FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

ABSTRACT OF DOCTORAL THESIS

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Synthetic geometry in various dimensions

Mathematical Institute of Charles University

Supervisor of the doctoral thesis: Mgr. Lukáš Krump, Ph.D.
Study programme: General Questions of Mathematics and Computer Science
Study branch: General Questions of Mathematics and Computer Science

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AUTOREFERÁT DISERTAČNÍ PRÁCE

Jakub Řada

Syntetická geometrie v různých dimenzích

Matematický ústav Univerzity Karlovy

Vedoucí disertační práce: Mgr. Lukáš Krump, Ph.D.
Studijní program: General Questions of Mathematics and Computer Science
Studijní obor: General Questions of Mathematics and Computer Science

Praha 2024

Anotace:

Tato disertační práce se zaměřuje na syntetickou geometrii v různých dimenzích, počínaje rovinnou geometrií. Cílem této práce je ukázat možnost využití syntetické geometrie v matematických důkazech. Dále je v práci popsán způsob, jak vyzualizovat n-dimenzionální prostor s hlavním zaměřením na n rovno čtyřem. V práci také popisujeme dva způsoby vizualizace 4-dimenzionálního prostoru (perspektiva a pravoúhlý průmět na dva navzájem kolmé podprostory). Ty jsou pak při aplikacích ve čtyřdimenzionálním prostoru.

Annotation:

This thesis focuses on synthetic geometry in various dimensions, starting with plane geometry to show the possibility of using synthetic geometry in proofs. Furthermore, the thesis describes visualisation of the n-dimensional space focusing on n equal to 4. Moreover, the thesis describes two ways how to visualise the 4-dimensional space (perspective and double orthogonal projection onto two mutually perpendicular subspaces). Finally, it provides applications of visualisation in 4-dimensions.

Abstrakt:

Tato disertační práce je zaměřena na syntetickou geometrii v různých dimenzích, počínaje rovinnou geometrií. Jejím cílem je ukázat možnost využití syntetické geometrie v matematických důkazech. Jako příklad používáme dva různé geometrické důkazy Pappus–Pascalovy věty, konstrukce oskulačních kružnic elipsy v libovolném bodě elipsy a grafické řešení kvadratické rovnice. Dále je v práci popsán způsob, jak vizualizovat n-dimenzionální prostor pomocí metody pohledu "za" a pomocí vrstevní ve směru kolmém. Práce se podrobně věnuje vizualizacím ve 4-dimenzionálním prostoru a konkrétně popisuje dva způsoby: zobecněné Mongeovo promítání (pravoúhlý průmět na dva navzájem kolmé podprostory) a zobecnění lineární perspektivy. Práce obsahuje také aplikace vizualizace 4-dimenzionálního prostoru. Využívá zobecněného Mongeova promítání a 4-dimenzionální perspektivy pro znázornění a práci s komplexní číselnou rovinou, pro syntetickou i algebraickou vizualizaci stínů ve 4-dimenzionálním prostoru a pro znázornění 4-dimenzionálních objektů pomocí 3D tisku.

Abstract:

This doctoral thesis focuses on synthetic geometry in various dimensions. We start with plane geometry to show how synthetic geometry can be used in proofs. We demonstrate the advantages of synthetic geometry on two different geometric proofs of the Pappus–Pascal theorem, the construction of the osculating circles of an ellipse at any point of the ellipse and the graphical solution of a quadratic equation. Moreover, the thesis describes visualisation of the n-dimensional space using the "behind" view method and perpendicular layering. Furthermore, the thesis focuses on visualisation of 4-dimensional space. It describes two possible methods: a generalisation of Monge's projection (orthogonal projection onto two mutually perpendicular subspaces) and a generalisation of linear perspective. Finally, the thesis contains applications of the visualisation of 4-dimensional space. For example, the usage of the generalised Monge's projection and 4-dimensional perspective for representation of the complex number plane. The application for visualisation of shadows in 4-dimensions synthetically and algebraically, and for representation of 4-dimensional objects using 3D printing.

Contents

1. Introduction – goals and structure

There are many reasons why to deal with geometry. Geometry is all around us. Even before humans were here, nature played with geometric shapes and patterns. We can find the golden spiral in a sunflower, a mollusc shell or a pineapple. The eggs we eat without noticing are made up of two different ellipsoids, and the filaments of caterpillars form gossamer webs close to hyperbolic paraboloids. Geometry surrounds us in beautiful patterns. There are two ways of approaching geometry, synthetic geometry and analytic geometry.

It is possible to represent almost any geometry with equations. This is useful with developed computer science, but it is a great pity because we miss out the visual aspect of geometry. Fortunately, with the advent of modern technology, synthetic geometry is making a comeback in computer graphics. Most of the work consists of published articles, and this thesis will focus on visual geometry. Of course, analytical geometry is also used in this work, as some constructions and proofs need to be supported by calculations. This thesis aims not to create a complete textbook of synthetic geometry bypassing analytical geometry. This thesis aims to use computer graphics to visualise the geometry and work with geometry synthetically in some areas of geometry especially in 4D geometry.

