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Black Hole Thermodynamics

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I would like to thank my supervisor for always guiding me through the learning journey that was this thesis. My sincere gratitude also goes to my friends who graciously listened to my ramblings through the whole process.

Title: Black Hole Thermodynamics

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Abstract: We use the basic techniques of general relativity to introduce black hole thermodynamics and directly address its relationship with classical thermodynamics. We give as many results explicitly calculated or directly cited as possible to make this thesis useful as a reference work. These results include calculation of surface gravity for a general spherical black hole and evaluation of Euclidean action for simple spacetimes, with the latter enabling one to fix the Bekenstein–Hawking formula for black hole entropy. We then consider the black hole phase transitions and give novel results showing that under simplifying assumptions Schwarzschild-AdS black hole is the only possible static black hole obeying the abridged virial expansion equation of state in the extended phase space.

Keywords: General Relativity, Black Hole Thermodynamics, Phase transitions

Název práce: Termodynamika černých děr

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Abstrakt: Základními technikami obecné relativity jsme představili termodynamiku černých děr a konkrétně zkoumali její vztah s klasickou termodynamikou. Co nejvíce výsledků jsme explicitně spočetli nebo přímo citovali, aby byla tato práce užitečná jako reference pro ostatní. Tyto výsledky zahrnují spočtení povrchové gravitace obecné sférické černé díry a Euklidovské akce jednoduchých prostoročasů, čímž jsme určili Bekenstein–Hawkingův vztah pro entropii černé díry. Poté jsme zkoumali fázové přechody černých děr a podali nové výsledky, které ukazují, že za zjednodušujících předpokladů je Schwarzschild-AdS jediná statická černá díra splňující stavovou rovnici ve tvaru zkráceného viriálního rozvoje pro rozšířený fázový prostor.

Klíčová slova: Obecná relativita, Termodynamika černých děr, Fázové přechody

Contents

Introduction

Since Bekenstein's landmark paper [1] offering a reconciliation of the second law or thermodynamics and the existence of black holes, there have been many efforts to apply the general study of systems embodied in thermodynamics to black holes. First seen only as a mathematical analogy [2], Hawkings's discovery of black hole radiation [3] convinced many that the laws governing black hole behavior, analogous to the laws of thermodynamics, are in fact the manifestation of the laws themselves. Apart from their beauty, they also hint at a connection between general relativity and quantum theory: being both intrinsically relativistic, as pertaining to black holes, and genuinely quantum, as fluctuating fields in the black hole background.

In this work we construct the equations known as the laws of black hole thermodynamics, describing their connection to the conventional laws of thermodynamics. Specifically, our aim is to describe the relationship for a reader familiar with general relativity equivalent to the customary "first course of general relativity", thus in some places we derive well-known results. The other aim is to provide a reference work for the important, but often scattered, results, thus we explicitly present detailed calculations wherever possible and refer to the literature for more tedious derivations.

In particular, the derivation of the laws themselves is often a fairly technical computation; thus in the main text they are only motivated, and the mathematics with all the necessary tools are gathered in the appendix. Due to the level intended for this work, we also only motivate the result for the temperature felt by an accelerated observer and, therefore, also a temperature of thermal radiation emitted by a black hole. To derive these results in a rigorous way, quantum field theory on curved spacetime is needed, which we do not presume the reader to be familiar with.

In the first part, we concern ourselves with the temperature of the black hole and its corresponding classical quantity – the surface gravity. We motivate their proportionality by means of a Rindler observer and purely thermodynamic considerations, also giving a classical resolution to the paradox of Geroch's engine in the process. Thus we land on the zeroth and third laws of black hole thermodynamics. The zeroth law is derived in part in the main text, and then more rigorously, employing the properties of geodesic congruences, in the appendix. By considering a stronger statement of the Raychaudhuri equation, we shorten the classical proof of Wald [4]. A rigorous derivation of the third law requires more than we presume of the reader, so instead we give a reference and a review of the discussion of its validity.

In the second part, we discuss the black hole entropy, proportional to the black hole horizon area, as first introduced by Bekenstein, and fix the proportionality constant, first derived by Hawking. We arrive at this constant of proportionality from multiple standpoints; for example, we employ the Euclidean path integral approach to thermodynamics to calculate the entropy for multiple spacetimes. Here we also discuss the first and second law.

In the last part we look at relatively recent developments regarding phase transitions of black hole spacetimes in anti-de Sitter (AdS) space, giving novel results for black holes described by the abridged virial expansion equation of state in the extended phase space. In particular, we give general procedure for finding the corresponding static metric and show that under a few simplifying assumptions only Schwarzschild-AdS (S-AdS) black hole spacetime is permitted by the imposed constraints.

Unless otherwise noted, we work in the geometrical units $c = \hbar = k_B = G = 1$ and signature $(-, +, +, +)$.

1 Premilinaries

1.1 The Rindler coordinates

In the article introducing the equivalence and general covariance principle Einstein wrote [5]:

> *"In the discussion that follows, we shall therefore assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system. This assumption extends the principle of relativity to the uniformly accelerated translational motion of the reference system. The heuristic value of this assumption rests on the fact that it permits the replacement of a homogeneous gravitational field by a uniformly accelerated reference system, the latter case being to some extent accessible to theoretical treatment."*

We shall then also start by discussing the uniformly accelerated (Rindler) observer. We are going to show that such an observer perceives the Mikowski vacuum as a thermal state of temperature proportional to the acceleration. Then any static observer near a black hole should be exactly the same as an accelerated observer and therefore see thermal radiation.

1.1.1 Linearly accelerated observer

Consider a Minkowski spacetime with inertial coordinates $x^\mu = (t, x)$, suppressing the remaining coordinates for brevity. Then, let a particle of rest mass m_0 be acted upon by a constant 3-force w.r.t. the above coordinates, such that it is in the direction of x and its magnitude is f, that is

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f = \text{const}\,. \tag{1.1}$$

It then follows that (using the relativistic formula for the momentum)

$$p = ft = \frac{m_0 v}{\sqrt{1-v^2}} \qquad \Rightarrow \qquad v = \frac{\frac{ft}{m_0}}{\sqrt{1+\frac{f^2t^2}{m_0^2}}}\,, \tag{1.2}$$

or, by integration:

$$x = \int_0^t \frac{\frac{ft}{m_0}}{\sqrt{1+\frac{f^2t^2}{m_0^2}}}\,\mathrm{d}t = \frac{m_0}{f}\sqrt{1+\frac{f^2t^2}{m_0^2}}\,, \tag{1.3}$$

where we have set integration constants to zero to describe acceleration from rest from the point $x(t=0) = \frac{m_0}{f}$.

Calculating the proper time τ of the observer by integrating

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1-v^2} = \frac{1}{\sqrt{1+\frac{f^2t^2}{m_0^2}}} \qquad \Rightarrow \qquad \frac{ft}{m_0} = \sinh\left(\frac{f\tau}{m_0}\right)\,, \tag{1.4}$$

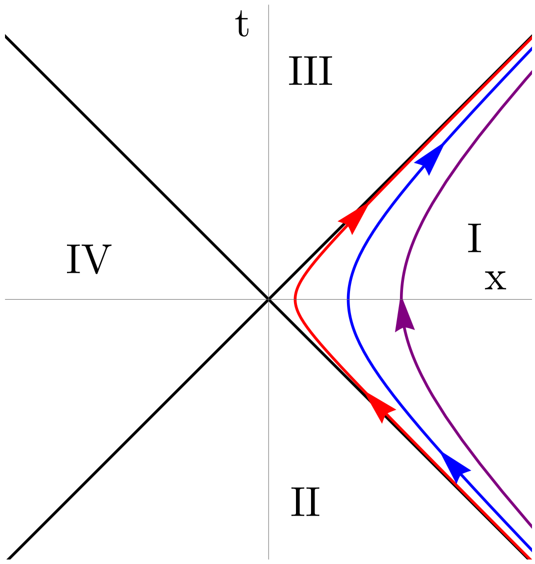

Figure 1.1: The Rindler coordinates with a few plotted lines of $X = \text{const}$

and substituting into the above relation (1.3) we obtain a trajectory of a uniformly accelerated observer:

$$t = \frac{1}{a} \sinh a\tau \,, \tag{1.5}$$

$$x = \frac{1}{a} \cosh a\tau \,, \tag{1.6}$$

where we have set $a \equiv f/m_0$.

We thus define the *Rindler coordinates* $X^\mu = (\tau, X)$, $\tau \in \mathbb{R}$, $X \in (0, \infty)$, using the relations

$$\begin{aligned} t &= X \sinh a\tau \,, \\ x &= X \cosh a\tau \,. \end{aligned} \tag{1.7}$$

Using the metric transformation rule, $g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu}$, the Minkowski metric now reads

$$\mathrm{d}s^2 = -X^2 a^2 \mathrm{d}\tau^2 + \mathrm{d}X^2 \,. \tag{1.8}$$

These coordinates cover only a region of spacetime called the (right) Rindler wedge, in Figure 1.1 denoted by I. Its boundary is the lightcone highlighted in the figure. Since light cones are invariant, no signals from region III can reach an accelerated observer, and any signal sent by her to region III cannot be returned. Thus, the light cone serves as a causal boundary – a *Rindler horizon*, formally described as $X = 0$.

Now we wish to make the the following identification:

$$\mathrm{i}\tau = \varphi \,. \tag{1.9}$$

Our motivation is threefold: for one it is fairly standard trick in special relativity in its early days, secondly the metric becomes a familiar one – in form identical to the Euclidean metric written in polar coordinates, and lastly it is Wick rotation, which will prove instrumental in our discussion of black hole temperature. With renaming $\rho \equiv X$ we get

$$ds^2 = \rho^2 \, d(a\varphi)^2 + d\rho^2 \,. \tag{1.10}$$

To ensure regularity at the origin, we must take φ to be ($\beta = 2\pi/a$)-periodic.[1] This observation will turn out to be important in Section 2.2.2, where we shall argue that the periodicity of the Wick-rotated time φ is related to the temperature as

$$T = \frac{1}{\beta} = \frac{a}{2\pi} \,. \tag{1.11}$$

It can be shown that this is precisely the temperature of thermal bath observed by an accelerated observer.

1.2 Black holes

Black holes are interesting objects in general relativity for a multitude of reasons. For one, they are simple macroscopic objects exactly described by general relativity, but they also show the relativistic features of spacetime to a measurable degree. First believed to be a purely mathematical construct, black holes have since been observed in multiple ways – for example, by direct interferometry producing the famous black hole image [7] or by observing the gravitational waves emitted by the merger of two black holes [8].

In general, a black hole (interior) is a region of spacetime causally disconnected from the rest, i.e., a signal cannot be sent from within a black hole to any outside observer. The boundary of a black hole is called the *event horizon*.

In this work, we shall focus on theoretical aspects of black holes, namely their thermodynamics. In this task, we shall restrict to *stationary*, *eternal* black holes. Stationarity is a feature of the whole spacetime and means the existence of a Killing vector timelike at infinity. A special case is a static spacetime that is invariant under “time” inversion, w.r.t. this Killing vector. By eternal we mean a black hole as a solution to the vacuum Einstein equations, not a feature of spacetime dynamically formed during its evolution. We shall also ignore the backreaction of various processes that occur in the black hole vicinity on the black hole geometry, unless explicitly stated.

