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Antonín Češík

Variational strategies in material sciences: Analysis & Numerics

Department of Mathematical Analysis

Supervisor of the doctoral thesis: doc. Sebastian Schwarzacher, Dr.
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In Prague, 31 July 2024

Antonín Češík

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Author: Antonín Češík

Department: Department of Mathematical Analysis

Supervisor: doc. Sebastian Schwarzacher, Dr., Department of Mathematical Analysis

Abstract: In this thesis, we present an application of the hyperbolic minimizing movements method to dynamical problems in continuum mechanics. In the presented papers, we treat largely deforming viscoelastic solids with collisions, fluid-structure interactions with the slip condition, as well as a full time-discretization of this approach.

First, we obtain the existence result for nonlinear viscoelastic solids in the large deformation regime with arbitrary collisions. For this we construct a physically correct measure-valued contact force. This result is extended also for solids with only Lipschitz regular boundaries.

Next, we investigate a version of the hyperbolic minimizing movements scheme which is fully discrete in time. For this we show stability results and a linear convergence rate. This result is presented in the context of nonlinear elastodynamics.

Finally, we show the existence of weak solutions for nonlinear viscoelastic bulk solid coupled to a Navier-Stokes equation with a full slip condition at the fluid-solid interface. We provide the necessary classes of test functions for the weak solution, and we also show its consistency with the corresponding strong formulation.

Keywords: Calculus of Variations, Solid Mechanics, Non-linear Elasticity, Fluid-Structure Interactions, Navier-Stokes Equations, Time Discretization

Contents

1. Introduction

In mechanics of continuous media, there are two prominent types of evolutionary problems, *quasistatic* and *dynamic*.[1]

The quasistatic problems are typically concerned with evolutions which are evolving slowly, such that *inertia* does not play a role (or can be reasonably neglected). Mathematically, these correspond to a *parabolic* partial differential equation, the prototypical example here is the heat equation

$$\partial_t u - \Delta u = f.$$

The dynamic problems take into account inertia in the form of Newton's second law of motion. In contrast to the quasistatic problems, it is meaningful here to talk about the *kinetic energy*. Mathematically, these correspond to a *hyperbolic*[2] partial differential equation. The prototypical example in this case is the wave equation

$$\partial_{tt} u - \Delta u = f$$

which has the corresponding natural energy equality

$$\underbrace{\frac{1}{2}\|\partial_t u(t)\|_{L^2}^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|\nabla u(t)\|_{L^2}^2}_{\text{potential energy}} = \underbrace{\frac{1}{2}\|\partial_t u(0)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(0)\|_{L^2}^2}_{\text{initial energy}} + \underbrace{\int_0^t \langle f(s), u(s)\rangle_{L^2}\, ds}_{\text{work done by external force}} .$$

In some cases the borderline between parabolic and hyperbolic problems is not clear cut. In particular, (as will be the case for us), this is the case when we have kinetic energy as well as dissipation effects. As a prototype one can think of the wave equation with damping, namely

$$\partial_{tt} u - \Delta u - \varepsilon \partial_t \Delta u = f.$$

Here we have a dissipation of energy,[3] the natural energy equality takes the form

$$\begin{aligned}\underbrace{\frac{1}{2}\|\partial_t u(t)\|_{L^2}^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\|\nabla u(t)\|_{L^2}^2}_{\text{potential energy}} + \underbrace{\varepsilon \int_0^t \|\partial_t \nabla u(s)\|_{L^2}^2\, ds}_{\text{dissipated energy}} \\ = \underbrace{\frac{1}{2}\|\partial_t u(0)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(0)\|_{L^2}^2}_{\text{initial energy}} + \underbrace{\int_0^t \langle f(s), u(s)\rangle_{L^2}\, ds}_{\text{work done by external force}} .\end{aligned}$$

We lean towards also calling this problem *hyperbolic*, because the presence of kinetic energy is the more important feature for us.

In this thesis, we investigate methods of *calculus of variations* [Dac08, Rin18] applied to dynamical problems of solid mechanics [KR19, Ngu00, Dog00], and fluid-structure interaction [BKS23, MC13, MČ15, MS22, BS18, BKS24, BS23].

[1] Note that some authors use the word "dynamic" for any time-dependent evolution. We shall reserve this word for the case where *inertial forces* are present, as described below.

[2] In the presence of dissipation, the equation is not hyperbolic in a strict sense, see below.

[3] In a full thermodynamical system dissipated energy is converted to heat. However in this thesis, we will not consider any thermal effects.

In contrast to more PDE-based methods such as Galerkin approximation [Rou13, Tem77] or the Rothe method [Kač86, Neu09], the variational time-stepping minimization approach naturally possesses the capacity to treat non-convex state spaces without additional difficulty. We shall now discuss the method, called *minimizing movements* [DG93] and its extension to hyperbolic problems [BKS23].

1.1 Minimizing movements

In the 90s, De Giorgi introduced his famous *minimizing movements* method [DG93]. It is a versatile method for solving time evolution problems in calculus of variations, differential equations and geometric measure theory, and in particular for parabolic equations. The unifying property of these problems is that they are driven by the energy contained in the system. A prominent example of such systems are *gradient flows*. These are evolutionary problems of the form

$$\partial_t x = -DF(x)$$

with the unknown $x\colon (0,T) \to X$ and given $F\colon X \to \mathbb{R}$. Examples include ordinary differential equations (when $X = \mathbb{R}^n$), or gradient flows in a Hilbert space X [Bra14, Chapter 7], with F sufficiently smooth so that "DF" can be understood as the Fréchet derivative. We will present this below, in a slightly more general setting.

1.1.1 Gradient flow with forcing, in a Banach space

We present here the minimizing movements method in the case of a separable reflexive Banach space X, which is densely embedded into a separable Hilbert space H. In particular

$$X \hookrightarrow H \simeq H^* \hookrightarrow X^* \tag{1.1}$$

form a Gelfand triple, i.e. both embeddings are dense and $H \simeq H^*$ denotes the identification by Riesz representation theorem. Let $F\colon X \to \mathbb{R}$ be

- bounded from below,[4]
- coercive in the sense that for some $\gamma > 0$ the functional $F(\cdot) + \gamma\|\cdot\|_H^2$ has bounded sublevel sets in X,
- weakly lower semicontinuous,
- (Fréchet) differentiable and let $DF\colon X \to X^*$ be bounded on bounded sets,

and further, let DF satisfy some kind of "monotonicity"-like property that we will specify later.

