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MASTER THESIS

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Wreath product of operadic categories

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Title: Wreath product of operadic categories

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Abstract: The concept of operadic categories was introduced in a 2015 paper "Operadic categories and duoidal Deligne's conjecture" by Batanin and Markl as a generalisation of various "operad-like" structures, including classical operads and their variants, versions of PROPs and other similar structures. This thesis introduces the concept of the wreath product of operadic categories and demonstrates that this construction satisfies the axioms of operadic categories. We show that the wreath product of operadic categories is non-commutative and associative. Furthermore, we show that a wreath product of Batanin's k-trees and l-trees produces a $k+l$-tree. Our work also establishes a relationship between the Boardman-Vogt product of single-colored operads and the wreath product of their operadic Grothendieck constructions. It is our hope that the wreath product of operadic categories will be a valuable tool in the understanding of the Boardman-Vogt tensor product of coloured operads.

Keywords: operad, operadic category

Název práce: Věncový součin operadických kategorií

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Abstrakt: Koncept operadických kategorií byl představen v roce 2015 v Bataninově a Marklově článku "Operadic categories and duoidal Deligne's conjecture" jako zobecnění různých "operadických" struktur, včetně klasických operád a jejich variant, různých forem PROPů a dalších podobných struktur. Tato práce zavádí pojem věncového součinu operadických kategorií a ukazuje, že zavedená konstrukce splňuje axiomy operadických kategorií. Ukazujeme, že věncový součin operadických kategorií je nekomutativní a asociativní. Dále ukážeme, že věncový součin Bataninových k-stromů a l-stromů vytváří $k+l$-strom. Naše práce také stanovuje vztah mezi Boardmanovým-Vogtovým součinem jednobarevných operád a věncovým součinem jejich operadických Grothendieckových konstrukcí. Doufáme, že věncový součin operadických kategorií se ukáže jako cenný nástroj při pochopení Boardmanova-Vogtova tenzorového součinu barevných operád.

Klíčová slova: operáda, operadická kategorie

Contents

Introduction

Operadic categories were introduced in [1] as a generalisation of various "operad-like" structures, including classical operads and their variants – cyclic, modular, etc. – as well as various forms of PROPs and other similar structures. We refer to [2] for an overview of these objects. These categories formalize the information determining an operad-like structure of a given type along with its algebras. Morphisms in operadic categories possess fibers, whose properties are modeled by the preimages of maps between finite sets. Each operadic category $\mathtt{O}$ has a corresponding category of its operads. An archetypal example of an operadic category is the skeletal category *sFSet* of finite sets, whose operads are classical single-coloured symmetric operads.

This thesis introduces the concept of the *wreath product* of operadic categories. For operadic categories $\mathtt{A}$ and $\mathtt{B}$, their wreath product, denoted by $\mathtt{A} \wr \mathtt{B}$, is a category with objects determined by an *object* $a \in \mathtt{A}$ and a suitable *sequence* of objects $b_1, ..., b_k \in \mathtt{B}$. The definition of this product was outlined by Michael Batanin during his tenure at the Mathematical Institute of the CAS in Prague, motivated by an apparent connection to the Boardman-Vogt product of symmetric operads [3].

Plan. In Chapter 1, we provide a number of auxiliary definitions for the category *sFSet* and recall the definitions of operadic categories and related notions. Some definitions have been revised to reflect minor corrections of the original sources, which are duly noted.

In Chapter 2, we define the wreath product of an operadic category $\mathtt{A}$ and a *connected* operadic category $\mathtt{B}$ and prove that the resulting category is also an operadic category. We discuss the necessity of the connectivity requirement for $\mathtt{B}$ and explore potential relaxations. We demonstrate that $\mathtt{A} \wr \mathtt{B}$ is a non-commutative associative operation and that for the categories of k-trees Ω_k and l-trees Ω_l,

$$\Omega_k \wr \Omega_l \cong \Omega_{k+l}.$$

The main objective of Chapter 3 is to establish a relationship between the wreath product of operadic categories and the Boardman-Vogt tensor product of single-coloured symmetric operads with a unit. We do this in Proposition 39 by stating that for single-coloured operads $\mathcal{X}, \mathcal{Y}$ there exists an *epimorphism* from $\mathcal{X} \otimes_{BV} \mathcal{Y}$ to an operad constructed from the wreath product of suitable categories.

Conventions. Operadic categories will be denoted by typewriter letters $\mathtt{A}, \mathtt{B}, \mathtt{O}$, etc. We denote by *sFSet* be the skeletal category of finite sets. The objects of this category are linearly ordered sets $\bar{n} = \{1, \ldots, n\}, n \in \mathbb{N}$, when it causes no confusion, we use n instead of $\bar{n}$. Morphisms are arbitrary maps between these sets. We use $\mathscr{V}$ to denote a complete, cocomplete closed symmetric monoidal category. Operads in $\mathscr{V}$ will be denoted by the calligraphic letters $\mathcal{X}, \mathcal{Y}$, etc.

In this context, the term 'classical' (colored) operad $\mathcal{X}$ refers to a symmetric (colored) operad with a unit. Let $\mathcal{X}$ be a classical single-colored operad in $\mathscr{V}$, we denote its composition maps by

$$\gamma_{\mathcal{X}} : \mathcal{X}(n) \otimes \mathcal{X}(k_1) \otimes ... \otimes \mathcal{X}(k_n) \to \mathcal{X}(k_1 + ... + k_n),$$

for $n \geq 1, k_1, ..., k_n \geq 0$, or simply by γ if there is no confusion, and its unit maps by $\eta_{\mathcal{Y}} : I \to \mathcal{Y}(1)$. If we wish to emphasize that $\mathcal{X}$ is a *sFSet*-operad, we denote by

$$\mu_{\mathcal{X}}(f) : \mathcal{X}(\bar{f}_1) \otimes ... \otimes \mathcal{X}(\bar{f}_m) \otimes \mathcal{X}(\bar{m}) \to \mathcal{X}(\bar{n}),$$

for a morphism $f : \bar{n} \to \bar{m}$ with respective fibers $\bar{f}_1, ..., \bar{f}_m$ in *sFSet*, the structure maps of $\mathcal{X}$ (or simply μ).

The monoidal structure on the category *Set* of sets and set maps is given by the Cartesian product and a one-point set $\mathbb{1} = \{*\}$ as a monoidal unit.

1 Preliminaries

1.1 Category *sFSet*

Let *sFSet* be the skeletal category of finite sets. The objects of this category are linearly ordered sets $\bar{n} = \{1, \ldots, n\}, n \in \mathbb{N}$. Morphisms are arbitrary maps between these sets.

For $\bar{n}_1 = \{1, ..., n_1\}, \bar{n}_2 = \{1, ..., n_2\} \in sFSet$, we introduce notation

$$\bar{n_1} \oplus \bar{n_2} = \{1, ..., n_1 + n_2\}. \tag{1.1}$$

Let $\bar{n}_1, ..., \bar{n}_k \in sFSet$, then we define several auxiliary maps.

1. An inclusion map, for each $1 \le j \le k$,

$$\begin{aligned} i_j : \bar{n}_j &\longrightarrow \bigoplus_{q=1}^{k} \bar{n}_q \\ x &\longmapsto x + \sum_{q=1}^{j-1} n_q. \end{aligned} \tag{1.2}$$

2. A projection map

$$p : \bigoplus_{q=1}^{k} \bar{n}_q \to \bar{k} \tag{1.3}$$

by the assignment

$$p(x) = j \text{ if and only if } x \in i_j(\bar{a}_j).$$

3. A renumbering partial map

$$\begin{aligned} r; \quad \bigoplus_{q=1}^{k} \bar{n}_q &\longrightarrow \bar{n}_j \\ x &\longmapsto i_j^{-1}(x). \end{aligned} \tag{1.4}$$

Example 1. Let $\bar{n}_1 = \bar{2}, \bar{n}_2 = \bar{1}, \bar{n}_3 = \bar{3}$, then

$$\begin{array}{ccc} i_3 : \ \bar{n}_3 & \to & \bigoplus_{q=1}^{3} \bar{n}_q \ , \\ 1 & \mapsto & 4 \\ 2 & \mapsto & 5 \\ 3 & \mapsto & 6 \end{array} \quad \begin{array}{ccc} p : \ \bigoplus_{q=1}^{3} \bar{n}_q & \to & \bar{3} \ , \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \\ 4 & \mapsto & 3 \\ 5 & \mapsto & 3 \\ 6 & \mapsto & 3 \end{array} \quad \begin{array}{ccc} r; \ \bigoplus_{q=1}^{3} \bar{n}_q & \to & \bar{n}_3 \ . \\ 4 & \mapsto & 1 \\ 5 & \mapsto & 2 \\ 6 & \mapsto & 3 \end{array}$$

Let moreover $\phi : \bar{k} \to \bar{l}$ be a map in *sFSet* and $\Big\{\phi_{i,j} : \bar{n}_i \to \bar{m}_j \Big| i \in \bar{k}, j = \phi(i)\Big\}$ a family of morphisms in *sFSet*. Then we can naturally define

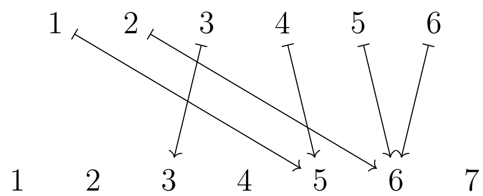
$$\begin{aligned} \bigoplus_{z=1}^{k} \phi_{z,\phi(z)} : \bigoplus_{q=1}^{k} \bar{n}_q &\longrightarrow \bigoplus_{s=1}^{l} \bar{m}_s \\ x &\longmapsto i_{\phi(p(x))} \circ \phi_{p(x),\phi(p(x))} \circ r(x), \end{aligned} \tag{1.5}$$

Remark 2. Denote by Δ_{alg} a category of finite ordinals (including the empty one) and order-preserving maps, Δ_{alg} is a subcategory of *sFSet*. Maps introduced in (1.2), (1.3), (1.4) are well-defined in Δ_{alg}, as well.

Example 3. In the notation of Example 1, let $\bar{m}_1 = \bar{4}, \bar{m}_2 = \bar{3}$ and $\phi : \bar{3} \to \bar{2}$ be a map given by $\phi(1) = 2, \phi(2) = 1, \phi(3) = 2$ and

$$\begin{array}{rccc} \phi_{1,2}: & \bar{n}_1 & \to & \bar{m}_2 \\ & 1 & \mapsto & 1 \\ & 2 & \mapsto & 2 \end{array} \qquad \begin{array}{rccc} \phi_{2,1}: & \bar{1} & \to & \bar{m}_1 \\ & 1 & \mapsto & 3 \end{array} \qquad \begin{array}{rccc} \phi_{3,2}: & \bar{3} & \to & \bar{m}_2 \\ & 1 & \mapsto & 1 \\ & 2 & \mapsto & 2 \\ & 3 & \mapsto & 2 \end{array}$$

Then $\bigoplus_{z=1}^{3} \phi_{z,\phi(z)} : \bigoplus_{q=1}^{3} \bar{n}_q \longrightarrow \bigoplus_{s=1}^{3} \bar{m}_s$ is the map of finite sets given by



We define the ith fiber $f^{-1}(i)$ of a morphism $f : T \to S$, $i \in S$, as the pullback of f along the map $\ulcorner i \urcorner : \bar{1} \to S$ which picks up the element i,

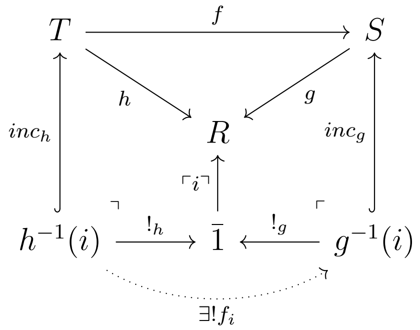
$$\begin{array}{ccc} f^{-1}(i) & \xrightarrow{\;!_f\;} & \bar{1} \\ {\scriptstyle inc_f}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \ulcorner i \urcorner} \\ T & \xrightarrow[\;f\;]{} & S \end{array}$$

so this is an object $f^{-1}(i) = \bar{n}_i \in$ *sFSet* which is isomorphic as a linearly ordered set to the preimage $\Big\{j \in T \mid f(j) = i\Big\}$. We observe that monomorphisms in *sFSet* are precisely injective maps, $\ulcorner i \urcorner : \bar{1} \to S$ is an injective map and monomorphisms are stable under pullbacks to conclude that $inc_f : f^{-1}(i) \to T$ is a monomorphism, as well. For clarity, we drop the index of the maps in the pullback if they are understood from the context.