The first part of the thesis focuses on plane geometry, where we show some graphical proofs. The purpose of this chapter is to make the reader wonder whether it is necessary to use analytic geometry directly for proofs, or whether there are other ways of doing proofs.

The second chapter is an introduction to n-dimensional geometry, especially 4D geometry. It tries to explain how the reader can understand and imagine 4D geometry.

The third chapter is a summary of the main principles of double orthogonal projection on two perpendicular three-dimensional spaces followed by the the fourth chapter, which explains the basic principles of the 4D perspective. The fourth chapter includes 3-sphere in a 4-perspective and cut of the 3-sphere with a 3-space.

Chapter five describes the use of 4D visualisations. This chapter begins with the visualisation of the complex number plane. The complex number plane could be a four-dimensional space so that we can visualise it with a double orthogonal projection onto two mutually perpendicular 3-spaces or with a 4D perspective. It is also essential to be able to imagine and grasp the 4D space. It jumps over two dimensions from paper to 4D space, but from 3D space to 4D space, it is only one dimension. This is why 3D printing is suitable for representing 4D space.

Throughout the text, we use many figures with the same constructions. It is, therefore, pointless to draw all from zero. Therefore, the appendix gives some hints on how to make the work easier when using GeoGebra. The thesis contains many figures created in GeoGebra using shortcuts from the article in the appendix. The rest of the figures are programmed in the computer program Mathematica.

The thesis begins by demonstrating the strength of the geometry. It is necessary to be able not only to visualise objects but also to work with displayed objects on paper (measuring, cutting, ...). The work then moves on to the main objective of the thesis, which is to introduce the reader to 4D geometry. This shows the reader how to perceive the 4D world and draw 4D space on paper. The work continues with the practical use of virtualising 4D geometry. In my diploma thesis it was very difficult to imagine the complex number plane. This work gives a way how to visualise and work with the complex number plane. Only a few papers are devoted to visualising the complex number plane. Most papers skip the complex representation and show only the real x, y plane. The work also includes algebraic shading. Many computer programs use point shading. They take a ray of light through each point of the scene and look at where each point is lighted. In this part, we use an algebraic representation of solids and look at their shadows algebraically. The shadows are done in 3D and 4D geometry.

2. Research part

The thesis is based on the published/preprint articles corresponding to the author's research, completed into one work. The thesis is divided into chapters according to topics.

2.1 Some geometric construction and proofs

The first chapter is focused on geometric proofs and constructions. Historically, but also nowadays, many proofs of geometric constructions are based on analytical calculations.

This chapter is divided into three sections. The first part is devoted to the Pappus–Pascal theorem. We prove this theorem using three different proofs (using homogeneous coordinates, using perspective view and using projectivity). The second part describes the construction of the osculating circle of any point of the ellipse. The third section of this chapter is focused on quadratic equations. In this section, we describe how to solve a quadratic equation graphically.

This chapter summarizes the research of other researchers reported in their books, articles etc.

