partial_t\eta\|_{L^2((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))}$$

and for some $\delta>0$ we have on Q_δ, a δ-neighborhood of Q,

$$\det\nabla\overline{\eta}\ge\varepsilon_0/2\quad\text{in }(0,T)\times Q_\delta$$

so that $\overline{\eta}$ is locally injective on Q_δ, and it is even globally injective for small enough δ, because η is. Moreover, δ depends only on E_0.

Extension of normal and tangent.

We assume the reference domain Q has C^∞ boundary, so we have the reference normal $n_Q\in C^\infty(\partial Q;\mathbb{R}^d)$, and then the deformed normal $n\colon\partial\eta(Q)\to\mathbb{R}^d$ to the deformed configuration $\eta(Q)$ is

$$n(y)=\frac{[\operatorname{cof}\nabla\eta(\eta^{-1}(y))]n_Q(\eta^{-1}(y))}{|[\operatorname{cof}\nabla\eta(\eta^{-1}(y))]n_Q(\eta^{-1}(y))|}$$

We now extend n_Q to P_δ, a δ-neighborhood of ∂Q, so that we have $\overline{n}_Q\in C^{k_0}(P_\delta;\mathbb{R}^d)$.

Then we define the extension of the deformed normal

$$\overline{n}(y)=\frac{[\operatorname{cof}\nabla\overline{\eta}(\overline{\eta}^{-1}(y))]\overline{n}_Q(\overline{\eta}^{-1}(y))}{|[\operatorname{cof}\nabla\overline{\eta}(\overline{\eta}^{-1}(y))]\overline{n}_Q(\overline{\eta}^{-1}(y))|}$$

for y in ε-neighborhood of $\eta(Q)$, where ε is given by δ and the energy bound E_0.

By Lemma 2.5 we see that $\overline{n}$ inherits all the regularity of $\overline{n}_Q$ up to $W^{k_0,2}$. In particular that $\overline{n}\in W^{k_0,2}(\overline{\eta}(Q_\delta);\mathbb{R}^d)$.

To extend the tangent field, recall that the projection to the tangent plane is $I - n \otimes n$, that is for a vector $v \in \mathbb{R}^d$

$$[v]_\tau = [I - n \otimes n]v = v - (v \cdot n)n$$

is the tangential component of v. Thus for the extended normal $\overline{n}$ and a vector field ξ, we consider its extended tangential component by

$$[\xi]_{\overline{\tau}}(y) = [\xi(y)]_{\overline{\tau}} = [I - \overline{n}(y) \otimes \overline{n}(y)]v(y) = v(y) - (v(y) \cdot \overline{n}(y))\overline{n}(y).$$

Here it is apparent that $[\xi]_{\overline{\tau}}$ inherits the regularity of $\overline{n}$ up to $W^{k_0,2}$ due to Lemma 2.5.

The approximations.

Proposition 2.9 (Approximation of fluid-only test functions)**.** *Let* $\eta^{(i)} \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d))$ *with* $E(\eta^{(i)}(t)) \leq E_0$ *and*

$$\begin{aligned} \eta^{(i)} &\overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty((0,T);W^{2,q}(Q;\mathbb{R}^d)) \\ \partial_t\eta^{(i)} &\rightharpoonup \partial_t\eta \quad \text{in } L^2((0,T);W^{1,2}(Q;\mathbb{R}^d)) \end{aligned}$$

so that also $\Omega \setminus \eta^{(i)}(t,Q) = \Omega^{(i)}(t) \to \Omega(t) = \Omega \setminus \eta(t,Q)$*. Let* $\xi \in C^\infty(\Omega(t);\mathbb{R}^d)$*,* $\xi \cdot n = 0$ *on* $\partial\Omega(t)$*, and* $\operatorname{div}\xi = 0$ *in* $\Omega(t)$*.*

(i) Then there is $\xi_\varepsilon^{(i)}$ *such that for every* $\varepsilon > 0$

$$\begin{aligned} \xi_\varepsilon^{(i)} &\overset{\eta}{\to} \xi_\varepsilon \quad \text{in } L^2((0,T);W^{1,2}(\Omega;\mathbb{R}^d),) \\ \partial_t\xi_\varepsilon^{(i)} &\overset{\eta}{\to} \partial_t\xi_\varepsilon \quad \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)) \end{aligned}$$

such that $\xi_\varepsilon^{(i)}(t) \in W^{1,2}(\Omega^{(i)}(t);\mathbb{R}^d)$*,* $\operatorname{div}\xi_\varepsilon^{(i)} = 0$ *in* $\Omega^{(i)}(t)$*,* $\xi_\varepsilon^{(i)} \cdot \overline{n}^{(i)} = 0$ *on an* ε*-neighborhood of* $\partial\Omega(t)$ *(that includes* $\partial\Omega^{(i)(t)}$ *for i large enough) and* $\xi_\varepsilon(t) \in W^{1,2}(\Omega(t);\mathbb{R}^d)$*,* $\xi_\varepsilon \cdot \overline{n} = 0$ *on an* ε*-neighborhood of* $\partial\Omega(t)$*with*

$$\begin{aligned} \xi_\varepsilon &\overset{\eta}{\to} \xi \quad \text{in } L^2((0,T);W^{1,2}(\Omega;\mathbb{R}^d)), \\ \partial_t\xi_\varepsilon &\overset{\eta}{\to} \partial_t\xi \quad \text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)). \end{aligned}$$

Moreover, the following estimates hold:

$$\begin{aligned} \|\xi_\varepsilon^{(i)}(t) - \xi_\varepsilon(t)\|_{W^{1,2}} &\leq c\varepsilon^{-2q(q-2)}\|\eta(t) - \eta^{(i)}(t)\|_{W^{2,q}} \\ \|\xi_\varepsilon(t) - \xi(t)\|_{W^{1,2}} &\leq c\varepsilon^{\frac{1}{2}}\|\xi(t)\|_{W^{1,\infty}} \end{aligned}$$

with c only depending on E_0*.*

(ii) If additionally

$$\eta^{(i)} \rightharpoonup \eta \quad \text{in } W^{1,2}((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))$$

then it also holds

$$\xi_\varepsilon^{(i)} \overset{\eta}{\to} \xi_\varepsilon \quad \text{in } L^2((0,T);W^{k_0,2}(\Omega;\mathbb{R}^d))$$

with the following estimates:

$$\|\xi_\varepsilon^{(i)}(t) - \xi_\varepsilon(t)\|_{W^{k_0,2}} \leq c\varepsilon^{-k_0}\|\xi(t)\|_{W^{k_0,2}}$$

where c depends only on E_0 *and* $\|\eta\|_{W^{k_0,2}}$*.*

If moreover, $\nabla^\ell(\xi \cdot n) = 0$*,* $\ell = 1, \ldots, k_0$*, then*

$$\xi_\varepsilon \overset{\eta}{\to} \xi \quad \text{in } L^2((0,T);W^{k_0,2}(\Omega;\mathbb{R}^d)$$

with the estimate

$$\|\xi_\varepsilon(t) - \xi(t)\|_{W^{k_0,2}} \leq c\varepsilon\|\xi(t)\|_{W^{k_0+1,2}}.$$

where c depends only on E_0 *and* $\|\eta\|_{W^{k_0,2}}$*.*

Proof. Let for now $\varepsilon > 0$ be fixed. Consider a smooth cutoff ψ_ε which is 1 on ε-neighborhood of $\partial\eta(Q)$ and vanishes outside its 2ε-neighborhood. That is, $\psi_\varepsilon \in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ such that $\psi_\varepsilon(y) = 1$ if $\operatorname{dist}(y, \partial\eta(Q)) \le \varepsilon$ and $\psi_\varepsilon(y) = 0$ if $\operatorname{dist}(y, \partial\eta(Q)) \ge 2\varepsilon$.

Let

$$\tilde{\xi}_\varepsilon^{(i)} = \psi_\varepsilon\left((\xi\cdot\overline{\tau})\overline{\tau}^{(i)}\right) + (1-\psi_\varepsilon)\xi \quad \text{in } \Omega^{(i)}(t)$$

and

$$\tilde{\xi}_\varepsilon = \psi_\varepsilon\left((\xi\cdot\overline{\tau})\overline{\tau}\right) + (1-\psi_\varepsilon)\xi \quad \text{in } \Omega(t).$$

These satisfy all the conditions except being divergence free.

So instead we put, where $\mathcal{B}$ is the universal Bogovskii operator from Theorem 2.8

$$\xi_\varepsilon^{(i)} = \tilde{\xi}_\varepsilon^{(i)} - \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon^{(i)})$$

so that then we get

$$\operatorname{div}\xi_\varepsilon^{(i)} = \varphi\int_{\Omega^{(i)}}\operatorname{div}\tilde{\xi}_\varepsilon^{(i)}\,dx = \varphi\int_{\partial\Omega^{(i)}}\tilde{\xi}_\varepsilon^{(i)}\cdot dn^{(i)} = 0$$

where the integral is zero because $\tilde{\xi}_\varepsilon^{(i)}\cdot n^{(i)} = 0$ on $\partial\Omega^{(i)}$. Define also

$$\xi_\varepsilon = \tilde{\xi}_\varepsilon - \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon)$$

so that then

$$\xi_\varepsilon - \xi_\varepsilon^{(i)} = \tilde{\xi}_\varepsilon - \tilde{\xi}_\varepsilon^{(i)} - \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon) + \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon^{(i)})$$

Now $\xi_\varepsilon, \xi_\varepsilon^{(i)}$ satisfy the given boundary conditions and divergence free condition as required. It is left to check the convergences and estimates.

(i): Since it holds

$$\tilde{\xi}_\varepsilon - \tilde{\xi}_\varepsilon^{(i)} = \psi_\varepsilon(\xi\cdot\overline{\tau})(\overline{\tau} - \overline{\tau}^{(i)}),$$

we compute

$$\begin{aligned}\|\nabla(\tilde{\xi}_\varepsilon - \tilde{\xi}_\varepsilon^{(i)})\|_{L^2} &= \left\|\nabla(\psi_\varepsilon(\xi\cdot\overline{\tau}))(\overline{\tau}-\overline{\tau}^{(i)}) + \psi_\varepsilon(\xi\cdot\overline{\tau})\nabla(\overline{\tau}-\overline{\tau}^{(i)})\right\|_{L^2}\\ \le \|\nabla(\psi_\varepsilon(\xi\cdot\overline{\tau}))\|_{L^{2q/(q-2)}}\|\overline{\tau}-\overline{\tau}^{(i)}\|_{L^q} &+ \|\psi_\varepsilon(\xi\cdot\overline{\tau})\|_{L^{2q/(q-2)}}\|\nabla(\overline{\tau}-\overline{\tau}^{(i)})\|_{L^q}\\ &\le C\varepsilon^{-2q/(q-2)}\|\eta-\eta^{(i)}\|_{W^{2,q}}\end{aligned}$$

so by the continuity of the Bogovskii operator from Theorem 2.8 we have the same estimate for ξ_ε, $\xi_\varepsilon^{(\tau)}$

$$\|\nabla(\xi_\varepsilon - \xi_\varepsilon^{(i)})\|_{L^2} \le C\varepsilon^{-2q/(q-2)}\|\eta-\eta^{(i)}\|_{W^{2,q}}$$

which is the desired estimate.

For the limit passage $\varepsilon \to 0$, we compute

$$\xi - \xi_\varepsilon = \xi - \tilde{\xi}_\varepsilon + \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon) = \psi_\varepsilon(\xi\cdot\overline{n})\overline{n} + \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon)$$

and so

$$\nabla(\tilde{\xi} - \tilde{\xi}_\varepsilon) = \nabla(\psi_\varepsilon(\xi\cdot\overline{n})\overline{n}) = \nabla\psi_\varepsilon(\xi\cdot\overline{n})\overline{n} + \psi_\varepsilon\nabla(\xi\cdot\overline{n})\overline{n} + \psi_\varepsilon(\xi\cdot\overline{n})\nabla\overline{n}.$$

To estimate this in $L^2(\Omega(t);\mathbb{R}^d)$, the cutoff ψ_ε satisfies

$$\|\nabla\psi_\varepsilon\|_{L^\infty} \le \frac{c}{\varepsilon},$$

the function $\xi\cdot\overline{n}$ has zero trace on $\partial\eta(Q)$ so that

$$\|\xi\cdot\overline{n}\|_{L^\infty} \le c\varepsilon,$$

and finally $\|\nabla\overline{n}\|_{L^q} \le C\|\eta\|_{W^{2,q}}$. So altogether with the fact that $|\operatorname{supp}\psi_\varepsilon| \le c\varepsilon$ we obtain

$$\|\nabla(\tilde{\xi} - \tilde{\xi}_\varepsilon)\|_{L^2} = \|\nabla\psi_\varepsilon(\xi\cdot\overline{n})\overline{n} + \psi_\varepsilon\nabla(\xi\cdot\overline{n})\overline{n} + \psi_\varepsilon(\xi\cdot\overline{n})\nabla\overline{n}\|_{L^2} \le C\varepsilon^{\frac{1}{2}}\|\eta\|_{W^{2,q}}\|\xi\|_{W^{1,\infty}}$$

and thus

$$\|\xi_\varepsilon - \xi_\varepsilon^{(\tau)}\|_{W^{1,2}} \le c\varepsilon^{\frac{1}{2}}.$$

Further, since $\operatorname{div}\tilde{\xi}_\varepsilon \to 0$ in $L^2((0,T);L^2(\Omega(t))$ we have also, by continuity of the operator $\mathcal{B}$ that $\mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon) \to 0$ in $L^2((0,T);L^2(\Omega(t);\mathbb{R}^d)$ and the same inequality holds for ξ, ξ_ε.

For convergence of the time derivative with $i \to \infty$, we write
$\partial_t(\tilde{\xi}_\varepsilon^{(i)} - \tilde{\xi}_\varepsilon) = \partial_t(\psi_\varepsilon(\xi \cdot \overline{\tau})(\overline{\tau} - \overline{\tau}^{(i)})) = \partial_t\psi_\varepsilon(\xi \cdot \overline{\tau})(\overline{\tau} - \overline{\tau}^{(i)}) + \psi_\varepsilon\partial_t(\xi \cdot \overline{\tau})(\overline{\tau} - \overline{\tau}^{(i)}) + \psi_\varepsilon(\xi \cdot \overline{\tau})\partial_t(\overline{\tau} - \overline{\tau}^{(i)})$
and use the estimates

$$\|\partial_t\psi_\varepsilon\|_{L^\infty} \le \frac{c}{\varepsilon}, \quad \|\overline{\tau} - \overline{\tau}^{(i)}\|_{L^\infty} \le c\|\eta - \eta^{(i)}\|_{W^{2,q}}, \quad \|\partial_t(\overline{\tau} - \overline{\tau}^{(i)})\|_{L^2} \le c\|\partial_t\eta - \partial_t\eta^{(i)}\|_{L^2}$$

along with $|\operatorname{supp}\psi_\varepsilon| \le c\varepsilon$ to obtain the estimate

$$\|\partial_t(\tilde{\xi}_\varepsilon^{(i)} - \tilde{\xi}_\varepsilon)\|_{L^2} \le \frac{C}{\varepsilon}\|\eta\|_{W^{2,q}}\|\eta - \eta^{(i)}\|_{W^{2,q}} + \|\partial_t\xi\|_{L^2}\|\eta - \eta^{(i)}\|_{W^{2,q}} + \varepsilon\|\xi\|_{L^\infty}\|\eta\|_{W^{2,q}}\|\partial_t\eta - \partial_t\eta^{(i)}\|.$$

Finally, for the time derivative with $\varepsilon \to 0$ we write

$$\partial_t(\tilde{\xi}_\varepsilon - \tilde{\xi}) = \partial_t(\psi_\varepsilon(\xi \cdot \overline{n})\overline{n}) = \partial_t\psi_\varepsilon(\xi \cdot \overline{n})\overline{n} + \psi_\varepsilon\partial_t(\xi \cdot \overline{n})\overline{n} + \psi_\varepsilon(\xi \cdot \overline{n})\partial_t\overline{n},$$

use the estimates (the second one following from $\xi \cdot n = 0$ on $\partial\Omega(t)$)

$$\|\partial_t\psi_\varepsilon\|_{L^\infty} \le \frac{c}{\varepsilon}, \quad \|\xi \cdot \overline{n}\|_{L^\infty} \le c\varepsilon, \quad \|\overline{n}\|_{L^\infty} \le c\|\eta\|_{W^{2,q}}, \quad \|\partial_t\overline{n}\|_{L^2} \le c\|\partial_t\eta\|_{L^2}$$

as well as $|\operatorname{supp}\psi_\varepsilon| \le c\varepsilon$ to obtain the estimate

$$\|\partial_t(\tilde{\xi}_\varepsilon - \tilde{\xi})\|_{L^2} \le C\varepsilon(\|\eta\|_{W^{2,q}} + \|\partial_t\xi\|_{L^2}\|\eta\|_{W^{2,q}} + \|\partial_t\eta\|_{L^2}).$$

(ii): For $i \to \infty$, compute

$$\tilde{\xi}_\varepsilon - \tilde{\xi}_\varepsilon^{(i)} = \psi_\varepsilon(\xi \cdot \overline{\tau})(\overline{\tau} - \overline{\tau}^{(i)})$$

Since $\psi_\varepsilon(\xi \cdot \overline{\tau}) \in C^{k_0}(\Omega; \mathbb{R}^d)$, we can calculate

$$\|\nabla^\ell(\tilde{\xi}_\varepsilon - \tilde{\xi}_\varepsilon^{(i)})\|_{L^2} = \left\| \sum_{j=0}^{\ell} \nabla^{\ell-j}(\psi_\varepsilon(\xi \cdot \overline{\tau}))\nabla^j(\overline{\tau} - \overline{\tau}^{(i)}) \right\|_{L^2} \le \|\nabla^j\psi_\varepsilon\|_{L^\infty}\|\nabla^{\ell-j}\xi \cdot \overline{\tau}\|_{C^{k_0}}\|\nabla(\overline{\tau} - \overline{\tau}^{(i)})\|_{L^2}$$
$$\le C\varepsilon^{-\ell}\|\xi \cdot \overline{\tau}\|_{C^{k_0}}\|\eta - \eta^{(i)}\|_{W^{k_0+1,2}}.$$

As before, due to the continuity of the Bogovskii operator, the same estimates for ξ_ε and $\xi_\varepsilon^{(\tau)}$.

For the limit passage $\varepsilon \to 0$, we compute

$$\xi - \xi_\varepsilon = \xi - \tilde{\xi}_\varepsilon + \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon) = \psi_\varepsilon(\xi \cdot \overline{n})\overline{n} + \mathcal{B}(\operatorname{div}\tilde{\xi}_\varepsilon)$$

$$\nabla^\ell(\psi_\varepsilon(\xi \cdot \overline{n})\overline{n}) = \sum_{i=0}^{\ell} \nabla^{\ell-i}\psi_\varepsilon \sum_{j=0}^{i} \nabla^{i-j}(\xi \cdot \overline{n})\nabla^j\overline{n}.$$

To estimate this in $L^2(\Omega(t); \mathbb{R}^d)$, the cutoff ψ_ε is chosen such that

$$\|\nabla^{\ell-i}\psi_\varepsilon\|_{L^\infty} \le \frac{c}{\varepsilon^{\ell-i}},$$

the function $\nabla^\ell(\xi \cdot \overline{n})$ has zero traces on $\partial\eta(Q)$ up to order k_0 so that

$$\|\nabla^{i-j}(\xi \cdot \overline{n}))\|_{L^\infty} \le c\varepsilon^{k_0-i+j},$$

and finally $\|\nabla^j\overline{n}\|_{L^\infty} \le C$. So altogether with the fact that $|\operatorname{supp}\psi_\varepsilon| \le c\varepsilon$ we obtain

$$\|\nabla^\ell(\psi_\varepsilon(\xi \cdot \overline{n})\overline{n})\|_{L^2} \le C\varepsilon.$$

This shows

$$\|\tilde{\xi}(t) - \tilde{\xi}_\varepsilon(t)\|_{L^2} \le C\varepsilon\|\eta(t)\|_{W^{k_0,2}}$$

and again by the continuity of the Bogovskii operator, the same holds for ξ and ξ_ε.

□

3. Strong and weak formulation

In this section formulate the fluid-structure interaction both in weak formulation and in the strong (pointwise) formulations. We further verify the consistency of the weak and strong solutions, namely that any sufficiently regular weak solution is also strong.

3.1. **Weak formulation.**

Definition 3.1 (Weak solution)**.** The deformation $\eta \in L^\infty((0,T),\mathcal{E}) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d))$ and the velocity field

$$\begin{aligned} v \in & L^2((0,T);W^{1,2}_{\operatorname{div},\eta}(\Omega(t);\mathbb{R}^d)) \\ & := \{u \in L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d)) : \operatorname{div} u = 0 \text{ in } \Omega(t), u\cdot n = (\partial_t\eta \circ \eta^{-1})\cdot n \text{ on } \partial\eta(t,Q)\} \end{aligned}$$

where $\Omega(t) = \Omega \setminus \eta(t,Q)$, are called a weak solution to the fluid-structure interaction problem if the following two weak equations hold:

Fluid-only equation. $\xi \in C^\infty([0,T]\times\overline{\Omega}(t);\mathbb{R}^d)$, $\xi\cdot n = 0$ on $\partial\Omega(t)$, $\operatorname{div}\xi = 0$, with $\xi(T)=0$ it holds

$$\int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt = \int_0^T \rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt + \rho_f\langle v_0,\xi(0)\rangle_{\Omega(0)}$$

Coupled equation. For all $\phi \in L^\infty((0,T);W^{2,q}(Q;\mathbb{R}^d)) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d))$, $\xi \in C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^d)$ with $\phi = \xi\circ\eta$ in Q, $\xi\cdot n = 0$ on $\partial\Omega$ (outer boundary, not the fluid-solid interface), $\operatorname{div}\xi = 0$ in $\Omega(t)$, $\xi(T)=0$, $\phi(T)=0$ it holds

$$\begin{aligned} &- \int_0^T \rho_s\langle\partial_t\eta,\partial_t\phi\rangle\,dt + \int_0^T DE(\eta)\langle\phi\rangle + D_2R(\eta,\partial_t\eta)\langle\phi\rangle dt \\ &+ \int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v,v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt \\ = &\int_0^T \rho_s\langle f,\phi\rangle dt + \int_0^T \rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt + \rho_s\langle\eta_*,\phi(0)\rangle + \rho_f\langle v_0,\xi(0)\rangle_{\Omega(0)}. \end{aligned}$$

We shall see now that from this weak formulation, the pressure can be reconstructed.

