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BACHELOR THESIS

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Hausdorff and Capacitary Dimensions

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Název práce: Vztah Hausdorffovy a kapacitární dimenze

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Abstrakt: Tato bakalářská práce má za cíl dokázat rovnost Hausdorffovy a kapacitární dimenze. Rovněž dokazujeme ekvivalenci Lebesgueovy a Hausdorffovy míry, k čemuž používáme především isodiametrickou nerovnost a Steinerovu symetrizaci. K důkazu rovnosti Hausdorffovy a kapacitární dimenze je nezbytné Frostmanovo lemma, jež uvádíme společně s důkazem.

Klíčová slova: Hausdorffova dimenze, Kapacitární dimenze, Hausdorffova míra, Lebesgueova míra, Frostmanovo lemma.

Title: Hausdorff and Capacitary Dimensions

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Abstract: This bachelor thesis aims to prove the equality of the Hausdorff and capacitary dimensions. Additionally, we establish the equivalence of Lebesgue and Hausdorff measures, which requires the Isodiametric inequality and Steiner Symmetrization. The proof of equality between Hausdorff and capacitary dimensions relies crucially on Frostman's lemma, which we introduce alongside its proof.

Key words: Hausdorff dimension, Capacitary dimension, Hausdorff measure, Lebesgue measure, Frostman's lemma.

Contents

Introduction

One may encounter Hausdorff and Capacitary dimensions in many branches of mathematics. The most prevalent ones are certainly Fractal Geometry and Geometric Measure Theory. Hausdorff dimension is capable of determining the dimension of objects with non-integer dimension, while the Capacitary dimension quantifies the energy distribution in those sets. This thesis aims to prove that the Hausdorff dimension is equal to Capacitary dimension.

To this end I have made a detailed introduction to Hausdorff measure and its most important properties. To further not only the goals of this thesis, but also the reader's understanding of the Hausdorff measure, I have shown how Hausdorff measure relates to the Lebesgue measure. These measures are, in fact, equivalent on $\mathbb{R}^n$. This is a paramount result with far-reaching applications in Mathematical Analysis. An important tool to show this equivalence is the Steiner Symmetrization Process alongside the Isodiametric Inequality. Both of these concepts can be found in this thesis including proofs. While this introduction might seem superfluous at first, it provides great insight into the intricate theory of Hausdorff measure, and consequently, Hausdorff dimension.

Besides defining both the Hausdorff and Capacitary dimensions and mentioning some of their fundamental properties, I needed a tool to prove that they are indeed equal. To prove this, one must prove both inequalities, where one is significantly more demanding than the other. A key ingredient to proving the more difficult inequality is a so called Frostman's Lemma, which was proved in this thesis. Essential theory has been either built up or referenced to prove this Lemma.

The main goal of this thesis was to provide a detailed and understandable summary of Hausdorff measure and its properties, the equivalence of Hausdorff and Lebesgue measure, and the equality of Hausdorff and Capacitary dimension. While these results are well-known, this thesis provides a direct approach to this equality and pays no mind to other aspects of Fractal Geometry or Geometric Measure Theory. It also delves deep into the Hausdorff measure and its properties to fully grasp its intricacy.

This thesis mainly draws from Pertti Mattila's book - Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability [3]. To cover the Hausdorff measure I have used the notes of Professor Rataj from Univerzita Karlova in Prag [4]. I have also used the book Measure Theory and Fine Properties of Functions [1] for Steiner Symmetrization, Isodiametric Inequality and the equivalence of Lebesgue measure and Hausdorff measure on $\mathbb{R}^n$.

Notation

$\varnothing$	Empty set
$\mathbb{N}$	Natural numbers
$\mathbb{R}^n$	Real space of dimension n
$\mathcal{P}(X)$	Power set of set X
$\mathcal{B}^n = \mathcal{B}(\mathbb{R}^n)$	σ-algebra of Borel sets
$\operatorname{diam}(A)$	Diameter of set A
$\lvert \cdot \rvert$	Euclidean norm in $\mathbb{R}^n$
$\operatorname{dist}(A, B)$	Distance between sets A and B
$\mu(A)$	Measure of set A
$\mu^*(A)$	Outer measure of set A
$\mathcal{H}^d(A)$	Hausdorff measure of dimension d of set A
$\mathcal{H}^d_\delta(A)$	δ-outer Hausdorff measure of dimension d of set A
$\lambda^n(A)$	Lebesgue measure of dimension n of set A
L^a_b	Line through point b in the direction of point a
P_a	Plane through the origin perpendicular to the point a
$S_a(A)$	Steiner Symmetrization of set A with respect to the plane P_a
$\langle x, y \rangle$	Scalar product of x and y
$\operatorname{supp}(\mu)$	Support of measure μ
X^*	The dual space of X
$\mathcal{C}_c(X)$	The space of compactly supported complex-valued continuous functions
$\xrightarrow{w^*}$	weak* convergence
$\dim_H A$	Hausdorff dimension of set A
$\dim_c A$	Capacitary dimension of set A

1 Hausdorff Measure

This chapter draws from Measure Theory by Evans and Gariepy [1], from lecture notes provided by Professor Rataj [4] and from lecture notes provided by Professor Tores from Purdue University in Indiana [7]. Lecture notes provided by Dr. Monica Torres have been especially helpful to prove Theorem 1.8.

We first motivate the definition of Hausdorff measure. Intuitively, we need to measure objects in a metric space which have a 'lower' dimension than the space itself. To this end, we shall use Hausdorff measure. We demonstrate the need for this measure in the following example:

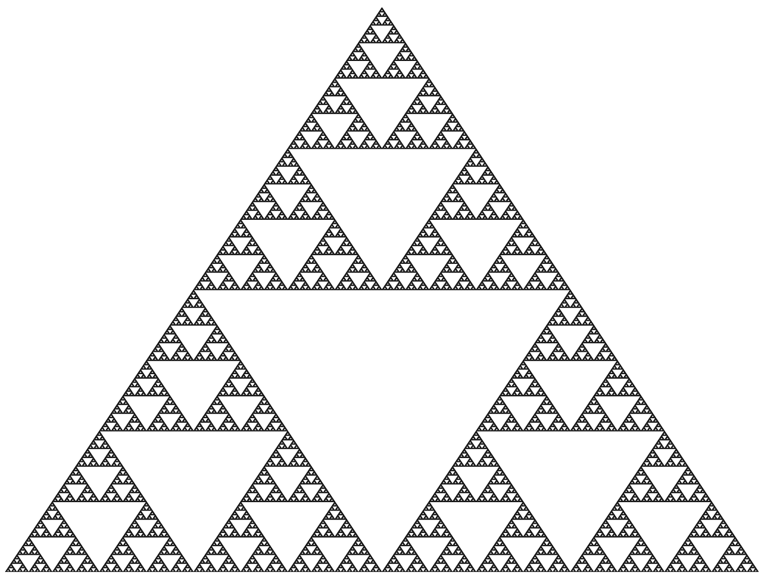

Figure 1: Sierpinski triangle

This is a so-called Sierpinski triangle. What is its dimension? It is possible to show that the Hausdorff dimension[1] is exactly $\frac{\log 3}{\log 2}$. This result is far from trivial and it is an important result in Fractal Geometry. We start with some elementary definitions.

1.1 Elementary Definitions and Properties

Definition 1.1 (Outer measure). Let X be a nonempty set. The function $\mu^* \colon \mathcal{P}(X) \to [0, \infty]$ is an outer measure on X if the following holds

1. $\mu^*(\varnothing) = 0$.
2. If $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$. In other words, the outer measure is monotone.
3. If $A_n \subset X$, $n \in \mathbb{N}$, then $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. We say that the outer measure possesses countable-subadditivity property.

[1]We will define the so-called Hausdorff dimension in the next section.

Definition 1.2 (Metric outer measure)**.** Let (X, ϱ) be a metric space. The outer measure μ^* is metric if and only if

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B),$$

for every $A, B \subset X$ with $\operatorname{dist}(A, B) > 0$.

