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MASTER THESIS

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Combinatorial structure of graph drawings

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Study programme: Computer Science
Study branch: Discrete models and algorithms

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Abstract: In this thesis, we study characterization by forbidden patterns of many classes of x-monotone drawings of complete graphs with various given restrictions. We generalize previously known characterizations of pseudolinear, semisimple, and simple drawings of K_n by showing that also bounded pseudoparabola drawings of K_n can be characterized by finite forbidden patterns. On the other hand, we show that there is no such finite characterization for extended pseudoparabola drawings of K_n. We strengthen our results even further to so-called (d_a, d_i)-degree drawings where given non-negative integers d_a and d_i represent a number of crossings between adjacent and independent edges, respectively. We provide a full characterization by forbidden patterns of each class of (d_a, d_i)-degree drawings.

Keywords: graph drawing, signotope, pseudolines, monotone curves

Contents

1. Introduction

1.1 Preliminaries

Let G be a graph without loops and multiple edges. A (bounded) *drawing* D of G in the plane is an image of a mapping that maps vertices to distinct points and edges to continuous arcs connecting the images of their endpoints. Formally, we have an injective function $f\colon V \to \mathbb{R}^2$, mapping vertices from V to the plane. Each edge $e = \{u, v\}$ corresponds to the image of the interval $[0, 1]$ and via the continuous mapping mapping $g\colon [0,1] \to \mathbb{R}^2$ such that $g(0) = f(u)$ and $g(1) = f(v)$.

An *extended drawing* D of G in the plane is the image of a mapping that maps vertices to distinct points and edges to continuous arcs connecting the images of their endpoints extended to infinity on both ends. Formally, the vertex function is the same again. However, for edges, we have different functions. Each edge $e = \{u, v\}$ corresponds to the image of the real line $\mathbb{R}$ via mapping $g\colon \mathbb{R} \to \mathbb{R}^2$ such that $g(0) = f(u)$ and $g(1) = f(v)$ and $g([0, 1])$ corresponds to the (bounded) part of the edge e. See an example of a bounded and an extended drawing in Figure 1.1.

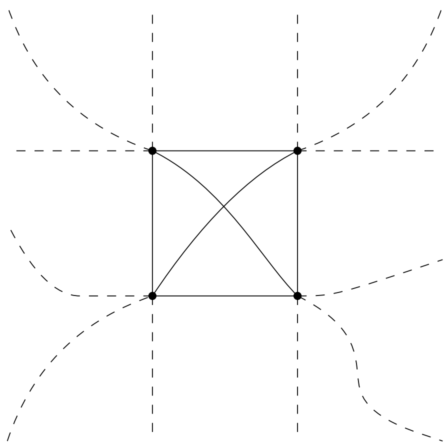


Figure 1.1: Full curves represent a bounded drawing of K_4. Together with the dashed parts they represent the extended drawing of K_4.

We sometimes do not distinguish between a graph and its drawing, in particular, we identify edges and the arcs representing them, and vertices with the points representing them as well. For simplicity, we assume that the following four conditions are satisfied:

1. no edge goes through a vertex that is not its endpoint,
2. no two edges touch at an interior point,
3. no three edges meet at one common interior point,
4. any two edges share a finite number of intersections.

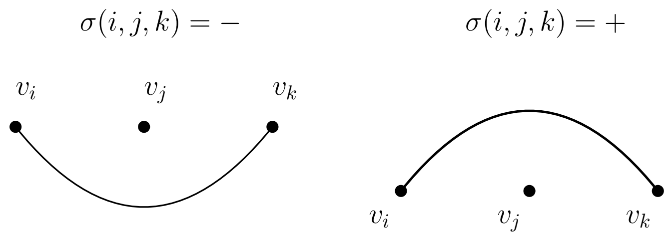


Figure 1.2: Encoding the position of an edge $\{v_i, v_k\}$ with respect to vertex v_j by a sign $\sigma(i, j, k)$ in signature σ.

A drawing of a graph is *rectilinear* if each edge is represented by a segment. An extended version is a linear drawing. *Linear* drawing of a graph is such that each edge is represented by a line. The generalization of linear drawings is pseudolinear drawing. A drawing is *pseudolinear* if each of two edges crosses exactly once (crossing at a vertex counts). A drawing of a graph is *x-monotone* if each edge is crossed by every vertical line at most once.

1.2 Sign functions

We would like to describe a combinatorial characterization of x-monotone drawings based on so-called signature functions. These and similar characterizations were used by many researchers [1, 2, 3, 4] as generalizations of order types of planar point sets.

Let T_n be the set of ordered triples (i, j, k) with $i < j < k$, from the set $[n] = 1, 2, \ldots, n$. Let Σ_n be the set of *signature functions*, that is, functions of the type $\sigma : T_n \to \{-, +\}$. Here we used $-$ and $+$ as abbreviations for -1 and $+1$, respectively. For positive integer k, we use the notation $(-)^k$ for $(-1)^k$ and $(+)^k$ for 1^k. In particular, inequality $- \leq +$ holds.

From now on, we consider only *complete graphs* K_n with n vertices and all $\binom{n}{2}$ edges. Take x-monotone drawing D of K_n with vertices $v_1, \ldots, v_n$ ordered by increasing x-coordinates, that is, $x(v_1) < x(v_2) < \cdots < x(v_n)$. We encode the drawing D with a signature function $\sigma \in \Sigma_n$ according to the following rule. For every edge $\{v_i, v_k\}$ and every integer $j \in (i, k)$, we define $\sigma(i, j, k) = +$ if edge $\{v_i, v_k\}$ goes above the point v_j, otherwise we set $\sigma(i, j, k) = -$; see Figure 1.2. To simplify, we use ξ to denote a general sign from $\{-, +\}$ and $\overline{\xi}$ to denote the opposite one. We then say that σ is realized by D.

It is easy to see that for every signature function $\sigma \in \Sigma_n$, there exists an x-monotone drawing D which induces σ. However, the interesting question is what is the minimal number of crossings for each pair of edges to have to realize such a σ.

A pattern of size k in a signature σ of an x-monotone drawing D of K_n is a signature function induced in σ by a subset of k vertices of D. Let $\mathcal{C}$ be a class of x-monotone drawings. For example, a class of unbounded drawings where edges are represented by lines. *A forbidden pattern* for $\mathcal{C}$ is a pattern that does not appear in a signature of any drawing from $\mathcal{C}$.

For integers $a, b, c, d \in [n]$ with $a < b < c < d$, signs $\xi_1, \xi_2, \xi_3, \xi_4 \in \{-, +\}$ and a signature function $\sigma \in \Sigma_n$, we say that the 4-tuple (a, b, c, d) *is of the form*

$\xi_1\xi_2\xi_3\xi_4$ *in* σ if

$$\sigma(a,b,c)=\xi_1,\ \sigma(a,b,d)=\xi_2,\ \sigma(a,c,d)=\xi_3,\ \text{and}\ \ \sigma(b,c,d)=\xi_4.$$

We sometimes use the abbreviation

$$\sigma(a,b,c)\sigma(a,b,d)\sigma(a,c,d)\sigma(b,c,d)$$

for the 4-tuple $(\sigma(a,b,c),\sigma(a,b,d),\sigma(a,c,d),\sigma(b,c,d))$.

We sometimes use a generalization of signature functions from triples to r-tuples. Let T_n^r be a set of all ordered r-tuples $(x_1,\ \ldots,\ x_r)$ with $x_1 < \cdots < x_r$ from the set $[n]$. Let Σ_n^r be the set of generalized signature functions, that is, functions of the type $\sigma : T_n^r \to \{-,+\}$. A *$r$-signotope* is a signature function $\sigma : \Sigma_n^r \to \{+,-\}$ such that every $(r+1)$-tuple $X=(x_1,\ldots,x_{r+1})$ of elements $x_i \in [n]$ with $x_1 < \ldots < x_n$ satisfies $\sigma(X^{r+1}) \le \sigma(X^r) \le \cdots \le \sigma(X^2) \le \sigma(X^1)$ or $\sigma(X^{r+1}) \ge \sigma(X^r) \ge \cdots \ge \sigma(X^2) \ge \sigma(X^1)$ where X^i is tuple X without the ith element. We call r the *order* of the signotope. If the order is omitted we mean 3-signotopes, where we are dealing with patterns of size 4.

Signotopes are a very natural class since there is a correspondence between signotopes and pseudolinear x-monotone drawings which was proved independently by Knuth [4], by Felsner, and Weil [3], and by Balko, Fulek, and Kynčl [2]. A similar characterization for pseudolinear drawings of K_n, which is based on so-called *CC systems*, was introduced by Knuth [4]; see [2] for more discussion.

Theorem 1 ([2, 3, 4])**.** *A signature function $\sigma \in \Sigma_n$ can be realized by a pseudolinear x-monotone drawing if and only if every ordered 4-tuple of indices from $[n]$ is of one of the forms*

$$\begin{gathered}++++,----,++--,--++,\\ ---+,+++-,+---,-+++\end{gathered}$$

in σ.

1.3 Our motivation

Similar characterizations as the one from Theorem 1 were obtained for other various classes of drawings of K_n and these constitute the main motivation of our work. Also, it turns out that, besides pseudolinear x-monotone drawings of K_n, several other geometric objects can be characterized by signotopes.

1.3.1 Realizations of signotopes

The characterization in Theorem 1 was generalized in various ways.

Miyata [5] introduced a similar characterization for k-intersecting pseudoconfiguration of points, sometimes called higher-order point configurations [6]. Eliáš and Matoušek [6] and Balko [7] considered various Erdős–Szekeres-type results. We now define this formally.

Definition 1. *[5] Let (P,L) be an arrangement of points P and curves L in a plane going trough these points. Then, (P,L) is a k-*intersecting pseudoconfiguration of points *if it satisfies the following three conditions:*

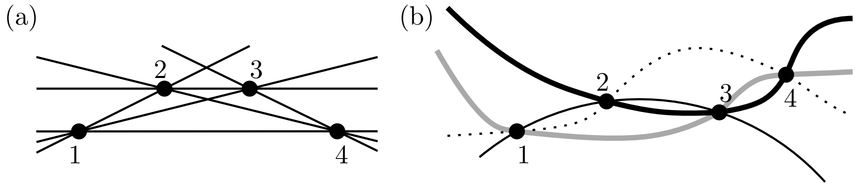


Figure 1.3: Examples of simple k-pseudoconfigurations of four points for $k = 1$ (part (a)) and $k = 2$ (part (b)). The sign function of the 1-pseudoconfiguration maps each triple of points to $-$. The sign function of the 2-pseudoconfiguration assigns $+$ to the only 4-tuple of points. This figure was taken from [7].

1. *for every $l \in L$, there are at least $k+1$ points of P lying on l,*
2. *for every $(k+1)$-tuple of distinct points of P, there is a unique curve l from L passing through each point of this $(k+1)$-tuple,*
3. *any two distinct curves from L cross at most k times.*

A k-pseudoconfiguration (P, L) of points is *simple* if each curve from L passes through exactly $k+1$ points of P; see Figure 1.3. If (P, L) is simple, we let $l_{i_1,\dots,i_{k+1}}$ be the curve from L passing through points $p_{i_1}, \dots, p_{i_{k+1}}$. Each curve l from L is a graph of a continuous function $f_l : \mathbb{R} \to \mathbb{R}$ and we let $l^- := \{(x, y) \in \mathbb{R}^2 : y < f_l(x)\}$. A *signature of a simple k-pseudoconfiguration* (P, L) is a function $f : \binom{P}{k+2} \to \{-, +\}$ such that, given $\{i_1, \dots, i_{k+2}\} \in \binom{P}{k+2}$ with $i_1 < \cdots < i_{k+2}$, we have $f(p_{i_1}, \dots, p_{i_{k+2}}) = -$ if and only if $p_{i_{k+2}} \in l^-_{i_1,\dots,i_{k+1}}$.

The signatures of simple k-pseudoconfigurations are characterized by the following result by Miyata [5].

Theorem 2 ([5]). *For $k, n \in \mathbb{N}$, there is a one-to-one correspondence between sign functions of simple k-pseudoconfigurations of n points and $(k+2)$-signotope. The $(k+2)$-signotope corresponding to a k-pseudoconfiguration P is the sign function of P.*

It was shown by Felsner and Weil [3] that a one-to-one correspondence exists between d-signotopes and arrangements of n pseudohyperplanes in $\mathbb{R}^{d-1}$ that allow a certain "sweeping procedure", for all $d \geq 3$. We now introduce the definition of these arrangements formally.

In $\mathbb{R}^d$, where $d \geq 2$, we define a *pseudohyperplane* H as a homeomorphic representation of a hyperplane in $\mathbb{R}^d$. Each pseudohyperplane H in $\mathbb{R}^d$ divides $\mathbb{R}^d$ into two connected components, each of which is homeomorphic to an open d-dimensional ball. Pseudohyperplanes H_1 and H_2 *cross* if H_1 (H_2, respectively) intersects both components of $\mathbb{R}^d \setminus H_2$ ($\mathbb{R}^d \setminus H_1$, respectively).

Now, turning our attention to arrangements of pseudohyperplanes in $\mathbb{R}^d$, referred to as *d-arrangements*, we define them as collections $\{H_1, \dots, H_n\}$ of pseudohyperplanes in $\mathbb{R}^d$. Within such an arrangement, any pair H_i and H_j crosses in a pseudohyperplane homeomorphic to $\mathbb{R}^{d-1}$. Additionally, the intersections $H_i \cap H_j$, for fixed i and $j \neq i$ ranging over $[n]$, form an arrangement of pseudohyperplanes in $H_j \cong \mathbb{R}^{d-1}$. A d-arrangement A is labeled as *simple* if any collection of $d+1$ pseudohyperplanes from A has an empty intersection.

We assume that every d-arrangement A of pseudohyperplanes $H_1, \dots, H_n$ is *normal*. This means that A is simple and embedded in $\mathbb{R}^d$ in the following manner.

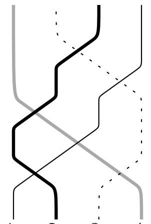


Figure 1.4: A C_2-arrangement of four pseudolines. Here, the sign function assigns $-$ to the triple $\{1, 2, 3\}$ and $+$ to the triple $\{2, 3, 4\}$. The figure was taken from [7].

Let A be embedded in the hypercube $[0, 1]^d$. For $i \in [d-1]$, we define I_i as the $(d-i)$-dimensional subspace of $\mathbb{R}^d$ containing the side of $[0, 1]^d$ obtained by setting the last i coordinates to 0. It is required that $A \cap I_i$ forms a $(d-i)$-arrangement of n pseudohyperplanes. Additionally, the pseudohyperplanes in A are labeled by their increasing first coordinates in their intersection with I_{d-1}.