Proposition 3.2 (Pressure reconstruction)**.** *Let $\eta \in L^\infty((0,T);\mathcal{E}) \cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d))$, $v \in L^2((0,T);W^{1,2}_{\operatorname{div},\eta}(\Omega(t);\mathbb{R}^d))$ be a weak solution to the fluid-structure interaction problem as in Definition 3.1. Then there exists a pressure $p \in \mathcal{D}'((0,T)\times\Omega)$ with* $\operatorname{supp} p \subset (0,T)\times\overline{\Omega(t)}$ *such that*

$$\begin{aligned} \int_0^T &-\rho_s\langle\partial_t\eta,\partial_t(\xi\circ\eta)\rangle + \langle DE(\eta) + D_2R(\eta,\partial_t\eta),\xi\circ\eta\rangle - \rho_s\langle f\circ\eta,\xi\circ\eta\rangle \\ &-\rho_f\langle v,\partial_t\xi\rangle - \rho_f\langle v\otimes v,\nabla\xi\rangle + \nu\langle\varepsilon v,\varepsilon\xi\rangle - \rho_f\langle f,\xi\rangle\,dt - \langle p,\operatorname{div}\xi\rangle = 0 \end{aligned} \tag{3.1}$$

holds for all $\xi \in C_0^\infty((0,T)\times\Omega;\mathbb{R}^d)$. Moreover, p has the regularity

$$p \in L^\infty((0,T);W^{-1,q'}(\Omega)) + W^{-1,\infty}((0,T);W_0^{1,2}(\Omega)) + L^a((0,T);L^b(\Omega))$$

for $1 < a < \infty$ and $b = \frac{ad}{ad-2}$.

Proof. The following weak equation is satisfied

$$\begin{aligned} &- \int_0^T \rho_s\langle\partial_t\eta,\partial_t(\xi\circ\eta)\rangle\,dt + \int_0^T DE(\eta)\langle\xi\circ\eta\rangle + D_2R(\eta,\partial_t\eta)\langle\xi\circ\eta\rangle dt \\ &+ \int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v,v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt \\ &= \int_0^T \rho_s\langle f\circ\eta,\xi\circ\eta\rangle dt + \int_0^T \rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt \end{aligned}$$

for $\xi \in C_0^\infty((0,T)\times\Omega;\mathbb{R}^d)$ with $\operatorname{div}\xi|_{\Omega(t)} = 0$, as defined above.

Define the functional Π by (in the end we will show $\Pi = \nabla p$)

$$\begin{aligned} \langle\Pi,\xi\rangle := \int_0^T &-\rho_s\langle\partial_t\eta,\partial_t(\xi\circ\eta)\rangle + \langle DE(\eta) + D_2R(\eta,\partial_t\eta),\xi\circ\eta\rangle - \rho_s\langle f\circ\eta,\xi\circ\eta\rangle \\ &-\rho_f\langle v,\partial_t\xi\rangle - \rho_f\langle v\otimes v,\nabla\xi\rangle + \nu\langle\varepsilon v,\varepsilon\xi\rangle - \rho_f\langle f,\xi\rangle\,dt, \quad \xi \in C_0^\infty((0,T)\times\Omega;\mathbb{R}^d). \end{aligned}$$

By the weak equation above, Π vanishes for $\xi \in C_0^\infty((0,T)\times\Omega;\mathbb{R}^d)$ with $\operatorname{div}\xi|_{\Omega(t)} = 0$. In particular Π vanishes for ξ with $\operatorname{supp}\xi \subset \eta(\cdot,Q)$, which means that $\operatorname{supp}\Pi \subset \Omega(t)$ (as a support of distribution $\Pi \in \mathcal{D}'((0,T)\times\Omega)^d$).

By interpolation of $|v|^2 \in L^\infty((0,T);L^1(\Omega(t))) \cap L^2((0,T);L^{\frac{d}{d-2}}(\Omega(t)))$ we have a bound on $v\otimes v$ in $L^a((0,T);L^b(\Omega(t);\mathbb{R}^{d\times d}))$. So from this we have for $\xi \in C_0^\infty([0,T]\times\overline{\Omega};\mathbb{R}^d)$ the estimate

$$
\begin{aligned}
|\langle \Pi,\xi\rangle| \le{}& \rho_s \|\partial_t\eta\|_{L^\infty((0,T);L^2(Q))}\|\partial_t(\xi\circ\eta)\|_{L^1((0,T);L^2(\eta(Q)))}\\
&+\|DE(\eta)\|_{L^\infty((0,T);W^{-2,q'}(Q))}\|\xi\circ\eta\|_{L^1((0,T);W^{2,q}(Q))}\\
&+\|D_2R(\eta,\partial_t\eta)\|_{L^2((0,T);W^{-1,2}(Q))}\|\xi\circ\eta\|_{L^2((0,T);W^{1,2}(\eta(Q)))}\\
&-\rho_s\|f\circ\eta\|_{L^2((0,T);L^2(Q;\mathbb{R}^d))}\|\xi\circ\eta\|_{L^2((0,T);L^2(Q;\mathbb{R}^d))}\\
-\rho_f\|v\|_{L^\infty((0,T);L^2(\Omega(t))}&\|\partial_t\xi\|_{L^1((0,T);L^2(\Omega(t)))} - \rho_f\|v\otimes v\|_{L^a((0,T);L^b(\Omega(t)))}\|\nabla\xi\|_{L^{a'}((0,T);L^{b'}(\Omega(t)))}\\
+\nu\|\varepsilon v\|_{L^2((0,T);L^2(\Omega(t)))}&\|\varepsilon\xi\|_{L^2((0,T);L^2(\Omega(t)))} - \rho_f\|f\|_{L^2((0,T);L^2(\Omega(t);\mathbb{R}^d))}\|\xi\|_{L^2((0,T);L^2(\Omega(t);\mathbb{R}^d))}\,dt.
\end{aligned}
$$

Recall that by Lemma 2.4 we have that $\xi\mapsto\xi\circ\eta$ is an isomorphism in the spaces above. This shows that indeed Π is defined (can be extended) for

$$\xi \in X := L^1((0,T);W_0^{2,q}(\Omega;\mathbb{R}^d)) \cap W_0^{1,1}((0,T);L^2(\Omega;\mathbb{R}^d)) \cap L^{a'}((0,T);W^{1,b'}(\Omega;\mathbb{R}^d))$$

and is bounded on this space. In other words, Π satisfies

$$\Pi \in X^* = L^\infty((0,T);W^{-2,q'}(\Omega;\mathbb{R}^d)) + W^{-1,\infty}((0,T);L^2(\Omega;\mathbb{R}^d)) + L^a((0,T);W^{-1,b'}(\Omega;\mathbb{R}^d)).$$

Consider now the divergence operator in Ω as a mapping

$$\operatorname{div}\colon X \to Y = L^1((0,T);W^{1,q}(\Omega)) \cap W^{1,1}((0,T);W^{-1,2}(\Omega)) \cap L^{a'}((0,T);L^{b'}(\Omega)).$$

We construct the fluid-structure Bogovskii operator

$$\mathcal{B}\colon C_0^\infty((0,T)\times\Omega) \to C_0^\infty((0,T)\times\Omega;\mathbb{R}^d)$$

as follows. Fix $b \in C_0^\infty((0,T)\times\eta(\cdot,Q))$ with $\int_{\eta(t,Q)} b(t)\,dx = 1$, $t\in(0,T)$. Then define

$$\mathcal{B}(\psi)(t) = \mathcal{B}_\Omega\left(\psi(t) - b(t)\int_\Omega \psi(t)\,dx\right), \quad \psi \in C_0^\infty([0,T]\times\Omega)$$

where $\mathcal{B}_\Omega$ is the Bogovskii operator for the fixed domain Ω. From this we see that

$$\operatorname{div}(\mathcal{B}\psi) = b\int_\Omega \psi\,dx, \quad \psi\in C_0^\infty([0,T]\times\Omega).$$

In fact we can see that it holds that it can be extended as a bounded operator $\mathcal{B}\colon Y\to X$.

Define now $p \in Y^*$ by setting

$$\langle p,\psi\rangle := \langle \Pi,\mathcal{B}\psi\rangle, \quad \psi\in Y.$$

This p is well defined and in Y^*, as it holds by Bogovskii estimates that $\mathcal{B}\colon Y\to X$ is bounded, and we already know that $\Pi\in X^*$. It remains to show that $\Pi = \nabla p$. We verify this as ∇ is a dual operator to div. For this we compute

$$\langle p,\operatorname{div}\psi\rangle = \langle\Pi,\mathcal{B}(\operatorname{div}\psi)\rangle = \langle\Pi,\psi\rangle + \langle\Pi,\mathcal{B}(\operatorname{div}\psi)-\psi\rangle = \langle\Pi,\psi\rangle, \quad \psi\in Y$$

where the last inequality holds since $\operatorname{div}(\mathcal{B}(\operatorname{div}\psi)-\psi) = 0$ in $\Omega(t)$ and thus it is annihilated by Π. This verifies that $\Pi = \nabla p$, and to conclude the regularity, we have

$$p\in Y^* = L^\infty((0,T);W^{-1,q'}(\Omega)) + W^{-1,\infty}((0,T);W_0^{1,2}(\Omega)) + L^a((0,T);L^b(\Omega))$$

By the definition of p, we directly verified that the weak equation with pressure

$$
\begin{aligned}
\int_0^T &-\rho_s\langle\partial_t\eta,\partial_t(\xi\circ\eta)\rangle + \langle DE(\eta)+D_2R(\eta,\partial_t\eta),\xi\circ\eta\rangle - \rho_s\langle f\circ\eta,\xi\circ\eta\rangle\\
&-\rho_f\langle v,\partial_t\xi\rangle - \rho_f\langle v\otimes v,\nabla\xi\rangle + \nu\langle\varepsilon v,\varepsilon\xi\rangle - \rho_f\langle f,\xi\rangle\,dt - \langle p,\operatorname{div}\xi\rangle = 0
\end{aligned}
$$

holds for all $\xi\in C_0^\infty((0,T)\times\Omega;\mathbb{R}^d)$ □

3.2. **Strong formulation.** Here we state the strong formulation of our problem. For this assume the solid energy and dissipation have a density, that is they can be written as

$$E(\eta) = \int_Q e\left(\nabla\eta, \nabla^2\eta\right)\, dx, \quad R\left(\eta, \partial_t\eta\right) = \int_Q r\left(\nabla\eta, \partial_t\nabla\eta\right)\, dx,$$

where $e \in C^3(\mathbb{R}^{d\times d}_{\det>0} \times \mathbb{R}^{d^3})$, $r \in C^2(\mathbb{R}^{d\times d}_{\det>0} \times \mathbb{R}^{d\times d})$. Further assume the boundary ∂Q is $C^{1,1}$, so that the normal to the boundary $n \in C^{0,1}(\partial Q; \mathbb{R}^d)$ has bounded mean curvature $\operatorname{div}_S n \in L^\infty(\partial Q)$.

Definition 3.3 (Strong solution)**.** $\eta \in C^4([0,T]\times\overline{Q}; \mathbb{R}^d) \cap L^\infty((0,T); \mathcal{E})$, $v \in C^2([0,T]\times\overline{\Omega}(t); \mathbb{R}^d)$ and $p \in C^1([0,T] \times \overline{\Omega}(t))$ is a *strong solution* if it satisfies the equations:
Bulk solid equation:

$$\begin{aligned}\rho_s\partial_{tt}\eta - \operatorname{div}_x \nabla_\xi e(\nabla\eta, \nabla^2\eta) + \operatorname{div}_x^2 \nabla_w e(\nabla\eta, \nabla^2\eta) - \operatorname{div}_x \nabla_\xi r(\nabla\eta, \partial_t\nabla\eta)&\\ - \operatorname{div}_x \nabla_z r(\nabla\eta, \partial_t\nabla\eta) &= \rho_s f \quad \text{in } (0,T)\times Q\end{aligned}$$

Bulk fluid equation:

$$\rho_f(\partial_t v + v\cdot\nabla v) = \nu\Delta v - \nabla p + \rho_f f \quad \text{in } (0,T)\times\Omega(t)$$
$$\operatorname{div} v = 0 \quad \text{in } (0,T)\times\Omega(t)$$

Initial conditions:

$$\begin{aligned} v(0,\cdot) &= v_0 \quad \text{in } \Omega(0)\\ \eta(0) &= \eta_0\\ \partial_t\eta(0) &= \eta_* \end{aligned}$$

Kinematic coupling on the interface:

$$v\cdot n = (\partial_t\eta\circ\eta^{-1})\cdot n \quad \text{on } (0,T)\times\partial\eta(t,Q)$$

Stresses and slipping on the interface:

$$\begin{aligned}&[\nabla_\xi e(\nabla\eta,\nabla^2\eta) - \operatorname{div}_x \nabla_w e(\nabla\eta,\nabla^2\eta) + \nabla_\xi r(\nabla\eta,\partial_t\nabla\eta) +\\ &\nabla_z r(\nabla\eta,\partial_t\nabla\eta) - \operatorname{div}_S \nabla_w e(\nabla\eta,\nabla^2\eta) - \operatorname{div}_S n \nabla_w e(\nabla\eta,\nabla^2\eta) n\otimes n]_n = [[\varepsilon v]n + pn]_n\\ &\qquad \text{on } (0,T)\times\partial Q\\ &[\nabla_\xi e(\nabla\eta,\nabla^2\eta) - \operatorname{div}_x \nabla_w e(\nabla\eta,\nabla^2\eta) + \nabla_\xi r(\nabla\eta,\partial_t\nabla\eta) +\\ &\nabla_z r(\nabla\eta,\partial_t\nabla\eta) - \operatorname{div}_S \nabla_w e(\nabla\eta,\nabla^2\eta) - (\operatorname{div}_S n)\nabla_w e(\nabla\eta,\nabla^2\eta) n\otimes n]_\tau = 0 \quad \text{on } (0,T)\times\partial Q\\ &([\varepsilon v]n)_\tau = 0 \quad \text{on } (0,T)\times\partial\Omega(t)\end{aligned}$$

Solid hyperstress:

$$\nabla_w e(\nabla\eta,\nabla^2\eta) : n\otimes n = 0 \quad \text{on } (0,T)\times\partial Q$$

Theorem 3.4 (Weak-strong compatibility)**.** *Let us assume we have a weak solution with* $\eta \in C^4([0,T]\times\overline{Q}; \mathbb{R}^d) \cap L^\infty((0,T); \mathcal{E})$, $v \in C^2([0,T]\times\overline{\Omega}(t); \mathbb{R}^d)$ *and* $p \in C^1([0,T]\times\overline{\Omega}(t))$. *Under this regularity, it is a strong solution in the sense of Definition 3.3.*

Proof. Throughout this proof, to aid reading, we include the calculations in vector notation, as well as written in (spatial) components. Component-wise calculations always use the Einstein summation convention, that is any repeating index (labelled i, j, k, l) is summed over $1, \dots, d$.

Fluid-only test function. By the weak formulation for the fluid-only test function, we have that for $\xi \in C^\infty([0,T]\times\overline{\Omega}(t); \mathbb{R}^d)$, $\xi\cdot n = 0$ on $\partial\Omega(t)$, with $\xi(T) = 0$ it holds by (3.1)

$$\int_0^T -\rho_f\langle v, \partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v, \varepsilon\xi\rangle_{\Omega(t)}\, dt - \langle p, \operatorname{div}\xi\rangle = \int_0^T \rho_f\langle f, \xi\rangle_{\Omega(t)}\, dt$$
$$+\rho_f\langle v_0, \xi(0)\rangle_{\Omega(0)}.$$

Time derivative: by our transport theorem, Theorem 2.1 we compute

$$\frac{d}{dt}\int_{\Omega(t)} v\cdot\xi\,dx = \int_{\Omega(t)}\partial_t v\cdot\xi\,dx + \int_{\Omega(t)} v\cdot\partial_t\xi\,dx + \int_{\partial\Omega(t)}(v\cdot\xi)\tilde{n}_t\,dS$$

integrate this in time (here $\xi(T)=0$)

$$-\langle v(0),\xi(0)\rangle_{\Omega(0)} = \int_0^T\left(\langle\partial_t v,\xi\rangle_{\Omega(t)} + \langle v,\partial_t\xi\rangle_{\Omega(t)} + \int_{\partial\Omega(t)}(v\cdot\xi)\tilde{n}_t\,dS\right)dt$$

Convective term: integrating by parts gives, thanks to $\operatorname{div} v = 0$ (i.e. $\partial_j v_j = 0$) and $v\cdot n = \tilde{n}_t$ (i.e. $v_j n_j = \tilde{n}_t$) on $\partial\Omega(t)$,

$$\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} = -\langle v\otimes v,\nabla\xi\rangle_{\Omega(t)} = \int_{\partial\Omega(t)}(v\cdot\xi)\underbrace{v\cdot dn}_{=\tilde{n}_t\,dS} - \int_{\Omega(t)} v\cdot\nabla v\cdot\xi\,dx$$

$$\int_{\Omega(t)} v_i v_j\partial_j\xi_i\,dx = \int_{\partial\Omega(t)} v_i\xi_i\underbrace{v_j\,dn_j}_{=\tilde{n}_t\,dS} - \int_{\Omega(t)}\partial_j v_i v_j\xi_i\,dx$$

Viscosity term: again using $\operatorname{div} v = 0$ (i.e. $\partial_j v_j = 0$)

$$\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)} = \int_{\partial\Omega(t)}[\varepsilon v]\xi\cdot dn - \int_{\Omega(t)}\frac{1}{2}\Delta v\xi\,dx$$

$$\int_{\Omega(t)}\frac{1}{4}(\partial_j v_i + \partial_i v_j)(\partial_j\xi_i + \partial_i\xi_j)\,dx = \int_{\partial\Omega(t)}\frac{1}{4}(\partial_j v_i + \partial_i v_j)(\xi_i n_j + \xi_j n_i)\,dS$$
$$- \int_{\Omega(t)}\frac{1}{4}\partial_{jj}v_i\xi_i + \frac{1}{4}\partial_{ii}v_j\xi_j\,dx$$

Finally for the pressure (the boundary term vanishing since $\xi\cdot n = 0$),

$$-\int_0^T\int_{\Omega(t)} p\operatorname{div}\xi\,dx\,dt = \int_0^T -\int_{\partial\Omega(t)} p\xi\cdot dn + \int_{\Omega(t)}\nabla p\cdot\xi\,dx\,dt = \int_0^T\int_{\Omega(t)}\nabla p\cdot\xi\,dx\,dt$$
$$-\int_0^T\int_{\Omega(t)} p\partial_i\xi_i\,dx\,dt = \int_0^T -\int_{\partial\Omega(t)} p\xi_i dn_i + \int_{\Omega(t)}\partial_i p\xi_i\,dx\,dt = \int_0^T\int_{\Omega(t)}\partial_i p\xi_i\,dx\,dt.$$

So altogether we obtain

$$\langle v(0),\xi(0)\rangle_{\Omega(0)} + \int_0^T\langle\partial_t v,\xi\rangle_{\Omega(t)} + \rho_f\int_{\Omega(t)} v\cdot\nabla v\cdot\xi\,dx$$
$$+\nu\left(\int_{\partial\Omega(t)}[\varepsilon v]\xi\cdot dn - \frac{1}{2}\int_{\Omega(t)}\Delta v\xi\,dx\right) + \int_{\Omega(t)}\nabla p\cdot\xi\,dx\,dt = \langle v_0,\xi(0)\rangle_{\Omega(0)} + \int_0^T\rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt$$

for all $\xi\in C^\infty([0,T]\times\overline{\Omega}(t);\mathbb{R}^d)$, $\xi\cdot n = 0$ on $\partial\Omega(t)$, with $\xi(T) = 0$. In the boundary term we use that $[\varepsilon v]\xi\cdot n = \xi\cdot[\varepsilon v]n$ and that ξ is an arbitrary tangential field. Thus in particular, we see that the tangential normal stress is zero, i.e.

$$([\varepsilon v]n)_\tau = 0 \quad \text{on } (0,T)\times\partial\Omega(t)$$

Continuous test function We have for $\phi\in L^\infty((0,T);W^{2,q}(Q;\mathbb{R}^d))\cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d))$, $\xi\in C^\infty([0,T]\times\overline{\Omega})$ with $\phi = \xi\circ\eta$ in Q, $\xi\cdot n = 0$ on $\partial\Omega$, $\xi(T) = 0$, $\phi(T) = 0$ that it holds by (3.1)

$$\begin{aligned}&-\int_0^T\rho_s\langle\partial_t\eta,\partial_t\phi\rangle\,dt + \int_0^T DE(\eta)\langle\phi\rangle + D_2R(\eta,\partial_t\eta)\langle\phi\rangle dt\\ &+\int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt - \langle p,\operatorname{div}\xi\rangle\\ &= \int_0^T\rho_s\langle f,\phi\rangle dt + \int_0^T\rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt + \rho_s\langle\eta_*,\phi(0)\rangle + \rho_f\langle v_0,\xi(0)\rangle_{\Omega(0)}\end{aligned}\tag{3.2}$$

So let us take such ξ and ϕ.

