FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

**BACHELOR THESIS**

Emil Čuřín

# Homological dimensions and special classes of rings

Department of Algebra

Supervisor of the bachelor thesis: Liran Shaul, Ph.D.
Study programme: Mathematics
Study branch: Mathematics for Information Technologies

Prague 2024

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ............. date .............                ....................................

Author's signature

Title: Homological dimensions and special classes of rings

Author: Emil Čuřín

Department: Department of Algebra

Supervisor: Liran Shaul, Ph.D., Department of Algebra

Abstract: In this thesis, we study commutative noetherian rings using homological methods. We characterize regular local rings as rings with finite global dimension and show that they are stable under localization. After this, we go on to generalize this result. We prove a generalization of the classical Auslander–Buchsbaum formula and an analogous characterization of Gorenstein rings.

Keywords: Regular ring, Gorenstein ring, Global dimension, Gorenstein homological algebra

# Contents

# Introduction

Points on an algebraic variety correspond to the maximal ideals of its coordinate ring. This allows us to use localization at a maximal ideal to study algebraic varieties at a point. An algebraic variety is smooth at some point if the local ring corresponding to it is regular. Geometric intuition tells us that localizing something smooth should again yield something smooth. Proving that regular local rings localize requires nontrivial tools, namely homological algebra.

This problem of localization of regular rings marked the first success of homological algebra in commutative algebra. It also began the study of commutative rings using homological methods. This thesis, although motivated by the localization problem is about the homological study of such rings. One of the major themes we shall explore is that even though we study homological properties, in actuality, we are still studying geometric properties.

We begin the second chapter by showing that regular local rings are integral domains and Cohen-Macaulay. Then we prove that if a ring has finite injective dimension as a module over itself, its injective dimension is equal to the Krull dimension; our first instance of the link between homological algebra and geometry. After this we prove the very famous characterization of regular local rings; the Auslander–Buchsbaum–Serre theorem, with this theorem it is simple to prove that regular local rings localize and to define regular rings.

A natural question that arises is how to generalize the Auslander–Buchsbaum–Serre theorem; this brings us to Gorenstein rings. Gorenstein rings, although less obviously geometric are ubiquitous in algebraic geometry. In the third chapter, we start by generalizing projective modules to a class of modules with good duality properties. After showing that this class of modules has many desirable properties we go on to prove an analogous characterization of Gorenstein rings and a generalization of the famous Auslander–Buchsbaum formula.

# 1. Preliminaries

We begin by defining our most important homological dimensions. First we define resolutions.

**Definition 1.1.** *Let $R$ be a ring and $M$ an $R$-module.*
*An exact complex $F\colon ... \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ with $F_i$ projective and* $\operatorname{Coker}\varphi_0 = M$ *is called a projective resolution for $M$.*

Injective resolutions are defined completely analogously.

**Definition 1.2.** *Let $R$ be a ring and $M$ an $R$-module.*
*An exact complex $I\colon E_0 \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} ...$ with $E_i$ injective $M$ and* $\operatorname{Ker}\varphi_1 = M$ *is called an injective resolution for $M$.*

Now that we know what resolutions are, we can define our first homological dimensions.

**Definition 1.3.** *Let $R$ be a ring and $M$ and $R$-module. We define the projective dimension of $M$ to be the minimal length of a projective resolution for $M$ and denote it by* $\operatorname{pd}_R M$. *The injective dimension* $\operatorname{injdim}_R M$ *is the minimal length of an injective resolution for $M$. For $R$ we define the global dimension* $\operatorname{gldim} R$ *to be the supremum of projective dimensions of $R$-modules.*

We begin with two basic lemmata. Any text concerning homological algebra would be incomplete without the following lemma. A proof can be found in Weibel [1994][Corollary 6.12]

**Lemma 1.1.** *Consider a commutative diagram of $R$-modules, where the rows are exact*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

*There is an exact sequence of kernels and cokernels:*

$$0 \to \operatorname{Ker}\phi' \to \operatorname{Ker}\phi \to \operatorname{Ker}\phi'' \to \operatorname{Coker}\phi' \to \operatorname{Coker}\phi \to \operatorname{Coker}\phi'' \to 0.$$

The next lemma is from Eisenbud [2013][Proposition A3.13.]. A proof for the lemma can be found there.

**Lemma 1.2.** *Let*

$$F\colon ... \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow ... \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

*and*

$$G\colon ... \longrightarrow G_i \xrightarrow{\psi_i} G_{i-1} \longrightarrow ... \longrightarrow G_1 \xrightarrow{\psi_1} G_0$$

*be complexes of $R$-modules, and set $M = H_0(F)$ and $N = H_0(G)$. If all the $F_j$ are projective and the homology of $G$ vanishes except for the zeroth one, then every map $\beta\colon M \to N$ is the map induced on $H_0$ by a map of complexes $\alpha\colon F \to G$.*

We shall work extensively with the derived functors Ext and Tor as defined in [Eisenbud, 2013, A3]. We will use the fact that right derived functors are characterized by the following 4 properties.

**Theorem 1.3.** *Let $F$ be an additive left exact functor on the category of modules over a ring $R$. The right-derived functors of $F$ are independent of the choice of resolution and are characterized by the following properties:*
*(a) $R^0 F = F$.*
*(b) If $E$ is an injective module, then $R^i F(E) = 0$ for all $i > 0$.*
*(c) For every short exact sequence*

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

*there is a long exact sequence*

$$... \xrightarrow{\delta_{i-1}} R^i F(A) \xrightarrow{R^i F(u)} R^i F(B) \xrightarrow{R^i F(v)} R^i F(C) \xrightarrow{\delta_i} ...$$

*(d) The connecting homomorphisms $\delta_i$ in the long exact sequence are natural: that is if*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

*is a commutative diagram with exact rows then the diagrams*

$$\begin{array}{ccc} R^i F(C) & \xrightarrow{\delta_i} & R^{i+1} F(A) \\ \downarrow{\scriptstyle R^i F(\gamma)} & & \downarrow{\scriptstyle R^{i+1} F(\alpha)} \\ R^i F(C') & \xrightarrow{\delta'_i} & R^{i+1} F(A') \end{array} \tag{1.1}$$

*commute.*

It is sometimes useful to have a lower bound for injective dimension, hence the following lemma.

**Lemma 1.4.** *If* $\operatorname{Ext}^i_R(M, N) \neq 0$ *for some $R$-modules $M$ and $N$, then* $\operatorname{injdim} N \geq i$*.*

*Proof.* Suppose there is an injective resolution $F$ of $N$ of length $n < i$. Computing $\operatorname{Ext}^i_R(M, N)$ from this resolution we see $\operatorname{Ext}^i_R(M, N) = 0$ because $\operatorname{Ext}^i_R(M, N)$ is just a quotient of a submodule of $\operatorname{Hom}(M, F_i = 0) = 0$.

□

The next lemma shows that projective resolutions localize. This fact will become important when we characterize regular local rings as rings with finite global dimension.

**Lemma 1.5.** *Let $R$ be ring and $P$ a prime ideal of $R$. If $F : ... \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ is a projective resolution for an $R$-module $M$, then $F_P = F \otimes_R R_P$ is a projective resolution for $M_P$.*

*Proof.* Localization is flat, so $F_P$ is also exact. Localization preserves cokernels, so the cokernel of $\varphi_1$ is $M_P$. The only thing left to show is that if $N$ is a projective $R$-module, then $N_P$ is a projective $R_P$ module. Using the characterization of projectives as direct summands of free modules and the fact that localization commutes with direct sums we are finished. □

A proof for the following theorem is found in [Eisenbud, 2013, Proposition 2.10].

**Theorem 1.6.** *Let $R$ be a noetherian ring and $M$ a finite $R$-module. Then for every $R$-module $N$ and prime $P$ it holds that*

$$\operatorname{Hom}_A(M, N)_P \cong \operatorname{Hom}_{A_P}(M_P, N_P).$$

*Moreover if $\varphi\colon F \to F'$, where $F$ and $F'$ are finite then*

$$\operatorname{Hom}_A(\varphi, N)_P \cong \operatorname{Hom}_{A_P}(\varphi_P, N_P).$$

Now that we know that localization commutes with taking projective resolutions, homs and quotients we can now see that localization commutes with Ext. This is further evidence that localization is very nice.

**Lemma 1.7.** *Let $R$ be a noetherian ring, $M$ a finite $R$-module and $N$ any $R$-module. Then for any prime $P$ of $R$ we have $\operatorname{Ext}^i_R(M, N)_P = \operatorname{Ext}^i_{R_P}(M_P, N_P)$.*

*Proof.* Take $F : \ldots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ to be projective resolution for $M$, since $R$ is noetherian we can assume all $F_i$ are finite. We know that localization is flat so $\operatorname{Ext}^i_R(M, N)_P = H^i(\operatorname{Hom}_A(F, N)_P) = H^i(\operatorname{Hom}_{A_P}(F_P, N_P))$ with the last equality being due to Theorem 1.6. By Lemma 1.5 we see $F_P$ is a projective resolution for $M_P$, so $\operatorname{Ext}^i_{R_P}(M_P, N_P) = H^i(\operatorname{Hom}_{A_P}(F_P, N_P))$. □

A proof for the following theorem can be found in [Matsumura and Reid, 1989, Lemma 19.2].

**Theorem 1.8.** *Let $R$ be a ring. If $\operatorname{pd}_R M \leq n$ for all finite $R$-modules $M$, then* $\operatorname{gldim} A \leq n$ *and* $\operatorname{injdim} M \leq n$ *for all $A$-modules $M$.*

# 2. Regular local rings

Every ring is commuative and noetherian. Unless stated otherwise let $(A, \mathfrak{m}, k)$ be a commuative and noetherian local ring.

## 2.1 Foundations

We begin with the geometric definition of regular local rings.

**Definition 2.1.** *A local ring (A,$\mathfrak{m}$,k) is called regular, if* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A)$.

The reason this definition is geometric is that the quotient $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space at the point of an algebraic variety corresponding to $\mathfrak{m}$. The number $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is called the embedding dimension of $A$, and it is useful to know that it is also just the length of any minimal generating set for $\mathfrak{m}$.

**Lemma 2.1.** *Let (A,$\mathfrak{m}$,k) be a local ring and $(x_1, ..., x_n)$ a minimal generating set for $\mathfrak{m}$. Then $n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.*

*Proof.* Let $(\overline{x}_1, ..., \overline{x}_n)$ be the image of the generating set in $\mathfrak{m}/\mathfrak{m}^2$. If $(\overline{x}_1, ..., \overline{x}_n)$ is not a basis, then some proper subset is. By Nakayama's lemma the preimage of this subset is a smaller generating set of $\mathfrak{m}$ which is a contradiction.

□

Knowing this we can use Krull's principal ideal theorem to infer $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(A)$. Another way to look at regular local rings is through regular sequences.

**Definition 2.2.** *Let A be a ring and M an A-module. An element $x \in A$ is said to be M-regular if $xa \neq 0$ for all $0 \neq a \in M$. A sequence $(x_1, ..., x_n)$ is called an M-sequence if $(x_1, ..., x_n)M \neq M$ and if $x_{i+1}$ is ($M/(x_1, ..., x_i)M$)-regular for all $1 \leq i \leq n$.*

In general the dimension of a local ring is greater than the maximal length of an $A$-sequence called the depth of $A$. A very powerful fact about regular local rings is that every minimal generating set forms a regular sequence. To prove this we need to know that regular local rings are integral domains. The following lemma is usually called prime avoidance.

**Lemma 2.2.** *Suppose $I_1, ..., I_n, J$ are ideals of A and at most two of the $I_j$ aren't prime. Then if $J \subseteq \bigcup_{j=1}^n I_j$, then $J \subseteq I_j$ for some j.*

*Proof.* The case when $n = 1$ is trivial. By induction we can now assume that $J$ is not contained in any smaller union. This means we can choose $x_1, ...x_n$ such that $x_i \in J - \bigcup_{j \neq i} I_j$. We assume $J \subseteq \bigcup_{j=1}^n I_j$, so it must hold that $x_i \in I_i$. If $n = 2$ then $x_1 + x_2$ is not in $I_1$ or $I_2$ which is a contradiction. Let $n > 2$ and let $I_1$ be prime, then $x_1 + x_2 x_3 ... x_n$ is not in $I_1$ which contradicts $J \subseteq \bigcup_{j=1}^n I_j$.