Let $x_0 \in X$ and $f \in L^2((0,T);H)$. We want to find a weak solution of the *forced* gradient flow[5]

$$\begin{aligned} \partial_t x + DF(x) &= f \\ x(0) &= x_0 \end{aligned} \tag{1.2}$$

[4]This bound is strictly speaking not necessary, it could be for instance replaced by $F(x) \geq -C(1 + \|x\|_H^2)$.

[5]Classically in the setting of gradient flows, a time-dependent term f is not treated.

by which we mean that $x \in C_w([0,T];X) \cap W^{1,2}((0,T);H)$ satisfies $x(0) = x_0$ and

$$\int_0^T \langle \partial_t x, y\rangle_H + \langle DF(x), y\rangle \, dt = \int_0^T \langle f, y\rangle_H \, dt, \quad y \in L^1((0,T);X) \cap L^2((0,T);H).$$

For this, we discretize the time at scale $\tau > 0$. So for[6] $k = 1, \dots, T/\tau$ we find $x_k^{(\tau)} \in X$ as a minimizer of

$$\mathcal{J}_k^{(\tau)}(x) = \frac{1}{2\tau} \left\| x - x_{k-1}^{(\tau)} \right\|_H^2 + F(x) - \langle f_k^{(\tau)}, x\rangle_H, \quad x \in X \tag{1.3}$$

with the initial condition $x_0^{(\tau)} := x_0$ and $f_k^{(\tau)} \in H$ the discretized version of f, defined by

$$f_k^{(\tau)} := \int_{(k-1)\tau}^{\tau k} f \, dt, \quad f^{(\tau)}(t) := f_k^{(\tau)} \quad \text{for } t \in ((k-1)\tau, k\tau].$$

The minimizer exists by the direct method of the calculus of variations [Dac08], due to our assumptions on F. Further it satisfies the Euler-Lagrange equation $D\mathcal{J}_k^{(\tau)}(x_k^{(\tau)}) = 0$, that is

$$\frac{x_k^{(\tau)} - x_{k-1}^{(\tau)}}{\tau} + DF(x_k^{(\tau)}) = f_k^{(\tau)}$$

in X^*. This in fact means that (recall that by (1.1) we have the embedding $X \hookrightarrow X^*$ through the scalar product on H)

$$\left\langle \frac{x_k^{(\tau)} - x_{k-1}^{(\tau)}}{\tau}, y \right\rangle_H + \langle DF(x_k^{(\tau)}), y\rangle = \langle f_k^{(\tau)}, y\rangle_H, \quad y \in X.$$

We denote by $x^{(\tau)} \colon (0,T) \to X$ the piecewise constant interpolation

$$x^{(\tau)}(t) := x_k^{(\tau)}, \quad t \in ((k-1)\tau, k\tau].$$

The aim is to pass with $\tau \to 0$ and to obtain the solution x as a limit of $x^{(\tau)}$.

The sufficient energy estimate is directly comparing in the minimization (1.3) with the previous step, namely $\mathcal{J}_k^{(\tau)}(x_k^{(\tau)}) \le \mathcal{J}_k^{(\tau)}(x_{k-1}^{(\tau)})$. This reads as

$$\frac{1}{2\tau} \left\| x_k^{(\tau)} - x_{k-1}^{(\tau)} \right\|_H^2 + F(x_k^{(\tau)}) \le F(x_{k-1}^{(\tau)}) + \langle f_k^{(\tau)}, x_k^{(\tau)} - x_{k-1}^{(\tau)}\rangle_H. \tag{1.4}$$

Using the Young inequality

$$\langle f_k^{(\tau)}, x_k^{(\tau)} - x_{k-1}^{(\tau)}\rangle_H \le \tau \|f_k^{(\tau)}\|_H^2 + \frac{1}{4\tau} \left\| x_k^{(\tau)} - x_{k-1}^{(\tau)} \right\|_H^2$$

and absorbing the last term, we see that

$$\frac{1}{4\tau} \left\| x_k^{(\tau)} - x_{k-1}^{(\tau)} \right\|_H^2 + F(x_k^{(\tau)}) \le F(x_{k-1}^{(\tau)}) + \tau \|f_k^{(\tau)}\|_H^2.$$

[6]For notational simplicity assume T is an integer multiple of τ.

Further, the estimate sums to

$$\frac{1}{4\tau}\sum_{i=1}^{k}\left\|x_i^{(\tau)}-x_{i-1}^{(\tau)}\right\|_H^2+F(x_k^{(\tau)})\le F(x_0)+\sum_{i=1}^{k}\tau\|f_i^{(\tau)}\|_H^2.$$

We further estimate

$$\begin{aligned}\frac{1}{4k\tau}\left\|x_k^{(\tau)}-x_0\right\|_H^2&\le\frac{1}{4k\tau}\left(\sum_{i=1}^{k}\left\|x_i^{(\tau)}-x_{i-1}^{(\tau)}\right\|_H\right)^2\\ \le\frac{1}{4\tau}\sum_{i=1}^{k}\left\|x_i^{(\tau)}-x_{i-1}^{(\tau)}\right\|_H^2&\le F(x_0)+\|f\|_{L^2((0,T);H)}^2\end{aligned}\tag{1.5}$$

where we have used the Jensen inequality $\sum_{i=1}^k\tau\|f_i^{(\tau)}\|_H^2\le\|f\|_{L^2((0,T);H)}^2$ so consequently

$$F(x^{(\tau)})+\frac{1}{4T}\|x^{(\tau)}\|_H^2\le F(x_0)+\|f\|_{L^2((0,T);H)}^2.$$