For the fluid terms, perform the manipulations as above to obtain

$$\begin{aligned}
&\int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)}+\rho_f\langle v,v\cdot\nabla\xi\rangle_{\Omega(t)}+\nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt-\langle p,\operatorname{div}\xi\rangle\\
&\quad-\int_0^T\rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt-\rho_f\langle v_0,\xi(0)\rangle_{\Omega(0)}=\langle v(0),\xi(0)\rangle_{\Omega(0)}\\
&\quad+\int_0^T\langle\partial_t v,\xi\rangle_{\Omega(t)}+\rho_f\int_{\Omega(t)}v\cdot\nabla v\cdot\xi\,dx+\nu\left(\int_{\partial\Omega(t)}[\varepsilon v]\xi\cdot dn-\frac{1}{2}\int_{\Omega(t)}\Delta v\xi\,dx\right)dt\\
&\quad+\int_{\Omega(t)}\nabla p\cdot\xi\,dx\,dt-\int_{\partial\Omega(t)}p\xi\cdot dn-\langle v_0,\xi(0)\rangle_{\Omega(0)}-\int_0^T\rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt
\end{aligned}$$

Now consider the solid terms from (3.2) , namely

$$-\int_0^T\rho_s\langle\partial_t\eta,\partial_t\phi\rangle\,dt+\int_0^T DE(\eta)\langle\phi\rangle+D_2R(\eta,\partial_t\eta)\langle\phi\rangle dt-\int_0^T\rho_s\langle f,\phi\rangle dt\\
-\rho_s\langle\eta_*,\phi(0)\rangle$$

We calculate (writing $e=e(\xi,w)$, $r=r(\xi,z)$) for a test function $\phi\in C^\infty([0,T]\times\overline{Q};\mathbb{R}^d)$ with $\phi(T)=0$

$$\begin{aligned}
DE(\eta)\langle\phi\rangle&=\int_Q\nabla_\xi e(\nabla\eta,\nabla^2\eta):\nabla\phi+\nabla_w e(\nabla\eta,\nabla^2\eta):\nabla^2\phi\,dx\\
D_2R(\eta,\partial_t\eta)\langle\phi\rangle&=\int_Q\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta):\nabla\phi+\nabla_z r(\nabla\eta,\partial_t\nabla\eta):\nabla\phi\,dx
\end{aligned}$$

in components:

$$\begin{aligned}
DE(\eta)\langle\phi\rangle&=\int_Q\partial_{\xi_{ij}}e(\nabla\eta,\nabla^2\eta)\partial_j\phi_i+\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\partial_{jk}\phi_i\,dx\\
D_2R(\eta,\partial_t\eta)\langle\phi\rangle&=\int_Q\partial_{\xi_{ij}}r(\nabla\eta,\partial_t\nabla\eta)\partial_j\phi_i+\partial_{z_{ij}}r(\nabla\phi,\partial_t\nabla\eta)\partial_j\phi_i\,dx
\end{aligned}$$

We can integrate by parts and obtain (in component notation denote $dn_i=n_i\,dS$ where S is the surface measure)

$$\begin{aligned}
DE(\eta)\langle\phi\rangle&=\int_{\partial Q}\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn-\int_Q\operatorname{div}_x\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx\\
&\quad+\int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta):\nabla\phi\cdot dn-\int_Q\operatorname{div}_x\nabla_w e(\nabla\eta,\nabla^2\eta):\nabla\phi\,dx\\
DE(\eta)\langle\phi\rangle&=\int_{\partial Q}\partial_{\xi_{ij}}e(\nabla\eta,\nabla^2\eta)\phi_i dn_j-\int_Q\partial_j\partial_{\xi_{ij}}e(\nabla\eta,\nabla^2\eta)\phi_i\,dx\\
&\quad+\int_{\partial Q}\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\partial_j\phi_i dn_k-\int_Q\partial_k\nabla_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\partial_j\phi_i\,dx
\end{aligned}$$

Integration by parts in the last term yields

$$\begin{aligned}
-\int_Q\operatorname{div}_x\nabla_w e(\nabla\eta,\nabla^2\eta):\nabla\phi\,dx&=-\int_{\partial Q}\operatorname{div}_x\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn+\int_Q\operatorname{div}_x^2\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx\\
-\int_Q\partial_k\nabla_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\partial_j\phi_i\,dx&=-\int_{\partial Q}\partial_k\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\phi_i\,dn_j+\int_Q\partial_j\partial_k\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\phi_i\,dx
\end{aligned}$$

Now, the second-to last term (namely $\int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta):\nabla\phi\cdot dn$) can be further rewritten as follows. Here we use the assumption of bounded mean curvature $\kappa=\frac{1}{2}\operatorname{div}_S n$. We denote

$$\begin{aligned}
&\nabla=(n\otimes n+(I-n\otimes n))\nabla=:n\otimes\partial_n+\nabla_S\\
&\partial_i=n_i n_j\partial_j+(\partial_i-n_i n_j\partial_j)
\end{aligned}$$

in other words for $u\colon \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned}\nabla u &= (n\cdot\nabla u)n + (\nabla u - (n\cdot\nabla u)n) = \partial_n u n + \nabla_S u\\ \partial_i u &= n_j\partial_j u n_i + (\partial_i u - n_j\partial_j u n_i)\end{aligned}$$

and for $\eta\colon \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{aligned}\nabla\eta &= (n\cdot\nabla\eta)n + (\nabla\eta - (n\cdot\nabla\eta)n) = \partial_n\eta\otimes n + \nabla_S\eta\\ \partial_j\eta_i &= n_k\partial_k\eta_i n_j + (\partial_j\eta_i - n_k\partial_k\eta_i n_j)\\ \operatorname{div}_S\eta &= \operatorname{Tr}\nabla_S\eta = \partial_i\eta_i - n_k\partial_k\eta_i n_i\end{aligned}$$

Rewriting the gradient to this normal and surface part, we obtain

$$\int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta) : \nabla\phi\cdot dn = \int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta) : n\otimes\partial_n\phi\cdot dn + \int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta) : \nabla_S\phi\cdot dn$$

$$\begin{aligned}\int_{\partial Q}\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)\partial_j\phi_i dn_k &= \int_{\partial Q}\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)n_l\partial_l\phi_i n_j dn_k\\ &\quad+\int_{\partial Q}\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)(\partial_j\phi_i - n_l\partial_l\phi_i n_j)dn_k\end{aligned}$$

Using the Gauss-Green formula on the surface ∂Q (which has empty relative boundary since it is compact) gives us

$$\begin{aligned}\int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta) : \nabla_S\phi\cdot dn &= -\int_{\partial Q}\operatorname{div}_S\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn\\ &\quad-\int_{\partial Q}(\operatorname{div}_S n)\nabla_w e(\nabla\eta,\nabla^2\eta) : n\otimes\phi\cdot dn\end{aligned}$$

$$\begin{aligned}\int_{\partial Q}\partial_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)(\partial_j\phi_i - n_l\partial_l\phi_i n_j)dn_k &= -\int_{\partial Q}(\partial_i\nabla_{w_{ijk}}e(\nabla\eta,\nabla^2\eta) - n_l\partial_l\nabla_{w_{ijk}}e(\nabla\eta,\nabla^2\eta)n_i)\phi_j\,dn_k\\ &\quad-\int_{\partial Q}(\partial_i n_i - n_k\partial_k n_i n_i)\nabla_{w_{ijk}}e(\nabla\eta,\nabla^2\eta))n_i\phi_k\,dn_j\end{aligned}$$

So thus in total we get for the energy

$$\begin{aligned}DE(\eta)\langle\phi\rangle &= \int_{\partial Q}\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn - \int_Q\operatorname{div}_x\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx + \int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta) : n\otimes\partial_n\phi\cdot dn\\ &\quad-\int_{\partial Q}\operatorname{div}_S\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn - \int_{\partial Q}(\operatorname{div}_S n)\nabla_w e(\nabla\eta,\nabla^2\eta) : n\otimes n\cdot\phi\cdot dn\\ &\quad-\int_{\partial Q}\operatorname{div}_x\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn + \int_Q\operatorname{div}_x^2\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx\end{aligned}$$

For the dissipation, integrating by parts in space

$$\begin{aligned}D_2R(\eta,\partial_t\eta)\langle\phi\rangle &= \int_{\partial Q}\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\cdot dn - \int_Q\operatorname{div}_x\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\,dx\\ &\quad+\int_{\partial Q}\nabla_z r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\cdot dn - \int_Q\operatorname{div}_x\nabla_z r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\,dx\\ D_2R(\eta,\partial_t\eta)\langle\phi\rangle &= \int_{\partial Q}\partial_{\xi_{ij}}r(\nabla\eta,\partial_t\nabla\eta)\phi_i\,dn_j - \int_Q\partial_j\partial_{\xi_{ij}}r(\nabla\eta,\partial_t\nabla\eta)\phi_i\,dx\\ &\quad+\int_{\partial Q}\partial_{z_{ij}}r(\nabla\eta,\partial_t\nabla\eta)\phi_i\,dn_j - \int_Q\partial_j\partial_{z_{ij}}r(\nabla\eta,\partial_t\nabla\eta)\phi_i\,dx\end{aligned}$$

For the inertial term, integrate by parts in time to obtain (remember that $\phi(T)=0$)

$$-\int_0^T\rho_s\langle\partial_t\eta,\partial_t\phi\rangle\,dt = \rho_s\langle\partial_t\eta(0),\phi(0)\rangle + \int_0^T\rho_s\langle\partial_{tt}\eta,\phi\rangle\,dt$$

Therefore in total the we get that

$$
\begin{aligned}
&-\int_0^T \rho_s\langle\partial_t\eta,\partial_t\phi\rangle\,dt+\int_0^T DE(\eta)\langle\phi\rangle+D_2R(\eta,\partial_t\eta)\langle\phi\rangle dt-\int_0^T\rho_s\langle f,\phi\rangle dt-\rho_s\langle\eta_*,\phi(0)\rangle=\\
&\int_0^T\rho_s\langle\partial_{tt}\eta,\phi\rangle+\int_{\partial Q}\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn\\
&-\int_Q \operatorname{div}_x\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx+\int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta):n\otimes\partial_n\phi\cdot dn\\
&-\int_{\partial Q}\operatorname{div}_S\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn-\int_{\partial Q}(\operatorname{div}_S n)\nabla_w e(\nabla\eta,\nabla^2\eta):n\otimes n\cdot\phi\cdot dn\\
&-\int_{\partial Q}\operatorname{div}_x\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn+\int_Q\operatorname{div}_x^2\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx\int_{\partial Q}\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\cdot dn\\
&-\int_Q\operatorname{div}_x\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\,dx\,dt+\int_{\partial Q}\nabla_z r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\cdot dn-\int_Q\operatorname{div}_x\nabla_z r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\,dx\\
&-\int_0^T\rho_s\langle f,\xi\rangle dt+\rho_s\langle\partial_t\eta(0),\phi(0)\rangle-\rho_s\langle\partial_t\eta(0),\phi(0)\rangle
\end{aligned}
$$

Altogether with the solid and fluid terms, it holds

$$
\begin{aligned}
&\langle v(0),\xi(0)\rangle_{\Omega(0)}+\int_0^T\langle\partial_t v,\xi\rangle_{\Omega(t)}+\rho_f\int_{\Omega(t)}v\cdot\nabla v\cdot\xi\,dx+\nu\left(\int_{\partial\Omega(t)}[\varepsilon v]\xi\cdot dn-\frac12\int_{\Omega(t)}\Delta v\xi\,dx\right)dt\\
&\quad+\int_{\Omega(t)}\nabla p\cdot\xi\,dx\,dt-\int_{\partial\Omega(t)}p\xi\cdot dn-\langle v_0,\xi(0)\rangle_{\Omega(0)}-\int_0^T\rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt+\int_0^T\rho_s\langle\partial_{tt}\eta,\phi\rangle+\\
&\int_{\partial Q}\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn-\int_Q\operatorname{div}_x\nabla_\xi e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx+\int_{\partial Q}\nabla_w e(\nabla\eta,\nabla^2\eta):n\otimes\partial_n\phi\cdot dn\\
&\quad-\int_{\partial Q}\operatorname{div}_S\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn-\int_{\partial Q}(\operatorname{div}_S)n\nabla_w e(\nabla\eta,\nabla^2\eta):n\otimes\phi\cdot dn\\
&-\int_{\partial Q}\operatorname{div}_x\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\cdot dn+\int_Q\operatorname{div}_x^2\nabla_w e(\nabla\eta,\nabla^2\eta)\cdot\phi\,dx\int_{\partial Q}\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\cdot dn\\
&-\int_Q\operatorname{div}_x\nabla_\xi r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\,dx\,dt+\int_{\partial Q}\nabla_z r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\cdot dn-\int_Q\operatorname{div}_x\nabla_z r(\nabla\eta,\partial_t\nabla\eta)\cdot\phi\,dx\\
&\quad-\int_0^T\rho_s\langle f,\xi\rangle dt+\rho_s\langle\partial_t\eta(0),\phi(0)\rangle-\rho_s\langle\partial_t\eta(0),\phi(0)\rangle=0
\end{aligned}
$$

for all $\phi\in L^\infty((0,T);W^{2,q}(Q;\mathbb{R}^d))\cap W^{1,2}((0,T);W^{1,2}(Q;\mathbb{R}^d))$, $\xi\in C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^d)$ with $\phi=\xi\circ\eta$ in Q, $\xi\cdot n=0$ on $\partial\Omega$, $\xi(T)=0$, $\phi(T)=0$.

From this, we can see that the strong formulation as in Definition 3.3 holds. □

4. Variational existence scheme

We will use the regularized energy and dissipation with $\kappa>0$, that is

$$E_\kappa(\eta)=E(\eta)+\kappa^{a_0}\|\nabla^{k_0+2}\eta\|^2\quad R_\kappa(\eta,b)=R(\eta,b)+\kappa\|\nabla^{k_0+2}b\|^2$$

where the exponent $a_0>0$ will be chosen later, and k_0 is chosen so large that $W^{k_0,2}$ embeds into $W^{2,q}$.

Now we fix $h>0$ and solve first the time delayed problem on $(0,h)$.

Definition 4.1 (Time-delayed solution). Let $w_f\in L^2((0,h)\times\Omega_0;\mathbb{R}^d)$, $w_s\in L^2((0,h)\times Q;\mathbb{R}^d)$ be given, where $\Omega_0=\Omega\setminus\eta_0(\overline{Q})$. Then η,v is called a weak solution to the time-delayed problem if $(\eta\circ\eta^{-1})\cdot n=v\cdot n$ on $\partial\eta(\cdot,Q)$, and satisfies the equations

Fluid-only equation.

$$\nu\langle \varepsilon v, \varepsilon\xi\rangle_{\Omega(t)} + \kappa\langle\nabla^{k_0} v, \nabla^{k_0}\xi\rangle_{\Omega(t)} + \rho_f\left\langle \frac{v\circ\Phi - w_f}{h}, \xi\circ\Phi\right\rangle_{\Omega_0} - \rho_f\langle f,\xi\rangle_\Omega = 0$$

for all test functions $\xi \in C^\infty((0,h)\times\overline{\Omega}(t);\mathbb{R}^d)$ with $\operatorname{div}\xi = 0$ in $\Omega(t)$, $\xi\cdot n = 0$ on $\partial\Omega(t)$.

Coupled equation.

$$\begin{aligned} DE(\eta)\langle\phi\rangle + D_2R(\eta,\partial_t\eta)\langle\phi\rangle + 2\kappa^{a_0}\langle\nabla^{k_0+2}\eta,\nabla^{k_0+2}\phi\rangle + 2\kappa\left\langle\nabla^{k_0+2}\partial_t\eta,\nabla^{k_0+2}\phi\right\rangle \\ +\rho_s\left\langle\frac{\partial_t\eta - w_s}{h},\phi\right\rangle - \rho_s\langle f\circ\eta,\phi\rangle \\ +\nu\langle \varepsilon v, \varepsilon\xi\rangle_{\Omega(t)} + \kappa\langle\nabla^{k_0} v, \nabla^{k_0}\xi\rangle_{\Omega(t)} + \rho_f\left\langle \frac{v\circ\Phi - w_f}{h}, \xi\circ\Phi\right\rangle_{\Omega_0} - \rho_f\langle f,\xi\rangle_\Omega = 0\end{aligned}$$

for all test functions $\xi\in C^\infty((0,h)\times\Omega;\mathbb{R}^d)$, $\phi\in C([0,h];W^{k_0+2,2}(Q;\mathbb{R}^d))$ with $\phi = \xi\circ\eta$ and $\operatorname{div}\xi = 0$ in $\Omega(t)$, $\nabla^\ell(\xi\cdot n) = 0$ on $\partial\Omega$, $\ell = 0,\dots,k_0$.

The flow map $\Phi\colon (0,T)\times\Omega_0$ is such that $\Phi_t := \Phi(t,\cdot)$ solves $\partial_t\Phi_t = v(t)\circ\Phi_t$ with $\Phi_0 = \mathrm{id}_{\Omega_0}$, here we denote $\Omega(t) = \Omega\setminus\eta(t,Q)$.

The velocities w_f resp. w_s will later represent the fluid resp. solid velocity in the previous h step. To solve this problem, we perform the following minimization scheme.

4.1. **Minimization.** For now let $\tau > 0$ be a fixed discretization step.

To ease the notation we write for the fluid terms $\|\cdot\|_{\Omega_k}$ for the $L^2(\Omega_k;\mathbb{R}^d)$ norm, and in the solid terms $\|\cdot\|$ for the $L^2(Q;\mathbb{R}^d)$ norm, similarly for scalar products on these spaces.

We discretize the right hand side

$$f_k^{(\tau)} = \int_{(k-1)\tau}^{k\tau} f(t)\,dt$$

and the velocities

$$w_{s,k}^{(\tau)} = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} w_s\,dt, \quad w_{f,k}^{(\tau)} = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} w_f\,dt.$$

Assume we have $\eta_{k-1}^{(\tau)}$, $v_{k-1}^{(\tau)}$, where $\eta_0^{(\tau)}\in\mathcal{E}\cap W^{k_0+2,2}(Q;\mathbb{R}^d)$ and $v_0^{(\tau)}\in W^{k_0,2}(\Omega_0;\mathbb{R}^d)$ is the given initial condition and the domain is $\Omega_{k-1}^{(\tau)} = \Omega\setminus\eta_{k-1}^{(\tau)}(\overline{Q})$.

The next step is found as

$$(\eta_k^{(\tau)}, v_k^{(\tau)}) \in \arg\min \mathcal{J}_k^{(\tau)}(\eta, v) \tag{4.1}$$

where the minimum is over $(\eta,v)\in\mathcal{E}\cap W^{k_0+2,2}(Q;\mathbb{R}^d)\times W^{k_0,2}_{\operatorname{div}}(\Omega_{k-1}^{(\tau)};\mathbb{R}^d)$ satisfying

$$\text{with } \frac{\eta-\eta_{k-1}^{(\tau)}}{\tau}\cdot n_{k-1}^{(\tau)} = v\circ\eta\cdot n_{k-1}^{(\tau)}, \text{ on } \partial Q, \text{ and } v\cdot n_{k-1}^{(\tau)} = 0 \text{ on } \partial\Omega, \text{ where}$$

and the functional $\mathcal{J}_k^{(\tau)}\colon W^{k_0+2,2}(Q;\mathbb{R}^d)\times W^{k_0,2}_{\operatorname{div}}(\Omega_{k-1}^{(\tau)};\mathbb{R}^d)\to\mathbb{R}$ is defined as

$$\begin{aligned}\mathcal{J}_k^{(\tau)}(\eta,v) := E(\eta) + \tau R\left(\eta_{k-1}^{(\tau)}, \frac{\eta-\eta_{k-1}^{(\tau)}}{\tau}\right) + \kappa^{a_0}\|\nabla^{k_0+2}\eta\|^2 + \kappa\tau\left\|\nabla^{k_0+2}\frac{\eta-\eta_{k-1}^{(\tau)}}{\tau}\right\|^2 \\ +\rho_s\frac{\tau h}{2}\left\|\frac{\frac{\eta-\eta_{k-1}^{(\tau)}}{\tau} - w_{s,k-1}^{(\tau)}}{h}\right\|^2 - \tau\rho_s\left\langle f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)}, \frac{\eta-\eta_{k-1}^{(\tau)}}{\tau}\right\rangle \\ +\frac{\tau\nu}{2}\|\varepsilon v\|^2_{\Omega_{k-1}^{(\tau)}} + \kappa\frac{\tau}{2}\|\nabla^{k_0}v\|^2_{\Omega_{k-1}^{(\tau)}} + \rho_f\frac{\tau h}{2}\left\|\frac{v\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h}\right\|^2_{\Omega_0} - \tau\rho_f\langle f_k^{(\tau)}, v\rangle_{\Omega_{k-1}^{(\tau)}}.\end{aligned}$$

The flow map begins as $\Phi_0^{(\tau)} := \mathrm{id}$ and for the next step is defined as

$$\Phi_k^{(\tau)} := (\mathrm{id} + \tau v_k^{(\tau)}) \circ \Phi_{k-1}^{(\tau)} \tag{4.2}$$

We will assume that it holds $\Omega_k^{(\tau)} := \Phi_k^{(\tau)}(\Omega_0)$, and moreover $\Phi_k^{(\tau)} \colon \Omega_0 \to \Omega_k^{(\tau)}$ is a diffeomorphism with bounded Jacobian, which is certainly true for $k = 0$ and for subsequent steps it will be shown below in Proposition 4.4.

