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OF MATHEMATICS
AND PHYSICS
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BACHELOR THESIS

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Weak saturation processes in multipartite hypergraphs

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Study programme: Computer Science

Prague 2024

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Here, I would like to express my gratitude to my supervisor, Mykhaylo Tyomkyn, Ph.D., for his guidance and the time he dedicated to our consultations.

Title: Weak saturation processes in multipartite hypergraphs

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Abstract: Given hypergraphs H and P, $wsat(H, P)$ denotes the smallest number of edges in a subgraph of H with the property that the missing edges can be sequentially added such that the addition of every edge creates a new copy of P. In 1985 Alon proved that $wsat(K_n, P)/n$ tends to a finite limit for any graph P. A generalisation of this Theorem to r-uniform hypergraphs was conjectured by Tuza in 1992 and proved by Shapira and Tyomkyn in 2021. In this thesis, we use the methodology introduced by Shapira and Tyomkyn to prove a similar theorem when H is a complete r-partite r-uniform hypergraph.

Keywords: wsat, weak saturation, hypergraph, extremal combinatorics

Název práce: Procesy slabej saturácie v multipartitných hypergrafoch

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Vedoucí bakalářské práce: Mykhaylo Tyomkyn, Ph.D., Katedra aplikovanej matematiky

Abstrakt: Dané hypergrafy H a P, $wsat(H, P)$ označuje najmenší počet hrán v podgrafe H s vlastnosťou, že chýbajúce hrany možno postupne pridať tak, že pridanie každej hrany vytvorí novú kópiu P. V roku 1985 Alon dokázal, že $wsat(K_n, P)/n$ konverguje k vlastnej limite pre akýkoľvek graf P. Tuza sa v roku 1992 domnieval, že platí zobecnenie tejto vety pre r-uniformné hypergrafy a dokázali ho Shapira a Tyomkyn v roku 2021. V tejto práci používame metodológiu, ktorú zaviedli Shapira a Tyomkyn, aby sme dokázali podobnuú vetu, v ktorej H je úplný r-partitný r-uniformný hypergraf.

Klíčová slova: wsat, slabá saturácia, hypergraf, extremálna kombinatorika

Contents

1 Introduction

Hypergraphs generalize the notion of a graph by allowing edges to connect more than two vertices. Formally, a hypergraph (V, E) is a structure composed of a finite set of vertices V and a set of edges E. Each edge $e \in E$ is a set of at least two vertices and we say that the vertices in this set are connected by the edge e. Given a hypergraph H, we use $V(H)$ to denote its set of vertices and $E(H)$ the set of edges.

We call a hypergraph H *r-uniform* if and only if each edge contains precisely r vertices. For brevity, we shorten r-uniform hypergraph to r-graph. In particular, 2-graphs are just ordinary graphs.

Further, we extend the notion of completeness from graphs to hypergraphs. An r-graph is said to be complete, if it contains all possible edges. In particular, the edge set of a complete r-graph with n-vertices has cardinality $\binom{n}{r}$.

For $k \geq r$, *k-partite r-graphs* are such hypergraphs, whose vertex set can be partitioned into k disjoint *partition classes* such that no edge contains more than one vertex from the same partition class. A *complete k-partite r-graph* is a k-partite r-graph containing all edges, which have at most one vertex in each partition class.

Hypergraphs have been objects of extensive study in *extremal combinatorics*, a field of mathematics, which aims to answer how big a combinatorial object has to be for it to certainly possess a given property or how small a combinatorial object possessing a certain property can be. Some examples of famous theorems of extremal combinatorics are *Turán's theorem* and the *Erdős-Ko-Rado theorem.*

Turán's theorem states that there does not exist graph with n vertices not containing K_{k+1} as a subgraph with more edges than the Turán graph $T(n, k)$. The Turán graph $T(n, k)$ is a complete k-partite graph on n vertices, with the vertices divided into partitions as evenly as possible.

The *Erdős-Ko-Rado theorem* concerns hypergraphs. It states, that an r-graph on n vertices, in which $n \geq 2r$ and each two edges have at least one vertex in common has at most $\binom{n-1}{r-1}$ edges.

In this thesis we study another extremal problem. *Weak saturation* was first defined by Bollobás in 1968 [1]. Given hypergraphs P, G and another hypergraph H, such that G is a subgraph of H on the same vertex set, we say that G is *weakly P-saturated* in the *host* hypergraph H with respect to the *pattern* hypergraph P, if the edges $E(H) \setminus E(G)$ can be added sequentially in such a way, that addition of every edge creates a new copy of P. This sequence of edges $e_1, e_2, ..., e_{|E(H)\setminus E(G)|}$ is called the *P-saturation process* of G in H. Define the *weak saturation number* of P in H, denoted by $wsat(H, P)$, to be the smallest possible number of edges in a weakly P-saturated hypergraph G. To practice the definition, note that $wsat(K_n, K_3) = n - 1$ for all n and the smallest weakly K_3-saturated graphs in a copy of K_n are precisely all trees. The search for exact values of this function is a challenging problem, both combinatorially and algorithmically.

To see why it is so, consider the following algorithmic problem, a natural extension of the example above, which is a considerable simplification of the general problem as we use $r = 2$ and $P = K_3$.

Problem. *Given a host graph H on n vertices, determine if $wsat(H, K_3) = n - 1$.*

Having studied this problem, we could not find any kind of monotonicity in the host graphs. Note that, the addition of a single edge can drastically change the structure of the graph with respect to this problem. Such an example can be seen in Figure 1.1.

Figure 1.1 *left:* the graph does not have a weakly K_3-saturated subgraph with 4 edges, *right:* dashed lines mark the weakly K_3-saturated subgraph

As is common in extermal combinatorics, we instead concentrate on the asymptotic growth of the function $wsat(H, P)$ as the size of H grows. In doing so, it is a natural setup to work within a "nice" class of host graphs such as complete partite hypergraphs.

There has been a lot of study of $wsat(K_n^r, H)$, where K_n^r is a complete r-graph of order n. In the paper [1] first introducing the concept of weak saturation, Bollobás proved for $3 \leq k \leq 7$ and all $n \geq k$ that

$$wsat(K_n, K_k) = (k-2)n - \binom{k-1}{2}.$$

This result was later proved for all k by Frankl [2] and, independently, Kalai [3] [4] with tools from algebra and geometry, however no purely combinatorial proof is known to date.

In [2] Frankl introduced the skew version of Bollobaś Two Families Theorem and the result follows as its application.

In 1984 Kalai proved it in [3] by showing that an embedding of weakly $K_{(d+2)}$-saturated graph into $\mathbb{R}^d$, such that its vertices are in general position is rigid and in 1985 in [4] by using tools from exterior algebra.

Continuing with graphs, but moving towards asymptotic results, in 1985 Alon [5] proved that for every graph P there exists a constant C_P such that

$$\lim_{n\to\infty} \frac{wsat(K_n, P)}{n} = C_P.$$

Alon's proof estabilishes that the sequence $\{wsat(K_n, P)\}_{n=|P|}^{\infty}$ is subadditive and subsequently applies Fekete's lemma. Extending this proof towards $r \geq 3$ does not seem easy, as for most graphs P, $wsat(K_n, P)$ is of order at least n^2 and we do not have an appropriate version of Fekete's lemma.

A first asymptotic result for hypergraphs has been obtained by Tuza. To state the result, we first need to introduce the following parameter. The *sparseness* of an r-graph P, denoted $s(P)$, is the size of the smallest vertex set contained in exactly one edge of P. Since a set of r vertices can be contained in at most one edge, $1 \leq s(P) \leq r$ always holds. Notice, that $s(P) = 1$ generalizes the familiar notion of a graph containing a leaf, that is, a vertex of degree 1, to hypergraphs.