Any commutative diagram in *sFSet*

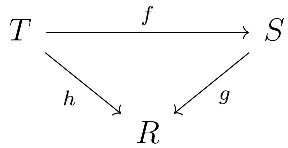
$$\begin{array}{ccc} T & \xrightarrow{\quad f \quad} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array}$$

then induces a unique map $f_i : h^{-1}(i) \to g^{-1}(i)$ of respective pullbacks



for any $i \in R$. This assignment is a functor $Fib_i : sFSet/R \to sFSet$. Moreover, the following lemma holds.

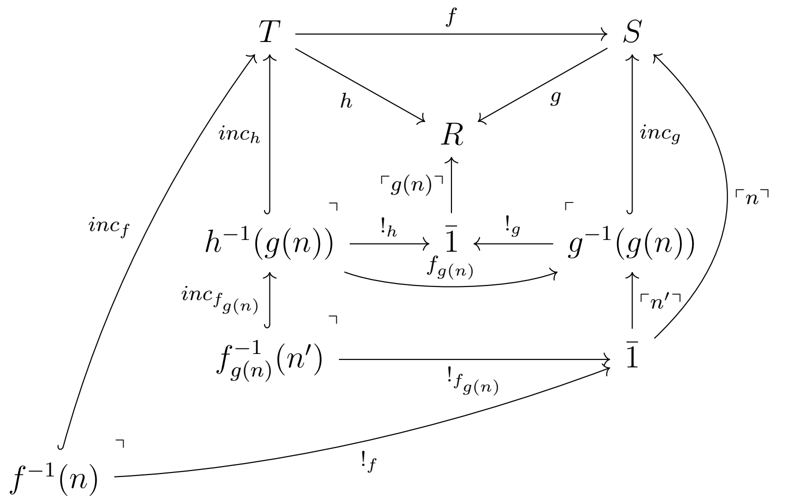
Lemma 4. *Let a commutative diagram*



in sFSet, $n \in S$ *and* $n' \in g^{-1}(g(n))$ *such that* $inc_g(n') = n$. *Then*

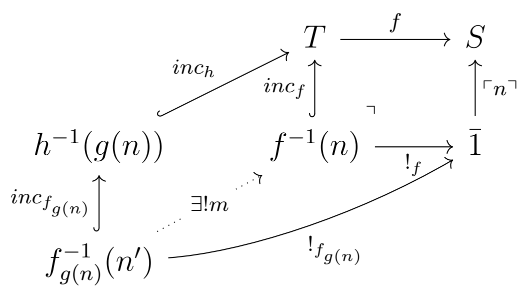
$$f^{-1}(n) = f^{-1}_{g(n)}(n').$$

Proof. We have a commutative diagram of pullbacks

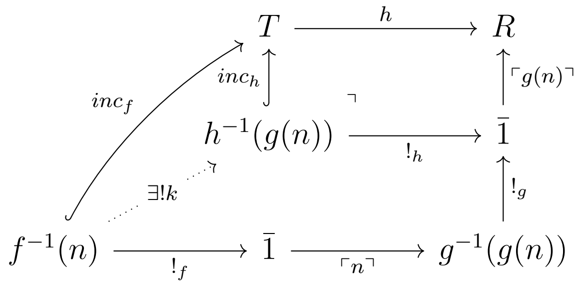


(1.6)

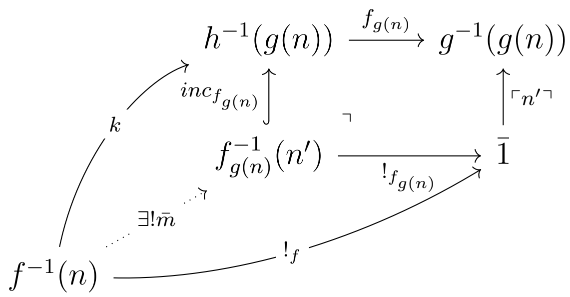
in *sFSet*. Then $(f^{-1}_{g(n)}(n'); inc_h \circ inc_{f_{g(n)}}, !_{f_{g(n)}})$ forms a cone above the following diagram and there is a unique map $m : f^{-1}_{g(n)}(n') \to f^{-1}(n)$ such that it is a morphism of respective cones.



Next, we observe that $(f^{-1}(n); inc_f, !_g \circ \ulcorner n' \urcorner \circ !_f)$ form a cone above the following diagram and hence there is a unique $k : f^{-1} \to h^{-1}(g(n))$ such that it is a morphism of respective cones.

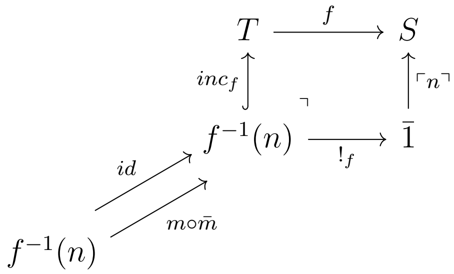


This gives rise to a cone $(f^{-1}(n); k, !_f)$ above the following diagram and a unique morphism $\bar{m} : f^{-1}(n) \to f^{-1}_{g(n)}(n')$ such that it is a morphism of respective cones.



To see that $(f^{-1}(n); k, !_f)$ is indeed a cone, it is enough to post-compose both its legs with a monomorphism $inc_g : g^{-1}(g(n)) \to S$.

Then we conclude that $m \circ \bar{m} = id$ by observing that both $id : f^{-1}(n) \to f^{-1}(n)$ and $m \circ \bar{m} : f^{-1}(n) \to f^{-1}(n)$ are morphisms from the same cone to a limiting cone.



Similarly, $\bar{m} \circ m = id$, which implies that both $m, \bar{m}$ are isomorphisms and since *sFSet* is skeletal, $f^{-1}(n) = f^{-1}_{g(n)}(n')$. ☐

We point out a technical remark that follows from the previous proof. It will be relevant later in Chapter 2.

Remark 5. In the situation of diagram (1.6), $inc_f = inc_h \circ inc_{f_{g(n)}}$.

The above structure on the category *sFSet* motivates the abstract definition of operadic categories which was introduced by Batanin and Markl in [1].

1.2 Operadic categories

We recall the original definition of an operadic category which was introduced in [1] with a minor correction in the statement of axioms (iv) and (v). We also recall several other definitions from the original paper and [4].

Definition 6. A strict operadic category is a category $\mathtt{O}$ equipped with a 'cardinality' functor $|\text{-}| : \mathtt{O} \to sFSet$ having the following properties. We require that each connected component of $\mathtt{O}$ has a chosen terminal object U_c, $c \in \pi_0(\mathtt{O})$. We also assume that for every $f : T \to S$ in $\mathtt{O}$ and every element $i \in |S|$ there is given an object $f^{-1}(i) \in \mathtt{O}$, which we will call *the i-th fiber* of f, such that $|f^{-1}(i)| = |f|^{-1}(i)$, denoted $f^{-1}(i) \triangleright f : T \to S$.

We also require that

(i) For any $c \in \pi_0(\mathtt{O})$, $|U_c| = 1$.

A trivial morphism $f : T \to S$ in $\mathtt{O}$ is a morphism such that, for each $i \in |S|$, $f^{-1}(i) = U_{d_i}$ for some $d_i \in \pi_0(\mathtt{O})$.

The remaining axioms for a strict operadic category are:

(ii) The identity morphism $id : T \to T$ is trivial for any $T \in \mathtt{O}$;

(iii) For any commutative diagram in $\mathtt{O}$

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array} \tag{1.7}$$

and every $i \in |R|$, one is given a map

$$f_i : h^{-1}(i) \to g^{-1}(i)$$

such that $|f_i| : |h^{-1}(i)| \to |g^{-1}(i)|$ is the map $|h|^{-1}(i) \to |g|^{-1}(i)$ of sets induced by

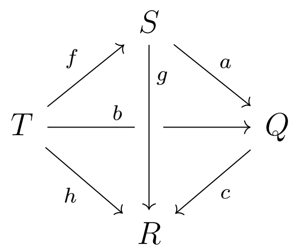
$$\begin{array}{ccc} |T| & \xrightarrow{|f|} & |S| \\ & {\scriptstyle |h|}\searrow \quad \swarrow{\scriptstyle |g|} & \\ & |R| & \end{array}$$

We moreover require that this assignment forms a functor $Fib_i : \mathtt{O}/R \to \mathtt{O}$. If $R = U_c$, the functor Fib_1 is required to be the domain functor $\mathtt{O}/R \to \mathtt{O}$.

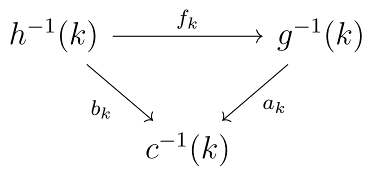
(iv) In the situation of (1.7), for any $i \in |S|$ and i' such that $i = inc(i')$ for $inc : |g|^{-1}(|g|(i)) \hookrightarrow |S|$, one has the equality

$$f^{-1}(i) = f^{-1}_{|g|(i)}(i'). \tag{1.8}$$

(v) Let



be a commutative diagram in $\mathtt{O}$ and let $i \in |Q|, k = |c|(i)$ and $i' \in |c^{-1}(k)|$ such that $inc(i') = i$ for $inc : |c^{-1}(k)| \hookrightarrow |Q|$. Then by axiom (iii) the diagram



commutes, so it induces a morphism $(f_k)_{i'} : b_k^{-1}(i') \to a_k^{-1}(i')$. By axiom (iv) we have

$$a^{-1}(i) = a_k^{-1}(i') \text{ and } b^{-1}(i) = b_k^{-1}(i').$$

We then require the equality

$$f_k = (f_k)_{i'}. \tag{1.9}$$

We will also assume that the set $\pi_0(\mathtt{O})$ of connected components is *small* with respect to a sufficiently big ambient universe.

Remark 7. It follows from axiom (iii) that the unique fiber of the canonical morphism $!_T : T \to U_c$ is T.

Example 8. The category Δ_{alg} of finite ordinals (including the empty one) and order-preserving maps has the obvious structure of an operadic category.

Example 9. Let $\mathfrak{C}$ be a set. A $\mathfrak{C}$-*bouquet* is a map $b : X+1 \to \mathfrak{C}$, where $X \in sFSet$. In other words, a $\mathfrak{C}$-bouquet is an ordered $(k+1)$-tuple $(c_1, ..., c_k; c), X = \bar{k}$, of elements of $\mathfrak{C}$. It can also be thought of as a planar corolla whose all edges (including the root) are colored. The extra color $b(1) \in \mathfrak{C}$ is called the *root color*. The finite set X is the *underlying set* of the bouquet b.

A map of $\mathfrak{C}$-bouquets $b \to c$ whose root colors coincide is an arbitrary map $f : X \to Y$ of their underlying sets. Otherwise there is no map between $\mathfrak{C}$-bouquets. We denote the resulting category of $\mathfrak{C}$-bouquets by $\mathtt{Bq}(\mathfrak{C})$.

The cardinality functor $|\text{-}| : \mathtt{Bq}(\mathfrak{C}) \to sFSet$ assigns to a bouquet $b : X+1 \to \mathfrak{C}$ its underlying set X. The fiber of a map $b \to c$ given by $f : X \to Y$ over an element $y \in Y$ is a $\mathfrak{C}$-bouquet whose underlying set is $f^{-1}(y)$, the root color coincides with the color of y and the colors of the elements are inherited from the colors of the elements of X. It is easy to see that $\mathtt{Bq}(\mathfrak{C})$ is an operadic category with $\mathfrak{C}$ its set of connected components.