Lemma 4.2 (Existence of a minimum). *The minimization problem* (4.1) *admits a minimum* $(\eta_k^{(\tau)}, v_k^{(\tau)})$.

Proof. We show that the minimum exists by the direct method of the calculus of variations. Firstly, the set over which we minimize is nonempty, because $(\eta_{k-1}^{(\tau)}, 0) \in \mathcal{E} \cap W^{k_0+2,2}(Q;\mathbb{R}^d) \times W^{k_0,2}_{\mathrm{div}}(\Omega_{k-1}^{(\tau)};\mathbb{R}^d)$ satisfies the coupling condition and is thus admissible. To see that the functional $\mathcal{J}_k^{(\tau)}$ is bounded from below, use the inequality $(a-b)^2 \geq a^2/2 - b^2$ in the terms

$$\rho_f \frac{\tau h}{2} \left\| \frac{v \circ \Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h} \right\|_{\Omega_0}^2 \geq \rho_f \frac{\tau}{4h} \|v \circ \Phi_{k-1}^{(\tau)}\|_{\Omega_0}^2 - \rho_f \frac{\tau}{2h} \|w_{f,k-1}^{(\tau)}\|^2$$

and

$$\rho_s \frac{\tau h}{2} \left\| \frac{\frac{\eta - \eta_{k-1}}{\tau} - w_{s,k-1}^{(\tau)}}{h} \right\|^2 \geq \rho_s \frac{\tau}{4h} \left\| \frac{\eta - \eta_{k-1}^{(\tau)}}{\tau} \right\|^2 - \rho_s \frac{\tau}{2h} \|w_{s,k-1}^{(\tau)}\|^2$$

and Young inequality in the f terms, namely

$$-\tau \rho_f \langle f, v \rangle_{\Omega_{k-1}} \geq -\rho_f \tau \|f\|_{\Omega_{k-1}^{(\tau)}}^2 - \rho_f \frac{\tau}{4} \|v\|_{\Omega_{k-1}^{(\tau)}}^2$$

$$-\tau \rho_s \left\langle f \circ \eta, \frac{\eta - \eta_{k-1}^{(\tau)}}{\tau} \right\rangle \geq -\rho_s \tau \|f \circ \eta\|^2 - \rho_s \frac{\tau}{4} \left\| \frac{\eta - \eta_{k-1}^{(\tau)}}{\tau} \right\|^2$$

and omitting the other non-negative terms, to get

$$\mathcal{J}_k^{(\tau)}(\eta, v) \geq E_{\min} - \rho_f \frac{\tau}{2h} \|w_{f,k-1}^{(\tau)}\|^2 - \rho_s \frac{\tau}{2h} \|w_{s,k-1}^{(\tau)}\|^2 - \rho_f \tau \|f\|_{\Omega_{k-1}^{(\tau)}}^2 - \rho_s \tau \|f \circ \eta\|^2$$

and, since the mapping $f \mapsto f \circ \eta$ is bounded in L^2 by Lemma 2.4 (since $E(\eta)$ is bounded) this gives a uniform bound and thus $\mathcal{J}$ is bounded from below.

Further $\mathcal{J}_k^{(\tau)}$ is coercive on $W^{k_0+2,2}(Q;\mathbb{R}^d) \times W^{k_0,2}(\Omega_{k-1}^{(\tau)};\mathbb{R}^d)$, as can be easily seen (the κ-terms are coercive and the f terms can be absorbed as above). Further, it is weakly lower semicontinuous due to the assumptions on E and R and the convexity of the other terms. So there exists a minimizing sequence with a weakly convergent subsequence.

Moreover the set over which we minimize in (4.1) is weakly closed by the following argument. The Ciralet-Necas condition is weakly closed, because weak convergence in $W^{k_0+2,2}(Q;\mathbb{R}^d)$ implies uniform convergence of η and $\nabla\eta$. Further, we stay strictly away from $\det \nabla\eta = 0$ by the assumption (E.2). Moreover the normal coupling is also a weakly closed condition, since continuous functions up to the boundary are compact in $W^{k_0+2,2}$ resp. $W^{k_0,2}$, we get that the trace and in particular the coupling is preserved in the weak limit. □

4.1.1. *The discrete weak Euler-Lagrange equation. Fluid-only equation.*

By $(\eta_k^{(\tau)}, v_k^{(\tau)})$ being a minimizer of $\mathcal{J}_k^{(\tau)}$ we get in particular that $v_k^{(\tau)}$ is a minimizer of $\mathcal{J}_k^{(\tau)}(\eta_k^{(\tau)}, \cdot)$ over v satisfying $v \circ \eta_k^{(\tau)} \cdot n_{k-1}^{(\tau)} = \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau} \cdot n_{k-1}^{(\tau)}$. Since the functional $\mathcal{J}_k^{(\tau)}(\eta_k^{(\tau)}, \cdot)$ is convex on $W^{k_0,2}(\Omega_{k-1}^{(\tau)};\mathbb{R}^d)$, we get that the following Euler-Lagrange equation for the minimizer

$v_k^{(\tau)}$ is satisfied:

$$\nu\langle \varepsilon v_k^{(\tau)}, \varepsilon\xi\rangle_{\Omega_{k-1}^{(\tau)}} + \kappa\langle \nabla^{k_0} v_k^{(\tau)}, \nabla^{k_0}\xi\rangle_{\Omega_{k-1}^{(\tau)}} + \rho_f \left\langle \frac{v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h}, \xi\circ\Phi_{k-1}\right\rangle_{\Omega_0} - \langle f_k^{(\tau)}, \xi\rangle_{\Omega_{k-1}^{(\tau)}} = 0 \tag{4.3}$$

holds for all $\xi \in C^\infty(\overline{\Omega}_k^{(\tau)};\mathbb{R}^d)$ with $\xi\cdot n_{k-1}^{(\tau)} = 0$ and $\operatorname{div}\xi = 0$ in $\Omega_{k-1}^{(\tau)}$.
Coupled equation

Let us now derive the discrete coupled equation. So we take functions $\xi \in C^\infty(Q;\mathbb{R}^d)$ and $\phi \in W^{k_0,2}(\Omega;\mathbb{R}^d)$ with $\operatorname{div}\xi = 0$ in $\Omega_{k-1}^{(\tau)}$ and $\phi = \xi\circ\eta_{k-1}^{(\tau)}$ in Q. Now let us take the perturbation with the scaling $(\phi, \xi/\tau)$ (this is the correct one that preserves the coupling condition on the interface.) That is, we differentiate

$$t \mapsto \mathcal{J}_k^{(\tau)}(\eta_k^{(\tau)} + t\phi, v_k^{(\tau)} + t\xi/\tau)$$

at $t = 0$. Here it is possible, in particular that the perturbation $(\eta_k^{(\tau)} + t\phi, v_k^{(\tau)} + t\xi/\tau)$ is admissible for small enough t.

The resulting equation is

$$\begin{aligned} &DE(\eta_k^{(\tau)})\langle\phi\rangle + D_2R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right)\langle\phi\rangle + 2\kappa^{a_0}\langle\nabla^{k_0+2}\eta_k^{(\tau)}, \nabla^{k_0+2}\phi\rangle + \\ &2\kappa\left\langle \nabla^{k_0+2}\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}, \nabla^{k_0+2}\phi\right\rangle + \rho_s\left\langle \frac{\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau} - w_{s,k-1}^{(\tau)}}{h}, \phi\right\rangle - \rho_s\left\langle f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)}, \phi\right\rangle \\ &+\nu\langle\varepsilon v_k^{(\tau)}, \varepsilon\xi\rangle_{\Omega_{k-1}^{(\tau)}} + \kappa\langle\nabla^{k_0}v_k^{(\tau)}, \nabla^{k_0}\xi\rangle_{\Omega_{k-1}^{(\tau)}} + \rho_f\left\langle \frac{v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h}, \xi\circ\Phi_{k-1}^{(\tau)}\right\rangle_{\Omega_0} \\ &-\rho_f\langle f_k^{(\tau)}, \xi\rangle_{\Omega_{k-1}^{(\tau)}} = 0 \end{aligned} \tag{4.4}$$

for all $\xi \in C^\infty(\overline{\Omega};\mathbb{R}^d)$, $\operatorname{div}\xi = 0$ in $\Omega_{k-1}^{(\tau)}$ and $\phi \in W^{k_0,2}(Q;\mathbb{R}^d)$ with $\phi = \xi\circ\eta_{k-1}^{(\tau)}$ in Q.

Lemma 4.3 (Discrete energy estimates)**.** *For the constructed η_k, v_k the following energy estimate holds:*

$$\begin{aligned} &E(\eta_k^{(\tau)}) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_k^{(\tau)}\|^2 + \sum_{j=1}^k\left[\kappa\tau\left\|\nabla^{k_0+2}\frac{\eta_j^{(\tau)} - \eta_{j-1}^{(\tau)}}{\tau}\right\|^2 + \tau R\left(\eta_{j-1}^{(\tau)}, \frac{\eta_j^{(\tau)} - \eta_{j-1}^{(\tau)}}{\tau}\right)\right. \\ &\quad\left. + \rho_s\frac{\tau}{8h}\left\|\frac{\eta_j^{(\tau)} - \eta_{j-1}^{(\tau)}}{\tau}\right\|^2 + \frac{\tau\nu}{2}\|\varepsilon v_j^{(\tau)}\|^2_{\Omega_{j-1}^{(\tau)}} + \kappa\frac{\tau}{2}\|\nabla^{k_0}v_j^{(\tau)}\|^2_{\Omega_{j-1}^{(\tau)}} + \rho_f\frac{\tau}{8h}\|v_j^{(\tau)}\circ\Phi_{j-1}^{(\tau)}\|^2_{\Omega_0}\right] \\ &\quad\le E(\eta_0) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_0\|^2 \\ &\quad + \sum_{j=1}^k\left[\rho_s\frac{\tau}{h}\left\|w_{s,j-1}^{(\tau)}\right\|^2 + \rho_f\frac{\tau}{h}\left\|w_{f,j-1}^{(\tau)}\right\|^2_{\Omega_0} + \rho_s 2\tau h\|f_j^{(\tau)}\circ\eta_{j-1}^{(\tau)}\|^2 + \rho_f 2\tau h\|f_j^{(\tau)}\|^2_{\Omega_{j-1}^{(\tau)}}\right] \end{aligned}$$

Proof. For the discrete energy estimates, we compare $(\eta_k^{(\tau)}, v_k^{(\tau)})$ with $(\eta_{k-1}^{(\tau)}, 0)$ in the minimization (4.1). That is,

$$\mathcal{J}_k^{(\tau)}(\eta_k^{(\tau)}, v_k^{(\tau)}) \le \mathcal{J}_k^{(\tau)}(\eta_{k-1}^{(\tau)}, 0).$$

Writing this out gives

$$E(\eta_k^{(\tau)}) + \tau R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_k^{(\tau)}\|^2 + \kappa\tau\left\|\nabla^{k_0+2}\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\|^2$$

$$+\rho_s\frac{\tau h}{2}\left\|\frac{\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau} - w_{s,k-1}^{(\tau)}}{h}\right\|^2 - \tau\rho_s\left\langle f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle$$

$$+\frac{\tau\nu}{2}\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \kappa\frac{\tau}{2}\|\nabla^{k_0}v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \rho_f\frac{\tau h}{2}\left\|\frac{v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h}\right\|^2_{\Omega_0} - \tau\rho_f\langle f_k^{(\tau)}, v_k^{(\tau)}\rangle_{\Omega_{k-1}^{(\tau)}}$$

$$\leq E(\eta_{k-1}^{(\tau)}) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_{k-1}^{(\tau)}\|^2 + \rho_s\frac{\tau h}{2}\left\|\frac{w_{s,k-1}^{(\tau)}}{h}\right\|^2 + \rho_f\frac{\tau h}{2}\left\|\frac{w_{f,k-1}^{(\tau)}}{h}\right\|^2_{\Omega_0}.$$

Now we sum this over $j = 1, \ldots, k$, so that we obtain the energy estimates

$$E(\eta_k^{(\tau)}) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_k^{(\tau)}\|^2 + \frac{1}{2}\sum_{j=1}^{k}\left[\kappa\tau\left\|\nabla^{k_0+2}\frac{\eta_j^{(\tau)} - \eta_{j-1}^{(\tau)}}{\tau}\right\|^2 + \tau R\left(\eta_{j-1}^{(\tau)}, \frac{\eta_j^{(\tau)} - \eta_{j-1}^{(\tau)}}{\tau}\right)\right.$$

$$+\rho_s\frac{\tau h}{2}\left\|\frac{\frac{\eta_j^{(\tau)}-\eta_{j-1}^{(\tau)}}{\tau} - w_{s,j-1}^{(\tau)}}{h}\right\|^2 - \tau\rho_s\left\langle f_j^{(\tau)}\circ\eta_{j-1}^{(\tau)}, \frac{\eta_j^{(\tau)} - \eta_{j-1}^{(\tau)}}{\tau}\right\rangle$$

$$\left.+\frac{\tau\nu}{2}\|\varepsilon v_j^{(\tau)}\|^2_{\Omega_{j-1}^{(\tau)}} + \kappa\frac{\tau}{2}\|\nabla^{k_0}v_j^{(\tau)}\|^2_{\Omega_{j-1}^{(\tau)}} + \rho_f\frac{\tau h}{2}\left\|\frac{v_j^{(\tau)}\circ\Phi_{j-1}^{(\tau)} - w_{f,j-1}^{(\tau)}}{h}\right\|^2_{\Omega_0} - \tau\langle f_j^{(\tau)}, v_j^{(\tau)}\rangle_{\Omega_{j-1}^{(\tau)}}\right]$$

$$\leq E(\eta_0) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_0\|^2 + C(h)\|w\|^2_{L^2(L^2)} + C\|f\|_{L^2(L^\infty)}$$

To get uniform in τ energy estimates, using and the Young inequality we get in inertial and forcing term in the solid

$$\rho_s\frac{\tau}{4h}\left\|\frac{\eta_k^{(\tau)} - \eta_{k-1}}{\tau}\right\|^2 - \rho_s\frac{\tau}{2h}\|w_{s,k-1}^{(\tau)}\|^2 - \rho_s\frac{\tau}{8h}^2\left\|\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\|^2 - \rho_s 2\tau h\|f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)}\|$$

$$\leq \rho_s\frac{\tau h}{2}\left\|\frac{\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau} - w_{s,k-1}^{(\tau)}}{h}\right\|^2 - \tau\rho_s\left\langle f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle$$

and in the fluid

$$\rho_f\frac{\tau}{4h}\|v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)}\|^2_{\Omega_0} - \rho_f\frac{\tau}{2h}\|w_{f,k-1}^{(\tau)}\|^2_{\Omega_0} - \rho_f\frac{\tau}{8h}\|v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} - \rho_f 2\tau h\|f_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}}$$

$$\leq \rho_f\frac{\tau h}{2}\left\|\frac{v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h}\right\|^2_{\Omega_0} - \tau\rho_f\langle f_k^{(\tau)}, v_k^{(\tau)}\rangle_{\Omega_{k-1}^{(\tau)}}.$$

As $\Phi_j^{(\tau)}$ is a diffeomorphism with bounded change of volume by by Proposition 4.4 below, we have $\|v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)}\|_{\Omega_0} = \|v_k^{(\tau)}\|_{\Omega_{k-1}^{(\tau)}}$.

So that we get

$$E(\eta_k^{(\tau)}) + \tau R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_k^{(\tau)}\|^2 + \kappa\tau\left\|\nabla^{k_0+2}\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\|^2$$

$$+\rho_s\frac{\tau}{8h}\left\|\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\|^2 + \frac{\tau\nu}{2}\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \kappa\frac{\tau}{2}\|\nabla^{k_0} v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \rho_f\frac{\tau}{8h}\|v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}\|^2_{\Omega_0}$$

$$\leq E(\eta_{k-1}^{(\tau)}) + \kappa^{a_0}\|\nabla^{k_0+2}\eta_{k-1}^{(\tau)}\|^2 + \rho_s\frac{\tau}{h}\left\|w_{s,k-1}^{(\tau)}\right\|^2 + \rho_f\frac{\tau}{h}\left\|w_{f,k-1}^{(\tau)}\right\|^2_{\Omega_0} + \rho_s 2\tau h\|f_k^{(\tau)} \circ \eta_{k-1}^{(\tau)}\|^2$$

$$+\rho_f 2\tau h\|f_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}}$$

Now we sum this over $j = 1, \dots, k$, so that we obtain the energy estimates in the statement of this lemma. □

Proposition 4.4 (Estimates of the flow map)**.** *For every $\varepsilon > 0$, the discrete flow map satisfies the estimate on its Jacobian*

$$\frac{1}{1+\varepsilon} \leq \det \nabla\Phi_k^{(\tau)} \leq 1 + \varepsilon$$

for τ small enough, in dependence of ε (i.e. for all $0 < \tau < \tau_0(\varepsilon)$). Consequently, it holds

$$\lim_{\tau\to 0} \det \nabla\Phi^{(\tau)} = 1$$

uniformly on $[0,T]$. Moreover the maps $\Phi^{(\tau)}$ are uniformly Lipschitz-continuous in space, so that the Lipschitz constant $\operatorname{Lip}\Phi^{(\tau)}(t)$ is independent of τ and t.

Proof. We recall that the definition (4.2) of $\Phi_k^{(\tau)}$, so that

$$\nabla\Phi_k^{(\tau)} = (I + \tau\nabla v_k^{(\tau)}) \circ \Phi_{k-1}^{(\tau)} \cdot \nabla\Phi_{k-1}^{(\tau)}$$

and consequently

$$\det \nabla\Phi_k^{(\tau)} = \det((I + \tau\nabla v_k^{(\tau)}) \circ \Phi_{k-1}^{(\tau)}) \det \nabla\Phi_{k-1}^{(\tau)} = \prod_{j=1}^{k} \det((I + \tau\nabla v_j^{(\tau)}) \circ \Phi_{k-1}^{(\tau)}).$$

Now compute, expanding the product by the definition of the determinant and assembling by powers of τ

$$\det(I + \tau\nabla v_j^{(\tau)}) = 1 + \tau \underbrace{\operatorname{Tr}\nabla v_j^{(\tau)}}_{=\operatorname{div} v_j^{(\tau)}=0} + \sum_{\ell=2}^{d} \tau^\ell M_\ell(\nabla v_j^{(\tau)})$$

where M_ℓ are homogeneous polynomials of degree ℓ, so that we have

$$\det \nabla\Phi_k^{(\tau)} \leq \prod_{j=1}^{k}\left(1 + \sum_{\ell=2}^{d} c_\ell\tau^\ell\|\nabla v_j^{(\tau)}\|^\ell_{L^\infty}\right) \leq \exp\left(\sum_{j=1}^{k}\sum_{\ell=2}^{d} c_\ell\tau^\ell\|\nabla v_j^{(\tau)}\|^\ell_{L^\infty}\right)$$

$$= \exp\left(\sum_{\ell=2}^{d} c_\ell\tau^{\ell/2}\sum_{j=1}^{k}\left(\tau\|\nabla v_j^{(\tau)}\|^2_{L^\infty}\right)^{\frac{\ell}{2}}\right) \leq \exp\left(\sum_{\ell=2}^{d} c_\ell\tau^{\frac{\ell}{2}}\sum_{j=1}^{k}\tau\|\nabla v_j^{(\tau)}\|^2_{L^\infty}\right) \leq \exp\left(\tau\sum_{\ell=2}^{d} c_\ell\mathcal{K}\right)$$

where we used $\sum_{j=1}^{d} a_j^p \leq \sum_{j=1}^{d} a_j$ for $p \geq 1$, and that we know from the energy estimate of Lemma 4.3, since we have the embedding $W^{k_0,2} \subset W^{1,\infty}$ with constant c, that

$$\sum_{j=1}^{k}\tau\|\nabla v_j^{(\tau)}\|^2_{L^\infty} \leq c^2\sum_{j=1}^{k}\tau\|\nabla^{k_0} v_j^{(\tau)}\|^2_{L^2} \leq \mathcal{K}$$

Choosing τ small enough we get the exponential smaller than $1 + \varepsilon$. Similarly we can estimate from below, using $1/(1-s) \leq 1+\varepsilon s$ for $0 \leq s \leq \varepsilon/(1+\varepsilon)$. The uniform convergence of Jacobians to 1 follows from this.

The Lipschitz continuity follows by (we use $1+t \leq e^t$)

$$\begin{aligned}\operatorname{Lip}\left(\Phi_k^{(\tau)}\right) &\leq \prod_{l=1}^{k}\left(1+\tau \operatorname{Lip}\left(v_l^{(\tau)}\right)\right) \leq \exp\left(\sum_{l=1}^{k}\tau \operatorname{Lip}\left(v_l^{(\tau)}\right)\right)\\ &\leq \exp\left(\sqrt{\sum_{l=1}^{k}\tau}\sqrt{\sum_{l=1}^{k}\tau \operatorname{Lip}\left(v_l^{(\tau)}\right)^2}\right) \leq \exp(\sqrt{h}\sqrt{\mathcal{K}}).\end{aligned}$$

□

4.1.2. *Weak limit $\tau \to 0$.* We define the piecewise constant and piecewise affine interpolations as

$$\begin{aligned}
&\eta^{(\tau)}(t,x) = \eta_k^{(\tau)}(x) && \text{for } \tau(k-1) \leq t < \tau k,\\
&\underline{\eta}^{(\tau)}(t,x) = \eta_{k-1}^{(\tau)}(x) && \text{for } \tau(k-1) \leq t < \tau k,\\
&\tilde{\eta}^{(\tau)}(t,x) = \frac{\tau k - t}{\tau}\eta_{k-1}^{(\tau)}(x) + \frac{t-\tau(k-1)}{\tau}\eta_k^{(\tau)}(x) && \text{for } \tau(k-1) \leq t < \tau k,\\
&v^{(\tau)}(y) = v_{k-1}^{(\tau)}(y) && \text{for } \tau(k-1) \leq t < \tau k, y \in \Omega_{k-1}^{(\tau)},\\
&\Phi^{(\tau)}(t,y) = \Phi_{k-1}^{(\tau)}(y) && \text{for } \tau(k-1) \leq t < \tau k,\\
&\tilde{\Phi}^{(\tau)}(t,y) = \frac{\tau k - t}{\tau}\Phi_{k-1}^{(\tau)}(x) + \frac{t-\tau(k-1)}{\tau}\Phi_k^{(\tau)}(x) && \text{for } \tau(k-1) \leq t < \tau k,
\end{aligned} \tag{4.5}$$

as well as $\Omega^{(\tau)}(t) = \Omega_{k-1}^{(\tau)}$ for $\tau(k-1) \leq t < \tau k$.