It has been proved by Tuza [6] that for every r-graph P with sparseness s, there exist two constants $0 < c'_P \leq C'_P$, such that

$$c'_P \cdot n^{s-1} \leq wsat(K_n^r, P) \leq C'_P \cdot n^{s-1}.$$

A full extension of Alon's theorem to all $r \geq 3$ was obtained in 2021 by Shapira and Tyomkyn [7], who proved, that for every r-graph P with sparseness s there exists a constant $\overline{C_P} > 0$, such that

$$\lim_{n\to\infty} \frac{wsat(K_n^r, P)}{n^{s-1}} = \overline{C_P}.$$

Note that, as $1 \leq s \leq r$, for every r-graph P this gives a constant $\overline{C_P^*} \geq 0$, such that

$$\lim_{n\to\infty} \frac{wsat(K_n^r, P)}{n^{r-1}} = \overline{C_P^*}.$$

An analogous result for multipartite host graphs In this thesis, we prove an analogous result for another natural class of host graphs. Namely, the class of r-partite r-graphs with all partitions of the same size n; we use the notation $K_{r\times n}^r$ to denote this r-graph. Our main theorem is

Theorem 1.1. *For every r-partite r-graph P of sparseness s, there is a constant $C_P > 0$, such that*

$$\lim_{n\to\infty} \frac{wsat(K_{r\times n}^r, P)}{n^{s-1}} = C_P.$$

As $1 \leq s \leq r$, we get the following corollary.

Corollary 1.2. *For every r-partite r-graph P, there exists a constant $C_P^* \geq 0$, such that*

$$\lim_{n\to\infty} \frac{wsat(K_{r\times n}^r, P)}{n^{r-1}} = C_P^*.$$

To prove Theorem 1.1, we follow the methodology used in [7], although we have to jump through some complications, such as applying Rödl's Approximate Designs Theorem in the setting of multipartite hypergraphs.

2 Notation and proof overview

Proof and thesis overview To prove Theorem 1.1 we first define *Template saturation* in Chapter 3. It is a tool which provides us a way to bound the weak saturation numbers $wsat(K^r_{r\times n}, P)$ from above for all pattern r-graphs P with certain parameters (sparseness, the size of the biggest partition class). It will be then extensively used to prove that various graphs are weakly P-saturated for all pattern r-graphs P with the given parameters. At the start of Chapter 4 we show that for all pattern r-graphs P with sparseness s, there exists a constant $c_P > 0$ such that $wsat(K^r_{r\times n}, P) \geq c_P \cdot n^{s-1}$. We set C_P to be the limes inferior of the sequence $\{wsat(K^r_{r\times n}, P)/n^{s-1}\}_{n=1}^{\infty}$ and in the rest of the chapter 4, we prove that for all $\varepsilon > 0$, there exists a threshold n_0 such that for all $n \geq n_0$, $wsat(K^r_{r\times n}, P) \leq (C_P + 4\varepsilon) \cdot n^{s-1}$. To do this, for all $\varepsilon > 0$, we construct such weakly P-saturated graphs with few edges for large enough n. The tools used are Rödl Approximate Designs Theorem and template saturation. In this chapter we introduce some key definitions used throughout the proof and demonstrate their usage by extending Alon's proof for complete host graphs towards complete bipartite host graphs. In other words, we prove Corollary 1.2 for $r = 2$.

Notation The notation $G \cong H$ denotes that G is a copy of H. The vertex sets $V_1(H), V_2(H), ..., V_r(H)$ are the partition classes of an r-partite r-graph H (so if $H \cong K^r_{r\times n}$, then $|V_i(H)| = n$ for all i). The symbol $\sqcup$ denotes disjoint union. For an r-partite r-graph we use $s(P)$ to denote the spareseness of P and $p(P)$ to denote the size of the biggest partition class of P.

Definition 2.1 (Division into clusters)**.** *Define division of H into m clusters, as a collection of sets $C_1, C_2, ..., C_m$, satisfying*

$$\bigsqcup_{1\leq i\leq m} C_i = V(H)$$

and

$$|C_i \cap V_j(H)| = |C_i \cap V_k(H)| \text{ for all } 1 \leq i \leq m, 1 \leq j, k \leq r.$$

Denote $C_i \cap V_j(H)$ as $C_{i,j}$ and call it the j-th partition of i-th cluster.

Definition 2.2 (Uniform division into clusters)**.** *Assuming m divides n, a division of H into m clusters $C_1, C_2, ..., C_m$ is defined to be uniform if*

$$|C_i \cap V_j(H)| = \frac{n}{m} \text{ for all } 1 \leq i \leq m, 1 \leq j \leq r.$$

Definition 2.3 (Rigid and loose vertices)**.** *Given $p \geq 1$, a hypergraph H and a division of H into m clusters, such that $|C_i|/r \geq p$ for all $1 \leq i \leq m$, take an arbitrary set of p vertices $R_{i,j} \subseteq C_{i,j}$ for each $1 \leq i \leq m$ and $1 \leq j \leq r$, and call these vertices rigid. Let $R = \bigcup R_{i,j}$ be the set of all rigid vertices. We call all other vertices loose. Let $L_{i,j} = C_{i,j} \setminus R_{i,j}$ be the set of all loose vertices in j-th partition of i-th cluster and let $L = \bigcup L_{i,j}$.*

To practice the definitions, let us prove Corollary 1.2 for $r = 2$ using subadditivity and Fekete's lemma, similarly to how Alon proved it in [5] for complete graphs. Notice that if $r = 2$, the host graph is a complete bipartite graph. We state the simplified theorem

Theorem 2.4. *For every bipartite graph P, there is a constant $C_P \geq 0$, such that*

$$\lim_{n\to\infty} \frac{wsat(K_{n,n}, P)}{n} = C_P.$$

To prove Theorem 2.4, we first define subadditivity and state Fekete's lemma.

Definition 2.5 (subadditivity)**.** *A sequence $\{a_n\}_{n=1}^{\infty}$ is called subadditive if for all $n, m \geq 1$*

$$a_{n+m} \leq a_n + a_m.$$

Proposition 2.6. *(Fekete's Lemma [8]) For every subadditive sequence $\{a_n\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to the infimum $\inf \frac{a_n}{n}$.*

Lemma 2.7. *Given a bipartite graph P, let $p \geq 1$ be the size of its vigger partition class. Then for all $1 \leq p \leq m \leq n$*

$$wsat(K_{n+m,n+m}, P) \leq wsat(K_{n,n}, P) + wsat(K_{m,m}, P) + 2p^2 \tag{2.1}$$

Proof. Partition the host graph H, a copy of $K_{n+m,n+m}$, into two clusters C_1, C_2, such that $|C_1|/2 = n$ and $|C_2|/2 = m$. Place a copy of the graph G_1 witnessing $wsat(K_{n,n}, P)$ on C_1 and a copy of the graph G_2 witnessing $wsat(K_{m,m}, P)$ on C_2. Run the saturation process within each of the clusters to add all the edges contained in precisely one cluster to the saturation process.

It remains to add the edges with endpoints in different clusters. Designate p rigid vertices in each partition of each cluster of H. Place a copy of $K_{p,p}$ on the vertex sets $R_{1,1} \cup R_{2,2}$ and $R_{1,2} \cup R_{2,1}$. We prove that the resulting graph is weakly H-saturated.

Note that we have added all edges with both rigid endpoints to the saturation process already. Let us show how to add all edges with one loose and one rigid endpoint. Without loss of generatility, assume that the edge uv we aim to add, has endpoints $u \in R_{1,1}$ and $v \in L_{2,2}$. Let $D \subseteq R_{2,2}$ be a set of arbitrary $p-1$ vertices. Notice that the vertex set $R_{1,1} \cup D \cup \{v\}$ induces a $K_{p,p}$ except the edge uv is missing. When we add this edge, a new copy of $K_{p,p}$ is created and thus also a new copy of P is created. This way, all edges with one rigid and one loose endpoint can be added.

Assume edges with less than two loose endpoints have been added and we show how to add edges with both endpoints loose. Without loss of generality, assume edge uv is missing with endpoints $u \in L_{1,1}$, $v \in L_{2,2}$. Let $D_1 \subseteq R_{1,1}$, $D_2 \subseteq R_{2,2}$ be arbitrary sets of $p-1$ vertices. The vertex set $D_1 \cup D_2 \cup \{u, v\}$ induces a $K_{p,p}$ except the edge uv is missing. Similarly to before, adding uv creates a new copy of $K_{p,p}$ and thus also a new copy of P. All edges with both loose endpoints can be added this way.

