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Products of Boolean Clones up to Minion Homomorphisms

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Abstract: This thesis takes known result about the classification of all boolean clones and extends this result into a lattice of all products of boolean clones modulo minion homomorphism. In the first part of the thesis we look what ordering comes solely from the order on boolean clones and in the second part we show that this describes the whole lattice.

Keywords: universal algebra, clones, multi-sorted Boolean clones, minion homomorphism, product of clones

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Introduction

In mathematics it is important to understand various types of symmetries. Group theory can be regarded as a field that investigates invertible symmetries. Universal algebra[10][6][7] generalizes this investigation to symmetries of larger arity.

For instance, given a relational structure the compatible functions can be regarded as its multivariate symmetries. collection of these functions form a structure called a clone. Understanding them tells us pieces of information about the relations. Because of that it is a very important classification project in universal algebra to understand all the clones and how they relate to each other.

All the clones on two element domain had been already fully understood by Post [11]. On the other hand the clones on three element domain are not understood yet. But there are some partial results like Zhuk's classification [14] of self-dual clones on three element domain.

It turns out that understanding all clones is a very hard task. However a weaker comparison of clones coming from so-called minion homomorphisms [1] is still useful in some context, such as for understanding the computational complexity of problems. Because it is weaker it makes the resulting lattices a lot more manageable.

For this reason we would like to understand all the clones modulo minion homomorphisms and this thesis provides a modest contribution towards this goal. In this thesis we look at clones that are products of boolean clones and we order all of them with respect to minion homomorphisms.

1 Multisorted clones

Definition 1.1 ($[n]$). By $[n]$ we will denote the set containing the first n positive integers that means the set $\{1, 2, \dots, n\}$.

1.1 Function clones

The notion of clones in universal algebra is used to study functions and their compostions.

Definition 1.2 (function composition). Let A be a set. Let f be a function $A^n \to A$ and $g_1, \dots, g_n$ be functions $A^m \to A$. Then we define the *composition* $f \circ (g_1, \dots, g_n)$ as function having arity m and the following definition

$$(f \circ (g_1, \dots, g_n))(x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

Definition 1.3 (projections). By the n-ary *projection* to the i-th coordinate on a set A we mean the function $A^n \to A$ defined by

$$\pi_i^n(x_1, \dots, x_n) = x_i$$

With these two definitions combined we are able to define the notion of a function clone.

Definition 1.4 (function clone). Let A be a set. Then we call a set C of functions on a set A a *function clone* if it contains all projections of all arities and is closed under composition.

Note. If not stated otherwise whenever we just say *clone* we are talking about a function clone.

1.2 Relations

We will give a notion of function being compatible with a relation, as this gives us a different way to characterize clones.

Definition 1.5 (compatibility with a relation). Let A be a set. Let R be a relation on A of arity m and f a function $A^n \to A$. Then we say that f is *compatible* with R if for all $a_{i,j} \in A$ with $i \le m, j \le n$ we have that $(a_{1,j}, \dots, a_{m,j}) \in R$ for all j implies that $(f(a_{1,1}, \dots, a_{1,n}), \dots, f(a_{m,1}, \dots, a_{m,n})) \in R$. This notion is summarized by the following diagram.

$$\begin{array}{ccccccccc} f(& a_{1,1} & a_{1,2} & \cdots & a_{1,n} &) & = & b_1 \\ f(& a_{2,1} & a_{2,2} & \cdots & a_{2,n} &) & = & b_2 \\ & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ f(& a_{m,1} & a_{m,2} & \cdots & a_{m,n} &) & = & b_m \\ & \in & \in & \cdots & \in & & & \in \\ & R & R & \cdots & R & & \Rightarrow & R \end{array}$$

Definition 1.6 (compatibility with set of relations). Let A be a set, f a function on A and $\mathcal{R}$ a set of relations on A. We say that f is *compatible* with $\mathcal{R}$ if it is compatible with every $R \in \mathcal{R}$.

Definition 1.7 (Pol). We denote the set of all functions compatible with $\mathcal{R}$ as $\mathrm{Pol}(\mathcal{R})$.

Theorem 1.8. Let A be a set and $\mathcal{R}$ be a set of relations on A. Then $\mathrm{Pol}(\mathcal{R})$ is a clone.

Proof. Proof is just a rutine check that whenever some functions satisfy given relation, then their composition does as well and that all projections satisfy every relation. For more details see [3].

It is known [5][8] that every clone on a finite set can be described as $\mathrm{Pol}(\mathcal{R})$ for some set of relations $\mathcal{R}$.

1.3 Multisorted clones

In this thesis we will work with special kind of clones called multisorted [12]. They give us a way to better handle some specific clones on bigger underlying sets.

Definition 1.9 (k-sorted set). A *k-sorted set* $A = (A_1, A_2, \dots, A_k)$ is a k-tuple of sets.

Definition 1.10 (sort). We will use the term *sort* to refer to some index of a k-sorted set.

Definition 1.11 (k-sorted function). Let $A = (A_1, A_2, \dots, A_k)$, $B = (B_1, B_2, \dots, B_k)$ be two k-sorted sets. Then a k-sorted function $A^n \to B$ is a tuple of k functions $f = (f_1, f_2, \dots, f_k)$, where f_i is a function $A_i^n \to B_i$.

This means that a k-sorted function f may be viewed as a special case of a function on products $(\prod A_i)^n \to \prod B_i$, where the sorts are independent. This observation gives us for free the notion of compositions and projections in the multisorted setting. But we still can view these in a multisorted way as shown in the following definitions.

Definition 1.12 (k-sorted composition). Let A be a k-sorted set. Further let $f = (f_1, \dots, f_k)$ be a k-sorted function $A^n \to A$. Finally let $g_1 = (g_{1,1}, \dots, g_{1,k}), \dots, g_n = (g_{n,1}, \dots, g_{n,k})$ be k-sorted functions of arity m. Then the arity m *composition* $f \circ (g_1, \dots g_n)$ is done component-wise, that is

$$f \circ (g_1, \dots g_n) = (f_1 \circ (g_{1,1}, \dots, g_{n,1}), \dots, f_1 \circ (g_{1,k}, \dots, g_{n,k}))$$

Definition 1.13 (k-sorted projection). Let A be a k-sorted set. The *k-sorted projection* of arity n onto i-th variable is $\pi_i^n = (\pi_i^n, \dots, \pi_i^n)$

Now similarly as before we may define the notion of a k-sorted clone. As said, this may be viewed as a clone using the notions of compostition and projection described here or as a special kind of a clone on $\prod A_i$, where for all functions the sorts are independent.

Definition 1.14 (k-sorted clone). Let A be a k-sorted set. Then we call a set C of k-sorted functions on A a *k-sorted clone* if it contains all the k-sorted projections of all arities and is closed under k-sorted composition.

1.4 Multisorted relations

In contrary to multisorted functions being independent on the sorts, multisorted relations give us a way to write some connections between the different sorts.

Definition 1.15 (k-sorted relation). Let $A = (A_1, \dots, A_k)$ be a k-sorted set and $i_1, \dots, i_m \in [k]$ be indicies. A *k-sorted relation* of arity m with type $(i_1, \dots, i_m)$ is a subset $R \subseteq A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$.

Note that the type $(i_1, \dots, i_m)$ is part of the definition of a relation, we cannot talk about a multisorted relation without knowing its type. And this type will be important even if all the A_i are the same set.

Definition 1.16 (compatibility with multisorted relation). Let $A = (A_1, \dots, A_k)$ be a k-sorted set, $f = (f_1, \dots, f_k)$ a k-sorted function $A^n \to A$ and R a k-sorted relation of arity m with type $(\ell_1, \dots, \ell_m)$. Then we say that f is *compatible* with R if for all $a_{i,j} \in A_{\ell_i}$ with $i \leq m$ and $j \leq n$ it holds that $(a_{1,j}, \dots, a_{m,j}) \in R$ for all j implies that

$$(f_{\ell_1}(a_{1,1}, \dots, a_{1,n}), \dots, f_{\ell_m}(a_{m,1}, \dots, a_{m,n})) \in R.$$

This is again best summarized by the following diagram.

$$\begin{array}{lcccccccc}
f_{\ell_1}(& a_{1,1} & a_{1,2} & \dots & a_{1,n} &) & = & b_1 \\
f_{\ell_2}(& a_{2,1} & a_{2,2} & \dots & a_{2,n} &) & = & b_2 \\
 & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\
f_{\ell_m}(& a_{m,1} & a_{m,2} & \dots & a_{m,n} &) & = & b_m \\
 & \in & \in & \dots & \in & & & \in \\
 & R & R & \dots & R & & \Rightarrow & R
\end{array}$$

In this thesis we will study special case of multisorted relation, where the type uses only one index. This means that all the f_{ℓ_i} in the above diagram would be the same function. And hence it is the same as bounding functions on each sort independently.

Definition 1.17 (simple k-sorted relation). By a *simple k-sorted relation* we mean any k-sorted relation of type $(i, i, \dots, i)$ for some i.

Definition 1.18 (clone product). Let $C_1, C_2, \dots, C_k$ be clones on the domain A. Then their *product* $\mathcal{C}(C_1, C_2, \dots, C_k)$ is the k-sorted clone containing all k-sorted functions $f = (f_1, \dots, f_k)$, where $f_i \in C_i$.

Lemma 1.19. Let R be a set of simple k-sorted relations. And let us denote R_i the subset of R containing all relations with type $(i, i, \dots, i)$. Finally let C be a k-sorted clone given by $C = \mathrm{Pol}(R)$. Then C is the same clone as the product of clones $\mathcal{C}(\mathrm{Pol}(R_1), \dots, \mathrm{Pol}(R_k))$.

Proof. This follows directly from the definition of compatibility with a relation, as simple relations on sort i exactly correspond to the relations satisfied by functions on the i-th sort.

In this thesis we want to study all multisorted clones on 2-element domain given as $\mathrm{Pol}(R)$ for some set of simple multisorted relations R. This translates into studying all the products of clones on 2-element domain. As all the clones on 2-element domain are well known we want to take all subsets of them and consider their products.

1.5 Homomorphisms

We want to talk about how different clones relate to each other and for that we will mostly use the notion of minion homomorphisms, but for completeness we also include the definition of clone homomorphisms.