□

This following proof is can be found in Matsumura and Reid [1989][Theorem 14.3].

**Theorem 2.3.** *A regular local ring is an integral domain.*

*Proof.* We show that 0 is a prime ideal by induction on $\dim(A)$; this shows there are no zerodivisors, so $A$ is an integral domain. If $\dim(A) = 0$ then $\mathfrak{m} = 0$ so $A$ is a field. Let $\dim(A) = 1$ and consider a minimal prime ideal $P$ of $A$ and $a \in P$. The maximal ideal $\mathfrak{m}$ is generated by just one element $x$ and $a \in P \subseteq \mathfrak{m}$ so $a = rx$ for some $r \in A$. But $P$ is a prime ideal and $x \notin P$ since $P \subset \mathfrak{m}$ so necessarily $r \in P$ so $\mathfrak{m}P = P$ and Nakayama's lemma gives $P = 0$. Now let $\dim(A) > 1$. Since $A$ is noetherian there are only finitely many minimal primes, see [Stacks Project Authors, 2018, Tag 00FR]. Lemma 2.2 gives an element $x \in \mathfrak{m}$ that is not in any of the minimal primes or $\mathfrak{m}^2$. Now the dimension of $B = A/xA$ is $\dim(A) - 1$ because $x$ is not in any minimal prime and any minimal generating set for $\mathfrak{m}B$ is also of length $\dim(A) - 1$ because $x \notin \mathfrak{m}^2$. Hence $B$ is regular, and by induction we know that B is a integral domain, so $xA$ is a prime. Suppose $\mathfrak{p}$ is a minimal prime contained in $xA$, then by the same argument as in the case $\dim(A) = 1$ shows $\mathfrak{p} = 0$ which proves $A$ is an integral domain.

□

Now we can prove the important property that the minimal generating set of the maximal ideal in a regular local ring enjoys.

**Theorem 2.4.** *Let $(A, \mathfrak{m})$ be a regular local ring. If $(x_1, ..., x_n)$ is a minimal generating set for $\mathfrak{m}$ then it is also an $A$-sequence.*

*Proof.* First let $n = 1$, then $x_1$ is $A$-regular because we now know $A$ is an integral domain and Nakayama's lemma gives $x_1A \neq A$. Now suppose the statement holds for all lengths shorter than $n > 1$. From Theorem 2.3 we see that the only minimal prime ideal of $A$ is just 0 and from $(x_1, ..., x_n)$ being a minimal generating set we see $x_1 \notin \mathfrak{m}^2$. Now taking $B = A/(x_1)$ we see that $B$ is regular, and hence from induction the images of $(x_2, ..., x_n)$ form a regular sequence in $B$. This by definition means $(x_1, ..., x_n)$ is a regular sequence.

□

We could use this statement to get an equivalent definition for regular local rings as those with depth equal to the embedding dimension. This result is very useful when used in conjunction with the Koszul complex. First we need to define minimal complexes.

**Definition 2.3.** *A complex over a local ring $(A, \mathfrak{m}, k)$ $F : ... \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ is called minimal if $\operatorname{Im}\varphi_n \subseteq \mathfrak{m}F_{n-1}$ for all $n$.*

The proof for the following theorem can be found in [Eisenbud, 2013, chap. 17].

**Theorem 2.5.** *If $(x_1, ..., x_n)$ is an $A$-sequence that generates $\mathfrak{m}$, then $K(x_1, ..., x_n)$ is a minimal free resolution for $k$.*

Theorem 2.4 also proves that regular local rings are Cohen-Macaulay. The Cohen-Macaulay property is local in the strong sense; that is, any localization of a Cohen-Macaulay ring at a prime is again Cohen-Macaulay. This brings us to the topic of the next section: a localization of a regular local ring at a prime is again regular. Geometric intuition tells us that this must hold, but the algebraic proof is nontrivial, and surprisingly, requires homological algebra techniques.

## 2.2 Injective dimension and Krull dimension

To characterize regular local rings as those with finite global dimension we will use a fact about injective dimension of $A$ as an $A$-module. For this we need a series of lemmata about the derived functor Ext that allow us to do inductive arguments. This next one exploits the fact that resolutions can be chosen to be minimal. It can be found in Matsumura and Reid [1989][Lemma 19.1].

**Lemma 2.6.** *Let $M$ be a finite nonzero $A$-module. If $\operatorname{pd} M = n$ for some $n$, then $\operatorname{Ext}^n_A(M, N) \neq 0$ for every finite $N \neq 0$.*

*Proof.* Let $0 \longrightarrow F_n \xrightarrow{d'} F_{n-1}$ be the end of a minimal projective resolution for M. Then $\operatorname{Ext}^n_A(M, N)$ is just the cokernel of the map $d$ obtained from applying $\operatorname{Hom}_A(-, N)$ to $d'$. If $d$ is not surjective the proof is done. The modules $F_i$ are free, so $\operatorname{Hom}_A(F_i, N)$ is isomorphic to a direct sum of copies of $N$ and the map $d$ is represented by the same matrix as $d'$. This implies $\operatorname{Im} d \subseteq \mathfrak{m}\operatorname{Hom}_A(F_n, N)$; since we assumed $N$ to be finitely generated we finish by Nakayama's lemma. □

An interesting corollary of the previous lemma is that if $A$ is of positive depth, then there are no finite injective modules(in general if $A$ has positive Krull dimension, then there are no finite injectives).

**Corollary 2.7.** *Let $x \in \mathfrak{m}$ be an $A$-regular element. There are no finite injective $A$-modules.*

*Proof.* Consider the projective resolution $0 \longrightarrow A \xrightarrow{x} A$ for the module $A/(x)$; this is just the Koszul complex, and $x$ is $A$-regular so it is a projective resolution. Now Lemma 2.6 shows $\operatorname{Ext}^1_A(A/(x), N) \neq 0$ for all finite $N$. This shows $\operatorname{injdim} N \geq 1$ for all finite $N$ by Lemma 1.4. □

The following lemma applies basic facts about associated primes to Ext. The proof of the existence of the filtration can be found in Matsumura and Reid [1989][Theorem 6.4].

**Lemma 2.8.** *Let $R$ be a noetherian ring and $M$ a finite $R$-module. There exists a chain $0 = M_0 \subset M_1 \subset ... \subset M_n = M$ of submodules of $M$ such that $M_i/M_{i-1} \cong R/P_i$, where $P_i$ is a prime ideal of $R$ with $\operatorname{ann}_R(M) \subseteq P_i$.*

This theorem is just writing out an argument used often in Matsumura and Reid [1989].

**Theorem 2.9.** *Let $R$ be a noetherian ring and $M$, $N$ finite $R$-modules. If $\operatorname{Ext}^i_R(R/P, M) = 0$ for all prime ideals $P$ with $\operatorname{ann}_R(N) \subseteq P$, then $\operatorname{Ext}^i_R(N, M) = 0$.*

*Proof.* Consider the filtration $0 = N_0 \subset N_1 \subset ... \subset N_n = N$ and $P_i$ with $\operatorname{ann}_R(N) \subseteq P_i$ given by Lemma 2.8. For each $m$ we get a short exact sequence

$$0 \to N_{m-1} \to N_m \to N_m/N_{m-1} = R/P_i \to 0.$$

Applying $\mathrm{Hom}_R(-, M)$ yields long exact sequences

$$... \to \mathrm{Ext}^i_R(R/P_m, M) \to \mathrm{Ext}^i_R(N_m, M) \to \mathrm{Ext}^i_R(N_{m-1}, M) \to ...$$

We assume $\mathrm{Ext}^i_R(R/P_i, M) = 0$ so the above exact sequence implies we may view $\mathrm{Ext}^i_R(N_m, M)$ as a submodule of $\mathrm{Ext}^i_R(N_{m-1}, M)$. Hence, we obtain a chain

$$\mathrm{Ext}^i_R(N_n = N, M) \subseteq \mathrm{Ext}^i_R(N_{n-1}, M) \subseteq ... \subseteq \mathrm{Ext}^i_R(N_0 = 0, M) = 0.$$

□

The following result is Matsumura and Reid [1989][Lemma 18.3].

**Lemma 2.10.** *Let $P$ be a prime ideal such that $\mathrm{ht}(\mathfrak{m}/P) = 1$. If $\mathrm{Ext}^{r+1}_A(k, M) = 0$ for some finite $M$, then $\mathrm{Ext}^r_A(A/P, M) = 0$.*

*Proof.* Take $x \in \mathfrak{m} - P$ and set $N = A/(P + Ax)$. Consider the short exact sequence

$$0 \longrightarrow A/P \xrightarrow{x} A/P \longrightarrow N \longrightarrow 0.$$

The only prime ideal that contains $P + Ax$ is $\mathfrak{m}$, hence from Theorem 2.9 we see $\mathrm{Ext}^{r+1}_A(N, M) = 0$. Inspecting the long exact sequence obtained by applying $\mathrm{Hom}_A(-, M)$ to the sequence above we see

$$... \longrightarrow \mathrm{Ext}^r_A(A/P, M) \xrightarrow{x} \mathrm{Ext}^r_A(A/P, M) \longrightarrow \mathrm{Ext}^{r+1}_A(N, M) = 0$$

so $x\mathrm{Ext}^r_A(A/P, M) = \mathrm{Ext}^r_A(A/P, M)$ which by Nakayama's lemma finishes the proof.

□

Our goal is to prove that if $A$ has finite injective dimension considered as a module over itself, then the injective dimension is equal to the Krull dimension. To do this we want to use induction on Krull dimension, the following two lemmata allows us to do just that. This following one is an argument used in the proof of Matsumura and Reid [1989][Theorem 18.1].

**Lemma 2.11.** *If $\dim A = n$, then $\mathrm{injdim}\, A \geq n$.*

*Proof.* Set $r = \mathrm{injdim}\, A$. We prove $\mathrm{Ext}^n_A(k, A) \neq 0$ by induction on $n$. If $n = 0$ then obviously $\mathrm{Hom}_A(A/\mathfrak{m}, A) \neq 0$ because $\mathfrak{m}$ is an associated prime of $A$. Now let $n > 0$ and $P$ be a prime such that $\mathrm{ht}(\mathfrak{m}/P) = 1$. From Lemma 2.10 we see that it suffices to show $\mathrm{Ext}^{n-1}_A(A/P, A) \neq 0$. Ext commutes with localization by Lemma 1.7 so

$$\mathrm{Ext}^{n-1}_A(A/P, A)_P = \mathrm{Ext}^{n-1}_{A_P}((A/P)_P, A_P) \neq 0$$

by induction.

□

What follows is Matsumura and Reid [1989][Lemma 18.2 (i)].

**Lemma 2.12.** *Let $x \in \mathfrak{m}$ be $A$-regular. If $M$ and $N$ are modules such that $x$ is $M$-regular and $xN = 0$ then $\mathrm{Ext}^{i+1}_A(N, M) \cong \mathrm{Ext}^i_B(N, M')$, where $B = A/xA$ and $M' = M/xM$.*

*Proof.* Remark that $N$ and $M'$ can be considered as $B$-modules; scalar multiplication for $N$ is given by $a'x = ax$, where $x \in M'$, $a' \in B$, and $a \in A$ is anything in the preimage of $a'$. Note that this is well defined since $xN = 0$ so the choice of $a$ does not matter, for $M'$ the definition is the same. We prove $T^i(N) = \mathrm{Ext}_A^{i+1}(N, M)$ has the 4 characteristic properties of the right derived functor of $\mathrm{Hom}_B(-, M')$, then the isomorphism follows from Theorem 1.3. First consider the long exact sequence obtained from applying $\mathrm{Ext}_A(N, -)$ to

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M' \longrightarrow 0.$$

Clearly $\mathrm{Hom}_A(N, M) = 0$, so

$$0 \longrightarrow \mathrm{Hom}_A(N, M') \longrightarrow T^0(N) \xrightarrow{x} T^0(N)$$

is exact. Computing $T^0(N)$ from an injective resolution for $M$ we see that $xT^0(N) = 0$, because $xN = 0$ so $x\,\mathrm{Hom}_A(N, C) = 0$ for all $C$. This proves $T^0(N) = \mathrm{Hom}_A(N, M') = \mathrm{Hom}_B(N, M')$. For any $L$ projective we have $T^i(L) = 0$ for $n > 0$ since $\mathrm{pd}_A B = 1$ and projectives are free over a local ring. The long exact sequence and naturality properties follow from those properties for $\mathrm{Ext}_A$.