Hawking showed [9] that any stationary black hole spacetime must be either static or axisymmetric. Thus, for a stationary rotating black hole, which is necessarily axisymmetric, there are two Killing vector fields t^μ, ϕ^μ, for which Hawking derived

$$k^\mu = t^\mu + \Omega_H \phi^\mu \,, \tag{1.12}$$

where Ω_H is the black hole’s angular frequency and k^μ is null at the horizon. In particular, for the static case $\Omega_H = 0$, so $k^\mu = t^\mu$. Therefore, any event horizon is a Killing horizon, i.e. a null hypersurface on which the norm of a Killing vector

[1] Why the regularity of the manifold should be preserved by taking a complex time is not obvious, more so when physical consequences are derived from it.

Physically, a different period would correspond to matter in the origin of the coordinates, but that is not the case by assumption. A mathematical argument can be made using theory beyond the scope of this work. For a formal treatment, see, e.g., [6].

is zero. Penrose [10] also showed that any event horizon is generated by null geodesics, that have no future end points.

Note that all future history of a spacetime must be known to determine the position of an event horizon. This “teleological” property makes it awkward to use, for example, in numerical general relativity. For this reason, terms such as apparent horizon or trapped surface (important especially in the dynamical settings) were devised in literature, but these notions go beyond the scope of this work.

2 Laws of black hole thermodynamics

2.1 Surface gravity

2.1.1 Definition of surface gravity and physical interpretation

For any stationary black hole, its horizon is generated by a Killing field k^μ. Since the horizon is a null surface and k^μ is normal to it $k^\mu k_\mu = 0 = \text{const}$, and therefore $(k^\mu k_\mu)_{;\nu}$ is also normal to the horizon. For some function κ it must be the case that

$$(k^\mu k_\mu)^{;\nu} = -2\kappa k^\nu \,. \tag{2.1}$$

Writing out the covariant derivative using the Leibniz rule and the fact that k^μ is a Killing vector

$$k_{\mu;\nu} k^\mu = -k_{\nu;\mu} k^\mu = -\kappa k_\nu \,, \tag{2.2}$$

from which we get a more conventional defining relationship for κ

$$k^\mu{}_{;\nu} k^\nu = \kappa k^\mu \,. \tag{2.3}$$

As is clear from the derivation, the quantities are defined only on the horizon. It turns out that κ has a physical interpretation, it is called the *surface gravity*. To see this, we first turn to some mathematical preliminaries [11].

Consider a family of surfaces $\Phi = \text{const}$ for a scalar function Φ, and a vector field orthogonal to it, which can therefore be written as

$$n_\mu = -\lambda \Phi_{,\mu} \tag{2.4}$$

differentiating this relationship gives

$$n_{\mu;\nu} = -\lambda \Phi_{;\mu\nu} - \lambda_{;\nu} \Phi_{;\mu} \tag{2.5}$$

Using these two results yields

$$n_{[\eta} n_{\mu;\nu]} = 0 \,, \tag{2.6}$$

which can be seen be writing out all the terms and using the fact that $\Phi_{;\mu\nu} = \Phi_{;\nu\mu}$. The result (2.6) and its converse are called the Frobenius theorem.

Since the horizon is a null hypersurface we get

$$k_{[\eta} k_{\mu;\nu]} = 0 \,. \tag{2.7}$$

Thanks to the Killing equation the expression simplifies to

$$k_\eta k_{\mu;\nu} + k_\nu k_{\eta;\mu} + k_\mu k_{\nu;\eta} = 0 \,, \tag{2.8}$$

contracting (2.8) by $k^{\mu;\nu}$ leads to

$$\begin{aligned} k_\eta k^{\mu;\nu} k_{\mu;\nu} &= -k_\nu k_{\eta;\mu} k^{\mu;\nu} - k_\mu k_{\nu;\eta} k^{\mu;\nu} \\ &= -k_\nu k_{\mu;\eta} k^{\nu;\mu} - k_\mu k_{\nu;\eta} k^{\mu;\nu} \\ &= \kappa k^\mu k_{\mu;\eta} + \kappa k^\nu k_{\nu;\eta} \\ &= -2\kappa^2 k_\eta \,, \end{aligned} \tag{2.9}$$

which gives another useful expression for κ

$$\kappa^2 = -\frac{1}{2} k^{\mu;\nu} k_{\mu;\nu} \,. \tag{2.10}$$

For every Killing vector it holds that

$$\begin{aligned} 3k^{[\eta} k^{\mu;\nu]} k_{[\eta} k_{\mu;\nu]} &= \frac{1}{3} \left[\left(k^\eta k^{\mu;\nu} + k^\nu k^{\eta;\mu} + k^\mu k^{\nu;\eta} \right) \left(k_\eta k_{\mu;\nu} + k_\nu k_{\eta;\mu} + k_\mu k_{\nu;\eta} \right) \right] \\ &= k^\eta k_\eta k^{\mu;\nu} k_{\mu;\nu} + k^\eta k^{\mu;\nu} k_\nu k_{\eta;\mu} + k^\eta k^{\mu;\nu} k_\mu k_{\nu;\eta} \\ &= k^\eta k_\eta k^{\mu;\nu} k_{\mu;\nu} - 2k^\eta k^{\mu;\nu} k_\nu k_{\mu;\eta} \,. \end{aligned} \tag{2.11}$$

On the horizon gradient of the original expression must vanish due to the Frobenius theorem, while the gradient of $k^\eta k_\eta$ does not (it is proportional to κ which we consider to be nonzero). Then by dividing the whole expression by $k^\eta k_\eta$, taking a limit to the horizon and applying the l'Hospital's rule, the LHS is zero. This yields

$$\kappa^2 = \lim \frac{-k^\mu{}_{;\nu} k^\nu k_{\mu;\eta} k^\eta}{k^\eta k_\eta} \,. \tag{2.12}$$

Let us denote $a^\mu \equiv k^\mu{}_{;\nu} k^\nu / (-k^\eta k_\eta)$, $a \equiv \sqrt{-a^\mu a_\mu}$, and $V \equiv \sqrt{-k^\eta k_\eta}$ so we finally obtain

$$\kappa = \lim V a \,. \tag{2.13}$$

This formula precisely gives direct physical interpretation of κ. To see this consider one observer staying in place near a black hole and another one at infinity.

If one wishes to define "staying in place" in a curved space time, sensible definition is that with local motion the local metric does not change. Then the 4-velocity is a (normalised) Killing vector, in our case $u^\mu = k^\mu / V$. Using the "comma-to-semicolon rule" we might define 4-acceleration as $a^\mu = u^\mu{}_{;\nu} u^\nu$. So

$$a^\mu = \frac{k^\mu{}_{;\nu} k^\nu V - \frac{1}{V^2} k^\mu k^\eta k_{\nu;\eta} k^\nu}{V^2} = \frac{1}{V} V^{;\mu} \,, \tag{2.14}$$

where we used the Killing equation. We also see that our two definitions of a^μ match.

Consider now a static spacetime, i.e., "$k^\mu \to \partial_t$", then we define energy at infinity of a particle as $E = -m k^\mu u_\mu$, m the mass as measured at infinity, in our case then $E = mV$. Let an observer at infinity "drop a string" to an observer near a black hole to produce the 4-acceleration. Force at infinity is simply

$$F_\infty = \left[(-E^{;\mu})(-E_{;\mu}) \right]^{\frac{1}{2}} = m \left[V^{;\mu} V_{;\mu} \right]^{\frac{1}{2}} \,. \tag{2.15}$$

For an observer/particle near the horizon it must be the case that $F^\mu = m a^\mu$, using (2.14)

$$F = \frac{m}{V} \left[V^{;\mu} V_{;\mu} \right]^{\frac{1}{2}} \,, \tag{2.16}$$

therefore $F_\infty/m = VF/m = Va$ which is near the horizon the same as expression (2.13)! We see that κ is the acceleration of a particle near a horizon as viewed by an observer in infinity, which is a good definition of surface gravity.[1]

2.1.2 Surface gravity as acceleration in the Rindler frame

Having identified surface gravity with a local acceleration viewed at infinity we can use locally Rindler frames near the horizon to describe the gravitational field as it is locally indistinguishable from an accelerated frame.

Concretely, consider a metric of the form

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{2.17}$$

where $\mathrm{d}\Omega^2$ is the line element on S_2. Clearly ∂_t is a Killing vector with norm $f(r)^{1/2}$. Therefore, its norm vanishes at some r_h such that $f(r_h) = 0$.

Using a Taylor expansion of $f(r)$ at r_h

$$f(r) = 0 + f'(r)(r - r_h) + \mathcal{O}\left((r - r_h)^2\right)\,. \tag{2.18}$$

Near the horizon, we can limit ourselves to the first order, finding the metric (2.17) takes the form

$$\mathrm{d}s^2 = -f'(r_h)(r - r_h)\,\mathrm{d}t^2 + \frac{1}{f'(r_h)(r - r_h)}\,\mathrm{d}r^2 + r_h^2\,\mathrm{d}\Omega^2 \tag{2.19}$$

Using a substitution $\rho = 2\sqrt{\frac{r-r_h}{f'(r_h)}}$ and denoting $\kappa = f'(r_h)/2$ we get

$$\mathrm{d}s^2 = -\rho^2\kappa^2\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + r_h^2\,\mathrm{d}\Omega^2\,, \tag{2.20}$$

which is exactly the metric (1.8) where κ plays the role of the acceleration as expected.

2.1.3 Calculating the surface gravity from definition

To dispel any doubts about the identification of κ in (2.20) as a surface gravity, we shall calculate it for the metric of the form (2.17) from definitions (2.3) and (2.10).

In the case of metric (2.17) the vector k^μ takes the form ∂_t or in coordinate expression $k^\mu = \delta^\mu_t$, so we get equation (2.3) as

$$\begin{aligned} k^\mu{}_{;\nu}k^\nu &= \left(\frac{\partial \delta^\mu_t}{\partial x^\nu} + \Gamma^\mu{}_{\nu\sigma}\delta^\sigma_t\right)\delta^\nu_t = \Gamma^\mu{}_{tt} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu t,t} + g_{t\nu,t} - g_{tt,\nu}\right) \\ &= -\frac{1}{2}g^{rr}g_{tt,r}\delta^\mu_r = \frac{f'(r)}{2}\left(f\partial_r\right)\,. \end{aligned} \tag{2.21}$$

One would expect the RHS of (2.21) to be proportional to ∂_t. It in fact is, although obscured by the degeneracy of the coordinates at the horizon – but then we claimed the validity of (2.3) only at the horizon.

[1] Locally the force is of course infinite. One might also wonder what to do if the space is not asymptotically flat or being at rest in infinity is not possible - we use the term surface gravity regardless to refer to κ defined more generally by (2.3).

To show this degeracy, we will follow the discussion of [11] and show that *at the horizon* $f\partial_r = \partial_t$.