Definition 1.20 (clone homomorphism). Let $\mathcal{A}$ be a clone on a set A and $\mathcal{B}$ a clone on a set B. Then mapping $\zeta\colon \mathcal{A} \to \mathcal{B}$ is a *clone homomorphism* if it satisfies the following:

- It preserves arities, that is, a function of arity n in $\mathcal{A}$ gets mapped to a function of arity n in $\mathcal{B}$.
- It preserves projections, that is, $\zeta(\pi_i^n) = \pi_i^n$.
- It preserves composition, that is, if $f \in \mathcal{A}$ is a function of arity n and $g_1, g_2, \ldots, g_n$ functions of arity m, then

$$\zeta(f \circ (g_1, \ldots, g_n)) = \zeta(f) \circ (\zeta(g_1), \ldots, \zeta(g_n))$$

When in the composition the functions g_i are projections, it turns out to be a very important special case. Hence it gets its own definition.

Definition 1.21 (minor). Let A be a set and f a function $A^n \to A$ and $i_1, i_2, \ldots, i_n \in [m]$ be arbitrary indicies. Then the composition $f \circ (\pi_{i_1}^m, \pi_{i_2}^m, \ldots, \pi_{i_n}^m)$ is called a *minor* of f.

An example of minor is function g defined as $g(x, y) = f(x, x, y)$. Here g is a minor of f. Note that "x" is the binary projection on the first coordinate and "y" is the binary projection on the second coordinate. And hence we could rewrite it as $g = f \circ (\pi_1^2, \pi_1^2, \pi_2^2)$.

Note that up to dummy variables and renaming variables this definition is interesting only if $m < n$.

With minors we will relax conditions of clone homomorphism to get a weaker notion of morphisms called minion homomorphisms that does not have to preserve composition.

Definition 1.22 (minion homomorphism). Let $\mathcal{A}$ be a clone on a set A and $\mathcal{B}$ a clone on a set B. Then mapping $\zeta\colon \mathcal{A} \to \mathcal{B}$ is a *minion homomorphism* if it satisfies the following:

- It preserves arities. Function of arity n in $\mathcal{A}$ gets mapped to a function of arity n in $\mathcal{B}$.
- Preserves composition with projections in other words it preserves minors. Let $f \in \mathcal{A}$ be a function of arity n and $i_1, i_2, \ldots, i_n \in [m]$ be arbitrary indicies. Then

$$\zeta(f \circ (\pi_{i_1}^m, \ldots, \pi_{i_n}^m)) = \zeta(f) \circ (\pi_{i_1}^n, \ldots, \pi_{i_n}^n)$$

Note. Because this definition comes from structure called minions [2], which are in general sets of functions between different sets, it does not make sense to talk about preserving projections, as in general there are no projections. But in our setting we may use the following lemma to assume they preserve projections, as we will be talking only about idempotent functions.

Definition 1.23 (idempotent function). Let A be a set and f a function $A^n \to A$. Then f is *idempotent* if for every $x \in A$ we have $f(x, x, \ldots, x) = x$.

Definition 1.24 (idempotent clone). Clone $\mathcal{A}$ is *idempotent* if every function $f \in \mathcal{A}$ is idempotent.

Lemma 1.25. Let $\mathcal{A}$, $\mathcal{B}$ be two idempotent clones and $\zeta\colon \mathcal{A} \to \mathcal{B}$ be a minion homomorphism. Then $\zeta(\pi_i^n) = \pi_i^n$.

Proof. Because they are idempotent, they have only one function of arity 1, the identity function $\tau(x) = x$. Thus it has to be mapped to itself by ζ. And note that $\pi_i^n = \tau \circ (\pi_i^n)$ and thus $\zeta(\pi_i^n) = \zeta(\tau) \circ (\pi_i^n) = \tau \circ (\pi_i^n) = \pi_i^n$. □

1.6 Preorder

From the notion of minion homomorphism we get a preorder on all clones.

Definition 1.26 ($\leq$). Let $\mathcal{A}$ and $\mathcal{B}$ be two clones. Then we say $\mathcal{A} \leq \mathcal{B}$ if there exists a minion homomorphism $\mathcal{A} \to \mathcal{B}$.

Lemma 1.27. The relation $\leq$ is a preorder.

Proof. This follows from two observations

- Identity is a minion homomorphism $\mathcal{A} \to \mathcal{A}$, thus the relation is reflexive.
- When we have minion homomorphism $\mathcal{A} \to \mathcal{B}$ and a minion homomorphism $\mathcal{B} \to \mathcal{C}$, then their composition gives us a minion homomorphism $\mathcal{A} \to \mathcal{C}$. Hence the relation is transitive. □

This preorder gives us equivalence in the usual way as written in the following definition.

Definition 1.28 (equivalence). Let $\mathcal{A}, \mathcal{B}$ be clones. Then we say $\mathcal{A}$ is *equivalent* to $\mathcal{B}$ denoted by $\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$.

We want to look at the order given by this preorder on the equivalence classes given by $\sim$. This order is already known for all clones on a 2-element domain. We will describe this order after we define our notation.

We will show a lemma which will be our proof strategy for proving that there does not exist a minion homomorphism from one clone to another. For that we need one more definition.

Definition 1.29 (Height one identity). *Height one identity* is an identity of the form

$$f_1 \circ (\pi_{i_1}, \dots, \pi_{i_n}) = f_2 \circ (\pi_{j_1}, \dots, \pi_{j_m}) = \cdots = f_c(\pi_{k_1}, \dots, \pi_{k_\ell})$$

Hence for example this identity is height one $f(x, x, y) = f(x, y, z) = g(x, z)$, but this one $f(f(x, y), y) = f(x, x)$ is not.

Lemma 1.30. Let S be set of height one identities. Let A, B be two clones. Suppose, that there exists some functions $f_1, \dots, f_n \in A$ satisfying all the identities in S, but there do not exist $g_1, \dots, g_n \in B$ satisfying all the identities in S. Then $A \not\leq B$.

Proof. We will prove this by a contradiction. Suppose $A \leq B$. Then we have some minion homomorphism $\zeta\colon A \to B$. Because $f_1, \dots, f_n$ satisfy all identities in S and the identities are height one, we get, from definition of minion homomorphism, that $\zeta(f_1), \dots, \zeta(f_n)$ satisfy all the identities, which is a contradiction.

□

Now we will look at few lemmas which will show how the product of clones behaves with respect to the minion homomorphisms. In a sense the product is the same as a free meet in the lattice given by $\leq$.

Lemma 1.31. Let $\mathcal{C}(X_1, X_2, \dots, X_n)$ be a product of clones. And let p be a permutation on the set $[n]$. Then $\mathcal{C}(X_1, X_2, \dots, X_n) \sim \mathcal{C}(X_{p(1)}, X_{p(2)}, \dots, X_{p(n)})$.

Proof. The mappings $(f_1, f_2, \dots, f_n) \mapsto (f_{p(1)}, f_{p(2)}, \dots, f_{p(n)})$ and its inverse are minion homomorphisms. □

Lemma 1.32. Let $\mathcal{C}(X_1, X_2, \ldots, X_n)$ and $\mathcal{C}(X_1, X_2, \ldots, X_m)$ be two products of clones such that there exists $j_1, j_2, \ldots, j_m \in [n]$ indicies for which $X_{j_i} \leq Y_i$ for every i. Then $\mathcal{C}(X_1, X_2, \ldots, X_n) \leq \mathcal{C}(X_1, X_2, \ldots, X_m)$

Proof. Every inequality $X_{j_i} \leq Y_i$ gives us a minion homomorphism $\zeta_i \colon X_{j_i} \to Y_i$. To prove the inequality we define new minion homomorphism ζ mapping $\mathcal{C}(X_1, X_2, \ldots, X_n)$ to $\mathcal{C}(X_1, X_2, \ldots, X_m)$ as follows:

$$\zeta(f_1, f_2, \ldots, f_n) = (\zeta_1(f_{j_1}), \zeta_2(f_{j_2}), \ldots, \zeta_m(f_{j_m}))$$

It is only a rutine check that this is really a minion homomorphism. □

Corollary 1.33. Let $\mathcal{C}(X_1, X_2, \ldots, X_n)$ be a product of clones. Suppose that $X_2 \leq X_1$, then

$$\mathcal{C}(X_1, X_2, \ldots, X_n) \sim \mathcal{C}(X_2, X_3, \ldots, X_n).$$

Proof. We will apply Lemma 1.32 twice. First we use indicies $j_1 = 2, j_2 = 3, \ldots, j_{n-1} = n$ to prove

$$\mathcal{C}(X_1, X_2, \ldots, X_n) \leq \mathcal{C}(X_2, X_3, \ldots, X_n).$$

And then we use the indicies $j_1 = 1, j_2 = 1, j_3 = 2, j_4 = 3, \ldots, j_n = n - 1$ to prove that

$$\mathcal{C}(X_2, X_3, \ldots, X_n) \leq \mathcal{C}(X_1, X_2, \ldots, X_n).$$

□

Corollary 1.34. By repeatably applying Corollary 1.33 and Lemma 1.31 we see that every product of clones is equivalent to one which is given only by incomparable elements. Thus if we want to study all products of clones modulo minion homomorphisms, it sufficies to look at products given by incomparable elements.

We want to describe the minion order for products of boolean clones. It turns out that it does not have any non-trivial collapses. For clarity of the proof we define the order that follows only from the order of the underlying clones.

Definition 1.35. ($\preceq$) Let $\mathcal{C}(X_1, X_2, \ldots, X_n), \mathcal{C}(Y_1, Y_2, \ldots, Y_m)$ be two products of clones. Then we say $\mathcal{C}(X_1, X_2, \ldots, X_n) \preceq \mathcal{C}(Y_1, Y_2, \ldots, Y_m)$ if there exist indices $j_1, j_2, \ldots, j_m \in [n]$ such that for every Y_i we have $X_{j_i} \leq Y_i$.

Definition 1.36. (trivially equivalent) Let $\mathcal{C}(X_1, X_2, \ldots, X_n), \mathcal{C}(Y_1, Y_2, \ldots, Y_m)$ be two products of clones. We say that they are *trivially equivalent* if

$$\mathcal{C}(X_1, X_2, \ldots, X_n) \preceq \mathcal{C}(Y_1, Y_2, \ldots, Y_m)$$
$$\mathcal{C}(Y_1, Y_2, \ldots, Y_m) \preceq \mathcal{C}(X_1, X_2, \ldots, X_n).$$

We denote this fact as $\mathcal{C}(X_1, X_2, \ldots, X_n) \approx \mathcal{C}(Y_1, Y_2, \ldots, Y_m)$

From Lemma 1.32 we have that whenever two products are trivially equivalent, they are equivalent.

Lemma 1.37. Let $X_1, X_2, \ldots, X_n$ be clones then in the order given by $\preceq$ on products of clones modulo trivial equivalence we have that $\mathcal{C}(X_1, X_2, \ldots, X_n)$ is the meet of $\mathcal{C}(X_1), \mathcal{C}(X_2), \ldots, \mathcal{C}(X_n)$.