□

The next lemma applies Baer's criterion for injectivity to Ext.

**Theorem 2.13** (Baer's criterion)**.** *Let $Q$ be an $R$-module. If for every ideal $I \subset R$, every homomorphism $\beta\colon I \to Q$ extends to $R$, then $Q$ is injective.*

This lemma is Matsumura and Reid [1989][Lemma 18.1].

**Lemma 2.14.** *Let $M$ be an $A$-module. If $\mathrm{Ext}_A^{n+1}(A/P, M) = 0$ for every prime $P$, then* $\mathrm{injdim}\, A \leq n$.

*Proof.* Theorem 2.9 shows $\mathrm{Ext}_A^{n+1}(N, M) = 0$ for all finite $N$. If $n = 0$, then from the exactness of

$$0 \to \mathrm{Hom}_A(A/I, M) \longrightarrow \mathrm{Hom}_A(A, M) \longrightarrow \mathrm{Hom}_A(I, M) \to 0 \qquad (2.1)$$

and Baer's criterion we see $M$ injective. If $n > 0$ consider an exact sequence

$$0 \to M \to N_0 \to \dots \to N_{n-1} \to N \to 0$$

with $N_i$ injective. It suffices to show $N$ is also injective; $0 = \mathrm{Ext}_A^{n+1}(A/I, M) \cong \mathrm{Ext}_A^1(A/I, N)$ so $N$ is injective.

□

Now that we have proven all the lemmata, proving the result about injective dimension is easy. This proof is taken from Matsumura and Reid [1989]. Even though we use the theorem to prove our desired result about regular local rings it is very interesting on its own and will be important in the next chapter about Gorenstein rings. It also shows that homological dimensions are deeply linked to geometric properties under favorable circumstances, which is in some sense what this thesis is about. The proof is taken from Matsumura and Reid [1989][Theorem 18.1].

**Theorem 2.15.** *If* $\dim A = n$ *and* $\mathrm{injdim}\, A < \infty$ *then* $\mathrm{injdim}\, A = n$.

*Proof.* Take $r$ to be the injective dimension of $A$. From Lemma 2.11 we get $n \leq r$. If $r = 0$ then $n = 0$. Let $r > 0$ from Lemma 2.14 we obtain a prime ideal $P$ such that $\operatorname{Ext}^r_A(A/P, A) \neq 0$. If $P \neq \mathfrak{m}$, then $\exists x \in \mathfrak{m} - P$ and from the exactness of $0 \longrightarrow A/P \xrightarrow{x} A/P \longrightarrow C \longrightarrow 0$ we get

$$\operatorname{Ext}^r_A(A/P, A) \xrightarrow{x} \operatorname{Ext}^r_A(A/P, A) \longrightarrow \operatorname{Ext}^{r+1}_A(C, A) = 0$$

is exact. Nakayama's lemma gives $\operatorname{Ext}^r_A(A/P, A) = 0$ which is a contradiction, so that $P = \mathfrak{m}$ and we get $\operatorname{Ext}^r_A(k, A) \neq 0$. Suppose $\mathfrak{m}$ contains no $A$-regular elements, then $\operatorname{depth} A = 0$, so that $A$ contains a copy of $A/\mathfrak{m} = k$. Hence, there is a short exact sequence

$$0 \longrightarrow k \longrightarrow A \longrightarrow A/k \longrightarrow 0$$

is exact, hence from applying $\operatorname{Hom}_A(-, A)$ to the sequence above we obtain

$$0 = \operatorname{Ext}^r_A(A, A) \longrightarrow \operatorname{Ext}^r_A(k, A) \longrightarrow \operatorname{Ext}^{r+1}_A(A/k, A) = 0$$

is exact too, with $0 = \operatorname{Ext}^r_A(A, A)$ because $A$ is free and $\operatorname{Ext}^{r+1}_A(A/k, A) = 0$ because $\operatorname{injdim} A = r$. This again contradicts $\operatorname{Ext}^r_A(k, A) \neq 0$. Choose $x \in \mathfrak{m}$ $A$-regular, then by Lemma 2.12 we get $\operatorname{Ext}^i_A(N, A) \cong \operatorname{Ext}^{i-1}_B(N, B)$ for $B = A/xA$ and all $B$-modules $N$ so $\operatorname{injdim} B = r - 1$. Using induction we get $\dim B = r - 1$ which shows $\dim A = r$.

□

## 2.3 Auslander–Buchsbaum–Serre's theorem

Recall that if $(A, \mathfrak{m})$ a local ring, the residue field is given by $k = A/\mathfrak{m}$. The residue field plays a vital role in commutative algebra. This next lemma shows that if $k$ has finite projective dimension, then all $A$-modules do. Very informally this states that in the favorable case when $\operatorname{pd} k < \infty$ we are only finitely far away from doing linear algebra. The other part of the lemma also shows that minimal resolutions are in fact minimal; in length that is.

**Lemma 2.16.** *If $M$ is a finite nonzero $A$-module, then* $\operatorname{pd} M$ *is the length of any minimal free resolution for $M$. If $i$ is the smallest integer for which* $\operatorname{Tor}^A_{i+1}(k, M) = 0$ *then* $\operatorname{pd} M = i$. *In particular if* $\operatorname{pd} k = n$ *then* $\operatorname{gldim} A = n$.

*Proof.* Computing $\operatorname{Tor}^A_i(k, M)$ from a minimal resolution $F$ for $M$ we get

$$\operatorname{Tor}^A_i(k, M) = k \otimes F_i$$

because the maps in $k \otimes F$ are 0. $M$ is finite, so all the $F_i$ can be chosen to be finite. Also $k \otimes F_i \cong F_i/\mathfrak{m}F_i$, so by Nakayama's lemma we have

$$\operatorname{Tor}^A_i(k, M) = 0 \iff F_i = 0.$$

This equivalence shows that if $\operatorname{Tor}^A_{n+1}(k, M) = 0$, then $F$ is of length smaller or equal than $n$. If $\operatorname{pd} M = n$, then clearly $\operatorname{Tor}^A_{n+1}(k, M) = 0$ and $F$ is of minimal length. If $\operatorname{pd} M = \infty$, then all $F_i$ are nonzero, so all the $\operatorname{Tor}^A_i(k, M)$ are nonzero too. The final statement follows from the second statement and Theorem 1.8.

□

**Corollary 2.17.** *If $(A, \mathfrak{m}, k)$ is a regular local ring, then* $\operatorname{gldim} A = \dim A$.

*Proof.* If $A$ is regular, take $(x_1, ..., x_n)$ to be a minimal generating set for $\mathfrak{m}$. Now by Theorem 2.5 $K(x_1, ..., x_n)$ is a minimal resolution for $k$ of length $n$, so by Lemma 2.16 $\operatorname{gldim} A = n$.

□

For our proof to work we have to prove that a minimal free resolution for $k$ is at least as long as the Koszul complex. The proof is taken from [Stacks Project Authors, 2018, Tag 065U].

**Lemma 2.18.** *If $n$ is the length of a minimal generating set of $\mathfrak{m}$, then* $\operatorname{pd} k \geq n$.

*Proof.* Let $(x_1, ..., x_n)$ be a minimal generating set for $\mathfrak{m}$ and $F$ be a minimal projective resolution for $k$. Consider the Koszul complex $K(x_1, ..., x_n)$; the cokernel of its last map is $k$ and all of its modules are free, so there exists a map $\varphi$ from $K(x_1, ..., x_n)$ to $F$ induced by the identity map $k \to k$ from Lemma 1.2. It suffices to show that all $\varphi_i \otimes k$ are injective, then $F_i \neq 0$ for $i \leq n$. The case when $n = 0$ is trivial, because $F_0$ is necessarily just $A$. Now suppose $i > 0$, consider the diagram

$$\begin{array}{ccc} \wedge^{n-i} A^n \otimes k & \xrightarrow{\ a\ } & \mathfrak{m}/\mathfrak{m}^2 \otimes \wedge^{n-i+1} k^n \\ \downarrow {\scriptstyle b=\varphi_i \otimes k} & & \downarrow {\scriptstyle c=\mathfrak{m}/\mathfrak{m}^2 \otimes \varphi_{i-1}} \\ F_i \otimes k & \xrightarrow{\ d\ } & \mathfrak{m}/\mathfrak{m}^2 \otimes F_{i-1} \end{array}$$

The injectivity of $c$ is obtained by induction, because $\mathfrak{m}/\mathfrak{m}^2$ is just a vector space. It now suffices to show that $a$ is injective too, because then $d \circ b$ is also injective implying $b$ is too. Let $\{x_j\}$ be a basis for $A^n$ and $\{e_j\}$ a basis for $k^n \cong \mathfrak{m}/\mathfrak{m}^2$. The map $a$ is the map on vector spaces induced by the differential $c \to x \wedge c$ in the Koszul complex. Thus $a(c) = \sum x_j \otimes e_j \wedge a$ with $x_j$ being linearly independent so it's enough that not all summands are 0 unless $a = 0$. If $e_j \wedge a = 0\ \forall j$ and $a \neq 0$ then $a$ has to be the tensor product of all the $e_j$, but $i > 0$ so $a = 0$.

□

The following result marks the first major success of homological algebra in commutative algebra. It was Jean-Pierre Serre who first proved it, but David Buchsbaum with Maurice Auslander also proved it not long after. As with Theorem 2.15 this also shows the importance of homological dimensions in geometry. The following proof uses the fact that when injective dimension is finite, it is the same as the dimension of the ring. This proof is one of my contributions to the thesis.

**Theorem 2.19.** *A noetherian local ring $(A, \mathfrak{m}, k)$ is regular if and only if*

$$\operatorname{gldim} A < \infty.$$

*Proof.* Let $(x_1, ..., x_n)$ be a minimal generating set for $\mathfrak{m}$.
First suppose $\operatorname{gldim} A < \infty$. From Lemma 2.18 we get $\operatorname{pd} k \geq n$. Lemma 2.6 tells us that $\operatorname{Ext}^r_A(k, A) \neq 0$ for some $r \geq n$ which by Lemma 1.4 implies $\operatorname{injdim} A \geq r$. Theorem 1.8 gives $\operatorname{injdim} A < \infty$ so by applying Theorem 2.15 and the fact that $\dim A$ is at most $n$ we get $\dim A = n$, so that $A$ is regular.
Conversely, assuming $A$ is regular local, then $\operatorname{gldim} A = n$ by Corollary 2.17.

□

Knowing this it is now easy to prove our desired result about localization.

**Theorem 2.20.** *If $P$ is a prime ideal of a regular local ring $A$, then $A_P$ is again regular local.*

*Proof.* From Theorem 2.19 we only need that $A_P$ has finite global dimension, and from Lemma 2.16 we only need $\operatorname{pd}_{A_P}(A_P/P_P) < \infty$. We have a finite projective resolution for $A/P$ in $A$ and by Lemma 1.5 we get a finite projective resolution for $A_P/P_P$ in $A_P$.

□

With this result in mind we can now define regular rings.

**Definition 2.4.** *A ring $R$ is called regular if for every prime $P$ the ring $R_P$ is regular local.*

We also get a result about the global dimension of a regular ring from Corollary 2.17. This proof is of my own doing.

