Štěpančík, Formalization of Homotopy Pushouts in Homotopy Type Theory: Referee report

Summary

Vojtěch has written a clear and detailed thesis on pushouts and their identity types in homotopy type theory. The thesis includes an introduction to homotopy type theory, descriptions of existing work that he has formalised for the `agda-unimath` library, and some original research, including a new result on identity systems for dependent types over pushouts, and multiple new proofs of known results. The presentation is very good, well structured, with diagrams where appropriate, clearly written explanations and the right level of detail. I only found one minor error on reading through. The written proofs are all correct as far as I can tell, despite being quite technical in some places. One consequence of putting in the work of producing a computer formalisation is that the formal proofs are guaranteed to be correct. The thesis is absolutely a suitable quality to be recognised as a Master's thesis. I have decided to recommend a grade of 1. The decision of recommending 1 rather than 2 was made on the basis of the very clear presentation and the original research in section 2.4.

Formalisation and explanation of known results

The attached diff files contain code added to the `agda-unimath` library of formalised mathematics (or in one case to be potentially added in the future), totalling to a net addition of 19,773 lines, which is not a small amount. This is mostly Agda code, but also includes documentation, as is standard for `agda-unimath`. Each file includes descriptions and occasionally ascii diagrams of the ideas used in the code. These are much more concise than the corresponding thesis sections, but I found that they are also very clearly written. As well as the main results of the thesis, this has included the descent theorem for the circle, descent for pushouts and also the results of Sojakova, Van Doorn and Rijke on sequential colimits, as described in section 3.2.

Many of the results are important foundational results in HoTT and will likely be used a lot by `agda-unimath` users and developers.

For people unfamiliar with libraries of formalised mathematics, I point out that although contributions to the libraries are not publications in a formal sense, they follow a similar process. Each contribution is packaged as a "pull request," which is peer reviewed by other developers to ensure it meets the standards expected, before being made publicly available as part of the larger library.

Original work

The best example of original research appears in section 2.4 with a connection between descent data for pushouts and identity systems. Namely, Vojtěch gives an explicit description of identity systems for dependent types over pushouts in terms of their descent data, and proof that it is equivalent to the usual notion.

As Vojtěch points out, this can be seen as a generalisation of a result of Kraus and von Raumer. The result is used later in the partial proof of Wärn's theorem in chapter 4. I think this is a promising approach to aid in both formalising and better explaining some implicit observations in Wärn's proof.

In addition to the above, chapter 2 also includes a new proof of the flattening lemma for pushouts. The result itself is by now well established in homotopy type theory. I would not say there is a very big difference to the proof for co-equalizers in [Uni13], but the proof does demonstrate a thorough understanding of the techniques used for this kind of result and there are some minor differences that may have helped make the formalisation as clear and efficient as possible.

The last chapter describes a mostly complete formalisation of a recent result due to Wärn. This is solution to a problem that was open for several years, of finding an explicit characterisation of identity types of pushouts. To give some indication why this problem was difficult, I point out that it includes as a special case loop spaces of n-dimensional spheres, which are known in homotopy theory to have rich structure.

Vojtěch gives a very clear account of the key ideas used in the proof, including the role of the "zigzag" construction and a description of his mostly complete formalisation. There are now two written accounts by Wärn of his result. The thesis follows the earlier version, with Vojtěch raising the valid point that that version may make it easier to obtain concrete descriptions in type theory of the equivalences involved. Vojtěch's proof differs from Wärn's in the use of identity systems as mentioned above, in filling in many details that were omitted from the original and in making some technical changes that make the proof easier to formalise in type theory. The proof requires working with four sequences of terms defined by simultaneous induction on the natural numbers, which is something that does lead to technical challenges. This is further complicated by the construction of higher coherences, which again are quite notorious for making formalisations technically difficult. Despite this, Vojtěch did a good job of understanding the problems involved and putting together a detailed plan for dealing with them. This was mostly carried out, with the remaining required coherence term stated clearly as a conjecture.

A minor error

In the summary I mentioned a minor error, so for completeness here it is. There appears to be a missing assumption in the statement of Corollary 1.0.17 or possibly "-1-truncated" was intended in place of "contractible" - as stated it includes the case where $A = 1$ and $P(x) = 2$, where the type of equivalences is empty.

As I mentioned, I didn't notice any other errors, which for a thesis this length indicates good attention to detail.

References

[Uni13] Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.