The category $\mathtt{Bq}(\mathfrak{C})$ has the following important property.

Proposition 10. *For each operadic category $\mathtt{O}$ with its set of connected components $\pi_0(\mathtt{O}) = \mathfrak{C}$, there is a canonical operadic 'arity' functor $Ar : \mathtt{O} \to \mathtt{Bq}(\mathfrak{C})$ giving rise to the factorization*

$$\begin{array}{ccc} \mathtt{O} & \xrightarrow{\;|\text{-}|\;} & sFSet \\ & {\scriptstyle Ar}\searrow \quad \nearrow{\scriptstyle |\text{-}|} & \\ & \mathtt{Bq}(\mathfrak{C}) & \end{array}$$

of the cardinality functor $|\text{-}| : \mathtt{O} \to sFSet$.

Proof. We cite the construction of the Ar functor presented in [1, Part I, Section 1]. Let the *source* $s(T)$ of $T \in \mathtt{O}$ be the set of fibers of the identity $id : T \to T$. We define $Ar(T) \in \mathtt{Bq}(\mathfrak{C})$ as the bouquet $b : s(T) + 1 \to \mathfrak{C}$, where b associates to each fiber $U_c \in s(T)$ the corresponding connected component $c \in \mathfrak{C}$, and $b(1) := \pi_0(T)$. The assignment $T \mapsto Ar(T)$ extends into an operadic functor. $\square$

Definition 11. A *strict operadic functor* between operadic categories is a functor $F : \mathtt{O} \to \mathtt{P}$ over $sFSet$ which preserves fibers in the sense that

$$F\Big(f^{-1}(i)\Big) = F(f)^{-1}(i),$$

for any $f : T \to S \in \mathtt{O}$ and $i \in |S| = |F(S)|$. We also require that F preserves the chosen local terminal objects, and that $F(f_i) = F(f)_i$ for f as in (1.7).

1.3 Operads over operadic categories

Let $\mathtt{O}$ be an operadic category, $\mathscr{V}$ be a complete, cocomplete closed symmetric monoidal category. We recall a definition of an $\mathtt{O}$-operad in $\mathscr{V}$ along with several examples which we will make use of in Chapter 3. A *$\mathtt{O}$-collection* in $\mathscr{V}$ is a collection $E = \{E(T)\}_{T \in \mathtt{O}}$ of objects of $\mathscr{V}$ indexed by the objects of the category $\mathtt{O}$. For a $\mathtt{O}$-collection E and a morphism $f : T \to S$ in $\mathtt{O}$ let

$$E(f) = \bigotimes_{i \in |S|} E(f^{-1}(i)).$$

The notion of $\mathtt{O}$-collections allows us to introduce $\mathtt{O}$-operads.

Definition 12. An $\mathtt{O}$-operad is an $\mathtt{O}$-collection $\mathcal{P} = \{\mathcal{P}(T)\}_{T\in\mathtt{O}}$ in $\mathscr{V}$ together with units

$$\eta_c : I \to P(U_c),\ c \in \pi_0(\mathtt{O}),$$

and structure maps

$$\mu_{\mathcal{P}}(f) : \mathcal{P}(f) \otimes \mathcal{P}(S) \to \mathcal{P}(T),\ f : T \to S,$$

satisfying the following axioms.

(i) Let $T \xrightarrow{f} S \xrightarrow{g} R$ be morphisms in $\mathtt{O}$ and $h := gf : T \to R$. Then the following diagram of structure maps of $\mathcal{P}$ combined with the canonical isomorphisms of products in $\mathscr{V}$ commutes:

$$\begin{array}{ccc} \bigotimes_{i\in|R|} \mathcal{P}(f_i) \otimes \mathcal{P}(g) \otimes \mathcal{P}(R) & \xrightarrow{\bigotimes_i \mu_{\mathcal{P}}(f_i)\otimes id} & \mathcal{P}(h) \otimes \mathcal{P}(R) \\ \downarrow{\scriptstyle id\otimes\mu_{\mathcal{P}}(g)} & & \downarrow{\scriptstyle \mu_{\mathcal{P}}(h)} \\ \bigotimes_{i\in|R|} \mathcal{P}(f_i) \otimes \mathcal{P}(S) \cong \mathcal{P}(f) \otimes \mathcal{P}(S) & \xrightarrow[\mu_{\mathcal{P}}(f)]{} & \mathcal{P}(T) \end{array}$$

(ii) The composition

$$\begin{array}{ccccc} \mathcal{P}(T) & \longrightarrow & \bigotimes_{i\in|T|} I \otimes \mathcal{P}(T) & \longrightarrow & \bigotimes_{i\in|T|} U_{c_i} \otimes \mathcal{P}(T) \\ & & & & \downarrow{=} \\ & & & & \mathcal{P}(id_T) \otimes \mathcal{P}(T) \xrightarrow{\mu_{\mathcal{P}}(id)} \mathcal{P}(T) \end{array} \tag{1.10}$$

is the identity for each $T \in \mathtt{O}$, as well as the identity is

(iii) the composition

$$\mathcal{P}(T) \otimes I \longrightarrow \mathcal{P}(T) \otimes \mathcal{P}(U_c) \xrightarrow{=} \mathcal{P}(!_T) \otimes \mathcal{P}(U_c) \xrightarrow{\mu_{\mathcal{P}}(!_T)} \mathcal{P}(T), c = \pi_0 T.$$

A *morphism* $\phi : \mathcal{P}' \to \mathcal{P}''$ of $\mathtt{O}$-operads in $\mathscr{V}$ is a collection $\{\phi_T\}_{T\in\mathtt{O}}$ of $\mathscr{V}$-morphisms $\phi_T : \mathcal{P}'(T) \to \mathcal{P}''(T)$ commuting with the structure operations. We denote by $\mathtt{Op}^{\mathscr{V}}\mathtt{O}$ the category of $\mathtt{O}$-operads in $\mathscr{V}$. Each operadic functor $F : \mathtt{O} \to \mathtt{P}$ obviously induces the restriction $F^* : \mathtt{Op}^{\mathscr{V}}\mathtt{P} \to \mathtt{Op}^{\mathscr{V}}\mathtt{O}$.

We mention a few examples of operads over various operadic categories.

Example 13. The category of operads over the category *sFSet* is isomorphic to the category of classical symmetric operads. Similarly, the category of operads over the category $\mathtt{Bq}(C), C \in Set$ is isomorphic to the category of classical symmetric C-colored operads.

Example 14. For an operadic category $\mathtt{O}$, its terminal operad over *Set*, $\zeta_{\mathtt{O}} \in \mathtt{Op}^{Set}\mathtt{O}$ is the collection $\zeta_{\mathtt{O}}(T) = \{T\}, T \in \mathtt{O}$.

1.4 The category of k-trees

We briefly recall the category Ω_k of k-trees, for $k \geq 0$, citing [1]; the details can be found in [5, Sec. 3, Example 8] or [6]. The category of 0-*trees* Ω_0 is the terminal category 1. Its unique object is denoted by U_0.

The category of 1-*trees* Ω_1 is the category of finite ordinals (n) also denoted $\{1, ..., n\}$, $n \geq 0$, and their order-preserving maps. As usual, we interpret $\{1, \ldots, n\}$ for $n = 0$ as the empty set. The terminal object of Ω_1 is $U_1 := (1)$. When the meaning is clear from the context, we will simplify the notation and denote the object $(n) \in \Omega_1$ simply by n. The category Ω_1 is isomorphic to the operadic category Δ_{alg} in Example 8.

Let $k \geq 2$. A k-tree is a chain

$$T = \left(n_k \stackrel{t_{k-1}}{\longrightarrow} n_{k-1} \stackrel{t_{k-2}}{\longrightarrow} \ldots \stackrel{t_1}{\longrightarrow} n_1\right) \tag{1.11}$$

of morphisms in Ω_1. A morphism

$$\sigma : \ (n_k \stackrel{t_{k-1}}{\longrightarrow} n_{k-1} \stackrel{t_{k-2}}{\longrightarrow} \ldots \stackrel{t_1}{\longrightarrow} n_1) \longrightarrow (m_k \stackrel{s_{k-1}}{\longrightarrow} m_{k-1} \stackrel{s_{k-2}}{\longrightarrow} \ldots \stackrel{s_1}{\longrightarrow} m_1) \tag{1.12}$$

of k-trees is a commutative diagram in *sFSet*[1]

$$\begin{array}{ccccccc} n_k & \xrightarrow{t_{k-1}} & n_{k-1} & \xrightarrow{t_{k-2}} & \ldots & \xrightarrow{t_1} & n_1 \\ \sigma_k \downarrow & & \sigma_{k-1} \downarrow & & & & \sigma_1 \downarrow \\ m_k & \xrightarrow{s_{k-1}} & m_{k-1} & \xrightarrow{s_{k-2}} & \ldots & \xrightarrow{s_1} & m_1 \end{array}$$

such that

(i) σ_1 is order preserving and

(ii) for any p, $k > p \geq 1$, and $i \in n_p$, the restriction of σ_{p+1} to the preimage $t_p^{-1}(i)$ is order-preserving.

We denote by Ω_k the category of k-*trees* and their morphisms as defined above. Its terminal object is the k-tree $U_k := (1 \to 1 \to \ldots \to 1)$.

An s-*leaf* (or a *leaf of height* s) of a k-tree T as in (1.11) is, for $s = k$, by definition an element of n_k. For $1 \leq s < k$ an s-leaf is an element $i \in n_s$ such that $t_s^{-1}(i) = \emptyset$. We denote by $L_s(T)$ the set of all s-leaves of T.

Let $\sigma : T \to S$ be a map of k-trees as in (1.12) and $i \in m_k = L_k(S)$ a k-leaf of S. Let us define the fiber $\sigma^{-1}(i)$ over i as the chain

$$\left(\sigma_k^{-1}(i) \xrightarrow{t_{k-1}} \sigma_{k-1}^{-1}(s_{k-1}(i)) \xrightarrow{t_{k-2}} \ldots \xrightarrow{t_1} \sigma_1^{-1}(s_1 \ldots s_{k-1}(i))\right) \tag{1.13}$$

[1]in [1], morphisms of k-trees are defined as commutative diagrams in *Set*. However, the fibers of the vertical morphisms σ_i, $i \in \bar{k}$ are computed as pullback in *sFSet*.

of the restrictions of the maps in (1.11).

The category Ω_k with the cardinality functor $|T| := L_k(T)$, with the fibers defined as above and the chosen terminal object the k-tree U_k, is an operadic category.

2 Wreath product of operadic categories

Throughout the following chapter, $\mathtt{A}$ and $\mathtt{B}$ will be operadic categories and $\mathtt{B}$ is connected[1].

2.1 The construction

In this section, we define the wreath product of operadic categories along with several supplemental notions.

Definition 15. Wreath product of operadic categories $\mathtt{A} \wr \mathtt{B}$ is a category whose:

- objects are strings $(a; a_1, \cdots, a_k)$, where $a \in \mathtt{A}$, $a_1, \cdots, a_k \in \mathtt{B}$ where $k = |a|$;
- morphisms $f : (a; a_1, \cdots, a_k) \to (b; b_1, \cdots, b_l)$, are tuples $(\phi; \Phi)$, where

  $$\phi : a \to b$$

  is a morphism in $\mathtt{A}$, and

  $$\Phi = \{\phi_{i,j} : a_i \to b_j |\ 1 \leq i \leq |a|, 1 \leq j \leq |b|, |\phi|(i) = j\}$$

  is a family of morphisms in $\mathtt{B}$.

Remark 16. The definition above indeed yields a category, the composition of morphisms is defined component-wise.