**Theorem 2.21.** *If $R$ is a regular ring, then $\dim R < \infty$ if and only if $\operatorname{gldim} R < \infty$. If the equivalent conditions are satisfied, then $\dim R = \operatorname{gldim} R$.*

*Proof.* Suppose $\dim R = n$, we show every finite $R$-module $M$ has projective dimension at most $n$. Let $i > n$ and $N$ be an $R$-module. Let $P$ be any prime of $R$, by Lemma 1.7 $\operatorname{Ext}^i_R(M, N)_P = \operatorname{Ext}^i_{R_P}(M_P, N_P) = 0$ since $\operatorname{gldim} R_P \leq n$ by Corollary 2.17- $R$ is regular so $R_P$ is regular local with $\dim R_P \leq n$. This implies $\operatorname{Ext}^i_R(M, N) = 0$, which in turn implies $\operatorname{pd} M \leq n$ and by applying Theorem 1.8 we get $\operatorname{gldim} R \leq n$. Now we prove $\dim R = \operatorname{gldim} R$. Let $\mathfrak{m}$ to be a maximal ideal such that $\operatorname{ht} \mathfrak{m} = n$ and $F$ a projective resolution for $R/\mathfrak{m}$ of minimal length $r$. From Lemma 1.5 we see that $F_\mathfrak{m}$ is a projective resolution for $(R/\mathfrak{m})_\mathfrak{m}$ in $R_\mathfrak{m}$, so $r \geq n$ by Corollary 2.17 and we get $\dim R = \operatorname{gldim} R$.
Conversely let $\operatorname{gldim} R = n$ and let $\mathfrak{m}$ be any maximal ideal. A projective resolution for $R/\mathfrak{m}$ of minimal length $r$ localizes to a projective resolution for $(R/\mathfrak{m})_m$ by Lemma 1.5. Again from Corollary 2.17 we see $n \geq r \geq \operatorname{ht} \mathfrak{m}$, so $\dim R < \infty$.

□

The condition that $\dim R < \infty$ is necessary.

# 3. Gorenstein rings

We saw in the previous chapter that having finite global dimension is a very strong condition on a ring. Naturally that makes us think about how to weaken it just enough, so we get something that is still reasonable. Most of this chapter is taken from [Christensen, 2000, Chapter 1]. We ultimately prove a similar result as in the previous chapter, but for Gorenstein local rings.

Just like in the previous chapter, all rings are commutative and noetherian.

**Definition 3.1.** *A local ring $A$ is Gorenstein, if* $\operatorname{injdim}_A A < \infty$*. By Theorem 2.15 this is equivalent to* $\operatorname{injdim}_A A = \dim A$*.*

## 3.1 The G-class

In this section we will introduce a class of finite modules, the G-class, that generalizes the class of finite projective modules. To talk about the G-class of modules we first need to define some standard maps. It is useful to know that these maps are natural.

**Definition 3.2.** *For a ring $R$ and $R$-modules $P, N, M$ we define the Hom evaluation map as the map $\theta_{PNM} \colon P \otimes_R \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(\operatorname{Hom}_R(P, N), M)$ given by $\theta_{PNM}(p \otimes \psi)(\nu) = \psi\nu(p)$.*

The main subject of this chapter is duality.

**Definition 3.3.** *For a ring $R$ and an $R$-module $M$ we define the dual of $M$ to be $M^* = \operatorname{Hom}_R(M, R)$. We define the biduality map $\delta_M \colon M \to M^{**}$ by $\delta_M(x)(\psi) = \psi(x)$.*

Now we can give Auslander's definition of the G-class. In linear algebra it is trivial that every finitely generated vector space is isomorphic to its double dual. The general theme of trying to be not too far away from linear algebra continues; the following class of modules shares this property.

**Definition 3.4.** *Let $R$ be a ring. We say that a finite $R$-module $M$ is in the G-class $\mathrm{G}(R)$, if it satisfies the following conditions*
*(1) $\operatorname{Ext}^i_R(M, R) = 0$ for $i > 0$;*
*(2) $\operatorname{Ext}^i_R(M^*, R) = 0$ for $i > 0$;*
*(3) The map $\delta_M$ is an isomorphism.*

It is immediate from the definition that the dual of a module in the G-class is also in the G-class. To talk about modules in the G-class it is useful to know when the Hom evaluation map is an isomorphism.

**Lemma 3.1.** *Let $R$ be a ring and $P, N, M$ $R$-modules. The Hom evaluation map $\theta_{PNM}$ is an isomorphism if one of the following holds:*
*(1) $P$ is finite and projective;*
*(2) $P$ is finite and $M$ is injective.*

*Proof.* First we prove that the map $\theta_{FNM}$ is an isomorphism when $F = R^n$. Since $F$ is finite direct sum of copies of $R$, it suffices to show that $\theta_{RNM}$ is an isomorphism. This follows from the fact that $\operatorname{Hom}_R(R, N) \cong N$. Now we get the desideratum for $F$ from the fact that finite direct sums commute with tensor products and homs. Consider a finite presentation $R^m \to R^k \to P \to 0$ for $P$. From this finite presentation we get a commutative diagram because the Hom evaluation map is natural

$$\begin{array}{ccccccc} R^m \otimes_R \operatorname{Hom}(N,M) & \longrightarrow & R^k \otimes_R \operatorname{Hom}(N,M) & \longrightarrow & P \otimes_R \operatorname{Hom}(N,M) & \longrightarrow & 0 \\ \downarrow \theta_{R^m NM} & & \downarrow \theta_{R^k NM} & & \downarrow \theta_{PNM} & & \\ \operatorname{Hom}(\operatorname{Hom}(R^m,N),M) & \longrightarrow & \operatorname{Hom}(\operatorname{Hom}(R^k,N),M) & \longrightarrow & \operatorname{Hom}(\operatorname{Hom}(P,N),M) & \longrightarrow & 0 \end{array}$$

The top row is always exact since tensoring is right exact. For the bottom row, consider the conditions:
(1) If $P$ is finite and projective, then the map $R^k \to P$ splits and since Hom is functorial we get that $\operatorname{Hom}(\operatorname{Hom}(R^k, N), M) \to \operatorname{Hom}(\operatorname{Hom}(P, N), M)$ also splits. This proves the bottom row is also exact.
(2) If $P$ is finite and $M$ is injective, we get that the functor $\operatorname{Hom}(-, M)$ is right exact. From the way we obtain the bottom row it follows that the bottom row is exact.
In both cases we get that the bottom row is exact so Lemma 1.1 yields $\theta_{PNM}$ is an isomorphism.

□

With this in mind it is now easy to show that the being in the G-class indeed generalizes projectivity for finite $R$-modules.

**Corollary 3.2.** *If $M$ is a finite projective $R$-module, then $M \in \mathrm{G}(R)$.*

*Proof.* The first condition is obvious. For the second it suffices that $\operatorname{Hom}(M, R)$ is again projective; this follows for example from the fact that Hom is additive. For the last condition consider the commutative square

$$\begin{array}{ccc} M & \xrightarrow{\delta_M} & \operatorname{Hom}(\operatorname{Hom}(M,R),R) \\ \downarrow \cong & & \uparrow \theta_{MRR} \\ M \otimes_R R & \xrightarrow{\cong} & M \otimes_R \operatorname{Hom}(R,R). \end{array} \tag{3.1}$$

We saw in Lemma 3.1 that the map $\theta_{MRR}$ is an isomorphism, so $\delta_M$ is an isomorphism too.

□

## 3.2 The G-dimension theorem

We define G-resolutions and G-dimension analogously to the projective case. The aim of this section is to show that G-dimension has many of the same properties as projective dimension.

**Definition 3.5.** *Let $R$ be a ring and $M$ an $R$-module.*
*An exact complex $F : ... \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ with $F_i \in \mathrm{G}(R)$ and $\operatorname{Coker}\varphi_0{=}M$ is called a G-resolution for $M$. The G-dimension $\text{G-dim}(M)$ is the minimal length of such a resolution.*

To prove results about G-dimension we need some basic facts about short exact sequences. The following is a restatment of Christensen [2000][1.1.10]

**Lemma 3.3.** *Let $0 \to K \to N \to M \to 0$ be a short exact sequence of finite $R$-modules.*
*(1) If $M \in \mathrm{G}(R)$, then the sequences*

$$0 \to M^* \to N^* \to K^* \to 0$$

*and*

$$0 \to K^{**} \to N^{**} \to M^{**} \to 0$$

*are exact.*
*(2) If $M \in \mathrm{G}(R)$, then $K \in \mathrm{G}(R)$ if and only if $N \in \mathrm{G}(R)$.*
*(3) If $N \in \mathrm{G}(R)$, then $\mathrm{Ext}^k_R(K,R) \cong \mathrm{Ext}^{k+1}_R(M,R)$.*
*(4) If the sequence splits, then $N \in \mathrm{G}(R)$ if an only if $M \in \mathrm{G}(R)$ and $K \in \mathrm{G}(R)$*

*Proof.* Set $T^i(-) = \mathrm{Ext}^i_R(-,R)$. And consider the long exact sequence

$$0 \to M^* \to N^* \to K^* \to T^1(M) \to ... \to T^i(M) \to T^i(N) \to T^i(K) \to ... \quad (3.2)$$

(1) The exactness of $0 \to M^* \to N^* \to K^* \to 0$ follows from $T^1(M) = 0$ and eq. (3.2). Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_K} & & \downarrow{\scriptstyle \delta_N} & & \downarrow{\scriptstyle \delta_M} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & N^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array} \quad (3.3)$$

The bottom row is exact at $K^{**}$ because it is obtained from

$$0 \to M^* \to N^* \to K^* \to 0$$

by applying a left exact functor. Exactness at $M^{**}$ follows from the fact that $\delta_M$ is an isomorphism and the diagram commutes.
(2) Suppose $K \in \mathrm{G}(R)$, then from 3.3 and Lemma 1.1 we see that $\delta_N$ is also an isomorphism. The condition about the vanishing of $T^i(N)$ is immediate from eq. (3.2), and $T^i(N^*) = 0$ follows from an analogous argument for

$$0 \to M^* \to N^* \to K^* \to 0.$$

(3) This follows at once from the eq. (3.2).
(4) Since the Hom functor is additive the bottom row in 3.3 also split. This shows that $\delta_N$ is an isomorphism if and only if $\delta_M$ and $\delta_K$ are. Ext is also additive so $T^i(N) \cong T^i(K) \oplus T^i(M)$ and $T^i(N^*) \cong T^i(K^*) \oplus T^i(M^*)$ which proves the equivalence.

□

**Corollary 3.4.** *Let ... $\to G_1 \to G_0$ be a G-resolution for an R-module M and* $K_n = \mathrm{Ker}(G_{n-1} \to G_{n-2})$. *There is an isomorphism*

$$\mathrm{Ext}^i_R(K_n, R) \cong \mathrm{Ext}^{i+n}_R(M, R).$$

*Proof.* Setting $K_1 = \mathrm{Im}(G_1 \to G_0)$ and $K_0 = M$ we get a short exact sequence $0 \to K_i \to G_{i-1} \to K_{i-1} \to 0$ for all $i > 0$. Inductively applying Lemma 3.3 (3) we get the isomorphism.

□

The following shows that if a module is of finite G-dimension, it is enough that condition (1) from the definition of the G-class holds. The lemma is from Christensen [2000][1.2.6].

**Lemma 3.5.** *Let M be a finite R-module of finite G-dimension. If* $\mathrm{Ext}^i_R(M, R) = 0$ *for all* $i > 0$, *then* $M \in \mathrm{G}(R)$.

*Proof.* Set $T^i(-) = \mathrm{Ext}^i_R(-, R)$.
We prove this on induction on G-dim $M$. If G-dim $M \leq 1$ then consider the short exact sequence

$$0 \to G_1 \to G_0 \to M \to 0.$$

From $T^i(M) = 0$ for $i > 0$ we see that the sequence

$$0 \to M^* \to G_0^* \to G_1^* \to 0$$

is also exact. Lemma 3.3 (2) tells us that $M^* \in \mathrm{G}(R)$ so $T^i(M^*) = 0$ for $i > 0$. As in the previous lemma we get the ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_{G_1}} & & \downarrow{\scriptstyle \delta_{G_0}} & & \downarrow{\scriptstyle \delta_M} & & \\ 0 & \longrightarrow & G_1^{**} & \longrightarrow & G_0^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array}$$

with the bottom row being exact due to $0 \to M^* \to G_0^* \to G_1^* \to 0$ being exact and Lemma 3.3 (1). Another application of Lemma 1.1 gives $\delta_M$ is an isomorphism.
Now let G-dim $M \leq n > 0$. Consider a G-resolution for $M$

$$0 \to G_n \to ... \to G_0$$

of length $n$ and set $K$ to be the kernel of the map $G_1 \to G_0$. Applying Lemma 3.3 (3) we see $T^i(K) = 0$ for $i > 0$. Also G-dim $K \leq n-1$ so $K \in \mathrm{G}(R)$ by induction. Now G-dim $M = 1$, so by induction $M \in \mathrm{G}(R)$.