The concept of a *sign function of a normal d-arrangement* A of n pseudohyperplanes $H_1, \ldots, H_n$ is introduced as a function $f : T_n^{d+1} \to \{-, +\}$, where for given $i_1 < \cdots < i_{d+1}$, $f(i_1, \ldots, i_{d+1}) = -$ if and only if the pseudoline $H_{i_3} \cap \cdots \cap H_{i_{d+1}}$, oriented away from I_1, intersects H_{i_1} before H_{i_2}.

A normal d-arrangement A is said to be a *C_d-arrangement* if the normal $(d-1)$-arrangement formed by $H \cap I_1$ for $H \in A$ has no $+$ sign in its sign function. It is important to note that while every normal arrangement of pseudolines is a C_2-arrangement, this does not hold for C_d-arrangements with $d \geq 3$.

We present a theorem, inspired by [3], stating that for $d \geq 2$ and $n \in \mathbb{N}$, there exists a bijection between sign functions of C_d-arrangements of n pseudohyperplanes in $\mathbb{R}^d$ and $(d+1)$-signotopes; see Figure 1.4.

Furthermore, besides the geometric interpretations of signotopes provided, there exists a third interpretation, as discovered by Ziegler [8], where signotopes can be viewed as extensions of the cyclic arrangement of hyperplanes with a pseudohyperplane.

Signotopes are sign functions that allow at most one change in the sign for each 4-tuple. It is therefore natural to generalize signotopes as follows. The *generalized signotopes* are sign functions that allow at most 2 changes of a sign. This structure was studied by Bergold et al. [9] and they proved that some well-known theorems also hold for this structure. Generalized signotopes are interesting because there is still not known their geometrical representation. Bergold et al. [9] also showed that sign functions of simple drawings of K_n are generalized signotopes, on the other hand, there are, asymptotically, more generalized signotopes than simple drawings of K_n.

1.3.2 Other classes of drawings

A drawing is *simple* if any two intersect at most once. That is adjacent edges cannot cross and two independent edges can share at most one crossing. A drawing is *semisimple* if any two adjacent edges cannot cross but independent edges can cross any number of times.

Theorem 1 is our main motivation and the main theorem we aim to generalize.

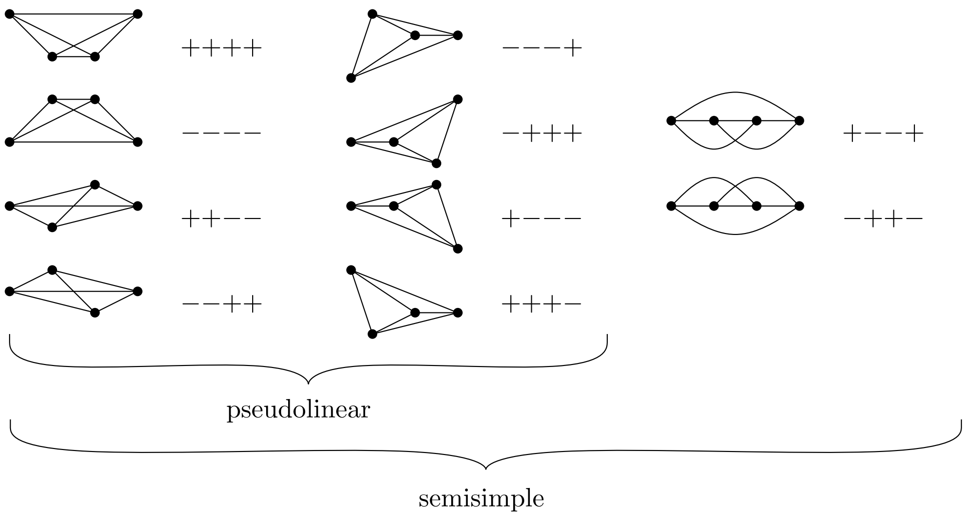


Figure 1.5: The 4-tuples in pseudolinear and semisimple drawings. This figure is taken from [2].

The following result by Balko, Fulek, and Kynčl [2] gives another example of a characterization of a class of x-monotone drawings of K_n by forbidden patterns of size four. Namely, we obtain this characterization for x-monotone semisimple drawings of K_n. For simple drawings, we also need some forbidden patterns of size five.

Theorem 3 ([2])**.** *A signature function $\sigma \in \Sigma_n$ can be realized by a semisimple x-monotone drawing if and only if every 4-tuple of indices from $[n]$ is of one of the forms*

$$++++, ----, ++--, --++, -++-, +--+,$$
$$---+, +++-, +---, -+++$$

in σ. The signature function σ can be realized by a simple x-monotone drawing if, in addition, there is no 5-tuple (a, b, c, d, e) with $a < b < c < d < e$ such that $\sigma(a, b, e) = \sigma(a, d, e) = \sigma(b, c, d) = \overline{\sigma(a, c, e)}$.

Note that the 4-tuples mentioned in Theorems 1 and 3 are the "allowed" patterns. However, one could equivalently list their complements of eight and six, respectively, forbidden patterns of size four. These two theorems are the basic building blocks of our work where we aim to obtain similar characterizations for broader classes of x-monotone drawings of K_n. See Figure 1.5 for a geometric realization of allowed patterns in Theorems 1 and 3.

We would also like to mention one more result about characterizations by forbidden patterns. Kynčl [10] showed that so-called complete *AT-graphs* (see [10] for definition) can be characterized by forbidden patterns of size at most six.

Theorem 4 ([10])**.** *Every complete AT-graph that is not simply realizable has an AT-subgraph on at most six vertices that is not simply realizable.*

Kynčl [10] also shows that six is a minimal size of forbidden patterns needed to characterize the simple realizability of complete AT-graphs. In our results, we also aim to use the forbidden patterns of minimal size.

Characterizations of classes of drawings by forbidden patterns of constant size are also useful from an algorithmic point of view as they provide polynomial time recognition algorithms for the corresponding classes of drawings. In particular, it follows from Theorem 1 that we can recognize x-monotone pseudolinear drawings of K_n in time $\mathcal{O}(n^4)$ by simply checking the signature of a given drawing for forbidden patterns by size four. Similarly, we obtain recognition algorithms for semisimple and simple x-monotone drawings of K_n with the worst-case running times $\mathcal{O}(n^4)$ and $\mathcal{O}(n^5)$, respectively. For AT-graphs, Theorem 4 analogously gives $\mathcal{O}(n^6)$-running time recognition algorithm.

1.4 Our contribution

Our goal is to develop similar characterizations based on small forbidden patterns for classes of x-monotone drawings where edges are allowed to cross multiple times. We consider several such classes depending on the maximum allowed number of crossings an edge can participate in. We mostly focus on a generalized variant of 1-intersecting pseudoconfigurations of points where we allow two distinct curves to cross more than once.

In Chapter 2, we generalize Theorem 1 to a characterization of *pseudoparabola drawing*, both bounded and unbounded; see Chapter 2 for definitions. First, we provide such a characterization based on forbidden patterns of size at most 6 for bounded pseudoparabola drawings (Theorem 5). In the case of unbounded pseudoparabola drawings, we, perhaps surprisingly, show that no characterization based on forbidden patterns of finite size is possible (Theorem 15).

We further generalize the characterization from Theorem 5 in Chapter 3 to so-called bounded *drawings of degree d*, where the edges are allowed to cross up to d times (Theorem 17). We also show that no characterization based on forbidden patterns of finite size is possible for unbounded drawings of degree d (Theorem 25).

Lastly in Chapter 4, we discuss x-monotone drawings where the maximum allowed number of crossings is different for pairs of adjacent edges and pairs of independent edges. For example, semisimple drawings fall within this framework. Our motivation here was to explore the border where the characterizations based on finite forbidden patterns stop working. We have found the border and the full characterization is in Theorems 27, 28, 29, 30.

2. Pseudoparabola x-monotone drawings

2.1 Introduction

Similarly to the combinatorial characterization of pseudolinear drawings of K_n from Theorem 1 proved by Balko, Fulek, and Kynčl [2], we would like to characterize higher-degree "pseudopolynomial" x-monotone drawings of K_n. A drawing D of a graph G is an *pseudoparabola drawing* if the edges of D can be drawn as simple curves that cross each other at most twice. Similarly, an extended drawing D of G is an *extended pseudoparabola drawing* if the edges of D can be drawn as unbounded simple curves that cross each other at most twice.

In this chapter, we will show that every x-monotone pseudoparabola drawing of K_n can be characterized combinatorically by forbidden 5-tuples and 6-tuples (Theorem 5) and disprove the existence of finite forbidden patterns for extended x-monotone pseudoparabolas (Theorem 15).

2.2 Bounded pseudoparabolas

In the following result we show that x-monotone pseudoparabola drawings can characterized by forbidden patterns of size 5 and 6; see Figure 2.1 for an illustration of this statement.

Theorem 5. *A signature function $\sigma \in \Sigma_n$ can be realized by bounded pseudoparabola x-monotone drawing if and only if there is no ordered 5-tuple $(a_1, \ldots, a_5)$ with $a_1 < \cdots < a_5$ that satisfies*

$$\sigma(a_1, a_2, a_5) = \overline{\sigma(a_1, a_3, a_5)} = \sigma(a_1, a_4, a_5) = \overline{\sigma(a_1, a_2, a_4)} = \sigma(a_1, a_3, a_4)$$

or

$$\sigma(a_1, a_4, a_5) = \overline{\sigma(a_1, a_3, a_5)} = \sigma(a_1, a_2, a_5) = \overline{\sigma(a_2, a_4, a_5)} = \sigma(a_2, a_3, a_5)$$

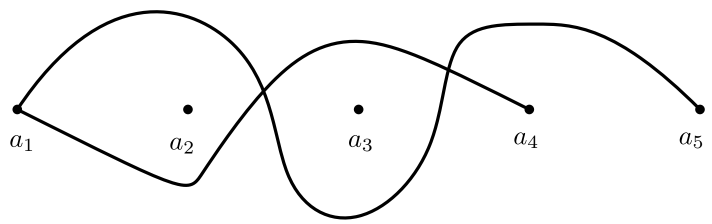
and, additionally, there is no ordered 6-tuple $(a_1, \ldots, a_6)$ with $a_1 < \cdots < a_6$ that satisfies

$$\begin{aligned}\sigma(a_1, a_2, a_5) &= \overline{\sigma(a_1, a_3, a_5)} = \sigma(a_1, a_4, a_5)\\ &= \sigma(a_2, a_3, a_6) = \overline{\sigma(a_2, a_4, a_6)} = \sigma(a_2, a_5, a_6)\end{aligned}$$

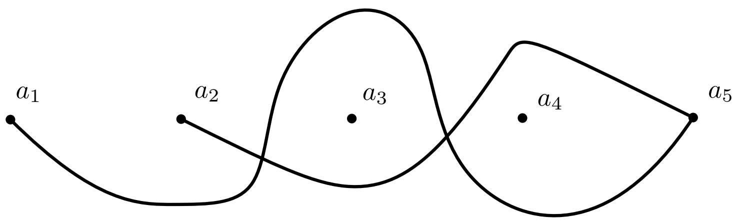
or

$$\begin{aligned}\sigma(a_1, a_2, a_6) &= \overline{\sigma(a_1, a_3, a_6)} = \sigma(a_1, a_4, a_6)\\ &= \overline{\sigma(a_1, a_5, a_6)} = \sigma(a_2, a_3, a_5) = \overline{\sigma(a_2, a_4, a_5)}.\end{aligned}$$

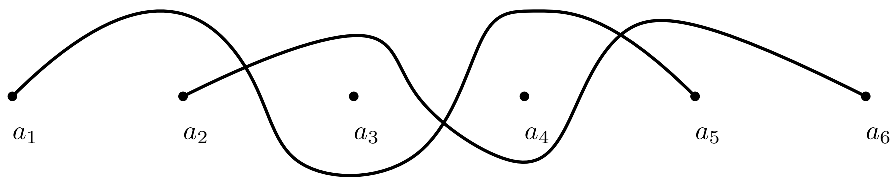
$$+ = \sigma(a_1,a_2,a_5) = \overline{\sigma(a_1,a_3,a_5)} = \sigma(a_1,a_4,a_5) = \overline{\sigma(a_1,a_2,a_4)} = \sigma(a_1,a_3,a_4)$$



$$- = \sigma(a_1,a_4,a_5) = \overline{\sigma(a_1,a_3,a_5)} = \sigma(a_1,a_2,a_5) = \overline{\sigma(a_2,a_4,a_5)} = \sigma(a_2,a_3,a_5)$$



$$+ = \sigma(a_1,a_2,a_5) = \overline{\sigma(a_1,a_3,a_5)} = \sigma(a_1,a_4,a_5) = \sigma(a_2,a_3,a_6) = \overline{\sigma(a_2,a_4,a_6)} = \sigma(a_2,a_5,a_6)$$



$$+ = \sigma(a_1,a_2,a_6) = \overline{\sigma(a_1,a_3,a_6)} = \sigma(a_1,a_4,a_6) = \overline{\sigma(a_1,a_5,a_6)} = \sigma(a_2,a_3,a_5) = \overline{\sigma(a_2,a_4,a_5)}$$

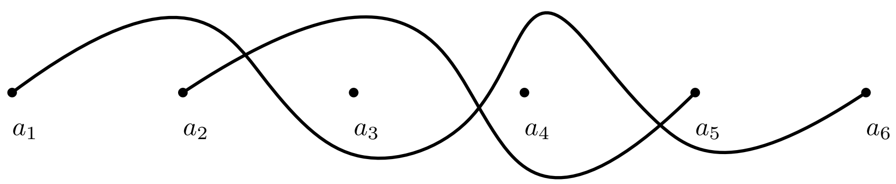


Figure 2.1: One example for each forbidden tuple in the statement of Theorem 5.

Proof. Let D be an x-monotone pseudoparabola drawing of K_n. It is clear that D does not contain any of the forbidden 5- or 6-tuples as otherwise there are two edges of D that cross more than twice; see Figure 2.1.

To prove the other implication, let σ be a sign function from Σ_n that does not contain any of the forbidden 5- or 6-tuples. We will construct an x-monotone pseudoparabola drawing D from σ. This construction will be also used later and we refer to it as a *pseudoparabola construction.*

Pseudoparabola construction. We start with the points $v_i = (i, 0)$ for $i \in [n]$ as vertices. We denote the vertical line going through vertices v_m as L_m for every $m \in [n]$. To determine the drawing D up to a combinatorial equivalence, it suffices to specify the right and left vertical orders of lines starting or ending in each v_m and also the relative orders of intersections of curves with vertical lines L_m. Then we place the crossings of edges of D with L_m so that they respect these orders and afterward we connect them with line segments.