Proof. Because $X_i \leq X_i$ we have that $\mathcal{C}(X_1, X_2, \ldots, X_n) \preceq \mathcal{C}(X_i)$. Now suppose that for some product of clones $\mathcal{C}(Z_1, Z_2, \ldots, Z_k)$ we have that $\mathcal{C}(Z_1, Z_2, \ldots, Z_k) \preceq \mathcal{C}(X_i)$ for all i. Then from definition of $\preceq$ we get for every X_i some Z_j such that $Z_j \leq X_i$. But these together give us that $\mathcal{C}(Z_1, Z_2, \ldots, Z_k) \preceq \mathcal{C}(X_1, X_2, \ldots, X_n)$. And hence $\mathcal{C}(X_1, X_2, \ldots, X_n)$ is a meet.

Lemma 1.38. Suppose that the lattice given by $\leq$ on single sorted clones can be covered by finite number of chains, then $\preceq$ induces complete lattice on all the finite products.

Proof. From Lemma 1.37 we have all finite meets. We will show what are the infinite meets. As all the elements are products, we have from Lemma 1.37 that they are meets of the single sorted clones. Hence WLOG we may assume we want to find meet of single sorted clones. Let n be the number of covering chains. We will denote element from the i-th chain with the upper index X^i. Suppose we have for every i some clones $X^i_1 \geq X^i_2 \geq \ldots$ and we want to find meet of all X^i_j. We claim the meet is the product of n clones

$$M = \mathcal{C}(\min(X^1_1, X^2_2, \ldots), \min(X^2_1, X^2_2, \ldots), \ldots \min(X^n_1, X^n_2, \ldots)),$$

where min is the infimum of the chain in the original lattice. Because we have $\min(X^i_1, X^i_2, \ldots) \leq X^i_j$ for all i, j, we get $M \preceq \mathcal{C}(X^i_j)$ for all i, j.

Now suppose we have some product of clones $\mathcal{C}(Z_1, Z_2, \ldots, Z_k)$ such that for every X^i_j we have $\mathcal{C}(Z_1, Z_2, \ldots, Z_k) \preceq \mathcal{C}(X^i_j)$. We will prove for every i that there is some Z_ℓ such that $Z_\ell \leq \min(X^i_1, X^i_2, \ldots)$. Fix some i. Then we have for every j that there exists some $h(j) \in [k]$ such that $Z_{h(j)} \leq X^i_j$.

Observation 1.39. Suppose that $X_1 \geq X_2, \ldots$ is totally ordered set and h a mapping it into a finite set $[k]$. Then there exists some ℓ such that the preimage of ℓ under h is unbounded. That means that for every X_j there exists some $j' \geq j$ such that $h(X_{j'}) = \ell$.

Proof. For contradiction suppose that no such ℓ exists. That means that for every $\ell \in [k]$ we have some j such that X_j is strictly smaller than every preimage of ℓ. But taking the minimum of these gives us element that is strictly less then preimage of any value, which is a contradiction.

Using the observation we have some Z_ℓ such that $Z_\ell \leq X^i_j$ for every j and thus $Z_\ell \leq \min(X^i_1, X^i_2, \ldots)$.

Because this holds for every i we have that $\mathcal{C}(Z_1, Z_2, \ldots, Z_k) \leq M$ and thus M is the meet. As we have all the infinite meets, the order is a complete lattice. □

2 Notation for clones on 2-element domain

From now on we will be working with 2-element domain $\{0,1\}$ with natural order $\leq$ and addition done modulo 2. Later we will use it in the multisorted setting.

Definition 2.1. (Notation for specific clones) Here we define the notation for clones and relations used in the later chapters. All the nontrivial clones we will be talking about are idempotent.

- ($\mathbf{R}_n$) The relation $\mathbf{R}_n$ for $n \geq 2$ is a relation of arity n given by $\mathbf{R}_n = \{0,1\}^n \setminus \{(1,1,\ldots,1)\}$.
- ($\mathbf{R}_\infty$) The set of relations $\mathbf{R}_\infty$ is a set containing all $\mathbf{R}_i$ for all i.
- (B_n) The clone B_n for $i \geq 2$ is $\mathrm{Pol}(\mathbf{R}_n)$.
- (B_∞) The clone B_∞ is $\mathrm{Pol}(\mathbf{R}_\infty)$.
- ($B_n^{\leq}$) The clone $B_n^{\leq}$ for $i \geq 2$ is $\mathrm{Pol}(\mathbf{R}_n, \leq)$.
- ($B_\infty^{\leq}$) The clone $B_\infty^{\leq}$ is $\mathrm{Pol}(\mathbf{R}_\infty, \leq)$.
- (A) The clone A is a clone that is generated by the binary function $\wedge$. Where $\wedge$ is a binary function returning minimum of its arguments.
- (C) The clone C is $\mathrm{Pol}(\neq)$.
- ($C^{\leq}$) The clone $C^{\leq}$ is $\mathrm{Pol}(\neq, \leq)$.
- (L) The clone L is the clone consiting of all idempotent linear functions, that is, functions $a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where odd number of a_i are equal to one and the others to zero.

Definition 2.2. (UP set) Let S be a subset of $\{0,1\}^n$. We use use $\leq$ on elements of $\{0,1\}^n$ componentwise, that means

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots y_n) \quad \Leftrightarrow \quad x_1 \leq y_1 \wedge x_2 \leq y_2 \wedge \cdots \wedge x_n \leq y_n$$

Then $\mathrm{UP}(S)$ is a subset of $\{0,1\}^n$ defined by $\{x \colon \exists y \in S, x \geq y\}$.

Definition 2.3. (function input notation) Instead of the usual $f(x_1, x_2, \ldots, x_n)$ we will often remove the commas and write only $f(x_1x_2 \ldots x_n)$. In addition to that we might sometimes want to optically distinguish some parts of the input. For that we will use small spaces, for example $f(100\ 100) = f(100100) = f(1,0,0,1,0,0)$.

Definition 2.4. (matrix set notation) Note that we will be using a matrix notation of sets, where we write the elements as rows of the matrix. This for example means that $\{001, 100\} = \left\{ \begin{smallmatrix} 011 \\ 100 \end{smallmatrix} \right\}$

Observation 2.5. The clones have the following properties.

- (B_n) Let f be a function, then $f \in B_n$ if and only if the following holds. Let us take n notnessesarly different k-tuples from $f^{-1}(1)$. Then there exists a coordinate on which all of them contain a 1.
- (B_∞) Similarly let f be a function then $f \in B_\infty$ if and only if there exists a coordinate such that all the elements from $f^{-1}(1)$ have it nonzero.
- ($B_n^{\leq}$) Suppose that f is a function in B_n and let O be the set $f^{-1}(1)$. Then function defined as $g(x) = 1$ if $x \in \mathrm{UP}(O)$ is in $B_n^{\leq}$.
- ($B_\infty^{\leq}$) Similarly let f be a function in B_∞ and let O be the set $f^{-1}(1)$. Then function defined as $g(x) = 1$ if $x \in \mathrm{UP}(O)$ is in $B_\infty^{\leq}$.
- (C) Let f be a function. Denote $\overline{x}$ the binary negation of x. Then $f \in C$ if and only if $f(\overline{x}) = \overline{f(x)}$ for every x.
- (C) Minority and majority lies in the clone C.
- ($C^{\leq}$) In $C^{\leq}$ are all the functions from C that are monotone.
- ($C^{\leq}$) Majority lies in $C^{\leq}$.

3 Lattice of clones on 2-element domain

We want to characterize all multisorted clones on 2-element domain modulo minion homomorphisms. From Lemma 1.19 this is the same as studying all products of clones on 2-element domain. All of these clones are known and characterized by Post's lattice[11], but in this thesis we want to look on them only modulo minion homomorphisms.

All clones on 2-element domain modulo minion homomorphism are also known [4]. The lattice of all these clones modulo minion homomorphism is shown in Figure 3.1

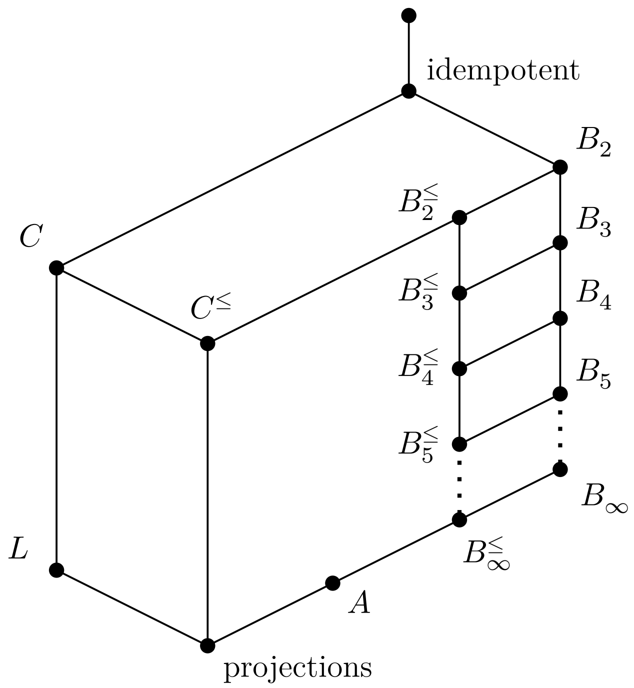


Figure 3.1 Lattice of boolean clones modulo minion homomorphisms

Now we want to add all the clone products into this lattice. First we look at them only modulo trivial equivalence.

Lemma 3.1. All products of boolean clones modulo trivial equivalence with the order $\preceq$ form a complete lattice with product being the meet.

Proof. Notice that the lattice in Figure 3.1 can be covered by four chains. And hence from Lemma 1.38 we have that all the finite products form a complete lattice. □

The lattice of all products of all boolean clones modulo trivial equivalence is drawn in Figure 3.2. In Figure 3.2 red points represent the sublattice from from Figure 3.1, that is all the single sorted clone products. The lattice is then formed by five triangles. In each such triangle, every vertex is the product of the top-left corner, the vertical arrow pointing to it and the horizontal arrow pointing to it. For example in the bottom-right triangle we have verticies of the form $\mathcal{C}(C^{\leq}, B_i, B_j^{\leq})$ for $3 \leq i \leq j$ and one vertex is of the form $\mathcal{C}(C^{\leq}, B_\infty, A) \approx \mathcal{C}(C^{\leq}, A)$.