□

We wish to prove that all G-resolutions contain a G-resolution of minimal length. For this we need to talk about mapping cones.

**Definition 3.6.** *Let* $\alpha \colon F \to G$ *be a map of complexes and* $\varphi$ *be the differential in* $F$, $\psi$ *in* $G$. *We define the mapping cone* $M(\alpha)$ *of* $\alpha$ *to be the complex where* $M(\alpha)_i = F_{i-1} \oplus G_i$ *and the differential* $F_{i-1} \oplus G_i \to F_{i-2} \oplus G_{i-1}$ *is given by* $(a, b) \mapsto (-\varphi_{i-1}(a), \alpha_{i-1}(a) + \psi_i(b))$.

**Definition 3.7.** *A map of complexes $\alpha\colon F \to G$, such that $H^n(\alpha)\colon H^n(F) \to H^n(G)$ is an isomorphism for all $n$ is called a quasi-isomorphism.*

The next theorem is from Weibel [1994][Corollary 1.5.4]

**Theorem 3.6.** *A map of complexes $\alpha\colon F \to G$ is a quasi-isomorphism if and only if the mapping cone $M(\alpha)$ is exact.*

**Corollary 3.7.** *If $F$, $G$ are exact complexes and $\alpha\colon F \to G$ is a map of complexes, then $M(\alpha)$ is exact.*

Note that proving an analogy of the following theorem for projective dimension is much simpler. For a module $M$ to be projective it suffices that $\operatorname{Ext}^i_R(M,N) = 0$ for $i > 0$ and all modules $N$; for the G-class it just is not that simple. Because we introduced the mapping cone in full generality, we can shorten the proof. This theorem can be found as Christensen [2000][1.2.7]

**Theorem 3.8.** *Let $M$ be a finite $R$-module and $n$ a natural number. The following conditions are equivalent:*
*(1)* $\operatorname{G-dim} M \leq n$.
*(2)* $\operatorname{G-dim} M < \infty$ *and* $\operatorname{Ext}^i_R(M,R) = 0$ *for* $i > n$.
*(3) If* $\ldots \to G_1 \to G_0$ *is a G-resolution for* $M$*, then* $\operatorname{Ker}(G_{n-1} \to G_{n-2}) \in \mathrm{G}(R)$ *(if $n = 0$ this is just the kernel of $G_0 \to M$).*

*Proof.* (1) $\Longrightarrow$ (2): If $\operatorname{G-dim} M \leq n$, then there exists a G-resolution

$$0 \to G_r \to \ldots \to G_1 \to G_0$$

for $M$, where $r \leq n$. From Corollary 3.4 we now see that $\operatorname{Ext}^{i+r}_R(M,R) \cong \operatorname{Ext}^i_R(K,R)$, where $K = \operatorname{Ker}(G_{r-1} \to G_{r-2})$. $K$ is in the G-class, hence it holds that $\operatorname{Ext}^m_R(M,R) = 0$ for $m > r \geq n$.
(2) $\Longrightarrow$ (1): Let

$$0 \to G_r \to \ldots \to G_1 \to G_0$$

be finite G-resolution for $M$. If $r \leq n$ we are done. Let $r > n$ and take $K = \operatorname{Ker}(G_{n-1} \to G_{n-2})$. Again from Corollary 3.4 we see $\operatorname{Ext}^{i+n}_R(M,R) \cong \operatorname{Ext}^i_R(K,R)$, and we assume $\operatorname{Ext}^i_R(M,R) = 0$, so from this and the fact that $\operatorname{G-dim} K < \infty$ we see $K \in \mathrm{G}(R)$ by Lemma 3.5.
(1) $\Longrightarrow$ (3): First we prove that if

$$F : 0 \to H_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$$

and

$$G : 0 \to K_n \to G_{n-1} \to \ldots \to G_0 \to M \to 0$$

are exact sequences with $P_i$ projective and $G_i \in \mathrm{G}(R)$, then $H_n \in \mathrm{G}(R) \iff K_n \in \mathrm{G}(R)$. All the $P_i$ are projective, so replacing $H_n$ by a projective module that surjects onto it gives a map $\alpha\colon F \to G$ induced by the identity on $M$ by Lemma 1.2. The mapping cone $M(\alpha)$ gives an exact sequence due to Corollary 3.7

$$0 \longrightarrow H_n \overset{\varphi_{n+1}}{\longrightarrow} K_n \oplus P_{n-1} \overset{\varphi_n}{\longrightarrow} G_{n-1} \oplus P_{n-2} \overset{\varphi_{n-1}}{\longrightarrow} \ldots \overset{\varphi_2}{\longrightarrow} G_1 \oplus P_0 \overset{\varphi_1}{\longrightarrow} G_0 \oplus M \overset{\varphi_0}{\longrightarrow} M \longrightarrow 0$$

Now letting $N_i = \operatorname{Ker} \varphi_{i-1}$ we get short exact sequences

$$0 \to N_{i+1} \to M(\alpha)_i \to N_i \to 0 \tag{3.4}$$

Observe that $M(\alpha)_i = G_i \oplus P_{i-1}$ for $n-1 \leq i \geq 2$ so $M(\alpha)_i \in \mathrm{G}(R)$ for $n-1 \geq i \geq 2$; this is because finite projectives are in the G-class by Corollary 3.2 and Lemma 3.3 (4). Also $N_1 = G_0$ so by inductively applying Lemma 3.3 (1) to eq. (3.4) we get $N_n \in \mathrm{G}(R)$. Finally the short exact sequence

$$0 \to H_n \to K_n \oplus P_{n-1} \to N_n \to 0$$

shows

$$H_n \in \mathrm{G}(R) \iff K_n \oplus P_n \in \mathrm{G}(R) \iff K_n \in \mathrm{G}(R)$$

with the last equivalence being due to Lemma 3.3 (4). Now choosing $G$ to be a G-resolution for $M$ of length $n$ we see that $H_n \in \mathrm{G}(R)$. Now if $G$ is any G-resolution for $M$ we see that $K_n \in \mathrm{G}(R)$ because $H_n \in \mathrm{G}(R)$.

□

A very useful corollary is that we can begin a G-resolution arbitrarily.

**Corollary 3.9.** *Suppose $M$ is finite $R$-module. Let $0 \to K \to G \to M \to 0$ be the first step of a G-resolution for $M$, that is $G \in \mathrm{G}(R)$. If $\text{G-dim}_R(M) = n$, then $\text{G-dim}_R(K) = n - 1$.*

*Proof.* First we observe that $\text{G-dim}_R K < \infty$. Continuing the resolution $... \to G = G_0 \to M \to 0$ we get that $\operatorname{Ker}(G_{n-1} \to G_{n-2}) \in \mathrm{G}(R)$ by Theorem 3.8 (3). This shows $\text{G-dim}_R K \leq n-1$ because $... \to G_1 \to K \to 0$ is a G-resolution for $K$. From Corollary 3.4 we get $\operatorname{Ext}^i_R(K, R) \cong \operatorname{Ext}^{i+1}_R(M, R)$, this shows $\operatorname{Ext}^i_R(K, R) = 0$ for $i > n-1$. Another application of Theorem 3.8 yields $\text{G-dim}_R K = n - 1$.

□

## 3.3 Regular elements and G-dimension

The following statement about the Hom Tensor adjunction can be found in Rotman [2008][Theorem 2.75].

**Theorem 3.10.** *Suppose $R$ and $S$ are rings. Let $A$ be an $R$-module, $C$ an $S$-module and $B$ an $(R, S)$ bimodule. Then*

$$\operatorname{Hom}_S(A \otimes_R B, C) \cong \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C)).$$

*The isomorphism is given by*

$$\tau\colon \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C)) \to \operatorname{Hom}_S(A \otimes_R B, C).$$

*Defined as $f \mapsto \tau(f)$, where $\tau(f)$ is the map induced by $a \oplus b \mapsto f(a)(b)$*

**Corollary 3.11.** *Suppose $R$ is a ring and $x \in R$. If $M$ is a finite $R$-module, then* $\operatorname{Hom}_R(M, \overline{R}) \cong \operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R})$*, where $\overline{M} = M/xM$ and $\overline{R} = R/(x)$. The isomorphism*

$$\tau_M \colon \operatorname{Hom}_R(M, \overline{R}) \to \operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R})$$

*is given by $\varphi \mapsto \tau(\varphi)$, where $\tau(\varphi)(a) = \varphi(a')$ where $a'$ is anything in the preimage of $a$.*

*Proof.* Applying Theorem 3.10 to $R = R$, $S = \overline{R}$, $A = M$ and $B = C = \overline{R}$ we see

$$\operatorname{Hom}_{\overline{R}}(M \otimes_R \overline{R}, \overline{R}) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{\overline{R}}(\overline{R}, \overline{R})).$$

Observing $M \otimes_R \overline{R} \cong \overline{M}$ and $\operatorname{Hom}_{\overline{R}}(\overline{R}, \overline{R}) \cong \overline{R}$ we get our first claim. The map $\tau$ given in Theorem 3.10 sends $\varphi \in \operatorname{Hom}_R(M, \overline{R})$ to the map $\tau(\varphi)$ defined by $\tau(\varphi)(a) = \tau(\varphi)(a' \oplus 1) = \varphi(a')(1) = \varphi(a')$, where $a \in \overline{M}$ and $a'$ is something in the preimage of $a$.

□

Most of the work in this chapter is about providing tools to be able to work with induction on depth. For this it is crucial to know that modules in the G-class share regular elements with $R$.

**Lemma 3.12.** *If $x \in R$ is $R$-regular, and $M$ is a finite $R$-module. The element $x$ is also $M^*$-regular. In particular, if $M \in \mathrm{G}(R)$, then $x$ is $M$-regular.*

*Proof.* Suppose $0 \neq \varphi \in M^*$, then there exists $a \in M$ such that $\varphi(a) \neq 0$. The element $x$ is $R$-regular so $\varphi(xa) = x\varphi(a) \neq 0$, and thus $x\varphi \neq 0$. The second statement follows from $(M^*)^* = M^{**}$.

□

Next we show that under good conditions it holds that the dual of a quotient is just the quotient of a dual. The next lemma is taken from Christensen [2000][1.3.4]

**Lemma 3.13.** *Let $M$ be a finite $R$-module and $x \in R$ is $M$-regular and $R$-regular. Set $\overline{R} = R/(x)$ and $\overline{M} = M/xM$.*
*(1)* $\operatorname{Tor}_i^R(M, R/(x)) = 0$ *for $i > 0$.*
*(2) If* $\operatorname{Ext}^1_R(M, R) = 0$ *then* $\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R}) \cong M^*/xM^*$*.*
*(3) If* $\operatorname{Ext}^1_R(M, R) = 0 = \operatorname{Ext}^1_R(M^*, R)$*, then*

$$\operatorname{Hom}_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R}), \overline{R}) \cong M^{**}/xM^{**}.$$

*Proof.* Set $\overline{R} = R/(x)$.
(1) There exists a free resolution of length 1 for $\overline{R}$, namely, the Koszul complex, so clearly $\operatorname{Tor}_i^R(M, \overline{R}) = 0$ for $i > 1$. The exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow \overline{M} \longrightarrow 0$ shows $\operatorname{Tor}_1^R(M, \overline{R}) = 0$ since it is obtained by tensoring $0 \longrightarrow R \xrightarrow{x} R \longrightarrow \overline{R} \longrightarrow 0$ with $M$.
(2) Because $\operatorname{Ext}^1_R(M, R) = 0$ we get a short exact sequence

$$0 \longrightarrow M^* \xrightarrow{x} M^* \longrightarrow \operatorname{Hom}_R(M, \overline{R}) \longrightarrow 0$$

which shows $\operatorname{Hom}_R(M, \overline{R}) \cong M^*/xM^*$. Hence from Corollary 3.11 we obtain $\operatorname{Hom}_R(M, \overline{R}) \cong \operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R})$.
(3) Lemma 3.12 shows $x$ is also $M^*$-regular. Applying (2) to $M^*$ yields

$$\operatorname{Hom}_{\overline{R}}(M^*/xM^*, \overline{R}) \cong M^{**}/xM^{**}.$$

In (2) we saw $\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R}) \cong M^*/xM^*$; this finishes the proof.