2.2 Across dimensions

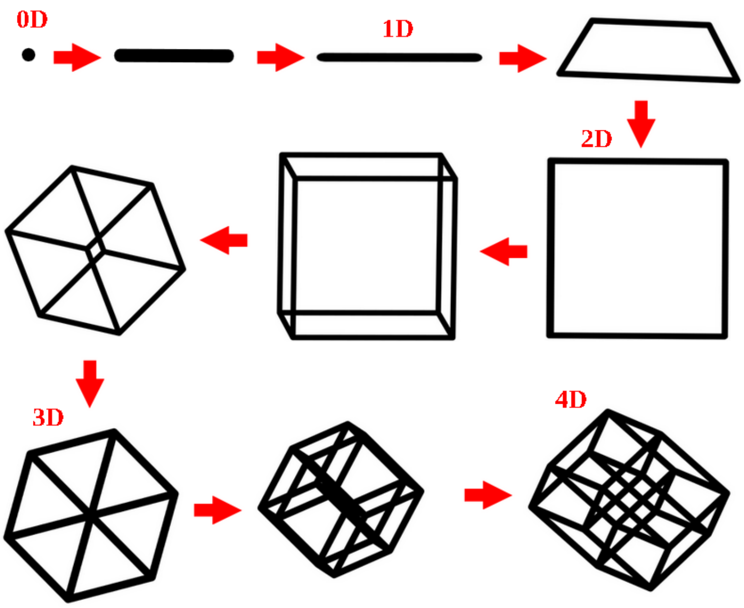


Figure 2.1: animation of looking behind

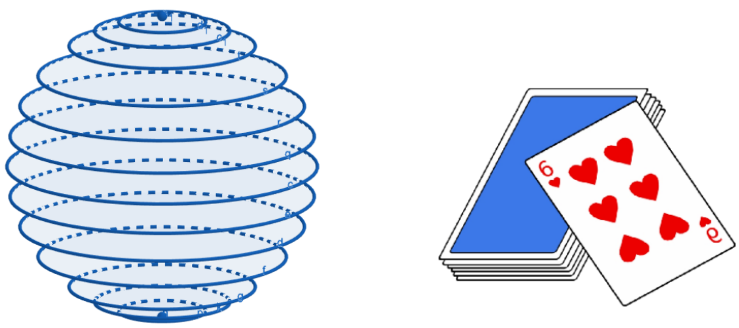
This chapter is divided into two sections. This chapter describes the basics of multidimensional geometry (especially 4D). We will use analogies to describe the understanding of multidimensional space. This chapter summarizes the knowledge of the book Flatland (Abbott [2015]), This chapter uses two approaches, how to visualise multidimensional space. The second part describes first light sight how to project space onto a paper. This part describes the main principles of Monge's projection and Linear perspective, which we will generalise to the 4D. This chapter serve as an introduction to the following chapters.

Look behind

First approach is to look behind. The main principle is, that something could be hidden behind the drawn construction. So we can discover an n-dimensional space. On the figure 2.1 is illustrated this approach up to 4D, which we are most interested in in the following chapters.

Using cut

Another way to introduce n-dimensional space is to use a cut. $n+1$ dimension is created by adding a direction perpendicular to the nth dimension. Therefore, for demonstration, we can easily set up a 3D space from the 2D by layering like deck of card or 3D printer (figure 2.2).



(a) sphere cuts

(b) deck of cards

Figure 2.2: visualization of 3D-space by using cuts

Double orthogonal projection

This chapter describes main principles of double orthogonal projection onto two mutually perpendicular 3-spaces. It is a generalization of Monge's projection. It describes, how to visualise a point, a line (figure 2.3), lines in special positions (figure 2.4), relative position of two lines (figure 2.5), plane (figure 2.6), planes **??** and basic shapes (Tesseract and 3-sphere).

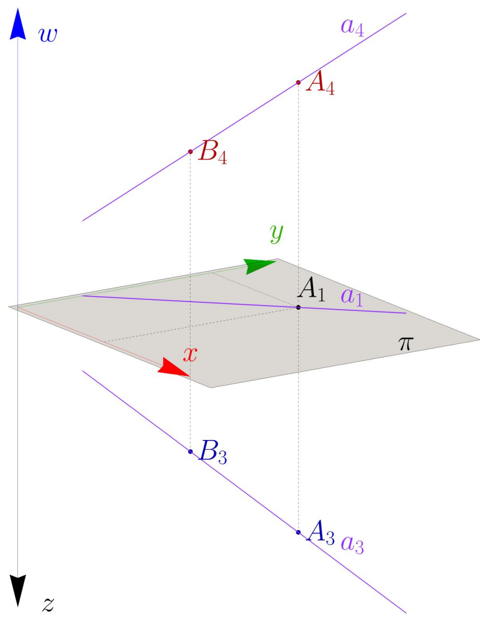


Figure 2.3: Projection of point $A = [x_A, y_A, z_A, w_A]$ and line $a = \overline{AB}$.

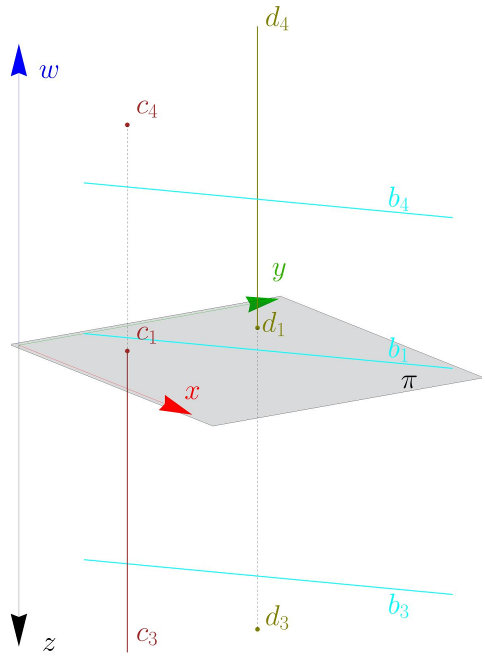


Figure 2.4: Lines in special positions.