For this we are going to use the Kruscal–Szekeres coordinates X^μ, which are non-singular at the horizon, defined as

$$U = -\frac{1}{\kappa}\mathrm{e}^{-\kappa u}\,, \quad V = \frac{1}{\kappa}\mathrm{e}^{\kappa v}\,, \tag{2.22}$$

where

$$u = t - r^\star\,, \quad v = t + r^\star\,; \quad r^\star \equiv \int \frac{dr'}{f(r')}\,, \quad \kappa = f'(r_h)/2\,. \tag{2.23}$$

The $r^\star$ is called "the tortoise coordinate" as it grows logarithmically. Then

$$\partial_t \equiv \frac{\partial}{\partial t} = \frac{\partial X^\mu}{\partial t}\frac{\partial}{\partial X^\mu} = \kappa\left(-U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}\right)\,, \tag{2.24}$$

$$\partial_r \equiv \frac{\partial}{\partial r} = \frac{\partial X^\mu}{\partial r}\frac{\partial}{\partial X^\mu} = \frac{1}{f}\frac{\partial X^\mu}{\partial r^\star}\frac{\partial}{\partial X^\mu} = \frac{\kappa}{f}\left(U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}\right)\,. \tag{2.25}$$

Noting that at the horizon $U = 0$, we get exactly

$$\begin{aligned}\partial_t|_{\text{horizon}} &= \kappa\left(-U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}\right)\bigg|_{\text{horizon}} = f\frac{\kappa}{f}\left(U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}\right)\bigg|_{\text{horizon}} = \\ &= f\partial_r|_{\text{horizon}}\,. \end{aligned}\tag{2.26}$$

Now equipped with using the equality (2.26), we can continue the equality (2.21)

$$k^\mu{}_{;\nu}\, k^\nu|_{\text{horizon}} = \frac{f'(r)}{2}\left(f\partial_r\right)\bigg|_{\text{horizon}} = \frac{f'(r_h)}{2}\left(\partial_t\right)\bigg|_{\text{horizon}} = \kappa k^\mu\;|_{\text{horizon}}\,, \tag{2.27}$$

where we used the definition (2.3), from which we immediately get

$$\kappa = \frac{f'(r_h)}{2}\,. \tag{2.28}$$

The alternative definition (2.10) allows us to compute κ directly from the form (2.17). Again $k^\mu = \delta^\mu_t$, $k_\mu = -f\delta^t_\mu$, so

$$k_{\mu;\nu} = \frac{\partial k_\mu}{\partial x^\nu} - \Gamma^\sigma{}_{\nu\mu}k_\sigma = -f_{,\nu}\delta^t_\mu - \Gamma_{t\nu\mu} \tag{2.29}$$

$$k^{\mu;\nu} = g^{\nu\sigma}k^\mu{}_{;\sigma} = g^{\nu\sigma}\left(\frac{\partial k^\mu}{\partial x^\sigma} + \Gamma^\mu{}_{\sigma\rho}k^\rho\right) = g^{\nu\sigma}\Gamma^\mu{}_{\sigma t} \tag{2.30}$$

$$\begin{aligned}k^{\mu;\nu}k_{\mu;\nu} &= g^{\nu\sigma}\Gamma^\mu{}_{\sigma t}\left(-f_{,\nu}\delta^t_\mu\right) - g^{\nu\sigma}\Gamma^\mu{}_{\sigma t}\Gamma_{t\nu\mu} \\ &= -\frac{1}{2}g^{rr}g^{tt}g_{tt,r}f_{,r} - 0 \\ &= -\frac{1}{2}\left(f'\right)^2 \end{aligned}\tag{2.31}$$

Using the formula (2.10) one exactly re-creates $\kappa = \pm f'(r)/2$ [2].

[2] Here we take the plus sign, but it is in fact WLOG. To see this, one can look at (2.3) and note that any scalar multiple of k fits the definition. In asymptotically flat spacetimes this issue is readily resolved by demanding $(k^\mu)^2 = -1$ at infinity.

2.1.4 Calculation in Eddington–Finkelstein coordinates

In the Schwarzschild-like coordinates (2.17), the horizon, located at $r = r_h$ is only covered as $t \to \infty$. Thus, one might have doubts about the above results. Let us then consider the Eddington-Finkelstein coordinates (v,r) where v is defined by (2.23). Since in terms of the Schwarzschild-like coordinates $\mathrm{d}v = \mathrm{d}t + f^{-1}\,\mathrm{d}r$

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2\,. \tag{2.32}$$

The inverse metric thus reads

$$\partial s^2 = 2\partial v\partial r + f(r)\partial r^2 + r^{-2}\partial\Omega^2\,. \tag{2.33}$$

To get the Eddington-coordinates expression of ∂_t consider

$$\frac{\partial}{\partial t} = \frac{\partial X^\mu}{\partial t}\frac{\partial}{\partial X^\mu} = \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = \frac{\partial}{\partial v}\,, \tag{2.34}$$

where X^μ denotes the new coordinates.

Finally using the definition (2.3), we have

$$k^\mu{}_{;\nu}k^\nu = \kappa k^\mu\,, \quad \delta^\mu_{t;\nu}k^\nu = \kappa\delta^\mu_t\,, \quad \Gamma^\mu{}_{\nu\sigma}\delta^\nu_v\delta^\sigma_v = \kappa\delta^\mu_v\,, \quad \Gamma^v{}_{vv} = \kappa\,. \tag{2.35}$$

But then

$$\Gamma^v{}_{vv} = \frac{1}{2}g^{v\sigma}\left(2g_{v\sigma,v} - g_{vv,\sigma}\right) = -\frac{1}{2}g^{vr}g_{vv,r} - \frac{1}{2}g^{vv}g_{vv,v} = -\frac{1}{2}g^{vr}\left(-f'\right) = \frac{f'}{2}$$
$$\kappa = \frac{f'}{2}\,, \tag{2.36}$$

exactly as expected.

2.1.5 0th law of black hole thermodynamics

There is a remarkable result: κ is constant on the event horizon of a stationary black hole. This fact, called the 0th law of black hole thermodynamics, should seem reasonable - it is exactly what we got in many ways for the metric (2.17).

To prove 0th law, people typically consider two possible sets of conditions [12]: i) Assuming that the horizon is a Killing horizon and the black hole is either static or axisymmetric and "$t - \phi$" reflection symmetric, in which case no field equations (or energy conditions) have to be imposed and ii) assuming only stationarity together with Einstein field equations and the dominant energy condition.

Here we shall show that κ is constant along orbits of k^μ [11], second part of the proof with the second set of assumptions can be found in the appendix.

Let us first derive a mathematical preliminary valid for all Killing vectors. Let k^μ to be a Killing vector. The Ricci identity reads

$$k_{\mu;\nu\eta} - k_{\mu;\eta\nu} = R_{\eta\nu\mu}{}^\rho k_\rho\,, \tag{2.37}$$

then, using the Killing equation and permutation, we see that

$$\begin{aligned} k_{\mu;\nu\eta} + k_{\eta;\mu\nu} &= R_{\eta\nu\mu}{}^\rho k_\rho\,,\\ -k_{\eta;\mu\nu} - k_{\nu;\eta\mu} &= -R_{\nu\mu\eta}{}^\rho k_\rho\,,\\ k_{\nu;\eta\mu} + k_{\mu;\nu\eta} &= R_{\mu\eta\nu}{}^\rho k_\rho\,. \end{aligned} \tag{2.38}$$

Summing the equations (and "adding a zero" at the end) leads to

$$2k_{\mu;\nu\eta} = (R_{\eta\nu\mu}{}^{\rho} + R_{\nu\mu\eta}{}^{\rho} + R_{\mu\eta\nu}{}^{\rho} - 2R_{\nu\mu\eta}{}^{\rho})k_{\rho}\,. \tag{2.39}$$

The sum of the first three terms on the RHS is zero due to the second Ricci identity, which finally yields

$$k_{\mu;\nu\eta} = -R_{\nu\mu\eta}{}^{\rho}k_{\rho}\,. \tag{2.40}$$

Consider now the change of κ^2 along k^μ given as a Killing generator of the horizon, calculated on the horizon. Using equations (2.10) and (2.40)

$$(\kappa^2)_{;\rho}k^{\rho} = -\frac{1}{2}(k^{\mu;\nu}k_{\mu;\nu})_{;\rho}k^{\rho} = -k^{\mu;\nu}k_{\mu;\nu\rho}k^{\rho} = k^{\mu;\nu}R_{\nu\mu\rho\sigma}k^{\rho}k^{\sigma} = 0\,, \tag{2.41}$$

where we used the (anti)symmetry of the Riemann tensor in the last two indices. Therefore κ is constant along the orbits of k^μ.[3] That the only change is along k^μ is proven in the appendix in section A.1.

[3] A skillful geometer can see this immediately by taking the Lie derivative of equation (2.1) and getting $\pounds_k\kappa = 0$.

2.2 Temperature

In this section we show that the temperature of a black hole can be identified as $T = \kappa/2\pi$. That the temperature of a black hole is proportional to its surface gravity can surprisingly be gleaned from purely classical treatment, but the full explanation requires QFT. There are three main ways to define temperature in thermodynamics that we know of:

1. an attribute of a system, that characterises thermal equilibrium, this way one gets *empirical temperature*,
2. a parameter of thermal baths between which a reversible engine operates, one gets *thermodynamic temperature*,
3. and an inverse of a derivative of the fundamental entropy equation w.r.t. internal energy.

We shall therefore be careful in what sense we talk about the temperature.

2.2.1 Geroch's engine

Consider now a particular example of a spherical metric (2.17), the familiar Schwarzschild metric, known to describe a vacuum spherically symmetric black hole [13]:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{2.42}$$

This metric is asymptotically flat with a timelike Killing vector ∂_t so energy is well defined – it is the projection of the 4-momentum onto ∂_t.

Geroch proposed [14] a thought experiment, which shows that the thermodynamic temperature of such black holes should be zero: Consider an observer at infinity slowly lowering an isolated body of rest mass m and nonzero temperature towards the black hole via a string, see Figure 2.1. When it is near the horizon, its energy as measured at infinity $E = m(1 - 2M/r)^{1/2}$ goes to zero (because of the redshift), thus the body has done work m.

Let the isolation then be undone and the body radiates energy Δm, then it is carried back to infinity by expending energy $m - \Delta m$.

In this procedure, the quantity of heat Δm has been completely converted to work, thus violating the second law of thermodynamics in Planck's formulation: "It is impossible to construct an engine which will work in a complete cycle, and produce no effect except the production of work and cooling of a heat reservoir".

This device is called the Geroch engine and has produced much discussion [15]. It seems to show that black hole must necessarily have zero temperature.[4], which turns out not to be the case, as we shall show.

[4]Efficiency of any engine working between two (thermodynamic) temperatures T_h, T_c, $T_c < T_h$ has efficiency $\eta \leq 1 - T_c/T_h$. Geroch's engine has $\eta = 1$ (all heat is transformed into work), which implies $T_c = 0$.

Roughly, we shall show that black holes behave as a canonical ensemble, which describes a system in equilibrium with a reservoir of constant temperature, which describes the *empirical* temperature. The connection of empirical and thermodynamic temperature rests on the possibility of connecting two such objects via a Carnot engine, which could very well mean that these two temperatures for black holes differ, since it is not obvious that such a connection is possible.[5]

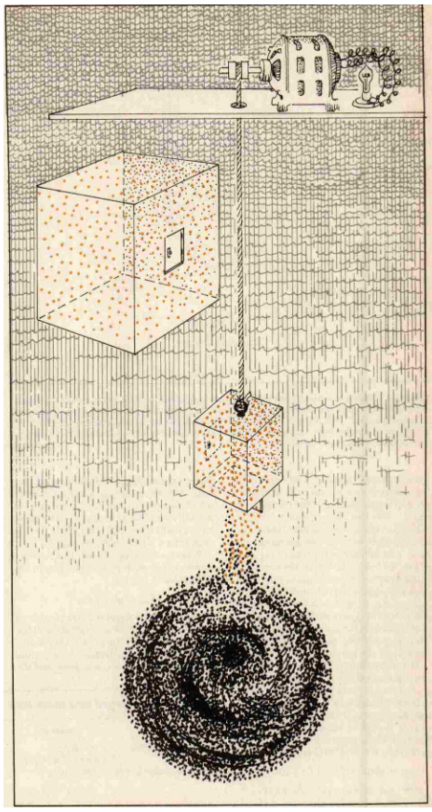

Figure 2.1: Artistic representation of the Geroch's engine [16]

There is a consensus that Geroch's result stems from an unphysical limit of lowering the mass to the horizon with various authors giving different solutions, e.g., [1][15]. At the classical level there is a resolution that points to the connection of κ with temperature that we shall describe here [12].