By coercivity of $F+\frac{1}{4T}\|\cdot\|_H^2$, we have $\{x^{(\tau)}\}_\tau$ is bounded in $L^\infty((0,T);X)$. So that (up to a subsequence of $\tau\to0$) we have for some $x\in L^\infty((0,T);X)$ that

$$x^{(\tau)}\overset{*}{\rightharpoonup}x\quad\text{in }L^\infty((0,T);X).\tag{1.6}$$

Now we estimate the time derivative. For this, introduce the piecewise affine interpolation

$$\hat{x}^{(\tau)}(t)=\frac{t-(k-1)\tau}{\tau}x_k+\frac{k\tau-t}{\tau}x_{k-1},\quad t\in((k-1)\tau,k\tau]$$

so that it holds $\partial_t\hat{x}^{(\tau)}=\frac{x^{(\tau)}-x^{(\tau)}(\cdot-\tau)}{\tau}$ on $(0,T)$ except for finitely many times $k\tau$. Note that since

$$\|\hat{x}^{(\tau)}-x^{(\tau)}\|_{L^\infty((0,T);X)}\le\max_{K=1,\dots T/\tau}\|x_k^{(\tau)}-x_{k-1}^{(\tau)}\|_X\le2\tau^{1/2}\sqrt{F(x_0)+\|f\|_{L^2((0,T);H)}^2},$$

it holds also

$$\hat{x}^{(\tau)}\overset{*}{\rightharpoonup}x\quad\text{in }L^\infty((0,T);X).$$

Then we see from (1.5) that

$$\|\partial_t\hat{x}^{(\tau)}\|_{L^2((0,k\tau);H)}^2=\sum_{i=1}^{k}\tau\left\|\frac{x_i-x_{i-1}}{\tau}\right\|_H^2\le4\left(F(x_0)+\sum_{i=1}^{k}\tau\|f_i^{(\tau)}\|_H^2\right)$$

so that we have a weak limit (for a subsequence)

$$\partial_t\hat{x}^{(\tau)}\rightharpoonup v\quad\text{in }L^2((0,T);H)$$

for some $v\in L^2((0,T);H)$. It is not hard to see that $v=\partial_tx$, as for any $y\in C_c^1((0,T);H)$ it holds

$$\int_0^T\left\langle\partial_t\hat{x}^{(\tau)},y\right\rangle_H\,dt\to\int_0^T\langle v,y\rangle_Hdt$$

with the left hand side equal to

$$=-\int_0^T\left\langle\hat{x}^{(\tau)},\frac{y-y(\cdot-\tau)}{\tau}\right\rangle_H\,dt\to-\int_0^T\langle x,\partial_ty\rangle_H\,dt.$$

Thus we have that the weak limit satisfies $x \in L^\infty((0,T);X) \cap W^{1,2}((0,T);H)$. In particular notice that this implies the Hölder continuity $x \in C^{0,1/2}([0,T];H)$ and also the weak continuity $x \in C_w([0,T];X)$. Using the boundedness of DF we have by (1.6) (for another subsequence)

$$DF(x^{(\tau)}) \xrightharpoonup{*} A \quad \text{in } L^\infty((0,T);X^*)$$

We thus see that **if it can be further proved** that

$$A = DF(x), \tag{1.7}$$

then x is a solution to the equation (1.2). To show (1.7), one usually employs the Minty method [Min63], which uses some monotonicity-like property of DF such as pseudomonotonicity [Rou13].

Example (parabolic p-Laplace equation). To show a concrete example, we demonstrate this on the parabolic p-Laplace equation with Neumann boundary. Let $X = W^{1,p}(\Omega)$, $H = L^2(\Omega)$, $u_0 \in W^{1,p}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain,

$$F(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx, \quad u \in W^{1,p}(\Omega)$$

and consider also the right hand side $f \in L^2((0,T) \times \Omega)$. Since F is convex, then DF is monotone. Then the minimizing movements procedure provides a solution to the corresponding gradient flow

$$\begin{aligned} \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f &\quad \text{in } (0,T) \times \Omega, \\ \partial_n u = 0 &\quad \text{on } (0,T) \times \partial\Omega, \\ u(0) = u_0. & \end{aligned}$$

Notice that here the functional F itself is *not* coercive on the space $X = W^{1,p}(\Omega)$, as there is no Poincaré inequality in this space.

Remark. The main advantage of the scheme is that it allows to treat nonconvexities, in particular in the state space. That is, it can be used to solve the problem with a constraint $x(t) \in \mathcal{E}$ for some weakly closed set $\mathcal{E} \subset X$. Note that in the minimization (1.3), we can simply minimize over $\mathcal{E}$ instead of X. This will be vital for our applications in large strain elasticity, as we want to forbid (self-)interpenetration of matter. The energy also need not be convex, although there still should be some way of ensuring (1.7).

1.1.2 Hyperbolic minimizing movements

The minimizing movements method works very well for problems that can be formulated as a gradient flow, as demonstrated above. These include various kinds of parabolic PDE. However, for hyperbolic problems, the scheme is not directly applicable. Let us demonstrate the issue on the wave equation.

Consider the wave equation in a Lipschitz domain $\Omega \subset \mathbb{R}^n$. That is, given $u_0 \in W^{1,2}(\Omega)$, $u_* \in L^2(\Omega)$ and $f \in L^2((0,T) \times \Omega)$, the problem

$$\begin{aligned} \partial_{tt} u - \Delta u = f &\quad \text{in } \Omega, \\ \partial_n u = 0 &\quad \text{in } \partial\Omega, \\ u(0) = u_0, & \\ \partial_t u(0) = u_*. & \end{aligned} \tag{1.8}$$

Let us mimic (1.3), so that we discretize the time with some time step $\tau > 0$ and approximate the second time derivative with a double difference quotient. Thus we find $u_k^{(\tau)}$ as a minimizer of

$$\mathcal{J}_k^{(\tau)}(u) = \frac{\tau^2}{2}\left\|\frac{u - 2u_{k-1}^{(\tau)} + u_{k-2}^{(\tau)}}{\tau^2}\right\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 - \langle f_k, u\rangle_{L^2}, \quad u \in W^{1,2}(\Omega)$$

where for the initial conditions we put $u_0^{(\tau)} = u_0$ and we appropriately define $u_{-1}^{(\tau)} = u_0 - \tau u_*$, so that $u_* = \frac{u_0 - u_{-1}^{(\tau)}}{\tau}$. Again by the direct method, this minimizer exists.