By the estimate of Lemma 4.3 we have (for a nonlabelled subsequence $\tau \to 0$) that

$$\begin{aligned}
\eta^{(\tau)} &\overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty((0,h); W^{k_0+2,2}(Q;\mathbb{R}^d))\\
\partial_t\tilde{\eta}^{(\tau)} &\rightharpoonup \partial_t\eta \quad \text{in } L^2((0,h); W^{k_0+2,2}(Q;\mathbb{R}^d))\\
v^{(\tau)} &\overset{\eta}{\rightharpoonup} v \quad \text{in } L^2((0,h); W^{k_0,2}(\Omega(t);\mathbb{R}^d)).
\end{aligned} \tag{4.6}$$

Since we have a compact embedding

$$C^{(1/2)^-}([0,h]; C^{1,\alpha^-}(Q;\mathbb{R}^d)) \hookrightarrow\hookrightarrow W^{1,2}((0,h); W^{k_0,2}(Q;\mathbb{R}^d))$$

we have that

$$\eta^{(\tau)} \to \eta \quad \text{in } C^{(1/2)^-}([0,h]; C^{1,\alpha^-}(Q;\mathbb{R}^d))$$

which in particular means that $\eta^{(\tau)} \rightrightarrows \eta$ on $[0,h]\times\partial Q$ and also $\det\nabla\eta^{(\tau)} \to \det\nabla\eta$ in $C([0,T]\times\overline{Q})$.

Now we verify that the coupling condition holds in the limit: We know that for the approximate deformations the following coupling condition holds

$$v^{(\tau)} \circ \eta^{(\tau)} \cdot n^{(\tau)} = \partial_t\eta^{(\tau)} \cdot n^{(\tau)} \quad \text{on } [0,T]\times\partial Q \tag{4.7}$$

Let us operate now in the Eulerian domain and let us fix an arbitrary test function $\psi \in C_0((0,T)\times\Omega)$. Now write the solid part using the divergence theorem, change of variables, and the above convergences:

$$\int_0^T \int_{\partial\Omega^{(\tau)}} \psi \partial_t \eta^{(\tau)} \circ (\eta^{(\tau)})^{-1} \cdot dn^{(\tau)}\, dt = \int_0^T \int_{\eta^{(\tau)}(Q)} \operatorname{div}(\psi \partial_t \eta^{(\tau)} \circ (\eta^{(\tau)})^{-1})\, dx\, dt$$

$$= \int_0^T \int_{\eta^{(\tau)}(Q)} \nabla\psi \cdot \partial_t \eta^{(\tau)} \circ (\eta^{(\tau)})^{-1} + \psi \nabla \partial_t \eta^{(\tau)} \circ (\eta^{(\tau)})^{-1} : \nabla^T (\eta^{(\tau)})^{-1}\, dx\, dt$$

$$= \int_0^T \int_Q \Big(\nabla\psi \circ \eta^{(\tau)} \cdot \partial_t \eta^{(\tau)} + \psi \circ \eta^{(\tau)} \nabla \partial_t \eta^{(\tau)} : \nabla^T (\eta^{(\tau)})^{-1} \circ \eta^{(\tau)} \Big) \det \nabla \eta^{(\tau)}\, dx\, dt$$

$$\to \int_0^T \int_Q \big(\nabla\psi \circ \eta \cdot \partial_t \eta + \psi \circ \eta \nabla \partial_t \eta : \nabla^T \eta^{-1} \circ \eta \big) \det \nabla \eta\, dx\, dt$$

$$= \int_0^T \int_{\eta(Q)} \nabla\psi \cdot \partial_t \eta \circ \eta^{-1} + \psi \nabla \partial_t \eta \circ \eta^{-1} : \nabla^T \eta^{-1}\, dt$$

$$= \int_0^T \int_{\eta(Q)} \operatorname{div}(\psi \partial_t \eta \circ \eta^{-1})\, dx\, dt = \int_0^T \int_{\partial\eta(Q)} \psi \partial_t \eta \circ \eta^{-1} \cdot dn\, dt$$

Similarly on the fluid domain, we can write by the divergence theorem

$$\int_0^T \int_{\partial\Omega^{(\tau)}(t)} \psi v^{(\tau)} \cdot dn^{(\tau)}\, dt = \int_0^T \int_{\Omega^{(\tau)}(t)} \nabla\psi \cdot v^{(\tau)} + \psi \operatorname{div} v^{(\tau)}\, dx\, dt$$

$$\to \int_0^T \int_{\Omega(t)} \nabla\psi \cdot v + \psi \operatorname{div} v\, dx\, dt = \int_0^T \int_{\partial\Omega(t)} \psi v \cdot dn\, dt.$$

Now recall that by (4.7) we have that the left hand sides of the last two equations are equal, so that this proves that the coupling condition holds in the limit, that is

$$v \cdot n = \partial_t \eta \circ \eta^{-1} \cdot n \quad \text{on } (0,T) \times \partial\Omega(t)$$

We also pass to the limit in the flow map. By the estimates of the previous section, Proposition 4.4, there is $\Phi \in C([0,T]; W^{1,\infty}(\Omega_0;\mathbb{R}^d))$ such that

$$\Phi^{(\tau)} \to \Phi \quad \text{in } C([0,T]; C^{0,\alpha}(\Omega_0;\mathbb{R}^d).$$

By Proposition 4.4 we know that $\det \nabla\Phi = 1$ a.e. Moreover, passing to the limit by 4.2 definition of $\Phi^{(\tau)}$ we have

$$\partial_t \Phi(t) = \lim_{\tau\to 0} \partial_t \tilde{\Phi}^{(\tau)}(t) = \lim_{\tau\to 0} u^{(\tau)}(t) \circ \Phi^{(\tau)}(t) = v(t) \circ \Phi(t). \tag{4.8}$$

From here it also follows that $\Phi_t \colon \Omega_0 \to \Omega(t)$ is a diffeomorphism, $t \in [0,T]$.

4.2. **Passing to the limit $\tau \to 0$ in the Euler-Lagrange equation.**

4.2.1. *Fluid only equation.*

Proposition 4.5. *The limit solution v from* (4.6) *satisfies the equation*

$$\int_0^h \nu\langle \varepsilon v, \varepsilon\xi\rangle_{\Omega(t)} + \kappa \langle \nabla^{k_0} v, \nabla^{k_0}\xi\rangle_{\Omega(t)} + \rho_f \left\langle \frac{v\circ\Phi - w^f}{h}, \xi\circ\Phi \right\rangle_{\Omega_0} - \langle f, \xi\rangle_{\Omega(t)}\, dt = 0$$

holds for all $\xi \in C^\infty([0,h]\times\overline{\Omega}(t);\mathbb{R}^d)$, $\nabla^\ell(\xi\cdot n) = 0$ on $\partial\Omega(t)$, $\operatorname{div}\xi = 0$ in $\Omega(t)$, with $\xi(h) = 0$ and $\nabla^\ell(\xi\cdot n) = 0$, $\ell = 1,\dots,k_0$

Proof. We have a fluid only equation as in (4.3), denoting this in the (τ)-notation of (4.5), this reads as

$$\int_0^h \nu\langle \varepsilon v^{(\tau)}, \varepsilon\xi^{(\tau)}\rangle_{\Omega^{(\tau)}} + \kappa \langle \nabla^{k_0} v^{(\tau)}, \nabla^{k_0}\xi^{(\tau)}\rangle_{\Omega^{(\tau)}} + \rho_f \left\langle \frac{v^{(\tau)}\circ\Phi^{(\tau)} - w^{(\tau),f}}{h}, \xi^{(\tau)}\circ\Phi^{(\tau)} \right\rangle_{\Omega_0} - \langle f^{(\tau)}, \xi^{(\tau)}\rangle_{\Omega^{(\tau)}}\, dt = 0 \tag{4.9}$$

where $\xi^{(\tau)} \equiv \xi_k^{(\tau)} \in C^\infty(\overline{\Omega}_{k-1}^{(\tau)}; \mathbb{R}^d)$ on $((k-1)\tau, k\tau]$ with $\xi_k^{(\tau)} \cdot n_k^{(\tau)} = 0$ on $\partial\Omega_k^{(\tau)}$ and $\operatorname{div} \xi_k^{(\tau)} = 0$ in $\Omega_k^{(\tau)}$.

Approximation of ξ: Let us have a fixed test function for the limit equation, that is $\xi \in C^\infty([0,h] \times \overline{\Omega}(t); \mathbb{R}^d)$, $\xi \cdot n = 0$ on $\partial\Omega(t)$, $\operatorname{div} \xi = 0$ in $\Omega(t)$, with $\xi(h) = 0$ and $\nabla^\ell(\xi \cdot n) = 0$, $\ell = 1, \dots, k_0$.

Let now $\varepsilon > 0$ be fixed. Consider a smooth cutoff ψ_ε which is 1 on ε-neighborhood of $\partial\eta(Q)$ and vanishes outside its 2ε-neighborhood. That is, $\psi_\varepsilon \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ such that $\psi_\varepsilon(y) = 1$ if $\operatorname{dist}(y, \partial\eta(Q)) \leq \varepsilon$ and $\psi_\varepsilon(y) = 0$ if $\operatorname{dist}(y, \partial\eta(Q)) \geq 2\varepsilon$.

We now define the sought approximation $\xi_\varepsilon^{(\tau)}$ and ξ_ε given by Proposition 2.9 (ii). By this lemma it is seen that $\xi_\varepsilon^{(\tau)}$ is an admissible test function for the equation (4.9). By the same proposition we see that for fixed $\varepsilon > 0$ we can pass to the limit in the equation as $\tau \to 0$ and see that the limit equation is satisfied with ξ_ε.

At this point we will comment in a bit more detail about the convergence $\tau \to 0$. For any ξ as above ($\xi \in C^\infty([0,T] \times \overline{\Omega}(t); \mathbb{R}^d)$, $\xi \cdot n = 0$ on $\partial\Omega(t)$, $\operatorname{div} \xi = 0$, with $\xi(T) = 0$), construct $\xi^{(\tau)}$ as above ($\xi^{(\tau)}(t) \in C^\infty(\overline{\Omega}_{k-1}^{(\tau)}; \mathbb{R}^d)$ on $t \in ((k-1)\tau, k\tau]$ with $\xi_k^{(\tau)} \cdot n_{k-1}^{(\tau)} = 0$ on $\partial\Omega_k^{(\tau)}$ and $\operatorname{div} \xi_k^{(\tau)} = 0$ in $\Omega_{k-1}^{(\tau)}$.) By the lemma we have

$$\xi_\varepsilon^{(\tau)} \xrightarrow{\eta} \xi_\varepsilon \quad \text{in } L^2((0,T); W^{k_0,2}(\Omega; \mathbb{R}^d))$$

holds in the sense that

$$(\nabla^\ell \xi_\varepsilon^{(\tau)})_0 \to (\nabla^\ell \xi_\varepsilon)_0 \quad \text{in } L^2((0,T); L^2(\Omega; \mathbb{R}^d))$$

for $\ell = 0, \dots, k_0$, where $(\cdot)_0$ is the extension by 0 to Ω. In particular this means

$$\int_0^T \|(\nabla^\ell \xi_\varepsilon^{(\tau)})_0 - (\nabla^\ell \xi_\varepsilon)_0\|_{L^2(\Omega;\mathbb{R}^d)}^2 \, dt \to 0$$

for $\ell = 0, \dots, k_0$..

Then we show in a standard way, that for every $u^{(\tau)} \in L^2((0,T); L^2(\Omega^{(\tau)}(t); \mathbb{R}^d))$ and $u \in L^2((0,T); L^2(\Omega(t); \mathbb{R}^d))$ with $u^{(\tau)} \rightharpoonup u$ it holds that

$$\int_0^T \langle u^{(\tau)}, \nabla^\ell \xi_\varepsilon^{(\tau)} \rangle_{\Omega^{(\tau)}(t)} \, dt \to \int_0^T \langle u, \nabla^\ell \xi_\varepsilon \rangle_{\Omega(t)} \, dt.$$

Indeed, we can write

$$\begin{aligned}
&\left| \int_0^T \langle u^{(\tau)}, \nabla^\ell \xi_\varepsilon^{(\tau)} \rangle_{\Omega^{(\tau)}(t)} - \langle u, \nabla^\ell \xi_\varepsilon \rangle_{\Omega(t)} \, dt \right| = \\
&\left| \int_0^T \langle u_0^{(\tau)}, (\nabla^\ell \xi_\varepsilon^{(\tau)})_0 - (\nabla^\ell \xi_\varepsilon)_0 \rangle_\Omega - \langle u_0 - u_0^{(\tau)}, (\nabla^\ell \xi_\varepsilon)_0 \rangle_\Omega \, dt \right| \leq \\
&\|u_0^{(\tau)}\|_{L^2((0,T);L^2(\Omega))} \|(\nabla^\ell \xi_\varepsilon^{(\tau)})_0 - (\nabla^\ell \xi_\varepsilon)_0\|_{L^2((0,T);L^2(\Omega;\mathbb{R}^d))}^2 + \left| \int_0^T \langle u_0 - u_0^{(\tau)}, (\nabla^\ell \xi_\varepsilon)_0 \rangle_\Omega \, dt \right| \to 0
\end{aligned}$$

where the convergence is respectively by boundedness of $u_0^{(\tau)}$, strong convergence of $(\nabla^\ell \xi_\varepsilon^{(\tau)})_0$ and weak convergence of $u_0^{(\tau)}$. In this the weak convergences in the equation should be understood. In subsequent usages of this approximation we will not comment more on this point.

$$\int_0^h \nu \langle \varepsilon v, \varepsilon \xi_\varepsilon \rangle_{\Omega(t)} + \kappa \langle \nabla^{k_0} v, \nabla^{k_0} \xi_\varepsilon \rangle_{\Omega(t)} + \rho_f \left\langle \frac{v \circ \Phi - w^f}{h}, \xi_\varepsilon \circ \Phi \right\rangle_{\Omega_0} - \langle f, \xi_\varepsilon \rangle_{\Omega(t)} \, dt = 0$$

By the same lemma we can now pass here with $\varepsilon \to 0$ with $\xi_\varepsilon \to \xi$ in $L^2((0,T); W^{k_0,2}(\Omega(t); \mathbb{R}^d))$ to obtain the desired limit equation.

□

4.2.2. *Coupled equation.*

Proposition 4.6. *For the limit η, v from (4.6) the following equation is satisfied*

$$\begin{aligned}\int_0^h DE(\eta)\langle\phi\rangle + D_2R(\eta,\partial_t\eta)\langle\phi\rangle + 2\kappa^{a_0}\langle\nabla^{k_0+2}\eta,\nabla^{k_0+2}\phi\rangle + 2\kappa\left\langle\nabla^{k_0+2}\partial_t\eta,\nabla^{k_0+2}\phi\right\rangle\\ +\rho_s\left\langle\frac{\partial_t\eta - w\circ\eta_0}{h},\phi\right\rangle^2 - \rho_s\langle f\circ\eta,\phi\rangle \qquad (4.10)\\ +\nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)} + \kappa\langle\nabla^{k_0}v,\nabla^{k_0}\xi\rangle_{\Omega(t)} + \rho_f\left\langle\frac{v\circ\Phi_t - w}{h},\xi\circ\Phi\right\rangle_{\Omega_0} - \rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt = 0\end{aligned}$$

for all $\xi\in C^\infty([0,h]\times\overline{\Omega}(t);\mathbb{R}^d)$, $\operatorname{div}\xi = 0$ in $\Omega(t)$ and $\phi\in W^{k_0+2,2}(Q;\mathbb{R}^d)$ with $\phi=\xi\circ\eta$ on Q. Further it satisfies the energy inequality

$$\begin{aligned}&E(\eta(h)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta(h)\|^2 + \frac{1}{2h}\int_0^h \rho_f\|v\|^2_{\Omega(t)} + \rho_s\|\partial_t\eta\|^2\,dt\\ &+\int_0^h 2R(\eta,\partial_t\eta) + 2\kappa\|\nabla^{k_0+2}\partial_t\eta\|^2 + \kappa\|\nabla^{k_0}v\|^2\,dt\\ &\le E(\eta(0)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta(0)\|^2 + \frac{1}{2h}\int_0^h \rho_f\|w_f\|^2_{\Omega_0} + \rho_s\|w_s\|^2\,dt + \int_0^t \rho_s\langle f\circ\eta,\partial_t\eta\rangle + \rho_f\langle f,v\rangle_{\Omega(t)}\,dt.\end{aligned} \qquad (4.11)$$

Proof. As for now, assume the minimizer $\eta_k^{(\tau)}$ is always in the interior of $\mathcal{E}^{k_0}$. Thus as above (4.4) it holds

$$\begin{aligned}DE(\eta_k^{(\tau)})\langle\phi\rangle + D_2R\left(\eta_{k-1}^{(\tau)},\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right)\langle\phi\rangle + 2\kappa^{a_0}\langle\nabla^{k_0+2}\eta_k^{(\tau)},\nabla^{k_0+2}\phi\rangle\\ +2\kappa\left\langle\nabla^{k_0+2}\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau},\nabla^{k_0+2}\phi\right\rangle + \rho_s\left\langle\frac{\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau} - w_{s,k-1}^{(\tau)}}{h},\phi\right\rangle - \rho_s\left\langle f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)},\phi\right\rangle\\ +\nu\langle\varepsilon v_k^{(\tau)},\varepsilon\xi\rangle_{\Omega_{k-1}^{(\tau)}} + \kappa\langle\nabla^{k_0}v_k^{(\tau)},\nabla^{k_0}\xi\rangle_{\Omega_{k-1}^{(\tau)}} + \rho_f\left\langle\frac{v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h},\xi\circ\Phi_{k-1}^{(\tau)}\right\rangle_{\Omega_0}\\ -\rho_f\langle f_k^{(\tau)},\xi\rangle_{\Omega_{k-1}^{(\tau)}} = 0\end{aligned}$$

for all $\xi\in C^\infty(\overline{\Omega};\mathbb{R}^d)$, $\operatorname{div}\xi=0$ in $\Omega_{k-1}^{(\tau)}$ and $\phi\in W^{k_0+2,2}(Q;\mathbb{R}^d)$ with $\phi=\xi\circ\eta_{k-1}^{(\tau)}$

Now we pass to the limit. We use as in Proposition 2.6 that the test functions can be approximated.

Let ξ be a test function for the limit equation, that is $\xi\in C^\infty([0,h]\times\overline{\Omega}(t);\mathbb{R}^d)$ with $\xi\cdot n=0$ on $\partial\Omega$ and $\operatorname{div}\xi=0$ in $\Omega(t)$. The corresponding solid test function ϕ is defined by $\phi:=\xi\circ\eta$.

Pick $\varepsilon>0$. Then for this ε, find ξ_ε and ϕ_ε by Proposition 2.6. As $\Omega^{(\tau)}(t)\to\Omega(t)$ in the Hausdorff distance we have that ξ_ε is divergence free on $\Omega^{(\tau)}$. Necessarily it holds $\phi_\varepsilon=\xi_\varepsilon\circ\eta$, so we will use this notation.

For τ small enough, ξ_ε, ϕ_ε is an admissible test function for the approximate equation. The (τ)-solution in (4.12) .

