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Two-dimensional integer trigonometry

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Abstract: In this thesis, we will formally define objects in Euclidean geometry, lattices and affine lattices and use them to describe objects in integer trigonometry. We will prove that the described objects in integer trigonometry are invariant under the action of the group of integer affine transformations and pose some similarities with Euclidean geometry in $\mathbb{R}^2$. We will prove geometric interpretations of definitions of said objects, their other properties and visualize them using concrete examples.

Keywords: integer trigonometry, lattices, Euclidean geometry,continued fractions

Contents

Introduction

Integer geometry has historically been studying the relationship between integer lattices and Euclidean geometry. By focusing on lattice invariants independently, it reveals a complex combinatorial framework. Trigonometric functions, which are crucial in Euclidean geometry, also manifest in integer geometry, as seen in the developments of integer lattice analogues for simplical cones in $\mathbb{R}^2$ in 2008 by Oleg Karpenkov. We are following the framework set up in [1].
Geometry is interpreted both in [1] and [2] as a set of objects related by congruence relations. In integer trigonometry, the objects of study are mainly integer points, rational angles, integer segments and integer triangles. The congruence relation is given by the group of integer affine transformations. We can compare it to the well known Euclidean geometry in $\mathbb{R}^2$, where the objects of study contain points, angles, line segments, triangles and the congruence relation is given by transformations on $\mathbb{R}^2$ that preserve Euclidean length.
The main source is [1], but only for the two-dimensional part. In this thesis, we provide most of the proofs for the propositions in the article. Next important source is [2], which has much more information about the subject.
We formally define and study the objects in integer trigonometry and prove their properties. If possible, we provide geometric interpretations or more intuitive approaches, which are thoroughly proven and give the reader better understanding, even amplified by visualization with concrete examples.

In Chapter 1, we will cover the necessary definitions and properties in Euclidean trigonometry. In Sections 2.1, 2.2, we introduce lattices and integer analogues to definitions from Chapter 1. In Sections 2.3, 2.4, we define integer length and integer area. We provide their geometric interpretations and more of their properties with proofs and examples. Next, in Section 2.5, we define integer distance, its interpretation and use it to prove some necessary lemmata. In Section 2.6, we introduce integer sines, sails and LLS sequences, their properties and provide examples for better understanding. In this section we also prove an important theorem (Theorem 2.38) about integer congruence of angles with the same LLS sequence. Then, in Section 2.7, we extend the notation of continued fractions and use it in Section 2.8, where we introduce the integer tangent using LLS sequence and continued fractions. In Section 2.9, we define integer arctangent and state its important properties in Lemma 2.48 and Lemma 2.49.
Then, in Chapter 3, in Section 3.1, we define adjacent and transpose angles in integer trigonometry and prove their trigonometric identities in Theorem 3.2 and Theorem 3.3. Then, in Sections 3.2, 3.3, we define right angles and summation of angles in integer trigonometry. Finally, in Section 3.4, we state a property of tangents of angles in triangles in Euclidean geometry and summarize a similar condition for integer tangents in integer triangles. The proofs of Lemma 2.48 and Theorem 3.9 require extra non-trivial notions and preliminary lemmata, so we do not include them in the thesis, and just provide references to their proofs in [2].

The author's input in this thesis is formally defining all the necessary tools in Chapter 1 and Section 2.1, providing proof of the geometric interpretation of integer length in Lemma 2.20, which is written as a definition in [2], but also as a geometric interpretation in [1]. Similarly, the author provides proof of the geometric interpretation of integer distance in Lemma 2.31 which was only written as an alternative definition of integer distance in [2]. The author also provides a different proof than the one in [2] for the affine invariance of integer length and integer area in Lemmata 2.21, 2.28, which are way more detailed and thorough. The author further proves the relation of indices of lattices in Lemmata 2.22, 2.23, which lead to proof of Lemma 2.34. This lemma was written as a definition in [2], but in [1] was said to follow directly from definition, which coincides with the definition in this thesis. We use these two lemmata to prove Lemma 2.30, which was left as an exercise for the reader in [2]. The author only revises proofs of Lemmata 2.25, 2.27 from [2] and explains them more thoroughly. The author also revises and explains Theorem 2.38 from [2] in more detail. The proof was changed to apply only for finite sails and some minor typing errors in [2] were corrected. The author also adds different proofs to Lemmata 2.43, 2.46, which discuss the value of the integer trigonometric functions. These lemmata were originally only one line long in [2]. The author only revises the proofs of Theorems 3.2, 3.3 from [2], all of the steps are now clearly explained and the proofs are longer, but not significantly. Finally, the author provides a more detailed proof for Lemma 3.5 than the one in [2]. Further the author illustrates these definitions and theorems with original examples and figures.

1. Definitions and setup in trigonometry

In this chapter, we create the setup for integer trigonometry, we first have to define the objects we will work with in $\mathbb{R}^2$ and then use them to define their integer analogues.

1.1 Angles, lines, triangles

Definition 1.1 (Point). *We define points as the elements of the vector space* $\mathbb{R}^2$ *over the field* $\mathbb{R}$.

Definition 1.2 (Line). *A line L passing through $A, B \in \mathbb{R}^2$ where $A \neq B$, or shortly line AB, is defined as $L = \{(t \cdot A + (1-t) \cdot B \mid t \in \mathbb{R}\}$.*

Definition 1.3 (Line segment). *Let $A, B \in \mathbb{R}^2$, where $A \neq B$. We define line segment AB as the set of linear combinations $\{t \cdot A + (1-t) \cdot B \mid 0 \leq t \leq 1\}$.*

Definition 1.4 (Angle). *Let $A, B, C \in \mathbb{R}^2$, where A, B, C are non-collinear. Then we define angle $\angle ABC$ as the set of convex combinations $\{B + s(B-A) + t(B-C) \mid s, t \in \mathbb{R} \;\; s, t > 0\}$, where line BA is its first boundary line and line BC is its second boundary line.*

Remark. In the definition of angle, we specify its boundary lines and give them order. We do this so that the angles have orientation thus the angles $\angle ABC$ and $\angle CBA$ are different.

Definition 1.5 (Measure of an angle). *Let $\angle ABC$ be an angle. We define its measure as the number*

$$a = \arccos\left(\frac{\langle B-A, B-C\rangle}{\|B-A\| \cdot \|B-C\|}\right),$$

where $\langle \cdot, \cdot \rangle$ is the standard dot product and $\| \cdot \|$ is the corresponding norm on $\mathbb{R}^2$. We then define: cosine of $\angle ABC$ as

$$\cos(\angle ABC) = \cos(a) = \frac{\langle B-A, B-C\rangle}{\|B-A\| \cdot \|B-C\|},$$

sine of $\angle ABC$ as

$$\sin(\angle ABC) = \sin(a) = \cos\left(\frac{\pi}{2} - a\right)$$

and tangent of $\angle ABC$ as

$$\tan(\angle ABC) = \tan(a) = \frac{\sin(a)}{\cos(a)}.$$

Definition 1.6 (Transpose angle). *Let $\angle ABC$ be an angle. The transpose angle to the angle $\angle ABC$ is the angle $\angle CBA$. We can see that if a denotes the measure of $\angle ABC$ and b denotes the measure of $\angle CBA$, then $a = b$.*

Definition 1.7 (Adjacent angle)**.** *Let $\angle ABC$ be an angle. The adjacent angle to the angle $\angle ABC$ is an angle $\angle CBA'$, where $A' = B - (A - B)$. We can see that if a denotes the measure of $\angle ABC$ and b denotes the measure of $\angle CBA'$, then $a + b = \pi$.*

Definition 1.8 (Triangle)**.** *Let $A, B, C \in \mathbb{R}^2$ that do not lie on one line. Triangle $\triangle ABC$ is the convex hull of three elements $\{A, B, C\}$, i.e. it is the set of all combinations $\{t \cdot A + s \cdot B + u \cdot C \mid 0 \leq t, s, u \leq 1, t + s + u = 1\}$.*

We will use volume and measure and their relation as defined and proved in [3], section 1.

Definition 1.9 (Volume)**.** *Let $a = \{a_1, a_2, \ldots, a_n\} \in \mathbb{R}^n, b = \{b_1, b_2, \ldots, b_n\} \in \mathbb{R}^n$. Let us have a set $W = \{x = \{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n \mid a_i < x_i < b_i, \ \forall i = 1, \ldots, n\}$. We define the volume of W as $vol(W) = \Pi_{i=1}^{n}(b_i - a_i)$.*

Definition 1.10 (Measure)**.** *Let $(X, \mathcal{A})$ be a measurable space. Then we define measure as a set function $\mu : \mathcal{A} \to [0, \infty]$ if it is not identically equal to ∞ and is σ-additive, i.e. if $A_k \in \mathcal{A}, k \in \mathbb{N}$ are pairwise disjoint, then $\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$. The triplet $(X, \mathcal{A}, \mu)$ is called a space with measure.*

Theorem 1.11. *There exists exactly one measure $\mathcal{L}_n$ on $\mathcal{B}(\mathbb{R}^n)$ such that for every W from the definition of volume it holds that $\mathcal{L}_n(W) = vol(W)$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel set, which is the smallest σ-algebra containing all open sets of $\mathbb{R}^n$.*

We will use this definition of measure for $\mathbb{R}^n$ to compute Euclidean area in trigonometry, i.e. for a measurable subset $A \subseteq \mathbb{R}^n$, we define its Euclidean area as $S(A) = \mathcal{L}_n(A)$.

To compute Euclidean length we will use the standard definition of the norm on $\mathbb{R}^n$, i.e. the Euclidean length of a line segment AB is defined as $\ell(AB) = \|B - A\|$.

Definition 1.12 (Congruence)**.** *Let $U, V \subseteq \mathbb{R}^n$. They are congruent if one can be transformed into the other using a sequence of translations, rotations and reflections.*

For example, two line segments are congruent if and only if they have the same length.

Definition 1.13 (Affine transformation)**.** *Let A be a matrix of type $n \times n$ with elements from $\mathbb{R}^n$, which represents linear mapping, let $c \in \mathbb{R}^n$ and let $x \in \mathbb{R}^n$ be arbitrary. Then, the transformation $x \mapsto A \cdot x + c$ is called an affine transformation.*

2. Definitions in integer trigonometry

In this chapter, we will define lattices and integer analogues to the definitions presented in the previous chapter. We will prove their essential properties and also try to develop intuitive approaches and connect these concepts with concepts in Euclidean trigonometry with focus on affine transformations.

2.1 Lattices

Definition 2.1 (Lattice). *A lattice is a discrete additive subgroup* Γ *of the vector space* $\mathbb{R}^n$*, i.e. it satisfies the following:*

(subgroup) : it is closed under addition and subtraction,

(discrete) : there exists $\varepsilon \in \mathbb{R},\ \varepsilon > 0$ *such that for every two elements* $x \neq y$ *in* Γ *it holds that* $\|x - y\| \geq \varepsilon$*.*

Definition 2.2 (Integer lattice). *A lattice is integer if it is a subgroup of* $\mathbb{Z}^n$*.*

In the case of this thesis, we will work with the lattice $\mathbb{Z}^2$. This is a lattice because it is closed under addition and subtraction, and the distance between any two different points in $\mathbb{Z}^2$ is at least 1.

Definition 2.3 (Integer point). *A point is called integer if it is an element of* $\mathbb{Z}^n$*.*

Definition 2.4 (Sublattice). *Let* $\Lambda,\ \Gamma$ *be lattices. We say that* Λ *is a sublattice of* Γ *if it is a subgroup of* Γ*, i.e.* $\Lambda \subseteq \Gamma$ *and for every* $A, B \in \Lambda$ *holds* $A + B \in \Lambda, A - B \in \Lambda$*.*

Definition 2.5 (Affine lattice). *Let* Λ *be a lattice and let* $A \in \mathbb{R}^n$*. Then*

$$\Gamma = \{A + t, t \in \Lambda\}$$

is an affine lattice. We say that Λ *is the underlying lattice of* Γ*, or that* Γ *is associated to* Λ*.*

Definition 2.6 (Index of an affine lattice in an affine lattice). *Let* Γ *be an associated affine lattice to lattice* Λ*. Then let* $\Gamma_1 \subseteq \Gamma$ *be an associated affine lattice to lattice* $\Lambda_1 \subseteq \Lambda$*. Then we define* $[\Gamma : \Gamma_1] = [\Lambda : \Lambda_1]$.

Definition 2.7 (Integer affine lattice). *An affine lattice* $\Gamma = \{A + t, t \in \Lambda\}$ *is integer, if* $A \in \mathbb{R}^n$ *is an integer point and if* Λ *is an integer lattice.*

The group Γ can be defined using any point $A \in \Gamma$. The underlying lattice Λ will always be the same, regardless of the choice of the point.