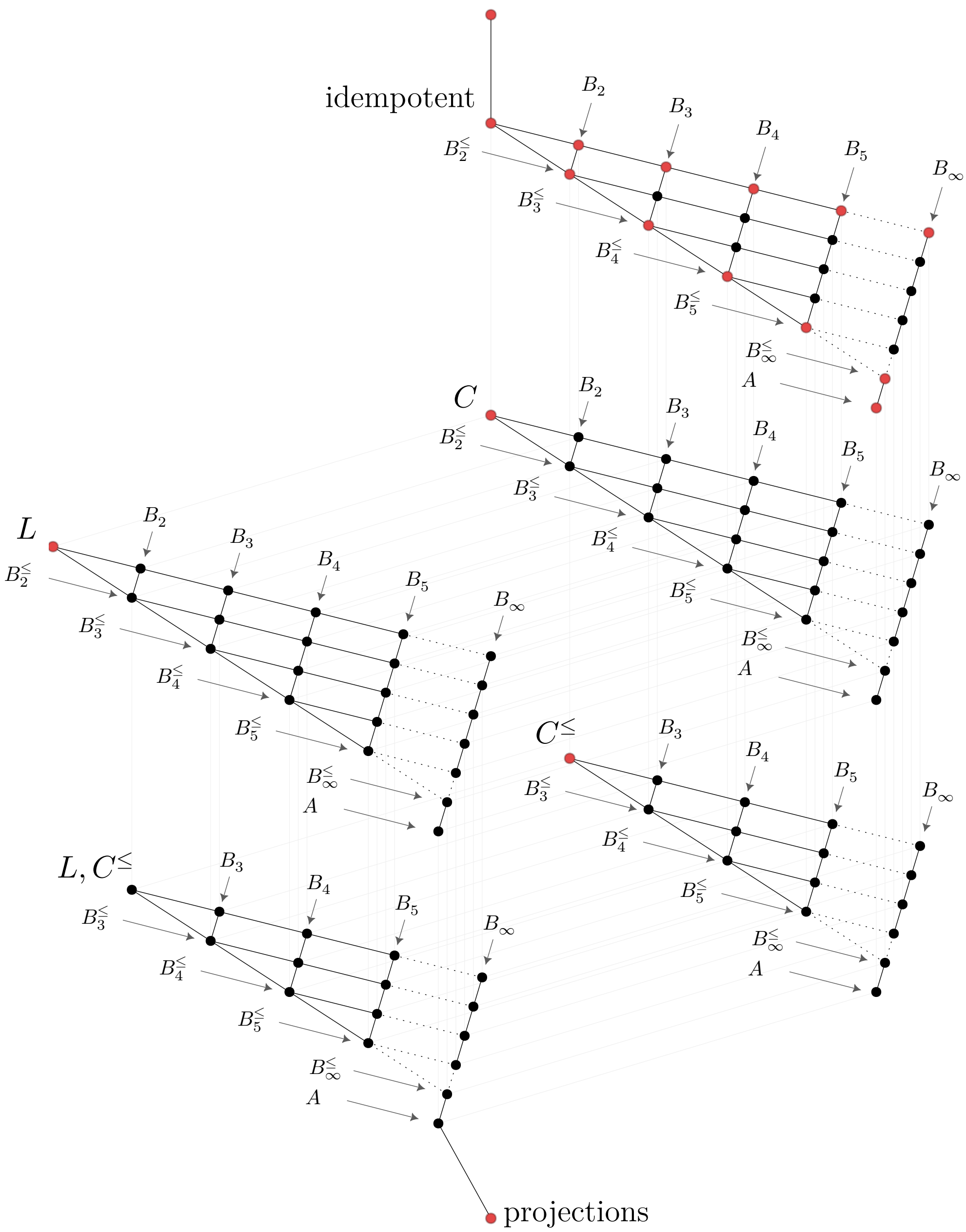


Figure 3.2 Lattice of products of boolean clones modulo trivial equivalence

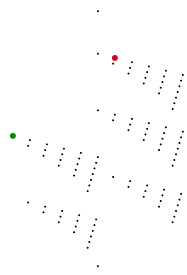
3.1 Proof preparation

We will show that this is also the lattice of all the products of boolean clones modulo minion homomorphisms. We already have that whenever there is $X \preceq Y$, then $X \leq Y$ from Lemma 1.32, thus we only have to show that when $X \not\preceq Y$, then $X \not\leq Y$. We will show this for some special pairs X and Y and then we will show how any other pair can be proven using these.

All the lemmas here will use the same proof strategy, we will show some concrete set of height one identities and then by using Lemma 1.30 we infer the non-inequality. Note that proving that product of clones has a function satisfying some identity by definition is the same as finding function in each of its components satisfying the identity.

Every lemma is proving one or more non-inequalities of the form $X \not\leq Y$ for some products of clones X and Y. Right of every lemma is a miniature of the Figure 3.2 with X being highlighted green and Y red.

Lemma 3.2. $\mathcal{C}(L) \not\leq \mathcal{C}(B_2)$



Proof. We will use identities

$$\begin{aligned} f(yyx) &= \\ = f(yxy) &= \\ = f(xyy) &= f(xxx) \end{aligned}$$

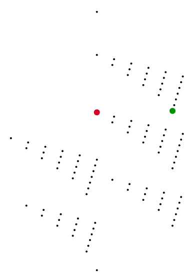
- $f \in L$. We define f as the sum of all three variables.
- $f \notin B_2$. Every such f satisfies

$$f(001) = f(111) = 1$$
$$f(010) = f(111) = 1$$

which contradicts the fact that f has to be compatible with the $\mathbf{R}_2$ relation.

Note that $f(xyy)$ in the identity was redundant, it is there only to make it symmetric.

Lemma 3.3. $\mathcal{C}(A) \not\leq \mathcal{C}(C)$

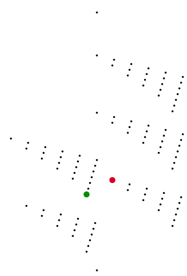


Proof. We will use identity

$$f(xy) = f(yx)$$

- $f \in \mathcal{C}(A)$. We define $f(xy) = x \wedge y$.
- $f \notin \mathcal{C}(C)$. From the condition we get $f(10) = f(01)$, but from compatibility with $\neq$ we have $f(10) \neq f(01)$, which is a contradiction.

Lemma 3.4. $\mathcal{C}(L, A) \not\leq \mathcal{C}(C^{\leq})$



Proof. We will use identities

$$f(xyyyy) = f(yxxxy) = f(xyyxx)$$

- $f \in \mathcal{C}(L)$. We define f as the sum of all variables.
- $f \in \mathcal{C}(A)$. We define f as the meet of all variables.
- $f \notin \mathcal{C}(C^{\leq})$. Suppose such f exists. We distinguish two cases
  - $f(10000) = 1$. Using compatibility with $\neq$ we get $f(01111) = 0$. Further using compatibility with $\leq$ we have

    $$f(01110) \leq f(01111) = 0,$$

    and thus

    $$f(01110) = 0 \neq 1 = f(10000),$$

    which contradicts the identity.
  - $f(10000) = 0$. Using identity we get $f(01110) = 0$. Now using compatibility with $\neq$ we have $f(10001) = 1$. Using compatibility with $\leq$ we have

    $$f(10011) \geq f(10001) = 1,$$

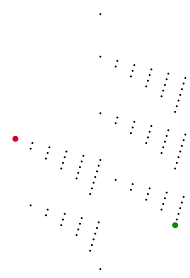
    hence we get

    $$f(10011) = 1 \neq 0 = f(10000),$$

    which contradicts the identity.

□

Lemma 3.5. $\mathcal{C}(C^{\leq}, A) \not\leq \mathcal{C}(L)$



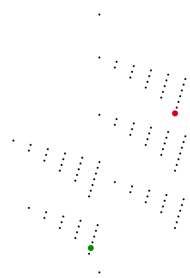
Proof. We will use identities

$$
\begin{aligned}
&1. \quad f(yyxxx) = f(xyxxx)\\
&2. \quad f(xyyxx) = f(xxyxx)\\
&3. \quad f(xxyyx) = f(xxxyx)\\
&4. \quad f(xxxyy) = f(xxxxy)\\
&5. \quad f(yxxxy) = f(yxxxx)
\end{aligned}
$$

- $f \in \mathcal{C}(C^{\leq})$. We define f as the majority on all five variables.
- $f \in \mathcal{C}(A)$. We define f as the meet of all variables.
- $f \notin \mathcal{C}(L)$. For contradiction suppose f is linear function satisfying the identities. Then let i be such that f depends on i-th variable. Then using the i-th identity we get a contradiction.

□

Lemma 3.6. $\mathcal{C}(L, C^{\leq}, B^{\leq}_{\infty}) \not\leq \mathcal{C}(A)$



Proof. We will use identities

$$
\begin{aligned}
p_1(xxy) &= p_1(xxx)\\
p_2(xxy) &= p_1(xyy)\\
q(xyy) &= p_2(xyy)\\
q(xxy) &= q(yyy)\\
p_i(xyx) &= p_i(xxx) \quad i \in \{1, 2\}
\end{aligned}
$$

- $p_1, p_2, q \in \mathcal{C}(L)$. We define $p_1(xyz) = p_2(xyz) = x$ and $q(xyz) = x + y + z$. Then the identities are satisfied as follows.

$$
\begin{aligned}
p_1(xxy) &= p_1(xxx) = x\\
p_2(xxy) &= p_1(xyy) = x\\
q(xyy) &= p_2(xyy) = x\\
q(xxy) &= q(yyy) = y\\
p_i(xyx) &= p_i(xxx) = x
\end{aligned}
$$

- $p_1, p_2, q \in \mathcal{C}(C^{\leq})$. We define $p_1(xyz)$ as the majority and $p_2(xyz) = q(xyz) = z$. Then the identities are as follows.

$$
\begin{aligned}
p_1(xxy) &= p_1(xxx) = x\\
p_2(xxy) &= p_1(xyy) = y\\
q(xyy) &= p_2(xyy) = y\\
q(xxy) &= q(yyy) = y\\
p_i(xyx) &= p_i(xxx) = x
\end{aligned}
$$

- $p_1, p_2, q \in \mathcal{C}(B^{\leq}_{\infty})$. We define $p_1(xyz) = x \wedge (y \vee z)$, $p_2(xyz) = z \wedge (x \vee y)$ and $q(xyz) = z$. Then the identities are as follows.

$$
\begin{aligned}
p_1(xxy) &= p_1(xxx) = x\\
p_2(xxy) &= p_1(xyy) = x \wedge y\\
q(xyy) &= p_2(xyy) = y\\
q(xxy) &= q(yyy) = y\\
p_i(xyx) &= p_i(xxx) = x
\end{aligned}
$$