Thus we get, after integrating over each τ intervals and summing this up over k, that

$$\int_0^h DE(\eta^{(\tau)})\langle \xi_\varepsilon \circ \underline{\eta}^{(\tau)}\rangle + D_2R\left(\underline{\eta}^{(\tau)}, \partial_t\tilde{\eta}^{(\tau)}\right)\langle \xi_\varepsilon \circ \underline{\eta}^{(\tau)}\rangle + 2\kappa^{a_0}\langle \nabla^{k_0+2}\overline{\eta}^{(\tau)}, \nabla^{k_0+2}(\xi_\varepsilon \circ \underline{\eta}^{(\tau)})\rangle$$
$$+2\kappa\left\langle \nabla^{k_0}\partial_t\tilde{\eta}^{(\tau)}, \nabla^{k_0}(\xi_\varepsilon \circ \underline{\eta}^{(\tau)})\right\rangle + \rho_s\left\langle \frac{\partial_t\tilde{\eta}^{(\tau)} - w^{(\tau)}_{s,k-1}}{h}, \xi_\varepsilon \circ \underline{\eta}^{(\tau)}\right\rangle - \rho_s\left\langle \overline{f}^{(\tau)} \circ \underline{\eta}^{(\tau)}, \xi_\varepsilon \circ \underline{\eta}^{(\tau)}\right\rangle$$
$$+\nu\langle \varepsilon v^{(\tau)}, \varepsilon\xi_\varepsilon\rangle_{\Omega^{(\tau)}} + \kappa\langle \nabla^{k_0}v^{(\tau)}, \nabla^{k_0}\xi_\varepsilon\rangle_{\Omega^{(\tau)}(t)} + \rho_f\left\langle \frac{v^{(\tau)} \circ \Phi^{(\tau)} - w^{(\tau)}_{f,k-1}}{h}, \xi \circ \Phi^{(\tau)}\right\rangle_{\Omega_0}$$
$$-\rho_f\langle \overline{f}^{(\tau)}, \xi_\varepsilon\rangle_{\Omega^{(\tau)}(t)}\, dt = 0 \tag{4.12}$$

Now we pass to the limit in this equation. It is possible since we have the following convergences

$$\begin{aligned}
\eta^{(\tau)} &\overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty((0,h); W^{k_0+2,2}(Q;\mathbb{R}^d))\\
\partial_t\tilde{\eta}^{(\tau)} &\rightharpoonup \partial_t\eta \quad \text{in } L^2((0,h); W^{k_0+2,2}(Q;\mathbb{R}^d))\\
\eta^{(\tau)} &\to \eta \quad \text{in } C^{(1/2)^-}([0,h]; C^{1,\alpha^-}(Q;\mathbb{R}^d))\\
v^{(\tau)} &\overset{\eta}{\rightharpoonup} v \quad \text{in } L^2((0,h); W^{k_0,2}(\Omega(t);\mathbb{R}^d))
\end{aligned}$$

The passage to the limit in the non linearity is not a problem, as thanks to the compact embedding $W^{k_0+2,2} \hookrightarrow\hookrightarrow W^{2,q}$ we have

$$\eta^{(\tau)} \to \eta \quad \text{in } L^2((0,h); W^{2,q}(Q;\mathbb{R}^d))$$

and thus by (E.5)

$$DE(\eta^{(\tau)}) \rightharpoonup DE(\eta) \quad \text{in } L^2((0,h); W^{2,q}(Q;\mathbb{R}^d)^*)$$

and also by (R.4)

$$D_2R(\underline{\eta}^{(\tau)}, \partial_t\tilde{\eta}^{(\tau)}) \rightharpoonup D_2R(\eta, \partial_t\eta) \quad \text{in } L^2((0,h); W^{1,2}(Q;\mathbb{R}^d)).$$

Thus one can pass to the limit $\tau \to 0$ and the limit equation in the statement is satisfied with ξ_ε. Since the functions $\xi_\varepsilon \to \xi$ in $L^2((0,T); W^{k_0,2}(\Omega(t);\mathbb{R}^d)$ from Proposition 2.6, we obtain equation in the statement for ξ.

Now let us derive the energy inequality. Note that (in contrast to [BKS23b, Lemma 4.8]) we derive already here on the discrete τ level. Instead we rely on our findings from [ČS23]. For this use in (4.12) the test function pair $\left(\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}, v_k^{(\tau)}\right)$ which satisfies the required coupling. This gives

$$\left(DE(\eta_k^{(\tau)}) + D_2R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right)\right)\left\langle \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle + 2\kappa^{a_0}\left\langle \nabla^{k_0+2}\eta_k^{(\tau)}, \nabla^{k_0+2}\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle$$
$$+2\kappa\left\|\nabla^{k_0+2}\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\|^2 + \rho_s\left\langle \frac{\frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau} - w^{(\tau)}_{s,k-1}}{h}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle - \rho_s\left\langle f_k^{(\tau)} \circ \eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle$$
$$+\nu\|\varepsilon v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \kappa\|\nabla^{k_0}v_k^{(\tau)}\|^2_{\Omega_{k-1}^{(\tau)}} + \rho_f\left\langle \frac{v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)} - w^{(\tau)}_{f,k-1}}{h}, v_k^{(\tau)} \circ \Phi_{k-1}^{(\tau)}\right\rangle_{\Omega_0} - \rho_f\langle f_k^{(\tau)}, v_k^{(\tau)}\rangle_{\Omega_{k-1}^{(\tau)}} = 0$$

Here we use the non convexity estimate (E.7)

$$DE(\eta_k^{(\tau)})\left\langle \frac{\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)}}{\tau}\right\rangle \geq \frac{1}{\tau}(E(\eta_k^{(\tau)}) - E(\eta_{k-1}^{(\tau)}) - C_1\|\nabla(\eta_k^{(\tau)} - \eta_{k-1}^{(\tau)})\|^2),$$

two-homogeneity of R (R.2)

$$D_2R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right)\left\langle \frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right\rangle = 2R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right),$$

we expand the term

$$2\kappa^{a_0}\langle \nabla^{k_0+2}\eta_k^{(\tau)}, \nabla^{k_0+2}\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\rangle = \frac{\kappa^{a_0}}{\tau}\left(\|\nabla^{k_0+2}\eta_k^{(\tau)}\|^2 + \|\nabla^{k_0+2}(\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)})\|^2 - \|\nabla^{k_0+2}\eta_{k-1}^{(\tau)}\|^2\right)$$

and use Young's inequality in the inertial terms

$$\rho_s\left\langle \frac{\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}-w_{s,k-1}^{(\tau)}}{h}, \frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right\rangle \geq \frac{\rho_s}{2h}\left\|\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right\|^2 - \frac{\rho_s}{2h}\|w_{s,k-1}^{(\tau)}\|^2$$

$$\rho_f\left\langle \frac{v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)} - w_{f,k-1}^{(\tau)}}{h}, v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)}\right\rangle_{\Omega_0} \geq \frac{\rho_f}{2h}\|v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)}\|_{\Omega_0}^2 - \frac{\rho_f}{2h}\|w_{f,k-1}^{(\tau)}\|_{\Omega_0}^2.$$

Altogether, multiplying by τ and using these estimates (the error from non convexity estimate is absorbed by dissipation for τ small enough) we obtain the estimate

$$\begin{aligned}
&E(\eta_k^{(\tau)}) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta_k^{(\tau)}\|^2 + \tau 2R\left(\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right) + \tau 2\kappa\left\|\nabla^{k_0+2}\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right\|^2\\
&+\tau\frac{\rho_s}{2h}\left\|\frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right\|^2 + \tau\kappa\|\nabla^{k_0}v_k^{(\tau)}\|_{\Omega_{k-1}^{(\tau)}}^2 + \tau\nu\|\varepsilon v_k^{(\tau)}\|_{\Omega_{k-1}^{(\tau)}}^2 + \tau\frac{\rho_f}{2h}\left\|v_k^{(\tau)}\circ\Phi_{k-1}^{(\tau)}\right\|_{\Omega_0}^2\\
&\qquad\leq E(\eta_{k-1}^{(\tau)}) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta_{k-1}^{(\tau)}\|^2 + \tau\frac{\rho_s}{2h}\|w_{s,k-1}^{(\tau)}\|^2 + \tau\frac{\rho_f}{2h}\|w_{f,k-1}^{(\tau)}\|_{\Omega_0}^2\\
&\qquad\quad+\tau\rho_s\left\langle f_k^{(\tau)}\circ\eta_{k-1}^{(\tau)}, \frac{\eta_k^{(\tau)}-\eta_{k-1}^{(\tau)}}{\tau}\right\rangle + \tau\rho_f\langle f_k^{(\tau)}, v_k^{(\tau)}\rangle_{\Omega_{k-1}^{(\tau)}}
\end{aligned}$$

which summing over k yields the energy inequality

$$\begin{aligned}
&E(\eta^{(\tau)}(h)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta^{(\tau)}(h)\|^2 + \frac{1}{2h}\int_0^h \rho_f\|v^{(\tau)}\|_{\Omega(t)}^2 + \rho_s\|\partial_t\tilde{\eta}^{(\tau)}\|^2\,dt\\
&\quad+\int_0^h 2R(\underline{\eta}^{(\tau)}, \partial_t\tilde{\eta}^{(\tau)}) + 2\kappa\|\nabla^{k_0+2}\partial_t\tilde{\eta}^{(\tau)}\|^2 + \kappa\|\nabla^{k_0}v^{(\tau)}\|^2\,dt\\
&\leq E(\eta^{(\tau)}(0)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta^{(\tau)}(0)\|^2 + \frac{1}{2h}\int_0^h \rho_f\|w_f\|_{\Omega_0}^2 + \rho_s\|w_s\|^2\,dt\\
&\quad+\int_0^t \rho_s\left\langle f\circ\eta^{(\tau)}, \partial_t\tilde{\eta}^{(\tau)}\right\rangle + \rho_f\langle f, v^{(\tau)}\rangle_{\Omega(t)}\,dt.
\end{aligned}$$

□

4.3. **Passing to limit with the delay $h \to 0$.** We construct the solution on intervals $(0,h)$, $(h,2h)$, ..., $(T-h,T)$, as in the previous section and glue this together to get a time-delayed solution on the entire time interval $(0,T)$.

More precisely, for $\ell = 0,\dots,T/h-1$ we let $\eta^{(h)}(\cdot+\ell h)$, $v^{(h)}(\cdot+\ell h)$ and $\Omega^{(h)}(\cdot+\ell h)$, $\Phi^{(h)}(\cdot+\ell h)$ to be inductively the solution of the time-delayed equation of Definition 4.1 constructed in the previous section, where we put first for $\ell = 0$

$$w_s(t) \equiv \eta_*, \quad w_f(t) \equiv v_0, \quad \eta_0 = \eta_0, \quad t \in [0,h).$$

and for the subsequent steps $\ell > 0$

$$w_s(t) = \partial_t\eta^{(h)}(t+(\ell-1)h), \quad w_f(t) = v^{(h)}(t+(\ell-1)h)\circ\Phi^{(h)}(t+(\ell-1)h), \quad t\in[0,h),$$
$$\Omega_0 = \Omega^{(h)}(\ell h), \quad \eta_0 = \eta^{(h)}(\ell h),$$

where all these quantities are given by the solution in the $(\ell-1)$-th step. Summing the time-delayed equations over $\ell = 0,\dots,T/h-1$, we have the following equations.
Fluid-only equation.

$$\int_0^T \nu\langle \varepsilon v^{(h)}, \varepsilon\xi\rangle_{\Omega^{(h)}(t)} + \kappa\langle \nabla^{k_0} v^{(h)}, \nabla^{k_0}\xi\rangle_{\Omega^{(h)}(t)} + \rho_f \left\langle \frac{v^{(h)}\circ\Phi_t^{(h)} - v^{(h)}(\cdot - h)\circ\Phi_{t-h}^{(h)}}{h}, \xi\circ\Phi^{(h)}\right\rangle_{\Omega_0} - \langle f,\xi\rangle_{\Omega^{(h)}(t)}\,dt = 0 \tag{4.13}$$

for all $\xi \in C^\infty([0,T]\times\overline{\Omega}^{(h)}(t)); \mathbb{R}^d$, $\xi\cdot n^{(h)} = 0$ on $\partial\Omega^{(h)}(t)$, $\operatorname{div}\xi = 0$ in $\Omega^{(h)}(t)$, with $\xi(T) = 0$.
Coupled equation.

$$\begin{aligned}\int_0^T DE(\eta^{(h)})\langle\phi\rangle &+ D_2R\left(\eta^{(h)},\partial_t\eta^{(h)}\right)\langle\phi\rangle + 2\kappa^{a_0}\langle\nabla^{k_0+2}\eta^{(h)},\nabla^{k_0+2}\phi\rangle\\ +2\kappa\left\langle\nabla^{k_0+2}\partial_t\eta^{(h)},\nabla^{k_0+2}\phi\right\rangle &+ \rho_s\left\langle\frac{\partial_t\eta^{(h)}-\partial_t\eta^{(h)}(\cdot-h)}{h},\phi\right\rangle - \left\langle f\circ\eta^{(h)},\phi\right\rangle\\ &+\nu\langle\varepsilon v^{(h)},\varepsilon\xi\rangle_{\Omega^{(h)}(t)} + \kappa\langle\nabla^{k_0}v^{(h)},\nabla^{k_0}\xi\rangle_{\Omega^{(h)}(t)}\\ +\rho_f\left\langle\frac{v^{(h)}\circ\Phi_t^{(h)} - v^{(h)}(t-h)\circ\Phi_{t-h}^{(h)}}{h},\xi\circ\Phi_t^{(h)}\right\rangle_{\Omega_0} &- \langle f,\xi\rangle_{\Omega^{(h)}(t)}\,dt = 0\end{aligned} \tag{4.14}$$

for all $\xi\in C^\infty([0,T]\times\overline{\Omega}^{(h)}(t);\mathbb{R}^d)$, $\operatorname{div}\xi = 0$ in $\Omega^{(h)}(t)$ and $\phi\in W^{k_0+2,2}(Q;\mathbb{R}^d)$ with $\phi = \xi\circ\eta$ on Q.

4.3.1. *Estimates and a weak limit.* By the energy inequality (4.11) we have the following estimate

$$\begin{aligned}E(\eta^{(h)}(t)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta^{(h)}(t)\|^2 + \frac{1}{2h}\int_{t-h}^t \rho_f\|v^{(h)}\|^2_{\Omega(t)} + \rho_s\|\partial_t\eta^{(h)}\|^2\,dt&\\ +\int_0^t 2R(\eta^{(h)},\partial_t\eta^{(h)}) + 2\kappa\|\nabla^{k_0+2}\partial_t\eta^{(h)}\|^2 + \|\nabla^{k_0}v^{(h)}\|^2\,dt&\\ \le E(\eta^{(h)}(0)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta^{(h)}(0)\|^2 + \frac{1}{2h}\int_{t-h}^t \rho_f\|w_f\|^2_{\Omega_0} + \rho_s\|w_s\|^2\,dt&\\ +\int_0^t \rho_s\left\langle f\circ\eta^{(h)},\partial_t\eta\right\rangle + \rho_f\langle f, v^{(h)}\rangle_{\Omega(t)}\,dt&\end{aligned} \tag{4.15}$$

and thus for (a subsequence of) $h\to 0$ we have weakly converging subsequences

$$\begin{aligned}\eta^{(h)} &\overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))\\ \partial_t\eta^{(h)} &\rightharpoonup \partial_t\eta \quad \text{in } L^2((0,T);W^{k_0+2,2}(Q;\mathbb{R}^d))\\ v^{(h)} &\overset{\eta}{\rightharpoonup} v \quad \text{in } L^2((0,T);W^{k_0,2}(\Omega(t);\mathbb{R}^d)).\end{aligned} \tag{4.16}$$

In order to pass to the limit in the inertial terms, we take an approximation of test functions for (4.13) as in Proposition 2.6 (resp. for (4.14) as in Proposition 2.9 (ii)) and pass to the limits $h\to 0$ in all of the terms in fluid-only equation and in the coupled equation in analogy to Proposition 4.5 and 4.6, except for the inertial terms that we shall deal with below.

For this we shall need an estimate on discrete versions of $\partial_{tt}\eta^{(h)}$ and $\partial_t v$. In particular, from the equation we get the estimate on h-difference quotients as follows.

Lemma 4.7 (Solid bounds with length h)**.** *The following bounds exists independent of h:*

$$\int_0^T \left\|\frac{\partial_t\eta^{(h)} - \partial_t\eta^{(h)}(t-h)}{h}\right\|^2_{W^{-k_0-2,2}(Q;\mathbb{R}^d)} dt \le C \tag{4.17}$$

Proof. To show (4.17) we use a test function $\phi \in W_0^{k_0+2,2}(Q;\mathbb{R}^d)$ in the equation (4.14) (the corresponding ξ is zero in the fluid domain), so we get

$$\begin{aligned}
\rho_s \left|\left\langle \frac{\partial_t \eta^{(h)}(t) - \partial_t \eta^{(h)}(t-h)}{h}, \phi \right\rangle\right| &\leq \left|\left\langle DE\left(\eta^{(h)}(t)\right), \phi\right\rangle\right| + \kappa^{a_0} \left|\left\langle \nabla^{k_0+2}\eta^{(h)}, \nabla^{k_0+2}\phi\right\rangle\right| \\
&+ \left|\left\langle D_2 R\left(\eta^{(h)}(t), \partial_t \eta^{(h)}(t)\right), \phi\right\rangle\right| + \kappa \left|\left\langle \nabla^{k_0+2}\partial_t\eta^{(h)}, \nabla^{k_0+2}\phi\right\rangle\right| + |\langle f_s(t), \phi\rangle| \\
&\leq \left(\left\|DE\left(\eta^{(h)}(t)\right)\right\|_{W^{-2,q'}} + \kappa^{a_0}\left\|\nabla^{k_0+2}\eta^{(h)}(t)\right\| + \|f_s\|_\infty\right)\|\phi\|_{W^{k_0+2,2}} \\
&+ \left(\left\|D_2 R\left(\eta^{(h)}(t), \partial_t\eta^{(h)}(t)\right)\right\|_{W^{-1,2}} + \kappa\left\|\nabla^{k_0}\partial_t\eta^{(h)}(t)\right\|\right)\|\phi\|_{W^{k_0+2,2}} \leq c(t)\|\phi\|_{W^{k_0+2,2}},
\end{aligned}$$

with the bound $c \in L^2((0,T))$ thanks to (4.15), after integration in time we get the estimate (4.17). □

For the solid, the issue of inertial term convergence is not difficult by the estimate we just proved. Let $b^{(h)}(t) = \frac{1}{h}\int_{t-h}^h \partial_t \eta^{(h)}$. By (4.15) it is uniformly bounded in $L^2((0,T); W^{k_0+2,2}(Q;\mathbb{R}^d))$ Since we have

$$\partial_t b^{(h)} = \frac{\partial_t \eta^{(h)}(\cdot) - \partial_t \eta^{(h)}(\cdot - h)}{h}$$

we see that Lemma 4.7 $\partial_t b^{(h)}$ is uniformly bounded in $L^2((0,T); W^{-k_0-2,2}(Q;\mathbb{R}^d))$ which immediately yields the convergence (for a subsequence)

$$b^{(h)} \to \partial_t \eta \quad \text{in } C([0,T]; L^2(Q;\mathbb{R}^d)).$$

The convergence of the fluid inertial term is a more delicate matter that we now shall deal with properly.

For the fluid, we shall use these additional flow map estimates

Proposition 4.8 (Flow map h-estimates)**.** *It holds*

$$\left\|\frac{\xi(t+h)\circ\Phi_h^{(h)} - \xi(t)}{h}\right\|_{L^\infty L^2} \leq C \tag{4.18}$$

for all $\xi \in C^\infty([0,T]\times\overline{\Omega}^{(h)}(t);\mathbb{R}^d)$, $\xi\cdot n^{(h)} = 0$ *on* $\partial\Omega^{(h)}(t)$, $\operatorname{div}\xi = 0$ *in* $\Omega^{(h)}(t)$.

Proof. Remember that $\partial_s \Phi_s^{(h)} = u(s)\circ\Phi_s^{(h)}$ from (4.8). We rewrite

$$\xi(t+h)\circ\Phi_h^{(h)} - \xi(t) = \int_0^h \partial_s(\xi(t+s)\circ\Phi_s^{(h)})\,ds = \int_0^h \partial_t\xi(t+s)\circ\Phi_s^{(h)} + \nabla\xi(t+s)\circ\Phi_s^{(h)}\cdot u^{(h)}(s)\,ds \tag{4.19}$$

we have

$$\left\|\frac{\xi(t+h)\circ\Phi_h^{(h)} - \xi(t)}{h}\right\|_{L^\infty L^2}^2 \leq C\,\mathrm{Lip}_t\,\xi + C\,\mathrm{Lip}_x\,\xi \sup_t \frac{1}{h}\int_t^{t+h} |v^{(h)}|^2\,dt \leq C$$

and we conclude by the L^∞ estimate of h-average of $v^{(h)}$ from (4.15). □

Estimate of the inertial term: We now work with the *global velocity field* $u^{(h)}$ defined by

$$u^{(h)}(t,y) = \begin{cases} v^{(h)}(t,y), & y \in \Omega^{(h)}(t) \\ \eta^{(h)}(t,(\eta^{(h)})^{-1}(t,y)), & y \in \eta^{(h)}(t,Q). \end{cases} \tag{4.20}$$

Note that $u^{(h)}$ has a tangential jump along $\partial\eta(\cdot,Q)$. For this global velocity field, we have the following estimate of its time derivative.

Lemma 4.9 (Fluid bounds with length h)**.** *Let $u^{(h)}$ be the global velocity field* (4.20)*. Then the for m large enough we have the estimate*

$$\int_0^T \left| \left\langle \frac{u^{(h)}(t) - u^{(h)}(t-h)}{h}, \xi(t) \right\rangle_{\Omega^{(h)}(t)} \right| dt \le C\|\xi\|_{L^2((0,T);W^{m,2}(\Omega^{(h)}(t);\mathbb{R}^d))}$$

for every $\xi \in C^\infty([0,T] \times \overline{\Omega}^{(h)}(t);\mathbb{R}^d)$, $\nabla^\ell(\xi \cdot n^{(h)}) = 0$ on $\partial\Omega^{(h)}(t)$, $\ell = 0,\dots,k_0$, $\operatorname{div}\xi = 0$ in $\Omega^{(h)}(t)$.