We have constructed a weakly P-saturated graph with at most $wsat(K_{n,n}, P) + wsat(K_{m,m}, P) + 2p^2$ edges and its saturation process. □

As a corollary of Lemma 2.7, we obtain the subadditivity of the sequence $\{wsat(K_{n,n}, P) + 2p^2\}_{n=p}^{\infty}$. To prove this, add $2p^2$ to the both sides of the inequality (2.1).

Thus the limit of $((wsat(K_{n,n}, P) + 2p^2)/n)_{n=p}^{\infty}$ exists by Proposition 2.6. And so, the limit of $((wsat(K_{n,n}, P))/n)_{n=1}^{\infty}$ also exists, and

$$\lim_{n\to\infty} \frac{wsat(K_{n,n}, P)}{n} = \lim_{n\to\infty} \frac{wsat(K_{n,n}, P) + 2p^2}{n}.$$

The proof of Theorem 2.4 is thus finished.

3 Template saturation

Template saturation is a tool defined and extensively used in [7]. Given $k \geq r \geq s \geq 1$, it provides a way to bound the weak saturation numbers $wsat(K_n^r, P)$ from above for all r-graphs P of order k and sparseness s, in that we in some sense pick the worst possible r-graph P. This allowed the authors of [7] to group the pattern graphs by their sparseness and prove that the same bound holds for all graphs from each group.

As this is a tool we need to use as well, we extend the concept of template saturation to the multipartite setting. That is, template saturation will provide us a way to bound the weak saturation numbers $wsat(K_{r\times n}^r, P)$ from above for all r-partite r-graphs P with sparseness s and the size of the biggest partition class p.

Template saturation is considered with respect to some special graph T described below. The difference between the template saturation defined in [7] and the one defined here is the graph we build T from is $K_{r\times p}^r$ where p is the biggest partition class of P. Let us proceed with the formal definition.

Given $1 \leq s \leq r \leq m$, let $T_{s,r,m}^-$ be an r-graph obtained from $F \cong K_{r\times m}^r$, by selecting a set $X \subseteq V(F)$, such that $|X| = s$ and $|V_i \cap X| \leq 1$ for all i, and deleting all edges containing X as a subset. Let $T_{s,r,m}$ be an r-graph formed from $T_{s,r,m}^-$ by adding back one of the deleted edges. Call this added edge f *marked*. Note that whichever set X we choose to remove and whichever edge f we add back, the graphs will be isomorphic to each other. Proof of this fact is given in Lemma 3.1.

Define the $T_{s,r,m}$*-template saturation process* of an r-graph G in host hypergraph $H \cong K_{r\times n}^r$, as an ordering of edges $e_1, e_2, ..., e_{|E(H)\setminus E(G)|}$ satisfying the following conditions:

1. for all i, there exists a copy of $T_{s,r,m}$ in $G \cup \{e_1, e_2, e_3, ..., e_i\}$ in which e_i plays the role of the marked edge f
2. $G \cup \{e_1, e_2, ..., e_{|E(H)\setminus E(G)|}\} = H$

A graph G is $T_{s,r,m}$*-template saturated* in H if it admits a $T_{s,r,m}$-template saturation process.

Lemma 3.1. *Let $F \cong K_{r\times m}^r$. Let $X_1, X_2 \subseteq V(F)$ be sets of size s such that $|V_i \cap X_1| \leq 1$ and $|V_i \cap X_2| \leq 1$ for all i. Let T_1 and T_2 be r-graphs formed from F by removing all edges containing X_1 and X_2 except some edges f_1 and f_2 respectively (f_1 is the marked edge of T_1 and f_2 is the marked edge of T_2). Then there exists an isomorphism $\Phi : V(T_1) \mapsto V(T_2)$ which maps f_1 to f_2.*

Proof. We construct such map Φ. To construct Φ, map bijectively X_1 to X_2, $f_1 \setminus X_1$ to $f_2 \setminus X_2$ and $V(T_1) \setminus f_1$ to $V(T_2) \setminus f_2$, such that if two vertices are in the same partition class of T_1, their images are in the same partition class of T_2. Φ is a well-defined bijection, as the sets X_1, $f_1 \setminus X_1$, $V(T_1) \setminus f_1$ have the same sizes of partition classes as X_2, $f_2 \setminus X_2$, $V(T_2) \setminus f_2$ respectively.

The marked edge f_1 is mapped to f_2 by Φ as required. To prove Φ is an isomorphism, consider an arbitrary edge $e \in E(T_1)$. If $e = f_1$, then e is mapped to $f_2 \in E(T_2)$. Otherwise e does not contain X_1 as a subset and is thus mapped to $h \subseteq V(T_2)$, such that h does not contain X_2 as a subset and does not contain two vertices in the same partition. Thus $h \in E(T_2)$ and we have proved that any edge $e \in E(T_1)$ is mapped to some edge $h \in E(T_2)$. As $|E(T_1)| = |E(T_2)|$, Φ is an isomorphism. □

Notice, that for any $r \geq s^* \geq s \geq 1$ and $m \geq 1$ the graph $T_{s,r,m}$ is a subgraph of $T_{s^*,r,m}$. To see this, let $T_{s,r,m}$ be constructed from $F \cong K^r_{r\times m}$ by removing all edges containing subset $X = \{v_1, v_2, ..., v_s\}$, where $v_i \in V_i(F)$ and adding back an edge $f = \{v_1, v_2, ..., v_s, u_{s+1}, u_{s+2}, ..., u_r\}$ for some $u_i \in V_i(F)$. The r-graph formed from F by removing all edges containing as a subset $Y = \{v_1, v_2, ..., v_s, u_{s+1}, u_{s+2}, ..., u_{s^*}\}$ and adding back f is $T_{s^*,r,m}$. As $Y \supseteq X$, all edges containing Y also contain X. Thus all edges removed from F in the formation of $T_{s^*,r,m}$ have also been removed in the formation of $T_{s,r,m}$. This shows $T_{s^*,r,m}$ is a supergraph of $T_{s,r,m}$. By Lemma 3.1, the choice of subsets of vertices X and Y and which edges we choose we choose to leave in the graph does not matter, as the resulting graphs are all isomorphic to each other.

By a similar argument, albeit much simpler, as we can ignore the part of the proof concerning marked edges, for any $r \geq s^* \geq s \geq 1$, $m \geq 1$, $T^-_{s,r,m}$ is a subgraph of $T^-_{s^*,r,m}$.

3.1 Monotonicity of template saturation

In this section we precisely describe how to use template saturation to bound weak saturation numbers from above. To do this, we first show how to embed an r-partite r-graph P into $T_{s,r,p}$ with appropriate parameters.

Lemma 3.2. *Let P be an r-partite r-graph with $s(P) = s$ and $p(P) = p$. By the definition of sparseness, there exists $S \subseteq V(P)$ with $|S| = s$ contained in exactly one edge. Let e be the edge containing S. Then there exists an embedding Φ of P into a T, a copy of $T_{s,r,p}$, such that Φ maps e to the marked edge f of T.*

Proof. The proof is similar to the proof of Lemma 3.1. We construct the map Φ as follows. Map the vertices of e to the marked edge f, such that S is mapped to X, where X is the set chosen in definition of $T_{s,r,p}$. Afterwards map the rest of $V(P)$ to $V(T)$ arbitrarily, under the condition, that if two vertices are in the same partition class of P, their images are in the same partition class of T and no two vertices of P are mapped to the same vertex of T. Φ is an embedding. To prove this, consider an arbitrary edge $c \in E(P)$. If $c = e$, then the image of c is $f \in E(T)$. Otherwise c does not contain S as a subset. Such edge c is mapped to some $h \subseteq V(T)$, such that h does not contain X as a subset and does not contain two vertices from the same partition class. Thus $h \in E(T)$ and we have proved that Φ is an embedding. □

We now use the Lemma 3.2 [just estabilished] to prove the anticipated Lemma, which allows us to bound weak saturation numbers from above by template saturation.