- $p_1, p_2, q \notin \mathcal{C}(A)$. From identity $q(xxy) = q(yyy) = y$ we get $q(xyz) = z$. Now we look at identities

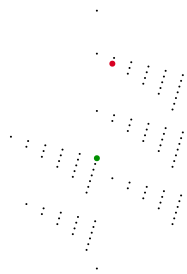
$$y = q(xyy) = p_2(xyy)$$
$$p_2(xyx) = p_2(xxx) = x,$$

From the first we have $p_2(xyz)$ is either y, z or $y \wedge z$. Plugging this into second we get the only option is $p_2(xyz) = z$. Now we look at identities

$$y = p_2(xxy) = p_1(xyy)$$
$$p_1(xyx) = p_1(xxx) = x,$$

Similarly as before from the first we get $p_1(xyz)$ is either y, z or $y \wedge z$. And again pluggin into the second one we get $p_1(xyz) = z$. But that contradicts the identity $p_1(xxy) = p_1(xxx) = x$. □

Lemma 3.7. $\mathcal{C}(L, B_\infty) \nleq \mathcal{C}(B_2^{\leq})$



Proof. We will use identities

$$f(y\ xy\ xy) = f(y\ yx\ yx)$$
$$f(y\ xy\ yy) = f(x\ xx\ xx)$$

- $f \in \mathcal{C}(L)$. We define f as the sum of all variables.
- $f \in \mathcal{C}(B_\infty)$. We define f as

$$f^{-1}(1) = \{1\ 11\ 11, 0\ 10\ 00\}$$

- $f \notin B_2^{\leq}$. From second identity we have

$$f(0\ 10\ 00) = f(1\ 11\ 11) = 1$$

Now using that f is monotone we have

$$f(0\ 10\ 10) = 1.$$

Similarly we have

$$f(1\ 01\ 11) = f(0\ 00\ 00) = 0$$

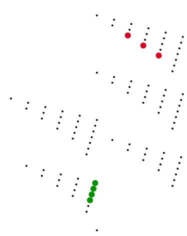
and from f being monotone we get

$$f(0\ 01\ 01) = 0$$

These results together contradict the first identity.

□

Lemma 3.8. For every $i \geq 3$ it holds that $\mathcal{C}(L, C^{\leq}, B_{i-1}^{\leq}, B_\infty) \nleq \mathcal{C}(B_i^{\leq})$.



Proof.

- When i is odd we use identities

$$f(\overbrace{yy\cdots yyx}^{i}\ \overbrace{xx\cdots xxy}^{i}\ \overbrace{xx\cdots xxy}^{i}) =$$
$$= f(yy\cdots yxy\ xx\cdots xyx\ xx\cdots xyx) =$$
$$\iddots \qquad \iddots \qquad \iddots$$
$$= f(xy\cdots yyy\ yx\cdots xxx\ yx\cdots xxx)$$

$$f(\overbrace{yy\cdots yyx}^{i}\ \overbrace{xx\cdots xxy}^{i}\ \overbrace{yy\cdots yyy}^{i}) = f(\overbrace{xx\cdots xxx}^{i}\ \overbrace{xx\cdots xxx}^{i}\ \overbrace{xx\cdots xxx}^{i})$$

- $f \in \mathcal{C}(L)$. We define f as the sum of all variables.
- $f \in \mathcal{C}(C^{\leq})$. We define f as the majority on the middle i variables.
- $f \in \mathcal{C}(B^{\leq}_{i-1})$. We define f as

$$f^{-1}(1) = \mathrm{UP}\left\{\begin{array}{l} \overbrace{00\cdots001}^{i}\ \overbrace{11\cdots110}^{i}\ \overbrace{00\cdots000}^{i} \\ 00\cdots010\ 11\cdots101\ 11\cdots101 \\ 00\cdots100\ 11\cdots011\ 11\cdots011 \\ \quad\ \iddots \qquad\quad \iddots \qquad\quad \iddots \\ 10\cdots000\ 01\cdots111\ 01\cdots111 \end{array}\right\}$$

- $f \in \mathcal{C}(B_{\infty})$. We define f as

$$f^{-1}(1) = \left\{\begin{array}{l} \overbrace{11\cdots111}^{i}\ \overbrace{11\cdots111}^{i}\ \overbrace{11\cdots111}^{i} \\ 00\cdots001\ 11\cdots110\ 00\cdots000 \end{array}\right\}$$

- $f \notin \mathcal{C}(B^{\leq}_{i})$. For every such f we have

$$1 = f(\overbrace{11\cdots111}^{i}\ \overbrace{11\cdots111}^{i}\ \overbrace{11\cdots111}^{i}) = f(\overbrace{00\cdots001}^{i}\ \overbrace{11\cdots110}^{i}\ \overbrace{00\cdots000}^{i})$$

From f being monotone we get

$$f(\overbrace{00\cdots001}^{i}\ \overbrace{11\cdots110}^{i}\ \overbrace{00\cdots000}^{i}) = 1 \Rightarrow f(\overbrace{00\cdots001}^{i}\ \overbrace{11\cdots110}^{i}\ \overbrace{11\cdots110}^{i}) = 1$$

And finally from the given identity we get

$$\begin{array}{c} f(\overbrace{00\cdots001}^{i}\ \overbrace{11\cdots110}^{i}\ \overbrace{11\cdots110}^{i}) = 1 \\ f(00\cdots010\ 11\cdots101\ 11\cdots101) = 1 \\ \iddots \qquad \iddots \qquad \iddots \qquad \vdots \\ f(10\cdots000\ 01\cdots111\ 01\cdots111) = 1 \end{array}$$

which contradicts the fact, that f is compatible with the $\mathbf{R}_i$ relation.

- When i is even we use identities

$$\begin{aligned} & f(y\ \overbrace{xx\cdots xxy}^{i}\ \overbrace{xx\cdots xxy}^{i}) = \\ = & f(y\ xx\cdots xyx\ xx\cdots xyx) = \\ & \vdots \qquad \iddots \qquad \iddots \\ = & f(y\ yx\cdots xxx\ yx\cdots xxx) \end{aligned}$$

$$f(y\ \overbrace{xx\cdots xxy}^{i}\ \overbrace{yy\cdots yyy}^{i}) = f(x\ \overbrace{xx\cdots xxx}^{i}\ \overbrace{xx\cdots xxx}^{i})$$

- $f \in \mathcal{C}(L)$. We define f as the sum of all variables.
- $f \in \mathcal{C}(C^{\leq})$. We define f as the majority on the first $i+1$ variables.
- $f \in \mathcal{C}(B^{\leq}_{i-1})$. We define f as

$$f^{-1}(1) = \mathrm{UP}\left\{\begin{array}{l} 0\ \overbrace{11\cdots110}^{i}\ \overbrace{00\cdots000}^{i} \\ 0\ 11\cdots101\ 11\cdots101 \\ 0\ 11\cdots011\ 11\cdots011 \\ \vdots \qquad \iddots \qquad\quad \iddots \\ 0\ 01\cdots111\ 01\cdots111 \end{array}\right\}$$

- $f \in \mathcal{C}(B_\infty)$. We define f as

$$f^{-1}(1) = \left\{ \begin{matrix} 1 \; \overbrace{11\cdots111}^{i} \; \overbrace{11\cdots111}^{i} \\ 0 \; 11\cdots110 \; 00\cdots000 \end{matrix} \right\}$$

- $f \notin \mathcal{C}(B_i^{\leq})$. For every such f we have

$$1 = f(1\; \overbrace{11\cdots111}^{i}\; \overbrace{11\cdots111}^{i}) = f(0\; \overbrace{11\cdots110}^{i}\; \overbrace{00\cdots000}^{i})$$

From f being monotone we get

$$f(0\; \overbrace{11\cdots110}^{i}\; \overbrace{00\cdots000}^{i}) = 1 \Rightarrow f(f(0\; \overbrace{11\cdots110}^{i}\; \overbrace{11\cdots110}^{i})) = 1$$

And finally from the given identity we get

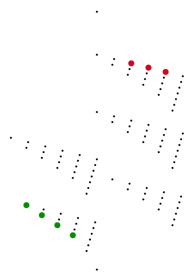
$$\begin{aligned} f(0\; \overbrace{11\cdots110}^{i}\; \overbrace{11\cdots110}^{i}) &= 1 \\ f(0\; 11\cdots101\; 11\cdots101) &= 1 \\ \vdots \quad \iddots \quad\quad \iddots & \\ f(0\; 01\cdots111\; 01\cdots111) &= 1 \end{aligned}$$

which contradicts the fact, that f is compatible with the $\mathbf{R}_i$ relation.

□

Lemma 3.9. $i \geq 3$ then $\mathcal{C}(L, C^{\leq}, B_{i-1}^{\leq}) \nleq \mathcal{C}(B_i)$

Proof.



- When i is odd we use identities

$$\begin{aligned} & f(y\; \overbrace{yy\cdots yyx}^{i}\; \overbrace{xx\cdots xxy}^{i}) = \\ &= f(y\; yy\cdots yxy\; xx\cdots xyx) = \\ & \quad \vdots \quad\quad \iddots \quad\quad\quad \iddots \\ &= f(y\; xy\cdots yyy\; yx\cdots xxx) = f(x\; xx\cdots xxx\; xx\cdots xxx) \end{aligned}$$

  - $\in \mathcal{C}(L)$. We define f as the sum of all variables.
  - $\in \mathcal{C}(C^{\leq})$. We define f as the majority on the last i variables.
  - $\in \mathcal{C}(B_{i-1}^{\leq})$. We define f as

$$f^{-1}(1) = \mathrm{UP} \left\{ \begin{matrix} 0\; \overbrace{00\cdots001}^{i}\; \overbrace{11\cdots110}^{i} \\ 0\; 00\cdots010\; 11\cdots101 \\ 0\; 00\cdots100\; 11\cdots011 \\ \vdots \quad \iddots \quad\quad \iddots \\ 0\; 10\cdots000\; 01\cdots111 \end{matrix} \right\}$$

  - $\notin \mathcal{C}(B_i)$. For every such f we have

$$\begin{aligned} f(1\; \overbrace{11\cdots111}^{i}\; \overbrace{11\cdots111}^{i}) = & f(0\; \overbrace{00\cdots001}^{i}\; \overbrace{11\cdots110}^{i}) = 1 = \\ & f(0\; 00\cdots010\; 11\cdots101) = 1 = \\ & \vdots \quad \iddots \quad\quad \iddots \\ & f(0\; 10\cdots000\; 01\cdots111) = 1 = \end{aligned}$$

which contradicts the fact, that f is compatible with the $\mathbf{R}_i$ relation.