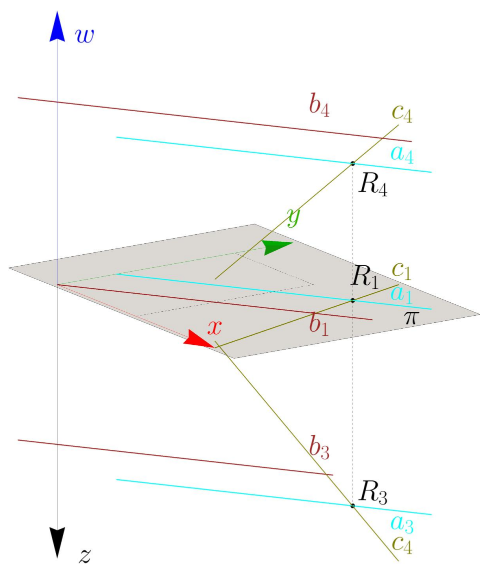


Figure 2.5: Relative position of lines a, b, c.

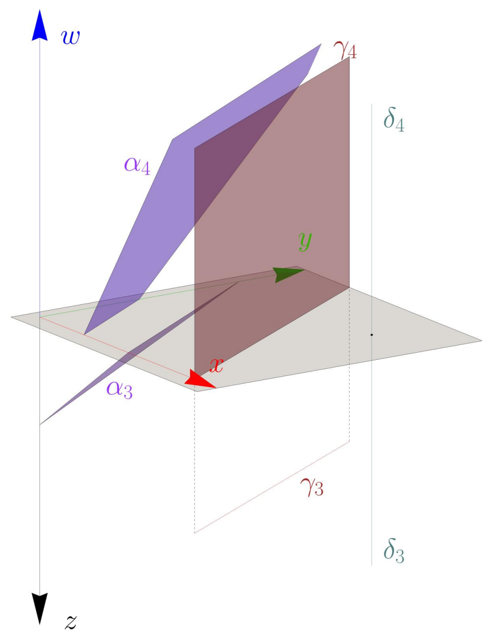


Figure 2.6: Plane α, γ, δ.

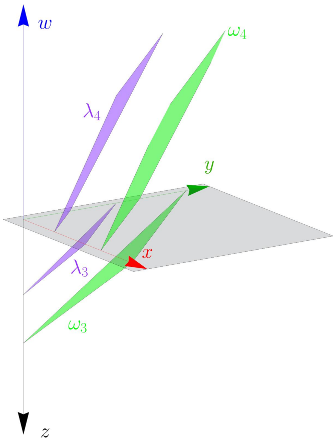


Figure 2.7: Planes λ, ω parallel in two direction.

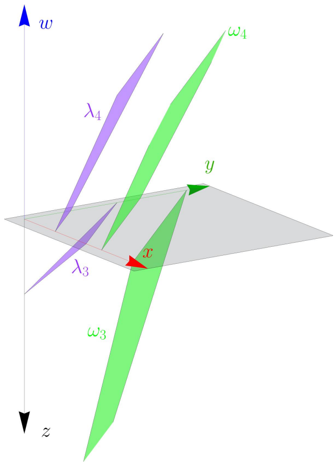


Figure 2.8: Planes λ, ω parallel in one direction.

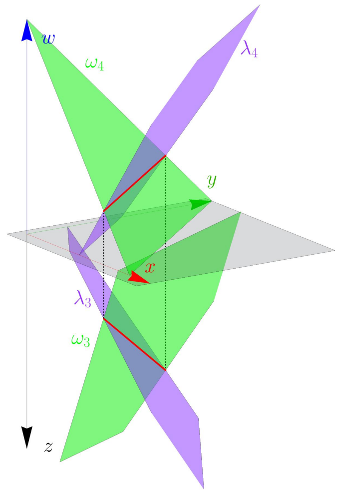


Figure 2.9: Planes λ, ω intersecting in a line.

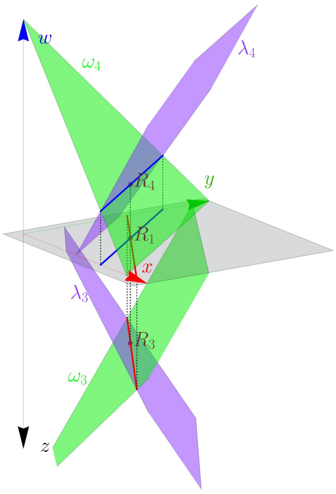


Figure 2.10: Planes λ, ω intersecting in a Point.