Consider two static black holes sufficiently far apart; in particular, let them be described at least in their neighborhood by a metric of the form (2.17), we denote their surface gravities κ_1, κ_2.

Let a massless box capture heat of rest mass m in a small distance d above the first black hole. By physical interpretation of κ, we know that the Killing energy must be $E_1 \approx m\kappa_1 d$. Then the box is slowly lifted and lowered to distance d above the second black hole, its energy $E_2 \approx m\kappa_2 d$. The useful work extracted is $E_1 - E_2$ with an input of E_1 near the first black hole, therefore the efficiency is $\eta = 1 - E_2/E_1$. Then, exactly as with the Carnot engine, we *define* $T_2/T_1 \equiv E_2/E_1$. Therefore, near the horizon $E_2/E_1 \approx \kappa_2/\kappa_1$, in limit $d \to 0$,[6]

$$\frac{T_2}{T_1} = \frac{\kappa_2}{\kappa_1} \implies T \propto \kappa \,. \tag{2.43}$$

2.2.2 Thermal states

For a more direct, but quantum, approach, we first show, that a thermal state in QFT corresponds to periodicity in complex time [4].

Consider a thermal density matrix for inverse temperature β

$$\hat{\rho} = \exp\Big(-\beta \hat{H}\Big)/Z \,, \tag{2.44}$$

[5]Strictly speaking, empirical temperature has the same value for object in thermal equilibrium but otherwise arbitrary. Here we naturally define that a body has a higher temperature if heat flows spontaneously from it to another - then above discussion holds.

[6]The unphysical nature of the limit in most discussion stems from quantum effects on the horizon, see sources above. On the classical level, we note that any physical rope snaps before reaching the horizon[17].

where $\hat{H}$ is the time-independent Hamiltonian[7], where $Z = \mathrm{Tr}\left(-\beta\hat{H}\right)$, then expected value of an observable A is

$$\left\langle \hat{A} \right\rangle_{\hat{\rho}} = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right), \tag{2.45}$$

and its time evolution is described (in Heisenberg picture) as

$$\hat{A}(t+t_0) = \mathrm{e}^{\mathrm{i}\hat{H}t}\hat{A}(t_0)\mathrm{e}^{-\mathrm{i}\hat{H}t}. \tag{2.46}$$

One notices the similarity between the evolution operator and the equation (2.44) given by $\beta = \mathrm{i}t$, which we will exploit. We define the thermal Feynman propagator at the inverse temperature β as

$$\mathrm{i}G_F(x_1, x_2) = \mathrm{Tr}[\hat{\rho}\mathrm{T}\phi(x_1)\phi(x_2)] = \frac{1}{Z}\mathrm{Tr}\left[\exp\left(-\beta\hat{H}\right)\mathrm{T}\phi(x_1)\phi(x_2)\right], \tag{2.47}$$

where $\phi(x_1)$ is the field observable operator, i.e., giving the value of the field, and T is the time ordered product. It can be shown that G_F is essential to the notion of particles in curved spacetime and therefore to thermal equilibrium [4].

If it is possible to analytically continue t, let $t = -\mathrm{i}\tau$, such that equations (2.46) and (2.47) continue to hold. Let then x_1' be the point x_1 translated in imaginary time by β and for imaginary times of x_1, x_2 denoted τ_1, τ_2, let $\tau_2 > \tau_1 > \tau_2 - \beta$ Then

$$\begin{aligned}
\mathrm{i}G_F(x_1', x_2) &= \frac{1}{Z}\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\mathrm{T}\phi(x_1')\phi(x_2)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\phi(x_1')\phi(x_2)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\mathrm{e}^{\beta\hat{H}}\phi(x_1)\mathrm{e}^{-\beta\hat{H}}\phi(x_2)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\phi(x_1)\mathrm{e}^{-\beta\hat{H}}\phi(x_2)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\mathrm{e}^{-\beta\hat{H}}\phi(x_2)\phi(x_1)\right] \\
&= \frac{1}{Z}\mathrm{Tr}\left[\exp\left(-\beta\hat{H}\right)\mathrm{T}\phi(x_1)\phi(x_2)\right] = \mathrm{i}G_F(x_1, x_2),
\end{aligned}$$

from which it is obvious that if $|\tau_1 - \tau_2| < \beta$ then $G_F(x_1, x_2)$ is β-periodic in complex time for both arguments - therefore this periodicity is characteristic for thermal states.[8]

Let us consider the formula (1.10). We precisely got $2\pi/a$-periodicity of the metric in the complex time which necessitates that every continuous function has this periodicity, in particular G_F which is analytic. Symbolically

$$T = \beta^{-1} = a/2\pi, \tag{2.48}$$

But then we identified in equation (2.20) a with κ so

$$T = \frac{\kappa}{2\pi}. \tag{2.49}$$

[7] As noted in [4], in QFT Hamiltonian cannot be rigorously defined "and, in any case, $\exp\{-\beta H\}$ would not define a normalizable density matrix", but equation (2.44) would still describe at least formally state of thermal equilibrium.

[8] For a more thorough explanation, see [6].

In SI units the temperature is $T = \hbar c^3 \kappa / 2\pi G k_B$, if we consider the Schwarzschild solution, we obtain $\kappa = 1/4M$ therefore $T = \hbar c^3 / 8\pi M G k_B$. For a black hole of mass $8{,}5 \cdot 10^{36}$ kg as is Sagittarius A*, the supermassive black hole in the center of the Milky Way, this turns out to be around 10^{-14} K, which is much lower than the temperature of CMB (about 2,7 K).

2.2.3 3rd law of Black Hole Mechanics

Broadly speaking, the third law of thermodynamics, sometimes called the Nernst postulate, says that the state of absolute zero cannot be reached in a finite number of steps. Looking at the analogies between thermodynamics and our version with black holes, it bars the black hole state of $\kappa \leq 0$.

Indeed, looking at the expression for the Schwarzschild black hole $\kappa = 1/4M$, so κ decreases with increasing mass, but it does not reach zero in finite mass. With κ being the surface gravity, it would also seem peculiar if the surface gravity of a body (as taken at infinity) were zero.

For the most general stationary solution, the Kerr-Newmann black hole, i.e. black hole with mass, spin, and charge the situation is more complicated. The solution describes a black hole for

$$m^2 \geq a^2 + e^2 \tag{2.50}$$

where m is the mass of the black hole, a its angular momentum per unit mass, and e its charge. Explicit calculation shows that this is equivalent to $\kappa \geq 0$ (with the same conditions for equality; see, e.g., [4]). The case of equality of (2.50) is an extreme case, as is infinite mass for the Schwarzschild solution, so such a black hole is called *extremal*.

The third law is closely tied to the cosmic censorship hypothesis, which roughly states that every singularity is enveloped by a horizon and therefore is "censored" from observation. This can be seen from the Kerr-Newmann solution with "$m^2 < a^2 + e^2$" describes a *naked* singularity, i.e., without a horizon.

The cosmic censorship hypothesis has not been proven, but no "reasonable" counterexample nor a disproving observation has been found.

Israel showed [18] that a non-extremal black hole cannot become extremal at finite time for a free-falling observer near the horizon in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the horizon, thus proving the law.

Related are the thought experiments of destroying a black hole, with the seminal work by Wald [19]. With a black hole satisfying (2.50) one can drop test particles with high charge and angular momentum to mass ratios to come arbitrarily close to equality. Wald therefore considered the extremal case and showed that any test particles that would cause the inequality to be violated will not be captured by the black hole.[9]

Another attempt of black hole destruction was due to Hubeny [20]. She showed that a non-rotating black hole can be overcharged by considering a close-to-extremal but non-extremal hole, in contrast to Wald. Crucially, this procedure

[9] It is worth noting that this paper came about 12 years before Israel's proof.

neglected the effects of backreaction on the particle. An analogous result with analogous setup has been obtained by Jacobson and Sotiriou for non-charged spinning black hole [21].

Chirco et al. [22] provided a reconciliation of these results and the Israel's proof by showing that while the body can reach the horizon at a finite proper time, if the process is considered to be continuous, the extremal black hole is not created in a finite time, i.e. the absorption of the particle must be assumed to last infinitely long.

Sorce and Wald [23] gave a resolution in which they generalized the result of [19] to arbitrary matter (with an energy condition on the stress-energy tensor) and by calculating the self-force effects, showed that thought experiments of the Hubeny type cannot succeed in violating inequality (2.50).

Some recent papers also suggest that extremal black holes are in fact more singular than expected and might therefore be un-physical [24].

To summarize, it seems that the third law of thermodynamics may also prevail in black hole thermodynamics.

2.3 Entropy

In his seminal paper on the generalised second law of thermodynamics Bekenstein wrote[1]:

> *"The area of a black hole appears to be the only one of its properties having this entropylike* [sic] *behavior which is so essential if the second law as we have stated it is to hold when entropy goes down a black hole."*

What he was chiefly worried about was that if an observer drops a package of entropy into the black hole, the entropy of the exterior decreases but once the black holes has settled to a new equilibrium, it has only the following degrees of freedom: mass, charge and angular momentum (as seen from the No Hair theorem). Therefore she cannot exclude the possibility that the total entropy of the universe has decreased and thus the second law becomes "transcended".[10]

He then proceeds to formulate the second law as "*Common entropy plus black-hole entropy never decreases*", where black-hole entropy is defined as proportional to its area.

Reasons to define it proportional to area were that Hawking proved that the surface area of a black hole never decreases [9] and Chistodoulou showed that it increases in most processes [25][26]. We shall motivate this definition from multiple perspectives and in doing so fix the proportionality constant of black-hole entropy and its area. We also give proof of the area increase theorem in the appendix.

2.3.1 1st law of black hole thermodynamics

The Schwarzschild metric (2.42) describes a one-parametric family of metrics, where the parameter M is easily associated with mass in the Newtonian limit.

In the metric the hypersurface $r = r_h = 2M$ describes the horizon, its area (at constant time) is $A = 4\pi r_h^2$, and one easily checks that $\kappa = 1/4M$. But then by the first relation $\mathrm{d}M = \mathrm{d}r_h/2$ and by the second $\mathrm{d}A = 8\pi r_h \,\mathrm{d}r_h$, so then

$$\mathrm{d}M = \frac{\kappa}{2\pi}\frac{\mathrm{d}A}{4}\,. \tag{2.51}$$

Since we identified the first fraction with temperature in previous sections and mass is equivalent with energy, then by analogy of the first law of thermodynamics $\delta E = T\delta S$ we must have

$$S_{BH} = \frac{A}{4}\,. \tag{2.52}$$

The equation (2.51) is the 1st law of black hole mechanics restricted to the Schwarzschild solution. More general version with physical derivation can be found in the appendix.

[10] Do note that Bekenstein does not claim that the entropy would be decreased, but only that an outside observer could not exclude the possibility, which then of course makes the second law observationally useless.

2.3.2 Path integral approach

In section 2.2.2 we saw a correspondence between the thermal state of a space and its time evolution. We will use this approach to calculate the partition function [27] from which we get entropy by the standard thermodynamic identities.