- When i is even we use identities

$$\begin{aligned} & f(y\,\overbrace{xx\cdots xxy}^{i}) = \\ = & f(y\ xx\cdots xyx) = \\ & \vdots \qquad \iddots \\ = & f(y\ yx\cdots xxx) = f(x\ xx\cdots xxx) \end{aligned}$$

  - $\in \mathcal{C}(L)$. We define f as the sum of all variables.
  - $\in \mathcal{C}(C^{\leq})$. We define f as the majority on all variables.
  - $\in \mathcal{C}(B_{i-1}^{\leq})$. We define f as

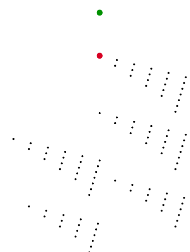
$$f^{-1}(1) = \mathrm{UP}\left\{\begin{matrix} 0\ \overbrace{11\cdots 110}^{i} \\ 0\ 11\cdots 101 \\ 0\ 11\cdots 011 \\ \vdots \qquad \iddots \\ 0\ 01\cdots 111 \end{matrix}\right\}$$

  - $\notin \mathcal{C}(B_i)$. For every such f we have

$$\begin{aligned} f(1\,\overbrace{11\cdots 111}^{i}) = & f(0\,\overbrace{11\cdots 110}^{i}) = 1 = \\ & f(0\ 11\cdots 101) = 1 = \\ & \vdots \qquad \iddots \\ & f(0\ 01\cdots 111) = 1 = \end{aligned}$$

    which contradicts the fact, that f is compatible with the $\mathbf{R}_i$ relation.

Lemma 3.10. $M \not\leq I$, where M is the boolean clone consisting of all functions and I is the clone consisting of all idempotent functions.



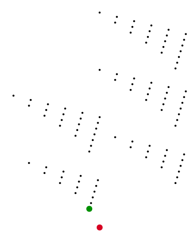
Proof. We will use the identity

$$f(x) = f(y)$$

- $f \in M$. Any constant function satisfies the condition.
- $f \notin I$. For idempotent function we have $f(x) = x \neq y = f(y)$. Thus no such f exists.

Lemma 3.11. $\mathcal{C}(L, C^{\leq}, A) \not\leq \Pi$, where Π is the clone consisting of all projections.

Proof. We will use the identities



$$\begin{aligned} f(yyyxxxx) &= f(xxyxxxx) \\ f(xxyyyxx) &= f(xxxxyxx) \\ f(xxxxyyy) &= f(xxxxxxy) \\ f(xxxxyyy) &= f(xxxxyxx) \end{aligned}$$

- $f \in L$. We define f as the sum of all variables.
- $f \in C^{\leq}$. We define f as the majority on all variables.
- $f \in A$. we define f as the meet of all variables.
- $f \notin \Pi$. We will show it cannot be projection on none of the coordinates.
  - Projection onto coordinate 1, or 2 contradicts the first identity.
  - Projection onto coordinate 3, or 4 contradicts the second identity.
  - Projection onto coordinate 5, or 6 contradicts the third identity.
  - Projection onto the last coordinate contradicts the last identity.

3.2 Finishing the proof

Now we will show why there does not exists any inequality $X \leq Y$ that is not drawn in Figure 3.2. For that we will show few more definitions and lemmas from lattice theory [13].

Definition 3.12. (critical pair) Let L be a lattice given by order $\preceq$ and x, y two of its elements. Then we call ordered pair (x, y) *critical* if it satisfies the following[1]

- x is minimal among all elements $\{e : e \npreceq y\}$.
- y is maximal among all elements $\{e : e \nsucceq x\}$.

Definition 3.13. (meet irreducible) Let L be a lattice, then its element x is *meet irreducible* if whenever $x = \bigwedge S$, then $x \in S$.

Definition 3.14. (join irreducible) Let L be a lattice, then its element x is *join irreducible* if whenever $x = \bigvee S$, then $x \in S$.

Lemma 3.15. Let $x \in L$. Then let y be such that y is maximum of the set $\{s : x \npreceq s\}$. Then y is meet irreducible.

Proof. Suppose for contradiction that $y = \bigwedge S$ and for every $s \in S$, $s > y$. Then because y is maximal among all elements not above x we get that for every $s \in S$ we have $s \succeq x$. But from that we have $x \preceq y$, which contradicts the fact that $x \npreceq y$. □

Lemma 3.16. Let $y \in L$. Then let x be such that x is minimum of the set $\{s : s \npreceq y\}$. Then x is join irreducible.

Proof. Proof is the same as in Lemma 3.15 when we interchange joins and meets. □

Corollary 3.17. Suppose (x, y) is a critical pair in a lattice L, then x is join irreducible and y is meet irreducible

Proof. Is a direct consequence of Lemma 3.15 and Lemma 3.16. □

Let L be the lattice of all products of boolean clones modulo trivial equivalence from Figure 3.2. Now we will finish the proof that the lattice of all products of boolean clones modulo minion homomorphisms is the lattice L. We will show it in two parts. Let us remind that to finish the proof we need to show that for every $X \npreceq Y$ we have $X \nleq Y$. First we will show that the lemmas from previous chapter show this for all critical pairs (X, Y). Then we will show why in this case that is sufficient.

Lemma 3.18. Let $Y \in L$ be meet irreducible. Then the following holds

- There exists unique X such that (X, Y) is critical. Let us fix such X.
- For every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$.
- It holds that $X \nleq Y$.

Proof. The following Figure 3.3 shows all the meet irreducible elements of L. Note that all such elements must be products of single sorted clones, as otherwise they would have been meet of all their component clones. As such, they have to be from the Figure 3.1. But not all of those are meet irreducible. The elements $B_\infty = \bigwedge B_i$, $B_\infty^{\leq} = \bigwedge B_i^{\leq}$ are not. Now we will go through all these meet irreducible elements Y and for each one will show all the elements X, such that (X, Y)

[1] Usually it is required to have x, y incomparable, we omit this part of the definition here.

is critical. For this for every case there are two figures, one figure is showing the critical pair (X, Y) in lattice L with X being green, Y red and all elements $Z\colon Z \preceq Y$ being blue. And the other figure shows the element Y in the lattice from Figure 3.1 with Y being red, all elements whose product is X are green and all elements $Z\colon Z \preceq Y$ are blue.

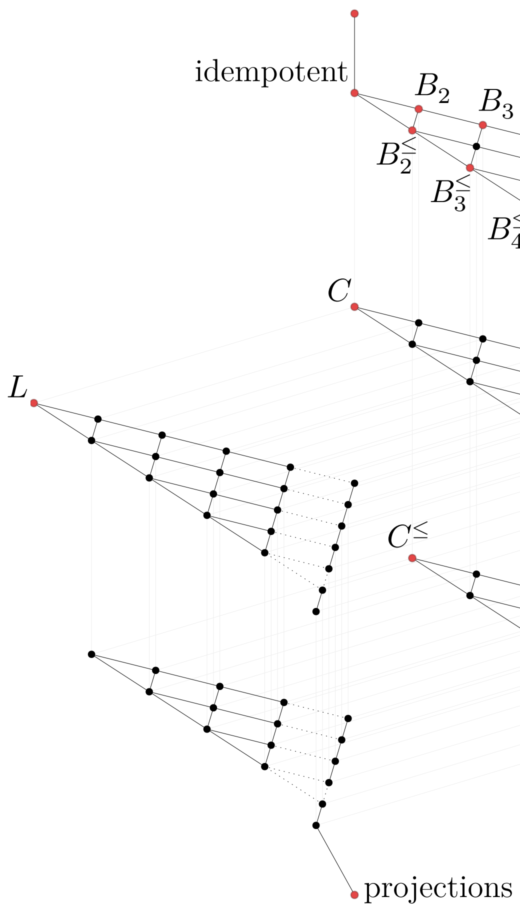


Figure 3.3 All meet irreducible elements of the lattice L.

- When $Y = B_i$ for $i \geq 3$, we get that there exists unique minimum of the set $\{s\colon s \npreceq Y\}$, which is $\mathcal{C}(L, C^{\leq}, B_{i-1}^{\leq})$. We see that $(\mathcal{C}(L, C^{\leq}, B_{i-1}^{\leq}), B_i)$ is unique critical pair for $Y = B_i$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.9 we have $X \npreceq Y$.

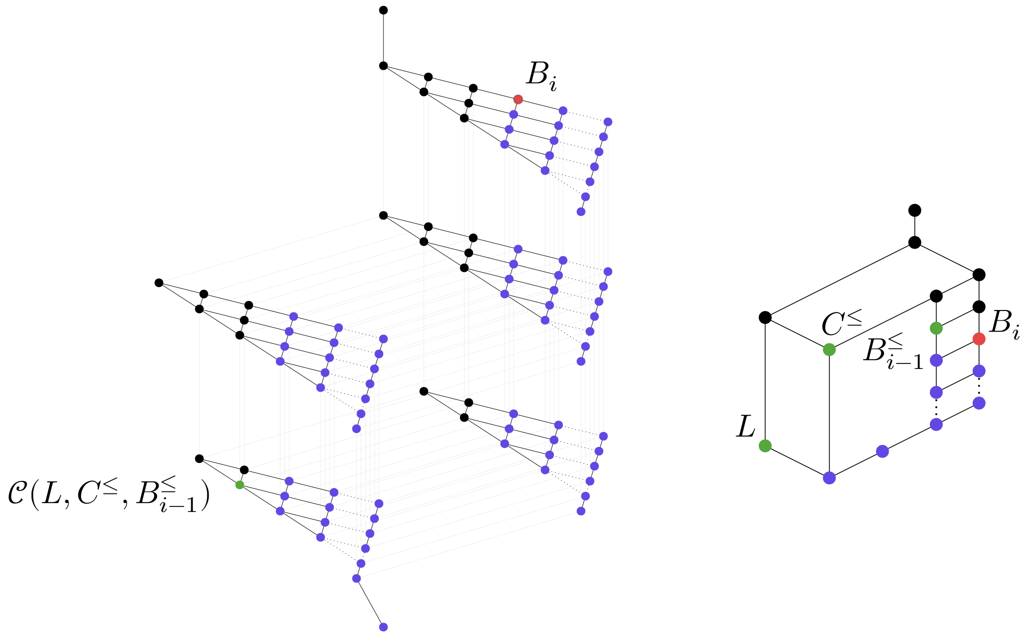


Figure 3.4 Lemma 3.9

- When $Y = B_2$, we get that there exists unique minimum of the set $\{s \colon s \npreceq Y\}$, which is L. We see that (L, B_2) is unique critical pair for $Y = B_2$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.2 we have $X \npreceq Y$.