Definition 2.8 (Generating a lattice). *Let* Λ *be a lattice in* $\mathbb{R}^n$*. We say that a set* $M \subseteq \Lambda$ *generates* Λ *if every* $a \in \Lambda$ *can be written as a linear combination of elements of* M *with integer coefficients.*

For example, if M is finite, then $M = \{m_1, m_2, \ldots, m_n\}$ and it generates Λ if and only if $\Lambda = m_1\mathbb{Z} + m_2\mathbb{Z} + \cdots + m_n\mathbb{Z}$.

Definition 2.9 (Basis of a lattice)**.** *Let Λ be a lattice in $\mathbb{R}^n$. We say that a set $M \subseteq \Lambda$ is a basis of Λ if M generates Λ and M is linearly independent in $\mathbb{R}^n$.*

Definition 2.10 (Generating an affine lattice)**.** *Let Γ be an affine lattice associated to a lattice Λ. We say that line segments $A_1B_1, A_2B_2, \ldots, A_nB_n$ generate Γ if $A_i, B_i \in \Gamma, \forall 1 \leq i \leq n$ and $\{B_1 - A_1, B_2 - A_2, \ldots, B_n - A_n\}$ generates Λ.*

Definition 2.11 (Basis of an affine lattice)**.** *Let Γ be an affine lattice associated to a lattice Λ. We say that line segments $A_1B_1, A_2B_2, \ldots, A_nB_n$ are a basis of Γ if $A_i, B_i \in \Gamma$, $\forall 1 \leq i \leq n$ and $\{B_1 - A_1, B_2 - A_2, \ldots, B_n - A_n\}$ is a basis of Λ.*

Lemma 2.12. *Let $v \in \mathbb{R}^n$ and let $\Gamma \subseteq \mathrm{Span}(v)$ be a lattice. Then there exists $\alpha \in \mathbb{R}, \alpha > 0$ such that $\alpha \cdot v \in \Gamma$ and $\beta \cdot v \notin \Gamma$ $\forall \beta \in \mathbb{R},\ 0 < \beta < \alpha$. Moreover, $\{\alpha \cdot v\}$ is a basis of Γ.*

Proof. Denote $M = \{\beta \in \mathbb{R} \mid \beta > 0, \beta \cdot v \in \Gamma\}$. M is bounded from below, therefore there exists $\alpha = \inf M$. We know that $\alpha \cdot v$ must be in Γ, otherwise $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0$ there would exist infinitely many elements of M in the interval $(\alpha, \alpha + \varepsilon)$, which would be a contradiction with Γ being discrete. If $\{\alpha \cdot v\}$ did not generate Γ, there would exist a β such that $n \cdot \alpha < \beta < (n + 1) \cdot \alpha$ for some $n \in \mathbb{Z}$. But then also $(\beta - n \cdot \alpha) \cdot v \in \Gamma$ and $0 < (\beta - n \cdot \alpha) < \alpha$, which is in contradiction with α being the infimum of M. □

2.2 Integer trigonometry definitions

Definition 2.13 (Integer angle)**.** *An angle $\angle ABC$ is integer, if its vertex B is an integer point.*

Definition 2.14 (Rational angle)**.** *Let $\angle ABC$ be an integer angle and let L be a line passing through points A, B and K be a line passing through points B, C. The integer angle $\angle ABC$ is rational if it has integer points distinct from B on both lines L and K.*

Definition 2.15 (Integer triangle)**.** *A triangle $\triangle ABC$ with vertices A, B, C is integer if all its vertices are integer points.*

Definition 2.16 (Integer segment)**.** *Let $A, B \in \mathbb{R}^2$. A line segment AB is called an integer segment if A, B are integer points.*

Definition 2.17 (Integer affine transformation)**.** *Let $A \in \mathrm{GL}(n, \mathbb{Z})$, let $c \in \mathbb{Z}^n$ and let $x \in \mathbb{R}^n$ be arbitrary. Then, the transformation $x \mapsto A \cdot x + c$ is called an integer affine transformation. The group of integer affine transformations is denoted as $\mathrm{Aff}(n, \mathbb{Z})$.*

Remark. Clearly every transformation in $\mathrm{Aff}(n, \mathbb{Z})$ maps $\mathbb{Z}^n$ to $\mathbb{Z}^n$.

Definition 2.18 (Integer congruent)**.** *Let $U, V \subseteq \mathbb{R}^n$. They are integer congruent if there exists an integer affine transformation $M \in \mathrm{Aff}(n, \mathbb{Z})$ such that $V = M(U)$.*

2.3 Integer length

Definition 2.19 (Integer length)**.** *Let L be a line that goes through integer points A and B. We will denote the affine lattice of all integer points contained on L as Γ. Let the integer segment AB generate a sublattice Γ_1 of Γ. Then the integer length $l\ell(AB)$ is the index of Γ_1 in Γ, i.e. $l\ell(AB) = [\Gamma : \Gamma_1]$.*

The following lemma proves the geometric interpretation of this definition.

Lemma 2.20. *The integer length of the integer segment AB can be computed as the number of integer points on the integer segment AB minus one, where the endpoints are included.*

Proof. Let us denote $v = B - A$ and let us define $w \in \mathbb{Z}^2$ with the use of Lemma 2.12, such that $\{w\}$ is the basis of lattice Λ, which is the lattice of all integer points in $\text{Span}(B - A)$. Then we define Γ as the affine lattice of all integer points on the integer segment AB as follows:

$$\Gamma = \{A + t \mid t \in \Lambda\}.$$

Then we denote Γ_0 as the affine lattice generated by the integer segment AB:

$$\Gamma_0 = \{A + o \cdot v \mid o \in \mathbb{Z}\}.$$

Let Λ_0 denote the underlying lattice of Γ_0.

Now we find $m \in \mathbb{Z}$ such that $B = A + m \cdot w$. Such m exists, because w is the basis of the lattice Λ. Since $B = A + m \cdot w$, we get that $v = m \cdot w$, thus m denotes the number of integer points on the integer segment AB minus one.

Clearly $\Lambda_0 \subseteq \Lambda$. We proceed to find sublattices that are cosets of Λ_0, denoted as

$$\Lambda_i = \{i \cdot w + \Lambda_0\}, \;\; i \in \{0, 1, 2, ..., m - 1\}.$$

We want to prove that every two cosets Λ_i, Λ_j are different for $i \neq j, 0 \leq i, j < m$. Let us prove this by contradiction, therefore, let $\Lambda_i = \Lambda_j$, thus $i \cdot w + \Lambda_0 = j \cdot w + \Lambda_0$. That happens if and only if $(j - i) \cdot w + \Lambda_0 = \Lambda_0 \Leftrightarrow (j - i) \cdot w \in \Lambda_0$. The basis of Λ_0 is $\{m \cdot w\}$ and because $(j - i) \cdot w \in \Lambda_0$, we get that $j \equiv i \pmod{m}$ and that can happen only if $j = i$. This way we found m different sublattices of Λ.
Now we need to prove that there are no other different sublattices of Λ, i.e. that the m different cosets of Λ_0 cover all of Λ. The cosets of Λ_0 are exactly $\Lambda_n = \{n \cdot w + \Lambda_0\}, n \in \mathbb{Z}$. Let for example $n = m + 1$, then $(m + 1) \cdot w + \Lambda_0$ and $w + \Lambda_0$ denote the same coset, because $m + 1 \equiv 1 \pmod{m}$, therefore $\Lambda_{m+1} = \Lambda_1$. Similarly for every other $n \in \mathbb{Z}$.
Thus

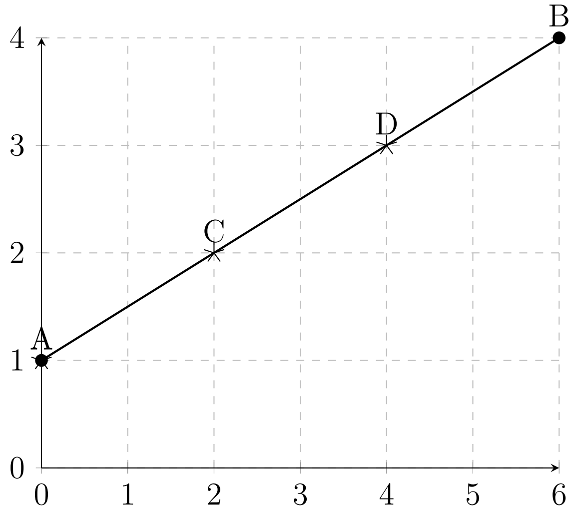
$$[\Lambda : \Lambda_0] = [\Gamma : \Gamma_0] = m.$$

Which is equal to the number of integer points on the integer segment AB minus one.

□

Example. Let AB be an integer segment with endpoints $A = (0, 1)$ and $B = (6, 4)$. We compute the length of the segment using Lemma 2.20. There are 4 integer points on the integer segment, including the endpoints, so the integer length is 3.

Figure 2.1: Computing integer length.



Lemma 2.21. *Integer length is invariant under integer affine transformations* $\mathrm{Aff}(2, \mathbb{Z})$.

Proof. We will prove that an integer affine transformation T maps inner integer points of an integer segment AB to inner integer points of an integer segment $A'B'$, where $A'B'$ is the image of AB. Then, by Lemma 2.20 and because the integer affine transformation is bijective, the statement will be proved.
Let $T : x \mapsto M \cdot x + g$, where $M \in \mathrm{GL}(2, \mathbb{Z}),\ g \in \mathbb{Z}^2$ as in the definition of an integer affine transformation. We will further assume $g = (0, 0)$ because the translation g cannot change the number of points. The matrix $M \in \mathrm{GL}(2, \mathbb{Z})$, therefore it denotes a linear transformation. Let AB be an integer segment and suppose C is its inner integer point. Then $M(AB)$ is also an integer segment because M sends integer points to integer points, since $M \in \mathrm{GL}(2, \mathbb{Z})$. Let us denote $M(A) = A',\ M(B) = B'$. Now, every inner point on the integer segment AB can be written as $C = t \cdot A + (1 - t) \cdot B$, where $0 < t < 1$. Then

$$M(C) = M(t \cdot A + (1 - t) \cdot B) = t \cdot M(A) + (1 - t) \cdot M(B) = t \cdot A' + (1 - t) \cdot B'.$$

Since $0 < t < 1$, we get that $M(C)$ is an inner point on the integer segment $M(AB)$. Since $M(C)$ must also be integer, it is an inner integer point of $M(AB)$. It is left to prove that every integer point X on integer segment $M(AB)$ has its pre-image Y, i.e. $M \cdot Y = X$. But because the matrix $M \in \mathrm{GL}(2, \mathbb{Z})$, it is invertible, therefore $Y = M^{-1} \cdot X$ and the transformation is surjective. Thus, we proved that integer length is invariant under integer affine transformations. □

Next, we introduce and prove two lemmata, which will be useful for proofs of Lemmata 2.30, 2.34.

Lemma 2.22. *Let* A, B, C *be integer points that do not lie on the same line. Let* Γ *be a lattice generated by* $\{A - B, C - B\}$. *Then let* Γ_1 *be a lattice generated by* $\{A_1 - B, C_1 - B\}$, *where* $A_1 \neq B$ *is the closest integer point to* B *on the integer segment* BA *and* $C_1 \neq B$ *is the closest integer point to* B *on the integer segment* BC. *Then*

$$[\Gamma_1 : \Gamma] = \frac{[\mathbb{Z}^2 : \Gamma]}{[\mathbb{Z}^2 : \Gamma_1]},$$

where all three indices of the groups are finite.