Proof. First we put in the flow map

$$\int_0^T \left\langle \frac{u^{(h)}(t) - u^{(h)}(t-h)}{h}, \xi(t) \right\rangle_{\Omega^{(h)}(t)} dt = \int_0^T \left\langle \frac{u^{(h)}(t) - u^{(h)}(t-h) \circ \Phi^{(h)}_{-h}(t)}{h}, \xi(t) \right\rangle_{\Omega^{(h)}(t)}$$
$$+ \left\langle \frac{u^{(h)}(t-h) \circ \Phi^{(h)}_{-h}(t) - u^{(h)}(t-h)}{h}, \xi(t) \right\rangle_{\Omega^{(h)}(t)} dt$$

We estimate by using the test function ξ in the fluid only equation (4.13). Then we estimate

$$\begin{aligned}\int_0^T \left\langle \frac{u^{(h)}(t) - u^{(h)}(t-h) \circ \Phi^{(h)}_{-h}(t)}{h}, \xi(t) \right\rangle_{\Omega^{(h)}(t)} dt = \\ = -\int_0^T \nu\langle \varepsilon u^{(h)}, \varepsilon\xi\rangle_{\Omega^{(h)}(t)} + \kappa\langle \nabla^{k_0} u^{(h)}, \nabla^{k_0}\xi\rangle_{\Omega^{(h)}(t)} - \langle f, \xi\rangle_{\Omega^{(h)}(t)}\, dt \\ \le C\|\xi\|_{L^\infty(C^{k_0})}\end{aligned} \tag{4.21}$$

and choose m so that $W^{m,2}$ embeds into C^{k_0}. □

4.3.2. *Limit $h \to 0$ in the fluid inertial term.* We now want to pass to the limit $h \to 0$ in

$$\int_0^T \rho_f \left\langle \frac{u^{(h)}(t) \circ \Phi^{(h)} - u^{(h)}(t-h)}{h}, \xi_\varepsilon^{(h)}(t) \right\rangle dt.$$

We want to do this in both the fluid-only equation (4.13), and also the coupled equation (4.14). We shall prove now a version of Aubin-Lions lemma for the fluid.

Denote

$$\tilde{m}^{(h)}(t) = \frac{1}{h}\int_{t-h}^t \rho v^{(h)}\, dt$$

Our aim now is to show the convergence

$$\int_0^T \langle u^{(h_i)}, A\tilde{m}^{(h_i)}\rangle\, dt \to \int_0^T \langle u, A\tilde{m}\rangle\, dt$$

where $A \in C_0^\infty((0,T) \times \Omega)$ is a cutoff function chosen that $u^{(h)}$ and also $u^{(h)}(\cdot + \sigma)$, $\sigma \in (0,h)$ is always well defined on $\operatorname{supp} A$.

First we show the Fluid Aubin-Lions in the form that

$$\int_0^T \langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)}\rangle\, dt \to \int_0^T \langle u_\delta, A\tilde{m}\rangle\, dt$$

We carefully construct our approximation of $(\cdot)_\delta$ by first truncating the functions on a δ strip. For that we use the cutoff ψ_δ defined in Proposition 2.9 and the uniform Bogovskii operator defined in Theorem 2.8 and take

$$(u^{(h_i)})^*_\delta := (1 - \psi_\delta)v^{(h_i)} - \mathcal{B}(\operatorname{div}((1 - \psi_\delta)v^{(h_i)})).$$

Please observe that since $v^{(h_i)} \in L^p((0,T) \times \Omega)$ for $p > 2$ we find (using the L^2-bound of the Bogovskii) that for $1 < q \le p$

$$\|(u^{(h_i)})^*_\delta - u^{(h_i)}\|_{L^q((0,T)\times\Omega)} \le C\delta^{\frac{1}{q}-\frac{1}{p}}\|(u^{(h_i)})^*_\delta - u^{(h_i)}\|_{L^2((0,T)\times\Omega)} \le C.$$

Note further that $(u^{(h_i)})^*_\delta$ is uniformly in $L^2(0,T;W^{1,2}_0(\Omega^{(h_j)}(t))$ for i,j large enough with bounds depending on δ. We take the standard mollifier $\phi_{\frac{\delta}{2}}$ and define

$$(u^{(h_i)})_\delta := (u^{(h_i)})^*_\delta * \phi_{\frac{\delta}{2}}. \tag{4.22}$$

Analogous to the proof of Proposition 2.6 one finds the respective bounds and convergence properties in dependence of δ. Moreover

$$\|(u^{(h_i)})_\delta - u^{(h_i)}\|_{L^q((0,T)\times\Omega)} \le C\delta^{\frac{1}{q}-\frac{1}{p}}.$$

Thus we formulate the fluid Aubin Lions in the following way

Theorem 4.10 (Fluid Aubin-Lions)**.** *For every $\delta > 0$ it holds that*

$$\int_0^T \langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)}\rangle\, dt \to \int_0^T \langle u_\delta, A\rho u\rangle\, dt,$$

where $(\cdot)_\delta$ is the approximation defined in (4.22).

Proof. Let $\delta > 0$. Observe that $(u^{(h_i)})_\delta$ is constructed so that the functions are divergence-free in a neighborhood in space-time. This means that $(u^{(h_i)})_\delta$ is a valid test functions in a small (but fixed) neighborhood of t uniformly for i large enough.

To obtain the desired convergence, we will show that $\int_0^T \langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)}\rangle\, dt$ is a Cauchy sequence. For this we write

$$\langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)}\rangle - \langle (u^{(h_j)})_\delta, A\tilde{m}^{(h_j)}\rangle = \langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)} - A\tilde{m}^{(h_j)}\rangle + \langle (u^{(h_i)})_\delta - (u^{(h_j)})_\delta, A\tilde{m}^{(h_j)}\rangle.$$

We now focus on the first term, namely

$$\langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)} - A\tilde{m}^{(h_j)}\rangle.$$

We partition the time with $\sigma > 0$ steps and replace the $A\tilde{m}^{(h_i)}$ with piecewise constant, i.e. we write

$$\begin{aligned} \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_i)}(t) - A\tilde{m}^{(h_j)}(t)\rangle &= \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_i)}(t) - A\tilde{m}^{(h_i)}(\sigma k)\rangle \\ + \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_i)}(\sigma k) - A\tilde{m}^{(h_j)}(\sigma k)\rangle &+ \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_j)}(\sigma k) - A\tilde{m}^{(h_j)}(t)\rangle \end{aligned} \tag{4.23}$$

Now we use that we have the uniform bound from Proposition 4.7

$$\|\tilde{m}^{(h_i)}(\sigma k)\|^2_{L^2} \le \frac{1}{h}\int_{\sigma k - h}^{\sigma k} \|\rho v(t)\|^2_{L^2}\, dt \le C.$$

So by the compact embedding

$$L^2(\Omega) \subset\subset (W^{1,2}_{\operatorname{div}}(\Omega_\delta))^*$$

we have for a subsequence

$$\tilde{m}^{(h_i)}(\sigma k) \to \tilde{m}(\sigma k) \quad \text{in } (W^{1,2}_{\operatorname{div}}(\Omega_\delta))^*$$

so that

$$\|(u^{(h_i)}(t))_\delta\|_{W^{1,2}(\Omega)} \|A\tilde{m}^{(h_i)}(\sigma k) - A\tilde{m}^{(h_j)}(\sigma k)\|_{(W^{1,2}_{\operatorname{div}}(\Omega_\delta))^*} \to 0$$

since that $\|(u^{(h_i)}(t))_\delta\|_{W^{1,2}(\Omega)} \le C$ by the apriori estimates (4.15).

It remains in (4.23) to estimate the term[1]

$$\int_0^T \langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_j)}(t) - A\tilde{m}^{(h_j)}(\sigma k)\rangle\, dt \tag{4.24}$$

To estimate it we apply the strategy to "replace σ-difference quotient with h-difference quotient" – with the aim to use then the bound on h-difference quotient. So for this to remove the σ-difference means we write

$$A\tilde{m}^{(h_j)}(t) - A\tilde{m}^{(h_j)}(\sigma k) = A\int_{\sigma k}^t \partial_\theta \tilde{m}^{(h_j)}(\theta)\, d\theta$$

[1]The other case if on the right there is i instead of j is dealt with in the same way.

so that we get in (4.24)

$$\langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_j)}(t) - A\tilde{m}^{(h_j)}(\sigma k)\rangle = \left\langle (u^{(h_i)}(t))_\delta, A\int_{\sigma k}^t \partial_\theta \tilde{m}^{(h_j)}(\theta)\, d\theta \right\rangle.$$

Realize that by definition of $\tilde{m}^{(h_j)}$ it is

$$\partial_\theta \tilde{m}^{(h_j)}(\theta) = \frac{\rho u^{(h_j)}(\theta) - \rho u^{(h_j)}(\theta - h_j)}{h_j}$$

So that now we have

$$\int_0^T \left\langle (u^{(h_i)}(t))_\delta, A\tilde{m}^{(h_j)}(t) - A\tilde{m}^{(h_j)}(\sigma k)\right\rangle dt = \int_0^T \left\langle (u^{(h_i)}(t))_\delta, A\int_{\sigma k}^t \frac{\rho u^{(h_j)}(\theta) - \rho u^{(h_j)}(\theta - h_j)}{h_j} d\theta \right\rangle dt.$$

Now comes a switch of the order of integration which results in

$$\leq \|A\|_\infty \int_0^\sigma \int_0^T \left| \left\langle (u^{(h_i)}(\theta + s))_\delta, \frac{\rho u^{(h_j)}(\theta) - \rho u^{(h_j)}(\theta - h_j)}{h_j} \right\rangle \right| ds\, d\theta$$

and we use the bound (4.21) to obtain, recall also the estimate from Proposition 2.6

$$\leq \|A\|_\infty \int_0^\sigma \|(u^{(h_i)})_\delta(\theta + \cdot)\|_{L^2 W^{k_0,2}_{\mathrm{div}}}\, d\theta \leq \|A\|_\infty C_\delta \sigma \|u^{(h_i)}\|_{L^2 W^{1,2}}$$

so this vanishes for $\sigma \to 0$, as $\|u^{(h_i)}\|_{L^2 W^{1,2}}$ is bounded by the energy inequality (4.15).

This proves the convergence

$$\langle (u^{(h_i)})_\delta, A\tilde{m}^{(h_i)}\rangle \to \langle u_\delta, A\tilde{m}\rangle$$

where $\tilde{m}^{(h_i)} \overset{*}{\rightharpoonup} \tilde{m}$ in $L^\infty((0,T); L^2(\Omega; \mathbb{R}^d))$ (the limit exists by the estimate (4.15)). It is not difficult to check $\tilde{m} = \rho u$, since for $\xi \in C_0((0,T) \times \Omega; \mathbb{R}^d)$ we have

$$\int_0^T \int_\Omega \tilde{m}^{(h_i)} \cdot \xi\, dx\, dt = \int_0^T \int_\Omega \rho u^{(h_i)} \cdot \frac{1}{h}\int_0^h \xi(t+s)\, ds\, dx\, dt \to \int_0^T \int_\Omega \rho u \cdot \xi\, dx\, dt$$

which concludes the proof.

□

Now we are equipped for the rest of the $h \to 0$ limit passage in the fluid inertial term.

Theorem 4.11 (Limit passage $h \to 0$)**.** *The limit η, h from* (4.16) *satisfies the following equations.*

Fluid-only equation*.*

$$\int_0^T \nu\langle \varepsilon v, \varepsilon \xi\rangle_{\Omega(t)} + \kappa \langle \nabla^{k_0} v, \nabla^{k_0}\xi\rangle_{\Omega(t)} - \rho_f \left\langle \partial_t v, \xi\right\rangle_{\Omega(t)} + \rho_f \langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} - \langle f, \xi\rangle_{\Omega(t)}\, dt = 0 \quad (4.25)$$

for all $\xi \in C^\infty([0,T] \times \overline{\Omega}(t)); \mathbb{R}^d)$, $\xi \cdot n = 0$ on $\partial\Omega(t)$, $\operatorname{div}\xi = 0$ in $\Omega(t)$, with $\xi(T) = 0$.

Coupled equation*.*

$$\int_0^T DE(\eta)\langle\phi\rangle + D_2 R(\eta, \partial_t\eta)\langle\phi\rangle + 2\kappa^{a_0}\langle \nabla^{k_0+2}\eta, \nabla^{k_0+2}\phi\rangle + 2\kappa\left\langle \nabla^{k_0+2}\partial_t\eta, \nabla^{k_0+2}\phi\right\rangle$$
$$-\rho_s\langle \partial_t\eta, \partial_t\phi\rangle - \langle f\circ\eta, \phi\rangle$$
$$+\nu\langle \varepsilon v, \varepsilon\xi\rangle_{\Omega(t)} + \kappa\langle \nabla^{k_0} v, \nabla^{k_0}\xi\rangle_{\Omega(t)} - \rho_f\left\langle \partial_t v, \xi\right\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} - \langle f, \xi\rangle_{\Omega(t)}\, dt = 0 \quad (4.26)$$

for all $\xi \in C^\infty([0,T] \times \overline{\Omega}(t); \mathbb{R}^d)$, $\operatorname{div}\xi = 0$ in $\Omega(t)$ and $\phi \in W^{k_0+2,2}(Q; \mathbb{R}^d)$ with $\phi = \xi \circ \eta$ on Q.

Further it satisfies the following energy inequality

$$
\begin{aligned}
&E(\eta(t)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta(t)\|^2 + \frac{\rho_f}{2}\|v(t)\|^2_{\Omega(t)} + \frac{\rho_s}{2}\|\partial_t\eta(t)\|^2 \\
&+ \int_0^t 2R(\eta,\partial_t\eta) + 2\kappa\|\nabla^{k_0+2}\partial_t\eta\|^2 + \|\nabla^{k_0}v\|^2\,dt \\
\le\ & E(\eta(0)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta(0)\|^2 + \frac{\rho_f}{2}\|v(0)\|^2_{\Omega_0} + \frac{\rho_s}{2}\|\partial_t\eta(0)\|^2\,dt + \int_0^t \rho_s\left\langle f\circ\eta,\partial_t\eta\right\rangle + \rho_f\langle f,v\rangle_{\Omega(t)}\,dt.
\end{aligned}
\tag{4.27}
$$

Proof. We are equipped to pass to the limit with the fluid inertial term, in particular

$$
\int_0^T \left\langle \frac{u^{(h)}(t)\circ\Phi_h^{(h)}(t-h) - u^{(h)}(t-h)}{h}, \xi(t)\circ\Phi_h^{(h)}(t-h)\right\rangle_{\Omega^{(h)}(t-h)} dt
$$

After discrete partial integration in time we have

$$
= -\int_0^T \left\langle u^{(h)}(t), \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)} dt
$$

We write only the term under the integral. Use the δ-divergence free approximation

$$
\begin{aligned}
&\left\langle u^{(h)}(t), \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)} \\
=&\left\langle (u^{(h)}(t))_\delta - u^{(h)}(t), \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)} + \left\langle (u^{(h)}(t))_\delta, \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)}
\end{aligned}
$$

In the first term use

$$
\le \left\|(u^{(h)}(t))_\delta - u^{(h)}(t)\right\|_{L^1L^2}\left\|\frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\|_{L^\infty L^2}
$$

as the former is by the δ-approximation of Proposition 2.6 bounded by

$$
\left\|(u^{(h)}(t))_\delta - u^{(h)}(t)\right\|_{L^1L^2} \le \delta^{\frac{d}{d+2}}\left\|u^{(h)}(t)\right\|_{L^2(W^{1,2})}
$$

and latter is bounded by Proposition 4.8, so that in total for the first term

$$
\left|\int_0^T \left\langle (u^{(h)}(t))_\delta - u^{(h)}(t), \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)} dt\right| \le \delta^{\frac{d}{d+2}}\tilde{C}.
$$

For the second term

$$
\left\langle (u^{(h)}(t))_\delta, \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)}
$$

we perform again the manipulations (4.19) so that

$$
\begin{aligned}
&\left\langle (u^{(h)}(t))_\delta, \frac{\xi(t+h)\circ\Phi^{(h)} - \xi(t)}{h}\right\rangle_{\Omega^{(h)}(t)}. \\
=&\left\langle (u^{(h)}(t))_\delta, \frac{1}{h}\int_0^h \Big(\partial_t\xi(t+s) + \nabla\xi(t+s)\cdot u^{(h)}(t+s)\Big)\circ\Phi_s^{(h)}\,ds\right\rangle
\end{aligned}
$$

This is now a sum where the first part is fine: after the limit $h\to 0$

$$
\int_0^T \left\langle (u^{(h)}(t))_\delta, \frac{1}{h}\int_0^h \partial_t\xi(t+s)\circ\Phi_s^{(h)}\,ds\right\rangle dt \to \int_0^T \left\langle (u(t))_\delta, \partial_t\xi(t)\right\rangle dt.
$$

The second term now needs care:

$$\int_0^T \left\langle (u^{(h)}(t))_\delta, \frac{1}{h}\int_0^h \Big(\nabla\xi(t+s)\cdot u^{(h)}(t+s)\Big)\circ \Phi_s^{(h)}\,ds\right\rangle dt$$

We can change the domain to obtain

$$= \int_0^T \frac{1}{h}\int_0^T \left\langle (u^{(h)}(t))_\delta \circ \Phi_{-s}^{(h)}, \nabla\xi(t+s)\cdot u^{(h)}(t+s)\right\rangle ds\,dt$$

$$= \int_0^T \frac{1}{h}\int_0^h \left\langle (u^{(h)}(t))_\delta \circ \Phi_{-s}^{(h)} - (u^{(h)}(t))_\delta, \nabla\xi(t+s)\cdot u^{(h)}(t+s)\right\rangle ds\,dt$$
$$+ \int_0^T \left\langle (u^{(h)}(t))_\delta, \frac{1}{h}\int_0^h \nabla\xi(t+s)\cdot u^{(h)}(t+s)\,ds\right\rangle dt$$

Convergence in the first term follows from Proposition 4.8

$$\left\|(u^{(h)}(t))_\delta \circ \Phi_{-s}^{(h)} - (u^{(h)}(t))_\delta\right\|_{L^2} \le ch\,\mathrm{Lip}_x(u^{(h)})_\delta \le hC_\delta \|u^{(h)}(t)\|_{W^{1,2}}$$

Convergence in the last term is obtained from

$$\int_0^T \left\langle (u^{(h)}(t))_\delta, \frac{1}{h}\int_0^h \nabla\xi(t+s)\cdot u^{(h)}(t+s)\,ds\right\rangle dt$$
$$= \int_0^T \left\langle (u^{(h)}(t))_\delta, \frac{1}{h}\int_0^h (\nabla\xi(t+s)-\nabla\xi(t))\cdot u^{(h)}(t+s)\,ds\right\rangle dt$$
$$+ \int_0^T \left\langle (u^{(h)}(t))_\delta, \nabla\xi(t)\cdot \frac{1}{h}\int_0^h u^{(h)}(t+s)\,ds\right\rangle dt$$

where in the first term we have

$$\|\nabla\xi(t+s)-\nabla\xi(t)\|_{L^\infty} \le h\|\partial_t\nabla\xi\|_{L^\infty} \to 0 \text{ with } h\to 0$$

and in the second we use the Theorem 4.10, where we take A to be an approximation of $\nabla\xi\chi_{\Omega(t)}$. More precisely, write

$$\int_0^T \left\langle (u^{(h)}(t))_\delta, \nabla\xi(t)\cdot \frac{1}{h}\int_0^h u^{(h)}(t+s)\,ds\right\rangle dt = \int_0^T \left\langle (u^{(h)}(t))_\delta, A_\delta(t)\cdot \frac{1}{h}\int_0^h u^{(h)}(t+s)\,ds\right\rangle dt$$
$$+ \int_0^T \left\langle (u^{(h)}(t))_\delta, \nabla(\xi(t)-A_\delta(t))\cdot \frac{1}{h}\int_0^h u^{(h)}(t+s)\,ds\right\rangle dt$$

and in the first term we have by Theorem 4.10

$$\int_0^T \left\langle (u(t))_\delta, A_\delta(t)\cdot \frac{1}{h}\int_0^h u^{(h)}(t+s)\,ds\right\rangle dt \to \int_0^T \left\langle (u^{(h)}(t))_\delta, A_\delta(t)\cdot u(t)\,ds\right\rangle dt.$$

In the second term we estimate by Hölder's inequality and Sobolev embedding for $a < d/(d-2)$

$$\left|\int_0^T \left\langle (u^{(h)}(t))_\delta, \nabla(\xi(t)-A_\delta(t))\cdot \frac{1}{h}\int_0^h u^{(h)}(t+s)\,ds\right\rangle dt\right|$$
$$\le \int_0^T \left\|\Big(u^{(h)}(t)\Big)_\delta\right\|_{L^{2a}(\Omega)} \frac{1}{h}\int_0^h \left\|u^{(h)}(t+s)\right\|_{L^a(\Omega)} \|A_\delta(t)-\chi_{\Omega^{(h)}}(t+s)\|_{L^{2a'}(\Omega)}\,dsdt$$
$$\le c\left\|\Big(u^{(h)}(t)\Big)_\delta\right\|_{L^2([0,T];W^{1,2}(\Omega))} \sup_{t\in T}\left(\frac{1}{h}\int_0^h \left\|u^{(h)}(t+s)\right\|^2 ds\right)^{1/2}$$
$$\times\left(\frac{1}{h}\int_0^h \|A_\delta(t)-\chi_{\Omega^{(h)}}(t+s)\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{1/2}$$
$$\le c\left(\frac{1}{h}\int_0^h \|A_\delta(\cdot)-\chi_{\Omega^{(h)}}(\cdot+s)\|^2_{L^2([0,T];L^{2a'}(\Omega))}\,ds\right)^{1/2}.$$

By the uniform convergence of $\eta^{(h)} \to \eta$, we find that

$$\lim_{h\to 0}\left(\frac{1}{h}\int_0^h \|A_\delta(\cdot) - \chi_{\Omega^{(h)}}(\cdot+s)\|^2_{L^2\left([0,T];L^{2a'}(\Omega)\right)}\,ds\right)^{1/2} = \|A_\delta - \chi_\Omega\|_{L^2\left([0,T];L^{2a'}(\Omega)\right)}.$$

Finally, choosing $A_\delta \in C([0,T];C_0^{k_0}(\Omega_{-\delta}))$ with $A_\delta \to \nabla\xi\chi_{\Omega(t)}$ and gives, collecting all above, that

$$\int_0^T \left\langle \frac{u^{(h)}(t)\circ\Phi^{(h)}(t-h) - u^{(h)}(t-h)}{h}, \xi(t)\circ\Phi^{(h)}(t-h)\right\rangle_{\Omega^{(h)}(t-h)} dt$$
$$\to \int_0^T \langle (u(t))_\delta, \partial_t\xi(t) + \nabla\xi(t)\cdot u(t)\rangle\, dt$$

to obtain the δ-regularized equations.