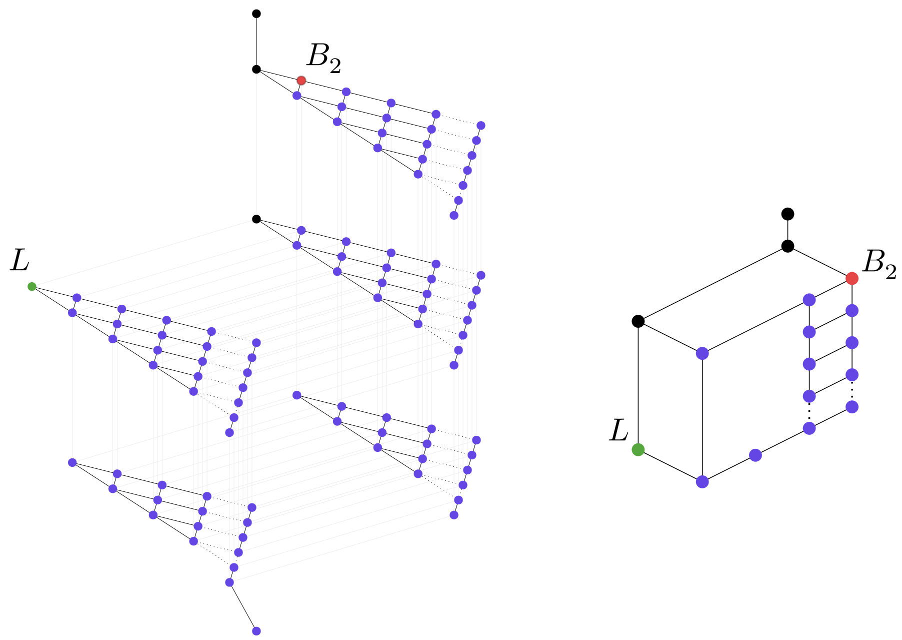


Figure 3.5 Lemma 3.2

- When $Y = B_2^{\leq}$, we get that there exists unique minimum of the set $\{s \colon s \npreceq Y\}$, which is $\mathcal{C}(L, B_\infty)$. We see that $(\mathcal{C}(L, B_\infty), B_2^{\leq})$ is unique critical pair for $Y = B_2^{\leq}$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.7 we have $X \npreceq Y$.

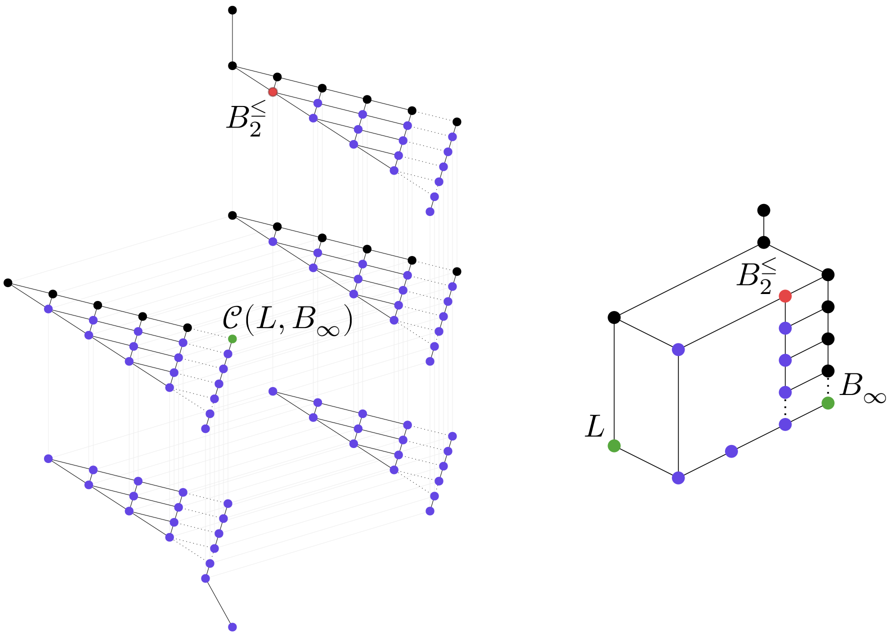


Figure 3.6 Lemma 3.7

- When $Y = B_i^{\leq}$, we get that there exists unique minimum of the set $\{s : s \npreceq Y\}$, which is $\mathcal{C}(L, C^{\leq}, B_\infty, B_{i-1}^{\leq})$. We see that $(\mathcal{C}(L, C^{\leq}, B_\infty, B_{i-1}^{\leq}), B_i^{\leq})$ is unique critical pair for $Y = B_i^{\leq}$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.8 we have $X \npreceq Y$.

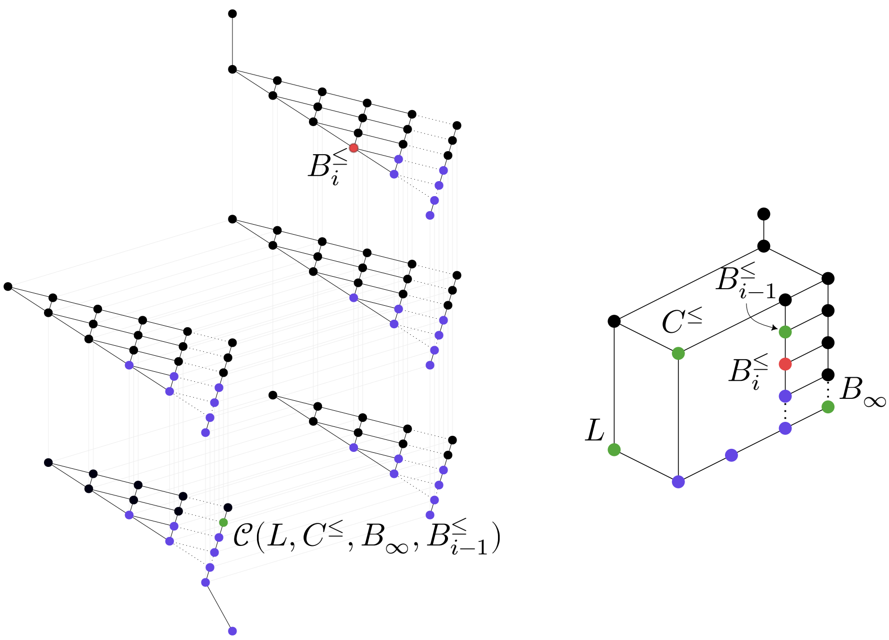


Figure 3.7 Lemma 3.8

- When $Y = A$, we get that there exists unique minimum of the set $\{s : s \npreceq Y\}$, which is $\mathcal{C}(L, C^{\leq}, B_\infty^{\leq})$. We see that $(\mathcal{C}(L, C^{\leq}, B_\infty^{\leq}), A)$ is unique critical pair for $Y = A$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.6 we have $X \npreceq Y$.

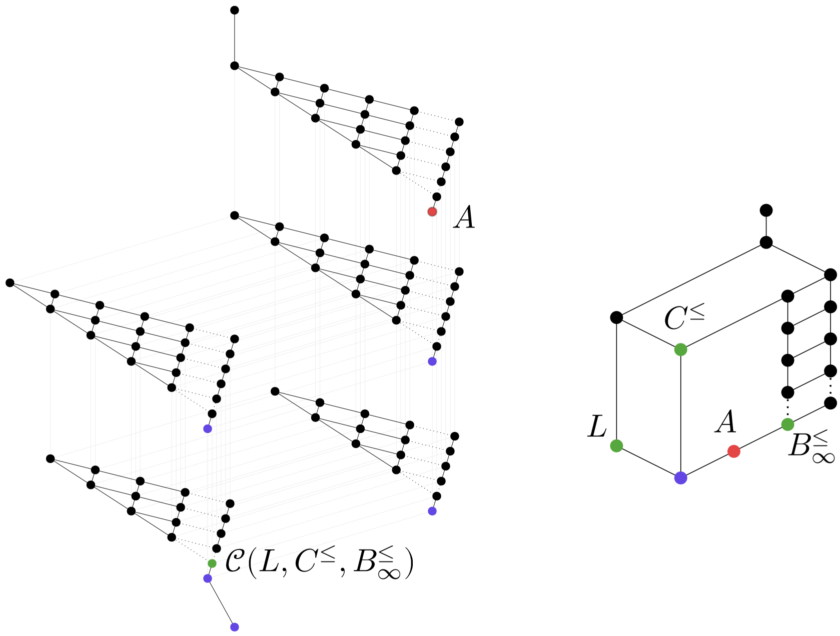


Figure 3.8 Lemma 3.6

- When $Y = C$, we get that there exists unique minimum of the set $\{s: s \npreceq Y\}$, which is A. We see that (A, C) is unique critical pair for $Y = C$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.3 we have $X \npreceq Y$.

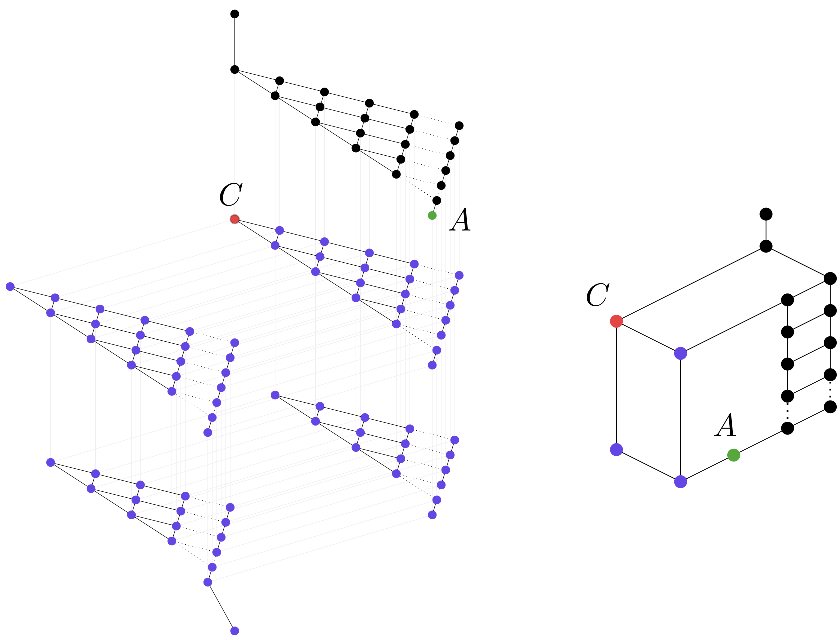


Figure 3.9 Lemma 3.3

- When $Y = C^{\leq}$, we get that there exists unique minimum of the set $\{s: s \npreceq Y\}$, which is $\mathcal{C}(L, A)$. We see that $(\mathcal{C}(L, A), C^{\leq})$ is unique critical pair for $Y = C^{\leq}$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.4 we have $X \npreceq Y$.