Definition 1.3 (Hausdorff measure)**.** Let (X, ϱ) be a metric space and $\delta > 0$. We define the δ-outer Hausdorff measure of dimension d by

$$\mathcal{H}^d_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \colon A \subset \bigcup_{i=1}^{\infty} U_i, \operatorname{diam} U_i < \delta \right\}.$$

We define the Hausdorff measure of dimension d by

$$\mathcal{H}^d(A) = \lim_{\delta \to 0^+} \mathcal{H}^d_\delta(A).$$

Observe that as $\delta \to 0^+$, the infinimum from the definition above is taken over a smaller and smaller class, meaning the right hand-side increases. Since the limit of a monotonic function always exists, the left hand-side is well-defined, albeit it may attain infinite values.

Remark. It is clear from the definition that we may assume later on that the sets U_i are closed, thus taking a so-called closed covering of set A.

We will now prove that both these measures are outer measures on X and, moreover, that $\mathcal{H}^d$ is metric.

Theorem 1.4. *Both $\mathcal{H}^d_\delta$ and $\mathcal{H}^d$ are outer measures on X.*

Proof. One can easily see that $\mathcal{H}^d_\delta(\varnothing) = 0 = \mathcal{H}^d(\varnothing)$. It follows directly from definition that the Hausdorff measure is monotone.

Now to prove the countable subadditivity. Let $A_i \subset X$ for all $i \in \mathbb{N}$. Assume $\mathcal{H}^d(A_i)$ is finite for all $i \in \mathbb{N}$. Were it not so, the inequality from the definition would follow trivially. Then $\mathcal{H}^d_\delta(A_i)$ is also finite for all $i \in \mathbb{N}$ and for all $\delta > 0$ as follows from the definition. We wish to show

$$\mathcal{H}^d_\delta \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} \mathcal{H}^d_\delta(A_i) \quad \text{for all } \delta > 0.$$

Proving this also shows that $\mathcal{H}^d$ is subadditive as we can interchange the sum and the limit thanks to the fact that the Hausdorff measure is non-negative. In other words, it holds that

$$\lim_{\delta \to 0^+} \sum_{i=1}^{\infty} \mathcal{H}^d_\delta(A_i) = \sum_{i=1}^{\infty} \lim_{\delta \to 0^+} \mathcal{H}^d_\delta(A_i) = \sum_{i=1}^{\infty} \mathcal{H}^d(A_i).$$

So in total we end up with

$$\lim_{\delta \to 0^+} \mathcal{H}^d_\delta \left(\bigcup_{i=1}^{\infty} A_i \right) = \mathcal{H}^d \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} \mathcal{H}^d(A_i)$$

thanks to the discussion above.

Let $\varepsilon > 0$. Let $\{U_j^i\}_{j=1}^\infty$ be a class such that $A_i \subset \bigcup_{j=1}^\infty U_j^i$ and

$$\sum_{j=1}^\infty (\operatorname{diam} U_j^i)^d < \mathcal{H}_\delta^d(A_i) + \frac{\varepsilon}{2^i}.$$

Then $\bigcup_{i=1}^\infty A_i \subset \bigcup_{i=1}^\infty \bigcup_{j=1}^\infty U_j^i$ and therefore

$$\mathcal{H}_\delta^d\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty \sum_{j=1}^\infty (\operatorname{diam} U_j^i)^d < \sum_{i=1}^\infty \mathcal{H}_\delta^d(A_i) + \varepsilon.$$

Sending $\varepsilon \to 0^+$ we get the desired result which concludes the proof. □

Theorem 1.5. *The outer measure $\mathcal{H}^d$ is metric.*

Proof. We only wish to show that

$$\mathcal{H}^d(A \cup B) \ge \mathcal{H}^d(A) + \mathcal{H}^d(B)$$

as the other inequality follows trivially from the fact that $\mathcal{H}^d$ is an outer measure. Once again we assume $\mathcal{H}^d(A \cup B)$ is finite, otherwise the inequality is trivial. Let $\varepsilon > 0$ and $0 < \delta < \operatorname{dist}(A, B)$. Let $\mathcal{U} = \{U_j\}_{j=1}^\infty$ be a cover of $A \cup B$ with $\operatorname{diam} U_i < \delta$ and

$$\sum_{i=1}^\infty (\operatorname{diam} U_i)^d \le \mathcal{H}^d(A \cup B) + \varepsilon.$$

Since $\operatorname{dist}(A, B) > \delta > \operatorname{diam} U_i$, then every set U_i may intersect only one set A or B. In such a way we get disjoint covers of A and B

$$\mathcal{V} = \{U_i \in \mathcal{U} \colon A \cap U_i \neq \varnothing\},$$
$$\mathcal{V}' = \{U_i \in \mathcal{U} \colon B \cap U_i \neq \varnothing\}.$$

Thus

$$\mathcal{H}_\delta^d(A) + \mathcal{H}_\delta^d(B) \le \sum_{U_i \in \mathcal{V}} (\operatorname{diam} U_i)^d + \sum_{U_i \in \mathcal{V}'} (\operatorname{diam} U_i)^d \le \sum_{i=1}^\infty (\operatorname{diam} U_i)^d < \mathcal{H}_\delta^d(A \cup B) + \varepsilon.$$

Taking $\varepsilon \to 0^+$ and then $\delta \to 0^+$ yields the result. □

Theorem 1.6. *The measure $\mathcal{H}^d$ is a Borel measure in the following sense: for all $A \subset X$ there exists $B \in \mathcal{B}^n$ such that $A \subset B$ and $\mathcal{H}^d(B) = \mathcal{H}^d(A)$.*

Proof. Let $A \subset \mathbb{R}^n$. If $\mathcal{H}^d(A) = \infty$ we can simply put $B = X$. Let us assume that $\mathcal{H}^d(A)$ is finite. Let $\{U_i^k\}_{i=1}^\infty$ be a closed covering of A, where $\operatorname{diam} U_i^k \le \frac{1}{k}$, $k \in \mathbb{N}$ is fixed, and

$$\sum_{i=1}^\infty (\operatorname{diam} U_i^k)^d \le \mathcal{H}_{\frac{1}{k}}^d(A) + \frac{1}{k}.$$

Set $B_k = \bigcup_{i=1}^{\infty} U_i^k$. This is a union of closed sets and is therefore a Borel set. Let $B = \bigcap_{k=1}^{\infty} B_k$. Then B is a Borel set and $A \subset B$. Thus $\mathcal{H}^d(A) \leq \mathcal{H}^d(B)$ is trivial. Now $B \subset \bigcup_{i=1}^{\infty} U_i^k$ and therefore we can write

$$\mathcal{H}^d_{\frac{1}{k}}(B) \leq \sum_{i=1}^{\infty} (\operatorname{diam} U_i^k)^d \leq \mathcal{H}^d_{\frac{1}{k}}(A) + \frac{1}{k}.$$

Letting $k \to \infty$ gives $\mathcal{H}^d(B) \leq \mathcal{H}^d(A)$ and concludes the proof. □

It is also worth noting that the Hausdorff measure is invariant under rotation and translation which can easily be seen from the fact that the diameter doesn't change under rotation or translation.

In the next subsection we discuss the equivalence of Lebesgue and Hausdorff measures.

1.2 Relation between Lebesgue and Hausdorff measures

Definition 1.7. We define the normalized Hausdorff measure as

$$\mathcal{H}^d_N(A) = \alpha_d \mathcal{H}^d(A) \quad \text{for all } A \subset \mathbb{R}^n,$$

where $\alpha_d = \frac{\pi^{\frac{d}{2}}}{2^d \Gamma\left(\frac{d}{2}+1\right)}$ with Γ denoting the Gamma function.

This is the exact measure for which we get $\mathcal{H}^n_N = \lambda^n$. It is worth noting that α_n is the volume of a unit sphere in $\mathbb{R}^n$, $n \in \mathbb{N}$. So, we get an interesting result in one dimension: $\mathcal{H}^1 = \lambda$.

Theorem 1.8 (Hausdorff and Lebesgue measure relation)**.** *It holds that* $\mathcal{H}^n_N = \lambda^n$ *on Lebesgue measurable subsets of* $\mathbb{R}^n$*,* $n \in \mathbb{N}$*.*

At first glance, it is far from clear on how one should proceed with the proof. In fact, we need a bit more theory.