Partition function

In QFT to get the amplitude to evolve from a field configuration ϕ_1 at time t_1 to ϕ_2 at t_2, one can calculate

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int \mathrm{d}[\phi] \exp(\mathrm{i}S[\phi])\,, \tag{2.53}$$

where $S[\phi]$ is the action and the integral is over all field configuration from ϕ_1 at t_1 to ϕ_2 at t_2. On the other hand for time independent Hamiltonian

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \left\langle \phi_2 \left| \exp\left(-\mathrm{i}\hat{H}(t_2 - t_1)\right) \right| \phi_1 \right\rangle\,. \tag{2.54}$$

If we now use the Wick rotation in the form $t_2 - t_1 = -\mathrm{i}\beta$ and consider only $\phi_1 = \phi_2$ then summing over all ϕ_1 (exactly the steps of taking a trace), we obtain

$$\int \mathrm{d}[\phi] \exp(\mathrm{i}S[\phi]) = \operatorname{Tr} \exp\left(-\beta \hat{H}\right), \tag{2.55}$$

where we take the integral over all field β-periodic in imaginary time. We then notice that the RHS is just the partition function Z for the canonical ensemble at inverse temperature β. For computation of the path integral it is advantageous to again complexify the metric by aN identification $\tau = -\mathrm{i}t$. We get

$$Z = \int \mathrm{d}[\phi] \exp(-S_E[\phi])\,,$$

where S_E is the euclidean action, i.e. action with the metric of "euclidean" signature $(+, +, +, +)$. Using the saddle point method we can approximate

$$Z \approx \exp\{-S_E[\phi_{\text{classic}}]\}\,, \tag{2.56}$$

where ϕ_{classic} is the classical (but euclidean) stationary metric of the spacetime. Due to the analogue of a canonical ensemble, we identify Helmholtz free energy as

$$F = -\frac{1}{\beta} \ln Z = \frac{S_E[\phi_{\text{classic}}]}{\beta}\,, \tag{2.57}$$

which implies the entropy given as

$$S = -\frac{\partial F}{\partial T}\,. \tag{2.58}$$

Calculation for the Rindler metric

The Euclidean action of general relativity over an open region Ω is given by

$$S_E = \int_\Omega \frac{\mathrm{d}^4 x \sqrt{g} R}{16\pi} - \int_{\partial\Omega} \frac{\mathrm{d}^3 y \sqrt{h} K}{8\pi}\,, \tag{2.59}$$

where g is the determinant of the metric, R is the Ricci scalar, h is the determinant of the induced metric on $\partial\Omega$, K is the second fundamental form, or extrinsic curvature, of $\partial\Omega$.

We take the metric (1.10) with $\varphi \in (0, \beta); \rho \in (0, \infty); y, z \in \mathbb{R}$. The region Ω is given by $\rho < C$, where C is a constant which we shall take to infinity. Thus, the parameterization covers the Rindler wedge and we are mainly interested in the "light cone horizon" $\rho = \text{const}$.

The metric is clearly flat, so the Riemann tensor identically vanishes, thus $R = 0$. For the hypersurface $\rho = C$ the normal is given by $n^\mu = \delta^\mu_\rho$, $\sqrt{h} = \rho a$, so

$$K \equiv n^\mu{}_{;\mu} = \frac{1}{\sqrt{g}} \left(\sqrt{g} n^\mu\right)_{,\mu} = \frac{1}{\rho a} \left(\rho a\, n^\mu\right)_{,\mu} = \frac{1}{\rho} \left(\rho\right)_{,\rho} = \frac{1}{\rho}\,, \tag{2.60}$$

from which

$$S_E = -\lim_{C\to\infty} \int_{\rho=C} \mathrm{d}\varphi\, \mathrm{d}y\, \mathrm{d}z \frac{\rho a}{\rho 8\pi} = -\frac{a}{2\pi} \frac{\beta A}{4}\,, \tag{2.61}$$

where we identified $A = \int \mathrm{d}y\, \mathrm{d}z$. Using equations (2.57), (2.48) and (2.58) we get

$$F = \frac{S_E}{\beta} = -\frac{a}{2\pi}\frac{A}{4} = -T\frac{A}{4}\,, \tag{2.62}$$

$$S = -\frac{\partial F}{\partial T} = \frac{A}{4}\,, \tag{2.63}$$

exactly as expected. Moreover, note that according to laws of thermodynamics, the free energy F is related to the internal energy (mass) as

$$F = M - TS\,. \tag{2.64}$$

This thermodynamic identity can easily be checked upon expecting (2.62) and noticing that for the Rindler space $M = 0$.

Calculation for the Schwarzschild metric

For the Schwarzschild metric (2.42) we take the boundary $r = r_0$, where we then consider the limit $r_0 \to \infty$. The action (2.59) turns out to be infinite but in the same way that an empty space ($M = 0$) would, so we consider a renormalized action

$$S_E = \int_\Omega \frac{\mathrm{d}^4 x \sqrt{g} R}{16\pi} - \int_{\partial\Omega} \frac{\mathrm{d}^3 y \sqrt{h} (K - K_0)}{8\pi}\,, \tag{2.65}$$

where K_0 is the extrinsic curvature of the hypersurface as embedded in a flat spacetime.[11] The R is again 0 thanks to the Einstein equations.

The Euclidean metric reads

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\varphi^2\,, \tag{2.66}$$

for the "Euclidean" coordinates, where $(r,\ \theta,\ \varphi)$ are the usual spherical coordinates and τ is the complex time in the interval $(0, \beta)$. We shall denote, as in

[11] This can be done thanks to the fact that the Schwarzschild metric is asymptotically flat.

previous discussion $f(r) = (1 - 2M/r)$. Square root of its determinant $\sqrt{g} = = r^2 \sin\theta$.

Then the induced metric on the surface $r = r_0$ reads

$$\mathrm{d}s^2_{r=r_0} = f(r_0)\,\mathrm{d}\tau^2 + r_0^2\,\mathrm{d}\theta^2 + r_0^2 \sin^2\theta\,\mathrm{d}\varphi^2\,, \tag{2.67}$$

thus the square root of its determinant $\sqrt{h} = \sqrt{f} r_0^2 \sin\theta$.

The unit normal is given as $n^\alpha = \sqrt{f}\delta_r^\alpha$ so then

$$\begin{aligned} K = n^\alpha{}_{;\alpha} &= \frac{1}{\sqrt{g}}\left(\sqrt{g}n^\alpha\right)_{,\alpha} \\ &= \frac{1}{r^2 \sin\theta}\left(r^2 \sin\theta \sqrt{f}\delta_r^\alpha\right)_{,\alpha} \\ &= \frac{1}{r^2}\left(r^2\sqrt{f}\right)_{,r} \\ &= \frac{2\sqrt{f}}{r} + \frac{f'}{\sqrt{f}}\,. \end{aligned} \tag{2.68}$$

For a flat metric $f = 1$, therefore $K_0 = 2/r$. Then

$$\begin{aligned} S_E &= \lim_{r_0\to\infty} -\frac{1}{8\pi}\int_0^\beta \int_{S^2} r_0^2 \sin\theta\sqrt{f}(K - K_0) \\ &= -\frac{\beta}{2}\lim_{r_0\to\infty} r_0^2(K - K_0) = \frac{\beta M}{2}\,, \end{aligned} \tag{2.69}$$

where we used the easily checked fact that for large r we can write $\sqrt{f}(K - K_0) = = -M/r^2 + \mathcal{O}(r^{-3})$. Thus by using equation (2.57) $F = M/2$. By earlier results, for the Schwarzschild black hole $T = 1/8\pi M = 1/4\pi r_h$, r_h being the black hole radius, which yields

$$F = \frac{1}{16\pi T}\,, \tag{2.70}$$

so then by equation (2.58)

$$S = -\frac{\partial F}{\partial T} = \frac{1}{16\pi T^2} = \pi r_h^2 = \frac{A}{4}\,. \tag{2.71}$$

de Sitter universe

The de Sitter universe is a vacuum solution to the Einstein equations with the cosmological constant $\Lambda > 0$. In a manner similar to the Schwarzschild solution, the metric can be written as [28]

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)\mathrm{d}t^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{2.72}$$

For simplicity it is useful to introduce $\ell \equiv \sqrt{3/\Lambda}$ which simplifies the metric to

$$\mathrm{d}s^2 = -\left(1 - \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{2.73}$$

We immediately see that this metric is singular at $r = \ell$ and with the same structure as the metric (2.17), in particular it is locally Rindler at, as such $r = \ell$ with temperature of

$$T = -\frac{f'(r=\ell)}{4\pi} = \frac{1}{2\pi\ell}\,. \tag{2.74}$$

The minus sign in the formula is added "by hand" to ensure a positive temperature. Physical calculations show that the temperature is actually positive [29], but recently there have been arguments that the temperature should be considered negative [30].

The de Sitter space is compact, thus the boundary term vanishes, but the bulk term does not. Since $\Lambda \neq 0$

$$S_E = -\int_\Omega \frac{\mathrm{d}^4x\sqrt{g}(R-2\Lambda)}{16\pi}\,, \tag{2.75}$$

The Einstein equations give $R = 4\Lambda$, taking $\Omega = \{x^\mu : r < \ell\}$

$$\begin{aligned} S_E &= -\frac{1}{16\pi}\int_0^\beta \int_{r<\ell} \sqrt{g}2\Lambda \\ &= -\frac{\Lambda}{8\pi}\beta\int_{r<\ell} r^2\sin\theta \\ &= -\frac{\Lambda}{8\pi}\beta\frac{4}{3}\pi\ell^3 \\ &= -\pi\ell^2\,, \end{aligned} \tag{2.76}$$

where we used the equation (2.74), the fact that $\beta = 1/T$ and the definition of ℓ.

The negative result could be surprising, but consider the thermodynamic case

$$\log Z = -\beta F = -\beta(E - TS) = S\,, \tag{2.77}$$

if $E = 0$ which is precisely the case here - the energy of a gravitational field is all in the boundary[31]. Therefore by the equation (2.56)

$$S = \pi\ell^2 = \frac{A}{4}\,, \tag{2.78}$$

where the area is naturally $A = 4\pi\ell^2$.

2.3.3 2nd law and Black Hole Mechanics

For the second law of black hole mechanics is usually taken the area increase theorem, but how does this integrate with the usual second law of thermodynamics $\mathrm{d}S > 0$?

As Bekenstein writes in his paper [1]

$$\mathrm{d}S \geq 0\,, \tag{2.79}$$

where S is the sum of entropies of black holes (proportional to their area) and the entropy of everything outside of the black holes, the "common entropy". The choice of direct proportionality with the area and not some monotonically increasing function of the area is interesting. He argues:

> *"We ensure that the total black-hole entropy of a system of black holes (the sum of individual $S_{\text{b.h.}}$) depends only on the total horizon area,"*

but this also preserves additivity of black hole entropy across "systems", the black holes, and therefore of entropy as a whole.

It is also illuminating to write the black hole entropy in proper units

$$S_{\text{BH}} = \frac{k_B}{L_{\text{P}}^2}\frac{A}{4} = \frac{k_B c^3}{G\hbar}\frac{A}{4}\,, \tag{2.80}$$

where k_B is the Boltzmann constant, L_{P} is the Planck length, c the speed of light, G the gravitational constant and $\hbar$ the reduced Planck constant. It is interesting that the natural units of measuring area turn out to be in Planck length squared; this makes the black hole entropy comparatively enormous. But maybe more importantly this formula has $\hbar$ in the denominator, thus the classical limit symbolically given as $\hbar \to 0$ cannot work, which points to fundamental connection between gravitation as given by general relativity and any quantum theory.

3 Black hole phase transitions

In classical thermodynamics the Van der Waals fluid is often used as an introductory example to phase transitions. It turns out that the Reissner–Norström-AdS (RN-AdS) black hole, i.e. charged black hole with negative cosmological constant Λ, exhibits classical critical behaviour analogous to Van der Waals fluids [32].