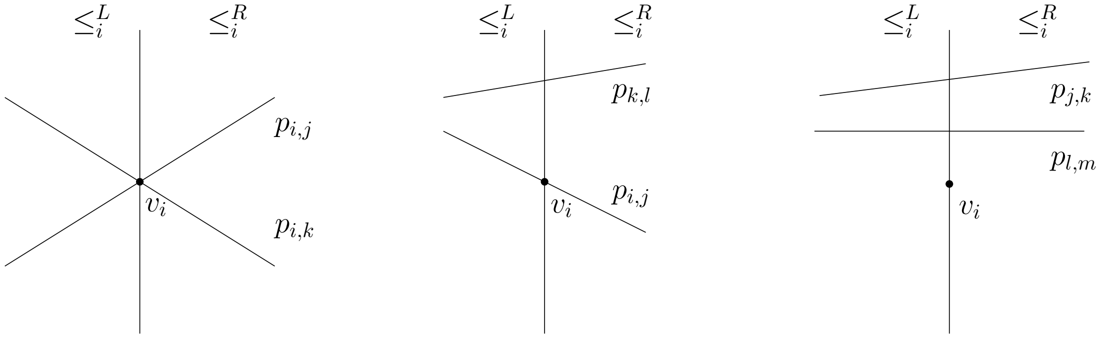


Figure 2.2: Illustration of the relative order on the vertical line L_i.

For distinct i and j from $[n]$, we denote by $p_{i,j}$ the resulting piece-wise linear curve representing the edge v_iv_j. Note that $p_{i,j} = p_{j,i}$; see Figure 2.2.

For each $i \in [n]$, we define two total orders $\leq_i^L$ and $\leq_i^R$ on the set $\mathcal{P} = \{p_{j,k} \colon j,k \in [n], j \neq k\}$ of all curves $p_{j,k}$. The order $\leq_i^L$ ($\leq_i^R$, respectively) represents the vertical left (right, respectively) order of curves from $\mathcal{P}$ at the neighborhood of L_i; see Figure 2.2. Let $\mathcal{P}_i = \{p_{i,j} \colon j \in [n], i \neq j\}$ be a set curves from $\mathcal{P}$ that contain the vertex v_i. We want the curves from $\mathcal{P} \setminus \mathcal{P}_i$ to be in the same relative order in $\leq_i^L$ and $\leq_i^R$.

We first specify the relative order of the curves from $\mathcal{P}_i$ in the orderings $\leq_i^L$ and $\leq_i^R$. Later, we also specify the relative order of the remaining curves from $\mathcal{P}$ in these two orders.

Definition 2. *For $i,j,k \in [n]$ with $j < k$ and $i \notin \{j,k\}$, the orders $\leq_i^L$ and $\leq_i^R$ on $\mathcal{P}_i$ are defined in the following way; see Figure 2.3:*

- *assume $i < j < k$, we set $p_{i,j} \leq_i^R p_{i,k}$ if $\sigma(i,j,k) = -$ and there is m satisfying $i < m < j$, $\sigma(i,m,j) = -$, and $\sigma(i,m,k) = +$. We set $p_{i,k} \leq_i^R p_{i,j}$ if $\sigma(i,j,k) = +$ and there is m satisfying $i < m < j$, $\sigma(i,m,j) = +$, and $\sigma(i,m,k) = -$. Otherwise, we set $p_{i,j} \leq_i^R p_{i,k}$ if $\sigma(i,j,k) = +$ and $p_{i,k} \leq_i^R p_{i,j}$ if $\sigma(i,j,k) = -$.*
- *assume $j < k < i$, we set $p_{j,i} \leq_i^L p_{k,i}$ if $\sigma(j,k,i) = -$ and there is m satisfying $j < m < k$, $\sigma(k,m,i) = -$ and $\sigma(j,m,i) = +$, We set $p_{k,i} \leq_i^L p_{j,i}$ if $\sigma(j,k,i) = +$ and there is m satisfying $j < m < k$, $\sigma(k,m,i) = +$ and $\sigma(j,m,i) = -$. Otherwise, we set $p_{j,i} \leq_i^L p_{k,i}$ if $\sigma(j,k,i) = +$ and $p_{k,i} \leq_i^L p_{j,i}$ if $\sigma(j,k,i) = -$.*

In both cases, the first variant is equivalent to one of the forbidden 4-tuples in pseudolinear drawings of K_n from [2] as one intersection is forced; see the first part of Figure 2.

It is easy to see that the relation $\leq_i^R$ on $\mathcal{P}_i$ is weakly anti-symmetric since we cannot have $p_{i,j} \leq_i^R p_{i,k}$ and $p_{i,k} \leq_i^R p_{i,j}$ for distinct curves $p_{i,j}, p_{i,k}$ as by Definition 2 we would have different signs $\sigma(i,j,k) = -$ and also $\sigma(i,j,k) = +$ which is impossible. Analogously, we can prove that $\leq_i^L$ is weakly anti-symmetric.

We will prove that the relations $\leq_i^L$ and $\leq_i^R$ described in Definition 2 are total orders at each point i.

Let $p_{i,j}$, $p_{i,k}$ be two distinct curves from $\mathcal{P}_i$ with $j < k$. To prove that $\leq_i^R$ is a total order, we will define for each pair $p_{i,j}$, $p_{i,k}$ new auxiliary relation $\leq_{i,a}^R$.

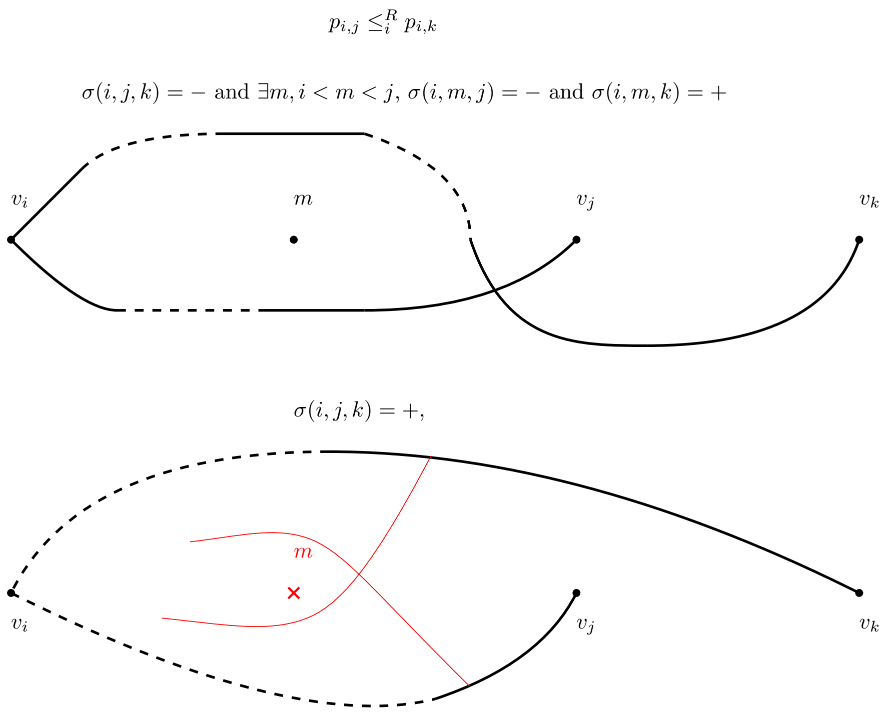


Figure 2.3: Examples of curves $p_{i,j}, p_{i,k}$ satisfying $p_{i,j} \leq_i^R p_{i,k}$ in Definition 2. The second case has no m satisfying $i < m < j$, $\sigma(i,m,j) = -$, and $\sigma(i,m,k) = +$.

The proof that $\leq_i^L$ can be done analogously due to vertical symmetry. The new relation $\leq_{i,a}^R$ is defined analogously as in Definition 2 but the first case considers with the only difference that the vertex m is restricted to lie only in the interval $a \leq m < j$, and not in the whole interval (i,j); see Figure 2.4.

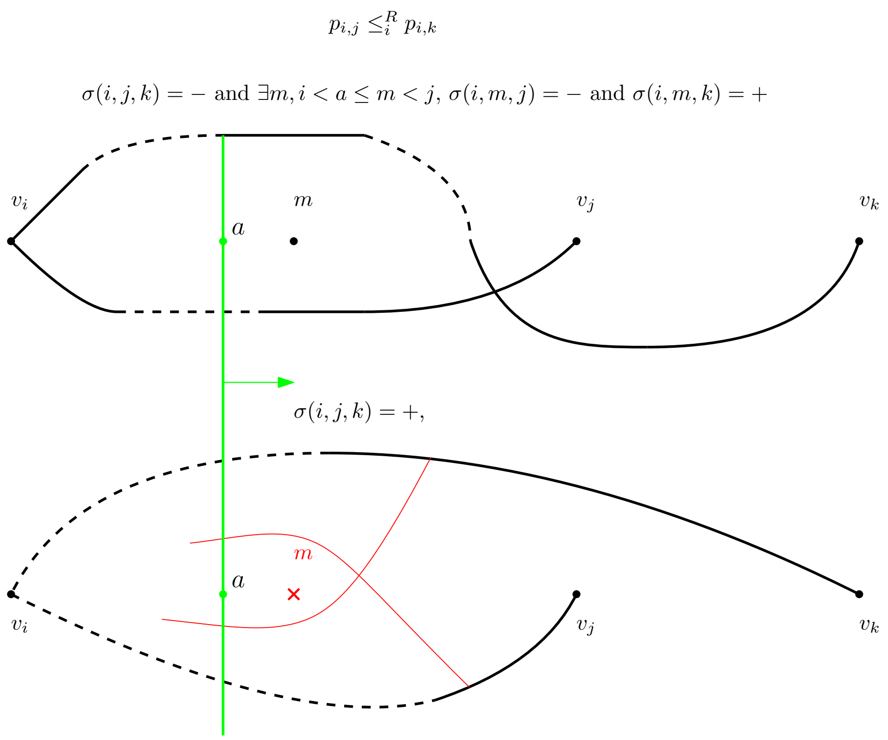


Figure 2.4: Example of $p_{i,j} \leq_{i,a}^R p_{i,k}$ in Definition 3.

Definition 3. *For $a, i, j, k \in [n]$ with $j < k$ and $i \notin \{j, k\}$, the orders $\leq^L_{a,i}$ and $\leq^R_{i,a}$ on $\mathcal{P}_i$ are defined in the following way; see Figure 2.4:*

- *assume $i \leq a \leq j < k$, we set $p_{i,j} \leq^R_{i,a} p_{i,k}$ if $\sigma(i,j,k) = -$ and there is m satisfying $a \leq m < j$, $\sigma(i,m,j) = -$, and $\sigma(i,m,k) = +$. We set $p_{i,k} \leq^R_{i,a} p_{i,j}$ if $\sigma(i,j,k) = +$ and there is m satisfying $a \leq m < j$, $\sigma(i,m,j) = +$ and $\sigma(i,m,k) = -$. Otherwise, we set $p_{i,j} \leq^R_{i,a} p_{i,k}$ if $\sigma(i,j,k) = +$ and $p_{i,k} \leq^R_{i,a} p_{i,j}$ if $\sigma(i,j,k) = -$.*

- *assume $j < k \leq a \leq i$, we set $p_{j,i} \leq^L_{a,i} p_{k,i}$ if $\sigma(j,k,i) = -$ and there is m satisfying $j < m \leq a$, $\sigma(k,m,i) = -$, and $\sigma(j,m,i) = +$. We set $p_{k,i} \leq^L_{a,i} p_{j,i}$ if $\sigma(j,k,i) = +$ and there is m satisfying $a \leq m < k$, $\sigma(k,m,i) = +$ and $\sigma(j,m,i) = -$. Otherwise, we set $p_{j,i} \leq^L_{a,i} p_{k,i}$ if $\sigma(j,k,i) = +$ and $p_{k,i} \leq^L_{a,i} p_{j,i}$ if $\sigma(j,k,i) = -$.*

As we mentioned above, we consider $p_{i,j} = p_{j,i}$ and for simplicity, for a triple (a,b,c) with $a < b < c$ and any triple (a',b',c') satisfying $\{a',b',c'\} = \{a,b,c\}$ we define $\sigma(a',b',c') = \sigma(a,b,c)$. We will now state an auxiliary lemma to prove the transitivity of $\leq^R_i$. Informally, this lemma says that two curves $p_{i,j}, p_{i,k}$ cross if and only if there is a vertex a between i and j with one of the curves above and the other one below while this relative order is opposite for $a + 1$.

Lemma 6. *For $a, i, j, k \in [n]$ with $i \leq a \leq j < k$, consider two curves $p_{i,j}$, $p_{i,k}$. If $p_{i,j} \leq^R_{i,a+1} p_{i,k}$, then we have $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ if and only if $p_{i,k} \leq^R_{i,a} p_{i,j}$.*

Similarly, if $p_{i,k} \leq^R_{i,a+1} p_{i,j}$, then we have $\sigma(i,a,j) = -$ and $\sigma(i,a,k) = +$ if and only if $p_{i,j} \leq^R_{i,a} p_{i,k}$.

Proof. By horizontal symmetry, we can without loss of generality assume that $\sigma(i,a,j) = +$. There are only two subcases:

1. Either $\sigma(i,j,k) = +$ and therefore a represents the vertex m in the first case of Definition 3 for $p_{i,k} \leq^R_{i,a} p_{i,j}$. It can also be seen that if $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ is not true, then there is no change of the relation $\leq^R_{i,a}$ in the case of Definition 3.

2. Or $\sigma(i,j,k) = -$ and there is m satisfying $a + 1 \leq m < j, \sigma(i,m,j) = -$ and $\sigma(i,m,k) = +$ and we have forbidden 5-tuple (i,a,m,j,k). On the other hand, if $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ is not true, then there is no change in the case of Definition 3.

□

Due to vertical symmetry, we obtain an analogous result for the $\leq^L_{a,i}$.

Lemma 7. *For $a, i, j, k \in [n]$ with $j < k \leq a \leq i$, consider two curves $p_{i,j}$, $p_{i,k}$. If $p_{j,i} \leq^L_{a-1,i} p_{k,i}$, then we have $\sigma(j,a,i) = +$ and $\sigma(k,a,i) = -$ if and only if $p_{k,i} \leq^L_{a,i} p_{j,i}$.*

Similarly, if $p_{k,i} \leq^L_{a-1,i} p_{j,i}$, then we have $\sigma(j,a,i) = -$ and $\sigma(k,a,i) = +$ if and only if $p_{j,i} \leq^L_{a,i} p_{k,i}$.