Definition 1.9 (Vitali Covering)**.** We say that covering of closed balls denoted $\mathcal{F}$ is a Vitali covering of set $A \subset \mathbb{R}^n$ if the following holds

For all $x \in A$ and for all $\varepsilon > 0$ there exists a ball $B \in \mathcal{F}$ such that $r(B) < \varepsilon$ and $x \in B$,

where $r(B)$ denotes the radius of ball B.

Lemma 1.10. *Let* $A \subset \mathbb{R}^n$ *and* $\mathcal{H}^d_\infty(A) = 0$*. Then* $\mathcal{H}^d(A) = 0$*.*

Proof. We know $\mathcal{H}^d_\infty(A) = 0$ and therefore for all $\varepsilon > 0$ there exists a covering $\{U_i\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \leq \varepsilon.$$

It follows that $\operatorname{diam} U_i \leq \varepsilon^{\frac{1}{d}}$. Let $\tilde{\delta} = \varepsilon^{\frac{1}{d}}$. Thus $\mathcal{H}^d_{\tilde{\delta}}(A) \leq \varepsilon$ and $\varepsilon \to 0^+$ and then $\tilde{\delta} \to 0^+$ gives our conclusion. □

Lemma 1.11 (Isodiametric inequality)**.** *Let $A \subset \mathbb{R}^n$. The following inequality holds*

$$\lambda^n(A) \le \alpha_n (\operatorname{diam} A)^n.$$

Theorem 1.12 (Vitali Covering Lemma)**.** *Let $A \subset \mathbb{R}^n$ and $\mathcal{G}$ be a Vitali covering of A. Then there exists a disjoint Vitali covering $\mathcal{F} \subset \mathcal{G}$ such that*

$$\lambda^n \left(A \setminus \bigcup_{B \in \mathcal{F}} B \right) = 0.$$

We shall prove these later. Now our goal is to prove the Theorem 1.8.

Proof of Theorem 1.8. Let us first prove $\lambda^n(A) \le \mathcal{H}^n_N(A)$. Let $\delta > 0$. We find a covering $\{U_i\}_{i=1}^{\infty}$ of A satisfying $\operatorname{diam} U_i < \delta$ such that we obtain

$$\lambda^n(A) \le \lambda^n \left(\bigcup_{i=1}^{\infty} U_i \right) \le \sum_{i=1}^{\infty} \lambda^n(U_i) \le \sum_{i=1}^{\infty} \alpha_n (\operatorname{diam} A)^n \le \alpha_n \mathcal{H}^n_\delta(A) + \delta,$$

where the third inequality follows from isodiametric inequality. Sending $\delta \to 0^+$ gives the desired result.

Now we prove the converse inequality $\mathcal{H}^n_N(A) \le \lambda^n(A)$. Let B be an open set, $A \subset B$ and

$$\lambda^n(B) \le \lambda^n(A) + \varepsilon.$$

We find a Vitali covering of set B by closed balls as in Definiton 1.9 and by Theorem 1.12 there exists a disjoint closed covering $\{B_i\}_{i=1}^{\infty}$ of set B with $\operatorname{diam} B_i < \delta$ such that

$$\lambda^n \left(B \setminus \bigcup_{i=1}^{\infty} B_i \right) = 0.$$

It holds

$$\alpha_n \mathcal{H}^n_\delta \left(\bigcup_{i=1}^{\infty} B_i \right) \le \sum_{i=1}^{\infty} \alpha_n (\operatorname{diam} B_i)^n = \sum_{i=1}^{\infty} \lambda^n(B_i) = \lambda^n \left(\bigcup_{i=1}^{\infty} B_i \right) = \lambda^n(B) \le \lambda^n(A) + \varepsilon.$$

The first equality holds thanks to the fact that α_n is the volume of a unit sphere in $\mathbb{R}^n$ and B_i are closed balls in $\mathbb{R}^n$. The balls B_i are disjoint and so the second equality is also true. The third equality follows from

$$0 = \lambda^n \left(B \setminus \bigcup_{i=1}^{\infty} B_i \right) = \lambda^n(B) - \lambda^n \left(\bigcup_{i=1}^{\infty} B_i \right).$$

It remains to show that $\mathcal{H}^n_\delta \left(B \setminus \bigcup_{i=1}^{\infty} B_i \right) = 0$. Let $\{C_i\}_{i=1}^{\infty}$ be a covering by open pairwise disjoint dyadic cubes of some open[2] set C. It is obvious that $\operatorname{diam} C_i = \sqrt{n} l(C_i)$, where $l(C_i)$ denotes the side length of cube C_i. From this we can easily derive

$$\alpha_n \mathcal{H}^n_\infty(C) \le \alpha_n \sum_{i=1}^{\infty} (\operatorname{diam} C_i)^n \le \alpha_n \sqrt{n}^n \sum_{i=1}^{\infty} (l(C_i))^n.$$

[2]Open set can be covered with pairwise disjoint dyadic cubes so that the measure of the set and the measure of the union of the cubes is equal.

This gives us
$$\alpha_n \mathcal{H}^n_\infty(C) \leq \alpha_n \sqrt{n}^n \lambda^n(C).$$
Observe that $\lambda^n(C)$ is truly equal to $\sum_{i=1}^\infty (l(C_i))^n$. On the right we have the volume of the cubes which cover the set C and thus we obtain its Lebesgue measure. Now for all $\varepsilon > 0$ we find an open set $\bigcup_{i=1}^\infty B_i$ so that $B \setminus \bigcup_{i=1}^\infty B_i \subset G$ and $\lambda^n \left(G \setminus (B \setminus \bigcup_{i=1}^\infty B_i)\right) < \varepsilon$. We use this to determine the following
$$\begin{aligned}\alpha_n \mathcal{H}^n_\infty \left(B \setminus \bigcup_{i=1}^\infty B_i\right) &\leq \alpha_n \mathcal{H}^n_\infty(G) \leq \alpha_n \sqrt[n]{n} \lambda^n(G) \leq \alpha_n \sqrt{n}^n \lambda^n \left(B \setminus \bigcup_{i=1}^\infty B_i\right) + \alpha_n \sqrt{n}^n \varepsilon \\ &= 0 + \alpha_n \sqrt{n}^n \varepsilon.\end{aligned}$$
Sending ε to 0^+ yields the desired conclusion. Note that
$$\alpha_n \mathcal{H}^n_\infty \left(B \setminus \bigcup_{i=1}^\infty B_i\right) = 0$$
implies
$$\alpha_n \mathcal{H}^n \left(B \setminus \bigcup_{i=1}^\infty B_i\right) = 0$$
as stated in Lemma 1.10 and therefore we obtain
$$\alpha_n \mathcal{H}^n_\delta \left(B \setminus \bigcup_{i=1}^\infty B_i\right) = 0.$$
In total we have
$$\begin{aligned}\alpha_n \mathcal{H}^n_\delta(A) &\leq \alpha_n \mathcal{H}^n_\delta \left(A \cap \bigcup_{i=1}^\infty B_i\right) + \alpha_n \mathcal{H}^n_\delta \left(A \setminus \bigcup_{i=1}^\infty B_i\right) \\ &\leq \alpha_n \mathcal{H}^n_\delta \left(\bigcup_{i=1}^\infty B_i\right) + \alpha_n \mathcal{H}^n_\delta \left(B \setminus \bigcup_{i=1}^\infty B_i\right) \leq \lambda^n(A) + \varepsilon.\end{aligned}$$
Sending $\varepsilon \to 0^+$ and then $\delta \to 0^+$ yields the desired inequality. □

1.2.1 Proof of the Isodiametric Inequality

To prove this inequality, we define the so-called Steiner symmetrization which transforms a compact set $A \subset \mathbb{R}^n$ into a symmetric set A^*. We say a set is symmetric if $A = -A$, i.e. if $a \in A$, then $-a \in A$ for all $a \in A$. We prove that a certain symmetric set which we shall denote $S_a(A)$ has the same Lebesgue measure as A but $\operatorname{diam} S_a(A) \leq \operatorname{diam} A$. This leads to the desired inequality.