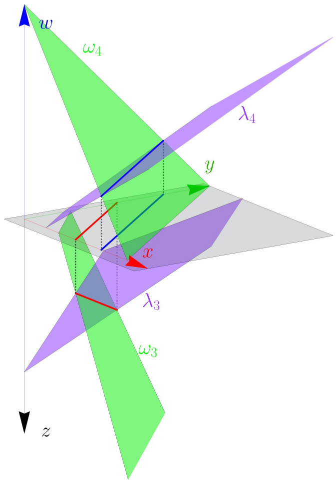


Figure 2.11: Planes λ, ω skew

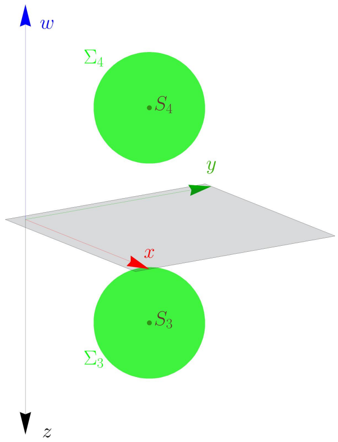
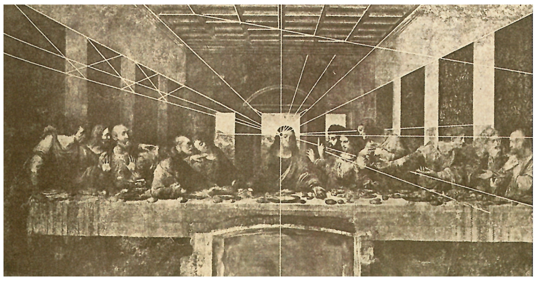


Figure 2.12: 3-Sphere Σ.

2.3 Perspective

To begin this chapter, a few stages in the development of the linear perspective are briefly described. How the vanishing points were not perfect in the beginning and the correctness in Leonardo paintings (figure 2.13). Later when the basic properties of linear perspective were already known, painters began to play with perspective. Paul Cézanne painted paintings with multiple principal points (figure 2.14) and Pablo Picasso took the violin to the basic elements (figure 2.15), which moves us to Cubism.

(a) The Apparition To Brother Augustin And The Bishop
Giotto di Bondone



(b) The Last Supper
Leonardo da Vinci

Figure 2.13: Kadeřávek's analysis of perspective in paintings, taken from the book (Kadeřávek [1922]).

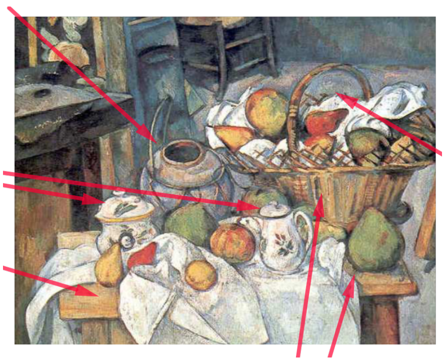


Figure 2.14: Kitchen Table - Paul Cézanne

Figure 2.15: Violin and Grapes
Pablo Picasso

Next part of this chapter focused on generalisation of linear perspective to 4D perspective. The main principles of 4D perspective are described (figure 2.16), visualisation of a point with associated 4DDOP (figure 2.17) and visualisation of shapes (Tesseract, hyperpyramid, 4D prism and 3-sphere including its cut)

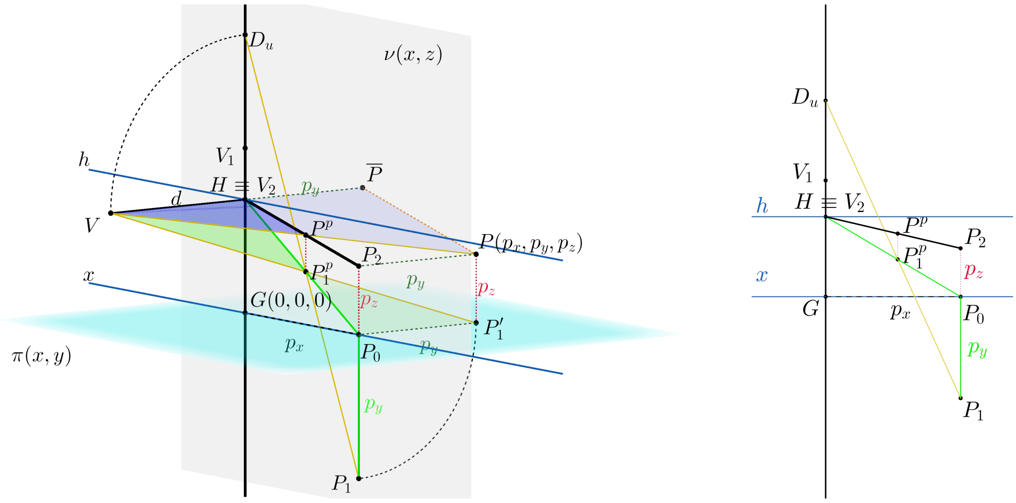


Figure 2.16: (Left) The principle of the 3-perspective construction of a point with associated Monge's projection. The 3D-image is projected in orthographic projection.(Right) The perspective image from the perspective center.