We say that these two curves $p_{i,j}, p_{i,k}$ from $\mathcal{P}_i$ *cross between* a *and* $a+1$ if $p_{i,j} \leq^R_{i,a+1} p_{i,k}$ and $\sigma(i,a,j) = + = \sigma(i,a,k)$ or if $p_{i,k} \leq^R_{i,a+1} p_{i,j}$ and $\sigma(i,a,j) = - = \sigma(i,a,k)$. By symmetry such a crossing between $a-1$ and a appears if $p_{j,i} \leq^L_{a-1,i} p_{k,i}$ and $\sigma(j,a,i) = + = \sigma(k,a,i)$ or if $p_{k,i} \leq^L_{a-1,i} p_{j,i}$ and $\sigma(j,a,i) = - = \sigma(k,a,i)$.

Corollary 8. For $a,i,j,k \in [n]$ with $i \leq a \leq j < k$ satisfying $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ following holds $p_{i,k} \leq^R_{i,a} p_{i,j}$.

Similarly, if $\sigma(i,a,j) = -$ and $\sigma(i,a,k) = +$ following holds $p_{i,j} \leq^R_{i,a} p_{i,k}$.

Proof. By horizontal symmetry, we can without loss of generality assume that $\sigma(i,a,j) = +$. Then the claim easily follows from Lemma 6 as either $p_{i,j} \leq^R_{i,a+1} p_{i,k}$ and there is a cross of $p_{i,j}, p_{i,k}$ and hence $p_{i,k} \leq^R_{i,a} p_{i,j}$ due to Lemma 6 or $p_{i,k} \leq^R_{i,a+1} p_{i,j}$ and there is no cross of $p_{i,j}, p_{i,k}$ and hence $p_{i,j} \leq^R_{i,a} p_{i,k}$ due to Lemma 6.

□

And again by vertical symmetry, we obtain the result for the relations $\leq^L_i$.

Corollary 9. For $a,i,j,k \in [n]$ with $k < j \leq a \leq i$ satisfying $\sigma(j,a,i) = +$ and $\sigma(k,a,i) = -$ following holds $p_{k,i} \leq^L_{i,a} p_{j,i}$.

Similarly, if $\sigma(j,a,i) = -$ and $\sigma(k,a,i) = +$ following holds $p_{j,i} \leq^L_{a,i} p_{k,i}$.

We are now ready to prove that the relations $\leq^R_{i,a}$ are transitive, which implies that the relations $\leq^R_i$ are also transitive. By vertical symmetry, we again obtain the transitivity also for the relations $\leq^L_i$ as all the Lemmas and Corollaries hold symmetrically for $\leq^R_{i,a}$ and $\leq^L_{a,i}$ too.

Lemma 10. *For every* $i, a \in [n]$ *with* $i \leq a$ *the relation* $\leq^R_{i,a}$ *is transitive.*

Proof. Let i,j,k,l be elements from $[n]$ with $i < j < k < l$. Each of the three curves $p_{i,j}$, $p_{i,k}$, $p_{i,l}$. Consider a satisfying $i < a \leq j$. We start with the base case with $a = j$ and then proceed by induction:

1. If $\sigma(i,j,k) = \xi$ and $\sigma(i,j,l) = \overline{\xi}$, then we can apply Corollary 8. This means that $p_{i,k} \leq^R_{i,j} p_{i,l}$ if $\xi = -$ and $p_{i,l} \leq^R_{i,j} p_{i,k}$ if $\xi = +$. Also $p_{i,j} \leq^R_{i,j} p_{i,l}$ ($p_{i,l} \leq^R_{i,j} p_{i,j}$, respectively) and $p_{i,k} \leq^R_{i,j} p_{i,j}$ ($p_{i,j} \leq^R_{i,j} p_{i,k}$, respectively) as there is no vertex m that is considered in Definition 3. Hence we end up with a transitive order $p_{i,k} \leq^R_{i,j} p_{i,j} \leq^R_{i,j} p_{i,l}$ ($p_{i,l} \leq^R_{i,j} p_{i,j} \leq^R_{i,j} p_{i,k}$, respectively).

2. If $\sigma(i,j,k) = \xi$ and $\sigma(i,j,l) = \xi$, then we can deduce that $p_{i,j} \leq^R_{i,j} p_{i,k}$ and $p_{i,j} \leq^R_{i,j} p_{i,l}$ if $\xi = +$ and $p_{i,k} \leq^R_{i,j} p_{i,j}$ and $p_{i,l} \leq^R_{i,j} p_{i,j}$ if $\xi = -$, as there is no vertex m that is considered in Definition 3. Then

   (a) either $p_{i,k} \leq^R_{i,j} p_{i,l}$ holds and we get $p_{i,j} \leq^R_{i,j} p_{i,k} \leq^R_{i,j} p_{i,l}$ ($p_{i,k} \leq^R_{i,j} p_{i,l} \leq^R_{i,j} p_{i,j}$, respectively) which gives the transitivity.

   (b) or $p_{i,l} \leq^R_j p_{i,k}$ and then $p_{i,j} \leq^R_j p_{i,l} \leq^R_j p_{i,k}$ ($p_{i,j} \leq^R_{i,j} p_{i,l} \leq^R_{i,j} p_{i,k}$, respectively) holds which gives the transitivity.

Now, we proceed with the induction step; see Figure 2.5. Assume we have the vertex m with $i < m < j$. We use integers a,b,c with $\{a,b,c\} = \{j,k,l\}$ to denote permutation of i,j,k. We assume that $p_{i,a} \leq^R_{i,m+1} p_{i,b} \leq^R_{i,m+1} p_{i,c}$ and $p_{i,a} \leq^R_{i,m+1} p_{i,c}$.

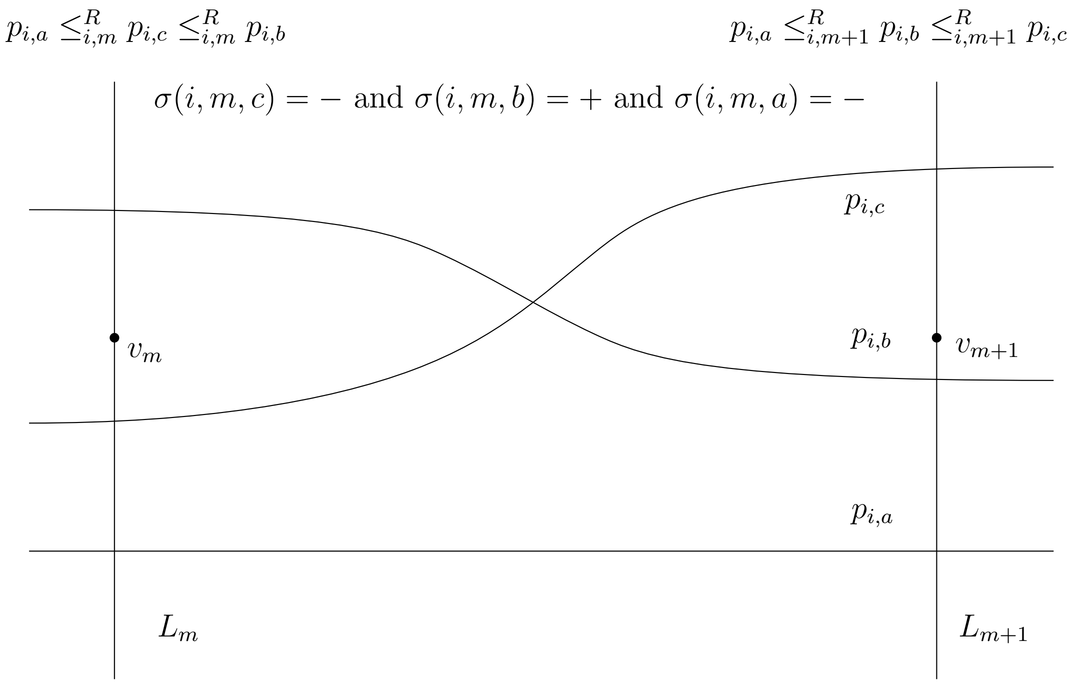


Figure 2.5: Illustration of case 2(a) of the induction step.

1. The simplest case happens when the signs of the curves $p_{i,a}, p_{i,b}, p_{i,c}$ with respect to vertex m do not cause a crossing between $m+1$ and m. Then none of these pairs changes the order, so we have $p_{i,a} \leq^R_{i,m} p_{i,b} \leq^R_{i,m} p_{i,c}$ and $p_{i,a} \leq^R_{i,m} p_{i,c}$.

2. Curves $p_{i,c}$ and $p_{i,b}$ cross, (that is $\sigma(i,m,c) = -$ and $\sigma(i,m,b) = +$) and then either

   (a) $\sigma(i,m,a) = -$ and hence the order between $p_{i,a}$, $p_{i,b}$ and $p_{i,a}$, $p_{i,c}$ stays the same and the order between $p_{i,a}$, $p_{i,b}$ changes due to Lemma 6. Then we have $p_{i,a} \leq^R_{i,m} p_{i,c} \leq^R_{i,m} p_{i,b}$ and $p_{i,a} \leq^R_{i,m} p_{i,b}$ as $p_{i,b}$ and $p_{i,c}$ is the only pair that changed the order.

   (b) $\sigma(i,m,a) = +$ and therefore $p_{i,c}$ crosses both $p_{i,b}$ and $p_{i,a}$. Due to Lemma 6 both orders are changed. The curves $p_{i,a}$ and $p_{i,b}$ do not cross due to Lemma 6. Then we have $p_{i,c} \leq^R_{i,m} p_{i,a} \leq^R_{i,m} p_{i,b}$ and $p_{i,c} \leq^R_{i,m} p_{i,a}$.

3. By horizontal symmetry, we can solve the case where $p_{i,b}$ crosses $p_{i,a}$ as they have the same relative order as $p_{i,c}$ and $p_{i,b}$.

4. If $p_{i,a}$ crosses $p_{i,c}$ then, also either $p_{i,b}$ crosses $p_{i,c}$ or $p_{i,a}$ depending on the sign $\sigma(i,m,b)$. Nevertheless, we have already solved both these cases in the second step, so we are done.

We have completed the induction step and the base case too. Therefore, we know that there is a transitive order of three curves $p_{i,j}, p_{i,k}, p_{i,l}$ at vertex $i+1$. Since we now know that $\leq^R_{i,i+1}$ is transitive and since $\leq^R_i = \leq^R_{i,i} = \leq^R_{i,i+1}$, the order $\leq^R_i$ from Definition 2 is transitive on $\mathcal{P}_i$. $\square$

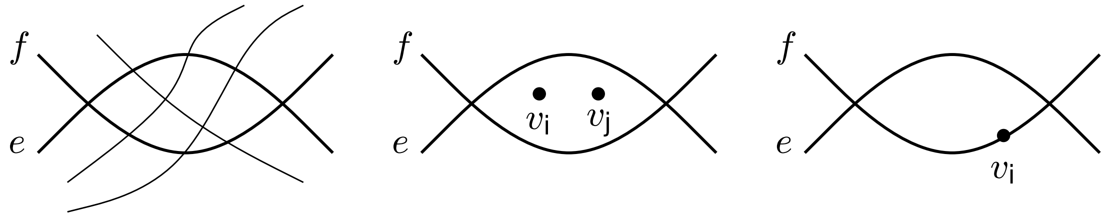


Figure 2.6: Three types of bigons formed by two curves e and f. A minimal empty bigon, a smooth bigon, and a bigon that is neither empty nor smooth. Figure was taken from [2].

Observe that the relations $\leq_m^R$ and $\leq_m^L$ are reflexive on $\mathcal{P}_m$. Altogether we know that $\leq_m^R$ and $\leq_m^L$ are reflexive, anti-symmetric, transitive, and total, thus they are total orders.

It is easy to see that orders $\leq_m^R$ and $\leq_m^L$ specify the relative position of curves with respect to vertices, but the relative position of the curves is still completely specified. For example, if we have two curves that do not contain vertex v_m but intersect L_m, the orders only specify a partition of these curves into two subsets, depending on whether the curves intersect L_m below or above v_m. We now choose a drawing D of K_n that obeys all the required conditions from $\leq_m^R$ and $\leq_m^L$ and additionally minimizes the total number of crossings of the curves. Combinatorially, this last condition is equivalent to minimizing the total number of inversions between pairs of permutations corresponding to $\leq_m^R$ and $\leq_{m+1}^L$ for all $m \in [n-1]$.

We assume that no three piece-wise linear curves in D that represent edges cross at a common interior point. We show that any two curves in D cross at most two times and thus they are indeed pseudoparabolas. Let e, f be two x-monotone curves from the drawing D. A *bigon* B formed by e and f is a closed topological disc bounded by two simple arcs e', f' that have common endpoints and disjoint relative interiors, and such that e' is a portion of e and f' is a portion of f. The common endpoints of e' and f' are the two vertices of B. Observe that if two curves e and f cross k times, then e and f form exactly $k-1$ bigons. A bigon B is *empty* if $B \cap \{v_1, v_2, \ldots, v_n\} = \emptyset$. Moreover, a bigon B is considered *smooth* if its boundary does not intersect $v_1, v_2, \ldots, v_n$; see to Figure 2.6.

Lemma 11. *No two curves from D form an empty bigon.*

Proof. Suppose, for the sake of contradiction, that two curves e and f form an empty bigon B. Let $e' \subseteq e$ and $f' \subseteq f$ be the two arcs forming the boundary of B. We consider B to be the minimal empty bigon with respect to inclusion among all such pairs of curves. Moreover, we assume that e' and f' are minimal among all pairs forming either the lower or the upper part of B. Next, any curve g, distinct from e and f, either does not intersect B or intersects both e' and f' the same number of times. Hence, we can redraw e alongside f' outside of B, reducing the number of crossings by two. Despite this modification, the orders $\leq_m^R$ and $\leq_m^L$ remain unaltered, as the neighborhoods of points v_i and the signature function of pseudoparabola curves and points v_i remain unchanged. However, this contradicts the assumption of the minimum number of crossings in D. $\square$

Corollary 12. Every smooth bigon formed by two curves contains at least one

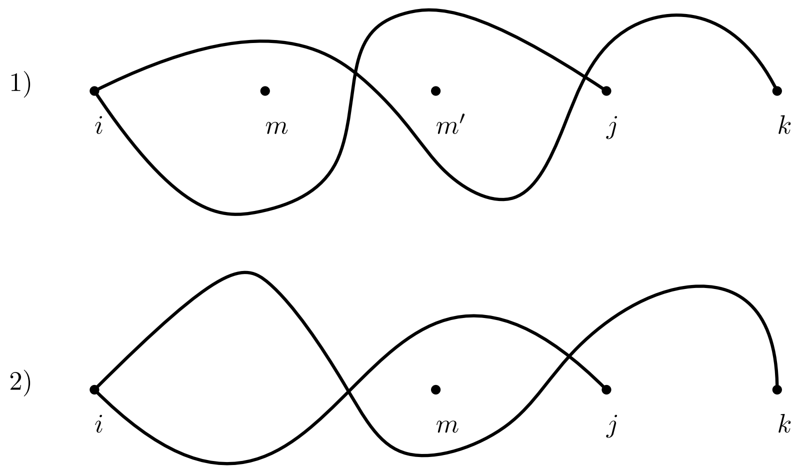


Figure 2.7: Illustration of the two cases in the proof of Claim 13.

point v_m in its interior.