Let $a, b \in \mathbb{R}^n$ and $|a| = 1$. We denote
$$\begin{aligned}L^a_b &= \{b + ta \colon t \in \mathbb{R}\}, \\ P_a &= \{x \in \mathbb{R}^n \colon \langle a, x\rangle = 0\}.\end{aligned}$$
The first set describes a line through point b in the direction of a. Set P_a denotes a plane through the origin perpendicular to a.

Definition 1.13 (Steiner Symmetrization)**.** Let $a \in \mathbb{R}^n$, $|a| = 1$ and $A \subset \mathbb{R}^n$. Steiner symmetrization of A with respect to the plane P_a is defined in the following way

$$S_a(A) = \bigcup_{\substack{b \in P_a \\ A \cap L_a^b \neq \varnothing}} \left\{ b + ta \colon |t| \leq \frac{1}{2} \lambda(A \cap L_b^a) \right\}.$$

Informally, we replace the set $L_b^a \cap A$ with line S_a which has the same length and is symmetric with respect to P_a. One can easily observe by the use of Fubini theorem that both sets have the same Lebesgue measures. We can see how this works in the following figure:

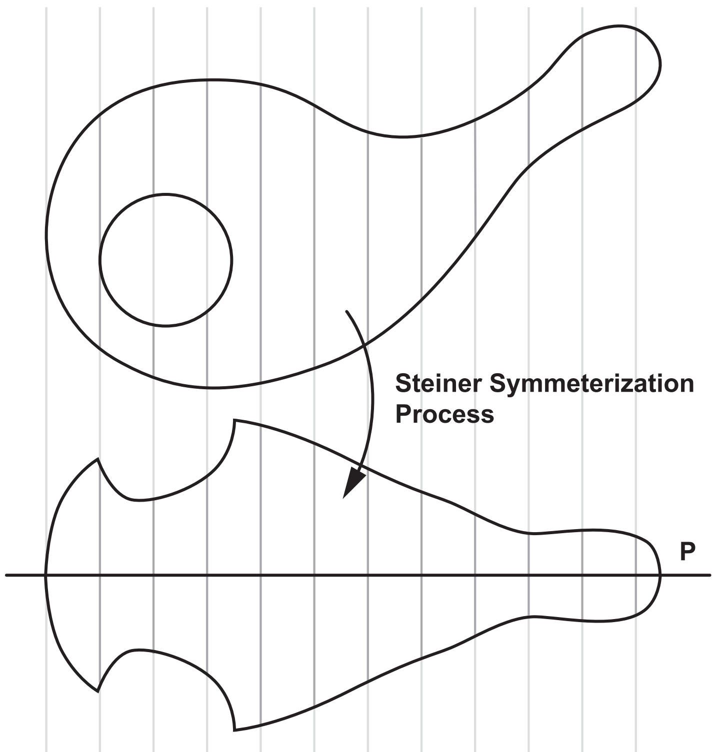

Figure 2: Steiner Symmetrization

Lemma 1.14. *Let $A \subset \mathbb{R}^n$. The following holds*

1. $\operatorname{diam} S_a(A) \leq \operatorname{diam} A$*.*

2. If A is measurable, then $\lambda^n(S_a(A)) = \lambda^n(A)$.

Proof. Recall that the Lebesgue measure is rotation invariant. Now we begin with the proof:

1. Assume $\operatorname{diam} A < \infty$ and that A is closed. If it was not so, we replace A with $\overline{A}$. Let $\varepsilon > 0$ and choose $x, y \in S_a(A)$ so that

$$\operatorname{diam} S_a(A) \leq |x - y| + \varepsilon.$$

Set $b = x - \langle a, x\rangle a$ and $c = y - \langle a, y\rangle a$. Using $|a| = 1$ we get $\langle x - \langle a, x\rangle a, a\rangle = 0$ and so $b \in P_a$. Showing $c \in P_a$ is analogous. We define

$$\begin{aligned} r &= \inf\{t \in \mathbb{R}\colon b + ta \in A\}, \\ s &= \sup\{t \in \mathbb{R}\colon b + ta \in A\}, \\ u &= \inf\{t \in \mathbb{R}\colon c + ta \in A\}, \\ v &= \sup\{t \in \mathbb{R}\colon c + ta \in A\}. \end{aligned}$$

Without any loss of generality we may assume that $v - r \geq s - u$, otherwise we would exchange the role of x and y. We may write

$$v - r \geq \frac{1}{2}(v - r) + \frac{1}{2}(s - u) = \frac{1}{2}(s - r) + \frac{1}{2}(v - u) \geq \frac{1}{2}\lambda(A \cap L_b^a) + \frac{1}{2}\lambda(A \cap L_c^a).$$

The last inequality follows from the fact that on the right hand-side we measure line segments, whereas on the left hand-side we have the length (measure) of a line connecting the first point of the first line segment and the last point of the last line segment. The equality happens if and only if it is a line segment. Since $x, y \in S_a(A)$ and we may write $x = b + ta$, where $b \in P_a$, $|a| = 1$ and $|t| \leq \frac{1}{2}\lambda(A \cap L_b^a)$, it is obvious that

$$|\langle a, x\rangle| \leq \frac{1}{2}\lambda(A \cap L_b^a)$$

and

$$|\langle a, y\rangle| \leq \frac{1}{2}\lambda(A \cap L_c^a).$$

It follows

$$v - r \geq |\langle a, x\rangle| + |\langle a, y\rangle| \geq |\langle a, x\rangle - \langle a, y\rangle|.$$

Thus we obtain

$$\begin{aligned} (\operatorname{diam} S_a(A) - \varepsilon)^2 &\leq |x - y|^2 = |b + \langle a, x\rangle a - (c + \langle a, y\rangle a)|^2 \\ &\leq |b - c|^2 + |\langle a, x\rangle - \langle a, y\rangle|^2 \leq |b - c|^2 + (v - r)^2 \\ &= |(b + ra) - (c + va)|^2 \leq (\operatorname{diam} A)^2. \end{aligned}$$

Since A is closed we have $b + ra \in A$ and $c + va \in A$. Sending $\varepsilon \to 0^+$ yields the result.

2. Without any loss of generality let $a = (0, \dots, 0, 1)^T$. Then $P_a = \mathbb{R}^{n-1}$. We define the map

$$f\colon \mathbb{R}^{n-1} \to \mathbb{R},\ f(b) = \lambda(A \cap L_b^a).$$

Note that $b \in P_a$. This map is measurable (proof can be found in [2]) and using Fubini's theorem we obtain

$$\lambda^n(A) = \int_{\mathbb{R}^{n-1}} f(b)\, \mathrm{d}b.$$

Therefore

$$S_a(A) = \left\{(b, y)\colon -\frac{f(b)}{2} \leq y \leq \frac{f(b)}{2}\right\} \setminus \{(b, 0)\colon A \cap L_b^a = \varnothing\}$$

is measurable since the area under a graph of a measurable function is measurable. Therefore

$$\lambda^n(S_a(A)) = \int_{\mathbb{R}^{n-1}} \int_{-\frac{f(b)}{2}}^{\frac{f(b)}{2}} 1 \,\mathrm{d}y \,\mathrm{d}b = \int_{\mathbb{R}^{n-1}} f(b) \,\mathrm{d}b = \lambda^n(A).$$

This concludes the proof. □

Now we can finally prove the isodiametric inequality.

Proof of Lemma 1.11. Let us assume that $\operatorname{diam} A < \infty$ and define the following sequence: $A_1 = S_{\mathbf{e}_1}(A)$, $A_2 = S_{\mathbf{e}_2}(A_1), \ldots, A_n = S_{\mathbf{e}_n}(A_{n-1})$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{R}^n$. We divide the proof into smaller steps:

1. We firstly show that $A^* = A_n$ is symmetric with respect to the origin. It is enough to show that it is symmetric with respect to $P_{\mathbf{e}_1}, \ldots, P_{\mathbf{e}_n}$. Indeed, let $(a_1, \ldots, a_n) \in A^*$ and let us assume that A^* is symmetric with respect to $P_{\mathbf{e}_1}, \ldots, P_{\mathbf{e}_n}$. Since A^* is symmetric with respect to $P_{\mathbf{e}_1}$, we may write $(-a_1, \ldots, a_n) \in A^*$. This set is also symmetric with respect to $P_{\mathbf{e}_2}$, so $(-a_1, -a_2, \ldots, a_n) \in A^*$. We proceed in the same manner by induction. A^* is symmetric with respect to $P_{\mathbf{e}_n}$ and thus $(-a_1, -a_2, \ldots, -a_n) \in A^*$. This is the same as saying that A^* is symmetric with respect to the origin.