□

The following corollary is left without proof in Christensen [2000].

**Corollary 3.14.** *Let $M$ be a finite $R$-module and $x$ an $R$-regular element. Set $\overline{R} = R/(x)$ and $\overline{M} = M/xM$. There is a commutative diagram*

$$\begin{array}{ccc} M \otimes_R \overline{R} & \xrightarrow{\delta_M \otimes id_{\overline{R}}} & M^{**} \otimes_R \overline{R} \cong M^{**}/xM^{**} \\ \downarrow \cong & & \downarrow a \\ \overline{M} & \xrightarrow{\delta_{\overline{M}}} & \operatorname{Hom}_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R}), \overline{R}). \end{array} \tag{3.5}$$

*Proof.* First we observe how the isomorphism $b_M \colon M^*/xM^* \to \operatorname{Hom}_R(M, \overline{R})$ given in the proof of Lemma 3.13 (2) works. The map $b_M$ is defined by $\psi \mapsto \pi \circ \psi$, where $\pi \colon R \to \overline{R}$ is the projection. Then we use the isomorphism $\tau_M$ given by Corollary 3.11. Now we can explicitly write down how $a$ works

$$a \colon M^{**}/xM^{**} \to \operatorname{Hom}_{\overline{R}}(M^*/xM^*, \overline{R}) \to \operatorname{Hom}_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R}), \overline{R})$$

is defined by $\varphi \mapsto \tau_{M^*}(b_{M^*}(\varphi)) \mapsto \tau_{M^*}(b_{M^*}(\varphi)) \circ b_M^{-1} \circ \tau_M^{-1}$.

Now consider where an element $m \otimes 1$ maps:

$$m \otimes 1 \mapsto \overline{\delta_M(m)} \mapsto \tau_{M^*}(b_{M^*}(\overline{\delta_M(m)})) \circ b_M^{-1} \circ \tau_M^{-1}.$$

Now take $\psi \in \operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R})$, then

$$\begin{aligned} a(\overline{\delta_M(m)})(\psi) = \tau_{M^*}(b_{M^*}(\overline{\delta_M(m)}))(b_M^{-1}(\tau_M^{-1}(\psi))) = \pi \circ \overline{\delta_M(m)}((b_M^{-1}(\tau_M^{-1}(\psi)))') = \\ = \pi \circ ((b_M^{-1}(\tau_M^{-1}(\psi)))')(m). \end{aligned} \tag{3.6}$$

Where $(b_M^{-1}(\tau_M^{-1}(\psi)))' \in M^*$ is any map in the preimage of $(b_M^{-1}(\tau_M^{-1}(\psi)))$. We compose this map with the projection $\pi$, so from the homomorphism theorem it holds that

$$\pi \circ ((b_M^{-1}(\tau_M^{-1}(\psi)))') = \pi \circ (b_M^{-1}(\tau_M^{-1}(\psi))) = \tau_M^{-1}(\psi)$$

Now we can clearly see

$$a(\overline{\delta_M(m)}) = \tau_M^{-1}(\psi)(m) = \psi(\overline{m}) = \delta_{\overline{M}}(\overline{m})(\psi).$$

□

This next result is taken from Matsumura and Reid [1989][Lemma 18.2. (ii)].

**Lemma 3.15.** *Let $R$ be a ring and $M$ and $R$-module. Suppose $x \in R$ is both $R$-regular and $M$-regular, and set $\overline{R} = R/(x)$ and $\overline{M} = M/xM$. Then there is an isomorphism $\operatorname{Ext}^i_R(M, \overline{R}) \cong \operatorname{Ext}^i_{\overline{R}}(\overline{M}, \overline{R})$ for all $i > 0$.*

*Proof.* Consider a free resolution

$$F : \ldots \to F_1 \to F_0 \to M \to 0$$

for $M$. From Corollary 3.11 we see

$$\operatorname{Ext}^i_R(M, \overline{R}) = H^i(\operatorname{Hom}_R(F, \overline{R})) \cong H^i(\operatorname{Hom}_{\overline{R}}(F \otimes_R \overline{R}, \overline{R})).$$

It now suffices to show that $F \otimes_R \overline{R}$ is a free resolution for $\overline{M}$ over $\overline{R}$. From Lemma 3.13 (1) we see $\operatorname{Tor}^A_i(M, \overline{R}) = 0$ for $i > 0$, hence the sequence

$$F \otimes_R \overline{R} : ... \to F_1 \otimes_R \overline{R} \to F_0 \otimes_R \overline{R} \to M \otimes_R \overline{R} = \overline{M} \to 0$$

is exact, hence a free resolution for $\overline{M}$ in $\overline{R}$.

□

Quotients of modules in the G-class by regular elements are again in the G-class. This is Christensen [2000][1.3.5]

**Lemma 3.16.** *Let $M$ be a finite $R$-module and $x$ an $R$-regular element. Again set $\overline{R} = R/(x)$ and $\overline{M} = M/xM$. If $M \in \mathrm{G}(R)$, then $\overline{M} \in \mathrm{G}(\overline{R})$.*

*Proof.* By Lemma 3.12 we see $x$ is $M$-regular. From this we see

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow \overline{M} \longrightarrow 0$$

is exact. Applying $\operatorname{Hom}_R(M, -)$ to it gives a long exact sequence

$$... \to \operatorname{Ext}^i_R(M, R) = 0 \to \operatorname{Ext}^i_R(M, \overline{R}) \to \operatorname{Ext}^{i+1}_R(M, R) = 0 \to ...$$

which shows $\operatorname{Ext}^i_{\overline{R}}(\overline{M}, \overline{R}) \cong \operatorname{Ext}^i_R(M, \overline{R}) = 0$ for $i > 0$, with the first isomorphism given by Lemma 3.15. Using the same argument for $M^*$ we get $\operatorname{Ext}^i_{\overline{R}}(M^*/xM^*, \overline{R}) = 0$. Now Lemma 3.13 (b) gives $\operatorname{Ext}^i_{\overline{R}}(\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{R}), \overline{R}) = 0$ for $i > 0$. Finally we prove $\delta_{\overline{M}}$ is an isomorphism. From Lemma 3.13 we get an isomorphism $a$, the top map $\delta_M \otimes id_{\overline{R}}$ is also an isomorphism since $\delta_M$ is an isomorphism. Now Corollary 3.14 shows $\delta_{\overline{M}}$ is also an isomorphism.

□

This theorem bounds G-dimension of a quotient by a regular element. It is taken from Christensen [2000][1.3.6]

**Theorem 3.17.** *Let $M$ be a finite $R$-module. If $x \in R$ is $M$-regular and $R$-regular, then*

$$\text{G-dim}_{R/(x)} M/xM \leq \text{G-dim}_R M.$$

*Proof.* It suffices to show that if $M$ has a G-resolution of length $n$, then $M/xM$ also has a resolution of length $n$ in $R/(x)$. Let

$$0 \to G_n \to ... \to G_0$$

be a G-resolution for M. By Lemma 3.12 $x$ is $G_i$ regular. Again set $K_i = \operatorname{Ker}(G_{i-1} \to G_{i-2})$ and $K_0 = M$, $K_1 = \operatorname{Ker}(G_0 \to M)$. We get short exact sequences

$$0 \to K_i \to G_{i-1} \to K_{i-1} \to 0.$$

We assume $x$ is $M$-regular, so $x$ is $K_0$ regular. All the $K_i$ are submodules of modules in the G-class for $i > 0$, hence $x$ is $K_i$ regular for all $i$, thus Lemma 3.13 (1) now gives

$$0 \to K_i/xK_i \to G_{i-1}/xG_{i-1} \to K_{i-1}/xK_{i-1} \to 0$$

is exact, so

$$0 \to G_n/xG_n \to ... \to G_0/xG_0$$

is a G-resolution by Lemma 3.16.

□

Over local rings a stronger version of Lemma 3.16 holds. This theorem is from Christensen [2000][1.4.4]

**Theorem 3.18.** *Let $A$ be a local ring, $x$ an $A$-regular element and $M$ a finite $R$-module. If $x$ is also $M$-regular, then $M \in \mathrm{G}(A)$ if and only if $M/xM \in \mathrm{G}(A/(x))$.*

*Proof.* Again set $\overline{A} = A/(x)$ and $\overline{M} = M/xM$.
If $M \in \mathrm{G}(A)$, then by Lemma 3.16 $\overline{M} \in \mathrm{G}(\overline{A})$. Now let $\overline{M} \in \mathrm{G}(\overline{A})$ and consider the defining conditions for the G-class:
(1) From Lemma 3.15 we see $\mathrm{Ext}^i_A(M, \overline{A}) \cong \mathrm{Ext}^i_{\overline{A}}(\overline{M}, \overline{A}) = 0$ for $i > 0$. From $x$ being $A$-regular we once again get the exact sequence

$$0 \longrightarrow A \xrightarrow{x} A \longrightarrow \overline{A} \longrightarrow 0. \tag{3.7}$$

Applying $\mathrm{Hom}_A(M, -)$ to the short exact sequence gives a long exact sequence

$$... \longrightarrow \mathrm{Ext}^i_A(M, A) \xrightarrow{x} \mathrm{Ext}^i_A(M, A) \longrightarrow \mathrm{Ext}^i_A(M, \overline{A}) = 0 \longrightarrow ...$$

This shows $x\mathrm{Ext}^i_A(M, A) = \mathrm{Ext}^i_A(M, A)$, and since $M$ is finite we see $\mathrm{Ext}^i_A(M, A) = 0$ for $i > 0$ by Nakayama's lemma.
(2) By Lemma 3.13 we get $M^*/xM^* \cong \mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{A})$ and again by Lemma 3.15 we see

$$\mathrm{Ext}^i_A(M^*, \overline{A}) \cong \mathrm{Ext}^i_{\overline{A}}(M^*/xM^*, \overline{A}) \cong \mathrm{Ext}^i_{\overline{A}}(\mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{A}), \overline{A}) = 0$$

for $i > 0$, the last equality being due to $\overline{M} \in \mathrm{G}(\overline{A})$. This now allows us to use the same argument as in (1) to conclude $\mathrm{Ext}^i_A(M^*, A) = 0$ for $i > 0$.
(3) Finally we show $\delta_M$ is an isomorphism. From Corollary 3.14 we see $\delta_M \otimes id_{\overline{R}}$ is an isomorphism. Let $K$ be the kernel of $\delta_M$ and $C$ its cokernel. There is an exact sequence $0 \longrightarrow K \longrightarrow M \xrightarrow{\delta_M} M^{**} \longrightarrow C \longrightarrow 0$. By tensoring with $\overline{R}$ we get an exact sequence

$$M/xM \xrightarrow{\delta_M \otimes id_{\overline{R}}} M^{**}/xM^{**} \longrightarrow C/xC \longrightarrow 0$$

which is exact at the right because tensoring is right exact. The map $\delta_M \otimes id_{\overline{R}}$ is surjective, so $C/xC = 0$ and since $M^{**}$ is finite Nakayama's lemma gives $C = 0$. There is now a short exact sequence $0 \longrightarrow K \longrightarrow M \xrightarrow{\delta_M} M^{**} \longrightarrow 0$. and tensoring now gives

$$0 \longrightarrow K/xK \longrightarrow M/xM \xrightarrow{\delta_M \otimes id_{\overline{R}}} M^{**}/xM^{**} \longrightarrow 0$$

is exact from Lemma 3.13 (1) and again this shows $K/xK = 0$ since $\delta_M \otimes id_{\overline{R}}$ is injective, and Nakayama's lemma finishes the proof.

□

With the previous theorem, we can now show that we can always study the G-dimension over a ring of lower depth. This means that we can use induction on depth effectively. The following is theorem is Christensen [2000][1.4.5]

**Theorem 3.19.** *Let $(A, \mathfrak{m}, k)$ be a local ring, $M$ a finite $A$-module and $x$ an $A$-regular element. If $x$ is $A$-regular, then* $\text{G-dim}_A M = \text{G-dim}_{A/(x)} M/xM$.