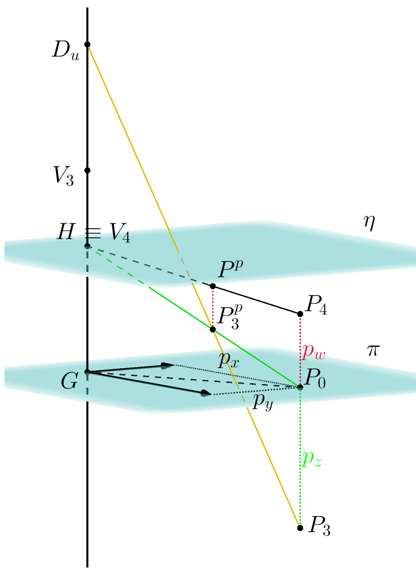


Figure 2.17: The principle of the 4-perspective construction of a point with associated 4DDOP in the modeling 3-space.

2.4 Use of 4D visualisation

Complex number plane

Last chapter is divided into three parts. First part is focused on complex number plane. To represent a complex number plane in euclidean plane is not easy and the consequent constructions are very difficult (Řada [2019]). Imaginary point lies somewhere up or below the euclidean plane. Therefore it is suitable to visualise complex number plane in double orthogonal projection or in 4D perspective. In this part we for example visualised the circle (Figure 2.18), hyperbola and etc (Figure 2.19). With the ability to simply visualise the complex number plane, we can easily construct any problem synthetically.

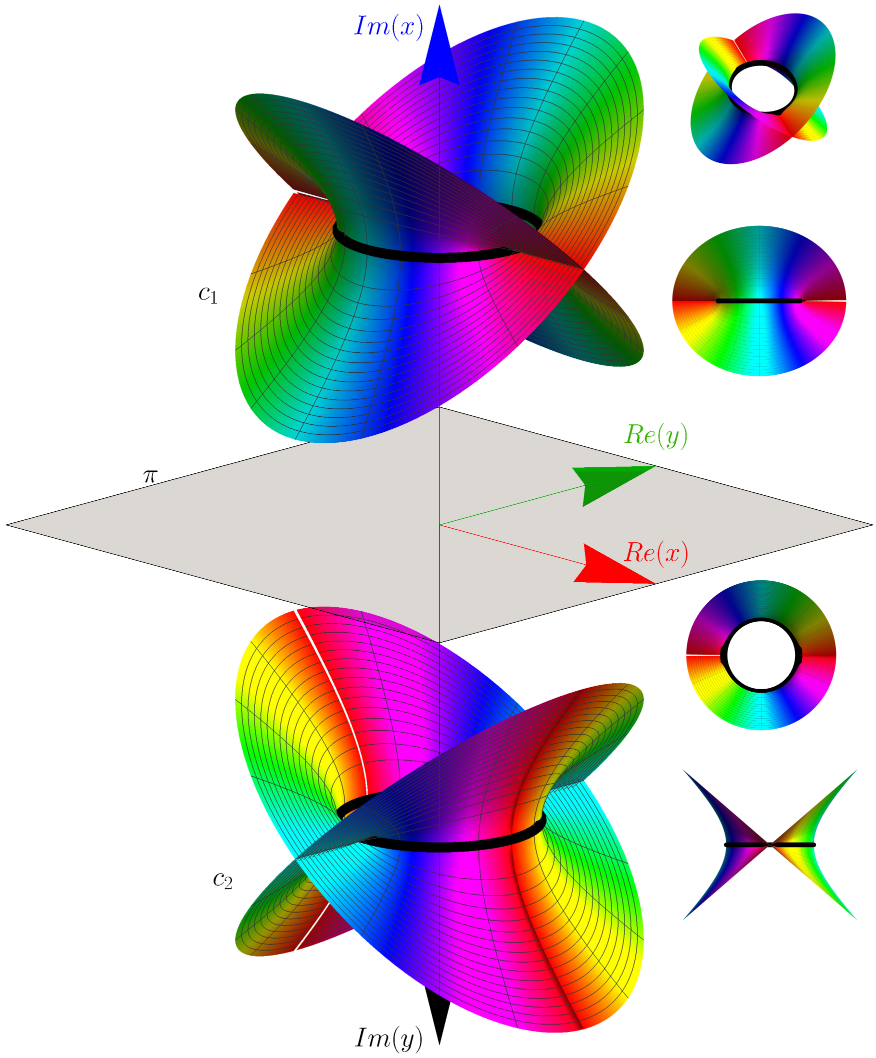
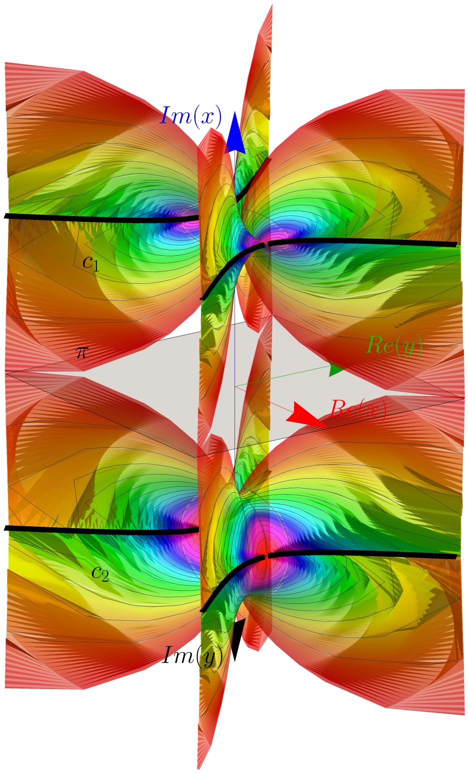
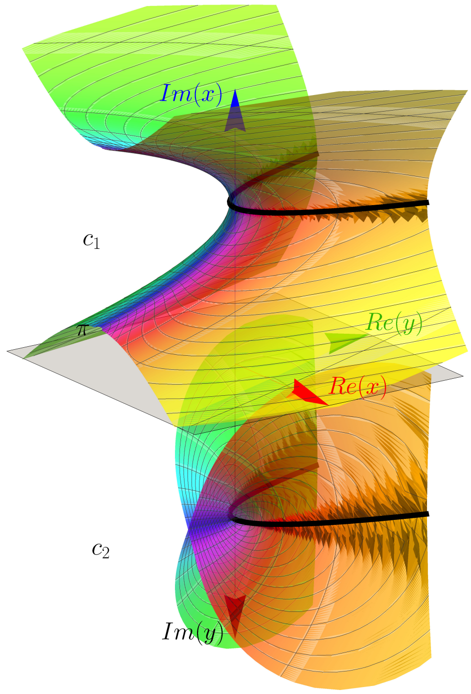