   We claim that A_k is symmetric with respect to $P_{\mathbf{e}_j}$ for $j \in \{1, \ldots, k\}$. We prove this statement by induction. We defined A_1 to be symmetric with respect to $P_{\mathbf{e}_1}$. Let $1 \leq k < n$ and assume A_k is symmetric with respect to $P_{\mathbf{e}_1}, \ldots, P_{\mathbf{e}_k}$. Again, by definition, $A_{k+1} = S_{\mathbf{e}_{k+1}}(A_k)$ is symmetric with respect to $P_{\mathbf{e}_{k+1}}$. Let $1 \leq j \leq k$ and S_j is Steiner Symmetrization with respect to $P_{\mathbf{e}_j}$. Let $b \in P_{\mathbf{e}_{k+1}}$. Set A_k is already symmetric since $1 \leq j \leq k$ and therefore $S_j(A_k) = A_k$ by induction hypothesis. The following holds

   $$\lambda\left(A_k \cap L_b^{\mathbf{e}_{k+1}}\right) = \lambda\left(A_k \cap L_{S_j(b)}^{\mathbf{e}_{k+1}}\right).$$

   It follows that

   $$\{t \in \mathbb{R} \colon b + t\mathbf{e}_{k+1} \in A_{k+1}\} = \{t \in \mathbb{R} \colon S_j(b) + t\mathbf{e}_{k+1} \in A_{k+1}\}$$

   and we get $S_j(A_{k+1}) = A_{k+1}$. That means A_{k+1} is symmetric with respect to $P_{\mathbf{e}_j}$. This gives us that A^* is symmetric with respect to $P_{\mathbf{e}_1}, \ldots, P_{\mathbf{e}_n}$.

2. We wish to show

   $$\lambda^n(A^*) \leq \alpha_n(\operatorname{diam} A^*)^n.$$

   Let $x \in A^*$. Set A^* is symmetric and so $-x \in A^*$ as follows from the previous part of the proof and $2|x| \leq \operatorname{diam} A^*$. Set A^* is thus contained in a ball with radius $\frac{\operatorname{diam} A^*}{2}$ and center in the origin. It follows that

   $$\lambda^n(A^*) \leq \lambda^n\left(B\left(0, \frac{\operatorname{diam} A^*}{2}\right)\right) = \alpha_n(\operatorname{diam} A^*)^n.$$

3. The final step is to get the isodiametric inequality itself. $\overline{A}$ is closed and bounded and therefore compact in $\mathbb{R}^n$. That means it is also measurable. Using Lemma 1.14 we get

$$\lambda^n(A) \leq \lambda^n\left(\overline{A}\right) = \lambda^n\left(\left(\overline{A}\right)^*\right) \leq \alpha_n\left(\operatorname{diam}\left(\overline{A}\right)^*\right)^n \leq \alpha_n\left(\operatorname{diam}\overline{A}\right)^n = \alpha_n(\operatorname{diam} A)^n.$$

□

1.2.2 Proof of the Vitali Covering Lemma

Before we begin with the proof itself, we show an additional lemma.

Lemma 1.15. *Let $A \subset \mathbb{R}^n$. Let $\mathcal{F}$ be a covering by closed balls of set A with $\sup_{B\in\mathcal{F}} r(B) < \infty$. There exists a disjoint covering of A by closed balls $\mathcal{F}'$ such that for all $B \in \mathcal{F}$ there exists a ball $C \in \mathcal{F}'$, $B \cap C \neq \varnothing$, and $B \subset 5C$, where $5C$ denotes a ball with five times the radius of ball C.*

Proof. Let $R = \sup_{B\in\mathcal{F}} r(B)$ and for all $k \in \mathbb{N}$ we define

$$\mathcal{F}_k = \left\{B \in \mathcal{F} \colon r(B) \in \left(\frac{R}{2^k}, \frac{R}{2^{k-1}}\right]\right\}.$$

Now we inductively define a sequence of systems of closed balls $\mathcal{B}_k \subset \mathcal{F}_k$, $k \in \mathbb{N}$, in the following way: $\mathcal{B}_1$ is an arbitrary maximum disjoint subsystem of $\mathcal{F}_1$. For $\mathcal{B}_1, \dots, \mathcal{B}_{k-1}$ defined $\mathcal{B}_k$ is an arbitrary maximum disjoint subsystem of

$$\{B \in \mathcal{F}_k \colon B \cap C = \varnothing \text{ for all } C \in \mathcal{B}_1 \cup \dots \cup \mathcal{B}_{k-1}\}.$$

Let $\mathcal{F} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$. This system of balls is pairwise disjoint and contains closed balls. It is also a subsystem of $\mathcal{F}$. Let $B \in \mathcal{F}$. Then there exists $k \in \mathbb{N}$ such that $B \in \mathcal{F}_k$ and therefore there also exists $C \in \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ with $B \cap C \neq \varnothing$. We wish to show that $B \subset 5C$. It is fairly obvious that $r(C) > \frac{R}{2^k}$. Thus $r(B) < 2r(C)$. Let $B = B(x_1, r_1)$ and $C = C(x_2, r_2)$. Let $y \in B$. It holds $\|x_1 - x_2\| \leq r_1 + r_2$ since the balls are not disjoint. It follows

$$\|y - x_2\| \leq \|y - x_1\| + \|x_1 - x_2\| \leq r_1 + r_1 + r_2 \leq 5r_2,$$

and therefore $y \in 5C$. This concludes the proof. □

Now we can prove the Vitali covering lemma.

Proof of Theorem 1.12. We may assume that our covering consists of balls with radius smaller than or equal to one since removing balls (of "big" radius) from Vitali covering does not change the fact that we still have a Vitali covering. Call this covering $\mathcal{G}$. By the previous lemma there exists a disjoint closed covering $\mathcal{F} \subset \mathcal{G}$ such that for all $B \in \mathcal{G}$ there exists $C \in \mathcal{F}$, $B \cap C \neq \varnothing$ and $B \subset 5C$. We wish to show

$$\lambda\left(\left(A \setminus \bigcup_{B\in\mathcal{F}} B\right) \cap B(0,r)\right) = 0 \quad \text{for all } r > 0.$$

Let $\varepsilon > 0$ and we take a subcovering $\mathcal{F}' \subset \mathcal{F}$ such that all balls in $\mathcal{F}'$ intersect $B(0,r)$. It is obvious that all balls of $\mathcal{F}'$ are contained in $B(0,r+2)$ since they intersect $B(0,r)$ and their radius is smaller than or equal to one. Therefore

$$\sum_{B(x,r')\in\mathcal{F}'} \lambda(B(x,r')) \leq \lambda(B(0,r+2)) < \infty. \tag{1}$$

Fix $\delta > 0$ such that

$$\sum_{\substack{B(x,r')\in\mathcal{F}' \\ r'<\delta}} \lambda(B(x,r')) < \frac{\varepsilon}{5^n}.$$

There are balls with $r' \geq \delta$ and there are only finitely many of them by (1). We set $G = \{B(x,r') \in \mathcal{F}' \colon r' \geq \delta\}$. Then G is a closed set as a finite union of closed sets. We have a Vitali cover and so for all $x \in \big(A \setminus \bigcup_{B\in\mathcal{F}} B\big) \cap B(0,r)$ there exists a ball $B_x \in \mathcal{G}$ such that $x \in B_x$, $B_x \subset B(0,r)$ and $B_x \cap G = \varnothing$. It is obvious that B_x intersects some ball $C_x \in \mathcal{F}'$ since $B_x \subset B(0,r)$. Also note that the radius of C_x is smaller than δ since $B_x \cap G = \varnothing$. Thus $B_x \subset 5C_x$ and we may write

$$\begin{aligned} \lambda\left(\left(A \setminus \bigcup_{B\in\mathcal{F}} B\right) \cap B(0,r)\right) &\leq \sum_{x\in\big(A\setminus\bigcup_{B\in\mathcal{F}} B\big)\cap B(0,r)} \lambda(B_x) \leq \sum_{x\in\big(A\setminus\bigcup_{B\in\mathcal{F}} B\big)\cap B(0,r)} \lambda(5C_x) \\ &\leq 5^n \sum_{\substack{B(x,r')\in\mathcal{F}' \\ r'<\delta}} \lambda(B(x,r')) < \varepsilon. \end{aligned}$$

Therefore $\big(A \setminus \bigcup_{B\in\mathcal{F}} B\big) \cap B(0,r)$ is a null set and the proof is finished. $\square$

2 Hausdorff Dimension

In this chapter we define the Hausdorff dimension. We draw from [1] and [3].