*Proof.* One more time set $\overline{A} = A/(x)$ and $\overline{M} = M/xM$. Theorem 3.17 allows us to assume $\text{G-dim}_{A/(x)} M/xM < \infty$ and proceed by induction on $n = \text{G-dim}_{\overline{A}} \overline{M}$. If $n = 0$, then by Theorem 3.18 $M \in \text{G}(A)$. Let $n > 0$ and consider the first step of a G-resolution

$$0 \to K \to G \to M \to 0.$$

By Lemma 3.13 (1) we get a short exact sequence

$$0 \to K/xK \to G/xG \to \overline{M} \to 0.$$

Since by Theorem 3.18 $G/xG \in \text{G}(\overline{A})$ we get by Lemma 3.12 $x$ is $G$-regular, so it is also $K$-regular. By Corollary 3.9 we have $\text{G-dim}_{\overline{A}} K/xK = n - 1$. Applying the induction hypothesis yields $\text{G-dim}_A K = n - 1$ which shows $\text{G-dim}_A M \leq n$ and we also have $\text{G-dim}_A M \geq n$ by Theorem 3.17 so the proof is finished. □

## 3.4 Depth and G-dimension

First we give two equivalent definitions of depth of a finite module. For the proof of their equivalence see Eisenbud [2013][Proposition 18.4].

**Definition 3.8.** *For a local ring $(A, \mathfrak{m}, k)$ and an $A$-module $M$ we define the depth of $M$ to be the smallest $n$ such that $\text{Ext}^n_A(k, M) \neq 0$. We denote depth of $M$ by $\text{depth}_A M$. If $M$ is finite then $\text{depth}_A M$ is the length of any maximal $M$-sequence.*

The following lemma is again left without proof in Christensen [2000]. It holds in general that $\text{Ass}_R(\text{Hom}_R(M, N)) = \text{Ass}_R(N) \cap \text{Supp}_R(M)$ for finite $M$ and $N$.

**Lemma 3.20.** *If $M$ is a finite $R$-module, then* $\text{Ass}_R(M^*) = \text{Ass}_R(R) \cap \text{Supp}_R(M)$.

*Proof.* Assume $P \in \text{Ass}_R(M^*)$, that is $\exists 0 \neq \varphi \in M^*$ such that $P = \text{ann}_R(\varphi)$. Clearly $\text{ann}_R(M) \subseteq P$ because $\text{ann}_R(M) \subseteq \text{ann}_R(\varphi)$; this shows $P \in \text{Supp}_R(M)$. We have $\text{ann}_R(\text{Im}(\varphi)) = P$, so $R/P$ is a submodule of $R$, which means $P \in \text{Ass}_R(R)$.

Conversely let $P \in \text{Ass}_R(R) \cap \text{Supp}_R(M)$. $P$ is an associated prime of $R$, so $R/P$ is a submodule of $R$. It now suffices to show $\text{Hom}_R(M, R/P) \neq 0$. Localizing at $P$ and using the fact that $M_P \neq 0$ we now get a composite map $M_P \to (M/PM)_P \to (R/P)_P$; Nakayama's lemma gives $(M/PM)_P \neq 0$ and the map $(M/PM)_P \to (R/P)_P$ is just a projection map of vector spaces. This shows $\text{Hom}_{R_P}(M_P, (R/P)_P) \neq 0$ which finishes the proof by Theorem 1.6. □

We have set the stage for the inductive step, now we only need to prove something about the base case. This lemma is from Christensen [2000][1.4.7]

**Lemma 3.21.** *Let $A$ be a local ring with $\text{depth}_A A = 0$. If a finite $A$-module $M$ has finite G-dimension, then $M \in \text{G}(A)$.*

*Proof.* We proceed by induction on $n = \text{G-dim}_A M$. Suppose $\text{G-dim}_A M \leq 1$, that is there exists a short exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with $G_i \in \mathrm{G}(A)$. Dualizing gives a long exact sequence

$$0 \to M^* \to G_0^* \to G_1^* \to \mathrm{Ext}_A^1(M, A) \to 0.$$

Dualizing once more we get

$$0 \to (\mathrm{Ext}_A^1(M, A))^* \to G_1^{**} \to G_0^{**}$$

is exact. Observe that the map $G_1^{**} \to G_0^{**}$ is injective; $\mathrm{Hom}(\mathrm{Hom}(-, A), A)$ is a functor, so the square

$$\begin{array}{ccc} G_1 & \longrightarrow & G_0 \\ \downarrow{\scriptstyle \delta_{G_1}} & & \downarrow{\scriptstyle \delta_{G_0}} \\ G_1^{**} & \longrightarrow & G_0^{**} \end{array}$$

commutes showing that the bottom map is injective. We have just shown $(\mathrm{Ext}_A^1(M, A))^* = 0$. Lemma 3.20 shows

$$\emptyset = \mathrm{Ass}_A\,(\mathrm{Ext}_A^1(M, A))^* = \mathrm{Ass}_A(A) \cap \mathrm{Supp}_A((\mathrm{Ext}_A^1(M, A))).$$

From the assumption that $\operatorname{depth} A = 0$ we see $\mathfrak{m} \in \mathrm{Ass}_A(A)$; all elements of $\mathfrak{m}$ are zerodivisors. Now we can conclude $\mathrm{Supp}_A((\mathrm{Ext}_A^1(M, A))) = \emptyset$, which shows $\mathrm{Ext}_A^1(M, A) = 0$. By Theorem 3.8 (2) we can now conclude $M \in \mathrm{G}(A)$.

Let $\text{G-dim}_A M \leq n > 1$ and let

$$0 \to G_n \to G_{n-1} \to \ldots \to G_0 \to M \to 0$$

be a G-resolution for $M$. We shall show that in fact, $\text{G-dim}_A M = 0$. The kernel $K = \mathrm{Ker}(G_{n-2} \to G_{n-3})$ is of G-dimension 1 because we assume $\text{G-dim}_A M = n$. By the induction hypothesis we get $K \in \mathrm{G}(A)$ which allows us shorten the resolution to $n - 1$, and conclude $M \in \mathrm{G}(A)$ by induction.

□

Over noetherian rings we can choose elements that are regular over a ring and a module at the same time.

**Lemma 3.22.** *Let $(A, \mathfrak{m})$ be a noetherian local ring with* $\operatorname{depth} A > 0$ *and $M$ a finite $A$-module with* $\operatorname{depth} M > 0$. *If* $\operatorname{depth} A > 0$, *then we can choose $x \in A$ that is both $A$-regular and $M$-regular.*

*Proof.* From $\operatorname{depth} A > 0$ and $\operatorname{depth} M > 0$ we see $\mathfrak{m}$ is not an associated prime of $A$ or $M$. Since both $M$ and $A$ have a finite set of associated primes we can by Lemma 2.2 find $x \in \mathfrak{m}$ such that $x$ is not contained in any of the associated primes of $A$ or $M$. This $x$ is now $M$-regular and $A$-regular, because the union of all associated primes is the set of zerodivisors and 0.

□

We want to do induction on the depth of a module, this means that the 0 depth case is undesirable. Fortunately taking one step of a G-resolution gets us back to positive depth.

**Lemma 3.23.** *Let* $0 \to K \to G \to M \to 0$ *be an exact sequence of finite modules over a local ring* $(A, \mathfrak{m}, k)$*. If* $\operatorname{depth} M = n$ *and* $\operatorname{depth} G > n$*, then* $\operatorname{depth} K = n + 1$.

*Proof.* We see $\operatorname{Ext}^0_A(k, K) = 0$, because $\operatorname{Ext}^0_A(k, G) = 0$ from $\operatorname{depth} G > n \geq 0$. Applying $\operatorname{Hom}_A(k, -)$ to the sequence we get a long exact sequence

$$... \to \operatorname{Ext}^i_A(k, G) \to \operatorname{Ext}^i_A(k, M) \to \operatorname{Ext}^{i+1}_A(k, K) \to \operatorname{Ext}^{i+1}_A(k, G) \to ...$$

For $i < n$ we get $0 = \operatorname{Ext}^i_A(k, M) \cong \operatorname{Ext}^{i+1}_A(k, K)$. For $i = n$ we get $0 \neq \operatorname{Ext}^n_A(k, M) \cong \operatorname{Ext}^{n+1}_A(k, K)$, hence $\operatorname{depth} K = n + 1$.

□

The following is a generalization of the classical Auslander-Buchsbaum formula. We shall show that the Auslander-Buchsbaum formula is a direct corollary of this result. This theorem is from Christensen [2000][1.4.8]

**Theorem 3.24** (Auslander-Bridger formula)**.** *Let* $A$ *be a local ring. If* $M$ *is a finite module of finite G-dimension, then*

$$\text{G-dim}\, M = \operatorname{depth} A - \operatorname{depth} M.$$

*Proof.* Again, we do induction on $\operatorname{depth} A$. If $\operatorname{depth} A = 0$, then by Lemma 3.21 $M \in \mathrm{G}(A)$ and it suffices to show that $\operatorname{depth} M = 0$. $M$ is isomorphic to $M^{**}$, so Lemma 3.20 gives

$$\operatorname{Ass}_A(M) = \operatorname{Ass}_A(M^{**}) = \operatorname{Ass}_A(A) \cap \operatorname{Supp}_R(M^*).$$

$M^*$ is nonzero, so $\mathfrak{m} \in \operatorname{Supp}_R(M^*)$ and from $\operatorname{depth} A = 0$ we get $\mathfrak{m} \in \operatorname{Ass}_A(A)$. We have shown $\mathfrak{m} \in \operatorname{Ass}_A(M)$, which proves $\operatorname{depth} M = 0$. Now let $\operatorname{depth} A = n > 0$ and consider the following cases:
(1) If $\operatorname{depth} M > 0$ we can choose an $M$-regular and $A$-regular element $a$ due to Lemma 3.22. By Theorem 3.19 we have the first equality

$$\begin{aligned} &\text{G-dim}_A\, M = \text{G-dim}_{A/(x)}\, M/xM = \\ &\quad = \operatorname{depth} A/(x) - \operatorname{depth}_{A/(x)} M/xM = (\operatorname{depth} A - 1) - (\operatorname{depth} M - 1) \end{aligned} \quad (3.8)$$

with the middle equality being due to the induction hypothesis.
(2) Now suppose $\operatorname{depth} M = 0$ and let $0 \to K \to G \to M \to 0$ be an exact sequence with $G \in \mathrm{G}(A)$. From Lemma 3.12 we see $\operatorname{depth} G > 0$ and Lemma 3.23 shows $\operatorname{depth} K = 1$. From Corollary 3.9 we get the first equality

$$\text{G-dim}(M) - 1 = \text{G-dim}(K) = \operatorname{depth} A - \operatorname{depth} K = \operatorname{depth} A - 1$$

with the middle equality being due to what we already proved in (1).

□

The following result is yet another corollary of Lemma 2.6.

**Theorem 3.25.** *Suppose* $A$ *is a local ring and* $M$ *is a finite* $A$*-module. If* $\operatorname{pd} M < \infty$*, then* $\operatorname{pd} M = \text{G-dim}\, M$.

*Proof.* Set $n = \operatorname{pd} M$, because projectives are in the G-class, we get $\operatorname{G-dim} M \leq n$. By Lemma 2.6 we get $\operatorname{Ext}^n_A(M, R) \neq 0$. Also from $n = \operatorname{pd} M$ we see $\operatorname{Ext}^i_A(M, R) = 0$ for $i > 0$. This shows $\operatorname{G-dim} M = n$; if $\operatorname{G-dim} M \leq n - 1$ then $\operatorname{Ext}^n_A(M, R) = 0$ by Theorem 3.8, which is a contradiction.

□

Now the famous Auslander-Buchsbaum formula is just a corollary.

**Corollary 3.26** (Auslander-Buchsbaum formula)**.** *Let $A$ be a local ring. If $M$ is a finite module of finite projective dimension, then*

$$\operatorname{pd} M = \operatorname{depth} A - \operatorname{depth} M.$$

## 3.5 Bass numbers

**Definition 3.9.** *Let $(A, \mathfrak{m}, k)$ be a local ring and $M$ an $A$-module. We define the Bass numbers as the numbers $\mu^m_A(M) = \dim_k \operatorname{Ext}^m_A(k, M)$.*

This lemma is found as implication (1') $\implies$ (2) in Matsumura and Reid [1989][Theorem 18.1].