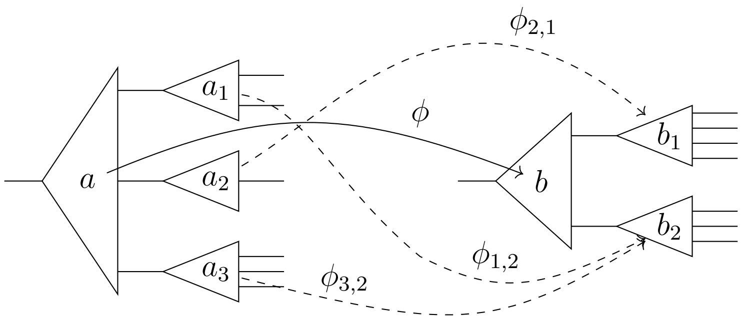


Figure 2.1: Visualization of objects and morphisms in $\mathtt{A} \wr \mathtt{B}$

Example 17. Let $a, b \in \mathtt{A}$ such that $|a| = \bar{3}$, $|b| = \bar{2}$ and $a_1, a_2, a_3, b_1, b_2 \in \mathtt{B}$. Then $(a; a_1, a_2, a_3), (b; b_1, b_2) \in \mathtt{A} \wr \mathtt{B}$.

[1]We provide a comment on this requirement in Remark 30.

Let $\phi : a \to b$ be a morphism in $\mathtt{A}$ such that $|\phi|(1) = 2, |\phi|(2) = 1, |\phi|(3) = 2$ and let $\phi_{1,2} : a_1 \to b_2, \phi_{2,1} : a_2 \to b_1, \phi_{3,2} : a_3 \to b_2$ be morphisms in $\mathtt{B}$. Then $(\phi; \{\phi_{1,2}, \phi_{2,1}, \phi_{3,2}\})$ is a morphism $(a; a_1, a_2, a_3) \to (b; b_1, b_2)$ in $\mathtt{A} \wr \mathtt{B}$.

Let $|a_1| = \bar{2}, |a_2| = \bar{1}, |a_3| = \bar{3}, |b_1| = \bar{4}, |b_2| = \bar{3}$. We illustrate objects of $\mathtt{A} \wr \mathtt{B}$ so that the bigger triangle (to the left) represents the object of $\mathtt{A}$, the smaller triangles represent objects $\mathtt{B}$ and the legs pointing out of the triangles represent the cardinality of the respective objects, see Figure 2.1.

Definition 18. We define the cardinality functor for $\mathtt{A} \wr \mathtt{B}$ in the following way. Let $(a; a_1, \cdots, a_k), (b; b_1, \cdots, b_l)$ be objects and $(\phi, \Phi) : (a; a_1, \cdots, a_k) \to (b; b_1, \cdots, b_l)$ be a morphism in $\mathtt{A} \wr \mathtt{B}$. Then we put

$$\begin{aligned} |-|: \quad \mathtt{A} \wr \mathtt{B} \quad &\longrightarrow \quad sFSet \\ (a; a_1, \cdots, a_k) &\longmapsto \bigoplus_{i=1}^{k} |a_i| \\ (\phi, \Phi) \quad &\longmapsto \bigoplus_{\substack{i=1 \\ j=\phi(i)}}^{k} |\phi_{i,j}|. \end{aligned}$$

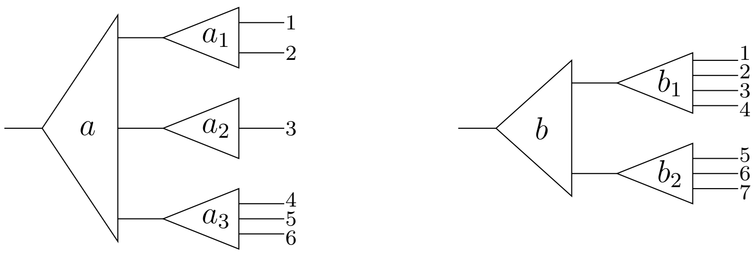


Figure 2.2: Visualization of the cardinalities of objects in $\mathtt{A} \wr \mathtt{B}$

Example 19. In the notation of Example 17, $|(a; a_1, a_2, a_3)| = \bar{6}$, $|(b; b_1, b_2)| = \bar{7}$. We identify elements in $|(a; a_1, a_2, a_3)|, |(b; b_1, b_2)|$ by enumerating legs pointing out of the smaller triangles, see Figure 2.2.

Definition 20. Let

$$f = (\phi, \Phi) : (a; a_1, \cdots, a_k) \to (b; b_1, \cdots, b_l)$$

be a morphism in $\mathtt{A} \wr \mathtt{B}$ and $x \in |(b; b_1, \cdots, b_l)|$ and $p_{(b;b_1,\cdots,b_l)}(x) = m$.

Then the x-th fiber $f^{-1}(x)$ is an object $(d; d_1, \cdots, d_n) \in \mathtt{A} \wr \mathtt{B}$, where

- $d = \phi^{-1}(m)$,
- $n = |\phi^{-1}(m)|$,
- $d_i = \phi^{-1}_{inc(i),m}(r_{(b;b_1,\cdots,b_l),m}(x))$, for $i \in |\phi^{-1}(m)|$ and $inc : |\phi^{-1}(m)| \hookrightarrow |a|$.

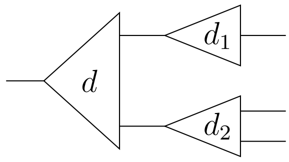


Figure 2.3: $f^{-1}(6) \triangleright (\phi; \{\phi_{1,2}, \phi_{2,1}, \phi_{3,2}\})$

Example 21. Let $(\phi, \Phi) : (a; a_1, a_2, a_3) \to (b; b_1, b_2)$ be as in Examples 17, 1, and $i = 6$. Then $f^{-1}(i) = (d; d_1, d_1)$, where $d = \phi^{-1}(2), |d| = 2, d_1 = \phi_{1,2}^{-1}(2), d_2 = \phi_{3,2}^{-1}(2))$, see Figure 2.3.

Corollary 22. *It is immediate from Definition 20 that the identity morphism* $id : (a; a_1, \cdots, a_k) \to (a; a_1, \cdots, a_k)$ *is trivial, for any* $(a; a_1, \cdots, a_k) \in \mathtt{A} \wr \mathtt{B}$.

2.2 The verification

In this section, we verify that the construction of the wreath product of operadic categories as presented in previous section is well-defined.

Proposition 23. *Let* $\mathtt{A}$*,* $\mathtt{B}$ *be operadic categories and* $\mathtt{B}$ *is connected. Then their wreath product* $\mathtt{A} \wr \mathtt{B}$ *is indeed an operadic category.*

We divide the proof of Proposition 23 into several lemmas.

Lemma 24. *Let* $\{U_c \in \mathtt{A} |\ c \in \pi_0\{\mathtt{A}\}\}$ *be a family of local terminal objects of* $\mathtt{A}$ *and* V *a terminal object of* $\mathtt{B}$*. Then* $\pi_0\{\mathtt{A} \wr \mathtt{B}\} \cong \pi_0\{\mathtt{A}\}$ *and* $\{(U_c; V)|\ c \in \pi_0\{\mathtt{A}\}\}$ *is a family of local terminal objects. Moreover,* $|(U_c; V)| = \bar{1}$ *for each such local terminal object.*

Proof. Immediate from the construction of $\mathtt{A} \wr \mathtt{B}$. □

Lemma 25. *Let*

$$\begin{array}{ccc} (a; a_1, \cdots, a_k) & \xrightarrow{f=(\theta;\Theta)} & (b; b_1, \cdots, b_l) \\ & {\scriptstyle h=(\psi;\Psi)} \searrow \quad \swarrow {\scriptstyle g=(\phi;\Phi)} & \\ & (c; c_1, \cdots, c_m) & \end{array} \tag{2.1}$$

be a commutative diagram in $\mathtt{A} \wr \mathtt{B}$*. Then the morphism*

$$f_n = (\delta; \Delta) : h^{-1}(n) \to g^{-1}(n), \tag{2.2}$$

where

- $\delta = \theta_t$,
- $\Delta = \left\{ (\theta_{inc(x),inc(y)})_{r(n)} \middle|\ x \in |\psi^{-1}(t)|, y = |\theta_t|(x) \right\}$.

is such that $|f_n| : |h^{-1}(n)| \to |g^{-1}(n)|$ *is the respective map of preimages as in axiom (iii) of Definition 6.*

Proof. Let $n \in |(c; w_1, \cdots, w_m)|$ and let $p(n) = t$, we explicitly describe the fibers

$$h^{-1}(n) = \left(\psi^{-1}(t); \left\{\psi^{-1}_{inc(k),t}(r(n))\right| \text{ for } k \in |\psi^{-1}(t)|\right\}\Big),$$

$$g^{-1}(n) = \left(\phi^{-1}(t); \left\{\phi^{-1}_{inc(k),t}(r(n))\right| \text{ for } k \in |\phi^{-1}(t)|\right\}\Big).$$

Diagram (2.1) gives rise to a commutative diagram $\in$ A,

$$\begin{array}{ccc} a & \xrightarrow{\theta} & b \\ & {}_{\psi}\searrow \quad \swarrow_{\phi} & \\ & c & \end{array}$$

which in turn induces a morphism $\theta_t : \psi^{-1}(t) \to \phi^{-1}(t)$ such that $|\theta_t|$ is a map of respective preimages $|\psi^{-1}(t)| \to |\phi^{-1}(t)|$ of $t \in |c|$ as shown in the following commutative diagram in *sFSet*.

$$\begin{array}{ccc} |\psi^{-1}|(t) & \xrightarrow{|\theta_t|} & |\phi^{-1}|(t) \\ {\scriptstyle inc}\big\downarrow & & \big\downarrow{\scriptstyle inc} \\ |a| & \xrightarrow[|\theta|]{} & |b| \end{array}$$

In particular, for every $x \in |\psi^{-1}(t)|, y \in |\phi^{-1}(t)|$, $|\theta_t|(x) = y$ if and only if $|\theta|(inc(x)) = inc(y)$.

Then, for every $x \in |\psi^{-1}(t)|, y = |\theta_t|(x) \in |\phi^{-1}(t)|$, there exists a component $\theta_{inc(x),inc(y)} : a_{inc(x)} \to b_{inc(y)}$ and the commutative diagram

$$\begin{array}{ccc} a_{inc(x)} & \xrightarrow{\theta_{inc(x),inc(y)}} & b_{inc(y)} \\ & {}_{\psi_{inc(x),t}}\searrow \quad \swarrow_{\phi_{inc(y),t}} & \\ & c_t & \end{array}$$

in B. Hence, there exists a map

$$(\theta_{inc(x),inc(y)})_{r(n)} : \psi^{-1}_{inc(x),t}(r(n)) \to \phi^{-1}_{inc(y),t}(r(n))$$

such that $|(\theta_{inc(x),inc(y)})_{r(n)}|$ is a respective map of the preimages

$$|\psi_{inc(x),t}|^{-1}(r(n)) \to |\phi_{inc(y),t}|^{-1}(r(n))$$

of $r(n) \in |w_t|$. Then there exists a morphism $f_n = (\delta; \Delta) : h^{-1}(n) \to g^{-1}(n)$, where

- $\delta = \theta_t$,

- $\Delta = \left\{ (\theta_{inc(x),inc(y)})_{r(n)} \middle| \; x \in |\psi^{-1}(t)|, y = |\theta_t|(x) \right\}$.

Moreover,

$$|f_n| = \bigoplus_{x \in \phi^{-1}(t)} |(\theta_{inc(x),inc(y)})_{r(n)}|,$$

so it is exactly the induced map of preimages $|h|^{-1}(n) \to |g|^{-1}(n)$ and therefore it is the desired morphism. ☐

Corollary 26. *For any $(b; b_1, \cdots, b_l) \in \mathtt{A} \wr \mathtt{B}$, $n \in |(b; b_1, \cdots, b_l)|$, the assignment*

$$\begin{array}{lccl} Fib_n : & (\mathtt{A} \wr \mathtt{B})/(b; b_1, \cdots, b_l) & \longrightarrow & \mathtt{A} \wr \mathtt{B} \\ & h : (a; a_1, \cdots, a_k) \to (b; b_1, \cdots, b_l) & \longmapsto & h^{-1}(n) \\ & \begin{array}{ccc} (a; a_1, \cdots, a_k) & \xrightarrow{\quad f \quad} & (b; b_1, \cdots, b_l) \\ & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle g} & \\ & (c; c_1, \cdots, c_m) & \end{array} & \longmapsto & f_n \end{array}$$

forms a functor. Moreover, if $R = (U_c; V)$, then Fib_1 is the domain functor.