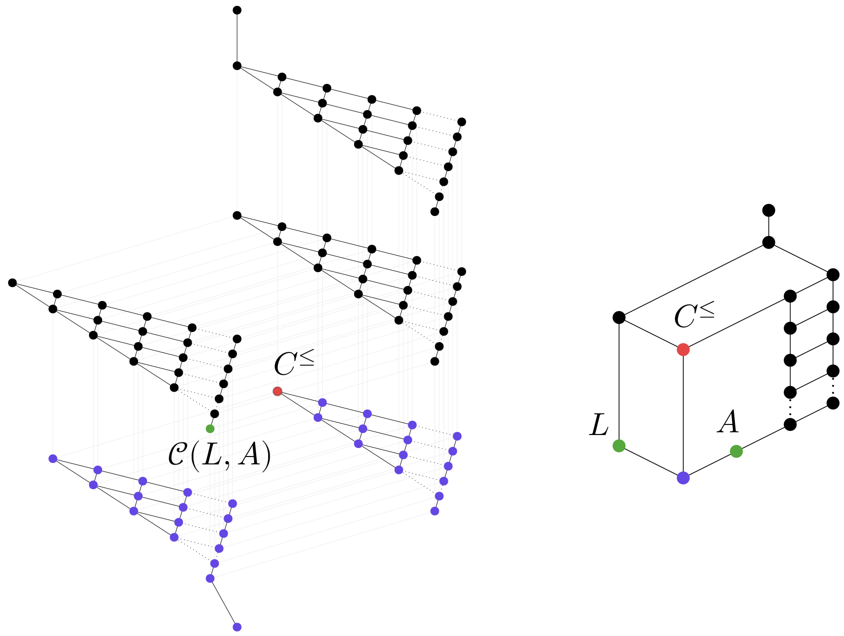


Figure 3.10 Lemma 3.4

- When $Y = L$, we get that there exists unique minimum of the set $\{s\colon s \not\preceq Y\}$, which is $\mathcal{C}(C^{\le}, A)$. We see that $(\mathcal{C}(C^{\le}, A), L)$ is unique critical pair for $Y = L$. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.5 we have $X \not\preceq Y$.

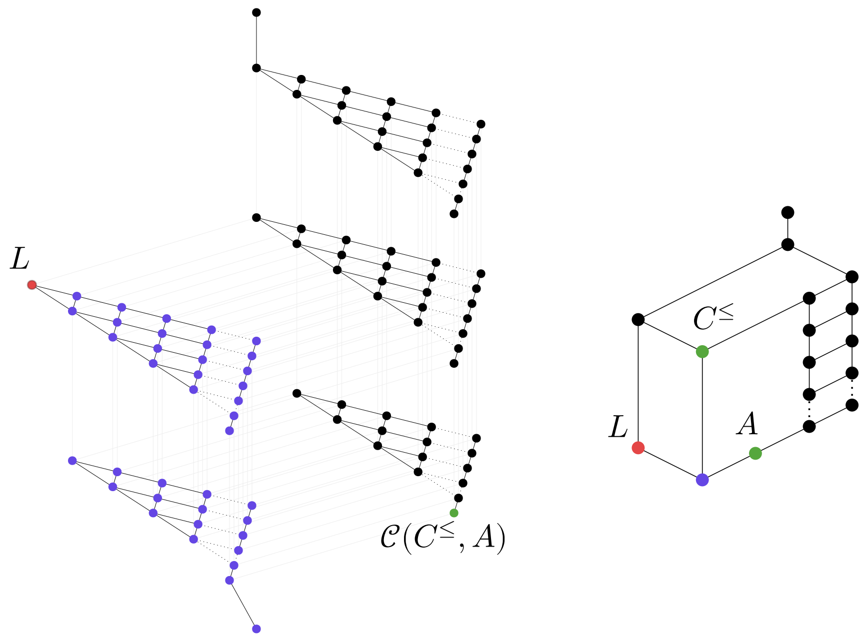


Figure 3.11 Lemma 3.5

- When Y is the clone consisting of all idempotent functions, we get that there exists unique minimum of the set $\{s\colon s \not\preceq Y\}$, which is the boolean clone consisting of all functions. From Lemma 3.10 we have $X \not\preceq Y$.

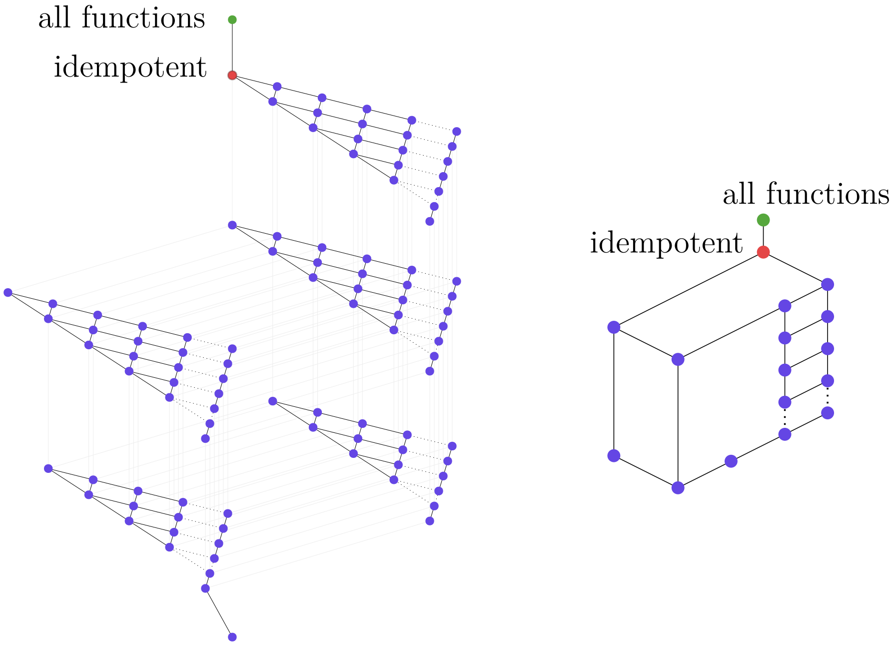


Figure 3.12 Lemma 3.10

- When Y = projections, the clone consisting of all projections, we get that there exists unique minimum of the set $\{s : s \npreceq Y\}$, which is $\mathcal{C}(L, C^{\leq}, A)$. We see that $(\mathcal{C}(L, C^{\leq}, A), \text{projections})$ is unique critical pair for Y = projections. And note that in this case for every element $Z \in L$ either $Z \preceq Y$ or $X \preceq Z$. From Lemma 3.11 we have $X \npreceq Y$.

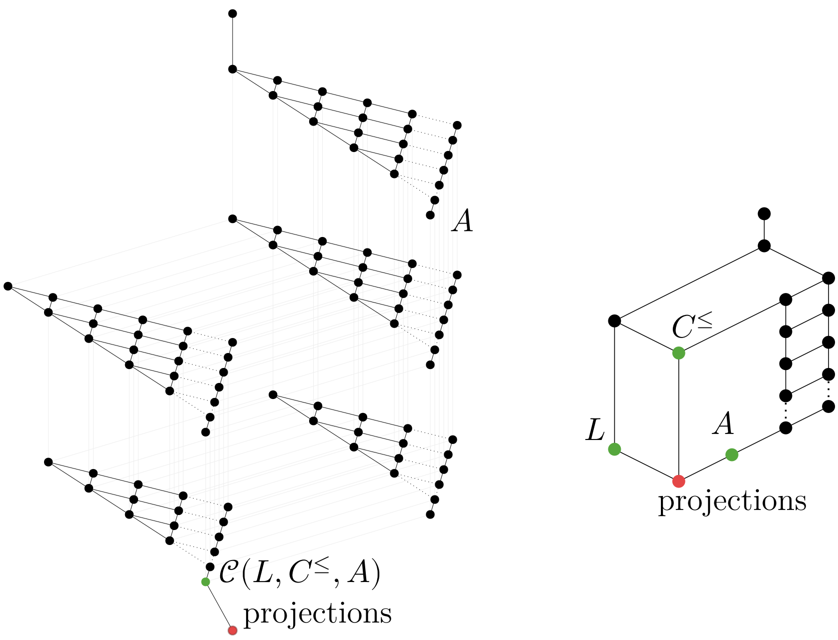


Figure 3.13 Lemma 3.11

□

Lemma 3.19. For every $X, Y \in L$ such that $X \npreceq Y$ we have $X \nleq Y$.

Proof. Let Y_1 be the maximum element of the set $\{s : s \succeq Y,\ X \npreceq s\}$, which exists because our lattice does not have upward infinite sequence. Note that $Y_1 \succeq Y$. Then observe that this is also the maximum of the set $\{s : X \npreceq s\}$, and thus from Lemma 3.15 we have Y_1 is meet irreducible. And thus from Lemma 3.18 there exists unique X_1 such that (X_1, Y_1) is critical and also from Lemma 3.18 because $X \npreceq Y_1$ we have $X \succeq X_1$.

Now suppose for contradiction that $X \leq Y$, then we have $X_1 \leq X \leq Y \leq Y_1$ and hence from transitivity we have $X_1 \leq Y_1$. But this contradicts that $X_1 \nleq Y_1$ which we know from Lemma 3.18. □

Theorem 3.20. The lattice of all products of boolean clones modulo minion homomorphism is the lattice L shown in Figure 3.2.

Proof. From Lemma 1.32 we have that when $X \preceq Y$ then $X \leq Y$ and from 3.19 we have when $X \npreceq Y$ then $X \nleq Y$. Combining these we get that

$$X \preceq Y \Leftrightarrow X \leq Y$$

This finishes the proof. □

Conclusion

In this thesis we showed the full lattice of all products of boolean clones modulo minion homomorphisms. To prove that it looks as in Figure 3.2 we have proven that there are no collapses on the critical pairs. It would be interesting to understand a bit more what are some easily checkable properties of the lattice that allowed us to check it only on critical pairs.

Additionaly to make working with the lattice easier it would help to have so-called minion cores of each described clone. These minion cores are in some sense the smallest representants of the equivalence classes. They no longer have to be clones, but they still give us a better way to talk about these equivalence classes. We have already put some work into studying these and will publish them in some later paper.

The next natural thing to study is to study all the multisorted boolean clones. There have already been some progress in understanding those multisorted clones that are determined by binary relations by Kapytka [9].

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