Theorem 2.1. *Let $0 \leq d < s$ and $A \subset \mathbb{R}^n$. The following holds:*

$$\mathcal{H}^d(A) < \infty \Rightarrow \mathcal{H}^s(A) = 0,$$
$$\mathcal{H}^s(A) > 0 \Rightarrow \mathcal{H}^d(A) = \infty.$$

Proof. Let $\mathcal{H}^d(A) < \infty$ and $\delta > 0$. We find a cover $\{U_i\}_{i=1}^{\infty}$ of A such that $\operatorname{diam} U_i < \delta$ and

$$\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \leq \mathcal{H}^d_\delta(A) + 1.$$

We know that $\mathcal{H}^d_\delta(A) \leq \mathcal{H}^d(A)$. Therefore

$$\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d \leq \mathcal{H}^d(A) + 1.$$

Let $s > d$. It follows

$$\mathcal{H}^s_\delta(A) \leq \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s = \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d (\operatorname{diam} U_i)^{d-s} \leq \delta^{d-s}(\mathcal{H}^d(A) + 1).$$

Letting $\delta \to 0^+$ yields $\mathcal{H}^s(A) = 0$. □

Definition 2.2 (Hausdorff dimension)**.** Let $A \subset \mathbb{R}^n$. We define the Hausdorff dimension as

$$\dim_H A = \inf\{d \geq 0 \colon \mathcal{H}^d(A) = 0\}.$$

We mention two important properties: Let $A \subset B \subset \mathbb{R}^n$ and $A_i \subset \mathbb{R}^n$, $i \in \mathbb{N}$, then

$$\dim_H A \leq \dim_H B,$$
$$\dim_H \bigcup_{i=1}^{\infty} A_i = \sup_{i \in \mathbb{N}} \dim_H A_i.$$

The first property follows from monotonicity of Hausdorff measure. If $\mathcal{H}^d(B) = 0$, then $\mathcal{H}^d(A) = 0$ and so $\inf\{t \colon \mathcal{H}^d(A) = 0\} \leq \inf\{t \colon \mathcal{H}^d(B) = 0\}$. The latter property follows from monotonicity and subadditivity: $\mathcal{H}^d(A_i) = 0$ for all $i \in \mathbb{N}$ if and only if $\mathcal{H}^d\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$.

3 Capacitary Dimension

This chapter relies almost entirely on Mattila's book [3], especially when it comes to proving the Frostman's Lemma (Theorem 3.13). Proofs have been completed or expanded upon.

Before we are able to define capacitary dimension we need to understand what a Radon measure is and what it means for a measure to have a compact support. With this knowledge in hand we are able to define the capacitary dimension and lay a foundation for Frostman's lemma which is key to proving the equality of capacitary and Hausdorff dimensions.

We start with some elementary definitions. We are working in $\mathbb{R}^n$, however a Radon measure, support of a measure and terms derived from this may be defined on a Hausdorff topological space.

3.1 Essential definitions and properties

In the following definitions we assume μ to be a measure on $\mathcal{B}^n := \mathcal{B}(\mathbb{R}^n)$.

Definition 3.1 (Inner regular measure)**.** We say that the measure μ is inner regular or tight if, for all U open it holds that

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ is compact}}} \mu(K).$$

Definition 3.2 (Locally finite measure)**.** The measure μ is said to be locally finite if for every point $x \in \mathbb{R}^n$ there exists $r > 0$ such that $\mu(B(x,r))$ is finite.

Definition 3.3 (Radon measure)**.** The measure μ is said to be a Radon measure if it is inner regular and locally finite.

Definition 3.4 (Support of a measure)**.** The support of measure μ is the smallest closed set $F \subset \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus F) = 0$. We denote the support of measure μ as $\operatorname{supp}(\mu)$.

Let us assume that μ is a Radon measure and denote

$$I_s(\mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^s} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y).$$

We shall soon see that this is key for the definition of capacitary dimension. Before that, we introduce two lemmas and Banach-Alaoglu theorem. Both of these lemmas will be proved. One is essential to prove Proposition 3.9 and the other to prove Frostman's lemma (Theorem 3.13). The Banach-Alaoglu theorem, which we need to prove Frostman's lemma, will not be proved as it is a well-known result from functional analysis and proving it would be above the scope of this thesis.

Lemma 3.5. *Let μ be a Borel measure and f a non-negative Borel function on a separable metric space X. Then*

$$\int_X f\,\mathrm{d}\mu = \int_0^\infty \mu(\{x \in X\colon f(x) \geq s\})\,\mathrm{d}s.$$

Proof. Let $A = \{(x,s) \in X \times [0,\infty)\colon f(x) \geq s\}$. By using Fubini's theorem we immediately see that

$$\begin{aligned}\int_0^\infty \mu(\{x \in X\colon f(x) \geq s\})\,\mathrm{d}s &= \int_0^\infty \mu(\{x \in X\colon (x,s) \in A\})\,\mathrm{d}s\\ &= \int_0^\infty \int_X \chi_A(x,s)\,\mathrm{d}\mu(x)\,\mathrm{d}s\\ &= \int_X \lambda(\{s \in [0,\infty)\colon (x,s) \in A\})\,\mathrm{d}\mu(x)\\ &= \int_X \lambda([0,f(x)])\,\mathrm{d}\mu(x) = \int_X f(x)\,\mathrm{d}\mu(x).\end{aligned}$$

Thus we obtain the desired result. □

Using this lemma we may obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{|x-y|^s}\,\mathrm{d}\mu(y) &= \int_0^\infty \mu\left(\left\{y\colon \frac{1}{|x-y|^s} \geq t\right\}\right)\mathrm{d}t = \int_0^\infty \mu\left(B\left(x, \frac{1}{t^{\frac{1}{s}}}\right)\right)\mathrm{d}t\\ &= s\int_0^\infty r^{-s-1}\mu(B(x,r))\,\mathrm{d}r. \qquad (2)\end{aligned}$$

Lemma 3.6. *Let $\mu_1, \mu_2, \ldots$ be Radon measures on a locally compact metric space. If $\mu_i \xrightarrow{w^*} \mu$ and $G \subset X$ is open, then*

$$\mu(G) \leq \liminf_{i\to\infty} \mu_i(G).$$

Proof. Let $\mu_i \xrightarrow{w^*} \mu$, then for any $f \in \mathcal{C}_c(X)$ it holds

$$\lim_{i\to\infty} \int_X f\,\mathrm{d}\mu_i = \int_X f\,\mathrm{d}\mu$$

Here we use a variation of the Riesz representation theorem from functional analysis. We formulate the theorem for clarity:

Let X be a locally compact Hausdorff space and let φ be a positive linear functional on $\mathcal{C}_c(X)$. Then there exists a unique Radon measure μ on X such that $\varphi(f) = \int_X f\,\mathrm{d}\mu$ for all $f \in \mathcal{C}_c(X)$.

Proof can be found in Danny Espejo's paper [6].