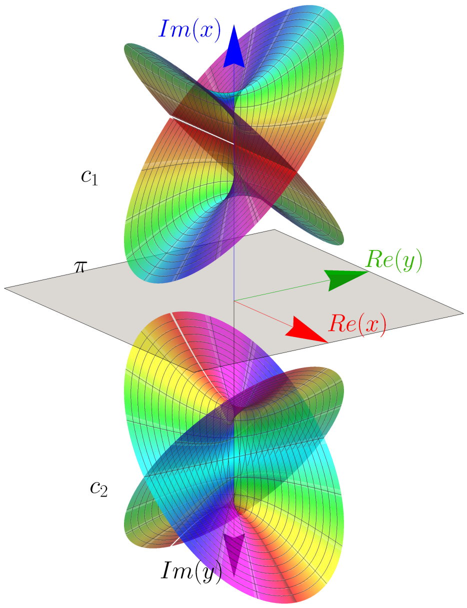
Figure 2.18: The surface of a real circle c generated by its complex points visualized in 4DDOP. Views in special positions are on the right side. The circle is shifted in the directions $Im(x)$ and $Im(y)$ so that images in the 3-spaces $(Re(x), Im(x), Re(y))$ and $(Re(x), Re(y), Im(y))$ do not overlap in the figure.



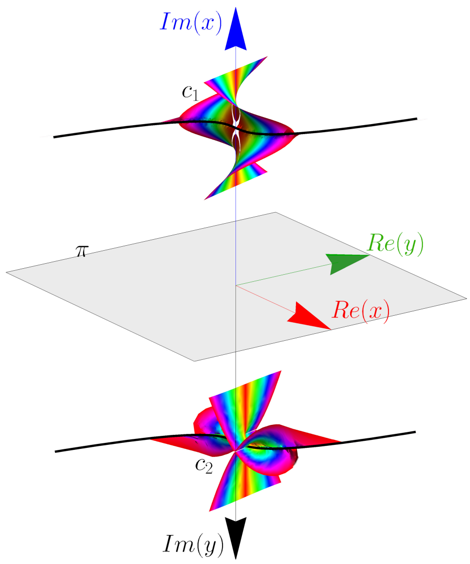
(a) Hyperbola: $x^2 - y^2 = 1$



(b) Parabola: $y = x^2$



(c) Imaginary regular conic: $x^2 + y^2 = -1$



(d) Cubic: $y = x^3$

Figure 2.19: Surfaces of curves generated by their complex points visualized in 4DDOP. All the surfaces are shifted in $Im(x)$ and $Im(y)$ directions so that they do not overlap.

3D Shadows of 4D Algebraic Hypersurfaces in a 4D Perspective

Second part focused on 3D Shadows of 4D Algebraic Hypersurfaces in a 4D Perspective. To explain this topic we first illustrated the shadows in 3D. Final Figure is figure 2.20. In addition, we made an animation where a point of light moves on a circle (see Attachment).