Proof. We know that $\Gamma \subseteq \Gamma_1$, because $l\ell(BA_1) = 1 = l\ell(BC_1)$ and $A - B = (A_1 - B) \cdot l\ell(BA)$, $C - B = (C_1 - B) \cdot l\ell(BC)$, therefore $(A - B), (C - B) \in \Gamma_1$. Also, $\Gamma \trianglelefteq \Gamma_1 \trianglelefteq \mathbb{Z}^2$, because all the groups are abelian. Therefore, by the second isomorphism theorem [4],

$$(\mathbb{Z}^2/\Gamma_1) \cong (\mathbb{Z}^2/\Gamma)/(\Gamma_1/\Gamma).$$

From the definition of factorgroups, we get that

$$|\mathbb{Z}^2/\Gamma_1| = [\mathbb{Z}^2 : \Gamma_1], \quad |\mathbb{Z}^2/\Gamma| = [\mathbb{Z}^2 : \Gamma], \quad |\Gamma_1/\Gamma| = [\Gamma_1 : \Gamma].$$

From the second isomorphism theorem, we also get that

$$|\mathbb{Z}^2/\Gamma_1| = |(\mathbb{Z}^2/\Gamma)/(\Gamma_1/\Gamma)|,$$

and because isomorphism is a bijection and because we know that all the indices are finite, then

$$(\mathbb{Z}^2/\Gamma),\ (\Gamma_1/\Gamma),\ (\mathbb{Z}^2/\Gamma_1)$$

are finite groups, therefore

$$[\mathbb{Z}^2 : \Gamma_1] = |\mathbb{Z}^2/\Gamma_1| = |(\mathbb{Z}^2/\Gamma)/(\Gamma_1/\Gamma)| = \frac{|(\mathbb{Z}^2/\Gamma)|}{|(\Gamma_1/\Gamma)|} = \frac{[\mathbb{Z}^2 : \Gamma]}{[\Gamma_1 : \Gamma]}$$

and that concludes the proof. □

Remark. This lemma can be also formulated for a lattice Γ_2, such that it is generated by $\{A_1 - B, C - B\}$, where $A_1 \neq B$ is the closest integer point to B on the integer segment BA. Then $[\Gamma_2 : \Gamma] = \frac{[\mathbb{Z}^2:\Gamma]}{[\mathbb{Z}^2:\Gamma_2]}$. The steps of the proof are the same as in the proof of 2.22.

Lemma 2.23. *Let A, B, C be integer points that do not lie on the same line. Let Γ be a lattice generated by $\{A - B, C - B\}$. Let Γ_1 be a lattice generated by $\{A_1 - B, C_1 - B\}$, where $A_1 \neq B$ is the closest integer point to B on the integer segment BA and $C_1 \neq B$ is the closest integer point to B on the integer segment BC. Then*

$$[\Gamma_1 : \Gamma] = l\ell(BA) \cdot l\ell(BC).$$

Proof. We know that $\{A - B, C - B\}, \{A_1 - B, C_1 - B\}$ form bases of the corresponding lattices. Since the elements of both bases are non-collinear, they are linearly independent. Because A_1, C_1 are defined as the closest integer points to B, we get that $l\ell(BA_1) = 1 = l\ell(BC_1)$. We can therefore write $A - B = (A_1 - B) \cdot l\ell(BA)$, $C - B = (C_1 - B) \cdot l\ell(BC)$. Let us denote $l\ell(BA) = n$, $l\ell(BC) = m$.
Therefore $\Gamma = \{k \cdot n \cdot (A_1 - B) + l \cdot m \cdot (C_1 - B) \mid k, l \in \mathbb{Z}\}$, $\Gamma_1 = \{k \cdot (A_1 - B) + l \cdot (C_1 - B) \mid k, l \in \mathbb{Z}\}$. From that we also get $\Gamma \subseteq \Gamma_1$.
The index $[\Gamma_1 : \Gamma]$ is equal to the number of different cosets $a + \Gamma$, $a \in \Gamma_1$. Let $a = k_1 \cdot (A_1 - B) + l_1 \cdot (C_1 - B)$, $b = k_2 \cdot (A_1 - B) + l_2 \cdot (C_1 - B)$, then $a + \Gamma = b + \Gamma \Leftrightarrow (a - b) \in \Gamma \Leftrightarrow k_1 \equiv k_2 \pmod{n}$, $l_1 \equiv l_2 \pmod{m}$. This way we proved that there exist $n \cdot m$ distinct cosets of Γ_1.

Now, there is left to prove that these cosets cover all of Γ_1. The cosets are exactly $\Gamma_{e,f} = \{e \cdot (A_1 - B) + f \cdot (C_1 - B) + \Gamma\}$ for $e, f \in \mathbb{Z}$. For every $e, f \in \mathbb{Z}$, we will find e', f', such that $e \equiv e' \pmod n$, $f \equiv f' \pmod m$, therefore $\Gamma_{e,f} = \Gamma_{e',f'}$. This way we have shown that there are $n \cdot m$ cosets of Γ in Γ_1, therefore

$$[\Gamma_1 : \Gamma] = n \cdot m = l\ell(BA) \cdot l\ell(BC).$$

□

Remark. This lemma can be also formulated for a lattice Γ_2, such that it is generated by $\{A_1 - B,\ C - B\}$, where $A_1 \neq B$ is the closest integer point to B on the integer segment BA. Then $[\Gamma_2 : \Gamma] = l\ell(BA)$. The steps of the proof are be the same as in the proof of 2.23.

2.4 Integer area

Definition 2.24 (Integer area)**.** *Let $\triangle ABC$ be an integer triangle, Γ_1 be the affine lattice generated by the integer segments AB and AC. Then the integer area $lS(\triangle ABC)$ of $\triangle ABC$ is equal to the index of Γ_1 in $\mathbb{Z}^2$, i.e. $lS(\triangle ABC) = [\mathbb{Z}^2 : \Gamma_1]$.*

As a geometric approach to computing the integer area of a given integer triangle $\triangle ABC$, we can sum the number of all integer points inside the parallelogram defined in Lemma 2.25.

Lemma 2.25. *The integer area of an integer triangle $\triangle ABC$ can be computed as the number of integer points P satisfying*

$$P = \alpha(B - A) + \beta(C - A), \tag{2.1}$$

where $0 \leq \alpha, \beta < 1$.

Proof. Let us first denote $v = B - A$, $w = C - A$. We want to prove that the index of $\Gamma_0 = \{mv + nw, m, n \in \mathbb{Z}\}$, which is the lattice generated by v and w, in $\mathbb{Z}^2$ is equal to the number of all integer points P satisfying 2.1. Let

$$Par = \{\alpha v + \beta w, 0 \leq \alpha, \beta < 1\}$$

be a parallelogram denoted by v and w. We want to show that for every $g \in \mathbb{Z}^2$ denoting a coset $g + \Gamma_0$ of Γ_0 there exists integer point $P \in Par$ such that $P \in g + \Gamma_0$ and thus $P + \Gamma_0 = g + \Gamma_0$.
Let

$$g = m_1 v + m_2 w, \ \ m_1, m_2 \in \mathbb{Q},$$

then

$$P = (m_1 - \lfloor m_1 \rfloor)v + (m_2 - \lfloor m_2 \rfloor)w$$

satisfies $P + \Gamma_0 = g + \Gamma_0$. And because

$$0 \leq m_1 - \lfloor m_1 \rfloor < 1, \quad 0 \leq m_2 - \lfloor m_2 \rfloor < 1,$$

the point P is inside the parallelogram and $P \in g + \Gamma_0$.
Then we have to prove the unambiguity of P, i.e. that for two different integer

points $P_1, P_2 \in Par$, the corresponding cosets $P_1 + \Gamma_0, P_2 + \Gamma_0$ are also different. Let us have $P_1, P_2 \in Par$, such that $P_1 \neq P_2$. For contradiction let $P_2 \in P_1 + \Gamma_0$. From this, we get that $p = P_2 - P_1 \in \Gamma_0$. We know that every P is of the type

$$\alpha v + \beta w, \quad 0 \leq \alpha, \beta < 1.$$

Therefore, p is of the type

$$\alpha v + \beta w, \quad -1 < \alpha, \beta < 1.$$

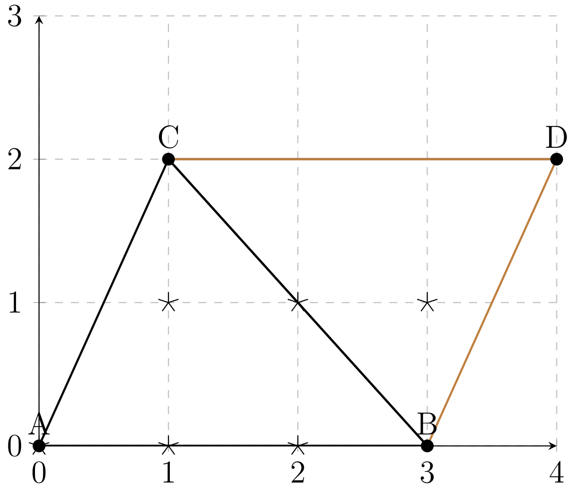
The only p of this type in Γ_0 is $p = (0, 0)$. Thus,

$$p = (0, 0) \implies P_1 = P_2.$$

We proved that the integer points in Par are in one-to-one integer correspondence to cosets of Γ_0 in $\mathbb{Z}^2$. □

Example. Let $A = (0, 0),\ B = (3, 0),\ C = (1, 2)$ be three integer vertices of an integer triangle $\triangle ABC$. We will compute its area using Lemma 2.25. On the graph below, we create a parallelogram $ABDC$, where $D = (4, 2)$. The integer points satisfying 2.1 are all integer points inside the parallelogram $ABDC$, all integer points on the integer segment AB excluding B and all integer points on the integer segment AC excluding C. These integer points are displayed as stars on the graph below, and if we sum them all, we get 6, which is the integer area of $\triangle ABC$.

Figure 2.2: Computing integer area.



Definition 2.26 (Empty integer triangle)**.** *An empty integer triangle is an integer triangle which does not contain any other integer points apart from its vertices.*

Let us have an integer triangle $\triangle ABC$, denoted as $S(\triangle ABC)$, where $A = (a_1, a_2),\ B = (b_1, b_2),\ C = (c_1, c_2)$. Then the Euclidean area of $\triangle ABC$ can be computed as

$$S(\triangle ABC) = \frac{1}{2}\left|\det\begin{pmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{pmatrix}\right| = \frac{1}{2}|\det(B - A, C - A)|.$$

The geometric meaning of the determinant of a matrix A of type 2×2 is the change of area during the linear transformation f_A. Determinant of u and v, that denote the two sides of the triangle, gives us the area of a parallelogram denoted by u and v. Thus, $\frac{1}{2} \det |(u, v)|$ is an area of a triangle, which is the exact half of the parallelogram.

Lemma 2.27. *Let $\triangle ABC$ be an integer triangle. Then the following statements are equivalent:*

(a) $\triangle ABC$ is empty;

(b) $lS(\triangle ABC) = 1$;

(c) $S(\triangle ABC) = \frac{1}{2}$.

Proof. $(a) \Rightarrow (b)$: Let $\triangle ABC$ be an empty triangle, and $Par = \{\alpha(B - A) + \beta(C - A) \mid 0 \leq \alpha, \beta \leq 1\}$ be a parallelogram denoted by the integer segments AB and AC. This parallelogram is also empty (i.e. does not contain any integer points apart from its vertices) because $\triangle ABC$ is empty. We can use Lemma 2.25., which states that there is only one coset of the subgroup of $\mathbb{Z}^2$ which is generated by the integer segments AB and AC, from which follows that

$$lS(\triangle ABC) = 1.$$

$(b) \Rightarrow (c)$: Let $lS(\triangle ABC) = 1$. From that, we know that the integer segments AB and AC generate the integer lattice $\mathbb{Z}^2$. That means that any integer point is an integer combination of $(B - A)$ and $(C - A)$, i.e. there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{Z}^2$ such that

$$(1, 0) = \lambda_1(B - A) + \lambda_2(C - A)$$

and

$$(0, 1) = \mu_1(B - A) + \mu_2(C - A).$$

Let us denote $(B - A) = (b_1, b_2)$ and $(C - A) = (c_1, c_2)$. We can put

$$J = \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}$$

therefore

$$L = JK = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The determinant of this matrix is equal to one, and because J and K are integer matrices, their determinants must be either 1 or -1 because $\det(J \cdot K) = 1$ and $\det(J)$, $\det(K)$ are integers. So the Euclidean area of $\triangle ABC$ is equal to 1/2 because $S(\triangle ABC) = \frac{1}{2}|\det(B - A, C - A)|$.
$(c) \Rightarrow (a)$: We will prove this by contradiction. Let us consider an integer triangle $\triangle ABC$ such that $S(\triangle ABC) = 1/2$. For contradiction, suppose it has an integer

point D in the interior or on its sides. Without loss of generality, we can assume that D will not lie on the integer segment AB. However then

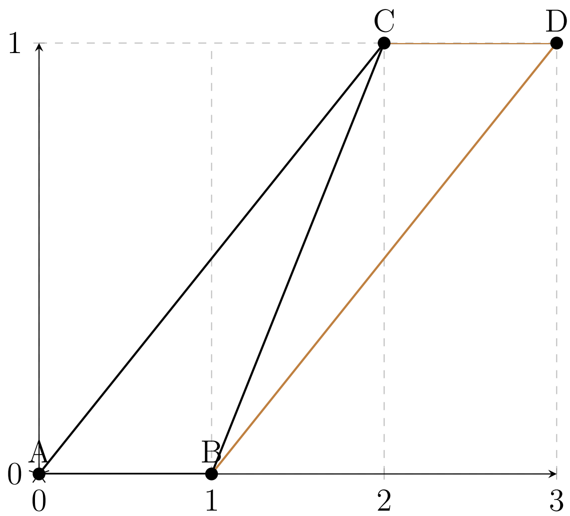
$$S(\triangle ADB) < S(\triangle ABC)$$

and that cannot happen. Otherwise, there would exist a number $a \in \mathbb{Z}$ such that $0 < a < 1$, because $S(\triangle ABC) = \frac{1}{2}|\det(B-A, C-A)| = 1/2$ and the $(B-A, D-A)$ matrix has only integer values, so its determinant is an integer, but it needs to be strictly smaller in absolute value than the determinant of the matrix $(B-A, C-A)$, which is equal to 1. $\square$

Example. We have an empty integer triangle $\triangle ABC$ with integer points $A = (0,0), B = (1,0), C = (2,1)$. We create a parallelogram by adding a point $D = (3,1)$ and compute the integer area. The integer area is 1, the only point is marked as a star, as defined in Definition 2.24. The Euclidean area is

$$\frac{1}{2}\left|\det\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\right| = 1/2.$$

Figure 2.3: Empty integer triangle and its integer area.