Lately, the variation of Λ in the first law of black hole thermodynamics has gained attention. This variation compares black holes with different asymptotics, as opposed to the AdS background being fixed. Thus we have to abandon the classical notion of Λ being the parameter of the theory. There are two reasons we shall mention here for this step: black hole thermodynamics can be interpreted as a study of a "solution phase space", where the variables are the parameters of the black hole. It can be shown that, for example, the first law still holds under reasonable assumptions [33]. One may consider theories where physical 'constants' are not a priori fixed but arise from some physical phenomena, in which case it is natural to consider their variations in the first law.

A natural interpretation of Λ is as pressure. Similarly as in the de Sitter case, we denote $\ell \equiv \sqrt{3/(-\Lambda)}$

$$P = -\frac{1}{8\pi}\Lambda = \frac{3}{8\pi}\frac{1}{\ell^2}\,, \tag{3.1}$$

with the natural "conjugate volume"[32]

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,J} = \frac{4}{3}\pi r_+^3\,, \tag{3.2}$$

where r_+ is the event horizon radius.

3.1 RN-AdS

In Schwarzchild-like coordinates, the metric of a 4-dimensional spherical RN-AdS black holes reads [32]

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2\,, \tag{3.3}$$

where

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}\,, \tag{3.4}$$

and the black hole event horizon is given as the largest root of f, $f(r_+) = 0$.

The black hole temperature can thus be identified as

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{\ell^2} - \frac{Q^2}{r_+}\right)\,, \tag{3.5}$$

and the entropy as

$$S = \frac{4\pi r_+^2}{4}\,. \tag{3.6}$$

Electric potential Φ at infinity with respect to the horizon is

$$\Phi = \frac{Q}{r_+}\,, \tag{3.7}$$

with P and V given as above we get the first law for RN-AdS black hole in the form

$$\mathrm{d}M = T\,\mathrm{d}S + \Phi\,\mathrm{d}Q + V\,\mathrm{d}P\,. \tag{3.8}$$

Note that with the first law in this form the natural interpretation of M is not the internal energy but *enthalpy*.

To investigate the phase transition, we shall consider the case $Q = \text{const}$, so from (3.5) we get an equation of state $P = P(V, T)$

$$P = \frac{T}{2r_+} - \frac{1}{8\pi {r_+}^2} + \frac{Q^2}{8\pi {r_+}^4}\,, \qquad r_+ = \left(\frac{3V}{4\pi}\right)^{1/3}\,, \tag{3.9}$$

where we used the fact that the identity $V = 4/3\pi r_+^3$ holds in these coordinates.

Consider the Van der Waals equation of state

$$P = \frac{RT}{v-b} - \frac{a}{v^2}\,, \tag{3.10}$$

where P is the pressure, R the gas constant, T the thermodynamic temperature, and v the molar volume, a, b being constants.

In proper units

$$\begin{aligned} \text{pressure} &\leftrightarrow \frac{hc}{L_p^2}P\,, \\ \text{temperature} &\leftrightarrow \frac{hc}{k_B}T\,, \end{aligned} \tag{3.11}$$

where L_p is the Planck length. Therefore, comparing the equations (3.9) and (3.10), we identify $2L_p^2 r_+ \leftrightarrow v$. Apart from the scaling factor of 2, it seems natural to have the reduced volume measured in L_p, the natural units of the black hole horizon area as seen in (2.80), and r_+, the independent length parameter.

From the viewpoint of thermodynamics, this identification is problematic as v in (3.10) is intensive, while r_+ increases with the size of the system in any reasonable definition of "the system".

Performing this identification, we obtain the "thermodynamic" equation of state as

$$P(T, v) = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}\,. \tag{3.12}$$

This equation can be used to study isotherms in the phase space. In particular, let us study the critical point, i.e. for some particular $T = T_c$, $v = v_c$

$$\begin{aligned} \left.\frac{\partial P}{\partial v}\right|_{T_c, v_c} = 0 \quad &\Rightarrow \quad -T_c\pi v_c^3 + v_c^2 - 8Q^2 = 0\,, \\ \left.\frac{\partial^2 P}{\partial v^2}\right|_{T_c, v_c} = 0 \quad &\Rightarrow \quad 2T_c\pi v_c^3 - 3v_c^2 + 40Q^2 = 0\,. \end{aligned} \tag{3.13}$$

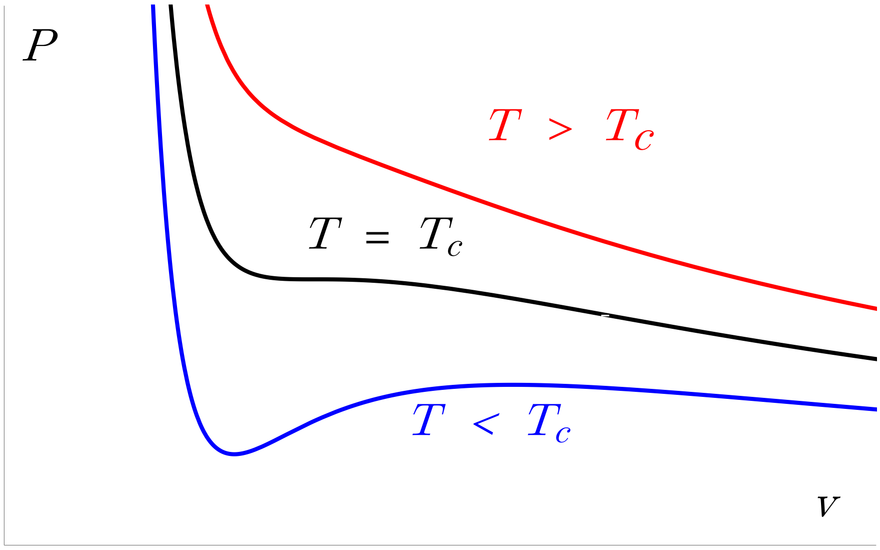

Figure 3.1: Plots of $P(v)$ for various T

Denoting $P(T_c, v_c) \equiv P_c$ the equations yield

$$v_c = 2\sqrt{6}Q,\, T_c = \frac{\sqrt{6}}{18\pi Q},\, P_c = \frac{1}{96\pi Q^2} \quad \Rightarrow \quad \frac{P_c v_c}{T_c} = \frac{3}{8}\,. \tag{3.14}$$

We note that the last fraction has the same value as in the study of the regular van der Waals fluid. The qualitative plot of few isotherms in figure 3.1 again shows behavior analogous to the van der Waals fluid. It can be shown that the Maxwell construction must again be performed to preserve the stability and that critical exponents of the transition have the same value as in the van der Waals case [32].

3.2 Virial expansion

More directly, one can look at a a virial expansion

$$\frac{Pv}{RT} = 1 + \frac{B(T)}{v} + \frac{C(T)}{v^2} + \frac{D(T)}{v^3} + \dots \tag{3.15}$$

or more suggestively

$$P/R = \frac{T}{v} + \frac{b(T)}{v^2} + \frac{c(T)}{v^3} + \frac{d(T)}{v^4} + \dots \tag{3.16}$$

which is exactly of the form found in (3.12).

Following [34], we shall here consider static ansatz and black hole thermodynamic requirements to seek a black hole spacetime that yields the equation of state in the virial expansion form.

We consider again spacetimes with negative cosmological constant Λ with identification of pressure as above

$$P \equiv -\frac{\Lambda}{8\pi} \equiv \frac{3}{8\pi\ell^2}\,. \tag{3.17}$$

For sake of simplicity we take the metric of the form

$$\mathrm{d}s^2 = -f(r,P)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r,P)} + r^2\,\mathrm{d}\Omega^2\,, \tag{3.18}$$

$$f(r,P) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} + h(r,P)\,, \tag{3.19}$$

for some function $h(r,P)$. If we consider $h = \mathcal{O}(1/r^2)$ we might identify the "thermodynamic" mass of the black hole as the M in (3.19) [35]. On the horizon $f(r_+,P) = 0$. This leads to

$$M = \frac{r_+}{2} + \frac{4}{3}\pi r_+^3 P + \frac{r_+}{2} h(r_+,P)\,. \tag{3.20}$$

Volume is the conjugate quantity of pressure, which yields

$$V \equiv \left.\frac{\partial M}{\partial P}\right|_{S,\ldots} = \frac{4}{3}\pi r_+^3 + \frac{r_+}{2}\frac{\partial h}{\partial P} \tag{3.21}$$

and lastly we place the size of the system to be proportional to the black hole area, in units,

$$N \equiv \frac{1}{k}\frac{A}{L_p^2} = \frac{4\pi r_+^2}{kL_p^2}\,, \tag{3.22}$$

for some constant k. Thus, for the specific volume v

$$v \equiv \frac{V}{N} = k\left(\frac{1}{3}r_+ + \frac{1}{8\pi r_+}\frac{\partial h}{\partial P}\right)\,. \tag{3.23}$$

The temperature of the black hole is

$$T = \frac{1}{4\pi}\left.\frac{\partial f}{\partial r}\right|_{r=r_+} = \frac{1}{4\pi}\left(\frac{2M}{{r_+}^2} + \frac{16\pi P}{3}r_+ + \left.\frac{\partial h}{\partial r}\right|_{r=r_+}\right)\,. \tag{3.24}$$

Thus for any equation of state $T(v,P)$ one may substitute (3.23), compare with (3.24) and solve for h.

Since this is in general difficult, let us consider a simple case of abridged virial case with constant coefficients b, c

$$P = \frac{T}{v} + \frac{b}{v^2} + \frac{c}{v^3}\,, \tag{3.25}$$

and $h = h(r)$. This has the benefit, that V is of the expected form (3.2) and (3.25) is invertible to get $T = T(v,P)$.

This gives equation

$$\frac{2M}{{r_+}^2} + \frac{16\pi P}{r_+} + \frac{\mathrm{d}h}{\mathrm{d}r} = 4\pi P v - \frac{4\pi b}{v} - \frac{4\pi c}{v^2}\,, \tag{3.26}$$

where M is given by (3.20) and v by (3.23) as $v = kr_+/3 \equiv \tilde{k}r_+$, where we rescaled the constant k, $\tilde{k} = k/3$. Similarly, let us denote $\tilde{b} = 4\pi b$ and $\tilde{c} = 4\pi c$. Then the solution of of (3.26) is given as

$$h(r) = \frac{-\tilde{k}r(\tilde{b}+\tilde{k}) - \tilde{c}\log(r) + \frac{4}{3}\pi(\tilde{k}-2)\tilde{k}^2 P r^3}{\tilde{k}^2 r} + \frac{c_1}{r}\,,$$

for some constant c_1.

Now we apply the asymptotic behaviour requirements to h. From here, the first term gives $\tilde{b} = -\tilde{k}$, second $\tilde{c} = 0$, third $\tilde{k} = 2$ and $c_1 = 0$. Note that $\tilde{k} = 2 \implies k = 6$ is in line with predictions by [36].

Thus we are left with $h \equiv 0$. Therefore, the only possible spacetime described by (3.19) is the one with $h \equiv 0$, that is, a Schwarzschild-AdS spacetime, which is a well-known and well-behaved solution of the Einstein equations.