Lemma 3.3. *Suppose the host graph $H \cong K^r_{r\times n}$ for some $n \geq 1$ and P is an r-partite r-graph with sparseness $s(P) = s$ and $p(P) = p$. If an r-graph G is $T_{s,r,p}$-template saturated in H, then G is also weakly P-saturated in H.*

Proof. We show that whenever an edge creating a new copy of a $T_{s,r,p}$ is added, such that the newly added edge can be mapped to the marked edge f, also a new copy of P is created.

Let $e_1, e_2, ...e_{|E(H)\setminus E(G)|}$ be the $T_{s,r,p}$-template saturation process of G in H and let T_i be the copy of $T_{s,r,p}$ created in the i-th step of the process. By the definition of sparseness, there exists $S \subseteq V(P)$ with $|S| = s$ contained in exactly one edge. Let e be the edge containing S. By Lemma 3.2 we can embed P into T_i, such that e is mapped to the marked edge f of T_i. Thus $G \cup \{e_1, e_2, ..., e_i\}$ contains a copy of P (subgraph of T_i) containing the edge e_i. □

The following lemmas allow us to construct weakly/template saturated graphs iteratively, that is, we start from an empty graph and then repeat adding some edges and running the saturation process within some part of the graph we built so far, until we get to the host graph, just like we did in the proof Theorem 2.4. For this, we prove that if we add edges to a weakly/template saturated graph G with respect to some graph P in H, it remains weakly/template saturated.

Lemma 3.4. *For any r-graphs $G \subseteq G' \subseteq H$ on the same vertex set and an r-partite r-graph P, if G is weakly P-saturated in H, then so is G'.*

Proof. We get a saturation process of G' in H from the saturation process of G in H by ignoring the edges already in G'. □

Lemma 3.5. *For any r-graphs $G \subseteq G' \subseteq H$ on the same vertex set, if G is $T_{s,r,p}$-template saturated in H, then so is G'.*

Proof. Exactly the same as Lemma 3.4. □

Lemma 3.6. *Let $H \cong K^r_{r\times p}$. For all $r \geq s^* \geq s \geq 1$ and r-graphs $G \subseteq H$, if G contains $T^-_{s^*,r,p}$ as a subgraph then G is $T_{s,r,p}$-template saturated in H.*

Proof. The graph $T^-_{s,r,p}$ is $T_{s,r,p}$-template saturated in $K^r_{r\times p}$. To see this, let f be an arbitrary missing edge from $T^-_{s,r,p}$. There is an isomorphism between $T^-_{s,r,p} \cup f$ and $T_{s,r,p}$ such that f is mapped to the marked edge of $T_{s,r,p}$. Thus any sequence of the missing edges forms a valid $T_{s,r,p}$-template saturation process.

As $T^-_{s,r,p}$ is a subgraph of $T^-_{s^*,r,p}$ and G, in turn, contains $T^-_{s^*,r,p}$ as a subgraph, G also contains $T^-_{s,r,p}$ as a subgraph. Applying Lemma 3.5 with $T^-_{s,r,p}$ and G playing the role of G and G' respectively, we conclude that G is $T_{s,r,p}$-template saturated in H. □

3.2 Small increases in host graph size

In this section, we show that small increases in host graph size lead to small increases in the weak saturation numbers.

Lemma 3.7. *Let $r \geq s \geq 2$, $a \geq b \geq 1$ and $a \geq p \geq 1$. Let $H \cong K^r_{r\times(a+b)}$. Suppose $V = A \sqcup B$ is the vertex set of H and $|V_i(H) \cap A| = a$ and $|V_i(H) \cap B| = b$ for all i. Let E be the set of all edges $e \in E(H)$ with $e \subseteq A$ (that is, (A, E) is complete r-partite). Then there exits a set of edges $E' \subseteq E(H)$ with $|E'| \leq r^{s-1}ba^{s-2}p^{r-s+1}$ such that $(V, E \cup E')$ is $T_{s,r,p}$-template saturated.*

Proof. We first describe the construction of E' and prove that $(V, E \cup E')$ is indeed $T_{s,r,p}$-template saturated in H. Then we prove the bound on $|E'|$.

Let $C \subseteq A$ be a set of $p \cdot r$ vertices, such that $|V_i(H) \cap C| = p$ for all i. Let

$$E' = \{e \in E(H) \setminus E : |e \setminus C| \leq s - 1\}.$$

We show that $G = (V, E \cup E')$ is $T_{s,r,p}$-template saturated in H. For an edge $f \in E(H)$, define $\lambda(f) = |f \setminus C|$. Note that edges with $\lambda(f) \leq s - 1$ are already included in G. To see this, note that edges with $\lambda(f) \leq s - 1$ are either included in E, or they are not in E. In the latter case they are in E' as the set E' is defined precisely to contain all edges f not in E with $\lambda(f) \leq s - 1$.

We show that there exists a $T_{s,r,p}$-template saturation process of $(V, E \cup E')$ in H by induction on λ, where $\lambda \leq s - 1$ is the base case discussed above.

Consider an arbitrary edge f with $\lambda(f) \geq s$ and assume, that all edges with $\lambda < \lambda(f)$ have been added to the process already. We show that f can be added as well. For each $i \leq r$, let $D_i \subseteq (V_i(H) \cap C) \setminus f$, be a set of arbitrary $p - 1$ vertices. Such sets exist as for each $V_i(H)$, f intersects $V_i(H) \cap C$ in at most one vertex and thus

$$|(V_i(H) \cap C) \setminus f| = |V_i(H) \cap C| - |f \cap V_i(H) \cap C| \geq |V_i(H) \cap C| - |f \cap V_i(H)| \geq p - 1.$$

Take $D = \bigcup_{i \leq r} D_i$. We prove that the subgraph induced by $D \cup f$ is $T_{s,r,p}$-template saturated in $K^r_{r\times p}$. All missing edges $e \subseteq D \cup f$, including f can thus be added to the saturation process of G in H, proving the statement. To see this, note that we can take the $T_{s,r,p}$-template saturation process of the subgraph induced by $D \cup f$ in $K^r_{r\times p}$ and append it to the saturation process of G in H we are currently constructing.

To prove that the subgraph induced by $D \cup f$ is $T_{s,r,p}$-template saturated in $K^r_{r\times p}$, note that all edges e not yet added to the process must contain $f \setminus C$ as a subset. This holds by induction hypothesis. If an edge $e \subseteq D \cup f$ does not contain $f \setminus C$ as a subset then $\lambda(e) < \lambda(f)$ since $D \setminus (f \setminus C) \subseteq C$. The subgraph induced by $D \cup f$ is thus isomorphic to a supergraph of $T^-_{\lambda(f),r,p}$ and as $\lambda(f) \geq s$, by Lemma 3.6 it is $T_{s,r,p}$-template saturated in $K^r_{r\times p}$.

We have shown that there exists the $T_{s,r,p}$-template saturation process of $(V, E \cup E')$ in H.