Using as in (1.4) the estimate $\mathcal{J}_k^{(\tau)}(u_k^{(\tau)}) \leq \mathcal{J}_k^{(\tau)}(u_{k-1}^{(\tau)})$ reads as

$$\begin{aligned}\frac{\tau^2}{2}\left\|\frac{u_k^{(\tau)} - 2u_{k-1}^{(\tau)} + u_{k-2}^{(\tau)}}{\tau^2}\right\|_{L^2}^2 + \frac{1}{2}\|\nabla u_k^{(\tau)}\|_{L^2}^2 \leq \frac{\tau^2}{2}\left\|\frac{u_{k-1}^{(\tau)} - u_{k-2}^{(\tau)}}{\tau^2}\right\|_{L^2}^2 &+ \frac{1}{2}\|\nabla u_{k-1}^{(\tau)}\|_{L^2}^2 \\ &+ \langle f_k, u_k^{(\tau)} - u_{k-1}^{(\tau)}\rangle.\end{aligned}$$

The gradient term telescopes as before, but it is not clear how to get any estimate on the discrete time derivative $\frac{u_k^{(\tau)} - u_{k-1}^{(\tau)}}{\tau}$ independently of τ. This estimate is needed if we hope to pass to the limit $\tau \to 0$ and solve the problem (1.8).

Fortunately, Benešová, Kampschulte and Schwarzacher have figured out how to modify this method in such a way that the hyperbolic problem can be solved [BKS23, Section 3]. Their crucial idea is to discretize time in two different scales. Namely, the *velocity scale* τ and the (possibly much larger) *acceleration scale* h. Keeping the acceleration scale fixed, one can solve this as a parabolic problem on the velocity scale, and pass with $\tau \to 0$ with no problem. Then, one can utilize the resulting equation to obtain sufficient estimates for passing with $h \to 0$ and thereby solving the hyperbolic problem. This we show below.

Continuing our wave equation example, this means that we first solve the *time-delayed problem*

$$\begin{aligned}\frac{\partial_t u - \partial_t u(\cdot - h)}{h} - \Delta u &= f \quad \text{in } \Omega, \\ \partial_n u &= 0 \quad \text{on } \partial\Omega, \\ u(0) &= u_0\end{aligned} \tag{1.9}$$

on each interval of length h, that is subsequently on on $(0, h), (h, 2h), \ldots, (T - h, T)$. Then $\partial_t u(\cdot - h)$ is not a part of the solution and is already known (for the first interval we put $\partial_t u(t) := u_*$ for $t \in (-h, 0)$). It can thus be treated as a given forcing term. So we view this as a gradient flow with forcing $\tilde{f} = f + \partial_t u(\cdot - h)/h$, which can be readily solved as in (1.2). Explicitly, given $\tau > 0$ now the next step $u_k^{(\tau)}$ is found by minimizing

$$\frac{1}{2\tau h}\|u - u_{k-1}^{(\tau)}\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 - \langle f_k^{(\tau)}, u\rangle - \frac{1}{h}\langle w_k^{(\tau)}, u\rangle, \quad u \in W^{1,2}(\Omega)$$

where $w_k^{(\tau)} = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \partial_t u(t - h)\, dt$ for $k > 0$ and $w_0 = u_*$. By this procedure one obtains solution to the time-delayed equation (1.9) as before.

The crucial observation now is that the time-delayed equation gives us enough estimates to pass with $h \to 0$. Let us denote by $u^{(h)}$ the solution of the time-delayed problem (1.9). The idea is to use $\partial_t u^{(h)}$ as a test function, however we run into the well-known regularity issue for hyperbolic equations, since we have only $\partial_t u^{(h)} \in L^2((0,T)\times\Omega)$. If it were possible to test with $\partial_t u^{(h)}$, one would obtain the energy inequality

$$\frac{1}{2h}\int_{t-h}^{t} \|\partial_t u^{(h)}\|_{L^2}^2\,dt + \frac{1}{2}\|\nabla u^{(h)}(t)\|_{L^2}^2 \le \frac{1}{2}\|u_*\|_{L^2}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \int_0^t \langle f, \partial_t u^{(h)}\rangle\,dt \tag{1.10}$$

which contains sufficient bounds for the limit passage $h \to 0$, solving the original problem (1.8). The issue of insufficient regularity can be solved by introducing artificial dissipation term in the time-delayed equation (1.9) which vanishes in the limit, for details see [BKS23, Section 3] and also (1.14) below.

1.2 Nonlinear viscoelasticity

One of the main focuses of this thesis is the inertial evolution of viscoelastic solids. Here we briefly touch upon some standard notions from the mathematical theory of elastic solids. The interested reader is referred to [KR19] for a more thorough exposition.

The standard description of solids is in the so-called *Lagrangian* coordinates. That is, there is the *reference configuration* $Q \subset \mathbb{R}^n$, the *deformation* $\eta\colon Q \to \mathbb{R}^n$ which maps it to the *deformed configuration* $\eta(Q) \subset \mathbb{R}^n$. The notation that we use here may differ from other authors. We choose to adopt the notation widely used in the area of *fluid-structure interaction*, where it is standard to denote the solid deformation by η, see Section 1.3 below.

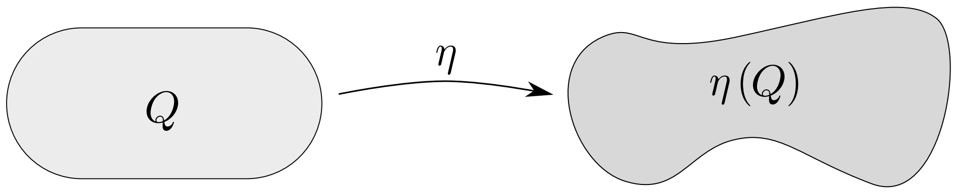

Figure 1.1: Elastic solid in Lagrangian coordinates.