Proof. Straightforward from construction (2.2) in Lemma 25. ☐

Remark 27. In combination, Lemma 25 and Corollary 26 prove that axiom (iii) from Definition 6 holds in $\mathtt{A} \wr \mathtt{B}$.

Lemma 28. *Axiom (iv) holds for $\mathtt{A} \wr \mathtt{B}$.*

Proof. In the situation of diagram (2.1), let $n \in |(b; b_1, \cdots, b_l)|$ such that $p(n) = t$, $n' \in g^{-1}(|g|(n))$ such that $inc(n') = n$ and $t' \in \phi^{-1}(t)$ such that $inc(t') = t$. Then by Lemma 25 there is a diagram

$$\begin{array}{ccccc} f^{-1}_{|g|(n)}(n') & \triangleright & h^{-1}(|g|(n)) & \xrightarrow{\quad f_{|g|(n)} \quad} & g^{-1}(|g|(n)) \\ & & \triangledown & & \triangledown \\ f^{-1}(n) & \triangleright & (a; a_1, \cdots, a_k) & \xrightarrow{\quad f \quad} & (b; b_1, \cdots, b_l) \\ & & & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle g} & \\ & & & (c; c_1, \cdots, c_m) & \end{array} \tag{2.3}$$

in $\mathtt{A} \wr \mathtt{B}$. We write the fibers explicitly following the construction (2.2).

$$f^{-1}(n) = \left(\theta^{-1}(t); \left\{ \theta^{-1}_{inc(k),t}(r(n)) \middle| \; \text{for } k \in |\theta^{-1}(t)| \right\} \right),$$

$$f^{-1}_{|g|(n)}(n') = \left(\theta^{-1}_{|\phi|(t)}(t'), \left\{ (\theta_{inc(inc(k)),inc(t')})^{-1}_{r(|g|(n))}(r(n')) \middle| \; \text{for } k \in |\theta^{-1}_{|\phi|(t)}(t')| \right\} \right).$$

Diagram 2.3 induces the following diagram

$$\begin{array}{ccccc}
\theta^{-1}_{|\phi|(t)}(t') & \rhd & \psi^{-1}(|\phi|(t)) & \xrightarrow{\theta_{|\phi|(t)}} & \phi^{-1}(|\phi|(t)) \\
\| & & \triangledown & & \triangledown \\
\theta^{-1}(t) & \rhd & a & \xrightarrow{\theta} & b \\
 & & & \searrow^{\psi} \quad \swarrow^{\phi} & \\
 & & & c &
\end{array}$$

in $\mathtt{A}$. Then by axiom (iv), the equality $\theta^{-1}_{|\phi|(t)}(t') = \theta^{-1}(t)$ holds.

Next, we modify the expression of $f^{-1}_{|g|(n)}(n')$ using that $j(t') = t$ and that by Remark 5

$$\begin{array}{ccc}
|\theta_t^{-1}(t')| & \overset{j}{\hookrightarrow} & |\psi^{-1}(\phi(t))| \\
\| & & \big\downarrow j \\
|\theta^{-1}(t)| & \underset{j}{\hookrightarrow} & |a|
\end{array}$$

commutes in *sFSet*. Hence,

$$f^{-1}_{|g|(n)}(n') = \left(\theta^{-1}_{|\phi|(t)}(t'), \left\{ (\theta_{inc(k),t})^{-1}_{r(|g|(n))}(r(n')) \Big| \text{ for } k \in |\theta^{-1}(t)| \right\} \right).$$

Moreover, for $x \in |\theta^{-1}(t)|$, diagram (2.3) induces the following diagram

$$\begin{array}{ccccc}
(\theta_{inc(x),t})^{-1}_{r(|g|(n))}(r(n')) & \rhd & {\psi_{inc(x),|\phi|(t)}}^{-1}(r(|g|(n))) & \xrightarrow{(\theta_{inc(x),t})_{r(|g|(n))}} & {\phi_{t,|\phi|(t)}}^{-1}(r(|g|(n))) \\
\| & & \triangledown & & \triangledown \\
{\theta_{inc(x),t}}^{-1}(r(n)) & \rhd & a_{inc(x)} & \xrightarrow{\theta_{inc(x),t}} & b_t \\
 & & & \searrow^{\psi_{inc(x),|\phi|(t)}} \quad \swarrow^{\phi_{t,|\phi|(t)}} & \\
 & & & c_{|\phi|(t)} &
\end{array}$$

in $\mathtt{B}$. Again by axiom (iv) the equality $(\theta_{inc(x),t})^{-1}_{r(|g|(n))}(r(n')) = {\theta_{inc(x),t}}^{-1}(r(n))$ holds.

□

Lemma 29. *Axiom (v) holds for* $\mathtt{A} \wr \mathtt{B}$.

Proof. Let

$$\begin{array}{ccccc}
 & & (s; s_1, \cdots, s_l) & & \\
 & \nearrow^{f=(\theta;\{\theta_{i,j}\})} & \big\downarrow g=(\phi;\{\phi_{i,j}\}) & \searrow^{a=(\alpha;\{\alpha_{i,j}\})} & \\
(t; t_1, \cdots, t_k) & & \xrightarrow{b=(\beta;\{\beta_{i,j}\})} & & (q; q_1, \cdots, q_n) \\
 & \searrow_{h=(\psi;\{\psi_{i,j}\})} & & \swarrow_{c=(\gamma;\{\gamma_{i,j}\})} & \\
 & & (w; w_1, \cdots, w_m) & &
\end{array}$$

be a commutative diagram in $\mathtt{A} \wr \mathtt{B}$, let $x \in |(q; q_1, \cdots, q_n)|$ such that $p(x) = z$, $y = |c|(x)$, then $p(y) = |\gamma|(z)$.

Then by corollary 26, there is a commutative triangle of fibers

$$\begin{array}{ccccc} & & g^{-1}(y) & & \\ & \overset{f_y}{\nearrow} & & \overset{a_y}{\searrow} & \\ h^{-1}(y) & & \xrightarrow[b_y]{\hspace{6em}} & & c^{-1}(y) \end{array}$$

Let $x' \in |c^{-1}(y)|$ be unique such that $j(x') = x$, than $p(x') = z'$, where $z' \in |\gamma^{-1}(|\gamma|(y))|$ unique such that $j(z') = z$.

The objective is to establish equality between the morphisms $(f_y)_{x'}$ and f_x. Lemma 28 indicates that their domains and codomains are equal.

$$\begin{array}{ccc} & & a_y^{-1}(x') \\ & \overset{(f_y)_{x'}}{\nearrow} & \parallel \\ b_y^{-1}(x') & & a^{-1}(x) \\ \parallel & \underset{f_x}{\nearrow} & \\ b^{-1}(x) & & \end{array} \tag{2.4}$$

We carefully expand Definition 20 and apply the construction (2.2), thereby uncovering that the equality of the morphisms on diagram (2.4) is equivalent to the equality of the following two expressions

$$(f_y)_{x'} = \left((\theta_{|\gamma|(z)})_{z'}; \left\{ (\theta_{j(u), |(\theta_{|\gamma|(z)})_{z'}|(j(u))})_{r(y)})_{r(x')} \right| \text{ for each } u \in |\beta^{-1}_{|\gamma|(z)}(z')| \right\} \right),$$

$$f_x = \left(\theta_{|\gamma|(z)}; \left\{ \theta_{j(u), j(\theta_{|\gamma|(z)}(u))})_{r(y)} \right| \text{ for each } u \in |\beta^{-1}(z)| \right\} \right).$$

In both expressions, the two components are equal, given that $\mathtt{A}$ and $\mathtt{B}$ are operadic and thus axiom (v) holds in both cases. □

This concludes the proof of Proposition 23.

Remark 30. The connectivity of $\mathtt{B}$ was necessary for the existance of local terminal objects in $\mathtt{A} \wr \mathtt{B}$. The remaining axioms of operadic categories are fulfilled without this assumption. This relaxed version of operadic categories was considered in [7].

2.3 Some properties

In this section, we explore some properties of the wreath product of operadic categories.

Proposition 31. *The wreath product of operadic categories is not commutative.*

Proof. Let $\mathtt{A}, \mathtt{B}$ be operadic categories and $\mathtt{B}$ be connected. In case $\mathtt{A}$ is not connected, the statement is immediate.

Otherwise, let $\mathtt{A} = \mathbf{1}$, i.e., the one-object category, $\mathtt{B} = \mathbf{2}$, i.e., the two-object category with one morphism between the objects, and equip both $\mathtt{A}$ and $\mathtt{B}$ with trivial cardinality functors. Then $\mathtt{A} \wr \mathtt{B} \cong \mathtt{A} \ncong \mathtt{B} \cong \mathtt{B} \wr \mathtt{A}$. ☐

Proposition 32. *The wreath product of operadic categories is associative.*

Proof. Let $\mathtt{A}$ be an operadic category and $\mathtt{B}$, $\mathtt{C}$ be connected operadic categories. Moreover, let

- $a, u \in \mathtt{A}$, $|a| = k$, $|u| = m$;
- $b_1, \cdots, b_k, v_1, \cdots, v_m \in \mathtt{B}$;
- $|(a; b_1, \cdots, b_k)| = |b_1| + \cdots + |b_k| = l$, $|(u; v_1, \cdots, v_m)| = |v_1| + \cdots + |v_m| = n$;
- $c_1, \cdots, c_l, w_1, \cdots, w_n \in \mathtt{C}$;
- $g : (a; b_1, \cdots, b_k) \to (u; v_1, \cdots, v_m)$ be a morphism in $\mathtt{A} \wr \mathtt{B}$ given by a morphism $\phi : a \to u$ in $\mathtt{A}$ and a family of morphisms $\{\phi_{i,j} : b_i \to v_j |\ i \in |a|, |\phi|(i) = j\}$ in $\mathtt{B}$;
- $G = \{g_{i,j} : c_i \to w_j |\ i \in |(a; b_1, \cdots, b_k)|, |g|(i) = j\}$ be a family of morphisms in $\mathtt{C}$.

This data defines objects

$$X = ((a; b_1, \cdots, b_k) ; c_1, \cdots, c_l) , Y = ((u; v_1, \cdots, v_m) ; w_1, \cdots, w_n)$$

and a morphism

$$f = (g; G) : X \to Y$$

in $(\mathtt{A} \wr \mathtt{B}) \wr \mathtt{C}$. Define an action on objects $F : (\mathtt{A} \wr \mathtt{B}) \wr \mathtt{C} \to \mathtt{A} \wr (\mathtt{B} \wr \mathtt{C})$ by

$$FX = \left(a; \left\{ (b_u; \left\{ c_d \middle|\ 1 + \sum_{d=1}^{u-1} |b_d| \leq d \leq \sum_{d=1}^{u} |b_d| \right\}) \middle|\ u \in |a| \right\} \right).$$

Next, we define the action on morphisms, which reassembles f into

$$Ff : FX \longrightarrow FY.$$

Following the previous definition,

$$FY = \left(u; \left\{ (v_h; \left\{ w_d \middle|\ 1 + \sum_{d=1}^{h-1} |v_d| \leq d \leq \sum_{d=1}^{h} |v_d| \right\}) \middle|\ h \in |u| \right\} \right).$$

We set the first component of Ff to be ϕ and the second component to be a family of pairs

$$\Big\{(\phi_{i,j};\Phi_{i,j}) \mid i \in |a|, |\phi|(i) = j\Big\}, \text{ where}$$

$$\Phi_{i,j} = \{g_{x,y} \mid z \in |b_i|, q = |\phi_{i,j}|(z), x = z + \sum_{d=1}^{i-1} |b_d|, y = q + \sum_{d=1}^{j-1} |b_d|\}.$$

We observe that Ff is well-defined in $\mathtt{A} \wr (\mathtt{B} \wr \mathtt{C})$, and that so defined action on morphisms preserves identities and compositions. Hence, F is a functor.