Conclusion

In this thesis we have introduced black holes and discussed their thermodynamic properties. Chapter 2 was devoted to the laws of black hole thermodynamics. First, we defined surface gravity of a black hole, motivated its name, and calculated it for a general static spherical black hole in multiple ways. We showed that it is constant on the event horizon of a stationary black hole. By considering a purely classical engine working between two black holes, we showed that the temperature of a black hole should be proportional to the surface gravity. Considering thermal quantum states with complex time, we fixed the proportionality constant as $T = \kappa/2\pi$. We then discussed the third law of thermodynamics vis-à-vis the extremal black holes and attempts to destroy black holes by various physical processes. We then considered black hole entropy, showing the Bekenstein–Hawking formula from multiple directions: i) as a quantity conjugate to temperature, ii) via the Euclidean path integral approach, and iii) by necessity of the second law of thermodynamics not being "transcended".

In chapter 3 we discussed phase transitions of black holes, showing a thermodynamic analogy between RN-AdS black holes and Van der Waals fluids, as well as presented novel results regarding the virial expansion equation of state on the extended phase space.

Critically, we have omitted quantum properties of black holes, namely the quantum origins of the black hole temperature, the *Hawking radiation*, and discussion of the black hole entropy and its (so far unknown) quantum microscopic origin. We have also ignored the black hole information problem and would like to refer the interested reader to [37] for a discussion of many recent developments on this issue.

We have found a unique solution for the virial expansion equation of state under various simplifying assumptions, some of which could presumably be lifted in upcoming work. In particular, one might look at the virial coefficients, analytically calculated for different models, and compare them with possible solutions. We hope to report on these in the nearby future.

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A Derivation of the laws

A.1 0th law derivation

Here we shall show that the gradient of κ is collinear with k^μ, thus finishing the proof from 2.1.5 [4].
Equation (2.3) reads

$$\kappa k_\mu = k_{\mu;\nu} k^\nu \,.$$

Applying $k_{[\sigma}\nabla_{\lambda]}$ gives

$$\begin{aligned} k_\mu \kappa_{;[\lambda} k_{\sigma]} + \kappa k_{\mu;[\lambda} k_{\sigma]} &= (k_{\mu;\nu} k^\nu)_{;[\lambda} k_{\sigma]} \\ &= k_{\mu;\nu} k^\nu{}_{;[\lambda} k_{\sigma]} + k^\nu k_{\mu;\nu[\lambda} k_{\sigma]} \\ &= k_{\mu;\nu} k^\nu{}_{;[\lambda} k_{\sigma]} - k^\nu R_{\nu\mu[\lambda}{}^\rho k_{\sigma]} k_\rho \,, \end{aligned} \tag{A.1}$$

where we used equation (2.40) in the last step. Re-arranging equation (2.8) gives

$$k_{\mu;\nu} k_\eta = -2 k_{\eta;[\mu} k_{\nu]} \,, \tag{A.2}$$

so then by this equation and the definition of surface gravity (2.3) we can express the first term on the RHS of (A.1) as

$$\begin{aligned} k_{\mu;\nu} k^\nu{}_{;[\lambda} k_{\sigma]} &= -\frac{1}{2} k_{\mu;\nu} k^\nu k_{\lambda;\sigma} \\ &= -\frac{1}{2} \kappa k_\mu k_{\lambda;\sigma} \\ &= \kappa k_{\mu;[\lambda} k_{\sigma]} \,. \end{aligned} \tag{A.3}$$

Therefore equation (A.1) reduces to

$$k_\mu \kappa_{;[\lambda} k_{\sigma]} = k^\nu R_{\mu\nu[\lambda}{}^\rho k_{\sigma]} k_\rho \,. \tag{A.4}$$

Let us apply $k_{[\sigma}\nabla_{\omega]}$ to equation (A.2)

$$k_{\mu;\nu} k_{\eta;[\omega} k_{\sigma]} + k_\eta k_{\mu;\nu[\omega} k_{\sigma]} = -2 k_{\eta;[\mu} k_{\nu];[\omega} k_{\sigma]} - 2 k_{\eta;[\mu|[\omega} k_{\sigma]|} k_{\nu]} \,. \tag{A.5}$$

The first term on the LHS equals the first term of RHS due to equation (A.2). Employing equation (2.40) on the remaining terms we get

$$-k_\eta R_{\nu\mu[\omega}{}^\rho k_{\sigma]} k_\rho = 2 k_{[\nu} R_{\mu]\eta[\omega}{}^\rho k_{\sigma]} k_\rho \,. \tag{A.6}$$

Contracting $\eta - \omega$ eliminitates the LHS and writing out the second bracket we obtain

$$-k_{[\nu} R_{\mu]}{}^\rho k_\sigma k_\rho = k_{[\nu} R_{\mu]\eta\sigma}{}^\rho k^\eta k_\rho \,. \tag{A.7}$$

The RHS the same structure as RHS of equation (A.4), so by comparison

$$\kappa_{;[\eta} k_{\sigma]} = -k_{[\sigma} R_{\eta]}{}^\rho k_\rho \,. \tag{A.8}$$

Assuming dominant energy condition and Einstein's equation the RHS vanishes. To show this we must first take a detour to geodesics.

A.1.1 Kinematics of geodesics

Consider a congruence of timelike geodesics ξ^μ parametrized by their proper time τ and normalized such that $\xi^\mu \xi_\mu = -1$. Let us then define tensor

$$B_{\mu\nu} \equiv \xi_{\mu;\nu} \, . \tag{A.9}$$

Consider η^μ the orthogonal deviation from an arbitrarily chosen geodetic γ_0. Then

$$\pounds_\xi \eta^\mu = 0 \, , \tag{A.10}$$

which can be expressed in coordinate notation as

$$\eta^\mu{}_{;\nu}\xi^\nu = \xi^\mu{}_{;\nu}\eta^\nu = B^\mu{}_\nu \eta^\nu \, , \tag{A.11}$$

from which one sees that $B^\mu{}_\nu$ measures the change on η^μ during a parallel transport — more intuitively, along γ_0 observer would see nearby geodesics as being deformed by the linear map $B^\mu{}_\nu$. $B_{\mu\nu}$ is "spatial" in the sense that

$$B_{\mu\nu}\xi^\mu = B_{\mu\nu}\xi^\nu = 0 \, , \tag{A.12}$$

and we define the spatial metric as

$$h_{\mu\nu} = g_{\mu\nu} + \xi_\mu \xi_\nu \, . \tag{A.13}$$

It is easy to check that $h^\mu{}_\nu$ is a projection operator to the normal subspace of ξ^μ.

In analogy to continuum mechanics we define expansion θ, shear $\sigma_{\mu\nu}$ and twist $\omega_{\mu\nu}$ of the congruence as

$$\theta = B^{\mu\nu} h_{\mu\nu} = B^\mu{}_\mu = \xi^{\mu;}{}_\mu \, , \tag{A.14}$$

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu} \, , \tag{A.15}$$

$$\omega_{\mu\nu} = B_{[\mu\nu]} \, . \tag{A.16}$$

Now using the Ricci identities

$$\begin{aligned} B_{\mu\nu;\sigma}\xi^\sigma &= \xi_{\mu;\nu\sigma}\xi^\sigma = \xi_{\mu;\sigma\nu}\xi^\sigma + R_{\sigma\nu\mu}{}^\rho \xi^\sigma \xi_\rho \\ &= (\xi_{\mu;\sigma}\xi^\sigma)_{;\nu} - \xi_{\mu;\sigma}\xi^\sigma{}_{;\nu} + R_{\sigma\nu\mu}{}^\rho \xi^\sigma \xi_\rho \\ &= -B_{\mu\sigma}B^\sigma{}_\nu + R_{\sigma\nu\mu}{}^\rho \xi^\sigma \xi_\rho \, . \end{aligned} \tag{A.17}$$

Taking the trace we get the Raychaudhuri's equation

$$\theta_{;\sigma}\xi^\sigma = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu \, . \tag{A.18}$$

In trying to retrace these steps for a congruence of null geodesics k^μ one runs into the problem of talking about the transversal direction, equivalent to spatial above, since null vectors are orthogonal with themselves. To work around this [38], one selects an auxiliary null field N_μ such that $k^\mu N_\mu = -1$.[1] This condition

[1] Consider the case of a Lorentz frame. Then one can consider null coordinates $u = t - x$, $v = t + x$. If then the motion movement is studied along one coordinate, the natural choice is the other for a transverse component.

does not determine N_μ uniquely, but the relevant quantities are independent of the choice of N_μ.[2]

We define the transverse metric by

$$h_{\mu\nu} = g_{\mu\nu} + 2N_{(\mu}k_{\nu)}\,. \tag{A.19}$$

It is then easy to see that it is purely transverse (null contraction by N^μ or k^μ in either index) and two-dimensional. We can continue as above - we define

$$B_{\mu\nu} = k_{\mu;\nu}\,, \tag{A.20}$$

and its transverse part

$$\tilde{B}_{\mu\nu} = h^\alpha_\mu h^\beta_\nu B_{\alpha\beta}\,. \tag{A.21}$$

Let the geodesics be parameterized such that $k^\mu{}_{;\nu}k^\nu = \kappa k^\mu$. Of course, one can always change the parameterization to get $\kappa = 0$, which will give standard results for equations below, but this nonstandard treatment shortens the proof. Let us then write equation (A.21) in explicit form

$$\begin{aligned}\tilde{B}_{\mu\nu} &= (g^\alpha_\mu + k^\alpha N_\mu + N^\alpha k_\mu)(g^\beta_\nu + k^\beta N_\nu + N^\beta k_\nu)B_{\alpha\beta}\\ &= (g^\alpha_\mu + k^\alpha N_\mu + N^\alpha k_\mu)(B_{\alpha\nu} + \kappa N_\nu k_\alpha + N^\beta k_\nu B_{\alpha\beta})\\ &= B_{\mu\nu} + N^\alpha k_\mu B_{\alpha\nu} + \kappa N_\nu k_\mu - \kappa N_\nu k_\mu + k_\nu B_{\mu\beta}N^\beta + k_\mu k_\nu B_{\alpha\beta}N^\alpha N^\beta\,.\end{aligned} \tag{A.22}$$

Using the same decomposition

$$\tilde{B}_{\mu\nu} = \frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}\,, \tag{A.23}$$

where $\theta = \tilde{B}^\mu{}_\mu$ is the expansion, $\sigma_{\mu\nu} = \tilde{B}_{(\mu\nu)} - 1/2\theta h_{\mu\nu}$ the shear, and $\omega_{\mu\nu} = \tilde{B}_{[\mu\nu]}$ is the twist. Explicitly

$$\theta = g^{\mu\nu}\tilde{B}_{\mu\nu} = g^{\mu\nu}B_{\mu\nu} - \kappa = k^\mu{}_{;\mu} - \kappa\,, \tag{A.24}$$

so θ does not depend on the choice of N^μ and in the case of affine parameterization the result is the same as for the timelike case.

Since θ characterises the expansion of the geodesics, it can be written in terms of cross-sectional area $\mathrm{d}S$ as [38]

$$\theta = \frac{1}{\mathrm{d}S}\frac{\mathrm{d}}{\mathrm{d}\lambda'}\,\mathrm{d}S\,, \tag{A.25}$$

where λ' is the curve parameter.

First, we define the cross section of null geodesic congruence. Consider one geodesic γ_0 and a point P on it such that $\lambda' = \lambda'_P$. Then let there be auxiliary curves with tangent N^μ parameterised by μ such that μ is constant on the geodesics. Let β be the auxiliary curve passing throug P and $\mu|_{\gamma_0} = \mu_{\gamma_0}$.

We define the cross section $\mathrm{d}\mathcal{S}(\lambda'_P)$ as a small neighborhood of P such that through each point P' in the neighborhood goes a geodesic from the congruence

[2]This of course should be rigorously proven for each quantity. We shall not do the calculations here, see e.g. [38].

and another auxiliary curve, and at each P' $\lambda' = \lambda'_P$, $\mu = \mu_{\gamma_0}$. We can then change the parameterization of γ_0 and β so that they intersect $\mathrm{d}\mathcal{S}(\lambda'_P)$ orthogonally.