Now we turn to proving the bound on $|E'|$. Define

$$E'_i = \{e \in E(H) \setminus E : |e \setminus C| \leq s - 1 \wedge |e \cap B| = i\}$$

and let us bound $|E'_i|$. Each edge intersects every partition class of H precisely in one element. There are $\binom{r}{r-s+1} = \binom{r}{s-1}$ ways to choose in which partitions an edge $e \in E'$ is guaranteed to intersect C satisfying the condition $|e \setminus C| \leq s - 1$ and once we fix them, there are p^{r-s+1} ways to select the vertices in which e intersects them. The rest $s - 1$ of the vertices in the edge e can intersect the vertex set anywhere in the not yet chosen partitions, under the condition that $|e \cap B| = i$ and this can be done in at most $\binom{s-1}{i} a^{s-1-i} b^i$ ways. We get

$$|E'_i| \leq \binom{r}{s-1} p^{r-s+1} \binom{s-1}{i} a^{s-1-i} b^i.$$

We continue by using the bound on $|E'_i|$ to bound $|E'|$. Note, that E'_0 is a subset of E, as each edge $e \in E'_0$ is a subset of A. Since

$$E' = \bigsqcup_{1 \le i \le s-1} E'_i,$$

we obtain

$$\begin{aligned}
|E'| &= \sum_{1 \le i \le s-1} |E'_i| \\
&\le \sum_{1 \le i \le s-1} \binom{r}{s-1} p^{r-s+1} \binom{s-1}{i} a^{s-1-i} b^i \\
&\le \sum_{1 \le i \le s-1} \binom{r}{s-1} p^{r-s+1} \binom{s-1}{i} a^{s-2} b \\
&= \binom{r}{s-1} p^{r-s+1} a^{s-2} b \sum_{1 \le i \le s-1} \binom{s-1}{i} \\
&\le r^{s-1} p^{r-s+1} a^{s-2} b
\end{aligned}$$

proving the desired bound. □

Corollary 3.8. *Let $p \ge r \ge s \ge 2$ and P be an r-partite r-graph with maximum size of a partition class p and sparseness s. Then for every $k_2 \le k_1$ we have*

$$wsat(K^r_{r\times(k_1+k_2)}, P) \le wsat(K^r_{r\times k_1}, P) + r^{s-1} p^{r-s+1} k_1^{s-2} k_2$$

Proof. To show the inequality, we create a weakly P-saturated graph in $H \cong K^r_{r\times(k_1+k_2)}$. Take a minimal weakly P-saturated graph G_1 in $K^r_{r\times k_1}$ and place its copy on the vertex set of H such that after running the saturation process within this copy, some vertex set $Z \subseteq V(H)$ induces $K^r_{r\times k_1}$. Add the remaining edges as described in Lemma 3.7 with Z playing the role of A and $V(H) \setminus Z$ playing the role of B. The resulting graph G_2 is $T_{s,r,p}$-template saturated by Lemma 3.7. By Lemma 3.3 we know that G_2 is also weakly P-saturated. Therefore, the graph G formed from the copy of G_1 placed on Z and the edges added by Lemma 3.7 is a weakly P-saturated graph in H with at most $wsat(K^r_{r\times k_1}, P) + r^{s-1} p^{r-s+1} k_1^{s-2} k_2$ edges. □

4 Proof of Theorem 1.1

4.1 Using the clusters

First we prove that for any r-partite r-graph P with sparsenss s, the sequence $\{wsat(K^r_{r\times n}, P)/n^{s-1}\}_{n=1}^{\infty}$ is bounded from below by some constant greater than 0. Then we prove the existence of the limit of the sequence above.

Lemma 4.1. *For every r-graph P with $s(P) = s$, there exists a constant $c'_P > 0$, such that*

$$wsat(K^r_{r\times n}, P) \geq c'_P \cdot n^{s-1}.$$

Proof. Let H be the host graph, a complete r-partite r-graph of order n. Every set $X \subseteq V(H)$ of $s-1$ vertices with $|X \cap V_i(H)| \leq 1$ for all i has to be included in at least one edge of any weakly P-saturated r-graph. As there are n^{s-1} such sets X and an edge can cover at most $\binom{r}{s-1}$ such sets, the smallest weakly P-saturated graph has at least $n^{s-1}/\binom{r}{s-1}$ edges. The constant $c'_P = 1/\binom{r}{s-1}$ thus satisfies the statement.

To prove that every such set has to be covered by an edge, suppose that there is a weakly P-saturated r-partite r-graph G in H and a set $X \subseteq V(H)$ of $s-1$ vertices, such that $|X \cap V_i(H)| \leq 1$ for all i, and there is no edge in G containing X as a subset. As $|X \cap V_i(H)| \leq 1$ for all i, there exists at least one edge $f \in E(H)$ such that $X \subseteq f$. Let

$$F = \{f \in E(H) : X \subseteq f\}.$$

As G is weakly P-saturated in H it has a saturation process $e_1, e_2, ..., e_{|E(H)\setminus E(G)|}$. All edges in F have to be added as a part of this process. Let

$$k = \min\{j : e_j \in F\}.$$

Now, by the definition of weak saturation, $G \cup \{e_1, e_2, ..., e_k\}$ contains a copy of P such that the edge e_k is mapped to an edge $z \in E(P)$. The set X is mapped to $W \subseteq z$. As e_k is the only edge in $E(G) \cup \{e_1, e_2, ..., e_k\}$ containing X, z is the only edge in $E(P)$ containing W. We have found a set W of size $s-1$ contained in exactly one edge of P, contradicting the assumption that sparseness of P is s. □

We continue by stating Rödl's Approximate Designs Theorem. When the host graph H is a complete r-graph of order n, given a pattern r-graph P with $s(P) = s$ and two integers $n \geq u$ such that $u \mid n$, the authors of [7] divided $V(H)$ into disjoint clusters of size u and shown how to find a small set of edges $\overline{E}$, using Rödl's Theorem such that all edges contained in at most $s-1$ clusters can be generated by a weak P-saturation process starting from $\overline{E}$.

It is not immediately clear how to do this in a multipartite setting. While it is natural to expect that some multipartite version of Rödl's Theorem is required, we show that using uniform partitions into clusters as defined in Definition 2.2 allows us to stay with the classical version of Rödl's Theorem.

Proposition 4.2. *(Rödl [9]). For every $k \geq t \geq 1$ and $\delta \geq 0$ and for all $N \geq N_0(k, t, \delta)$ the following holds. There exists a set family $\mathcal{F} \subseteq \binom{[N]}{k}$ of size at most $(1+\delta)\binom{N}{t}/\binom{k}{t}$ such that every $A \in \binom{[N]}{t}$ is a subset of some $F_A \in \mathcal{F}$.*

In the following lemma, we show how to do find such set of edges $\overline{E}$ in the multipartite setting. Let the host graph $H \cong K^r_{r\times n}$ and let P be an r-partite r-graph with $s(P) = s$. Given two integers $n \geq u$ such that $u \mid n$ we uniformly divide H into n/u clusters as defined in Definition 2.2 and show how to find a small set of edges $\overline{E}$ such that starting from $\overline{E}$ all edges contained in at most $s-1$ such clusters can be generated by a P-saturation process. We bound the set with respect to a variable m dependent on u as it will give us a bound we can directly apply later on in the proof.

Lemma 4.3. *Let $H \cong K^r_{r\times n}$ and P be r-partite r-graph with $S(P) = s$. Given $m > 0$ such that m is a perfect $(s-1)$-st power, $u = m^{1/(s-1)}$, $\delta > 0$, and a uniform division into n/u clusters of H, define $E^+ \subseteq E(H)$ to be a set of edges intersecting at most $s-1$ clusters. Then for sufficiently large n ($n \geq n_0(m, s, \delta)$) satisfying $u \mid n$, there exists a set of edges $\overline{E} \subseteq E(H)$ such that*

1. $$|\overline{E}| \leq (1 + 2\delta)\frac{n^{s-1}}{m^{s-1}} wsat(K^r_{r\times m}, P).$$

2. *The r-graph $(V(H), \overline{E})$ is weakly P-saturated in $(V(H), E^+)$.*

Proof. Let $C_1, C_2, ..., C_{n/u}$ be a uniform division into n/u clusters of H. Apply Proposition 4.2, with m/u,$s-1$ and δ playing the role of k,t and δ, respectively with the ground set $[n/u]$. For sufficiently large n, we get a set family $\mathcal{F}$ with

$$\begin{aligned}
|\mathcal{F}| \leq (1+\delta)\frac{\binom{n/u}{s-1}}{\binom{m/u}{s-1}} &= (1+\delta)\frac{(n/u)!(m/u-s+1)!(s-1)!}{(m/u)!(n/u-s+1)!(s-1)!} \\
&= (1+\delta)\frac{(n/u)(n/u-1)...(n/u-s+2)}{(m/u)(m/u-1)...(m/u-s+2)} \\
&= (1+\delta)\frac{(n/u)^{s-1}+o((n/u)^{s-1})}{(m/u)^{s-1}+o((m/u)^{s-1})} \\
&\leq (1+2\delta)\frac{n^{s-1}}{m^{s-1}}, \qquad (4.1)
\end{aligned}$$

and every $A \in \binom{[n/u]}{s-1}$ is contained in some $F_A \in \mathcal{F}$.