Next, we show that F is an operadic functor. It is straightforward to see that F preserves cardinalities of objects and morphisms, as well as chosen local terminal objects. To verify that F preserves fibers, let

$$f = (g;G) : ((a;b_1,\cdots,b_k);c_1,\cdots,c_l) \to ((u;v_1,\cdots,v_m);w_1,\cdots,w_n)$$

as described before, and $i \in |((u;v_1,\cdots,v_m);w_1,\cdots,w_n)|$ such that

$$p(i) = s \in \bigoplus_{d=1}^{l} |w_d| \text{ and } p(s) = t \in \bigoplus_{d=1}^{m} |v_d|.$$

Then

$$\begin{aligned} f^{-1}(i) &= \Big(g^{-1}(s); \{g^{-1}_{inc(e),s}(r(i)) \mid e \in |g^{-1}(s)|\}\Big) \\ &= \Big((\phi^{-1}(t); \{\phi^{-1}_{inc(h),t}(r(s)) \mid h \in |\phi^{-1}(t)|\}); \{g^{-1}_{inc(e),s}(r(i)) \mid e \in |g^{-1}(s)|\}\Big). \end{aligned}$$

and consequently

$$\begin{aligned} F(f^{-1}(i)) &= \left(\phi^{-1}(t); \Big\{(\phi^{-1}_{inc(u),t}(r(s)); G_u) \mid u \in |\phi^{-1}(t)|\Big\}\right) \\ &= (Ff)^{-1}(i), \end{aligned}$$

where

$$G_u = \{g^{-1}_{inc(e),s}(r(i)) \mid 1 + \sum_{d=1}^{u-1} |\phi^{-1}_{inc(d),t}(r(s))| \le e \le \sum_{d=1}^{u} |\phi^{-1}_{inc(d),t}(r(s))|\}.$$

Analogously, F preserves induced morphisms between fibers defined in (1.7).

Lastly, F has a clear operadic inverse. □

Proposition 33. *Let $k \in \mathbb{N}$, then $\Omega_k \wr \Omega_1 \cong \Omega_{k+1}$.*

Proof. For an object $N = ((n_k \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_1} n_1); (a_1), \cdots, (a_{n_k}))$ of $\Omega_k \wr \Omega_1$, we define

$$n_{k+1} = \bigoplus_{i=1}^{n_k} (a_i)$$

as in (1.1) and a map

$$t_k : (n_{k+1}) \longrightarrow (n_k) \tag{2.5}$$

by the assignment $t_k(x) = p(x)$ as was defined in (1.3). We notice that t_k is an order-preserving morphism, so it is a morphism in Ω_1.

Let $M = ((m_k \xrightarrow{s_{k-1}} \cdots \xrightarrow{s_1} m_1); (b_1), \cdots, (b_{m_k}))$ also be an object in $\Omega_k \wr \Omega_1$ and

$$(\theta, \Theta) : N \to M$$

a morphism in $\Omega_k \wr \Omega_1$, i.e. it is a family of morphisms

$$\{\theta_i : (n_i) \to (m_i) |\ i \in k\}$$

in *sFSet* satisfying (1.12) together with a family of morphisms

$$\{\theta_{i,j} : (a_i) \to (b_j) |\ i \in n_k, j = \theta_k(i)\}$$

in Ω_1. For such a morphism (θ, Θ), we define a map

$$\theta_{k+1} = \bigoplus_{z=1}^{n_k} \theta_{z,\theta_k(z)}$$

as in (1.5). We notice that, for $z \in n_k$

$$\theta_{k+1} \restriction t_k^{-1}(z) = \theta_{k+1} \restriction i_z(a_z)$$

is order-preserving. We now can define a functor

$$\begin{array}{rrcl} F: & \Omega_k \wr \Omega_1 & \longrightarrow & \Omega_{k+1} \\ & N & \longmapsto & (n_{k+1} \xrightarrow{t_k} n_k \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_1} n_1) \\ & (\theta, \Theta) & \longmapsto & \theta \cup \{\theta_{k+1}\}. \end{array}$$

The functor F preserves cardinalities since

$$|N| = \Sigma_{i=1}^{n_k} |a_i| = \Sigma_{i=1}^{n_k} a_i = |n_{k+1}| = |FN|,$$

$$|(\theta, \Theta)| = \theta_{k+1} = |F(\theta, \Theta)|.$$

To show that F preserves fibers, let $x \in |M| = |FM|$ such that $p(x) = y$. Then the x-th fiber $(\theta, \Theta)^{-1}(x)$ equals

$$\left(\left(\theta_k^{-1}(y) \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_1} \theta_1^{-1}(s_1 \cdots s_{k-1}(y)) \right); \left\{ \theta_{inc(k),y}^{-1}(r(x)) \middle|\ k \in |\theta^{-1}(y)| \right\} \right),$$

and consequently $F((\theta, \Theta)^{-1}(x))$ is equal to

$$\bigoplus_{k \in \theta_k^{-1}(y)} \theta_{inc(k),y}^{-1}(r(x)) \xrightarrow{t_k} \theta_k^{-1}(y) \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_1} \theta_1^{-1}(s_1 \cdots s_{k-1}(y)).$$

On the other hand, $(F(\theta, \Theta))^{-1}(x)$ is equal to

$$\theta_{k+1}^{-1}(x) \xrightarrow{t_k} \theta_k^{-1}(t_k(x)) \xrightarrow{s_{k-1}} \cdots \xrightarrow{t_1} \theta_k^{-1}(s_1 \cdots s_k(x)),$$

where we observe that

- $\theta_{k+1}^{-1}(x)$ is isomorphic (as a linearly ordered set) to the preimage of x under the map θ_{k+1}, which in turn is isomorphic (as a linearly ordered set) to

$$\left\{(k, x') \middle| \; k \in \theta_k^{-1}(y), x' \in \theta_{inc(k),y}^{-1}(r(x))\right\}$$

together with lexicographic order. Finally, the last linearly ordered set is isomorphic to $\bigoplus_{k \in \theta_k^{-1}(y)} \theta_{inc(k),y}^{-1}(r(x))$.

- $s_k(x) = y$ by definition.

By construction, the functor F preserves the terminal object. To show that F preserves induced morphisms between fibers as in (1.7), let

$$L = ((l_k \xrightarrow{q_{k-1}} \cdots \xrightarrow{q_1} l_1); (c_1), \cdots, (c_{l_k})) \in \Omega_k \wr \Omega_1,$$

$x \in |L|$ such that $p(x) = y$, and

$$\begin{array}{ccc} N & \xrightarrow{(\theta,\Theta)} & M \\ & {\scriptstyle (\psi,\Psi)} \searrow \quad \swarrow {\scriptstyle (\phi,\Phi)} & \\ & L & \end{array}$$

a commutative diagram in $\in \Omega_k \wr \Omega_1$, we write the induced morphism

$$(\theta, \Theta)_x : (\psi, \Psi)^{-1}(x) \to (\phi, \Phi)^{-1}(x)$$

explicitely

$$\begin{array}{c} \left(\left(\psi_k^{-1}(y) \xrightarrow{t_{k-1}} \cdots \xrightarrow{t_1} \psi_1^{-1}(t_1 \cdots t_{k-1}(y))\right); \left\{\psi_{inc(d),y}(r(x)) \middle| \; d \in \psi_k^{-1}(y)\right\}\right) \\ \Big\downarrow {\scriptstyle \left(\theta_y; \left\{ \left(\theta_{inc(d),inc(e)}\right)_{r(x)} \middle| \; d \in \psi_k^{-1}(y), e = (\theta_k)_y(d) \right\}\right)}, \\ \left(\left(\phi_k^{-1}(y) \xrightarrow{s_{k-1}} \cdots \xrightarrow{s_1} \phi_1^{-1}(s_1 \cdots s_{k-1}(y))\right); \left\{\phi_{inc(e),y}(r(x)) \middle| \; e \in \phi_k^{-1}(y)\right\}\right) \end{array}$$

where the morphism θ_y is a commutative diagram

$$\begin{array}{ccccccc} \psi_k^{-1}(y) & \xrightarrow{t_{k-1}} & \psi_{k-1}^{-1}(q_{k-1}(y)) & \xrightarrow{t_{k-2}} & \cdots & \xrightarrow{t_1} & \psi_1^{-1}(q_1 \cdots q_{k-1}(y)) \\ {\scriptstyle (\theta_k)_y}\downarrow & & \downarrow{\scriptstyle (\theta_{k-1})_{q_{k-1}(y)}} & & & & \downarrow{\scriptstyle (\theta_1)_{q_1 \cdots q_{k-1}(y)}} \\ \phi_k^{-1}(y) & \xrightarrow[s_{k-1}]{} & \phi_{k-1}^{-1}(q_{k-1}(y)) & \xrightarrow[s_{k-2}]{} & \cdots & \xrightarrow[s_1]{} & \phi_1^{-1}(q_1 \cdots q_{k-1}(y)) \end{array}$$

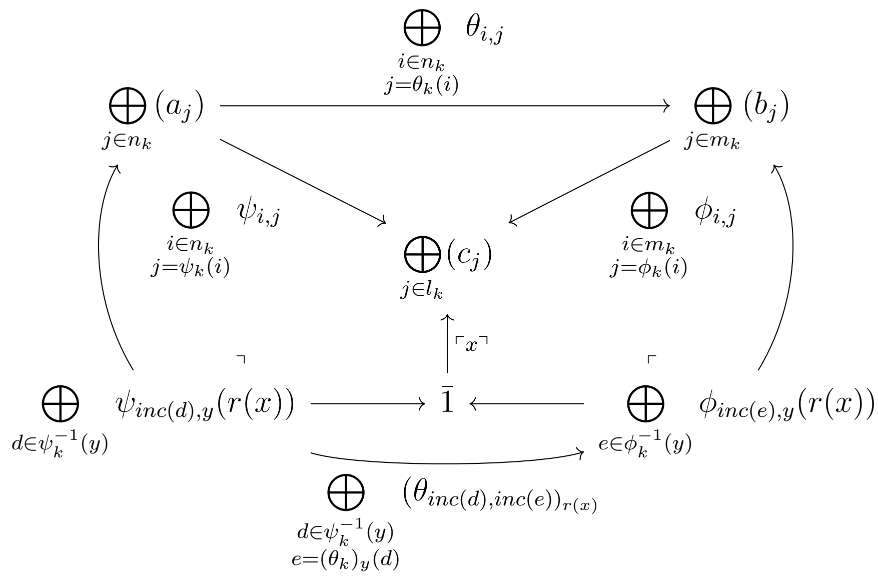
in *sFSet*. The induced morphism between fibers

$$\begin{array}{ccc} F(\psi, \Psi)^{-1}(x) & \xrightarrow{F(\theta,\Theta)_x} & F(\phi, \Phi)^{-1}(x) \\ \triangledown & & \triangledown \\ FN & \xrightarrow{F(\theta,\Theta)} & FM \\ & {\scriptstyle F(\psi,\Psi)} \searrow \quad \swarrow {\scriptstyle F(\phi,\Phi)} & \\ & FL & \end{array}$$

equals

$$\begin{array}{ccccccc}
\psi_{k+1}^{-1}(x) & \xrightarrow{t_k} & \psi_k^{-1}(y) & \xrightarrow{t_{k-1}} & \cdots & \xrightarrow{t_1} & \psi_1^{-1}(q_1 \cdots q_{k-1}(y)) \\
\big\downarrow{\scriptstyle (\theta_{k+1})_x} & & \big\downarrow{\scriptstyle (\theta_k)_y} & & & & \big\downarrow{\scriptstyle (\theta_1)_{q_1 \cdots q_{k-1}(y)}} \\
\phi_{k+1}^{-1}(x) & \xrightarrow[s_k]{} & \phi_k^{-1}(y) & \xrightarrow[s_{k-1}]{} & \cdots & \xrightarrow[s_1]{} & \phi_1^{-1}(q_1 \cdots q_{k-1}(y))
\end{array}.$$

Showing that the functor F preserves induced morphisms between fibers reduces to showing that the induced morphism between pullbacks $\psi_{k+1}^{-1}(x)$ and $\phi_{k+1}^{-1}(x)$ equals to $\bigoplus_{\substack{d \in \psi_k^{-1}(y) \\ e=(\theta_k)_y(d)}} (\theta_{inc(d),inc(e)})_{r(x)}$. Since the diagram



commutes in *sFSet*, we are done. Lastly, F has a clear operadic inverse. □

The next statement follows from Propositions 32 and 33.