**Lemma 3.27.** *If $(A, \mathfrak{m}, k)$ is a local ring with* $\operatorname{injdim} A = n$*, then $\mu^i_A(A) = 0$ for all $i < n$ and $\mu^n_A(A) \neq 0$. In particular,* $\operatorname{depth} A = n$*.*

*Proof.* We proceed by induction on $n$. If $n = 0$, then $\mathfrak{m} \in \operatorname{Ass}_A A$, so there exists a map $k \to A$. This proves $\mu^0_A(A) = \dim_k(\operatorname{Hom}_A(k, A)) \neq 0$. Suppose $n > 0$. In the proof of Theorem 2.15 we saw that $\mathfrak{m}$ contains an $A$-regular element $x$; clearly $\operatorname{Hom}_A(k, A) = 0$, so $\mu^0_A(A) = 0$. Lemma 2.12 shows $\mu^n_A(A) = \mu^{n-1}_{A/(x)}(A/(x))$ which by induction proves the lemma if $\operatorname{injdim}_{A/(x)} A/x = n - 1$. Again from Lemma 2.12 we see $\operatorname{Ext}^n_B(N, B) \cong \operatorname{Ext}^{n+1}_A(N, A)$ for all $B$-modules $N$, and hence by Lemma 2.14 it follows that $\operatorname{injdim}_{A/(x)} A/x = n - 1$.

□

An interesting application of the previous lemma is another original proof for the hard implication of the Auslander-Buchsbaum-Serre theorem. In fact this proof uses less tools than the one given in the previous chapter.

**Theorem 3.28.** *If $(A, \mathfrak{m}, k)$ is a local ring with* $\operatorname{gldim} A < \infty$*, then $A$ is regular.*

*Proof.* Set let $n$ be the length of a minimal generating set for $\mathfrak{m}$ and let $r = \operatorname{pd}_A k$. From Lemma 2.18 we see $r \geq n$ and Lemma 2.6 we have $\mu^r_A(A) \neq 0$. We can also use Theorem 1.8 to see that $\operatorname{injdim} A$ is finite. Finally, Lemma 3.27 gives $\operatorname{depth} A = r \geq n$. This shows $A$ is regular, because depth is bounded by dimension, hence $\dim A = n$.

□

Recall that a local ring $A$ is Cohen-Macaulay if $\operatorname{depth} A = \dim A$.

**Corollary 3.29.** *Local Gorenstein rings are Cohen-Macaulay.*

*Proof.* This follows immediately from Lemma 3.27 and the definition of depth. □

The following is from Matsumura and Reid [1989][Lemma 18.4]

**Corollary 3.30.** *Let $P$ be a prime ideal of a local ring $(A, \mathfrak{m}, k)$ such that the height of $\mathfrak{m}/P$ is $d$. If $\operatorname{Ext}_A^{r+d}(k, M) = 0$ for some finite $M$, then*

$$\operatorname{Ext}^r_{A_P}((A/P)_P, M_P) = 0.$$

*Proof.* Induction on $d$: the case $d = 1$ is covered by Lemma 2.10. Suppose $d > 1$ and take $P_1$ such that $P \subsetneq P_1 \subsetneq \mathfrak{m}$ such that $\operatorname{ht}(\mathfrak{m}/P_1) = 1$. Now by induction we see $\operatorname{Ext}^r_A(A/P_1, M) = 0$ and localizing gives $\operatorname{Ext}^r_{A_{P_1}}((A/P_1)_{P_1}, M_{P_1}) = 0$ by Lemma 1.7. Now we have $\operatorname{ht}((P_1/P)_{P_1}) = d - 1$, so induction gives $\operatorname{Ext}^r_{A_P}((A/P)_P, M_P) = 0$. □

**Lemma 3.31.** *If $f$ is an injective endomorphism of an artinian module $M$, then $f$ is an isomorphism.*

*Proof.* Suppose $f(M) \subset M$. Then $f^i(M) \subset f^{i-1}(M)$ for $i > 0$, because $f$ is injective. This gives rise to a descending chain of submodules of $M$, which has to terminate at some point because $M$ is artinian; this is a contradiction. □

This is a more powerful result about Bass numbers than the one used in Christensen [2000] and it will allow us to simplify the proof of the Gorenstein theorem. This theorem is implication (3) $\Longrightarrow$ (1) in Matsumura and Reid [1989][Theorem 18.1]

**Theorem 3.32.** *If $(A, \mathfrak{m}, k)$ is an $n$-dimensional local ring and $\mu^i_A(A) = 0$ for some $i > n$, then $A$ is Gorenstein. By Theorem 2.15 this is the same as saying $\operatorname{injdim} A = n$.*

*Proof.* By induction: if $n = 0$, then Lemma 2.14 gives $\operatorname{injdim} A < i$ because $\mathfrak{m}$ is the only prime ideal. Suppose $n > 0$ and set $T^i(-) = \operatorname{Ext}^i_A(-, A)$. We wish to show that $T^i(A/P) = 0$ for all prime ideals $P$; this shows $\operatorname{injdim} A < i$ by Lemma 2.14 again. Let $P \neq \mathfrak{m}$ be any prime ideal and let $d = \operatorname{ht}(\mathfrak{m}/P)$ and $B = A_P$. From Corollary 3.30 we see $\operatorname{Ext}^{i-d}_B((A/P)_P, B) = 0$, and since $\dim B = n-d < i-d$ we can use induction to conclude $\operatorname{injdim} A = n-d < n < i$. We have just shown that for all prime ideals $P \neq \mathfrak{m}$ and finite $A$-modules $M$ we have $T(M)_P = 0$, that is $\operatorname{Supp}_A(T(M)) \subseteq \{\mathfrak{m}\}$. Now the filtration from Lemma 3.20 is a composition series for $T(M)$, so $T(M)$ is artinian. Assume, for sake of contradiction, that $P$ is prime ideal of $A$ maximal such that $T(A/P) \neq 0$. We assume $T(k) = 0$, so $P \neq \mathfrak{m}$; this allows us to choose $x \in \mathfrak{m} - P$. $A/P$ is an integral domain so $x$ is a not a zerodivisor on $A/P$. Consider the short exact sequence

$$0 \longrightarrow A/P \xrightarrow{x} A/P \longrightarrow A/(P + x) \longrightarrow 0.$$

Theorem 2.9 gives $T(A/(P + x)) = 0$, because any prime ideal that contains $A/(P + x)$ is strictly larger than $P$. Hence the sequence

$$0 \longrightarrow T(A/P) \xrightarrow{x} T(A/P)$$

is exact. This shows multiplication by $x$ is injective, but we've shown $T(A/P)$ is artinian so by Lemma 3.31 we now have $xT(A/P) = T(A/P)$. $T(A/P)$ is finite, so by Nakayama's lemma $T(A/P) = 0$.

□

## 3.6 Auslander's theorem

Although stated differently, this is analogous to the Auslander–Buchsbaum–Serre theorem. Replacing Gorenstein with regular and G-dimension with projective dimension gives a stronger yet equivalent result to the one given in the previous chapter. The theorem is a slightly modified version of Christensen [2000][1.4.9]. We make the remark that follows after it in Christensen [2000] a part of the statement.

**Theorem 3.33.** *Let $(A, \mathfrak{m}, k)$ be a $n$-dimensional local ring. The following conditions are equivalent:*
*(1) $A$ is Gorenstein.*
*(2)* $\text{G-dim}_A\, k < \infty$.
*(2')* $\text{G-dim}_A\, k = n$.
*(3)* $\text{G-dim}_A\, M < \infty$ *for all finite modules $M$.*
*(3')* $\text{G-dim}_A\, M \leq n$ *for all finite modules $M$.*

*Proof.* (1) $\implies$ (3'): Suppose $A$ is Gorenstein, by Corollary 3.29 we have $n = \operatorname{depth} A$. With all the tools we have, we can proceed by induction on $n$. If $n = 0$, then $A$ is an injective $A$-module, so $\operatorname{Ext}^i_A(-, A) = 0$ for $i > 0$. From eq. (3.1) and Lemma 3.1 we see $\delta_M$ is an isomorphism for all finite $A$-modules $M$; this shows that all finite $A$-modules are in the G-class.

Suppose $n > 0$ and let $M$ be a finite $A$-module. Consider the following cases:
(a) If $\operatorname{depth} M > 0$, we can choose an element $x$ that is both $M$-regular and $R$-regular due to Lemma 3.22. Theorem 3.19 shows the first equality $\text{G-dim}_A\, M = \text{G-dim}_{A/(x)}\, M/xM \leq n - 1$ and the second equality is given by induction.
(b) If $\operatorname{depth} M = 0$ we can do the same as in the proof of Theorem 3.24 and take one step of a G-resolution for $M$, getting a short exact sequence

$$0 \to K \to G \to M \to 0.$$

Now $\operatorname{depth} K = 1$ by Lemma 3.23, so from (a) we have $\text{G-dim}\, K \leq n - 1$. This proves $\text{G-dim}\, M \leq n$.

(2) $\implies$ (1): Suppose $r = \text{G-dim}_A\, k < \infty$. Theorem 3.8 yields $\mu^i_A(A) = 0$ for all $i > r$ which by Theorem 3.32 shows $\operatorname{injdim} A = n$.
(2) $\implies$ (2'): We proved that (2) implies $A$ is Gorenstein, so by Theorem 3.24 we have $\text{G-dim}_A\, k = \operatorname{depth} A = n$.

□

For this theorem to be completely analogous to the one for regular local rings we would want a notion of Gorenstein global dimension. This leads us to the question of generalizing modules in the G-class to infinitely generated modules.

Unfortunately, the theory of Gorenstein projective modules is beyond the scope of this thesis.

We can use this theorem to prove that the Gorenstein property localizes. Note that unlike for regular local rings, this theorem is not necessary.

**Theorem 3.34.** *The localization of a local Gorenstein ring* $(A, \mathfrak{m}, k)$ *at a prime ideal* $P$ *is again a Gorenstein ring.*

*Proof.* We saw in Lemma 1.5 that projective resolutions localize. This same proof can be used for G-resolutions, if we show that modules in the G-class localize too. Suppose $M \in \mathrm{G}(A)$ and $P$ is a prime ideal of $A$. From Lemma 1.7 we see the that $\mathrm{Ext}^i_{A_P}(M_P, A_P) = 0$ and $\mathrm{Ext}^i_{A_P}((M^*)_P = (M_P)^*, A_P) = 0$ for $i > 0$. Isomorphism localize to isomorphism so $(\delta_M)_P$ is an isomorphism and the commutative diagram

$$\begin{array}{ccc} M_P & \xrightarrow{(\delta_M)_P} & \mathrm{Hom}(\mathrm{Hom}(M,R),R)_P \\ \downarrow{=} & & \downarrow{a} \\ M_P & \xrightarrow{\delta_{M_P}} & \mathrm{Hom}(\mathrm{Hom}(M_P,R_P),R_P) \end{array}$$

where $a$ is the isomorphism given by Theorem 1.6 shows $\delta_{M_P}$ is an isomorphism.

Now let $(A, \mathfrak{m}, k)$ be a local Gorenstein ring and $P$ a prime ideal.
Theorem 3.33 shows $\text{G-dim}_A(A/P) < \infty$, hence the previous paragraph shows $\text{G-dim}_{A_p}((A/P)_P) < \infty$ and another application of Theorem 3.33 yields $A_P$ is Gorenstein.

□

# Conclusion

Throughout this text we have seen that homological dimensions are deeply linked to geometric dimensions, and hence, rings from algebraic geometry. My main contribution to this thesis was proving the Auslander–Serre–Buchsbaum theorem using facts about injective dimension. In the third chapter I added some proofs that were omitted in the source material and modified some. Lastly I would like to thank my advisor for being very resourceful and kind.

# Bibliography

Lars W Christensen. *Gorenstein dimensions*. Number 1747. Springer Science & Business Media, 2000.

David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media, 2013.

H. Matsumura and M. Reid. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989. ISBN 9780521367646. URL https://books.google.cz/books?id=yJwNrABugDEC.

J.J. Rotman. *An Introduction to Homological Algebra*. Universitext. Springer New York, 2008. ISBN 9780387683249. URL https://books.google.cz/books?id=P2HV4f8gyCgC.

The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

C.A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994. ISBN 9780521559874. URL https://books.google.cz/books?id=flm-dBXfZ_gC.