Lemma 2.28. *Integer area is invariant under integer affine transformations* $\mathrm{Aff}(2,\mathbb{Z})$.

Proof. We will use Lemma 2.25, therefore, we will prove that the inner parallelogram integer points are mapped to inner parallelogram integer points. Because the integer affine transformation is bijective, it will be sufficient to prove this. Let $T : x \mapsto M \cdot x + g$, where $M \in \mathrm{GL}(2,\mathbb{Z}),\ \ g \in \mathbb{Z}^2$ as in Definition 2.17. T maps integer points to integer points, since $M \in \mathrm{GL}(2,\mathbb{Z}),\ \ g \in \mathbb{Z}^2$. The transformation also preserves the parallelism of lines because the matrix $M \in \mathrm{GL}(2,\mathbb{Z})$, therefore, it is a linear transformation and $g \in \mathbb{Z}^2$ is a translation, so it cannot interfere with the parallelism of lines. We will further assume $g = (0,0)$ because the translation g cannot change the number of integer points. Therefore, the transformation T maps an integer parallelogram to an integer parallelogram.

Now let E be an inner integer point of the parallelogram made from an integer triangle $\triangle ABC$. Then $E = \alpha(B - A) + \beta(C - A), 0 \leq \alpha, \beta < 1$. Let

$$M(A) = A', \;\; M(B) = B', \;\; M(C) = C'.$$

Then $M(B - A) = B' - A'$, similarly $M(C - A) = C' - A'$, because M is linear. Then by computation,

$$M(E) = M(\alpha(B - A) + \beta(C - A)) = \alpha M(B - A) + \beta M(C - A) =$$

$$= \alpha(B' - A') + \beta(C' - A').$$

Because $0 \leq \alpha, \beta < 1$, M preserves the inner parallelogram integer points. Now it is left to prove that every integer point X has its pre-image Y, i.e. $M \cdot X = Y$. But because $M \in \mathrm{GL}(2, \mathbb{Z})$, it is invertible, therefore $X = M^{-1} \cdot Y$. Thus we proved that integer area is invariant under T. □

2.5 Integer distance

Definition 2.29 (Integer distance)**.** *Let A, B, C be three non-collinear integer points. The integer distance from the integer point A to the integer segment/line BC is the index of an affine lattice generated by the integer segments AB, BB_1, where $B_1 \neq B$ is the closest integer point to B on the line segment BC, in the integer lattice $\mathbb{Z}^2$. It is denoted by $ld(A, BC)$.*

If the points A, B, C lie on the same line, then we define $ld(A, BC) = 0$.

Lemma 2.30. *For any integer triangle $\triangle ABC$ it holds that*

$$lS(\triangle ABC) = l\ell(AB) \cdot ld(C, AB).$$

Proof. Let Γ be a lattice with basis $\{AB, AC\}$, then $lS(\triangle ABC) = [\mathbb{Z}^2 : \Gamma]$. Let Γ_0 be a lattice with basis $\{AA_1, AC\}$, where $A_1 \neq A$ is the closest integer point to A on the integer segment AB. Then $ld(C, AB) = [\mathbb{Z}^2 : \Gamma_0]$. From Lemma 2.22, we get that

$$[\Gamma_0 : \Gamma] = \frac{[\mathbb{Z}^2 : \Gamma]}{[\mathbb{Z}^2 : \Gamma_0]} = \frac{lS(\triangle ABC)}{ld(C, AB)}.$$

And from Lemma 2.23 and Definition 2.6, we get that

$$[\Gamma_0 : \Gamma] = l\ell(AB).$$

This concludes the proof. □

Remark. Because both integer length and integer area are invariant under integer affine transformations, so is the integer distance.

The geometric interpretation of the definition of integer distance is as follows.

Lemma 2.31. *Let $A, B, C \in \mathbb{Z}^2$ that are non-collinear. Let us draw all integer parallel lines to the line BC and denote AA' the integer parallel line to BC that contains the point A. Let m denote the number of integer parallel lines to the line BC in the region bounded by BC and AA', excluding BC and AA'. Then $ld(A, BC) = m + 1$.*

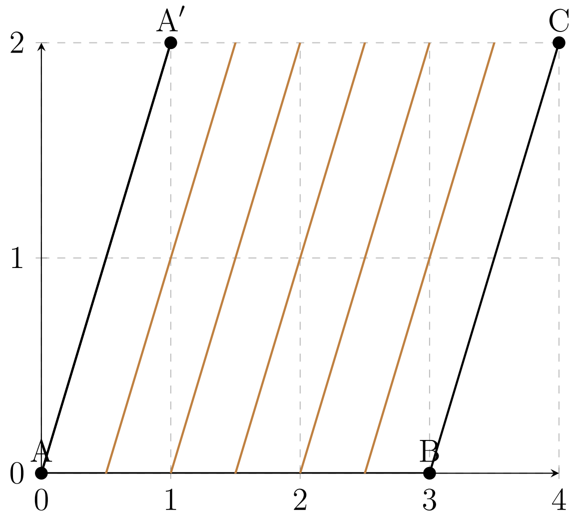
Proof. From Lemma 2.30, we know that $lS(\triangle ABC) = l\ell(BC) \cdot ld(A, BC)$. Let us have $B_1 \neq B$ as the closest integer point to B on the line segment BC. Then from Lemmata 2.22, 2.23, we know that

$$ld(A, BC) = \frac{lS(\triangle ABC)}{l\ell(BC)} = lS(\triangle ABB_1).$$

From Lemma 2.25, the integer area of $\triangle ABB_1$ is equal to the number of integer points inside a parallelogram ABB_1A'. We need to prove that there is one integer point for every integer parallel line and that for every integer point there exists an integer line parallel to BB_1, such that it passes through the integer point. Let us have a parallel line with two integer points X_1, X_2 inside the parallelogram. Since the line X_1X_2 is parallel to the line BB_1, WLOG $X_2 - X_1 = \alpha \cdot (B_1 - B)$, for some $0 < \alpha < 1$. Because $X_2 - X_1$ is an integer point, it is a contradiction with B_1 being the closest integer point to B on the line segment BC. Therefore, the number of integer parallel lines in the region plus one is equal to the number of integer points inside the parallelogram. □

Example. Let us have integer points $A = (0, 0), B = (3, 0), C = (4, 2)$. From Figure 2.4, we can see that there are five integer parallel lines in the region bounded by BC, AA'. Therefore, the integer distance is $ld(A, BC) = 5 + 1 = 6$.

Figure 2.4: Computing integer distance.



2.6 Integer sines, sails, LLS sequences

Definition 2.32 (Integer sine)**.** *Let $\angle ABC$ be a non-trivial rational angle with vertex B. Then let $A_1 \neq B$ be the integer point on the line segment BA, which is the closest to B. Let $C_1 \neq B$ be the integer point on the line segment BC, which is the closest to B. Then the integer sine of $\angle ABC$ is equal to $lS(\triangle A_1BC_1)$ and is denoted by $l\sin(\angle ABC)$. If $\angle ABC$ is a trivial angle, $l\sin(\angle ABC) = 0$.*

Remark. Integer sine is invariant under integer affine transformations because the integer area is invariant under integer affine transformations.

The sine function $\sin\alpha$ attains values from -1 to 1. For comparison, the value of integer sine is an integer greater than 1, as proved in Lemma 2.33.

Lemma 2.33. *For a non-trivial rational angle $\alpha = \angle ABC$, let $A_1 \neq B$ be the closest integer point to B on the line segment AB and let $C_1 \neq B$ be the closest integer point to B on the line segment BC. Then $l\sin\alpha \geq 1$ and equality holds if and only if $\triangle A_1BC_1$ is an empty triangle.*

Proof. From the definition, integer sine is equal to the integer area of the integer triangle $\triangle A_1BC_1$. The non-trivial integer triangle with the smallest integer area is an empty integer triangle, and its integer area is equal to 1, as proven in Lemma 2.27. Other integer triangles have integer area greater than one. Thus, if α is non-trivial, then the integer triangle $\triangle A_1BC_1$ is either empty and then $l\sin\alpha = 1$, or it is not empty, so its integer area is greater than one and then $l\sin\alpha > 1$. □

Lemma 2.34. *The integer sine satisfies*

$$l\sin(\angle ABC) = \frac{lS(\triangle ABC)}{l\ell(AB) \cdot l\ell(BC)}.$$

Proof. Let Γ be a lattice with basis $\{BA, BC\}$, then $lS(\triangle ABC) = [\mathbb{Z}^2 : \Gamma]$. Let Γ_0 be a lattice with basis $\{BA_1, BC_1\}$, where $A_1 \neq B$ is the closest integer point to B on the integer segment BA and $C_1 \neq B$ is the closest integer point to B on the integer segment BC. Then $l\sin(\angle ABC) = [\mathbb{Z}^2 : \Gamma_0]$. From Lemma 2.22, we get that

$$[\Gamma_0 : \Gamma] = \frac{[\mathbb{Z}^2 : \Gamma]}{[\mathbb{Z}^2 : \Gamma_0]} = \frac{lS(\triangle ABC)}{l\sin(\angle ABC)}.$$

From Lemma 2.23 and Definition 2.6, we get that

$$[\Gamma_0 : \Gamma] = l\ell(BA) \cdot l\ell(BC).$$

This concludes the proof. □

Next, we mention an analogy of the sine rule from Euclidean geometry in $\mathbb{R}^2$, where the following holds for an angle $\angle ABC$:

$$\frac{\sin(\angle ABC)}{\ell(AC)} = \frac{\sin(\angle BCA)}{\ell(AB)} = \frac{\sin(\angle CAB)}{\ell(BC)} = \frac{2 \cdot S(\triangle ABC)}{\ell(AB) \cdot \ell(BC) \cdot \ell(AC)}.$$

The last equality follows from the fact that the Euclidean area of a triangle can be computed as $S(\triangle ABC) = \frac{\ell(BC)\cdot\ell(AX)}{2}$, where AX is the altitude of the triangle. Then, the sine function can be geometrically computed as $\sin(\angle ABC) = \frac{\ell(AX)}{\ell(AB)}$, and therefore, the last equality holds.
From this we also get an Euclidean analogy for Lemma 2.34, because

$$\sin(\angle ABC) = \frac{\ell(AX)}{\ell(AB)} = \frac{2 \cdot S(\triangle ABC)}{\ell(BC) \cdot \ell(AB)}.$$

Lemma 2.35. *Let $\angle ABC$ be a rational angle, then*

$$\frac{l\sin(\angle ABC)}{l\ell(AC)} = \frac{l\sin(\angle BCA)}{l\ell(AB)} = \frac{l\sin(\angle CAB)}{l\ell(BC)} = \frac{lS(\triangle ABC)}{l\ell(AB) \cdot l\ell(AC) \cdot l\ell(BC)}.$$

Proof. From Lemma 2.34. we get that

$$\frac{l\sin(\angle ABC)}{l\ell(AC)} = \frac{lS(\triangle ABC)}{l\ell(AB)\cdot l\ell(AC)\cdot l\ell(BC)},$$

$$\frac{l\sin(\angle BCA)}{l\ell(AB)} = \frac{lS(\triangle BCA)}{l\ell(AB)\cdot l\ell(CA)\cdot l\ell(BC)},$$

$$\frac{l\sin(\angle CAB)}{l\ell(BC)} = \frac{lS(\triangle CAB)}{l\ell(AB)\cdot l\ell(CA)\cdot l\ell(BC)}.$$

From definitions of integer length and integer area, we get that

$$lS(\triangle ABC) = lS(\triangle BCA) = lS(\triangle CAB)$$

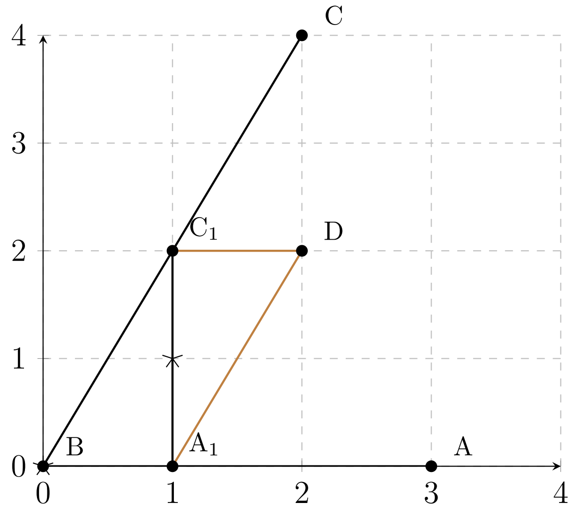
and

$$l\ell(BC) = l\ell(CB),\ \ l\ell(AB) = l\ell(BA),\ \ l\ell(AC) = l\ell(CA).$$

This concludes the proof. □

Example. Let us have a rational angle $\angle ABC$, where $A = (3,0),\ B = (0,0),\ C = (2,4)$. We search for the closest integer point $C_1 \neq B$ to B on the integer segment BC and we find that it is $C_1 = (1,2)$. We search for the closest integer point $A_1 \neq B$ to B on the integer segment BA and we find that it is $A_1 = (1,0)$. Therefore the $l\sin\angle ABC = lS(\triangle A_1BC_1)$, which is 2.