Now let

$$G_k = \left\{x \in X\colon \operatorname{dist}(x, X \setminus G) \leq \frac{1}{k}\right\}$$

for some $\varepsilon > 0$ and set $G = \bigcup_{k=1}^\infty G_k$. Then $\mu(G) = \lim_{k\to\infty} \mu(G_k)$ as $G_1 \subset G_2 \subset \ldots$ and G and G_k are open (thus measurable) for all $k \in \mathbb{N}$. Let f_k be a continuous function with compact support such that $0 \leq f_k \leq 1$ on X, $f_k = 1$ on G_k and $f_k = 0$ outside G. Existence of f_k follows directly from Urysohn's lemma. This lemma can indeed be used as every metric space is normal. We obtain

$$\mu(G_k) \leq \int_X f_k \,\mathrm{d}\mu = \lim_{i\to\infty} \int_X f_k \,\mathrm{d}\mu_i \leq \liminf_{i\to\infty} \mu_i(G). \tag{3}$$

The last inequality holds as $\int_X f_k \,\mathrm{d}\mu_i \leq \mu_i(G)$ and we apply limit inferior to both sides. Using $k \to \infty$ in (3) yields the desired result. □

Theorem 3.7 (Banach-Alaoglu). *Let X be a normed separable space. Then every bounded sequence in the dual space X^* has a weak* convergent subsequence.*

Let us just point out that this is not the standard formulation of the Banach-Alaoglu theorem. It is however a direct corollary. The proof of the standard version can be found in [5].

3.2 Capacitary dimension and Frostman's lemma

Definition 3.8. Let $A \subset \mathbb{R}^n$. The capacitary dimension of A is defined in the following way

$$\begin{aligned}\dim_c A &= \sup\{d\colon \text{There exists } \mu \in \mathcal{M}(A) \text{ such that } \mu(B(x,r)) \leq r^d, x \in \mathbb{R}^n, d > 0\}\\ &= \sup\{s\colon \text{There exists } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(A) = \{\mu\colon &\mu \text{ is a Radon measure with compact support},\\ &\operatorname{supp}(\mu) \subset A, 0 < \mu(\mathbb{R}^n) < \infty\}.\end{aligned}$$

The following proposition shows that the equality in the preceding definition holds.

Proposition 3.9. *The following equality holds*

$$\begin{aligned}&\sup\{d\colon \text{There exists } \mu \in \mathcal{M}(A) \text{ such that } \mu(B(x,r)) \leq r^d, x \in \mathbb{R}^n, d > 0\}\\ &= \sup\{s\colon \text{There exists } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.\end{aligned}$$

Proof. If $\mu \in \mathcal{M}(A)$ and we have some $d > s$ for which $\mu(B(x,r)) \leq cr^d$, where $x \in \mathbb{R}^n$ and $r > 0$, then

$$\begin{aligned}I_s(\mu) &= \int_{\mathbb{R}^n} s \int_0^\infty r^{-s-1}\mu(B(x,r)) \,\mathrm{d}r \,\mathrm{d}\mu(x)\\ &\leq \int_{\mathbb{R}^n} s \underbrace{\int_0^1 r^{d-s-1} \,\mathrm{d}r}_{<\infty} \,\mathrm{d}\mu(x) + \int_{\mathbb{R}^n} \mu(X) \cdot s \int_1^\infty r^{-s-1} \,\mathrm{d}r \,\mathrm{d}\mu(x) < \infty,\end{aligned}$$

for $d > s$, where we use (2) from discussion above in the first inequality.

Now let us assume $\mu \in \mathcal{M}(A)$ and define a Borel set

$$A = \left\{ x \colon \int_{\mathbb{R}^n} \frac{1}{|x-y|^d} \, \mathrm{d}\mu(y) \leq M \right\}.$$

If $I_d(\mu) < \infty$, then there exists some constant M for which the set A has positive measure. Let $\nu = \mu\big|_A$. It holds that

$$\frac{1}{r^d}\nu(B(x,r)) \leq \int_{B(x,r)} \frac{1}{|x-y|^d} \, \mathrm{d}\nu(y) \leq M,$$

where $x \in A$ and $r > 0$. Let $x \in \mathbb{R}^n$ and $r > 0$. If $B(x,r) \cap A = \varnothing$, then $\nu(B(x,r)) = 0$. If there exists $z \in B(x,r) \cap A$, then

$$\frac{1}{r^d}\nu(B(x,r)) \leq 2^d \frac{1}{(2r)^d}\nu(B(z,2r)) \leq 2^d M.$$

Choosing measure $\eta = \frac{1}{2^d M}\nu$ yields the desired result. □

Definition 3.10. Let $s > 0$. The Riesz s-capacity of $A \subset \mathbb{R}^n$ is defined by

$$C_s(A) = \sup\{I_s(\mu)^{-1} \colon \mu \in \mathcal{M}(A), \mu(\mathbb{R}^n) = 1\}.$$

We also define $C_s(\varnothing) = 0$.

Definition 3.11. Let $s > 0$ and $A \subset \mathbb{R}^n$. Then we define

$$\dim_c A = \sup\{s \colon C_s(A) > 0\} = \inf\{s \colon C_s(A) = 0\}.$$

Theorem 3.12. *Let $A \subset \mathbb{R}^n$.*

1. *If $s > 0$ and $\mathcal{H}^s(A) < \infty$, then $C_s(A) = 0$.*
2. *$\dim_c A \leq \dim_H A$.*

Proof. We consecutively prove both statements:

1. We prove this by contradiction. Let us assume that $C_s(A) > 0$, $A \subset \mathbb{R}^n$. There exists $\mu \in \mathcal{M}(A)$ so that $\mu(A) = 1$ and $I_s(\mu) < \infty$. Since

$$I_s(\mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|x-y|^s} \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) < \infty,$$

then for almost all $x \in \mathbb{R}^n$ we obtain

$$\lim_{r \to 0^+} \int_{B(x,r)} \frac{1}{|x-y|^s} \, \mathrm{d}\mu(y) = 0.$$

Therefore, for an arbitrary $\varepsilon > 0$, there exists $B \subset A$ and $\delta > 0$ such that $\mu(B) > \frac{1}{2}$ and

$$\mu(B(x,r)) \leq r^s \int_{B(x,r)} \frac{1}{|x-y|^s} \, \mathrm{d}\mu(y) \leq \varepsilon r^s \quad \text{for all } x \in B \text{ and } 0 < r \leq \delta.$$

Indeed, let

$$A_k = \left\{ x \in A\colon \int_{B(x,r)} \frac{1}{|x-y|^s} \,\mathrm{d}\mu(y) \le r^s \text{ for all } r \in \left(0, \frac{1}{k}\right) \right\}$$

Then $A_k \nearrow A$ and thus there exists $k \in \mathbb{N}$ such that $\mu(A_k) > \frac{1}{2}\mu(A)$. We put $B = A_k$ and $\delta = \frac{1}{k}$.

Now we find sets $B_1, B_2, \dots$ so that $B \subset \bigcup_{i\in\mathbb{N}} B_i$, $B \cap B_i \neq \varnothing$, $\operatorname{diam} B_i \le \delta$ for all $i \in \mathbb{N}$ and

$$\sum_{i\in\mathbb{N}} (\operatorname{diam} B_i)^s \le \mathcal{H}^s(A) + 1.$$

Let $x_i \in B \cap B_i$ and set $r_i = \operatorname{diam}(B_i)$, then we obtain

$$\frac{1}{2} < \mu(B) \le \sum_{i\in\mathbb{N}} \mu(B(x_i, r_i)) \le \varepsilon \sum_{i\in\mathbb{N}} r_i^s \le \varepsilon(\mathcal{H}^s(A) + 1).$$

Sending $\varepsilon \to 0^+$ we get $\mathcal{H}^s(A) = \infty$, which is the contradiction.

2. This follows directly from the first statement.

This conclude the proof. □

Now we prove the Frostman's lemma which is key to proving $\dim_H A \le \dim_c A$, $A \subset \mathbb{R}^n$ is a Borel set. As we have already proven the converse inequality in the previous theorem, we eventually get $\dim_H A = \dim_c A$, which is our objective.