A main feature such elastic solids is the existence of a *Piola-Kirchhoff stress tensor* representing the elastic response of the material.

One of the standard assumptions is that of *hyperelasticity* [Tou64]. The defining attribute of hyperelastic materials is the existence of a *stored energy*. In the most classical way, it means that the stress tensor is given by the derivative of the stored energy, which depends on $\nabla\eta$. There are other models, namely *non-simple materials*, where the stress tensor may depend also on $\nabla^2\eta$, or even higher gradients of the deformation η. For us, this will mainly play the mathematical role of providing higher regularity.

In evolution problems, it is also possible to talk about *viscosity*. As with the elastic stress, the viscous stress can be standardly assumed to arise from a dissipation potential. The classical case is the *Kelvin-Voigt viscoelastic material*, where the stress is the sum of the elastic and the viscous stress. In the setting of nonsimple materials, there is also the *hyperstress* coming from the second gradient. These are the materials that we will use due to their desirable mathematical properties.

Our work is focused on the variational point of view and not so much on the modelling and continuum mechanics. We thus prefer to write all our equations with operator notation. That is, for a deformation η the elastic stress is given by an elastic energy potential E and equal to $DE(\eta)$. Then $DE(\eta) \in X^*$ where X is the space of admissible deformations. Likewise, the viscosity will be arising from the dissipation potential R and the viscous stress for a deformation η is now given by $D_2R(\eta, \partial_t\eta)$,[7] where D_2 represents the derivative with respect to the second variable.

The typical equation is then of the form

$$\underbrace{\rho\partial_{tt}\eta}_{\text{change of momentum}} + \underbrace{DE(\eta)}_{\text{elastic stress}} + \underbrace{D_2R(\eta, \partial_t\eta)}_{\text{viscous stress}} = \underbrace{f}_{\text{external force}} \tag{1.11}$$

where ρ is the (Lagrangian) density.

1.2.1 Large deformations

We are in the setting of finite strain, so we allow for *large deformations*, and do not resort to any linearizations of the problem (1.11). For these reasons, the physically desirable injectivity of the deformation becomes a problem.

A standard physical requirement in elasticity is the positivity of the Jacobian of the deformation [Bal76, Cia88], namely $\det \nabla\eta > 0$ in Q. A closely related question to this is the existence of the Euler-Lagrange equation for the minimizer of an elastic energy. In classical nonlinear elasticity, it is a famous open problem to show the satisfaction of the Euler-Lagrange equation for the static minimizer [Bal02], along with the uniform positivity of the Jacobian. A way to circumvent this issue is the usage of nonsimple materials. In particular, [HK09] have shown that under suitable growth condition on the Jacobian, an upper bound on the energy gives a uniform positive lower bound on the Jacobian of the deformation

$$E(\eta) \leq E_0 \implies \det \nabla\eta(x) \geq \varepsilon_0 > 0, \quad x \in Q \tag{1.12}$$

where ε_0 depends on E_0. This then ensures locally the existence of the Euler-Lagrange equation for minimizers of E.

As the body is allowed to deform largely, it may happen that two far away parts of the body would overlap. As this interpenetration of matter is not physically reasonable, we impose the non-interpenetration as an extra condition. The by-now classical analytical way is the Ciarlet-Nečas condition [CN87], which asserts that

$$|\eta(Q)| = \int_Q \det \nabla\eta(x)dx. \tag{1.13}$$

This condition ensures that the deformation with positive Jacobian is injective, except possibly at the boundary.

[7] Here the dependence of R also on η is necessary to allow the physically desirable property of frame indifference, as was observed by [Ant98].

Altogether, the set of admissible deformations in our applications will be

$$\mathcal{E} = \left\{ \eta \in W^{2,q}\left(Q;\mathbb{R}^n\right) : \eta(Q) \subset \Omega, \det \nabla\eta > 0,\ |\eta(Q)| = \int_Q \det \nabla\eta(x) dx \right\},$$

where $q > n$ and $\Omega \subset \mathbb{R}^n$ is a given Eulerian container (including the possibility $\Omega = \mathbb{R}^n$), and the equation (1.11) is solved under the condition that $\eta(t) \in \mathcal{E}$.

Now we discuss how this this setting makes it well-suited for the hyperbolic minimizing movements scheme. The minimization may be performed only over $\mathcal{E}$, which is weakly closed in $W^{2,q}(Q;\mathbb{R}^n)$. By virtue of (1.12), the Jacobian of the deformation is positive, consequently the only way for the minimization to reach $\eta \in \partial\mathcal{E}$ is that there is a collision. If there is a collision, it corresponds to a Lagrange multiplier in the minimization (1.3). The method is also well-suited for studying such multipliers. The time-delayed equation then can be solved by minimizing movements. The passage to the limit in the time-delayed problem requires testing with $\partial_t\eta$, which necessitates usage of regularized energy and dissipation (otherwise there is not enough regularity for this testing). Those can be chosen of the form

$$E_h(\eta) = E(\eta) + h^{a_0} \left\|\nabla^{k_0}\eta\right\|_{L^2}^2, \quad R_h(\eta, b) = R(\eta, b) + h \left\|\nabla^{k_0} b\right\|_{L^2}^2, \tag{1.14}$$

where $W^{k_0,2}(Q;\mathbb{R}^n) \hookrightarrow W^{2,q}(Q;\mathbb{R}^n)$. The appropriately chosen regularizations vanish in the limit, for the full proof see [BKS23, Section 3] or [ČGK24].

1.3 Fluid-Structure interactions

In the final part, we shall deal with *fluid-structure interactions*. These are problems where an elastic structure is mechanically coupled to a fluid. The mutual interaction poses many mathematical challenges.