Figure 2.5: Computing $l\sin\angle ABC$.



Definition 2.36 (Sail)**.** *Let $\alpha = \angle ABC$ be an integer angle with a vertex B. The sail of α is the boundary of the convex hull of all integer points inside the angle without B.*

If the angle $\alpha = \angle ABC$ is rational, then the main part of the sail will be determined by a finite set of integer points $A_0, \ldots, A_n$, where both the endpoints A_0, A_n lie on the angle rays, namely A_0 lies on line BA, A_n lies on line BC. We choose points $A_0, \ldots, A_n$ such that they are extremal, i.e. no three consecutive points $A_{i-1}A_iA_{i+1}$ lie on one line. From these endpoints, the sail continues to infinity on both sides of the angle but does not have any more extremal vertices.

We will call this piecewise linear curve a broken line $A_0 \dots A_n$ of the sail of α.
If $\angle ABC$ is not rational, then there will be infinitely many extremal vertices of the sail (on one or both sides of the sail).
Further in this thesis, we will only work with rational angles.

Definition 2.37 (Lattice length sine sequence)**.** *Let α be a rational angle and let $A_0 A_1 ... A_n$ be the broken line of the sail of α. Then the lattice length sine sequence of α, also denoted as LLS sequence, is the sequence defined as follows:*

$$a_{2k} = l\ell(A_k A_{k+1}),$$

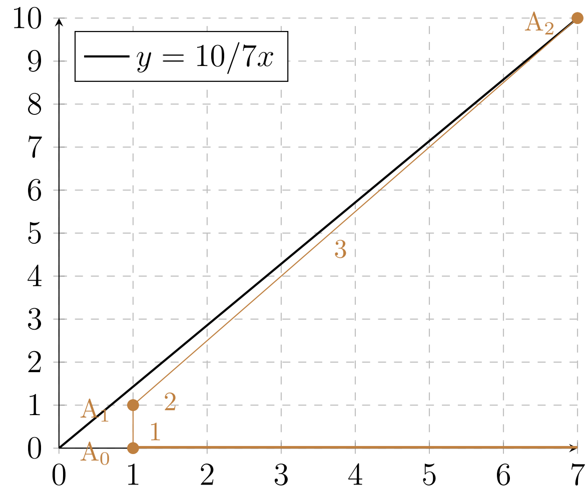
$$a_{2k-1} = l\sin(\angle A_{k-1} A_k A_{k+1}),$$

where $0 \leq k \leq n-1$ for a_{2k} and $1 \leq k \leq n-1$ for a_{2k-1}. We denote this LLS sequence as $LLS(\alpha)$.

The LLS sequence can be defined for an angle that is not rational, then the LLS sequence can be infinite on one or both sides, depending on whether the sail consists of an infinite broken line on one or both sides. In the case of rational angles, the LLS sequence is always finite because the sail consists of finite broken line.
The LLS sequence of an angle is invariant under integer affine transformations of the plane because integer length and integer sine are invariant under integer affine transformations, and convex hulls are preserved by the elements of $\text{Aff}(2, \mathbb{Z})$.

Example. Let us compute the sail and LLS sequence of an angle formed by two lines $\{(x, 0) \mid x \geq 0\}$ and $\{(x, 10/7x) \mid x \geq 0\}$. The sail consists of a broken line $A_0 A_1 A_2$, where $A_0 = (1, 0)$, $A_1 = (1, 1)$, $A_2 = (7, 10)$, the continuation from C to infinity in the same direction as the ray of the angle and the continuation from A to infinity in the same direction as the ray of the angle. The LLS sequence is $(1, 2, 3)$, because $l\ell(A_0 A_1) = 1$, $l\sin(\angle A_0 A_1 A_2) = 2$ and $l\ell(A_1 A_2) = 3$.

Figure 2.6: Computing sail and LLS sequence.



Theorem 2.38. *Two rational angles are integer congruent if and only if their LLS sequences are the same.*

Proof. We already know that the LLS sequence is invariant under integer affine transformations. Thus, two integer congruent angles have the same LLS sequences. Therefore, we need to prove that if two LLS sequences of two angles coincide, then these angles will be integer congruent. Let us have two integer angles $\alpha,\ \beta$ with corresponding broken lines $\{A_i\},\ \{B_i\}$, LLS sequences $(a_i),\ (b_i)$ and integer vertices $O_\alpha,\ O_\beta$. Now let us assume that the LLS sequence (b_i) coincides with the LLS sequence (a_i).

From the definition of sail we know that $l\ell(O_\beta B_0) = l\ell(O_\beta B_1) = l\ell(O_\alpha A_0) = l\ell(O_\alpha A_1) = 1$. Let $B' \neq B_0$ be the closest integer point to B_0 on the integer segment B_0B_1. Similarly, let $A' \neq A_0$ be the closest integer point to A_0 on the integer segment A_0A_1. Since $l\ell(A_0A') = l\ell(B_0B') = 1$ and because the integer triangle $\triangle O_\beta B_0 B'$ is empty, we know that the integer segments $O_\beta B_0,\ B_0B'$ generate the whole lattice $\mathbb{Z}^2$, thus they are the basis of $\mathbb{Z}^2$. Similarly the integer segments $O_\alpha A_0,\ A_0A'$ generate the whole lattice $\mathbb{Z}^2$ and form its basis. Because of that, there exists an affine transformation ξ such that it maps $O_\beta B_0 \mapsto O_\alpha A_0,\ B_0B' \mapsto A_0A'$. It especially maps $\triangle O_\beta B_0 B' \mapsto \triangle O_\alpha A_0 A'$, which are both empty triangles. The transformation ξ is an integer affine transformation because it is the change of basis transformation, and both are the bases of $\mathbb{Z}^2$. Finally, we want to prove that ξ maps $B_1 \mapsto A_1$. We know that $lS(\triangle O_\beta B_0 B_1) = a_0 = l\ell(B_0B_1) = l\ell(A_0A_1)$. We therefore know that $A_1 = (A' - A_0)\cdot a_0 + A_0$ and $B_1 = (B' - B_0)\cdot a_0 + B_0$. Since ξ is an integer affine transformation, it will map $B_1 \mapsto A_1$, because both A_0, A', A_1 and B_0, B', B_1 lie on one line and affine transformations preserve affine combinations.

Now let β be transformed to an angle γ using the integer affine transformation ξ. The angle γ has its integer vertex O_α, sail $\{C_i\}$ and $C_0 = A_0, C_1 = A_1$.

We proceed to show that $A_0A_1\dots A_n$ and $C_0C_1\dots C_n$ coincide by induction. Let $A_0A_1\dots A_{k-1}$ coincide with $C_0C_1\dots C_{k-1}$, we want to prove that $A_k = C_k$. Firstly let

$$l\sin(\angle A_{k-2}A_{k-1}A_k) = l\sin(C_{k-2}C_{k-1}C_k) = a_{2k-3},$$
$$l\ell(A_{k-1}A_k) = l\ell(C_{k-1}C_k) = a_{2k-2}.$$

Then with the help of Lemma 2.30 and Lemma 2.34 we can compute

$$ld(A_k, A_{k-2}A_{k-1}) = \frac{lS(\triangle A_k A_{k-2} A_{k-1})}{l\ell(A_{k-2}A_{k-1})} = l\sin(\angle A_{k-2}A_{k-1}A_k)\cdot l\ell(A_{k-1}A_k) =$$
$$= a_{2k-3}\cdot a_{2k-2} = ld(C_k, C_{k-2}C_{k-1}).$$

Both $\angle A_{k-2}A_{k-1}A_k$ and $\angle C_{k-2}C_{k-1}C_k$ are parts of sails, since sail is defined as the boundary of a convex hull, therefore A_k, C_k lie on a different halfspace than O_α with respect to the line $A_{k-2}A_{k-1} = C_{k-2}C_{k-1}$. Thus A_k, C_k lie on a line ℓ_1 that is parallel to the line $A_{k-2}A_{k-1}$ and this line contains integer points that are at integer distance of $a_{2k-3}\cdot a_{2k-2}$ from the line $A_{k-2}A_{k-1}$, because as shown above, $ld(A_k, A_{k-2}A_{k-1}) = a_{2k-3}\cdot a_{2k-2}$.

Now we continue. Because of the definition of sail, we know that $l\ell(O_\alpha A_{k-1}) = l\ell(O_\alpha C_{k-1}) = 1$ and that the only integer points of the integer triangle (apart from its vertices) $\triangle A_k O_\alpha A_{k-1}$ are the integer points on the integer segment $A_{k-1}A_k$. Therefore, by Lemma 2.25, $lS(\triangle C_k O_\alpha C_{k-1}) = lS(\triangle A_k O_\alpha A_{k-1}) = l\ell(A_{k-1}A_k) = l\ell(C_{k-1}C_k) = a_{k-2}$. Thus,

$$ld(A_k, O_\alpha A_{k-1}) = \frac{lS(\triangle A_k O_\alpha A_{k-1})}{l\ell(O_\alpha A_{k-1})} = \frac{lS(\triangle C_k O_\alpha C_{k-1})}{l\ell(O_\alpha C_{k-1})} = ld(C_k, O_\alpha C_{k-1})$$

and

$$ld(A_k, O_\alpha A_{k-1}) = a_{2k-2} = ld(C_k, O_\alpha C_{k-1}).$$

We know that $O_\alpha A_{k-1} = O_\alpha C_{k-1}$ and $A_k,\ C_k$ lie on a different halfspace than A_{k-2} with respect to the line $O_\alpha A_{k-1}$, because the construction of sail as a convex hull implies that all points in the angle lie on one side of any line segment of the sail. Therefore $A_k,\ C_k$ lie on a line ℓ_2, which is parallel to the line $O_\alpha A_{k-1}$ and this line contains points that are at integer distance of a_{2k-2} from the line $O_\alpha A_{k-1}$. Since the lines ℓ_1, ℓ_2 are parallel to $A_{k-2}A_{k-1}$ and $O_\alpha A_{k-1}$ respectively, they are not parallel and intersect at a unique point $A_k = C_k$. Thus, we proved that $A_0A_1 \dots A_n$ and $C_0C_1 \dots C_n$ coincide. □

2.7 Continued fractions

In this subchapter and further, we will use continued fractions, their notation and properties as defined and proved in [5]. We extend the notation in [5] to also allow finite continued fractions to contain negative integers if the corresponding rational number is well defined.

Definition 2.39 (Continued fraction)**.** *Let $a_0, a_1, \dots, a_n \in \mathbb{Z}$. We say that $[a_0, a_1, \dots, a_n]$ is a continued fraction of a number $\xi \in \mathbb{Q}$, if the fraction*

$$[a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_n}}},$$

is well defined and equals ξ.

Definition 2.40 (Odd and even continued fractions)**.** *We define odd and even continued fractions, continued fraction $[a_0, \dots, a_n]$ is odd (resp. even), when n is even (resp. odd), i.e. the number of elements in the fraction is odd (resp. even).*

Definition 2.41 (Regular continued fraction)**.** *A continued fraction is called regular if the first element a_0 is an integer and all of the other elements are positive integers.*

Remark. Every rational number has exactly two regular continued fractions, one odd and one even:

$$[a_0, \dots a_n] = [a_0, \dots, a_n - 1, 1].$$

This leads us to an example of a non-regular continued fraction, which will be used more in the next chapter.