Theorem 3.13 (Frostman's lemma)**.** *Let $A \subset \mathbb{R}^n$ be a Borel set. Then $\mathcal{H}^s(A) > 0$ if and only if there exists nonzero $\mu \in \mathcal{M}(A)$ such that $\mu(B(x,r)) \le r^s$ for $x \in \mathbb{R}^n$ and $r > 0$. Moreover, we can find μ so that $\mu(A) \ge c\mathcal{H}^s_\infty(A)$ where $c > 0$ depends only on n.*

Proof. Let us first assume that $\mu(B(x,r)) \le r^s$ for $x \in \mathbb{R}^d$ and $r > 0$ arbitrary. Let $\varepsilon > 0$. We find a covering of A by closed balls $\{B_i\}$ so that $A \subset \bigcup_{i\in\mathbb{N}} B_i$, $A \cap B_i \neq \varnothing$, $\operatorname{diam} B_i \le \delta$ for all $i \in \mathbb{N}$ and

$$\sum_{i\in\mathbb{N}} (\operatorname{diam} B_i)^s \le \mathcal{H}^s_\delta(A) + \varepsilon \le \mathcal{H}^s(A) + \varepsilon.$$

Choose $x_i \in A \cap B_i$ and set $r_i = \operatorname{diam}(B_i)$. Thus we obtain

$$0 < \mu(A) \le \sum_{i\in\mathbb{N}} \mu(B(x_i, r_i)) \le \sum_{i\in\mathbb{N}} r_i^s \le \mathcal{H}^s(A) + \varepsilon.$$

Letting $\varepsilon \to 0^+$ we get $\mathcal{H}^s(A) > 0$.

It remains to show the converse implication. Let us assume that A is compact and that A is contained in a dyadic cube. Since $\mathcal{H}^s(A) > 0$, then $\mathcal{H}^s_\infty(A) > 0$. Therefore for any covering of A by cubes $\{Q_i\}_{i\in\mathbb{N}}$ there exists a constant $c > 0$ depending on $n \in \mathbb{N}$ so that

$$\sum_{i\in\mathbb{N}} (\operatorname{diam} Q_i)^s \ge c\mathcal{H}^s_\infty(A).$$

For $m \in \mathbb{N}$ denote $\mathcal{Q}_m$ the family of dyadic cubes of $\mathbb{R}^n$ with side-length $\frac{1}{2^m}$. For $Q \in \mathcal{Q}_m$ we define the measure μ_m^m in the following way:

$$\begin{aligned} \mu_m^m|_Q &= \tfrac{1}{2^{ms}} \tfrac{1}{\lambda^n(Q)} \lambda^n|_Q, \quad \text{if } A \cap Q \neq \varnothing, \\ \mu_m^m|_Q &= 0, \qquad\qquad\quad \text{if } A \cap Q = \varnothing, \end{aligned}$$

Furthermore, for $Q \in \mathcal{Q}_{m-1}$ we define

$$\begin{aligned} \mu_{m-1}^m|_Q &= \mu_m^m|_Q, \qquad\qquad\qquad \text{if } \mu_m^m(Q) \leq \tfrac{1}{2^{(m-1)s}}, \\ \mu_{m-1}^m|_Q &= \tfrac{1}{2^{(m-1)s}} \tfrac{1}{\mu_m^m(Q)} \mu_m^m|_Q, \quad \text{if } \mu_m^m(Q) > \tfrac{1}{2^{(m-1)s}}. \end{aligned}$$

We proceed in this manner. For $Q \in \mathcal{Q}_{m-k-1}$ we get μ_{m-k-1}^m from μ_{m-k}^m as follows:

$$\mu_{m-k-1}^m|_Q = \eta(Q) \mu_{m-k}^m|_Q,$$

where $\eta(Q) = \min\{1, \frac{1}{2^{(m-k-1)s}} \frac{1}{\mu_{m-k}^m(Q)}\}$. We finish this process if $A \subset Q$ for some $Q \in \mathcal{Q}_{m-k_0}$ and set $\mu^m = \mu_{m-k_0}^m$. It is vital to realize that the measure of the dyadic cubes does not increase, and so $\mu^m(Q) \leq \frac{1}{2^{(m-k)s}}$ for any $Q \in \mathcal{Q}_{m-k}$, $k \in \mathbb{N}$. Also, for all $x \in A$ there exist $k \in \mathbb{N}$ and $Q \in \mathcal{Q}_{m-k}$ such that $x \in Q$ and

$$\mu^m(Q) = \frac{1}{2^{(m-k)s}} = \left(\frac{1}{\sqrt{n}}\right)^s \left(\frac{\sqrt{n}}{2^{m-k}}\right)^s = \frac{1}{n^{\frac{s}{2}}} (\operatorname{diam} Q)^s. \tag{4}$$

For each $x \in A$ we choose the largest such Q. Since A is compact, we obtain a finite covering of A by cubes $Q_1, \dots, Q_k$ so that

$$\mu^m(\mathbb{R}^n) = \sum_{i=1}^k \mu^m(Q_i) = \frac{1}{n^{\frac{s}{2}}} \sum_{i=1}^k (\operatorname{diam} Q_i)^s \geq \frac{c}{n^{\frac{s}{2}}} \mathcal{H}_\infty^s(A). \tag{5}$$

Set $\nu^m = \frac{\mu^m}{\mu^m(\mathbb{R}^n)}$, then $\nu^m(\mathbb{R}^n) = 1$ and for $Q \in \mathcal{Q}_{m-k}$ using (4) and (5) we get

$$\nu^m(Q) \leq \frac{1}{2^{(m-k)s}} \frac{n^{\frac{s}{2}}}{c\mathcal{H}_\infty^s(A)}. \tag{6}$$

The sequence $\{\nu^m\}$ has a weakly convergent sub-sequence $\{\nu^{m_i}\} \xrightarrow{w} \nu$, which follows from Theorem 3.7. From this it follows that $\nu \in \mathcal{M}(A)$ and $\nu(A) = 1$. Indeed, let φ be a continuous function with compact support, $0 \leq \varphi \leq 1$ and $\varphi = 1$ on a neighbourhood of A. Then

$$\int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu = \lim_{m \to \infty} \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu^m = \lim_{m \to \infty} \nu^m(\mathbb{R}^n) = 1.$$

As ν is Radon, we get $\inf_\varphi \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu = \nu(A)$.

For all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ it holds that $B(x,r) \subset \operatorname{int} \bigcup_{i=1}^{2^n} Q_i$, where $Q \in \mathcal{Q}_p$ with $\operatorname{diam} Q_i = \frac{n^{\frac{1}{2}}}{2^p} \leq 4 n^{\frac{1}{2}} r$. Therefore, for all $m \geq p$ using (6) we obtain

$$\nu^m\left(\operatorname{int} \bigcup_{i=1}^{2^n} Q_i\right) \leq \frac{2^n}{c\mathcal{H}_\infty^s(A)} \frac{n^{\frac{s}{2}}}{2^{ps}} \leq \frac{2^{n+2s} n^{\frac{s}{2}} r^s}{c\mathcal{H}_\infty^s(A)}.$$

Finally, by Lemma 3.6 we get

$$\nu(B(x,r)) \leq \nu^m \left(\operatorname{int} \bigcup_{i=1}^{2^n} Q_i \right) \leq \liminf_{i \to \infty} \nu^{m_i} \left(\operatorname{int} \bigcup_{i=1}^{2^n} Q_i \right) \leq \frac{2^{n+2s} n^{\frac{s}{2}} r^s}{c\mathcal{H}^s_\infty(A)}.$$

Thus $\nu(B(x,r)) \leq Cr^s$, as we wanted. Setting $\mu = \frac{c\mathcal{H}^s_\infty(A)}{2^{n+2s} n^{\frac{s}{2}}} \nu$ finishes the proof. □

Corollary. Let $A \subset \mathbb{R}^n$ be a Borel set. If $s > 0$ and $C_s(A) = 0$, then $\mathcal{H}^d(A) = 0$ for $d > s$. From this also follows that $\dim_H(A) \leq \dim_c(A)$

Proof. We prove this by contrapositive. If $\mathcal{H}^d(A) > 0$, then from Theorem 3.13 there exists $\mu \in \mathcal{M}(A)$ such that $\mu(B(x,r)) \leq r^d$. We already know from the proof of Proposition 3.9, that for $0 < s < d$ we have $I_s(\mu) < \infty$. Therefore $C_s(A) > 0$ and the proof is finished. □

Theorem 3.14 (Equality of Hausdorff and Capacitary dimension)**.** *Let $A \subset \mathbb{R}^n$ be a Borel set. Then $\dim_c A = \dim_H A$.*

Proof. This follows directly from the previous corollary and Theorem 3.12. □

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