Claim 13. *Any two curves sharing a point v_i cross at most twice.*

Proof. Suppose that for some $j, k \in [n] \setminus \{i\}$ with $j < k$, the curves $p_{i,j}$ and $p_{i,k}$ cross more than twice. By symmetry, we may assume that $i < k$ and that $p_{i,j} \preceq_i^R p_{i,k}$.

We have only two cases due to Corollary 12 as there cannot be any empty bigon except for the first one containing vertex i; see Figure 2.7.

1. There are at least 2 consecutive bigons with points m, m' in their interiors, and therefore (i, m, m', k, l) forms one of the forbidden 5-tuples from the statement of Theorem 5.

2. We have only two bigons. One bigon is without a point in its interior and the other one contains a vertex m inside due to Corollary 12. However, this contradicts $p_{i,j} \preceq_i^R p_{i,k}$ as $\sigma(i, m, j) = +$, $\sigma(i, m, k) = -$ and $\sigma(i, j, k) = +$. □

Claim 14. *For any pairwise distinct integers i, j, k, l from $[n]$, the curves $p_{i,j}$ and $p_{k,l}$ do not intersect more than twice.*

Proof. Suppose for contradiction that $p_{i,j}$ and $p_{k,l}$ cross at least 3 times and hence form at least two smooth bigons B_1, and B_2, numbered from left. Hence, by Corollary 12, B_1 and B_2 contain points m_1 and m_2 inside, respectively. Without loss of generality assume that $i < k$. Then

1. either $j < l$ holds. Consequently, $\sigma(i, k, j) = \overline{\sigma(i, m_1, j)} = \sigma(i, m_2, j) = \sigma(k, m_1, l) = \overline{\sigma(k, m_2, l)} = \sigma(k, j, l)$, which forms a forbidden 6-tuple (i, k, m_1, m_2, j, l).

2. or $l < j$ holds. Then, $\sigma(i, k, j) = \overline{\sigma(i, m_1, j)} = \sigma(i, m_2, j) = \overline{\sigma(i, l, j)} = \sigma(k, m_1, l) = \overline{\sigma(k, m_2, l)}$, which forms a forbidden 6-tuple (i, k, m_1, m_2, l, j). □

Altogether, we see that any two curves from D cross at most twice. Thus, we have finished the proof of the other implication from the statement of Theorem 5. $\square$

All the forbidden 5-tuples and 6-tuples from the statement of Theorem 5 are necessary. We call a signature function σ *realizable by a pseudoparabola drawing* if there is a pseudoparabola drawing with σ as its signature function.

Consider a signature that is not realizable by a pseudoparabola drawing with one of the forbidden 5-tuples from the statement of Theorem 5 on the first 5 vertices and with the rest of the signs set to $+$. This signature does not contain any forbidden 6-tuple from the statement of Theorem 5 as there are no two independent curves with alternating signs. Hence, the forbidden 6-tuples are not sufficient by themselves.

Analogously, we can show that the forbidden 5-tuples are also not sufficient by themselves. Consider a signature that is not realizable by a pseudoparabola drawing with one of the forbidden 6-tuples from the statement of Theorem 5 on the first 6 vertices and with the rest of the signs set to $+$. This signature does not contain any forbidden 5-tuple from the statement of Theorem 5 as there are no two adjacent curves with alternating signs.

2.3 Extended pseudoparabolas

Since we have Theorem 5 and since there is characterization of pseudoline drawings based on forbidden 4-tuples (Theorem 1), which is similar to the characterization of x-monotone simple and semisimple drawings (Theorem 3), one could expect that there is characterization of extended pseudoparabola drawing by finite forbidden configurations. Perhaps surprisingly, we show in this section that this is not the case. That is, we prove the following result.

Theorem 15. *For extended pseudoparabola drawing, there does not exist a set of forbidden t-tuples, where $t \leq C$ for some fixed constant C.*

We start by proving the following auxiliary result about extending pseudoparabola drawings of K_n.

Lemma 16. *Every extended pseudoparabola D of a graph G on n vertices can be extended into an extended pseudoparabola drawing of K_n.*

Proof. We start with an informal sketch of the proof. By applying a suitable homeomorphism of the plane, we can assume that all vertices of D lie on a horizontal line. Consider shrinking D so that it lies in a small neighborhood of this horizontal line. Then, each curve $p_{i,j}$ representing an edge of K_n that is not in G can be composed of line segments and half-lines to form a piece-wise linear curve in the following way; see Figure 2.8.

We start drawing in the minus infinity on the x-axis and below D going horizontally until we almost reach the x-coordinate of the vertex v_i and then we draw an almost vertical line going through v_i up to above D. Then, we draw the rest of the curve as a horizontal line until we almost reach the x-coordinate of the vertex v_j where we draw an almost vertical line down through vertex v_j. Lastly, we draw a horizontal line to the right up to infinity on the x-axis. For every i,

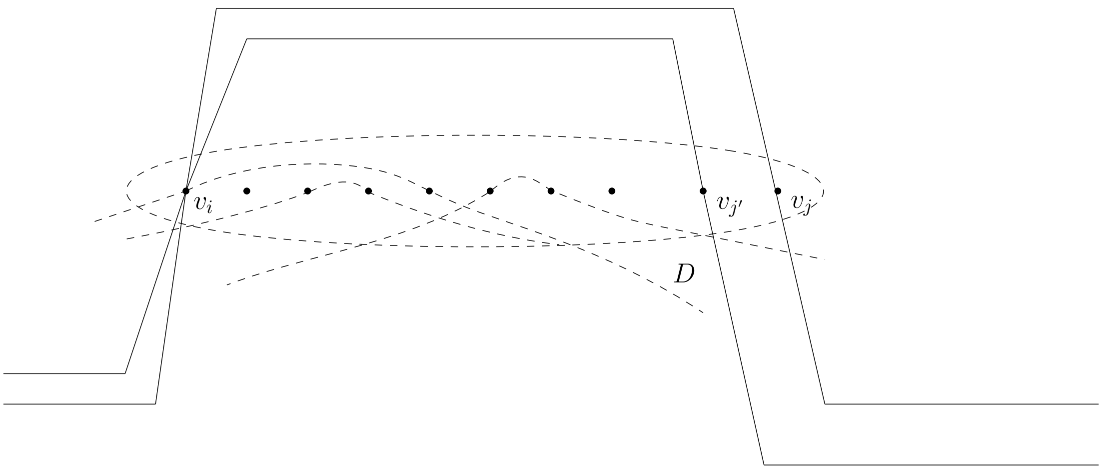


Figure 2.8: Extension of subdrawing in Lemma 16.

we draw the curves $p_{i,j}$ so that they intersect only at v_i. We will now describe this more formally.

We describe relations $\leq_i^R$ similarly as in the proof of Theorem 5. We will first define set $\mathcal{P} = \{p_{j,k} \colon j, k \in [n], j \neq k\}$ of all curves $p_{j,k}$. Let $\mathcal{P}_i = \{p_{i,j} \colon j \in [n], i \neq j\}$. We have already drawn D. That means we have already realized curves $p_{i,j}$ for some $i, j \in [n]$ with $i < j$. We denote $\mathcal{P}_D$ as the set of all such curves $p_{i,j}$ contained in D. The drawing D already naturally defines $\leq_m^R$ on the curves of $\mathcal{P}_D$. Consider $i, j, j' \in [n]$ with $i < j' < j$ and extend the orders $\leq_i^R$ in the following way.

- For $p_{i,j}, p_{i,j'} \in \mathcal{P}_i \setminus \mathcal{P}_D$ we set $p_{i,j'} \leq_i^R p_{i,j}$.
- For $p_{i,j}, \in \mathcal{P}_D, p_{i,j'} \in \mathcal{P}_i \setminus \mathcal{P}_D$ we set $p_{i,j} \leq_i^R p_{i,j'}$.
- For $p_{i,j'} \in \mathcal{P}_D, p_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_D$ we set $p_{i,j'} \leq_i^R p_{i,j}$.

It is easy to verify that this relation is a total order on $\mathcal{P}_i$. For two new curves, the order depends only on the relative order of the endpoints of the curves. For pairs made of new and old curves, the order is given.

Lastly, we will define the remaining signs for curves in $\mathcal{P} \setminus \mathcal{P}_D$. We define the signs similarly as before, in other words, for $i, j, k \in [n]$ with $i < j < k$, we define $\sigma(i, j, k) = +$. Among all such drawings of K_n extending D satisfying relations $\leq_i^R$ and the prescribed signs from the previous paragraph, we choose one with a minimal number of crossings. One can observe that the formal definition now merges with the description in the sketch of the proof from the second paragraph.

When we draw the remaining curves in this way, we can easily see that any two curves have at most 2 crossings. This is obvious for the curves contained in D as D is an extended pseudoparabola drawing. Next, an intersection of two new curves can only appear on the almost vertical parts as we chose the curves so that we have a minimal number of crossings. Therefore, new curves can cross at most once. The last option is to consider a crossing between a new curve and an original curve of D. This also can happen at most two times as horizontal parts are away from D and we have only two (almost) vertical parts of each new curve that can be crossed at most once each as curves are x-monotone. $\square$

We are now ready to prove Theorem 15.

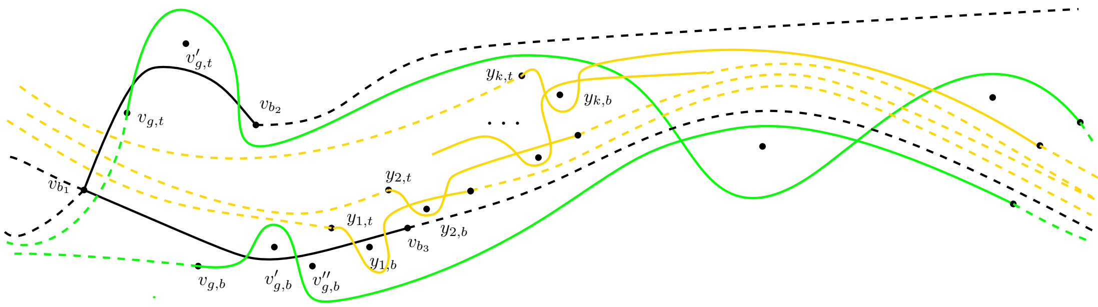


Figure 2.9: Full curves are given by signs of σ and dashed parts are extensions.

Proof of Theorem 15. Suppose for contradiction that there is a positive integer C such that for every positive integer n all extended pseudoparabola drawings of K_n can be characterized by forbidden t-tuples with $t \leq C$. For some positive integers k and m with $m > k > t$, we construct an extended drawing D of a graph on m vertices that is not an extended psedoparabola drawing and contains k vertices $y_{1,b}, \ldots, y_{k,b}$ such that by deleting any of them, we obtain an extended pseudoparabola drawing D; see Figure 2.9 for the construction. By Lemma 16 each of the smaller drawings on $m-1$ vertices can be extended to an extended pseudoparabola drawing of K_{m-1}. Since D is not an extended pseudoparabola drawing, there is a t-tuple T of its vertices that forms one of the forbidden patterns. Let $y_{c,b}$ be a vertex of D that is not contained among these t vertices. The vertex $y_{c,b}$ exists as $k > t$. Let D' be a drawing obtained from D by removing $y_{c,b}$. We will show that D' is an extended pseudoparabola drawing but still contains the forbidden pattern T. We will then apply Lemma 16 to this extended pseudoparabola drawing to obtain an extended pseudoparabola drawing D'' of K_{m-1}. However, D'' then contains the forbidden pattern T on t vertices which contradicts the assumption that these patterns characterize the extended pseudoparabola drawings of K_n.

Consider the x-monotone drawing D from Figure 2.9 and let σ be its signature function. We now show that σ is not realizable by an extended pseudoparabola drawing as extensions of its curves are forced to cross too many times.

We name each yellow and green curve according to its left vertex, that is, a curve with left vertex v is called $\gamma(v)$. Note that each yellow and green curve in Figure 2.9 is uniquely determined by its left vertex. For convenience, we divide each curve into three parts — its left extension, its inner part, and its right extension. For yellow curves, the left extension of a curve $\gamma(y_{i,t})$ stays below curve $\gamma(y_{i+1,t})$ for $i \in \{1, \ldots, k-1\}$ as they have already crossed twice in the inner part (given by σ).

Similarly, the left extension of the curve $\gamma(y_{1,t})$ stays above curve $v_{b_1}v_{b_3}$ as they also already crossed twice (again given by σ). Moreover, the left extension of $\gamma(v_{g,t})$ stays below the left extension of the curve $v_{b_1}v_{b_2}$ as they already crossed twice. Lastly, the left extension of the curve $\gamma(y_{k,t})$ stays below the inner part of $\gamma(v_{g,t})$ as they already crossed twice. Hence, the left extensions of $\gamma(v_{k,t})$ and $\gamma(v_{g,t})$ are forced to cross in the neighborhood of the vertex v_{b_1} as can be seen in Figure 2.9. Therefore, D is not an extended pseudoparabola drawing as the curves $\gamma(v_{k,t})$ and $\gamma(v_{g,t})$ intersect 3 times.

To finish the proof, it remains to find the extended pseudoparabola drawing

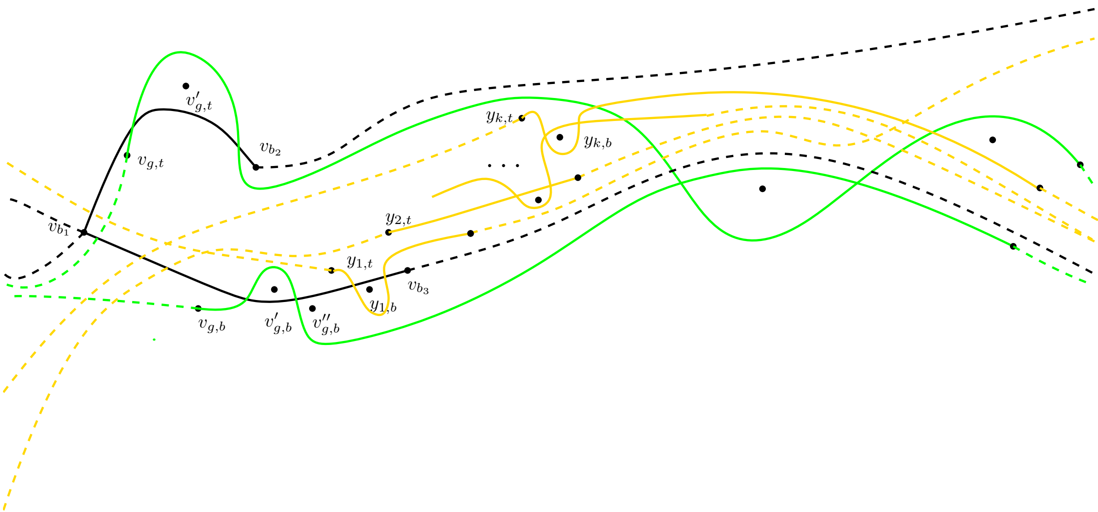


Figure 2.10: Redrawn figure 2.9 so that left extension of all yellow curves $\gamma(y_{j,t})$ for $j \geq c$ goes down trough inner curve segment $v_{b_1}v_{b_3}$. On the other hand, all right extensions of yellow curves $\gamma(y_{j,t})$ for $j < c$ go up through the yellow curve $\gamma(v_{y_k,t})$.