This area of research has been very active in the recent years. There have been results with fluids interacting with lower-dimensional structures such as plates or shells [BS18, SS22, MC13, MČ15, KSS23, LR14], see also [BGN14] for an overview with applications. More recent result handle also the case of bulk solids (i.e. of full dimension), results here include [BKS23, BKS24, KMT24]. We shall treat the latter case of a bulk solid immersed in a fluid and describe the setting here.

The solid deformation is described at each time t by a Lagrangian map $\eta(t)\colon Q \to \Omega$. The fluid occupies the rest of the fixed container $\Omega \subset \mathbb{R}^n$, so that the fluid domain is $\Omega(t) = \Omega \setminus \eta(t, Q)$. The fluid is determined by its velocity $v(t)\colon \Omega(t) \to \mathbb{R}^n$ and pressure $p\colon \Omega(t) \to \mathbb{R}$.

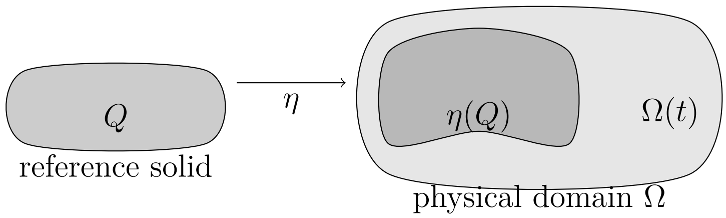

Figure 1.2: Geometrical description of the fluid-structure interaction problem.

The solid is viscoelastic in Section 1.2, and the fluid is an incompressible Navier-Stokes fluid. Then the equations read as

$$\begin{aligned} \rho_s \partial_{tt}\eta + DE(\eta) + D_2R(\eta, \partial_t\eta) &= \rho_s f && \text{in } Q, \\ \rho_f(\partial_t v + v\cdot\nabla v) &= \nu\Delta v - \nabla p + \rho_f f && \text{in } \Omega(t), \\ \operatorname{div} v &= 0 && \text{in } \Omega(t) \end{aligned} \tag{1.15}$$

along with a boundary condition at the fluid-solid interface.

The kinematic coupling through a no-slip boundary condition

$$\partial_t\eta = v\circ\eta \quad \text{on } \partial Q \tag{1.16}$$

is treated in [BKS23]: this paper brings, additionally to the Hyperbolic minimizing movements, two other novelties.

First, it is the construction of a minimizing movement scheme which solves the corresponding *parabolic* fluid-structure interaction problem [BKS23, Section 2]. A prominent feature is that the fluid domain $\Omega(t)$ is time-dependent. For this, in constructing the minimizing movements approximation η_k, v_k, one needs to construct the fluid domain Ω_k as well. These need to constructed in such a way that in the limit one solves the parabolic equation and also recovers the boundary coupling condition (1.16). The suitable approximation η_k, v_k, Ω_k satisfies the coupling

$$\begin{gathered} \eta_k \in \mathcal{E}, v_k \in W^{1,2}_{\operatorname{div}}(\Omega^{(\tau)}_{k-1};\mathbb{R}^n), \Omega_{k-1} = \Omega\setminus\eta_{k-1}(\overline{Q}) \\ \text{with } \; v|_{\partial\Omega} = 0 \text{ and } \frac{\eta_k-\eta_{k-1}}{\tau} = v_k\circ\eta_{k-1} \text{ on } \partial Q \end{gathered}$$

and solves an iterative minimization scheme analogous to (1.3). Sufficient bounds to pass to the weak limits are obtained by comparison of (η_k, v_k) with $(\eta_{k-1}, 0)$, in analogy with (1.4). One point worth mentioning here is the global velocity field. Although the approximate fluid velocity v_k is defined on a varying domain Ω_k, one may define a global velocity field u_k which is equal to v_k on Ω_{k-1} and to $\partial_t\eta_k\circ(\eta_{k-1})^{-1}$ on $\eta_{k-1}(Q)$. Thanks to the no slip condition, $u_k \in W^{1,2}(\Omega;\mathbb{R}^n)$ and thus in this way one can work on the fixed domain Ω in the estimates.

The second novelty is the fluid-structure interaction in the fully dynamical setting [BKS23, Section 4]. In contrast to the parabolic problem, one needs to construct, already at the discrete level, a *flow map* Φ which captures the movements of each particle of the fluid. In the limit, the flow map has the property $\partial_t\Phi_t = v(t)\circ\Phi_t$ and $\Phi_0 = \mathrm{id}$. This is vital for the Navier-Stokes problem since then the time-delayed problem contains the fluid term

$$\frac{v\circ\Phi_h - v(\cdot-h)}{h}$$

which is a time-delayed approximation of the material time derivative $\partial_t v + v\cdot\nabla v$. The energy estimates for $h\to 0$ again use $(\partial_t\eta, v)$ as a test function in the time-delayed problem (analogously to (1.9)). Here regularity issues are solved by adding extra $W^{k_0,2}$-dissipation terms for the solid (as in (1.14)) as well as the fluid. For all the details see [BKS23, Section 4].

2. Results of the thesis

Here we describe the contents and results of the papers contained in this thesis.

2.1 Papers I and II – Collisions of elastic solids

In [ČGK24] we solve the dynamical problem for viscoelastic solids in presence of collisions. Our approach treats the fully inertial problem (1.11) and contrary to previous results, the solution is global in time and continues after the time of collision. We thus solve the equation

$$\begin{aligned} \rho\partial_{tt}\eta + DE(\eta) + D_2R(\eta, \partial_t\eta) &= f + \sigma, \\ \eta(0) &= \eta_0, \\ \partial_t\eta(0) &= \eta_* \end{aligned} \tag{2.1}$$

for given initial values η_0, η_*, where σ is the contact force, and the reference domain Q has $C^{1,\alpha}$ boundary. We emphasize that the contact force σ is not given a priori, we construct it as a consequence of the no-interpenetration condition (1.13). This force is a measure supported at the points of contact, and acting in a normal direction to the boundary. In this way we claim the force σ to be physically reasonable. Moreover the solution satisfies the energy inequality

$$E(\eta(t)) + \frac{\rho}{2}\left\|\partial_t\eta(t)\right\|_{L^2}^2 + \int_0^t 2R\left(\eta, \partial_t\eta\right)dt \leq E\left(\eta_0\right) + \frac{\rho}{2}\left\|\eta^*\right\|_{L^2}^2 + \int_0^t \left\langle f, \partial_t\eta\right\rangle_{L^2} dt.$$

We construct the solution as follows. The minimizing movements scheme produces an approximation even if there is contact i.e. when $\partial\mathcal{E}$ is reached. Now only some directions are admissible (i.e. those that do not result in overlap), we have the *variational inequality* for these directions.