Let G be the r-partite r-graph witnessing $wsat(K^r_{r\times m}, P)$. For every $F \in \mathcal{F}$, take the vertex set

$$D_F = \bigcup_{i\in F} C_i.$$

and notice that for each $j \leq r$

$$D_F \cap V_j(H) = (m/u)u = m.$$

Place a copy of G on every D_F, and let $\overline{E}$ be the union of sets of edges added this way. As the subgraph induced by D_F is weakly P-saturated, we run the weak P-saturation process within each D_F. Afterwards, each D_F induces a copy of $K^r_{r\times m}$. The sequence formed by concatenating all of these saturation processess and removing all occurences of an edge except the first is the saturation process of the r-graph $(V(H), \overline{E})$ in $(V(H), E^+)$ satisfying condition 2. We only need to prove that all edges f contained in at most $s-1$ clusters have been added as a part of some saturation process. To this end, take an arbitrary such edge f. Let

$$B = \{i : f \cap C_i > 0\}.$$

As $|B| \leq s-1$, there exists a set $A \in \binom{[n/u]}{s-1}$ such that $B \subseteq A$. Furthermore, there exists $F_A \in \mathcal{F}$ containing A as a subset and thus $f \subseteq D_{F_A}$. After running the saturation processes within D_F for all $F \in \mathcal{F}$, D_F induces $K^r_{r\times m}$ and thus f is either in $\overline{E}$ or has been added in the corresponding weak P-saturation process within D_{F_A}.

Now, by inequality (4.1),

$$|\overline{E}| \leq |\mathcal{F}| wsat(K^r_{r\times m}, P) \leq (1+2\delta)\frac{n^{s-1}}{m^{s-1}} wsat(K^r_{r\times m}, P).$$

□

Given an r-partite r-graph P with $s(P) = s$ and a uniform division into clusters of $H \cong K^r_{r\times n}$, and an edge set E such that all edges intersecting at most $s-1$ clusters can be generated as a part of a P-saturation process, the next two lemmas describe a construction of an edge set E^*, such that $(V(H), E \cup E^*)$ is weakly P-saturated and E^* does not contain too many edges.

The first lemma describes a construction of such edge set E^* described above, assuming the host graph H is partitioned into exactly s clusters. The set is denoted by E'. The proof is rather long, but it contains three distinct parts, which can be understood one by one.

We designate p rigid vertices in each partition of each cluster of H as defined in Definition 2.3, aiming to show the existence of $T_{s,r,p}$-template saturation process of $(V(H), E \cup E')$. The first part of the proof shows how to add missing edges with exactly $s-1$ loose vertices to the saturation process. The second part of the proof shows how to add missing edges with at least s loose vertices to the saturation process assuming all other edges have been added already. The third part proves a bound on the size of E'.

Lemma 4.4. *Given $n \geq p \geq 1$, $s \geq 2$ satisfying $s \mid n$ and $n/s \geq p$, let $C_1, C_2, ..., C_s$ be a uniform division of $H \cong K^r_{r\times n}$, into s clusters. Let the set $E \subseteq E(H)$ contain all edges intersecting at most $s-1$ clusters. Designate p rigid vertices in each partition of each cluster of H as defined in Definition 2.3. Let*

$$E' = \{e \in E(H) \setminus E : |e \cap L| \leq s-2.\}$$

Then $(V(H), E \cup E')$ is $T_{s,r,p}$-template saturated and

$$|E'| \leq s^r \binom{r}{s-2} \left(\frac{n}{s}\right)^{s-2} p^{r-s-2}.$$

Proof. We show that there exists a $T_{s,r,p}$-template saturation of the r-graph $(V(H), E \cup E')$ in H. First note that edges $f \in E(H)$ with at most $s-2$ loose vertices are already contained in $E \cup E'$. To see this, consider an edge f with at most $s-2$ loose vertices. If $f \in E$ then it has been added already. If $f \notin E$ then $f \in E'$ as E' is defined to contain precisely all edges not in E with at most $s-2$ loose vertices.

Next, let us show how to add to the saturation process edges with exactly $s-1$ loose vertices. Consider such an edge f. By pigeonhole, there exists a cluster C_i, such that f does not contain a loose vertex from this cluster and thus we can define

$$\rho(f) = \min_{1\le i\le s}\{|(\bigcup_{1\le j\le r} R_{i,j}) \cap f| : |(f \cap C_i) \cap L| = 0\}.$$

as the set we take minimum over is nonempty.

We now aim to show, by induction on ρ, that every edge with exactly $s-1$ loose vertices can be added to the saturation process. Edges f with $\rho(f) = 0$ are missing one of the clusters and are thus in E. So, we can treat $\rho(f) = 0$ as the base case. For the induction step, consider an arbitrary edge f with $\rho(f) > 0$ and suppose that all edges with smaller ρ are already present. Let $\gamma(f)$ be the index of the cluster attaining minimum in the definition of ρ and i be an arbitrary index of a cluster, such that $i \neq \gamma(f)$. For all $z \le r$, let $D_z \subseteq R_{i,z} \setminus f$ be a set of $p-1$ vertices. For all $z \le r$, the sets D_z are well-defined, as

$$|R_{i,z} \setminus f| = |R_{i,z}| - |R_{i,z} \cap f| \ge |R_{i,z}| - 1 = p - 1.$$

Let $D = \bigcup_{1\le z\le r} D_z$. We show that the subgraph induced by $D \cup f$ is $T_{s,r,p}$-template saturated. For this, note that missing edges $e \subseteq (f \cup D)$ must contain $(f \cap L) \cup (C_{\gamma(f)} \cap f)$ as a subset. This is because missing edges have to contain both all loose vertices in $D \cup f$ as edges with at most $s-1$ loose vertices have been added already, and all vertices from $(C_{\gamma(f)} \cap f)$ as ones that do not contain them, have lower ρ and have been added by inductive assumption. The sets $(f \cap L)$ and $(C_{\gamma(f)} \cap f)$ are disjoint, $|(f \cap L)| = s - 1$ and $|(C_{\gamma(f)} \cap f)| = \rho(f) > 0$. Thus $|(f \cap L) \cup (C_{\gamma(f)} \cap f)| = s - 1 + \rho(f) \ge s$.

Define $s' = s - 1 + \rho(f)$. The graph $T_{s'}$ obtained from the subgraph induced by vertices $D \cup f$ by removing all edges containing $(f \cap L) \cup (C_{\gamma(f)} \cap f)$ as a subset is isomorphic to $T^-_{s',r,p}$ and as $s' \ge s$, the subgraph induced by $D \cup f$ is $T_{s,r,p}$-template saturated by Lemma 3.6. All missing edges contained in $D \cup f$, including the edge f can thus be added to the process, as the $T_{s,r,p}$-template saturation process of the subgraph induced by $D \cup f$ can be appended to it.

We continue by adding edges with at least s loose vertices to the process. Consider such an edge f and define $\lambda(f) = |f \cap L|$. We show that every edge with at least s loose vertices can be added to the process by induction on λ. Let $\lambda(f) = s - 1$ be the base case of the induction.

For the induction step, consider an arbitrary edge f with $\lambda(f) \ge s$ and suppose that all edges with $\lambda < \lambda(f)$ have been added to the process already. For all $z \le r$, let $D_z \subseteq R_{1,z} \setminus f$ be a set of $p-1$ vertices. For all $z \le r$, the sets D_z are well-defined, as

$$|R_{1,z} \setminus f| = |R_{1,z}| - |R_{1,z} \cap f| \ge |R_{1,z}| - 1 = p - 1.$$

Let $D = \bigcup_{z\le r} D_z$. We show that the subgraph induced by $D \cup f$ is $T_{s,r,p}$-template saturated. For this, note that missing edges $e \subseteq (D \cup f)$ must contain $f \cap L$ as a subset otherwise e intersects L in fewer than $|f \cap L| = \lambda(f)$ vertices and thus already has been added to the process by inductive assumption. The r-partite r-graph $T_{\lambda(f)}$ obtained from the subgraph induced by $D \cup f$ by removing all edges containing $f \cap L$ is isomorphic to $T^-_{\lambda(f),r,p}$ and as the subgraph induced by $D \cup f$

is supergraph of $T_{\lambda(f)}$ and $\lambda(f) \geq s$, it is $T_{s,r,p}$-template saturated by Lemma 3.6. All missing edges contained in $D \cup f$, including the edge f can thus be added to the process.