D' with signature function σ'. Each curve $\gamma(y_{i,t})$ has two vertices $y_{i,t}$ and $y_{i,b}$ associated with it. There is at least one vertex $y_{i,b}$ which is not included in T as $k > C \geq t$. Therefore, there exists c for which the vertex $y_{c,b}$ is not in T. Hence, we can redraw D so that curves $\gamma(y_{c,t})$ and $\gamma(y_{c-1,t})$ do not cross in the inner part of the curves; see Figure 2.10 where this is illustrated for $c = 2$. This means that σ' is realizable with an extended pseudoparabola drawing D'. Now we can apply Lemma 16 to obtain also a pseudoparabola drawing D'' of K_{m-1} which is a contradiction as D'' contains T. $\square$

Theorem 15 hence implies that there is no algorithm that checks the realizability of a signature function by extended pseudoparabola drawing that is based on checking forbidden patterns of fixed size. One would need to check patterns of size dependent on the number of vertices n. We recall that such algorithms exist for pseudoline drawings and simple x-monotone drawings of complete graphs.

3. Higher degree curves x-monotone drawings

3.1 Introduction

In this chapter, we generalize our results from the previous chapter to "higher-degree pseudopolynomial" x-monotone drawings of K_n. An x-monotone drawing D of a graph G is a *d-degree drawing* if the edges of D can be drawn as simple x-monotone curves that intersect each other at most d times. Similarly, an extended x-monotone drawing D of G is an *extended d-degree drawing* if the edges of D can be drawn as unbounded simple x-monotone curves that intersect each other at most d-times. One can easily see that extended 1-degree drawings are equivalent to pseudoline drawings and 2-degree drawings are equivalent to pseudoparabola drawings.

We will prove that a d-degree drawings of K_n with a fixed degree d also admit a characterization with forbidden patterns of constant size $f(d)$ depending on d and therefore we can easily obtain a brute force algorithm that checks the realizability of a d-degree drawing of K_n of the given signature in $\mathcal{O}(n^{f(d)})$. Similarly as for extended pseudoparabola drawings of K_n, extended higher degree drawings do not admit such a characterization.

3.2 Bounded d-degree curves

In the following result, we show a similar result to Theorem 5. Theorem 17 states that d-degree drawings can characterized by forbidden patterns of size $d+3$ and $d+4$; see Figure 2.1 for an illustration of this statement.

Theorem 17. *A signature function $\sigma \in \Sigma_n$ can be realized by bounded d-degree drawing if and only if there is no ordered $(d+3)$-tuple $(a_1, \dots, a_{d+3})$ with $a_1 < \dots < a_{d+3}$ that satisfies, for all feasible $i, j, k, l \in [n]$,*

$$\sigma(a_1, a_{2i}, a_{d+3}) = \overline{\sigma(a_1, a_{2j+1}, a_{d+3})} = \overline{\sigma(a_1, a_{2k}, a_{d+2})} = \sigma(a_1, a_{2l+1}, a_{d+3})$$

or

$$\sigma(a_1, a_{2i}, a_{d+3}) = \overline{\sigma(a_1, a_{2j+1}, a_{d+3})} = \overline{\sigma(a_2, a_{2k}, a_{d+3})} = \sigma(a_2, a_{2l+1}, a_{d+3})$$

and, additionally, there is no ordered $(d+4)$-tuple $(a_1, \dots, a_{d+4})$ with $a_1 < \dots < a_{d+4}$ that satisfies, for all feasible $i, j, k, l \in [n]$,

$$\sigma(a_1, a_{2i}, a_{d+3}) = \overline{\sigma(a_1, a_{2j+1}, a_{d+3})} = \overline{\sigma(a_2, a_{2k}, a_{d+4})} = \sigma(a_2, a_{2l+1}, a_{d+4})$$

or

$$\sigma(a_1, a_{2i}, a_{d+4}) = \overline{\sigma(a_1, a_{2j+1}, a_{d+4})} = \overline{\sigma(a_2, a_{2k}, a_{d+3})} = \sigma(a_2, a_{2l+1}, a_{d+3}).$$

Proof. Let D be an x-monotone d-degree drawing of K_n. It is clear that D does not contain any of the forbidden $(d+3)$- or $(d+4)$-tuples as otherwise there are two edges of D that cross more than d times; see Figure 2.1 for $d = 2$.

To prove the other implication, let σ be a sign function from Σ_n that does not contain any of the forbidden $(d+3)$- or $(d+4)$-tuples. We will start with Pseudoparabola construction 2.2 that states basic requirements on relations $\leq_i^L$ and $\leq_i^R$ which represents relative order of lines $p_{j,k} \in \mathcal{P} = \{p_{j,k} \colon j, k \in [n], j \neq k\}$ at each L_m. To recall, $\mathcal{P}_i = \{p_{i,j} \colon j \in [n], i \neq j\}$ is a set curves from $\mathcal{P}$ that contain the vertex v_i.

We first specify the relative order of the curves from $\mathcal{P}_i$ in the orderings $\leq_i^L$ and $\leq_i^R$. Later, we also specify the relative order of the remaining curves from $\mathcal{P}$ in these two orders.

Definition 4. *For $i, j, k \in [n]$ with $j < k$ and $i \notin \{j, k\}$, the orders $\leq_i^L$ and $\leq_i^R$ on $\mathcal{P}_i$ are defined in the following way; see Figure 2.3*

- *assume $i < j < k$ then consider maximal $d' \in [d-1]$ such that,*

  1. *there are d' vertices $m_1, \ldots, m_{d'}$ satisfying $i < m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} < j$, $\sigma(i, m_{2a+1}, j) = -$, $\sigma(i, m_{2a+1}, k) = +$, $\sigma(i, m_{2a}, j) = +$, and $\sigma(i, m_{2a}, k) = -$ for all feasible nonnegative integers a. Lastly $\sigma(i, j, k) = (-)^{d'}$.*
  2. *or there are d' vertices $m_1, \ldots, m_{d'}$ satisfying $i < m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} < j$, $\sigma(i, m_{2a+1}, j) = +$, $\sigma(i, m_{2a+1}, k) = -$, $\sigma(i, m_{2a}, j) = -$, and $\sigma(i, m_{2a}, k) = +$ for all feasible nonnegative integers a. Lastly $\sigma(i, j, k) = (-)^{d'+1}$.*

  We set $p_{i,j} \leq_i^R p_{i,k}$ in the first case and $p_{i,k} \leq_i^R p_{i,j}$ in the second case.

- *assume $k < j < i$ then consider maximal $d' \in [d-1]$ such that,*

  1. *there are d' vertices $m_1, \ldots, m_{d'}$ satisfying $j < m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} < i$, $\sigma(j, m_{2a+1}, i) = -$, $\sigma(k, m_{2a+1}, i) = +$, $\sigma(j, m_{2a}, i) = +$, and $\sigma(k, m_{2a}, i) = -$ for all feasible nonnegative integers a. Lastly $\sigma(k, j, i) = (-)^{d'}$.*
  2. *or there are d' vertices $m_1, \ldots, m_{d'}$ satisfying $j < m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} < i$, $\sigma(j, m_{2a+1}, u) = +$, $\sigma(k, m_{2a+1}, i) = -$, $\sigma(j, m_{2a}, i) = -$, and $\sigma(k, m_{2a}, i) = +$ for all feasible nonnegative integers a. Lastly $\sigma(k, j, i) = (-)^{d'+1}$.*

  We set $p_{j,i} \leq_i^L p_{k,i}$ in the first case and $p_{k,i} \leq_i^L p_{j,i}$ in the second case.

Is is easy that for $d' = 0$ one of the two conditions is satisfied and hence the relations between $p_{i,j}$ and $p_{i,k}$ is set too.

One can easily verify that the relation $\leq_i^R$ on $\mathcal{P}_i$ is weakly anti-symmetric since we cannot have $p_{i,j} \leq_i^R p_{i,k}$ and $p_{i,k} \leq_i^R p_{i,j}$ for distinct curves $p_{i,j}, p_{i,k}$ as by Definition 4 we would have different signs $\sigma(i, j, k) = -$ and also $\sigma(i, j, k) = +$ which is impossible. Analogously, we can prove that $\leq_i^L$ is weakly anti-symmetric.

We will prove that the relations $\leq_i^L$ and $\leq_i^R$ described in Definition 4 are total orders at each point i.

Let $p_{i,j}$, $p_{i,k}$ be two distinct curves from $\mathcal{P}_i$ with $j < k$. To prove that $\leq_i^R$ is a total order, we will again define for each pair $p_{i,j}$, $p_{i,k}$ new generalized auxiliary relation $\leq_{i,a}^R$. The proof that $\leq_i^L$ can be done analogously due to vertical symmetry. The new relation $\leq_{i,a}^R$ is defined analogously as in Definition 4 but the first case considers with the only difference that the vertices $m_1, \dots, m_p$ is restricted to lie only in the interval $a \leq m_p < j$ for $p \in \{1, \dots, d'\}$, and not in the whole interval (i, j); see Figure 2.4.

Definition 5 (Generalized Definition 3)**.** *For $a, i, j, k \in [n]$ with $j < k$ and $i \notin \{j, k\}$, the orders $\leq_{a,i}^L$ and $\leq_{i,a}^R$ on $\mathcal{P}_i$ are defined in the following way; see Figure 2.4 for $d = 2$:*

- *assume $i \leq a \leq j < k$ then consider maximal $d' \in [d-1]$ such that,*
  1. *there are d' vertices $m_1, \dots, m_{d'}$ satisfying $a \leq m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} < j$, $\sigma(i, m_{2p+1}, j) = -$, $\sigma(i, m_{2p+1}, k) = +$, $\sigma(i, m_{2p}, j) = +$, and $\sigma(i, m_{2p}, k) = -$ for all feasible non negative integers p. Lastly $\sigma(i, j, k) = (-)^{d'}$.*
  2. *or there are d' vertices $m_1, \dots, m_{d'}$ satisfying $a \leq m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} < j$, $\sigma(i, m_{2p+1}, j) = +$, $\sigma(i, m_{2p+1}, k) = -$, $\sigma(i, m_{2p}, j) = -$, and $\sigma(i, m_{2p}, k) = +$ for all feasible non negative integers p. Lastly $\sigma(i, j, k) = (-)^{d'+1}$.*

  We set $p_{i,j} \leq_i^R p_{i,k}$ in the first case and $p_{i,k} \leq_i^R p_{i,j}$ in the second case.

- *assume $k < j \leq a \leq i$ then consider maximal $d' \in [d-1]$ such that,*
  1. *there are d' vertices $m_1, \dots, m_{d'}$ satisfying $j < m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} \leq a$, $\sigma(j, m_{2p+1}, i) = -$, $\sigma(k, m_{2p+1}, i) = +$, $\sigma(j, m_{2p}, i) = +$, and $\sigma(k, m_{2p}, i) = -$ for all feasible non negative integers p. Lastly $\sigma(k, j, i) = (-)^{d'}$.*
  2. *or there are d' vertices $m_1, \dots, m_{d'}$ satisfying $j < m_1 < m_2 < \cdots < m_{d'-1} < m_{d'} \leq a$ and also $\sigma(j, m_{2p+1}, u) = +$, $\sigma(k, m_{2p+1}, i) = -$, $\sigma(j, m_{2p}, i) = -$, and $\sigma(k, m_{2p}, i) = +$ for all feasible non negative integers p. Lastly $\sigma(k, j, i) = (-)^{d'+1}$.*

  We set $p_{j,i} \leq_i^L p_{k,i}$ in the first case and $p_{k,i} \leq_i^L p_{j,i}$ in the second case.

As we mentioned above, we consider $p_{i,j} = p_{j,i}$ and for simplicity, for a triple (a, b, c) with $a < b < c$ and any triple (a', b', c') satisfying $\{a', b', c'\} = \{a, b, c\}$ we define $\sigma(a', b', c') = \sigma(a, b, c)$. We will again state an auxiliary lemma, generalization of Lemma 6, to prove the transitivity of $\leq_i^R$. Informally, this lemma says that two curves $p_{i,j}, p_{i,k}$ cross if and only if there is a vertex a between i and j with one of the curves above and the other one below while this relative order is opposite for $a + 1$.

Lemma 18 (Generalized Lemma 6)**.** *For $a, i, j, k \in [n]$ with $i \leq a \leq j < k$, consider two curves $p_{i,j}$, $p_{i,k}$. If $p_{i,j} \leq_{i,a+1}^R p_{i,k}$, then we have $\sigma(i, a, j) = +$ and $\sigma(i, a, k) = -$ if and only if $p_{i,k} \leq_{i,a}^R p_{i,j}$.*

Similarly, if $p_{i,k} \leq_{i,a+1}^R p_{i,j}$, then we have $\sigma(i, a, j) = -$ and $\sigma(i, a, k) = +$ if and only if $p_{i,j} \leq_{i,a}^R p_{i,k}$.

Proof. By horizontal symmetry, we can without loss of generality assume the first case $p_{i,j} \leq^R_{i,a+1} p_{i,k}$, $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$. Consider maximal d' witnessing $p_{i,j} \leq^R_{i,a+1} p_{i,k}$ together with d' vertices $m'_1, \dots, m'_{d'}$. There are only two subcases:

1. Either $d' < d$ and therefore a represents the vertex m_1 in the first case of Definition 5 for $p_{i,k} \leq^R_{i,a} p_{i,j}$ for $d'+1$ with $m_1 = a$ and $m_p = m'_{p-1}$ for $p \in \{2, \dots, d'+1\}$. It can also be seen that if $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ is not true, then there is no change of the relation $\leq^R_{i,a}$ in the case of Definition 5.

2. Or $d' = d$ and we have forbidden $(d+3)$-tuple $(i, a, m'_1, m'_2, \dots, m'_{d'}, j, k)$. On the other hand, if $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ is not true, then there is no change in the case of Definition 5. $\square$

Due to vertical symmetry, we obtain an analogous result for the relation $\leq^L_{a,i}$.