Drawing from the static case in [PH17] one can characterize this inequality by a measure. The idea is, roughly speaking, to see how large gap can one create with how much energy. Then, separating these reachable and unreachable distances with a hyperplane, we get the contact force. Here having a “normal direction” is what enables to obtain σ as a measure.

Note that one can, instead of this geometrical argument, use directly the equation and and get σ in the dual space $W^{-2,q'}$. This weaker result has been for the quasistatic case found in [KR20].

Next, passing to the limit in time-delayed variational inequality proceeds after we check that $-\partial_t\eta$ is an admissible test direction. Intuitively, this should be the case, as this is the direction from which the body came from, and indeed we prove this to be the case. For this we include a characterization of admissible test functions.

For passing to the limit with the contact force, we use our *compactness-closure result.* It says that if a sequence of deformations of bounded energy converges in C^1, each having a contact force bounded as measure, then a subsequence of these contact forces converges to a limit contact force. We emphasize here that this in particular means that the *normal direction is preserved* in the limit.

2.1.1 Lipschitz boundary

In the subsequent work [ČGK23], we extend these results to solids which have only a Lipschitz boundary in the reference configuration Q.

As described above, the notion of a "normal direction" is crucial for constructing the measure σ. The main contribution of this work is identifying the correct generalization of a normal for Lipschitz boundaries which allows for the entire procedure.

For this, we use the notions of tangent and normal cones from variational analysis [RW98] that we shall describe here. It may be at first glance tempting to define the tangent cone at the point $x \in \partial Q$ as the limit of vectors pointing inside Q, in the paper we call this the *tangent cone* $T_Q(x)$. The drawback is that this notion is not stable with respect to convergence of deformations. But it turns out that the *regular tangent cone* $\widehat{T}_Q(x)$ is sufficiently stable. This is constructed, loosely speaking, as the vectors that are also close to tangent vectors for nearby points.

Finally the *(convexified) normal cone* $\overline{N}_Q(x)$ is the dual cone to the regular tangent cone. This we claim to be the correct generalization of the normal for our problem. Using this dual description of cones, we go carefully through the approximation and recover most of the steps. The static case of constructing the contact force measure for Lipschitz domain has been investigated by [Pal19].

The one step that we do not recover, is the testing with $\partial_t \eta$. It turns out to be challenging to verify that this is an admissible direction, the reasons are described in the paper. Instead of this, to get the energy estimate, we use the *Moreau-Yosida approximation* [AGS05] in the minimizing movements. This refined estimate is sufficient for limit passage and solving the problem (2.1).

2.2 Paper III – Full time-discretization

The paper [ČS23] investigates a fully time-discrete approximation scheme for equations of elastodynamics of the type

$$\begin{aligned} &\partial_{tt}\eta(t,x) + DE(\eta(t,x)) = f(t,x) \text{ for } (t,x) \in [0,T] \times Q \\ &\eta(0,x) = \eta_0(x), \quad \partial_t \eta(0,x) = \eta_*(x) \text{ for } x \in Q, \end{aligned}$$

in the spirit of [BKS23]. As demonstrated in Section 1.1.2, the hyperbolic minimizing movements scheme of lies in usage of two different time scales, velocity scale τ and the acceleration scale h. It is shown that by this method, one can first pass with $\tau \to 0$ and obtain the time-delayed equation, which yields sufficient estimates for the final passage $h \to 0$. It is thus in essence a subsequent limit passage of two discretization parameters.

As a step towards making this scheme amenable to numerical methods, we investigate here the behavior of this scheme when both scales are kept discrete, and pass with $h, \tau \to 0$ simultaneously. In this way $\partial_{tt}\eta$ is approximated as

$$\partial_{tt}\eta(t) \approx \frac{\frac{\eta(t)-\eta(t-\tau)}{\tau} - \frac{\eta(t-h)-\eta(t-h-\tau)}{\tau}}{h}.$$

For this, we need a discrete (in τ) estimates, which are independent of h. In particular note that the τ-estimates (1.4) are h-dependent and conversely the energy inequality (1.10) depends on having already passed with $\tau \to 0$. So here we attempt to imitate the continuous estimate while remaining discrete in τ: Instead of $\partial_t \eta$ we test with the *time difference quotient* (of length τ). A technical difficulty is that in this case is that the chain rule

$$\langle DE(\eta), \partial_t \eta \rangle = \frac{d}{dt} E(\eta)$$

cannot be used. The inequality

$$\left\langle DE(\eta_1), \frac{\eta_1 - \eta_0}{\tau} \right\rangle \geq \frac{E(\eta_1) - E(\eta_0)}{\tau}$$

would be a suitable replacement, however in general it holds only for convex energies E. Since we want to include energies that are necessarily not convex, we obtain the desired results for energies where the non-convexity can be sufficiently estimated. We thus rely on what we call the *non-convexity estimate*[1]

$$\langle DE(\eta_1), \eta_1 - \eta_0 \rangle \geq E(\eta_1) - E(\eta_0) - C \|\eta_1 - \eta_0\|^2 . \tag{2.2}$$

Note that this estimate is energy-dependent, that is, the constant C depends on $\max(E(\eta_0), E(\eta_1))$. It is thus a remarkable property of the scheme that using first the minimizing movements estimate (see (1.4)) is sufficient to get the constant C independent of h and thus (2.2) yields the desired h-independent estimate.