Since $\mathrm{d}\mathcal{S}(\lambda'_P)$ is a two-dimensional space, we can introduce coordinates on it by labeling each point by θ^A, $(A = 2, 3)$. Since exactly one geodesic goes through every point, we labeled the geodesics themselves. By requiring geodesics to keep their labels, we have constructed coordinates for every cross section $\mathrm{d}\mathcal{S}(\lambda')$.

Thus, we have constructed a coordinate system (λ, μ, θ^A) in the neighborhood of γ_0 with a well-defined transformation to arbitrary different coordinates x^μ, which allows us to write

$$k^\mu = \left(\frac{\partial x^\mu}{\partial \lambda'}\right)_{\mu,\theta^A}, \tag{A.26}$$

and vectors

$$e^\mu_A = \left(\frac{\partial x^\mu}{\partial \theta^A}\right)_{\lambda',\mu}, \tag{A.27}$$

are tangent to the cross section. By this definition it is clear that

$$\pounds_k e^\mu_A = 0 \tag{A.28}$$

and on γ_0 $k_\mu e^\mu_A = N_\mu e^\mu_A = 0$.

We define the metric on the cross section $\mathrm{d}\mathcal{S}(\lambda')$ as $\sigma_{AB} \equiv g_{\mu\nu} e^\mu_A e^\nu_B$ and its area $\mathrm{d}S \equiv \sqrt{\sigma}\,\mathrm{d}^2\theta$, σ being the determinant of σ_{AB}. Let the inverse metric σ^{AB} be such that $h^{\mu\nu} = \sigma^{AB} e^\mu_A e^\nu_B$.

Using these definition the RHS of equation (A.25) reads

$$\frac{1}{\sqrt{\sigma}} \frac{\mathrm{d}}{\mathrm{d}\lambda'} \sqrt{\sigma} = \frac{1}{2} \sigma^{AB} \frac{\mathrm{d}\sigma_{AB}}{\mathrm{d}\lambda'}. \tag{A.29}$$

But

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{AB}}{\mathrm{d}\lambda'} &= \left(g_{\mu\nu} e^\mu_A e^\nu_B\right)_{;\rho} k^\rho, \\
&= g_{\mu\nu} \left(e^\mu_{A;\rho} k^\rho\right) e^\nu_B + g_{\mu\nu} e^\mu_A \left(e^\nu_{B;\rho} k^\rho\right), \\
&= g_{\mu\nu} \left(k^\mu{}_{;\rho} e^\rho_A\right) e^\nu_B + g_{\mu\nu} e^\mu_A \left(k^\nu{}_{;\rho} e^\rho_B\right), \\
&= \left(B_{\mu\nu} + B_{\nu\mu}\right) e^\mu_A e^\nu_B,
\end{aligned} \tag{A.30}$$

where we used equation (A.28) on the second line. Then to conclude the proof

$$\sigma^{AB} \frac{\mathrm{d}\sigma_{AB}}{\mathrm{d}\lambda'} = \left(B_{\mu\nu} + B_{\nu\mu}\right) \sigma^{AB} e^\mu_A e^\nu_B = 2B_{\mu\nu} h^{\mu\nu} = 2B_{\mu\nu} g^{\mu\nu} = 2\theta. \tag{A.31}$$

To get the analogue of Raychaudhuri's equation

$$\begin{aligned} B_{\mu\nu;\sigma}k^\sigma &= k_{\mu;\nu\sigma}k^\sigma = k_{\mu;\sigma\nu}k^\sigma + R_{\sigma\nu\mu}{}^{\rho}k^\sigma k_\rho \\ &= (k_{\mu;\sigma}k^\sigma)_{;\nu} - k_{\mu;\sigma}k^\sigma{}_{;\nu} + R_{\sigma\nu\mu}{}^{\rho}k^\sigma k_\rho \\ &= (\kappa k_\mu)_{;\nu} - B_{\mu\sigma}B^\sigma{}_\nu + R_{\sigma\nu\mu}{}^{\rho}k^\sigma k_\rho\,, \end{aligned} \tag{A.32}$$

and taking the trace

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda'} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}k^\mu k^\nu\,. \tag{A.33}$$

with possible substitution from the Einstein's equation

$$R_{\mu\nu}k^\mu k^\nu = 8\pi T_{\mu\nu}k^\mu k^\nu\,. \tag{A.34}$$

A.1.2 Finishing the proof

Equation (A.2) reads $k_{\mu;\nu}k_\eta = -2k_{\eta;[\mu}k_{\nu]}$, contracting it with arbitrary vectors m^μ, n^ν tangent to the horizon leads to

$$k_{\mu;\nu}m^\mu n^\nu = 0 \tag{A.35}$$

or in the notation of previous section $\tilde{B}_{\mu\nu} = 0$, thus θ, $\sigma_{\mu\nu}$, $\omega_{\mu\nu}$ vanish and from equation (A.33) we get

$$R_{\mu\nu}k^\mu k^\nu = 0\,. \tag{A.36}$$

The dominant energy condition states that $-T^\mu{}_\nu k^\nu$ must be future directed timelike or null, but equations (A.34) and (A.36) imply $T^\mu{}_\nu k^\nu k_\mu = 0$. Therefore $-T^\mu{}_\nu k^\nu$ must be collinear with k^μ, thus

$$k_{[\mu}T_{\nu]\sigma}k^\sigma = 0 \tag{A.37}$$

so using Einstein's equation shows that the RHS of equation (A.8) vanishes. This yields

$$\kappa_{[;\eta}k_{\sigma]} = 0\,, \tag{A.38}$$

from which we conclude that the gradient of κ is collinear with k^μ and thus κ is constant on the horizon. □

A.2 Area increase theorem

Let us consider the Raychaudhuri's equation for null geodesics (equation (A.33)) for affinely parameterized congruence ($\kappa = 0$) that is hypersurface orthogonal ($\omega_{\mu\nu} = 0$ by the Frobenius theorem, holds at horizon due to equation (A.35)). The value of $\sigma_{\mu\nu}\sigma^{\mu\nu}$ is positive since $\sigma_{\mu\nu}$ is transverse, so it is a sum of squares. Then by the Einstein equations

$$R_{\mu\nu}k^\mu k^\nu = 8\pi(T_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu})k^\mu k^\nu = 8\pi T_{\mu\nu}k^\mu k^\nu\,, \tag{A.39}$$

the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$ implies that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \leq -\frac{1}{2}\theta^2\,. \tag{A.40}$$

By separating variables and integrating from λ_0 to λ we obtain

$$\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta(\lambda_0)} + \frac{1}{2}\lambda\,. \tag{A.41}$$

This is the focusing theorem. It says that if the geodesics are converging at λ_0, i.e. $\theta < 0$, then $\theta \to -\infty$ in finite λ. This generally means that some geodesics come together in a point - a caustic. Finally we note that every event horizon is generated by null geodesics with no future end points, thus they may never create a caustic.

Then necessarily $\theta > 0$ everywhere on the event horizon for all λ and increasing λ means increasing time for a timelike observer so the horizon's surface area cannot decrease. A more rigorous proof with all the necessary details can be found in [4].

A.3 First law derivation

We shall prove the first law in the form [38]

$$\delta M = \frac{\kappa}{2\pi}\frac{\delta A}{4} + \Omega\delta J\,. \tag{A.42}$$

Consider the Killing timelike vector ξ^μ and the vector field

$$\varepsilon^\mu = -T^\mu{}_\nu \xi^\nu\,, \tag{A.43}$$

where $T_{\mu\nu}$ is a stress-energy tensor. Then ε^μ can be interpreted as a energy density flux vector, more so when one consider that

$$\varepsilon^\mu{}_{;\mu} = -T^\mu{}_{\nu;\mu}\xi^\nu - T^\mu{}_\nu \xi^\nu{}_{;\mu} = 0\,, \tag{A.44}$$

where we used the fact that the stress tensor is divergence-free and symmetric, and the Killing equation. This then implies conservation by the usual Gauss's law argument and

$$\Delta M = -\int_\Sigma T^\mu{}_\nu \xi^\nu \,\mathrm{d}\Sigma_\mu\,, \tag{A.45}$$

where ΔM is mass transferred through the hypersurface Σ. Similar procedure for the Killing vector of axial symmetry ψ^μ yield

$$\Delta J = \int_\Sigma T^\mu{}_\nu \psi^\nu \,\mathrm{d}\Sigma_\mu\,, \tag{A.46}$$

for the transferred angular momentum ΔJ.

Consider the null hypersurface of the event horizon H. We take its normal to be k^μ and parameterize it by the non-affine parameter of k^μ-generated geodesics λ' together with θ^A, which label the generators. Then

$$\mathrm{d}\Sigma_\mu = -k_\mu \,\mathrm{d}S\,\mathrm{d}\lambda'\,, \tag{A.47}$$

where $\mathrm{d}S$ is the surface element of the two-surface of constant λ', which we shall denote $\mathcal{H}(\lambda')$.

Let us drop a small amount of matter into a stationary black hole, described by and infinitesimal stress-energy tensor $T_{\mu\nu}$ such that the change of the black hole mass and angular momentum change as given by equations (A.45) and (A.46) (replacing Δ with δ). We assume, that the black holes becomes stationary again in the future.

Since the perturbation is small, we take Raychaudhuri's equation (A.33) only in the first order, i.e. neglecting the θ^2 and $\sigma_{\mu\nu}\sigma^{\mu\nu}$ with $\omega_{\mu\nu}$ coming only from the curvature which can be again neglected in the firt order we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda'} = \kappa\theta - 8\pi T_{\mu\nu}k^\mu k^\nu \,. \tag{A.48}$$

Then (taking $k^\nu = \xi^\nu + \Omega\psi^\nu$)

$$\begin{aligned}
\delta M - \Omega\delta J &= \int_H T_{\mu\nu}(\xi^\nu + \Omega\psi^\nu)k^\mu \,\mathrm{d}S\,\mathrm{d}\lambda' \\
&= \int \mathrm{d}\lambda' \int_{\mathcal{H}(\lambda')} T_{\mu\nu}k^\mu k^\nu \\
&= -\frac{1}{8\pi}\int \mathrm{d}\lambda' \int_{\mathcal{H}(\lambda')} \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda'} - \kappa\theta\right)\mathrm{d}S \\
&= -\frac{1}{8\pi}\int_{\mathcal{H}(\lambda')} \theta\,\mathrm{d}S\bigg|_{-\infty}^{+\infty} + \frac{\kappa}{8\pi}\int \mathrm{d}\lambda' \int_{\mathcal{H}(\lambda')} \theta\,\mathrm{d}S \\
&= \frac{\kappa}{8\pi}\int \mathrm{d}\lambda' \int_{\mathcal{H}(\lambda')} \left(\frac{1}{\mathrm{d}S}\frac{\mathrm{d}}{\mathrm{d}\lambda'}\,\mathrm{d}S\right)\mathrm{d}S \\
&= \frac{\kappa}{8\pi}\int_{\mathcal{H}(\lambda')} \mathrm{d}S\bigg|_{-\infty}^{+\infty} = \frac{\kappa}{2\pi}\frac{\delta A}{4}\,,
\end{aligned} \tag{A.49}$$

where on the first line we used the definition of H and k^ν, on the second line equation (A.48), on the third line the fact that the black hole is stationary before and after perturbation so $\theta(\lambda' = \pm\infty) = 0$, and on the fourth the interpretation of θ as given by (A.25). □