Finally, let us prove the bound on $|E'|$. To this end, note that an edge $e \in E'$ intersects each partition precisely once. There are s^r ways to choose the mapping $\Phi : \{1, 2, ..., r\} \mapsto \{1, 2, ..., s\}$ which asserts that e intersects the i-th partition in cluster $C_{\Phi(i)}$. Once this map has been chosen, there are $\binom{r}{r-s+2}$ ways to choose which partitions e must intersect in rigid vertices, and for each of those, there are p^{r-s+2} ways to choose which rigid vertices it intersects and $(\frac{n}{s})^{s-2}$ ways to choose which vertices it intersects in the other partitions – they can be either rigid or loose. We have counted some edges with more than $r - s + 2$ rigid vertices more than once, but the important thing is we have counted each edge at least once. We get the desired bound

$$|E'| \leq s^r \binom{r}{s-2} \left(\frac{n}{s}\right)^{s-2} p^{r-s-2}.$$

□

The following lemma uses Lemma 4.4 to construct the desired set E^*, even when the host graph is divided into more than s clusters.

Lemma 4.5. *Given $n \geq u \geq p \geq 1$, $u \mid n$, let $C_1, C_2, ..., C_{n/u}$ be a uniform division of $H \cong K^r_{r\times n}$, into n/u clusters and let the set $E \subseteq E(H)$ contain all edges intersecting at most $s - 1$ clusters. Then there exists a set of edges $E^* \subseteq E(H)$ such that $(V(H), E \cup E^*)$ is $T_{s,r,p}$-template saturated and*

$$|E^*| \leq \binom{n/u - 1}{s-1} s^r \binom{r}{s-2} u^{s-2} p^{r-s-2}.$$

Proof. Designate p rigid vertices in each partition of each cluster of H. Let

$$E^* = \bigcup_{Q \in \binom{\{2,3,...n/u\}}{s-1}} E'(Q \cup \{1\}),$$

where for a set $M \subseteq \{1, 2, ..., n/u\}$ of size $s - 1$, $E'(M) \subseteq E(H)$ denotes the set of all edges e containing at most $s - 2$ loose vertices satisfying $e \subseteq \bigcup_{i\in M} C_i$. The construction of $E'(M)$ is exactly the same as the construction of E' in Lemma 4.4. We now show that $(V(H), E \cup E^*)$ admits a $T_{s,r,p}$-template saturation process in H.

We have constructed E^*, such that each subgraph induced by a set of s clusters containing cluster C_1 is $T_{s,r,p}$-saturated by Lemma 4.4, thus all edges contained in exactly s clusters can be added to the process as long as one of the clusters is C_1.

Now, for any edge f define $\Lambda(f) = f \setminus C_1$ and $\lambda(f) = |\Lambda(f)|$. We show that we can add to the process the remaining missing edges by induction on $\lambda(f)$. Let $\lambda(f) \leq s - 1$ be the base case, as these edges are in E.

Consider an edge f with $\lambda(f) \leq s$ and suppose that all edges e with $\lambda(e) < \lambda(f)$ have been added already. For all $z \leq r$ let $D_z \subseteq (C_{1,z} \setminus f)$ be arbitrary set of $p-1$ vertices and $D = \bigcup_{z\leq r} D_z$. For all $z \leq r$ the sets D_z are well-defined as

$$|C_{1,z} \setminus f| = |C_{1,z}| - |C_{1,z} \cap f| = u - |C_{1,z} \cap f| \geq p - 1.$$

All edges $e' \subseteq D \cup f$ not yet added must have $\Lambda(f)$ as a subset, otherwise e' contains fewer vertices outside of C_1 and is already added by induction hypothesis. The r-partite r-graph $T_{\lambda(f)}$ formed from the subgraph induced by $D \cup f$ by removing all edges containing $\Lambda(f)$ is isomorphic to $T^-_{\lambda(f),r,p}$. As the subgraph induced by $D \cup f$ is a supergraph of $T_{\lambda(f)}$ and $\lambda(f) \geq s$, it is $T_{s,r,p}$-saturated by Lemma 3.6. All edges $e' \subseteq D \cup f$, including the edge f can thus be added to the process. To finish the proof, let us bound $|E^*|$. As

$$E^* = \bigcup_{Q \in \binom{\{2,3,\dots,n/u\}}{s-1}} E'(Q \cup \{1\})$$

and for each $E'(Q \cup \{1\})$ in the formula, the bound from Lemma 4.4 for $|E'|$ holds with $u \cdot s$ playing the role of n, we get

$$\begin{aligned}
|E^*| &\leq \sum_{Q \in \binom{\{2,3,\dots,n/u\}}{s-1}} |E'(Q \cup \{1\})| \\
&\leq \sum_{Q \in \binom{\{2,3,\dots,n/u\}}{s-1}} s^r \binom{r}{s-2} \left(\frac{us}{s}\right)^{s-2} p^{r-s-2} \\
&\leq \binom{n/u-1}{s-1} s^r \binom{r}{s-2} u^{s-2} p^{r-s-2},
\end{aligned}$$

proving the bound as desired. □

4.2 Putting it all together

In this section we combine everything we have proven to complete the proof of Theorem 1.1.

First, assume $s(P) = 1$. We show that the sequence $\{wsat(K^r_{r\times n}, P)\}^\infty_{n=p}$ is non-increasing. Hence, as it is bounded from below by Lemma 4.1, it is convergent. To do so, we show that for any $n_2 \geq n_1 \geq p$, we can build a weakly P-saturated graph in H, a copy of $K^r_{r\times n_2}$, with $wsat(K^r_{r\times n_1}, P)$ edges. For this, place the graph G witnessing $wsat(K^r_{r\times n_1}, P)$ on a vertex set Z formed by taking arbitrary n_1 vertices from each partition class of H. Then run the saturation process within Z.

By the definition of sparseness, there exists a vertex $u \in V(P)$ contained in exactly one edge $f \in E(P)$. Now it's possible to pick any vertex $v \in V(H) \setminus Z$ and add all edges $e \subseteq Z \cup \{v\}$, by mapping u to v, f to e and the rest of the vertices of P to Z, such that if two vertices of P are in the same partition class, their images are in the same partition class of H. We have embedded a P into $Z \cup \{v\}$ such that the only missing edge of P is f. Thus e can be added as a part of the P-saturation process. Subsequently, repeat this for each vertex to obtain the complete saturation process.

Let us now prove that the Theorem 1.1 holds for sparseness $s(P) > 1$. Let

$$C_P = \liminf_{n\to\infty} \frac{wsat(K^r_{r\times n}, P)}{n^{s-1}}.$$

It follows by Lemma 4.1 that $C_P > 0$. We now claim that the limit in Theorem 1.1 exists and equals C_P.