Example. Let us have the rational number $\frac{11}{8}$. Then the regular continued fraction is

$$\frac{11}{8} = 1 + \cfrac{1}{2 + \frac{1}{1+\frac{1}{2}}} = [1, 2, 1, 2] = [1, 2, 1, 1, 1].$$

The non-regular continued fraction is for example

$$\frac{11}{8} = 1 + \cfrac{1}{2 + \frac{1}{3+\frac{1}{-1+\frac{1}{3}}}} = [1, 2, 3, -1, 3].$$

We can also write

$$\frac{11}{8} = 1 + \frac{1}{2 + \frac{1}{4+\frac{1}{-1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}}} = [1, 2, 4, -1, 1, 1, 2]$$

and we can see that non-regular fractions of rational numbers can be written in different ways.

2.8 Integer tangents, integer cosines

Definition 2.42 (Integer tangent)**.** *Let $(a_0, ..., a_{2n})$ be a LLS sequence of a rational angle α. The integer tangent $l\tan\alpha$ of α is defined as follows:*

$$l\tan\alpha = a_0 + \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_{2n}}}} = [a_0, a_1, \ldots, a_{2n}].$$

If α is a trivial angle, then $l\tan\alpha = 0$.

The integer tangent can be also defined for all angles $\alpha = \angle ABC$, such that there is a distinct integer point $C \neq B$ on the line BA but not on the line BC. Then α is not a rational angle and the LLS sequence is infinite. Therefore, we will have an infinite continued fraction.

Remark. Integer tangent is invariant under integer affine transformations, as is the LLS sequence.

Lemma 2.43. *Let $\alpha = \angle ABC$ be a non-trivial rational angle. Then let $A_1 \neq B$ be the integer point on the line segment BA, which is the closest to B. Let $C_1 \neq B$ be the integer point on the line segment BC, which is the closest to B. Then $l\tan\alpha \geq 1$ and equality holds if and only if the integer segments BA_1, BC_1 generate the whole lattice $\mathbb{Z}^2$.*

Proof. Because α is a non-trivial angle, the first element of its corresponding LLS sequence is non-zero. That proves that $l\tan\alpha \geq 1$. If $l\tan\alpha = 1$, then, from the definition of integer tangent, the LLS sequence of α is equal to (1). That means that the broken line of its sail consists of only two points with integer length 1, therefore the corresponding triangle $\triangle A_1BC_1$ is empty and therefore the index of an affine lattice generated by $BA_1,\ BC_1$ in $\mathbb{Z}^2$ is 1, thus it generates the whole lattice $\mathbb{Z}^2$. □

Corollary 2.44. *Two integer angles α, β are integer congruent if and only if $l\tan\alpha = l\tan\beta$.*

Proof. Every number ξ has a unique odd regular continued fraction representation, as mentioned in Section 2.7. Then, from the definition of integer tangent, we know that the LLS sequence is uniquely defined by integer tangent. Then the statement holds because of Theorem 2.38. □

Definition 2.45 (Integer cosine)**.** *Let α be a non-trivial rational angle. The integer cosine $l\cos\alpha$ of α is defined as follows*

$$l\cos\alpha = \frac{l\sin\alpha}{l\tan\alpha}.$$

For a trivial angle, the integer cosine is defined to be 1.

Lemma 2.46. *Let $\alpha = \angle ABC$ be a non-trivial rational angle. Then let $A_1 \neq B$ be the integer point on the line segment BA, which is the closest to B. Let $C_1 \neq B$ be the integer point on the line segment BC, which is the closest to B. Then $l\cos\alpha \leq l\sin\alpha$ and equality holds if and only if the integer segments BA_1, BC_1 generate the whole lattice $\mathbb{Z}^2$.*

Proof. As shown in Lemma 2.33 and Lemma 2.43, $l\sin\alpha \geq 1$ and $l\tan\alpha \geq 1$. If $l\tan\alpha = 1$, then $l\cos\alpha = l\sin\alpha$ and from Lemma 2.43, if $l\tan\alpha = 1$, we know that the integer segments BA_1, BC_1 generate the whole lattice $\mathbb{Z}^2$. If $l\tan\alpha > 1$, then $l\cos\alpha < l\sin\alpha$, because $l\sin\alpha > 1$. □

2.9 Integer arctangents

Definition 2.47 (Integer arctangent)**.** *Let $q = \frac{m}{n} \geq 1$ be a rational number, where $m \geq n > 0$ are relatively prime integers. The integer arctangent $l\arctan q$ of q is a rational angle α with a vertex in the origin and edges passing through the points $(1, 0)$ and (n, m).*

Lemma 2.48. *For every $q \geq 1$ rational, it holds that $l\tan(l\arctan q) = q$.*

We omit the proof of this Lemma, see [2], Proposition 5.4.

Lemma 2.49. *For every rational angle α, it holds that $l\arctan(l\tan\alpha) \cong \alpha$, where $\cong$ denotes integer congruence.*

Proof. From the definition of integer arctangent, both angles $l\arctan(l\tan\alpha)$ and α have the same LLS sequences, because $l\tan(l\arctan(l\tan(a))) = l\tan(a)$ by Lemma 2.48. Therefore, they are integer congruent by Theorem 2.38. □

Remark. For every non-trivial rational integer angle there exists a unique integer arctangent that is integer congruent to this angle.

Lemma 2.50. *The integer sine and integer cosine are relatively prime positive integers for any rational angle.*

Proof. Because of Lemma 2.49 and because the integer sine and integer cosine are invariant under integer affine transformations, it is sufficient to prove this for angles $l\arctan q$, where $q \geq 1$ is rational.
Now let $q = \frac{m}{n} \geq 1$, where m, n are relatively prime integers. The integer distance between the point (n, m) and the line $y = 0$ is clearly equal to m. Therefore, by Lemma 2.30 and Lemma 2.34, $l\sin(l\arctan\frac{m}{n}) = \frac{m}{1} = m$. From Lemma 2.48, we get that $l\tan(l\arctan\frac{m}{n}) = \frac{m}{n}$, therefore from Definition 2.45, $l\cos(l\arctan\frac{m}{n}) = n$. □

Remark. If $q \neq 1$, it is possible to rewrite the edges of integer arctangent as columns in a matrix in a following way:

$$\begin{pmatrix} 1 & l\cos\alpha \\ 0 & l\sin\alpha \end{pmatrix},$$

where α is a non-trivial angle and $0 < l\cos\alpha < l\sin\alpha$, because we are excluding the case $l\sin\alpha = 1$. From Lemma 2.48 and Definition 2.47, $l\tan\alpha = \frac{m}{n}$ and from

Definition 2.45, we can also rewrite integer tangent as $l\tan\alpha = \frac{l\sin\alpha}{l\cos\alpha}$.
In the case of $q = 1$, the matrix above is defined as an identity matrix, but from Lemma 2.48, $l\tan\alpha = l\tan(l\arctan 1) = 1$, therefore from Lemma 2.43, the rays of the angle generate the whole lattice, thus from Lemma 2.33, $l\sin\alpha = 1$. Finally, from Definition 2.45, we get that $l\cos\alpha = 1$. Hence $l\sin\alpha = l\cos\alpha = l\tan\alpha = 1$.

3. Angles in integer trigonometry

In this chapter, we will study the relationships between different angles in integer trigonometry. We will define right angles and the summation of angles in integer trigonometry and compare the conclusions to properties of angles in Euclidean geometry. Finally, we will show a property for integer angles inside triangles.

3.1 Adjacent and transpose angles

Recall the definition of transpose and adjacent angle.

Definition 3.1 (Transpose and adjacent angle)**.** *Let $\alpha = \angle BAC$ be a rational angle. Then:*

- *the angle $\angle CAB$ is the transpose angle, denoted as α^t*
- *the angle $\angle CAB'$, where $B' = A - (B - A)$ is the adjacent angle, denoted as $\pi - \alpha$.*

Theorem 3.2. *For a non-trivial rational angle α:*

$$l\sin\alpha^t = l\sin\alpha,$$

$$l\cos\alpha^t \cdot l\cos\alpha \equiv 1 \pmod{l\sin\alpha}.$$

Moreover, if $l\tan\alpha = [a_0, \ldots, a_{2n}]$ is the regular odd continued fraction, then $l\tan\alpha^t = [a_{2n}, \ldots, a_0]$.

Proof. Let α be an rational angle, then let $l\tan\alpha = \frac{p}{q}$, where p, q are relatively prime. From Lemma 2.49 we get that $l\arctan(\frac{p}{q}) \cong \alpha$.
If $\frac{p}{q} = 1$, then $l\sin\alpha = l\cos\alpha = 1$ and the case is trivial.
Now let $\frac{p}{q} > 1$. Let $A = (1, 0)$, $B = (q, p)$, $O = (0, 0)$. Suppose an integer point $C = (q', p')$ on the sail of the angle $l\arctan\frac{p}{q}$, such that it is the closest integer point to B with respect to Euclidean distance such that it lies inside the angle, not on its sides. It is always possible to choose C such that $p', q' > 0$. The triangle $\triangle BOC$ is empty because BO, CO form a basis of $\mathbb{Z}^2$, because of the choice of C. From that, and since the orientation of the tuples of differences of integer points $(A - O, B - O)$ and $(B - O, C - O)$ is different, we get that

$$\det\begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = -1.$$

We can then define a linear transformation ξ_1 as

$$\xi_1 = \begin{pmatrix} p - p' & -q + q' \\ p & -q \end{pmatrix}.$$

The determinant of this matrix is equal to $-q(p-p') - p(-q+q') = q\cdot p' - p\cdot q' = -1$, therefore ξ_1 is an integer affine transformation. Let us compute what ξ_1 does to

$(l\arctan\frac{p}{q})^t$. From definition, $(l\arctan\frac{p}{q})^t$ is an angle with edges passing through the points (q,p) and $(1,0)$:

$$\begin{pmatrix} p-p' & -q+q' \\ p & -q \end{pmatrix} \cdot \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} p-p' & -q+q' \\ p & -q \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p-p' \\ p \end{pmatrix}.$$

We can see that the transformation gave us an angle with edges passing through the points $(1,0)$ and $(p-p',p)$, which is the angle $l\arctan\frac{p}{p-p'}$. Because (p,p') are relatively prime due to the determinant being -1 and $p > p-p'$, we can use Lemma 2.48, so that

$$l\tan\left(\xi_1\left(\left(l\arctan\frac{p}{q}\right)^t\right)\right) = l\tan\left(l\arctan\frac{p}{p-p'}\right) = \frac{p}{p-p'}.$$

Because integer tangent is invariant under $\mathrm{Aff}(2,\mathbb{Z})$, we get that $l\tan\alpha^t = \frac{p}{p-p'}$ and from the definition of $l\cos\alpha$, we get that $l\tan\alpha^t = \frac{l\sin\alpha^t}{l\cos\alpha^t}$. In this case,

$$\frac{p}{p-p'} = \frac{l\sin\alpha^t}{l\cos\alpha^t}$$

and it follows from Lemma 2.50 that $l\sin\alpha^t = p$ and $l\cos\alpha^t = p-p'$. In the same manner, it follows that $l\sin\alpha = p$ and $l\cos\alpha = q$, because

$$l\tan\left(l\arctan\frac{p}{q}\right) = \frac{p}{q},\ \frac{p}{q} = l\tan\alpha = \frac{l\sin\alpha}{l\cos\alpha},$$

p, q are relatively prime and $p > q$.
Because $q\cdot p' - p\cdot q' = -1$, we get that $q\cdot p' \equiv -1 \pmod{p}$. Thus,

$$l\cos\alpha^t \cdot l\cos\alpha = q\cdot(p-p') \equiv 1 \pmod{p}.$$

Hence

$$l\sin\alpha^t = l\sin\alpha$$

$$l\cos\alpha^t\cdot l\cos\alpha \equiv 1 \pmod{l\sin\alpha}.$$

For the second part of the theorem, if we know that $l\tan\alpha = [a_0,\ldots,a_{2n}]$, then the LLS sequence of α is $(a_0,\ldots,a_{2n})$. The LLS sequence from definition just alternates integer length and integer sine. Because the sail is the same for both α,α^t, the LLS sequence of α^t will therefore be the same as for α, but just reversed. Thus $l\tan\alpha^t = [a_{2n},\ldots,a_0]$. □

Example. Let $\angle ABC$ be a rational angle, where $A=(7,-5), B=(0,0), C=(3,5)$. The LLS sequence of $\angle ABC$ is $(1,1,2,1,1,1,2)$. The LLS sequence of the transpose angle $\angle CBA$ is the reversed one for the angle $\angle ABC$, $(2,1,1,1,2,1,1)$. The integer sines are

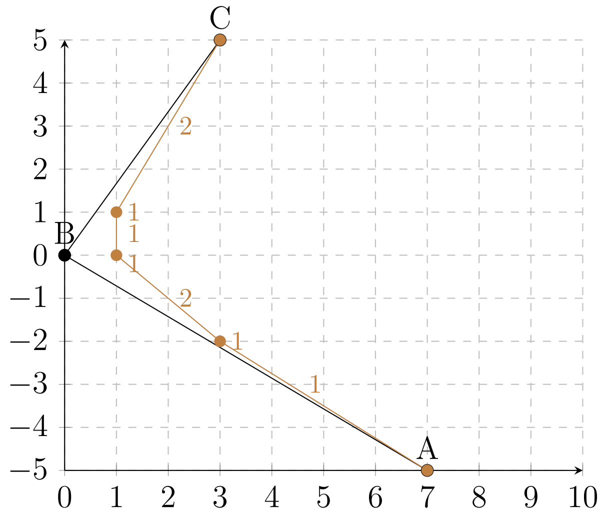
$$l\sin(\angle ABC) = 50 = l\sin(\angle CBA).$$

And the integer cosines are

$$l\cos(\angle ABC) = 29, \quad l\cos(\angle CBA) = 19,$$

because $[1,1,2,1,1,1,2] = \frac{50}{29}, [2,1,1,1,2,1,1] = \frac{50}{19}$ and $19 \cdot 29 = 551 \equiv 1 \pmod{50}$.