Lemma 19. *For $a, i, j, k \in [n]$ with $j < k \leq a \leq i$, consider two curves $p_{i,j}$, $p_{i,k}$. If $p_{j,i} \leq^L_{a-1,i} p_{k,i}$, then we have $\sigma(j,a,i) = +$ and $\sigma(k,a,i) = -$ if and only if $p_{k,i} \leq^L_{a,i} p_{j,i}$.*

Similarly, if $p_{k,i} \leq^L_{a-1,i} p_{j,i}$, then we have $\sigma(j,a,i) = -$ and $\sigma(k,a,i) = +$ if and only if $p_{j,i} \leq^L_{a,i} p_{k,i}$.

Analogously to pseudoparabola drawings, we say that these two curves $p_{i,j}, p_{i,k}$ from $\mathcal{P}_i$ *cross between a and $a+1$* if $p_{i,j} \leq^R_{i,a+1} p_{i,k}$ and $\sigma(i,a,j) = + = \overline{\sigma(i,a,k)}$ or if $p_{i,k} \leq^R_{i,a+1} p_{i,j}$ and $\sigma(i,a,j) = - = \overline{\sigma(i,a,k)}$. By symmetry, such a crossing between $a-1$ and a appears if $p_{j,i} \leq^L_{a-1,i} p_{k,i}$ and $\sigma(j,a,i) = + = \overline{\sigma(k,a,i)}$ or if $p_{k,i} \leq^L_{a-1,i} p_{j,i}$ and $\sigma(j,a,i) = - = \overline{\sigma(k,a,i)}$. $\square$

Corollary 20. (Generalized Corollary 8) For $a, i, j, k \in [n]$ with $i \leq a \leq j < k$ satisfying $\sigma(i,a,j) = +$ and $\sigma(i,a,k) = -$ following holds $p_{i,k} \leq^R_{i,a} p_{i,j}$.

Similarly, if $\sigma(i,a,j) = -$ and $\sigma(i,a,k) = +$ following holds $p_{i,j} \leq^R_{i,a} p_{i,k}$.

Proof. By horizontal symmetry, we can without loss of generality assume that $\sigma(i,a,j) = +$. Then the claim easily follows from Lemma 18 as either $p_{i,j} \leq^R_{i,a+1} p_{i,k}$ and there is a cross of $p_{i,j}, p_{i,k}$ and hence $p_{i,k} \leq^R_{i,a} p_{i,j}$ due to this lemma or $p_{i,k} \leq^R_{i,a+1} p_{i,j}$ and there is no cross of $p_{i,j}, p_{i,k}$. $\square$

And again, by vertical symmetry, we obtain the result for the relations $\leq^L_i$.

Corollary 21. For $a, i, j, k \in [n]$ with $k < j \leq a \leq i$ satisfying $\sigma(j,a,i) = +$ and $\sigma(k,a,i) = -$ following holds $p_{k,i} \leq^L_{i,a} p_{j,i}$.

Similarly, if $\sigma(j,a,i) = -$ and $\sigma(k,a,i) = +$ following holds $p_{j,i} \leq^L_{a,i} p_{k,i}$.

We are now ready to prove that the relations $\leq^R_{i,a}$ are transitive, which implies that the relations $\leq^R_i$ are also transitive. By vertical symmetry, we again obtain the transitivity also for the relations $\leq^L_i$ as all the lemmas and corollaries hold symmetrically for $\leq^R_{i,a}$ and $\leq^L_{a,i}$ too.

Lemma 22 (Generalized Lemma 10)**.** *For every $i, a \in [n]$ with $i \leq a$ the relation $\leq^R_{i,a}$ is transitive.*

Proof. The proof is analogous to the proof of Lemma 10. Let i, j, k, l be elements from $[n]$ with $i < j < k < l$. Each of the three curves $p_{i,j}$, $p_{i,k}$, $p_{i,l}$. Consider a satisfying $i < a \leq j$. We start with the base case with $a = j$ and then proceed by induction:

1. If $\sigma(i, j, k) = \xi$ and $\sigma(i, j, l) = \overline{\xi}$, then we can apply Corollary 20. This means that $p_{i,k} \leq^R_{i,j} p_{i,l}$ if $\xi = -$ and $p_{i,l} \leq^R_{i,j} p_{i,k}$ if $\xi = +$. Also $p_{i,j} \leq^R_{i,j} p_{i,l}$ ($p_{i,l} \leq^R_{i,j} p_{i,j}$, respectively) and $p_{i,k} \leq^R_{i,j} p_{i,j}$ ($p_{i,j} \leq^R_{i,j} p_{i,k}$, respectively) as there is no vertex m that is considered in Definition 5. Hence we end up with a transitive order $p_{i,k} \leq^R_{i,j} p_{i,j} \leq^R_{i,j} p_{i,l}$ ($p_{i,l} \leq^R_{i,j} p_{i,j} \leq^R_{i,j} p_{i,k}$, respectively).

2. If $\sigma(i, j, k) = \xi$ and $\sigma(i, j, l) = \xi$, then we can deduce that $p_{i,j} \leq^R_{i,j} p_{i,k}$ and $p_{i,j} \leq^R_{i,j} p_{i,l}$ if $\xi = +$ and $p_{i,k} \leq^R_{i,j} p_{i,j}$ and $p_{i,l} \leq^R_{i,j} p_{i,j}$ if $\xi = -$, as there is no vertex m that is considered in Definition 5. Then

   (a) either $p_{i,k} \leq^R_{i,j} p_{i,l}$ holds and we get $p_{i,j} \leq^R_{i,j} p_{i,k} \leq^R_{i,j} p_{i,l}$ ($p_{i,k} \leq^R_{i,j} p_{i,l} \leq^R_{i,j} p_{i,j}$, respectively) which gives the transitivity.

   (b) or $p_{i,l} \leq^R_j p_{i,k}$ and then $p_{i,j} \leq^R_j p_{i,l} \leq^R_j p_{i,k}$ ($p_{i,j} \leq^R_{i,j} p_{i,l} \leq^R_{i,j} p_{i,k}$, respectively) holds which gives the transitivity.

Now, we proceed with the induction step; see Figure 2.5. Assume we have the vertex m with $i < m < j$. We use integers a, b, c with $\{a, b, c\} = \{j, k, l\}$ to denote permutation of i, j, k. We assume that $p_{i,a} \leq^R_{i,m+1} p_{i,b} \leq^R_{i,m+1} p_{i,c}$ and $p_{i,a} \leq^R_{i,m+1} p_{i,c}$.

1. The simplest case happens when the signs of the curves $p_{i,a}, p_{i,b}, p_{i,c}$ with respect to vertex m do not cause a crossing between $m + 1$ and m. Then none of these pairs changes the order, so we have $p_{i,a} \leq^R_{i,m} p_{i,b} \leq^R_{i,m} p_{i,c}$ and $p_{i,a} \leq^R_{i,m} p_{i,c}$.

2. Curves $p_{i,c}$ and $p_{i,b}$ cross, (that is $\sigma(i, m, c) = -$ and $\sigma(i, m, b) = +$) and then either

   (a) $\sigma(i, m, a) = -$ and hence the order between $p_{i,a}$, $p_{i,b}$ and $p_{i,a}$, $p_{i,c}$ stays the same due to Lemma 18. Then we have $p_{i,a} \leq^R_{i,m} p_{i,c} \leq^R_{i,m} p_{i,b}$ and $p_{i,a} \leq^R_{i,m} p_{i,b}$ as $p_{i,b}$ and $p_{i,c}$ is the only pair that changed the order.

   (b) $\sigma(i, m, a) = +$ and therefore $p_{i,c}$ crosses both $p_{i,b}$ and $p_{i,a}$. The curves $p_{i,a}$ and $p_{i,b}$ do not cross due to Lemma 18. Then we have $p_{i,c} \leq^R_{i,m} p_{i,a} \leq^R_{i,m} p_{i,b}$ and $p_{i,c} \leq^R_{i,m} p_{i,a}$.

3. By horizontal symmetry, we can solve the case where $p_{i,b}$ crosses $p_{i,a}$ as they have the same relative order as $p_{i,c}$ and $p_{i,b}$.

4. If $p_{i,a}$ crosses $p_{i,c}$ then, also either $p_{i,b}$ crosses $p_{i,c}$ or $p_{i,a}$ depending on the sign $\sigma(i, m, b)$. Nevertheless, we have already solved both these cases the second step, so we are done.

We have completed the induction step and the base case too. Therefore, we know that there is a transitive order of three curves $p_{i,j}, p_{i,k}, p_{i,l}$ at vertex $i + 1$. Since we now know that $\leq^R_{i,i+1}$ is transitive and since $\leq^R_i = \leq^R_{i,i} = \leq^R_{i,i+1}$, the order $\leq^R_i$ from Definition 4 is transitive on $\mathcal{P}_i$.

We have completed the induction step and the base case too. Therefore, we know that there is a transitive order of three curves $p_{i,j}, p_{i,k}, p_{i,l}$ at vertex $i+1$. Since we now know that $\leq^R_{i,i+1}$ is transitive and since $\leq^R_i = \leq^R_{i,i} = \leq^R_{i,i+1}$, the order $\leq^R_i$ from Definition 4 is transitive on $\mathcal{P}_i$. □

Observe that the relations $\leq^R_m$ and $\leq^L_m$ are reflexive on $\mathcal{P}_m$. Altogether we know that $\leq^R_m$ and $\leq^L_m$ are reflexive, anti-symmetric, transitive, and total, thus they are total orders.

Similarly to pseudoparabola drawings, it is easy to see that orders $\leq^R_m$ and $\leq^L_m$ specify the relative position of curves with respect to vertices, but the relative position of the curves is still completely specified. We again choose a drawing D of K_n that obeys all the required conditions from $\leq^R_m$ and $\leq^L_m$ and additionally minimizes the total number of crossings of the curves.

The proofs of Lemma 11 and Corollary 12 depend only on the minimality of number of crossing of D hence these results also hold for the d-degree drawing D with a minimal number of crossings without any change in the proofs.

Claim 23. *Any two curves that share a point v_i cross at most d times.*

Proof. Suppose that for some $j, k \in [n] \setminus \{i\}$ with $j < k$, the curves $p_{i,j}$ and $p_{i,k}$ cross more than d times. By symmetry, we may assume that $i < k$ and that $p_{i,j} \leq^R_i p_{i,k}$.

We have only two cases due to Lemma 11 as there cannot be any empty bigon except for the first one containing vertex i; see Figure 2.7 for $d = 2$.

1. There are at least d consecutive bigons with points $m_1, \ldots, m_d$ in their interiors and therefore $(i, m_1, \ldots, m_d, k, l)$ forms one of the forbidden $(d+3)$-tuples from the statement of Theorem 17.

2. We have d bigons. The first bigon from left is without a point in its interior and the other $d-1$ bigons contain vertices $m_1, \ldots, m_{d-1}$, respectively, in their interior due to Corollary 12. However, this contradicts $p_{i,j} \leq^R_i p_{i,k}$ as $\sigma(i, m_{2p+1}, j) = +$, $\sigma(i, m_{2p+1}, k) = -$ and $\sigma(i, m_{2p}, j) = -$, $\sigma(i, m_{2p}, k) = +$ for all feasible nonnegative integers p and lastly $\sigma(i, j, k) = (-)^{d-1}$. The $d-1$ is greatest possible d' to consider in Definition 4 hence $m_1, \ldots, m_d$ truly witness the maximum. □

Claim 24. *For any pairwise distinct integers i, j, k, l from $[n]$, the curves $p_{i,j}$ and $p_{k,l}$ do not intersect more than d times.*

Proof. Suppose for contradiction that $p_{i,j}$ and $p_{k,l}$ cross at least $d+1$ times and hence form at least d smooth bigons $B_1, B_2, \ldots, B_d$ numbered from left. Hence, by Corollary 12, $B_1, B_2, \ldots, B_d$ contain points $m_1, m_2, \ldots, m_d$ inside, respectively. Without loss of generality, assume that $i < k$. Then

1. $j < l$ and d is even. Consequently, we have $\sigma(i, k, j) = \overline{\sigma(i, m_1, j)} = \sigma(i, m_2, j) = \cdots = \overline{\sigma(i, m_{d-1}, j)} = \sigma(i, m_d, j) = \sigma(k, m_1, l) = \overline{\sigma(k, m_2, l)} = \cdots = \sigma(k, m_{d-1}, l) = \overline{\sigma(k, m_d, l)} = \sigma(k, j, l)$, which forms a forbidden $(d+4)$-tuple $(i, k, m_1, \ldots, m_d, j, l)$.

2. $j < l$ and d is odd. Consequently, we have $\sigma(i,k,j) = \overline{\sigma(i,m_1,j)} = \sigma(i,m_2,j) = \cdots = \sigma(i,m_{d-1},j) = \overline{\sigma(i,m_d,j)} = \sigma(k,m_1,l) = \overline{\sigma(k,m_2,l)} = \cdots = \overline{\sigma(k,m_{d-1},l)} = \sigma(k,m_d,l) = \overline{\sigma(k,j,l)}$, which forms a forbidden $(d+4)$-tuple $(i,k,m_1,\ldots,m_d,j,l)$.

3. $l < j$ and d is even. Consequently, $\sigma(i,k,j) = \overline{\sigma(i,m_1,j)} = \sigma(i,m_2,j) = \cdots = \overline{\sigma(i,m_{d-1},j)} = \sigma(i,m_d,j) = \overline{\sigma(i,l,j)} = \sigma(k,m_1,l) = \overline{\sigma(k,m_2,l)} = \cdots = \overline{\sigma(i,m_{d-1},j)} = \sigma(i,m_d,j)$, which forms a forbidden $(d+4)$-tuple $(i,k,m_1,m_2,\ldots,m_d,l,j)$.

4. $l < j$ and d is odd. Consequently, $\sigma(i,k,j) = \overline{\sigma(i,m_1,j)} = \sigma(i,m_2,j) = \cdots = \sigma(i,m_{d-1},j) = \overline{\sigma(i,m_d,j)} = \sigma(i,l,j) = \sigma(k,m_1,l) = \overline{\sigma(k,m_2,l)} = \cdots = \sigma(i,m_{d-1},j) = \overline{\sigma(i,m_d,j)}$, which forms a forbidden $(d+4)$-tuple $(i,k,m_1,m_2,\ldots,m_d,l,j)$. $\square$

Altogether, we see that any two curves from D cross at most d times. Thus, we have finished the proof of the other implication from the statement of Theorem 17.

All the forbidden $(d+3)$-tuples and $(d+4)$-tuples from the statement of Theorem 17 are necessary. We call a signature function σ *realizable by a d-degree drawing* if there is a d-degree drawing with σ as its signature function.