Corollary 34. *Let $l, k \in \mathbb{N}$, then $\Omega_l \wr \Omega_k \cong \Omega_{l+k}$.*

3 Application to Boardman-Vogt tensor product of operads

3.1 Operadic Grothendieck construction

We recall the *operadic Grothendieck construction* [1]. Let $\mathtt{O}$ be an operadic category and an $\mathtt{O}$-operad $\mathcal{P} \in \mathtt{Op}^{Set}\mathtt{O}$. One then has the category $\int_{\mathtt{O}} \mathcal{P}$ whose objects are $t \in \mathcal{P}(T)$ for some $T \in \mathtt{O}$. A morphism $\sigma : t \to s$ from $t \in \mathcal{P}(T)$ to $s \in \mathcal{P}(S)$ is a pair (ε, f) consisting of a morphism $f : T \to S$ in $\mathtt{O}$ and of some $\varepsilon \in \times_{i \in |S|} \mathcal{P}(f^{-1}(i))$, such that

$$\mu_{\mathcal{P}}(f)(\varepsilon, s) = t,$$

where $\mu_{\mathcal{P}}$ is the structure map of the operad $\mathcal{P}$. Compositions of morphisms are defined in the obvious manner. The category $\int_{\mathtt{O}} \mathcal{P}$ thus constructed is clearly an operadic category. We use notation $\mathbb{I}(\mathcal{P}) = \int_{\mathtt{O}} \mathcal{P}$.

Lemma 35. *Let $\mathcal{Y}$ be an sFSet-operad in Set. Then $\mathbb{I}(\mathcal{Y})$ has a terminal object and therefore is a connected operadic category.*

Proof. The operad $\mathcal{Y}$ is provided with a unit $\eta : \mathbb{1} \to \mathcal{Y}(\bar{1})$, let $y = \eta(*)$ and $x \in \mathcal{Y}(\bar{n})$. We show that there is a unique morphism $!_x : x \to y$ in $\mathbb{I}(Y)$.

Let $!_{\bar{n}} : \bar{n} \to \bar{1}$ be the unique morphism to the terminal object. By Remark 7, its unique fiber $!_{\bar{n}}^{-1}(1) = \bar{n}$. From condition (1.10), it immediately follows that (x) is unique ε such that $\mu_{\mathcal{Y}}(!_{\bar{n}})(\varepsilon, y) = x$. Put $!_x = (!_{\bar{n}}, (x))$. Therefore y is the terminal object of $\mathbb{I}(Y)$. □

3.2 The adjunction between categories of operads

By Proposition 10, any operadic category $\mathtt{O}$ with $C = \pi_0(\mathtt{O})$ comes with a canonical factorisation of the cardinality functor.

$$\begin{array}{ccc} \mathtt{O} & \xrightarrow{|\text{-}|} & sFSet \\ & {\scriptstyle Ar}\searrow \quad \nearrow{\scriptstyle |\text{-}|} & \\ & \mathtt{Bq}(C) & \end{array}$$

In case $Ar : \mathtt{O} \to \mathtt{Bq}(C)$ is a so-called *discrete operadic fibration* [1, Definition 2.1.], it induces an adjoint pair

$$Op^{\mathcal{O}}(Set) \underset{Ar^*}{\overset{Ar_!}{\rightleftarrows}} Op^{\mathtt{Bq}(C)}(Set) \ .$$

In this section, we do not assume that $Ar : \mathtt{O} \to \mathtt{Bq}(C)$ is a discrete operadic fibration and build a similar adjoint pair of functors between corresponding categories of operads.

We will use the following fact from [8].

Theorem 36 (Theorem 20.3.22.)**.** *There is a free-forgetful adjunction*

$$\mathscr{V}^{Prof(\mathfrak{C})\times\mathfrak{C}} \underset{U^{\Sigma}}{\overset{F^{\Sigma}}{\rightleftarrows}} Operad^{\Sigma(\mathfrak{C})(\mathscr{V})} ,$$

where $\mathscr{V}^{Prof(\mathfrak{C})\times\mathfrak{C}}$ is the category of $Prof(\mathfrak{C}) \times \mathfrak{C}$-colored objects [8, Example 9.4.4.] and $Operad^{\Sigma(\mathfrak{C})(\mathscr{V})}$ is the category of $\mathfrak{C}$-colored symmetric operads.

Let $\mathcal{Q} \in Op^{\mathtt{O}}(Set)$ together with structure maps $\mu_{\mathcal{Q}}(f)$ for every morphism f in $\mathtt{O}$. We define a $\mathtt{Bq}(\mathfrak{C})$-collection in *Set* and notice that $\mathtt{Bq}(\mathfrak{C})$-collections correspond to $Prof(\mathfrak{C}) \times \mathfrak{C}$-colored objects in the notation of [8].

$$E_{\mathcal{Q}}(T) = \coprod_{Ar(t)=T} \mathcal{Q}(t).$$

Then we construct a free operad

$$\mathcal{F}_{\mathcal{Q}} = F^{\Sigma}(E_{\mathcal{Q}}),$$

this is a $\mathfrak{C}$-colored operad, hence a $\mathtt{Bq}(\mathfrak{C})$-operad together with structure maps $\mu_{\mathcal{F}}(g)$ for every morphism g in $\mathtt{Bq}(\mathfrak{C})$.

For $N \in \mathtt{Bq}(\mathfrak{C})$, we generate an equivalence relation $\sim_N$ in the following way. Let $F_1, \cdots, F_k, M$ be bouquets in $\mathtt{Bq}(\mathfrak{C})$ such that $|M| = k$, and let $g : N \to M$ be a morphism with respective fibers $g^{-1}(i) = F_i, 1 \leq i \leq k$. Then there is structure map

$$\mu_{\mathcal{F}}(g) : \times_{i\in|M|}\mathcal{F}_{\mathcal{Q}}(F_i) \times \mathcal{F}_{\mathcal{Q}}(M) \to \mathcal{F}_{\mathcal{Q}}(N).$$

Let moreover $f_1, \cdots, f_k, n, m \in \mathtt{O}$ and a morphism $f : \bar{n} \to \bar{m}$ with respective fibers $f^{-1}(i) = f_i, 1 \leq i \leq k$ such that $Ar(f) = g$, and elements $x_i \in \mathcal{Q}(f_i)$ for $1 \leq i \leq k$ and $y \in \mathcal{Q}(m)$. Again, there is structure map

$$\mu_{\mathcal{Q}}(f) : \times_{i\in|m|}\mathcal{Q}(f_i) \times \mathcal{Q}(m) \to \mathcal{Q}(n).$$

We put

$$\mu_{\mathcal{F}}(g)(x_1, \cdots, x_m, y) \sim_n \mu_{\mathcal{Q}}(f)(x_1, \cdots, x_m, y) \sim_N .$$

This lets us define a functor $Ar_! : Op^{\mathtt{O}}(Set) \to Op^{\mathtt{Bq}(C)}(Set)$ by its action on objects

$$Ar_!(\mathcal{Q}) = {}^{\mathcal{F}_{\mathcal{Q}}}\!/\!_{\sim} \tag{3.1}$$

and the induced action on morphisms.

Lemma 37. *The functor* $Ar_! : Op^{\mathtt{O}}(Set) \to Op^{\mathtt{Bq}(C)}(Set)$ *is the left adjoint to the restriction functor* $Ar^* : Op^{\mathtt{Bq}(C)}(Set) \to Op^{\mathtt{O}}(Set)$*.*

Proof. We need to establish, for $\mathcal{Q} \in Op^{\mathtt{O}}(Set)$ and $\mathcal{P} \in Op^{\mathtt{Bq}(C)}(Set)$, a natural isomorphism

$$Op^{\mathtt{O}}(Set)(\mathcal{Q}, Ar^*(\mathcal{P})) \cong Op^{\mathtt{Bq}(C)}(Set)(Ar_!(\mathcal{Q}), \mathcal{P}). \tag{3.2}$$

Let $\phi : \mathcal{Q} \to Ar^*(\mathcal{P})$ be a morphism of $\mathtt{O}$-operads. It is a collection of morphisms

$$\phi_t : \mathcal{Q}(t) \to Ar^*(\mathcal{P})(t),$$

for each $t \in \mathtt{O}$. By definition, $Ar^*(\mathcal{P})(t) = \mathcal{P}(Ar(t))$. By the universal property of the coproduct, we then have a map

$$\psi_T : \coprod_{Ar(t)=T} \mathcal{Q}(t) \to \mathcal{P}(T),$$

for each $T \in \mathtt{Bq}(\mathfrak{C})$. The collection $\{\psi_T\}_{T \in \mathtt{Bq}(C)}$ is a map of $Prof(\mathfrak{C}) \times \mathfrak{C}$-colored objects

$$\psi : E_{\mathcal{Q}} \to U^{\Sigma}(\mathcal{P}),$$

by adjunction in Theorem 36 there exists a naturally corresponding morphism of $\mathtt{Bq}(C)$-operads

$$\bar{\psi} : \mathcal{F}_{\mathcal{Q}} \to \mathcal{P}$$

and the following diagram commutes,

$$\begin{array}{ccc} E_{\mathcal{Q}} & & \\ {\scriptstyle \eta_{E_0(\mathcal{Q})}} \downarrow & \searrow^{\psi} & \\ U^{\Sigma}F^{\Sigma}(E(\mathcal{Q})) & \xrightarrow[U^{\Sigma}(\bar{\psi})]{} & U^{\Sigma}(\mathcal{P}) \end{array}$$

and therefore,

$$\sim_N \subseteq Ker(\psi_N),$$

for each $N \in \mathtt{Bq}(\mathfrak{C})$. This implies that there exists a unique $\bar{\phi} : Ar_!(\mathcal{Q}) \to \mathcal{P}$ such that the following diagram commutes

$$\begin{array}{ccc} F_Q & \xrightarrow{\bar{\psi}} & P \\ \downarrow & \nearrow_{\bar{\phi}} & \\ Ar_!(Q) & & \end{array} .$$

We define the isomorphism in (3.2) by the assignment $\phi \mapsto \bar{\phi}$. The naturality of this isomorphism comes from the universality of our constructions. $\square$

Remark 38. In case, $Ar : \mathtt{O} \to \mathtt{Bq}(C)$ is an operadic fibration, the construction we describe in (3.1) is isomorphic to the one described in [1, Proposition 2.3.].

We introduce notation $\mathbb{A}(\mathtt{O}) = Ar_!(\zeta_{\mathtt{O}})$, where $\zeta_{\mathtt{O}}$ is the terminal $\mathtt{O}$-operad. The object $Ar_!(\zeta_{\mathtt{O}})$ is a $\mathtt{Bq}(\mathfrak{C})$-operad, hence it can be identified with a $\mathfrak{C}$-colored classical operad.

3.3 Application of the wreath product to monocolored classical operads

Let $\mathcal{X}$, $\mathcal{Y}$ be classical monocolored operads in *Set*, we identify them with *sFSet*-operads with respective structure maps $\mu_{\mathcal{X}}, \mu_{\mathcal{Y}}$ and unit maps $\eta_{\mathcal{X}}, \eta_{\mathcal{Y}}$. Applying the construction in Chapter 2, we obtain the wreath product $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. It is a category consisting of objects

$$(x \in \mathcal{X}(\bar{n}); y_1 \in \mathcal{Y}(\bar{m}_1), \cdots, y_n \in \mathcal{Y}(\bar{m}_n)).$$

Let $(z \in \mathcal{X}(\bar{k}); w_1 \in \mathcal{Y}(\bar{l}_1), \cdots, w_k \in \mathcal{Y}(\bar{l}_k))$ be another object in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. A morphism

$$(\phi, \Phi) : (x; y_1, \cdots, y_n) \longrightarrow (z; w_1, \cdots, w_k)$$

is given by

- a morphism $f : \bar{n} \to \bar{k}$ and a tuple $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_k)$ such that $\mu_{\mathcal{X}}(f)(\varepsilon, z) = x$. The pair (f, ε) determines ϕ.
- a family of morphisms $\Phi = \{\phi_{i,j} : y_i \to w_j |\ i \in |x| = \bar{n}, j = |\phi|(i) = f(i)\}$. Each $\phi_{i,j}$ is similarly given by a morphism $f_{i,j} : \bar{m}_i \to \bar{l}_j$ and a tuple $\sigma = (\sigma_1, \cdots, \sigma_{l_j})$ such that $\mu_{\mathcal{Y}}(f_{i,j})(\sigma, w_j) = y_i$.