Let $\varepsilon > 0$. First, we show that there exist arbitrarily large perfect $(s-1)$-st powers m, such that

$$wsat(K^r_{r\times m}, P) \le (C_P + 2\varepsilon)\cdot m^{s-1}.$$

Lemma 4.6. *Let $m_0 > 0$ and let $m = \lceil m_0^{1/(s-1)}\rceil^{s-1}$ be the next largest perfect $(s-1)$-st power. Then*

$$wsat(K^r_{r\times m}, P) - wsat(K^r_{r\times m_0}, P) = o(m^{s-1}).$$

Proof. By the binomial theorem,

$$\begin{aligned} m - m_0 &\le \lceil m_0^{1/(s-1)}\rceil^{s-1} - \lceil m_0^{1/(s-1)} - 1\rceil^{s-1} \\ &= \lceil m_0^{1/(s-1)}\rceil^{s-1} - (\lceil m_0^{1/(s-1)}\rceil - 1)^{s-1} \\ &= \lceil m_0^{1/(s-1)}\rceil^{s-1} - \sum_{i=0}^{s-1}(-1)^{s-1-i}\binom{s-1}{i}\lceil m_0^{1/(s-1)}\rceil^{i} \\ &= \sum_{i=0}^{s-2}(-1)^{s-i}\binom{s-1}{i}\lceil m_0^{1/(s-1)}\rceil^{i} \\ &= O(m^{(s-2)/(s-1)}). \end{aligned}$$

Hence, using Corollary 3.8, with m_0,$m - m_0$ playing the role of k_1,k_2 respectively, we get

$$\begin{aligned} wsat(K^r_{r\times m}, P) &\le wsat(K^r_{r\times m_0}, P) + r^{s-1}p^{r-s+1}m_0^{s-2}(m - m_0) \\ &= wsat(K^r_{r\times m_0}, P) + O(m^{s-2}m^{(s-2)/(s-1)}). \end{aligned}$$

Rearranging,

$$wsat(K^r_{r\times m}, P) - wsat(K^r_{r\times m_0}, P) = O(m^{s-2}m^{(s-2)/(s-1)}) = o(m^{s-1}).$$

□

By the properties of limes inferior, there exist arbitrarily large values of m_0, such that

$$wsat(K^r_{r\times m_0}, P) \le (C_p + \varepsilon)m_0^{s-1}.$$

By Lemma 4.6, there exists a threshold m_1, such that for all $m_0 \ge m_1$,

$$wsat(K^r_{r\times\lceil m_0^{1/(s-1)}\rceil^{s-1}}, P) - wsat(K^r_{r\times m_0}, P) \le \varepsilon\lceil m_0^{1/(s-1)}\rceil^{s-1}.$$

We can thus pick arbitrarily large $m_0 \ge m_1$ satisfying the first condition and then getting the desired perfect $(s-1)$-st power by taking $m := \lceil m_0^{1/(s-1)}\rceil^{s-1}$. We get

$$\begin{aligned} wsat(K^r_{r\times m}, P) &\le (wsat(K^r_{r\times m}, P) - wsat(K^r_{r\times m_0}, P)) + wsat(K^r_{r\times m_0}, P) \\ &\le \varepsilon m^{s-1} + (C_P + \varepsilon)m_0^{s-1} \\ &\le (C_P + 2\varepsilon)m^{s-1}. \end{aligned} \tag{4.2}$$

Fix such m, satisfying $m > g(\varepsilon, H)$, where g is a function to be specified later.

We continue by proving the following lemma.

Lemma 4.7. *Let $n_0, u > 0$ and let $n = u\lfloor n_0/u \rfloor$. Then*

$$wsat(K^r_{r\times n_0}, P) - wsat(K^r_{r\times n}, P) = o(n^{s-1}).$$

Proof. It follows from Corollary 3.8, that

$$\begin{aligned} wsat(K^r_{r\times n_0}, P) &\le wsat(K^r_{r\times n}, P) + r^{s-1}p^{r-s+1}n^{s-2}(n_0 - n) \\ &\le wsat(K^r_{r\times n}, P) + r^{s-1}p^{r-s+1}n^{s-2} \\ &= wsat(K^r_{r\times n}, P) + o(n^{s-1}) \end{aligned}$$

By rearranging, we get

$$wsat(K^r_{r\times n_0}, P) - wsat(K^r_{r\times n}, P) = o(n^{s-1}).$$

□

By Lemma 4.7, given some constant u, there exists n_1, such that for all $n \ge n_1$

$$wsat(K^r_{r\times n}, P) - wsat(K^r_{r\times u\lfloor n/u\rfloor}, P) \le \varepsilon(u\lfloor n/u \rfloor)^{s-1} \le \varepsilon n^{s-1}.$$

We can thus choose some u and it suffices to prove that there exists n_2, such that for all $n \ge n_2 - u$ satisfying $u \mid n$,

$$wsat(K^r_{r\times n}, P) \le (C_P + 3\varepsilon) \cdot n^{s-1}.$$

Then it will follow that for all $n \ge \max(n_1, n_2)$,

$$wsat(K^r_{r\times n}, P) \le (C_P + 4\varepsilon) \cdot n^{s-1},$$

proving Theorem 1.1.

Let us do just that. Pick $u = m^{1/(s-1)}$ (recall that m is a perfect $(s-1)$-st power satisfying (4.2)). We shall prove that for all n large enough, such that $u \mid n$,

$$wsat(K^r_{r\times n}, P) \le (C_P + 3\varepsilon) \cdot n^{s-1}.$$

We may assume that n is large enough to satisfy the requirement of Lemma 4.3 with m, ε/C_P playing the role of m, δ respectively. For such n satisfying $u \mid n$, $H \cong K^r_{r\times n}$ and a uniform division into n/u clusters of H, it provides us with an edge set $\overline{E}$ such that $(V(H), \overline{E})$ is weakly P-saturated in $(V(H), E^+)$, where E^+ is a set of edges intersecting at most $s-1$ clusters, satisfying

$$\begin{aligned} |\overline{E}| &\le (1 + 2\varepsilon/C_P)\frac{n^{s-1}}{m^{s-1}} wsat(K^r_{r\times m}, P) \\ &\le (1 + 2\varepsilon/C_P)\frac{n^{s-1}}{m^{s-1}}(C_P + 2\varepsilon)m^{s-1} \\ &\le (C_P + 2\varepsilon)n^{s-1}. \end{aligned}$$

Now, we can use Lemma 4.5 with n,u and p playing the role of n,u and p respectively. It provides us with a set E^*, precisely constructed, such that $(V(H), E^+ \cup E^*)$ is $T_{s,r,p}$-template saturated in H and

$$|E^*| \le \binom{n/u - 1}{s-1} s^r \binom{r}{s-2} u^{s-2} p^{r-s-2} = O((n/u)^{s-1}u^{s-2}) = O(n^{s-1}/u). \quad (4.3)$$

By Lemma 3.3, $(V(H), E^+ \cup E^*)$ is a fortiori weakly P-saturated in H. Since $(V(H), E^+ \cup E^*)$ is weakly P-saturated, $(V(H), \overline{E} \cup E^*)$ is too, as E^+ was generated from $\overline{E}$ by sequentially adding edges which give rise to a new copy of P. The precise saturation process of $(V(H), \overline{E} \cup E^*)$ is formed by concatenating the sequence generating E^+ from $\overline{E}$ given by Lemma 4.3 and the saturation process of $(V(H), E^+ \cup E^*)$.

To finish the proof, we need to show $\overline{E} \cup E^*$ is small.

$$\begin{aligned} |\overline{E} \cup E^*| &\leq |\overline{E}| + |E^*| \\ &\leq (C_P + 2\varepsilon)n^{s-1} + O(n^{s-1}/u) \\ &= (C_P + 2\varepsilon)n^{s-1} + O(n^{s-1}/m^{1/(s-1)}) \\ &\leq (C_P + 3\varepsilon)n^{s-1}. \end{aligned}$$

We get the last inequality by specifying $g(\varepsilon, H)$ in such a way that for $m > g(\varepsilon, H)$ the function bounding $|E^*|$ (4.3) is at most $\varepsilon \cdot n^{s-1}$ (recall that $u = m^{1/(s-1)}$ in (4.3)).

Conclusion

In this thesis, we investigated the limiting constant behaviour of weak saturation processes in complete multipartite host hypergraphs.

Our main contribution is the proof of Theorem 1.1, an extension of a related result in complete hypergraphs [7].

Further result we would like to see is an analysis of Problem 1. Is there a polynomial time algorithm or can we prove that it is NP-complete? Does some property of the host graph, such as planarity, make the problem easier?

In [10], the authors proved that the sequences $\{wsat(K_{n,n}, P)/2n\}_{n=1}^{\infty}$ and $\{wsat(K_n, P)/n\}_{n=1}^{\infty}$ converge to the same finite limit for any bipartite graph P. Is it possible to appropriately extend this theorem to r-uniform r-partite hypergraphs with $r > 2$?

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