Figure 3.1: Transpose angle.



Theorem 3.3. *Let α be a non-trivial rational angle. Then the following holds for adjacent angles:*

$$l\sin(\pi - \alpha) = l\sin\alpha,$$

$$l\cos(\pi - \alpha) \cdot l\cos\alpha \equiv -1 \pmod{l\sin\alpha}.$$

Proof. Let α be a non-trivial rational angle and let $l\tan\alpha = \frac{p}{q}$, where p, q are relatively prime. From Lemma 2.49 we get that $l\arctan(\frac{p}{q}) \cong \alpha$. If $\frac{p}{q} = 1$, then $\pi - \alpha \cong \pi - l\arctan(1)$ and $\alpha \cong l\arctan(1)$ and the case is trivial, because $l\sin(\pi - \alpha) = l\sin\alpha = 1$.

Now let $\frac{p}{q} > 1$. Let $A = (1,0)$, $B = (q,p)$, $O = (0,0)$. Suppose an integer point $C = (q', p')$ on the sail of the angle $\pi - l\arctan(\frac{p}{q})$, such that it is the closest integer point to B, with respect to Euclidean distance, such that it lies inside the angle, not on its sides. It is always possible to choose C such that $p', q' > 0$. Similarly as in the proof of Theorem 3.2, we get that the triangle $\triangle BOC$ is empty and the orientation of the tuples of differences of integer points $(A - O, B - O)$ and $(B - O, C - O)$ is the same, therefore we get that

$$\det\begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = 1.$$

We can then define a linear transformation ξ_2 as

$$\xi_2 = \begin{pmatrix} -p + p' & q - q' \\ -p & q \end{pmatrix}.$$

The determinant of this matrix is equal to $q(-p+p') - (-p(q-q')) = q \cdot p' - p \cdot q' = 1$, therefore ξ_2 is an integer affine transformation that preserves orientation. Let's compute what ξ_2 does to $\pi - l\arctan(\frac{p}{q})$. From definition, $\pi - l\arctan(\frac{p}{q})$ is an angle with edges passing through the points (q,p) and $(-1,0)$:

$$\begin{pmatrix} -p+p' & q-q' \\ -p & q \end{pmatrix} \cdot \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -p+p' & q-q' \\ -p & q \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} p-p' \\ p \end{pmatrix}.$$

We can see that the transformation gave us an angle with edges passing through the points $(1,0)$ and $(p-p',p)$, which is the angle $l\arctan\frac{p}{p-p'}$. Because (p,p') are relatively prime and $p > p-p'$, we can use Lemma 2.48, so that

$$l\tan\left(\xi_2\left(\pi - l\arctan\frac{p}{q}\right)\right) = l\tan\left(l\arctan\frac{p}{p-p'}\right) = \frac{p}{p-p'}.$$

Because integer tangent is invariant under $\text{Aff}(2,\mathbb{Z})$, we get that $l\tan(\pi-\alpha) = \frac{p}{p-p'}$ and from the definition of $l\cos\alpha$, we get that $l\tan(\pi-\alpha) = \frac{l\sin(\pi-\alpha)}{l\cos(\pi-\alpha)}$. In this case,

$$\frac{p}{p-p'} = \frac{l\sin(\pi-\alpha)}{l\cos(\pi-\alpha)}$$

and from Lemma 2.50 it follows that $l\sin(\pi-\alpha) = p$ and $l\cos(\pi-\alpha) = p-p'$. In the same manner, it follows that $l\sin\alpha = p$ and $l\cos\alpha = q$, because

$$l\tan\left(l\arctan\frac{p}{q}\right) = \frac{p}{q},\ \frac{p}{q} = l\tan\alpha = \frac{l\sin\alpha}{l\cos\alpha},$$

p,q are relatively prime and $p > q$.
Because $q\cdot p' - p\cdot q' = 1$, we get that $q\cdot p' \equiv 1 \pmod{p}$. Thus,

$$l\cos\alpha\cdot l\cos(\pi-\alpha) = q\cdot(p-p') \equiv -1 \pmod{p}.$$

Hence

$$l\sin(\pi-\alpha) = l\sin\alpha$$
$$l\cos(\pi-\alpha)\cdot l\cos\alpha \equiv -1 \pmod{l\sin\alpha}.$$

□

Example. Let us have an angle α formed by two lines $\{(x,0) \mid x \geq 0\}$ and $\{(x,\frac{10}{7}x) \mid x \geq 0\}$, as in Figure 2.6. The LLS sequence of α is $(1,2,3)$. The adjacent angle $\pi-\alpha$ is formed by two lines $\{(x,\frac{10}{7}x) \mid x \geq 0\}$ and $\{(-x,0) \mid x \geq 0\}$ and the LLS sequence of $\pi-\alpha$ is $(1,2,3)$ The integer sine functions are

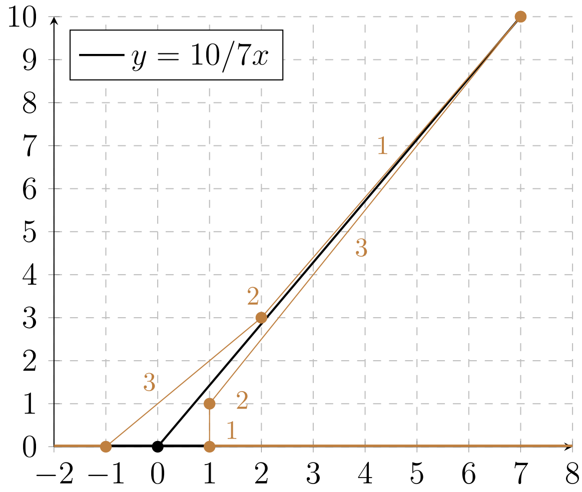
$$l\sin\alpha = 10 = l\sin(\pi-\alpha),$$

the integer cosines are

$$l\cos\alpha = 7,\ \ l\cos(\pi-\alpha) = 7,$$

because $[1,2,3] = \frac{10}{7}$ and $7\cdot 7 \equiv -1 \pmod{10}$. It is just a coincidence that the LLS sequences are the same, and it is not a general property of adjacent angles.

Figure 3.2: Adjacent angle.



In Euclidean geometry in $\mathbb{R}^2$, the trigonometric identities posed in Theorems 3.2, 3.3 are as follows:

$$\sin\alpha = \sin\alpha^t, \quad \cos\alpha = \cos\alpha^t$$

and

$$\sin(\pi - \alpha) = \sin\alpha,$$
$$\cos(\pi - \alpha) = -\cos\alpha.$$

3.2 Right angles

Definition 3.4 (Right angle)**.** *A rational angle $\angle ABC$ is right if it is integer congruent to both its adjacent angle and its transpose angle.*

Lemma 3.5. *Every right integer angle is integer congruent to either $l\arctan 1$ or $l\arctan 2$.*

Proof. Let α be a rational right angle. Then from the definition of right angle and from Theorem 3.2, we get that

$$l\cos\alpha^2 \equiv l\cos\alpha \cdot l\cos\alpha^t \equiv 1 \pmod{l\sin\alpha}.$$

Then from Theorem 3.3, we get that

$$l\cos\alpha^2 \equiv l\cos\alpha \cdot l\cos(\pi - \alpha) \equiv -1 \pmod{l\sin\alpha}.$$

Therefore

$$-1 \equiv 1 \pmod{l\sin\alpha},$$

thus $l\sin\alpha$ is equal to either 1 or 2.
If $l\sin\alpha = 1$, then $l\cos\alpha = 1$, therefore also $l\tan\alpha = 1$ and from Lemma 2.49, $l\arctan(l\tan\alpha) = l\arctan(1) \cong \alpha$.
If $l\sin\alpha = 2$, then from Lemma 2.46 $l\cos\alpha = 1$, therefore $l\tan\alpha = 2$ and from Lemma 2.49 $l\arctan(l\tan\alpha) = l\arctan(2) \cong \alpha$.

∎

Remark. Both $l\arctan 1$ and $l\arctan 2$ are right angles. In the example below, we verify that $l\arctan 2$ is a right integer angle, for $l\arctan 1$ the steps are the same.

Example. Let us verify that $l\arctan 2$ indeed is a right angle. From definition, $2 = \frac{m}{n}$ and therefore $l\arctan 2$ is an angle with edges passing through integer points $(1,0),(1,2)$. Denote $\angle ABC = l\arctan 2$. The transpose angle to $\angle ABC$ is the angle $\angle CBA$ with edges passing through integer points $(1,2),(1,0)$. To ensure that $\angle ABC$ and $\angle CBA$ are integer congruent, we are trying to find a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that its determinant is equal to either 1 or -1 and which maps $(1,0) \mapsto (1,2)$ and $(1,2) \mapsto (1,0)$. From $(1,0) \mapsto (1,2)$, we get that $a = 1, c = 2$ and then

$$\begin{pmatrix} 1 & b \\ 2 & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2b \\ 2+2d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

thus $b = 0, d = -1$ and $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$. The determinant of A is equal to -1, therefore the angle $\angle ABC$ is integer congruent to its transpose angle $\angle CBA$.

The adjacent angle to $\angle ABC$ is the angle $\angle CBA'$, where $A' = (-1,0)$. In the same manner, we are searching for a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that its determinant is equal to either $1, -1$ and which maps $(1,0) \mapsto (1,2)$ and $(1,2) \mapsto (-1,0)$. From $(1,0) \mapsto (1,2)$, we get that $a = 1, c = 2$ and then

$$\begin{pmatrix} 1 & b \\ 2 & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2b \\ 2+2d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

thus $b = -1, d = -1$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$. The determinant of B is equal to 1, therefore the angle $\angle ABC$ is integer congruent to its adjacent angle $\angle CBA'$. We have just verified that $l\arctan 2$ is a right angle.

3.3 Summation of angles

Angle summation is not defined uniquely up to integer conjugacy classes. For example, if we sum two angles integer congruent to $l\arctan 2$, we can obtain a straight line, but also a non-trivial angle.

Figure 3.3: Summation of angle integer congruent to $l\arctan 2$, first option.

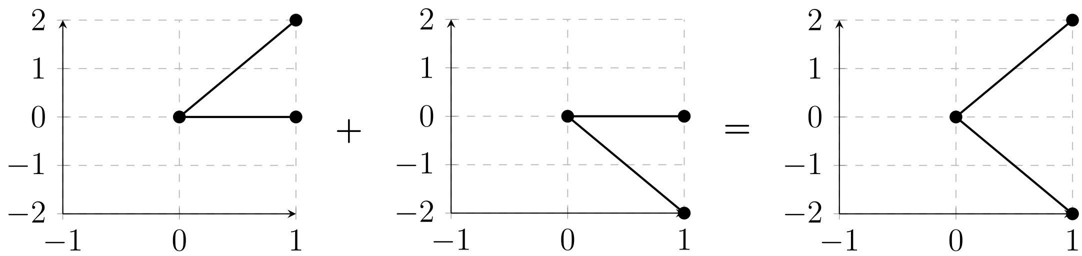
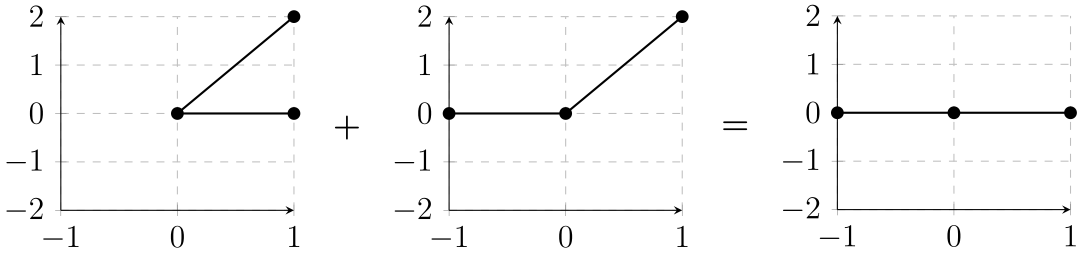


Figure 3.4: Summation of angle integer congruent to $l \arctan 2$, second option.