The morphisms in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$ induce relations in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$. We highlight special types morphisms in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$ and derive relations in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$. Using these relations, we will be able to prove the following proposition.

Proposition 39. *Let $\mathcal{X}$, $\mathcal{Y}$ be classical monocolored operads in Set. Then there is an epimorphism of operads*

$$\phi : \mathcal{X} \otimes_{BV} \mathcal{Y} \longrightarrow \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})).$$

Denote by $u = \eta_X(*) \in \mathcal{X}(\bar{1})$ and $v = \eta_Y(*) \in \mathcal{Y}(\bar{1})$, by Lemma 35, u, v are the terminal objects of the categories $\mathbb{I}(\mathcal{X}), \mathbb{I}(\mathcal{Y})$, respectively. We observe that for any $\bar{n}, \bar{m} \in sFSet, x \in \mathcal{X}(\bar{n}), y \in \mathcal{Y}(\bar{m})$, the objects $(u; y)$ and $(x; v, \cdots v)$ belong to $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$ and look into three types of morphisms that occur in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$.

Type 1. Let $f : \bar{n} \to \bar{m}$ be a morphism in *sFSet* with fibers $\bar{f}_i = f^{-1}(i), i \in \bar{m}$. Then the operad $\mathcal{X}$ is equipped with a structure map

$$\mu_{\mathcal{X}}(f) : \mathcal{X}(\bar{f}_1) \times \cdots \times \mathcal{X}(\bar{f}_m) \times \mathcal{X}(\bar{m}) \to \mathcal{X}(\bar{n}).$$

Let $\varepsilon_i \in \mathcal{X}(\bar{f}_i), i \in \bar{m}, x \in \mathcal{X}(\bar{n}), z \in \mathcal{X}(\bar{m})$ be such that

$$\mu_{\mathcal{X}}(f)(\varepsilon_1, \cdots, \varepsilon_m, z) = x.$$

Then there is a morphism

$$\Big(\Big((\varepsilon_1, \cdots, \varepsilon_m), f\Big); I_v\Big) : (x; v_1, \cdots, v_n) \longrightarrow (z; v_1, \cdots, v_m),$$

where $v_i = v, 1 \leq i \leq \max(n, m)$ and $I_v = \{id : v_i \to v_j |\ i \in \bar{n}, j = f(i)\}$, in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. For $i \in \bar{m}$, the i-th fiber $\Big(\Big((\varepsilon_1, \cdots, \varepsilon_m), f\Big); I_v\Big)^{-1}(i)$ equals to $(\varepsilon_i; v, \cdots, v)$.

This implies that the equality

$$\gamma((z; v, \cdots, v), (\varepsilon_1; v, \cdots, v), \cdots, (\varepsilon_m; v, \cdots, v)) = (x; v, \cdots, v) \tag{3.3}$$

holds in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$.

Type 2. For the same morphism $f : \bar{n} \to \bar{m}$ in *sFSet* with respective fibers $\bar{f}_i = f^{-1}(i), i \in \bar{m}$, the operad $\mathcal{Y}$ is equipped with a structure map

$$\mu_{\mathcal{Y}}(f) : \mathcal{Y}(\bar{f}_1) \times \cdots \times \mathcal{Y}(\bar{f}_m) \times \mathcal{Y}(\bar{m}) \to \mathcal{Y}(\bar{n}).$$

Let $\sigma_i \in \mathcal{Y}(\bar{f}_i), i \in \bar{m}, y \in \mathcal{Y}(\bar{n}), w \in \mathcal{Y}(\bar{m})$ be such that

$$\mu_{\mathcal{Y}}(f)(\sigma_1, \cdots, \sigma_m, w) = y.$$

Then there is a morphism

$$\Big(id_u; \Big((\sigma_1, \cdots, \sigma_m), f\Big)\Big) : (u; y) \longrightarrow (u; w)$$

in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. For $i \in \bar{m}$, the i-th fiber $\Big(id_u; \Big((\sigma_1, \cdots, \sigma_m), f\Big)\Big)^{-1}(i)$ equals $(u; \sigma_i)$.

This implies that the equality

$$\gamma((u; w), (u; \sigma_1), \cdots, (u; \sigma_m)) = (u; y). \tag{3.4}$$

holds in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$.

Type 3. Let $y_1 \in \mathcal{Y}(\bar{m}_1), \cdots, y_n \in \mathcal{Y}(\bar{m}_n)$, then there is a morphism

$$(id_x; J) : (x; y_1, \cdots, y_n) \to (x; v, \cdots, v), \text{ where } J = \{!_{y_i} : y_i \to v |\ i \in \bar{n}\}$$

in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. For $i \in \bar{n}$, i-th fiber $(id_x; J)^{-1}(i) = (u; y_i)$. This implies that the equality

$$\gamma((x; v, \cdots, v), (u; y_1), \cdots, (u; y_n)) = (x; y_1, \cdots, y_n). \tag{3.5}$$

holds in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$.

In the case $\bar{m}_1 = \cdots = \bar{m}_n = \bar{m}$ and $y_1 = \cdots = y_n = y$, there is also a morphism

$$(!_x; I_y) : (x; y, \cdots, y) \to (u; y), \text{ where } I_y = \{id_y : y \to y |\ i \in \bar{n}\}$$

in $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. For $j \in \bar{m}$, the j-th fiber $(id; I)^{-1}(j)$ equals $(x, v, \cdots, v)$. This implies that the equality

$$\begin{aligned} \gamma((x; v, \cdots, v), (u; y), \cdots, (u; y)) &= (x; y, \cdots, y) \\ &= \gamma((u; y), (x; v, \cdots, v), \cdots, (x; v, \cdots, v)) \end{aligned} \tag{3.6}$$

holds in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$.

Proof of Proposition 39. Denote by $\gamma_{\mathcal{X}}, \gamma_{\mathcal{Y}}$ the composition maps of $\mathcal{X}$, $\mathcal{Y}$, respectively, and by $\eta_{\mathcal{X}}, \eta_{\mathcal{Y}}$ the units of $\mathcal{X}$, $\mathcal{Y}$, respectively. Let $u = \eta_X(*), v = \eta_Y(*)$. We define two morphisms of operads

$$\begin{array}{rrcl} \phi_{\mathcal{X}} : & \mathcal{X} & \longrightarrow & \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})) \\ \phi_{\mathcal{X}}(n) : & \mathcal{X}(n) & \longrightarrow & \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))(n) \\ & x & \longmapsto & (x; v, \cdots, v), \end{array}$$

and

$$\begin{array}{rrcl} \phi_{\mathcal{Y}} : & \mathcal{Y} & \longrightarrow & \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})) \\ \phi_{\mathcal{Y}}(m) : & \mathcal{Y}(m) & \longrightarrow & \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))(m) \\ & y & \longmapsto & (u; y). \end{array}$$

Equalities (3.3) and (3.4) imply that $\phi_{\mathcal{X}}, \phi_{\mathcal{Y}}$ are indeed morphisms operads. Equality (3.6) implies that $\phi_{\mathcal{X}}, \phi_{\mathcal{Y}}$ extend to a morphism of operads

$$\phi : \mathcal{X} \otimes_{BV} \mathcal{Y} \longrightarrow \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})),$$

which is surjective on each its component

$$\phi_n : (\mathcal{X} \otimes_{BV} \mathcal{Y})(n) \longrightarrow (\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})))(n)$$

since (3.5) holds. □

Conclusion

The main aim of this thesis was to define the wreath product $\mathtt{A} \wr \mathtt{B}$ of an operadic category $\mathtt{A}$ and a connected operadic category $\mathtt{B}$ and to define a structure of an operadic category on $\mathtt{A} \wr \mathtt{B}$. Section 2.2 is devoted to verifying that $\mathtt{A} \wr \mathtt{B}$ fulfills the axioms of operadic categories. Furthermore, it has been demonstrated that this product is not commutative, but it is associative. In the specific case where $\mathtt{A} = \Omega_k$ and $\mathtt{B} = \Omega_l$, $\mathtt{A} \wr \mathtt{B}$ is shown to be isomorphic to Ω_{k+l}.

Furthermore, in Chapter 3 we have established in Proposition 39 the existence of an epimorphism from the Boarman-Vogt tensor product of single-colored operads $\mathcal{X} \otimes_{BV} \mathcal{Y}$ to $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$. In the course of our work, we have demonstrated that

- $\mathbb{I}(\mathcal{X}), \mathbb{I}(\mathcal{Y})$ have terminal objects, as shown in in Lemma 35, so the wreath product $\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y})$ is well-defined. These terminal objects then were crucial to establish equalities which hold in $\mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$.

- For an operadic category $\mathtt{O}$ and an $\mathtt{O}$-operad $\mathcal{Q}$, the canonical functor $Ar : \mathtt{O} \to \mathtt{Bq}(\pi_0\mathtt{O})$ gives rise to a $\mathtt{Bq}(\pi_0\mathtt{O})$-operad $Ar_!(\mathcal{Q})$. Moreover, this action is functorial and is a left adjoint to the restriction Ar^*.

We believe that the wreath product of operadic categories opens a path to future research. We wish to extend the result stated in Proposition 39 to colored operads, additionally seeking to ascertain the conditions under which an epimorphism $\phi : \mathcal{X} \otimes_{BV} \mathcal{Y} \longrightarrow \mathbb{A}(\mathbb{I}(\mathcal{X}) \wr \mathbb{I}(\mathcal{Y}))$ of operads gives rise to an isomorphism.

Bibliography

1. Batanin, Michael; Markl, Martin. Operadic categories and Duoidal Deligne's conjecture. *Advances in Mathematics*. 2014, vol. 285. Available from doi: 10.1016/j.aim.2015.07.008.
2. Markl, Martin. Operads and PROPs. In: Hazewinkel, M. (ed.). North-Holland, 2008, vol. 5, pp. 87–140. Handbook of Algebra. issn 1570-7954. Available from doi: https://doi.org/10.1016/S1570-7954(07)05002-4.
3. Boardman, John Michael; Vogt, Rainer. *Homotopy Invariant Algebraic Structures on Topological Spaces*. Springer Berlin, Heidelberg, 2006. Available from doi: https://doi.org/10.1007/BFb0068547.
4. Batanin, Michael; Markl, Martin. Operadic categories as a natural environment for Koszul duality. *Compositionality*. 2023, vol. 3. Available from doi: 10.32408/compositionality-5-3.
5. Batanin, Michael. Monoidal Globular Categories As a Natural Environment for the Theory of Weak n-Categories. *Advances in Mathematics*. 1998, vol. 136, no. 1, pp. 39–103. issn 0001-8708. Available from doi: https://doi.org/10.1006/aima.1998.1724.
6. Batanin, Michael; Street, Ross. The universal property of the multitude of trees. *Journal of Pure and Applied Algebra*. 2000, vol. 154, no. 1, pp. 3–13. issn 0022-4049. Available from doi: https://doi.org/10.1016/S0022-4049(99)00184-X. Category Theory and its Applications.
7. Batanin, Michael; Markl, Martin. Operads, Operadic Categories and the Blob Complex. *Applied Categorical Structures*. 2024, vol. 32. Available from doi: 10.1007/s10485-023-09759-4.
8. Yau, Donald. *Colored Operads*. American Mathematical Society, 2017.