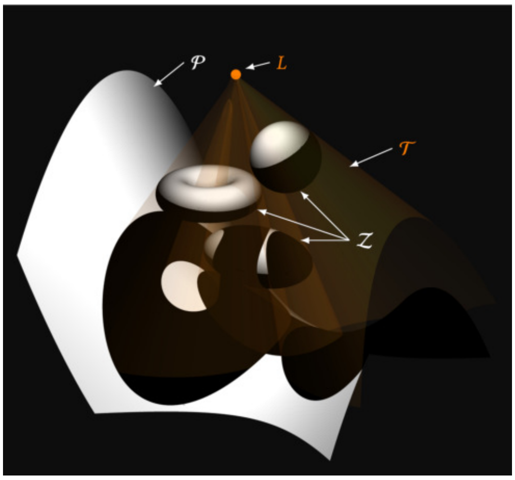


Figure 2.20: The final visualization of a scene with a sphere, torus, and ellipsoid casting shadows on themselves and on a hyperbolic paraboloid.

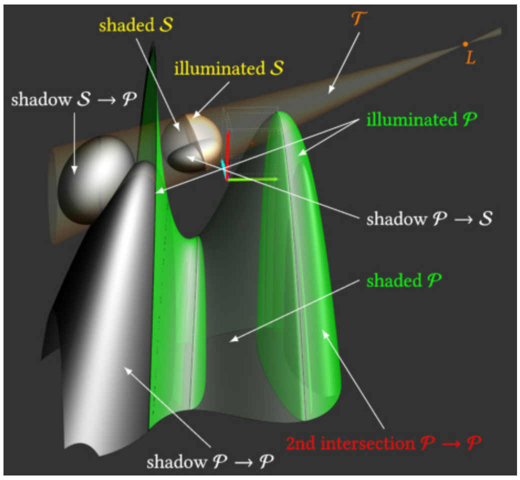


Figure 2.21: An illumination of a 3-sphere $\mathcal{S}$ and its shadow on a 3-surface $\mathcal{P}$ of degree 3 from a point light source L in a 4-D perspective. The figure contains the excess intersection of the tangent cone (not visualized) of $\mathcal{P}$, and the self-shaded region of $\mathcal{P}$ is not excluded from the illuminated part.

After a detailed explanation of how shadows are created, including an explanation of the algorithm, we created further visualizations in 4D (for example figure 2.21).

3D printed models of a tesseract in the double orthogonal projection and 4D perspective

Final part of this chapter focused on better understanding of 4D though 3D printing. The 3D printed model is easier to look at from different angles and to touch, unlike the visualisation on a computer. Therefore we construct a 3D printed model of a Tesseract in double orthogonal projection and 4D perspective.

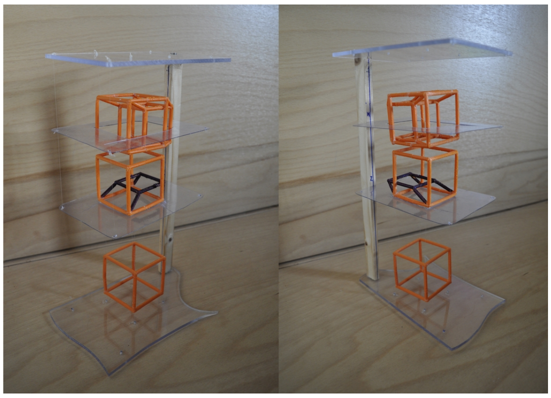


Figure 2.22: 3D printed models of a tesseract in the double orthogonal projection and 4D perspective.

3. Conclusion

This thesis is divided into five chapters. The first chapter of this thesis deals with plane geometry. There are many geometric theorems which are usually proved algebraically. Therefore, we take some theorems and show geometrical proofs of them. We aimed to show that, in many cases, geometric theorems can be proved with geometric proofs. This work focuses on synthetic geometry in different dimensions, so the second chapter is focused on dimensions. In the second chapter of this thesis, we introduce n-dimensional geometry and how to visualise n-dimensional geometry, focusing mainly on four-dimensional geometry. For this, we use two typical geometrical generalisations of how to describe higher dimensions. The third part of this thesis aims to generalise Monge's projection to double orthogonal projection onto two mutually perpendicular 3-spaces. We explain the basic principles of how to draw a point, a line, a plane and a shape. This part of the thesis also briefly introduces the next section. The fourth part aims to generalise linear perspective to 4D perspective using double orthogonal projection. After a brief introduction to the historical development of perspective, the basic principles of 4D perspective are explained. The principles of visualising basic shapes are described. The final part of this thesis uses double orthogonal projection and 4D perspective in practice. This section shows several methods of using 4D visualisation. The first is to visualise and solve problems in the complex number plane. The complex number plane is visualised as a four-dimensional space. The central part is devoted to an intersection of a real circle with sets of lines. In other words, we visualise the circle with its complex parts. The second part describes shadows in 4D algebraically. We describe how shadows work algebraically in 3D space and then generalise this approach to 4D space. The last part describes how we can better understand and visualise 4D space through 3D printing. There has been a huge development in 3D printers in recent years. Therefore, 3D printers are a good and cheap way to look at 4D space from any angle and to touch it ourselves using the methods of double orthogonal projection and 4D perspective.

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