Example. Let us use the angle defined in Figure 3.1 and divide it into two angles $\angle ABD$ and $\angle DBC$, where $D = (2, 1)$. The LLS sequence of the angle $\angle ABD$ is $(1, 1, 2, 2, 1)$ and the LLS sequence of the angle $\angle DBC$ is $(1, 2, 2)$. The LLS sequence of the combined angle $\angle ABC$ is $(1, 1, 2, 1, 1, 1, 2)$. Then

$$l \tan(\angle ABC) = \frac{50}{29} = [1, 1, 2, 1, 1, 1, 2] = [1, 1, 2, 2, 1, -1, 1, 2, 2].$$

We can see that the angle $\angle ABC$ can have a LLS sequence composed of the LLS sequences of the angles $\angle ABD$ and $\angle DBC$ and between them we see an inserted integer. That is not a coincidence, in [2], there is defined angle summation up to an integer parameter s. Here $\angle ABC = \angle ABD +_{-1} \angle DBC$. For more see [2], Chapter 16.

Figure 3.5: LLS sequences of two angles $\angle ABD$ and $\angle DBC$.

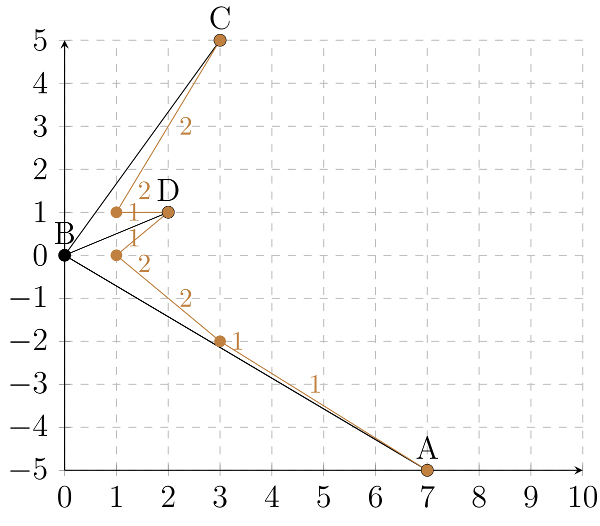
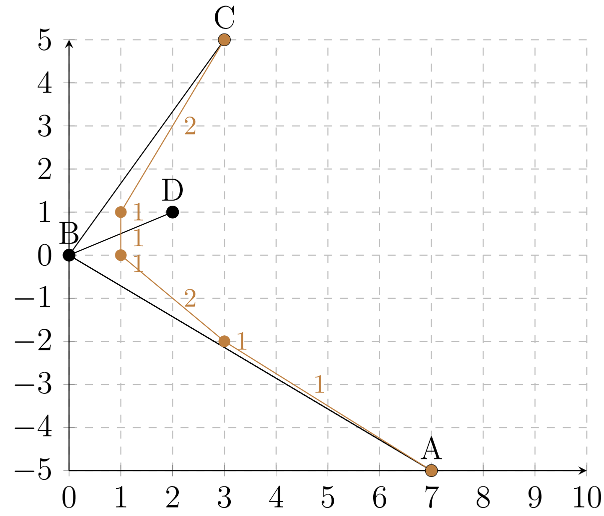


Figure 3.6: LLS sequences of the summed angle $\angle ABC = \angle ABD +_{-1} \angle DBC$.



3.4 Angles in triangles

In Euclidean geometry, it holds that all the inner angles α, β, γ in a triangle sum up to π. Because angle summation is defined not uniquely up to integer conjugacy classes, this rule from Euclidean geometry cannot be generalised to integer trigonometry. In this subchapter, we summarize a similar condition but for integer tangents. We present Theorem 3.9 without proof, see [2].

Definition 3.6 (Acute angle)**.** *Let $\angle ABC$ be an angle and a its measure. The angle is acute, if $0 < a < \frac{\pi}{2}$.*

Theorem 3.7. *Let α, β, γ be angles where α is acute. There exists a triangle with inner angles (α, β, γ), if and only if the following conditions hold:*

$$\tan(\alpha + \beta + \gamma) = 0,$$

$$\tan(\alpha + \beta) \notin [0, \tan \alpha].$$

Proof. $\Rightarrow$: The first condition follows from the fact that all inner angles in a triangle sum up to π and $\tan \pi = 0$.
For the proof of the second condition, let us assume for contradiction that $0 \leq \tan(\alpha + \beta) \leq \tan \alpha$. Since α is an acute angle, $\tan \alpha > 0$. Now if $\alpha + \beta$ is acute, then $\tan(\alpha + \beta) > 0$. Because the tangent function is increasing on the interval $(0, \frac{\pi}{2})$, we know that $(\alpha + \beta) \leq \alpha$ cannot happen, because $\beta > 0$. If $\alpha + \beta$ is an obtuse angle, then $\tan(\alpha + \beta) < 0$, therefore we have also come to a contradiction. Finally, if $\alpha + \beta$ is a right angle, then $\tan(\alpha + \beta)$ is not defined therefore it cannot be in the interval $[0, \tan \alpha]$.
$\Leftarrow$: We know that $0 < \alpha, \beta, \gamma < \pi$. Therefore $0 < \alpha + \beta + \gamma < 3\pi$. There exists a triangle with angles α, β, γ if and only if $\alpha + \beta + \gamma = \pi$. From the first condition we get that $\alpha + \beta + \gamma = \pi$ or $\alpha + \beta + \gamma = 2\pi$. Let us assume that $\alpha + \beta + \gamma = 2\pi$. Now we know that α is acute, therefore $0 < \alpha < \frac{\pi}{2}$. Then if $\alpha + \beta + \gamma = 2\pi$, we get $0 < \alpha + \beta < \frac{3\pi}{2}$ and $\pi < 2\pi - \gamma < 2\pi$, thus $\pi < \alpha + \beta < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$. Therefore β is obtuse. Let us assume

$\tan(\alpha+\beta) \notin [0, \tan\alpha]$, therefore $\tan(\alpha+\beta) < 0$ or $\tan(\alpha+\beta) > \tan\alpha$. We know that $\tan\alpha < \tan(\alpha+\beta)$ if and only if $\beta \in \left(0, \frac{\pi}{2} - \alpha\right)$ or $\beta \in \left(\pi, \frac{3\pi}{2} - \alpha\right)$. Since $\beta \in \left(\frac{\pi}{2}, \pi\right)$, this cannot happen and therefore $\tan(\alpha+\beta) < 0$. We know that $\tan(\alpha+\beta) = \tan(2\pi - \gamma) = -\tan\gamma < 0$ and $\tan\gamma > 0$ and γ is acute, therefore $0 < \gamma < \frac{\pi}{2}$. But then $\alpha + \beta + \gamma < \frac{\pi}{2} + \pi + \frac{\pi}{2} = 2\pi$ and that is a contradiction, therefore $\alpha + \beta + \gamma = \pi$.

This concludes the proof. □

Definition 3.8 (Sequence of continued fractions)**.** *Let $q_1, \ldots, q_n$ be a sequence of rational numbers with odd regular continued fractions such that*

$$q_i = [a_{i,0}, a_{i,1}, \ldots, a_{i,2k_i}]$$

for $1 \leq i \leq n$. The we define the sequence of continued fractions as

$$]q_1, q_2, \ldots, q_k[= [a_{1,0}, a_{1,1}, \ldots, a_{1,2k_1}, a_{2,0}, a_{2,1}, \ldots, a_{2,2k_2}, \ldots, a_{n,0}, a_{n,1}, \ldots, a_{n,2k_n}].$$

Theorem 3.9. *Let $(\alpha_1, \alpha_2, \alpha_3)$ be an ordered triple of angles. There exists an integer triangle with consecutive angles integer congruent to $\alpha_1, \alpha_2, \alpha_3$ if and only if there exists $i \in \{1, 2, 3\}$, such that the angles $\alpha = \alpha_i,\ \beta = \alpha_{i+1 \pmod 3},\ \gamma = \alpha_{i+2 \pmod 3}$ satisfy the following conditions:*

$$]l\tan\alpha, -1, l\tan\beta, -1, l\tan\gamma[= 0,$$

$$]l\tan\alpha, -1, l\tan\beta[\quad \notin [0, l\tan\alpha].$$

For the proof of this theorem, see [2], the formulation in Theorem 6.9 and the proof in Section 16.3.

Remark. Let us have an integer triangle $\triangle ABC$. Consecutive angles are for example $\alpha = \angle BAC,\ \gamma = \angle ACB,\ \beta = \angle CBA$. The orientation of angles is important, for example, angles $\alpha = \angle CAB,\ \gamma = \angle ACB,\ \beta = \angle CBA$ are not consecutive.

Example. We will provide an example of an integer triangle satisfying the conditions from Theorem 3.9. Let us denote three integer points $A = (0,0),\ C = (2,4),\ B = (6,2)$ and their corresponding angles $\alpha = \angle BAC,\ \gamma = \angle ACB,\ \beta = \angle CBA$. Their LLS sequences are $(2,1,1),\ (1,1,2),\ (1,3,1)$, respectively. We can compute that

$$[2,1,1,-1,1,1,2,-1,1,3,1] = 0,$$

$$[2,1,1,-1,1,1,2] = 5 \notin \left[0, \frac{5}{2}\right].$$

Thus, by Theorem 3.9 there exists an integer triangle $\triangle ABC$ with these given angles, as displayed in Figure 3.7.

Now, we will show that the orientation of the angles is important, let us have, for example, $\alpha^t = \angle CAB$ and ordering as above, therefore $\alpha^t,\ \gamma = \angle ACB,\ \beta = \angle CBA$ and their corresponding LLS sequences $(1,1,2),\ (1,1,2),\ (1,3,1)$. Then

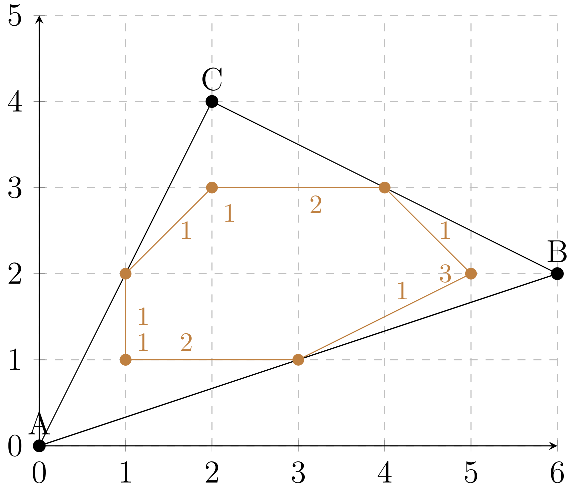
$$[1,1,2,-1,1,1,2,-1,1,3,1] = \frac{5}{4} \neq 0$$

and

$$[1,1,2,-1,1,1,2] = 0 \in \left[0, \frac{5}{3}\right].$$

These angles do not satisfy the conditions from the theorem.

Figure 3.7: Angles in triangle $\triangle ABC$.



Conclusion

In this thesis, we have presented the definitions of the most important objects in integer trigonometry in $\mathbb{Z}^2$, we have proved their main properties and provided their geometric interpretations. We have proven an important Theorem 2.38 stating that two angles with the same LLS sequences are integer congruent. We have also proven the geometric interpretations of definitions of integer length in Lemma 2.20, integer area in Lemma 2.25 and integer distance in Lemma 2.31.

For further extension to this thesis, we could add the proof of Lemma 2.48, which requires a few other theorems and more theory about continued fractions. We could also prove Theorem 3.9, but its proof requires many more definitions and helpful lemmata. Both of these proofs can be found in [2].
Naturally, integer trigonometry also exists in $\mathbb{Z}^n$ for $n > 2$, which is thoroughly described in [2].

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