In fact we can pass to the limit $\delta \to 0$ and obtain the limiting equation. Finally, using $(\partial_t\eta, v)$ as a test function (we still have enough regularity for that due to the regularizing terms) we get, as in (4.11) the energy inequality. □

4.4. **Estimate of the flow map and No contact.** We will now see that since we keep at this point the regularizing terms with $\kappa > 0$, that this in fact means that the flow map remains Lipschitz regular. Indeed, by the estimate of Proposition 4.4 we have that

$$\operatorname{Lip}\Phi(t) \le \exp(\sqrt{T}\sqrt{\|\nabla^{k_0} v\|_{L^2((0,T);L^2(\Omega(t);\mathbb{R}^d))}})$$

and the norm on the right is by (4.27) finite (although depending on κ). Then we can argue that $\Phi(t)\colon \Omega_0 \to \Omega(t)$ is a diffeomorphism. This in particular means that there is *no change of topology* and consequently no contact between any solid parts.

Corollary 4.12 (No contact with regularization)**.** *The solution obtained in Theorem 4.11 does not reach a collision.*

4.5. **Passing to the limit with the regularization** $\kappa \to 0$**.** We now reveal the dependence of v, η in Theorem 4.11 on κ, so we write $v^{(\kappa)}, \eta^{(\kappa)}$. Recall that so far we have shown $v^{(\kappa)}, \eta^{(\kappa)}$ to be a solution of the equation with $W^{k_0,2}$ (resp. $W^{k_0+2,2}$)-regularizer depending on $\kappa > 0$.

Note that by Corollary 4.12 we know that the deformation $\eta^{(\kappa)}$ never reaches a collision, as long as $\kappa > 0$. However this is no longer guaranteed after $\kappa \to 0$. Thus below in Theorem 4.13, to get a limiting weak equation, we take the absence of collisions in the limit η as an assumption (which is true at least for short times, see [BKS23b]). We aim to incorporate the possibility of collisions and description of the corresponding Lagrange multiplier in a future work.

Until now we have had a regularized initial conditions, so we need to approximate the initial conditions now. That is, for given initial conditions

$$\eta_0 \in \mathcal{E}, \quad \eta_* \in W^{1,2}(Q;\mathbb{R}^d), \quad v_0 \in W^{1,2}(\Omega_0;\mathbb{R}^d)$$

we approximate it by

$$\eta_0^{(\kappa)} \in \mathcal{E}\cap W^{k_0+2,2}(Q;\mathbb{R}^d), \quad \eta_*^{(\kappa)} \in W^{k_0+2,2}(Q;\mathbb{R}^d), \quad v_0^{(\kappa)} \in W^{k_0,2}(\Omega_0;\mathbb{R}^d)$$

such that as $\kappa \to 0$ we have

$$\begin{aligned} \eta_0^{(\kappa)} &\to \eta_0 \quad \text{in } W^{2,q}(Q;\mathbb{R}^d),\\ \eta_*^{(\kappa)} &\to \eta_* \quad \text{in } W^{1,2}(Q;\mathbb{R}^d),\\ v_0^{(\kappa)} &\to v_0 \quad \text{in } W^{1,2}(\Omega_0;\mathbb{R}^d) \end{aligned}$$

and below the $v^{(\kappa)}$, $\eta^{(\kappa)}$ solution will be corresponding to these initial conditions.

Theorem 4.13 (Full problem)**.** *There exists a subsequence $\kappa \to 0$ such that the limit is a weak solution to the full problem as defined in Definition 3.1, until the time of the first collision.*

Proof. Fluid-only equation.

The fluid-only equation (4.25) now reads as

$$\int_0^T -\rho_f\langle v^{(\kappa)},\partial_t\xi\rangle_{\Omega^{(\kappa)}(t)} + \rho_f\langle v^{(\kappa)}, v^{(\kappa)}\cdot\nabla\xi\rangle_{\Omega^{(\kappa)}(t)} + \nu\langle\varepsilon v^{(\kappa)},\varepsilon\xi\rangle_{\Omega^{(\kappa)}(t)} + \kappa\langle\nabla^{k_0}v^{(\kappa)},\nabla^{k_0}\xi\rangle\,dt$$
$$= \int_0^T \rho_f\langle f,\xi\rangle_{\Omega^{(\kappa)}(t)}\,dt + \rho_f\langle v_0^{(\kappa)},\xi(0)\rangle_{\Omega^{(\kappa)}(0)} \tag{4.28}$$

for all $\xi\in C^\infty([0,T]\times\overline{\Omega}^{(\kappa)}(t))$, $\xi\cdot n^{(\kappa)}=0$ on $\partial\Omega^{(\kappa)}(t)$, $\operatorname{div}\xi=0$ in $\Omega^{(\kappa)}(t)$, with $\xi(T)=0$.

We further have the uniform bounds on $v^{(\kappa)}$ in $L^2((0,T);W^{1,2}(\Omega^{(\kappa)}(t);\mathbb{R}^d))$ and a (κ-independent) bound

$$\sqrt{\kappa}\|\nabla^{k_0}v^{(\kappa)}\|_{L^2((0,T);L^2(\Omega^{(\kappa)}(t);\mathbb{R}^d))}\le C.$$

Thus for a subsequence $\kappa\to 0$ we have

$$v^{(\kappa)}\overset{\eta}{\rightharpoonup} v \quad\text{in } L^2((0,T);W^{1,2}(\Omega(t);\mathbb{R}^d)).$$

Moreover, we can read the estimate on $\partial_t v^{(\kappa)}$ in distributional sense, as from (4.28) we have

$$\left|\int_0^T\langle v,\partial_t\xi\rangle\,dt\right|\le C\|\xi\|_{L^\infty(C^{k_0})}.$$

Thus from the Aubin-Lions lemma [BKS24, Corollary 2.9] we get the strong convergence

$$v^{(\kappa)}\overset{\eta}{\to} v \quad\text{in } L^2((0,T);L^2(\Omega(t);\mathbb{R}^d)). \tag{4.29}$$

We desire to show that the limiting equation holds, that is

$$\int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)} + = \int_0^T \rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt + \rho_f\langle v_0,\xi(0)\rangle_{\Omega(0)}$$

for all $\xi\in C^\infty([0,T]\times\overline{\Omega}(t);\mathbb{R}^d)$, $\xi\cdot n=0$ on $\partial\Omega(t)$, $\operatorname{div}\xi=0$ in $\Omega(t)$, with $\xi(T)=0$.

So for this let us now fix a test function for the limit, that is $\xi\in C^\infty([0,T]\times\overline{\Omega}(t));\mathbb{R}^d$, $\xi\cdot n=0$ on $\partial\Omega(t)$, $\operatorname{div}\xi=0$ in $\Omega(t)$, with $\xi(T)=0$.

For this ξ find $\xi_\varepsilon^{(\kappa)}$ as defined in Proposition 2.9 (i), this $\xi_\varepsilon^{(\kappa)}$ is now a valid test function for (4.28). We thus have

$$\int_0^T -\rho_f\langle v^{(\kappa)},\partial_t\xi_\varepsilon^{(\kappa)}\rangle_{\Omega^{(\kappa)}(t)} + \rho_f\langle v^{(\kappa)}, v^{(\kappa)}\cdot\nabla\xi_\varepsilon^{(\kappa)}\rangle_{\Omega^{(\kappa)}(t)} + \nu\langle\varepsilon v^{(\kappa)},\varepsilon\xi_\varepsilon^{(\kappa)}\rangle_{\Omega^{(\kappa)}(t)}$$
$$+\kappa\langle\nabla^{k_0}v^{(\kappa)},\nabla^{k_0}\xi_\varepsilon^{(\kappa)}\rangle\,dt = \int_0^T\rho_f\langle f,\xi_\varepsilon^{(\kappa)}\rangle_{\Omega^{(\kappa)}(t)}\,dt + \rho_f\langle v_0^{(\kappa)},\xi_\varepsilon^{(\kappa)}(0)\rangle_{\Omega^{(\kappa)}(0)}$$

Now we use the weak convergence of $v^{(\kappa)}\rightharpoonup v$ strong convergence $\xi_\varepsilon^{(\kappa)}\overset{\eta}{\to}\xi_\varepsilon$ and the estimate (which follows from (4.28))

$$\int_0^T\kappa|\langle\nabla^{k_0}v^{(\kappa)},\nabla^{k_0}\xi_\varepsilon^{(\kappa)}\rangle_{\Omega^{(\kappa)}(t)}|\,dt\le\kappa\|\nabla^{k_0}v^{(\kappa)}\|_{L^2((0,T);L^2(\Omega^{(\kappa)}(t);\mathbb{R}^d)}\|\nabla^{k_0}\xi_\varepsilon^{(\kappa)}\|_{L^2((0,T);L^2(\Omega^{(\kappa)}(t);\mathbb{R}^d)}$$
$$\le\sqrt{\kappa}CC_\varepsilon\to 0$$

with $\kappa\to 0$ and $\varepsilon>0$ fixed. So that after passing to $\kappa\to 0$ we have for all $\varepsilon>0$ (passing to the limit in $\int_0^T\langle v, v\cdot\nabla\xi_\varepsilon\rangle\,dt$ is due to the strong convergence (4.29))

$$\int_0^T -\rho_f\langle v,\partial_t\xi_\varepsilon\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi_\varepsilon\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi_\varepsilon\rangle_{\Omega(t)}\,dt = \int_0^T\rho_f\langle f,\xi_\varepsilon\rangle_{\Omega(t)}\,dt + \rho_f\langle v_0,\xi_\varepsilon(0)\rangle_{\Omega(0)}$$

Since then we can as before pass to $\varepsilon\to 0$ and see that we have the desired limiting equation

$$\int_0^T -\rho_f\langle v,\partial_t\xi\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} + \nu\langle\varepsilon v,\varepsilon\xi\rangle_{\Omega(t)}\,dt = \int_0^T\rho_f\langle f,\xi\rangle_{\Omega(t)}\,dt + \rho_f\langle v_0,\xi(0)\rangle_{\Omega(0)}.$$

By a density argument we can see that this continues to hold for $\xi \in W^{1,2}((0,T); L^2(\Omega(t);\mathbb{R}^d)) \cap L^2((0,T); W^{1,2}_n(\Omega(t);\mathbb{R}^d))$.

Coupled equation.

The coupled equation (4.26) is now

$$\int_0^T DE(\eta^{(\kappa)})\langle\phi^{(\kappa)}\rangle + D_2R\left(\eta^{(\kappa)}, \partial_t\eta^{(\kappa)}\right)\langle\phi^{(\kappa)}\rangle + 2\kappa^{a_0}\langle\nabla^{k_0+2}\eta^{(\kappa)}, \nabla^{k_0+2}\phi^{(\kappa)}\rangle +$$
$$2\kappa\left\langle\nabla^{k_0+2}\partial_t\eta^{(\kappa)}, \nabla^{k_0+2}\phi^{(\kappa)}\right\rangle - \rho_s\left\langle\partial_t\eta^{(\kappa)}, \partial_t\phi^{(\kappa)}\right\rangle - \rho_s\left\langle f\circ\eta^{(\kappa)}, \phi^{(\kappa)}\right\rangle$$
$$+\nu\langle\varepsilon v^{(\kappa)}, \varepsilon\xi\rangle_{\Omega^{(\kappa)}(t)} + \kappa\langle\nabla^{k_0}v^{(\kappa)}, \nabla^{k_0}\xi\rangle_{\Omega^{(\kappa)}(t)} - \rho_f\langle\partial_t v^{(\kappa)}, \xi\rangle_{\Omega(t)} + \rho_f\langle v^{(\kappa)}, v^{(\kappa)}\cdot\nabla\xi\rangle_{\Omega(t)}$$
$$-\rho_f\langle f, \xi\rangle_{\Omega^{(\kappa)}(t)}\,dt = 0$$

for all $\xi \in C^\infty([0,T]\times\overline{\Omega}^{(\kappa)}(t);\mathbb{R}^d)$, $\operatorname{div}\xi = 0$ in $\Omega^{(\kappa)}(t)$ and $\phi^{(\kappa)} \in W^{k_0+2,2}(Q;\mathbb{R}^d)$ with $\phi^{(\kappa)} = \xi\circ\eta^{(\kappa)}$ on Q.

We shall now pass to the limit $\kappa \to 0$. We have estimates by the energy inequality (4.27) which now read as

$$E(\eta^{(\kappa)}(t)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta^{(\kappa)}(t)\|^2 + \frac{\rho_f}{2}\|v^{(\kappa)}(t)\|^2_{\Omega(t)} + \frac{\rho_s}{2}\|\partial_t\eta^{(\kappa)}(t)\|^2$$
$$+\int_0^t 2R(\eta^{(\kappa)}, \partial_t\eta^{(\kappa)}) + 2\kappa\|\nabla^{k_0+2}\partial_t\eta^{(\kappa)}\|^2 + \|\nabla^{k_0}v^{(\kappa)}\|^2\,dt$$
$$\leq E(\eta^{(\kappa)}(0)) + 2\kappa^{a_0}\|\nabla^{k_0+2}\eta^{(\kappa)}(0)\|^2 + \frac{\rho_f}{2}\|v^{(\kappa)}(0)\|^2_{\Omega_0} + \frac{\rho_s}{2}\|\partial_t\eta^{(\kappa)}(0)\|^2\,dt$$
$$+\int_0^t \rho_s\left\langle f\circ\eta^{(\kappa)}, \partial_t\eta^{(\kappa)}\right\rangle + \rho_f\langle f, v^{(\kappa)}\rangle_{\Omega(t)}\,dt$$

to obtain weak convergences

$$\begin{aligned}
\eta^{(\kappa)} &\overset{*}{\rightharpoonup} \eta && \text{in } L^\infty((0,T); W^{2,q}(Q;\mathbb{R}^d)),\\
\partial_t\eta^{(\kappa)} &\rightharpoonup \partial_t\eta && \text{in } L^2((0,T)W^{1,2}(Q;\mathbb{R}^d)),\\
v^{(\kappa)} &\overset{\eta}{\rightharpoonup} v && \text{in } L^2((0,T); W^{1,2}(\Omega(t);\mathbb{R}^d)).
\end{aligned}$$

Moreover as in [BKS23b, Lemma 3.9], which also explains the choice of a_0, one can argue by the Minty property (E.6) that

$$\eta^{(\kappa)}(t) \to \eta(t) \quad \text{in } W^{2,q}(Q;\mathbb{R}^d) \text{ for a.a. } t\in(0,T).$$

and as above we have

$$v^{(\kappa)} \overset{\eta}{\to} v \quad \text{in } L^2((0,T); L^2(\Omega(t);\mathbb{R}^d)).$$

Passing now to the limit in the coupled equation, we obtain that

$$\int_0^T \kappa^{a_0}|\langle\nabla^{k_0+2}\eta^{(\kappa)}, \nabla^{k_0+2}\phi^{(\kappa)}\rangle|\,dt \leq \kappa^{a_0}\|\nabla^{k_0+2}\eta^{(\kappa)}\|_{L^\infty L^2}\|\nabla^{k_0+2}\phi^{(\kappa)}\|_{L^1L^2} \leq \kappa^{a_0}C$$
$$\int_0^T \kappa\left\langle\nabla^{k_0+2}\partial_t\eta^{(\kappa)}, \nabla^{k_0+2}\phi^{(\kappa)}\right\rangle\,dt \leq \kappa\|\nabla^{k_0+2}\partial_t\eta^{(\kappa)}\|_{L^2L^2}\|\nabla^{k_0+2}\phi^{(\kappa)}\|_{L^2L^2} \leq \kappa C$$

so that these regularizing terms vanish as $\kappa \to 0$.

We have thus shown enough to pass to the limit $\kappa \to 0$ and solve the limit problem

$$\int_0^T DE(\eta)\langle\phi\rangle + D_2R(\eta, \partial_t\eta)\langle\phi\rangle - \rho_s\langle\partial_t\eta, \partial_t\phi\rangle - \rho_s\langle f\circ\eta, \phi\rangle$$
$$+\nu\langle\varepsilon v, \varepsilon\xi\rangle_{\Omega(t)} - \rho_f\left\langle\partial_t v, \xi\right\rangle_{\Omega(t)} + \rho_f\langle v, v\cdot\nabla\xi\rangle_{\Omega(t)} - \rho_f\langle f, \xi\rangle_{\Omega(t)}\,dt = 0.$$

□

Acknowledgements. The authors acknowledge the support of the ERC-CZ grant LL2105. S. S. further acknowledges the support of the VR-Grant 2022-03862 by the Swedish Research Council. A. Č. and M.K further acknowledge the support of the Czech Science Foundation (GAČR) grant No. 23-04766S and Charles University Research Centre program No. UNCE/24/SCI/005.

References

[Bal02] John M. Ball. Some Open Problems in Elasticity. In Paul Newton, Philip Holmes, and Alan Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 3–59. Springer-Verlag, New York, 2002.

[BKS23a] B. Benešová, M. Kampschulte, and S. Schwarzacher. Variational methods for fluid–structure interaction and porous media. *Nonlinear Analysis: Real World Applications*, 71:103819, June 2023.

[BKS23b] Barbora Benešová, Malte Kampschulte, and Sebastian Schwarzacher. A variational approach to hyperbolic evolutions and fluid-structure interactions. *Journal of the European Mathematical Society*, June 2023.

[BKS24] Dominic Breit, Malte Kampschulte, and Sebastian Schwarzacher. Compressible fluids interacting with 3D visco-elastic bulk solids. *Mathematische Annalen*, May 2024.

[BS18] Dominic Breit and Sebastian Schwarzacher. Compressible Fluids Interacting with a Linear-Elastic Shell. *Archive for Rational Mechanics and Analysis*, 228(2):495–562, May 2018.

[BS23] Dominic Breit and Sebastian Schwarzacher. Navier-Stokes-Fourier fluids interacting with elastic shells. *ANNALI SCUOLA NORMALE SUPERIORE - CLASSE DI SCIENZE*, pages 619–690, June 2023.

[ČGK24] Antonín Češík, Giovanni Gravina, and Malte Kampschulte. Inertial evolution of non-linear viscoelastic solids in the face of (self-)collision. *Calculus of Variations and Partial Differential Equations*, 63(2):55, February 2024.

[ČS23] Antonín Češík and Sebastian Schwarzacher. Stability and convergence of in time approximations of hyperbolic elastodynamics via stepwise minimization, June 2023.

[GHH06] Matthias Geißert, Horst Heck, and Matthias Hieber. On the Equation div u = g and Bogovskii's Operator in Sobolev Spaces of Negative Order. In Erik Koelink, Jan Van Neerven, Ben De Pagter, Guido Sweers, Annemarie Luger, and Harald Woracek, editors, *Partial Differential Equations and Functional Analysis*, pages 113–121. Birkhäuser Basel, Basel, 2006.

[HK09] Timothy J. Healey and Stefan Krömer. Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(4):863–871, October 2009.

[KMT24] Malte Kampschulte, Boris Muha, and Srdan Trifunović. Global weak solutions to a 3D/3D fluid-structure interaction problem including possible contacts. *Journal of Differential Equations*, 385:280–324, March 2024.

[KR19] Martin Kružík and Tomáš Roubíček. *Mathematical Methods in Continuum Mechanics of Solids.* Interaction of Mechanics and Mathematics. Springer International Publishing, Cham, 2019.

[KSS23] Malte Kampschulte, Sebastian Schwarzacher, and Gianmarco Sperone. Unrestricted deformations of thin elastic structures interacting with fluids. *Journal de Mathématiques Pures et Appliquées*, 173:96–148, May 2023.

[LMN24] Yadong Liu, Sourav Mitra, and Šárka Nečasová. On a compressible fluid-structure interaction problem with slip boundary conditions, May 2024.

[LR14] Daniel Lengeler and Michael Růžička. Weak Solutions for an Incompressible Newtonian Fluid Interacting with a Koiter Type Shell. *Archive for Rational Mechanics and Analysis*, 211(1):205–255, January 2014.

[MC13] Boris Muha and Suncica Canić. Existence of a Weak Solution to a Nonlinear Fluid–Structure Interaction Problem Modeling the Flow of an Incompressible, Viscous Fluid in a Cylinder with Deformable Walls. *Archive for Rational Mechanics and Analysis*, 207(3):919–968, March 2013.

[MČ15] Boris Muha and Sunčica Čanić. Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy. *Interfaces and Free Boundaries*, 17(4):465–495, 2015.

[MMN+22] Václav Mácha, Boris Muha, Šárka Nečasová, Arnab Roy, and Srđan Trifunović. Existence of a weak solution to a nonlinear fluid-structure interaction problem with heat exchange. *Communications in Partial Differential Equations*, 47(8):1591–1635, August 2022.

[MS22] Boris Muha and Sebastian Schwarzacher. Existence and regularity of weak solutions for a fluid interacting with a non-linear shell in three dimensions. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 39(6):1369–1412, June 2022.

[SS22] Sebastian Schwarzacher and Matthias Sroczinski. Weak-Strong Uniqueness for an Elastic Plate Interacting with the Navier–Stokes Equation. *SIAM Journal on Mathematical Analysis*, 54(4):4104–4138, August 2022.

Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675, Prague, Czech Republic
Email address: cesik@karlin.mff.cuni.cz

Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675, Prague, Czech Republic
Email address: kampschulte@karlin.mff.cuni.cz

Department of Mathematics, Uppsala University, Box 480 751 06 Uppsala & Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675, Prague, Czech Republic
Email address: schwarz@karlin.mff.cuni.cz