In the paper [ČS23], we show two properties of the fully time-discrete minimization scheme[2]

$$\eta_k^\ell = \operatorname*{argmin}_{\eta \in \mathcal{E}} \frac{\tau h}{2} \left\| \frac{\frac{\eta - \eta_{k-1}^\ell}{\tau} - \frac{\eta_k^{\ell-1} - \eta_{k-1}^{\ell-1}}{\tau}}{h} \right\|_{L^2}^2 + E(\eta) + \frac{c\tau^2}{2} \left\| \nabla \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|_{L^2}^2 - \left\langle f_k^\ell, \eta \right\rangle_{L^2} .$$

First, we prove its *stability* in [ČS23, Section 2]. Stability means that the energy of the discrete solution does not grow, resp. that it only grows as a result of work done by the external force f.

Second, under convexity of highest-order terms in E, we show a *linear rate of convergence* in [ČS23, Section 3]. To obtain the convergence rate results, we use regularity results that we also develop there, namely in [ČS23, Subsection 3.1]. Linear convergence rate then means that the discrete solution converges to the regular solution linearly with respect to the discretization step length.

2.3 Paper IV – Fluid-Structure interaction with slip

In the last paper [ČKS24] we extend the previous results to the case of visco-elastic bulk solid interacting with an incompressible Navier-Stokes fluid (1.15) to the case of a full *slip condition* at the fluid-solid interface. The main novelty here

[1]This property is by some authors called λ-convexity [AGS05].

[2]For the precise role of the artificial dissipation $\frac{c\tau^2}{2} \left\| \nabla \frac{\eta_k^\ell - \eta_{k-1}^\ell}{\tau} \right\|_{L^2}^2$ we refer to the paper itself, and just note here that it vanishes with $\tau \to 0$.

is the treatment of the impermeability condition, that is, (1.16) is replaced by the kinematic condition

$$v \cdot n = (\partial_t \eta \circ \eta^{-1}) \cdot n \quad \text{on } \partial\eta(t, Q) \tag{2.3}$$

where n denotes the normal vector to the fluid-solid interface $\partial\eta(t,Q)$ (the deformed configuration). For this coupled system we prove the existence of weak solutions.

The *coupled* test functions are continuous over the entire fluid-solid domain Ω and are divergence free in the fluid part $\Omega(t)$. These are the very same test functions employed by [BKS23] (Note that if only these test functions were employed, only the no-slip condition can be treated). This means that we require that for $\xi \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^n)$ with $\phi := \xi \circ \eta$ in Q and $\operatorname{div} \xi = 0$ in $\Omega(t)$, $\xi(T) = 0$, $\phi(T) = 0$ it holds

$$\begin{aligned}
&- \int_0^T \rho_s \langle \partial_t \eta, \partial_t \phi \rangle \, dt + \int_0^T DE(\eta)\langle \phi \rangle + D_2 R(\eta, \partial_t \eta)\langle \phi \rangle dt \\
&\quad + \int_0^T -\rho_f \langle v, \partial_t \xi \rangle_{\Omega(t)} + \rho_f \langle v, v \cdot \nabla \xi \rangle_{\Omega(t)} + \nu \langle \varepsilon v, \varepsilon \xi \rangle_{\Omega(t)} \, dt \\
&\qquad = \int_0^T \rho_s \langle f, \phi \rangle dt + \int_0^T \rho_f \langle f, \xi \rangle_{\Omega(t)} \, dt + \rho_s \langle \eta_*, \phi(0) \rangle + \rho_f \langle v_0, \xi(0) \rangle_{\Omega(0)}.
\end{aligned}$$

The *fluid-only* test functions are defined in the fluid part of the domain only, and have zero normal component at the fluid boundary. This then allows the tangential jump of the velocity to be seen by the test functions, and in this way the slip condition is obtained. This means that for all $\xi \in C^\infty([0,T] \times \overline{\Omega}(t); \mathbb{R}^n)$, $\xi \cdot n = 0$ on $\partial\Omega(t)$, $\operatorname{div} \xi = 0$, with $\xi(T) = 0$ it holds

$$\begin{aligned}
\int_0^T -\rho_f \langle v, \partial_t \xi \rangle_{\Omega(t)} + \rho_f \langle v, v \cdot \nabla \xi \rangle_{\Omega(t)} + \nu \langle \varepsilon v, \varepsilon \xi \rangle_{\Omega(t)} \, dt = \int_0^T \rho_f \langle f, \xi \rangle_{\Omega(t)} \, dt \\
+ \rho_f \langle v_0, \xi(0) \rangle_{\Omega(0)}.
\end{aligned}$$

We include also the strong pointwise formulation of the problem, including all the boundary conditions coming from the stress, the slipping law, and the solid hyperstress, and show its compatibility with the weak formulation. Namely, we show that provided a weak solution is sufficiently regular, it is also a strong solution. Further for the weak formulation, we show how pressure can be reconstructed.

A technical point in our construction is that the global velocity field is no longer a $W^{1,2}(\Omega; \mathbb{R}^n)$ function due to the tangential jump between fluid and solid. For this reason, we treat the velocities separately and treat the convergences in spaces of functions on moving domains.

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List of publications

Publications listed without a journal name are preprints as of July 2024.

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Paper I: Antonín Češík, Giovanni Gravina, and Malte Kampschulte. Inertial evolution of non-linear viscoelastic solids in the face of (self-)collision. *Calculus of Variations and Partial Differential Equations*, 63(2):55, February 2024

Paper II: Antonín Češík, Giovanni Gravina, and Malte Kampschulte. Inertial (self-)collisions of viscoelastic solids with Lipschitz boundaries. (arXiv:2312.00431), December 2023

Paper III: Antonín Češík and Sebastian Schwarzacher. Stability and convergence of in time approximations of hyperbolic elastodynamics via stepwise minimization. (arXiv:2305.19880), June 2023

Paper IV: Antonín Češík, Malte Kampschulte, and Sebastian Schwarzacher. Fluid-structure interactions with slip. 2024

Publications not used in the thesis

- Antonín Češík. Convex hull property for elliptic and parabolic systems of PDE. *Nonlinear Analysis*, 245:113554, August 2024