Consider a signature that is not realizable by a d-degree drawing with one of the forbidden $(d+3)$-tuples from the statement of Theorem 17 on the first $d+3$ vertices and with the rest of the signs set to $+$. This signature does not contain any forbidden $(d+4)$-tuple from the statement of Theorem 17 as there are no two independent curves with alternating signs. Hence, the forbidden $(d+4)$-tuples are not sufficient by themselves.

Analogously, we can show that the forbidden $(d+3)$-tuples are also not sufficient by themselves. Consider a signature that is not realizable by a d-degree drawing with one of the forbidden $(d+4)$-tuples from the statement of Theorem 17 on the first $d+4$ vertices and with the rest of the signs set to $+$. This signature does not contain any forbidden $(d+3)$-tuple from the statement of Theorem 17 as there are no two adjacent curves with alternating signs.

3.3 Extended d-degree x-monotone drawings

Similarly to Theorem 15, one could expect that there is no characterization of extended d-degree drawing by finite forbidden configurations. Indeed, we will show that there is no such characterization. The following theorem generalizes Theorem 15 but needs to distinguish cases for odd and even d.

Theorem 25 (Generalized Theorem 15)**.** *For extended d-degree drawing, there does not exist a set of forbidden t-tuples, where $t \leq C$ for some fixed constant C.*

Proof. Suppose for contradiction that there is a positive integer C such that for every positive integer n all extended d-degree drawings of K_n can be characterized by forbidden t-tuples with $t \leq C$. For some positive integers k and m with $m > k > t$, we construct an extended drawing D of a graph on m vertices that is not an extended d-degree drawing and contains k vertices $y_{1,b_1}, \ldots, y_{k,b_1}$ such that

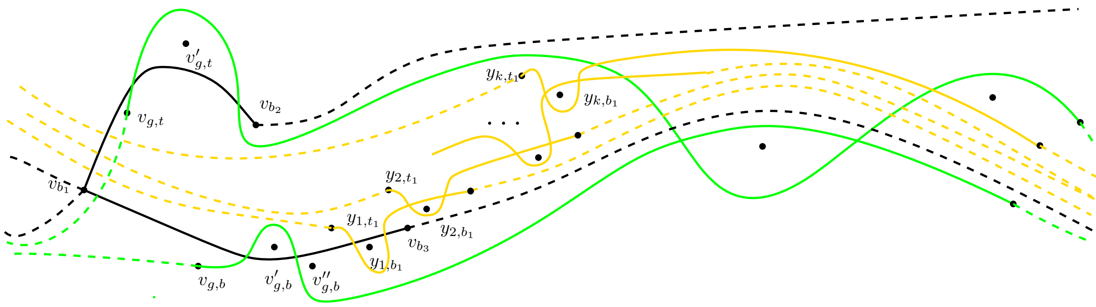


Figure 3.1: Full curves are given by signs of σ and dashed parts are extensions for even d.

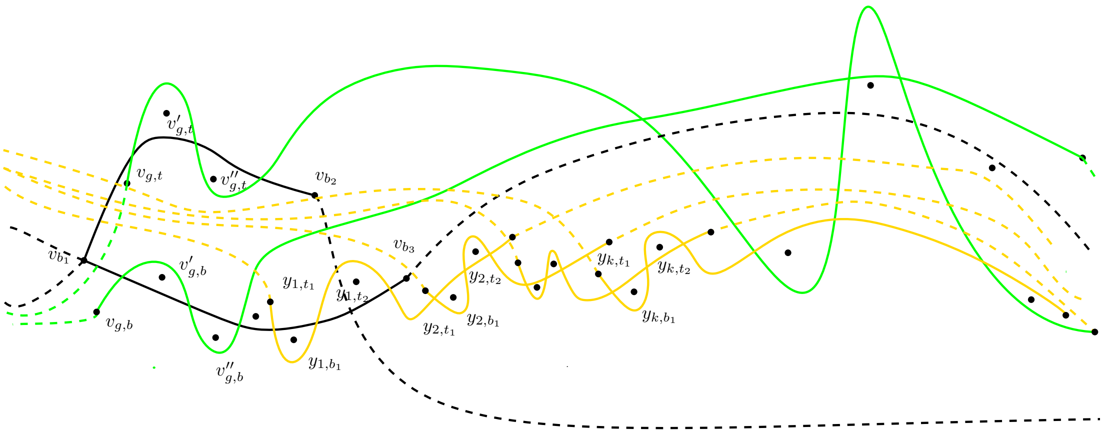


Figure 3.2: Full curves are given by signs of σ and dashed parts are extensions for odd d.

by deleting any of them, we obtain an extended d-degree drawing; see Figure 2.9 for the construction. By Lemma 16 each of the smaller drawings on $m-1$ vertices can be extended to an extended d-degree drawing of K_{m-1}. Since D is not an extended d-degree drawing, there is a t-tuple of its vertices that forms one of the forbidden patterns. Let y_{c,b_1} be a vertex of D that is not contained among these t vertices. The vertex y_{c,b_1} exists as $k > t$. Let D' be a drawing obtained from D by removing y_{c,b_1}. We will show that D' is an extended d-degree drawing. We will then apply Lemma 16 to this extended d-degree drawing to obtain an extended d-degree drawing D'' of K_{m-1}. However, D'' then contains the forbidden pattern T on t vertices which contradicts the assumption that these patterns characterize the extended d-degree drawings of K_n.

Consider the x-monotone drawing from Figure 3.1 (Figure 3.2) and let σ be its signature function. We now show that σ is not realizable by an extended d-degree drawing as extensions of its curves are forced to cross too many times.

We name each yellow and green curve according to its left vertex, that is, a curve with left vertex v is called $\gamma(v)$. Note that each yellow and green curve in Figure 3.1 (Figure 3.2) is uniquely determined by its left vertex. For convenience, we divide each curve into three parts — its left extension, its inner part, and its right extension. For yellow curves, the left extension of a curve $\gamma(y_{i,t_1})$ stays below curve $\gamma(y_{i+1,t_1})$ for $i \in \{1, \ldots, k-1\}$ as they have already crossed d times in the inner part (given by σ).

Similarly, the left extension of the curve $\gamma(y_{1,t_1})$ stays above curve $v_{b_1}v_{b_3}$ as they also already crossed d times (again given by σ). Moreover, the left extension of $\gamma(v_{g,t})$ stays below the inner part of the curve $v_{b_1}v_{b_2}$ as they already crossed d times. Lastly, the left extension of the curve $\gamma(y_{k,t_1})$ stays below the inner part $\gamma(v_{g,t})$ as they already crossed d times. Hence, the left extensions of $\gamma(v_{k,t_1})$ and $\gamma(v_{g,t})$ are forced to cross in the neighborhood of the vertex v_{b_1} as can be seen in Figure 3.1 (Figure 3.2). Therefore, D is not an extended d-degree drawing as the curves $\gamma(v_{k,t_1})$ and $\gamma(v_{g,t})$ intersect $d+1$ times.

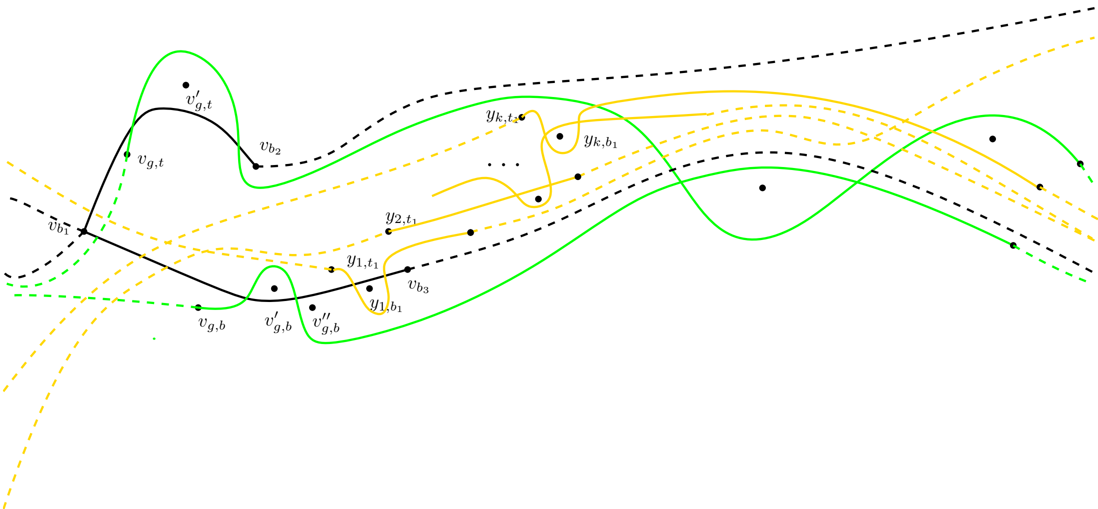


Figure 3.3: Redrawn figure 3.1 so that left extension of all yellow curves $\gamma(y_{j,t_1})$ for $j \geq c$ goes down trough inner curve segment $v_{b_1}v_{b_3}$. On the other hand, all right extensions of yellow curves $\gamma(y_{j,t_1})$ for $j < c$ go up through the yellow curve $\gamma(v_{y_k,t_1})$.

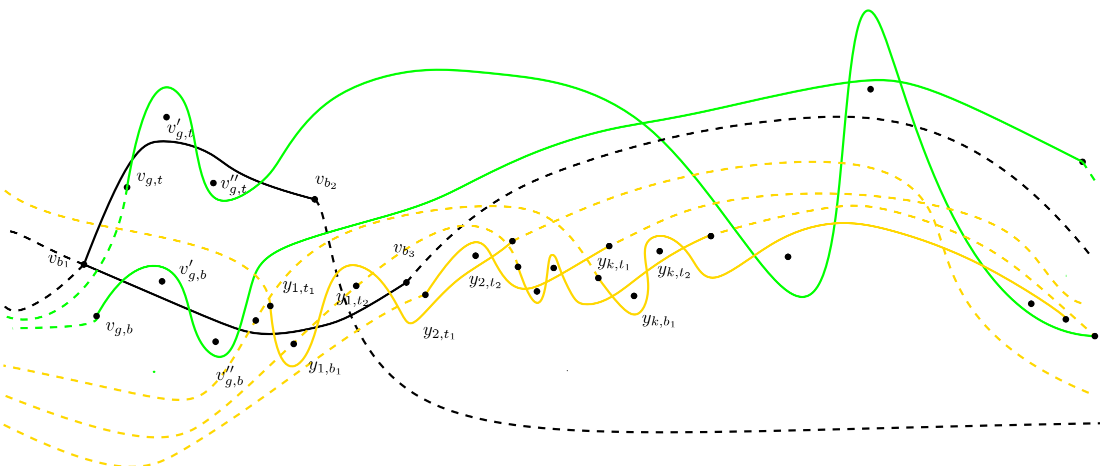


Figure 3.4: Redrawn figure 3.2 so that left extension of all yellow curves $\gamma(y_{j,t_1})$ for $j \geq c$ goes down trough inner curve segment $v_{b_1}v_{b_3}$. On the other hand, all right extensions of yellow curves $\gamma(y_{j,t_1})$ for $j < c$ go down through the yellow curve $\gamma(v_{y_k,t_1})$.

To finish the proof, it remains to find the extended d-degree drawing D' with its signature function σ'. Each curve $\gamma(y_{i,t_1})$ has d vertices $y_{i,t_1}, y_{i,b_1}, \ldots, y_{i,t_{d/2}}, y_{i,b_{d/2}}$ ($y_{i,t_1}, y_{i,b_1}, \ldots, y_{i,t_{\lceil d/2 \rceil}}$) associated with it. There is at least one vertex y_{i,b_1} which is not included in T as $k > C \geq t$. Therefore, there exists c for which the vertex y_{c,b_1} is not in T. Hence, we can redraw D so that curves $\gamma(y_{c,t_1})$ and $\gamma(y_{c-1,t_1})$ do not cross in the inner part of the curves; see Figure 3.3 (Figure 3.4) where this is illustrated for $c = 2$. This means that σ' is realizable with an extended d-degree drawing D'. Then we can apply Lemma 16 to obtain the extended d-degree drawings D'' which is a contradiction as D'' contains T. $\square$

Similarly as before, Theorem 25 hence implies that there is no algorithm that checks the realizability of a signature function by extended d-degree drawing that is based on checking forbidden patterns of fixed size. One would need to check patterns of size dependent on the number of vertices n.

4. Independent generalisation x-monotone drawings

4.1 Introduction

In this chapter, we generalize our results from the previous chapter even more to "higher-degree pseudopolynomial" x-monotone drawings of K_n where the allowed number of crossings for adjacent and independent edges can be different.

For integers $d_i \geq 0$ and $d_a \geq 1$, an x-monotone drawing D of a graph G is a (d_a, d_i)-*degree drawing* if the edges of D can be drawn as simple x-monotone so that adjacent curves intersect each other at most d_a times and independent curves intersect each other at most d_i times. Similarly, an x-monotone drawing D of a graph G is a *extended* (d_a, d_i)-*degree drawing* if the edges of D can be drawn as unbounded simple x-monotone so that adjacent curves intersect each other at most d_a times and independent curves intersect each other at most d_i times.

We will first show the result for (extended) $(d, 0)$-degree drawings as it turns out that this class is rather trivial.

4.2 (Extended) $(d, 0)$-degree drawings

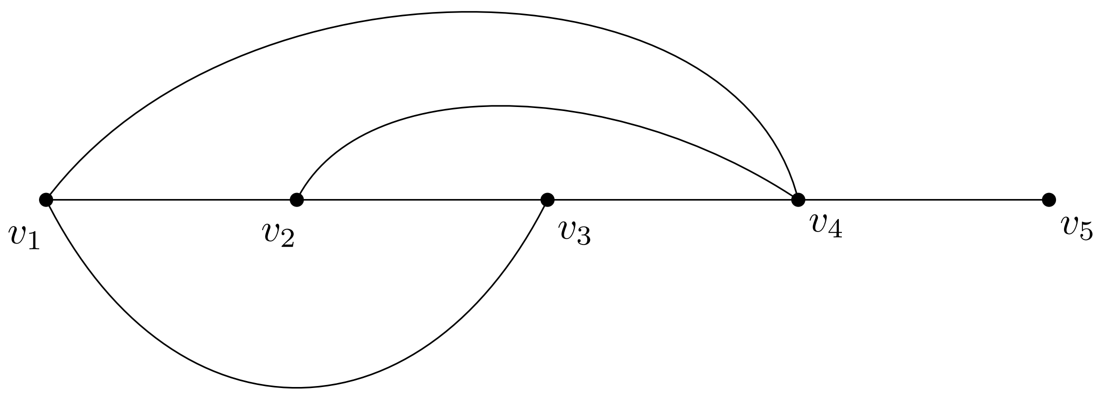


Figure 4.1: Up to vertical and horizontal symmetries due to Lemma 26 unique $(d, 0)$-degree drawing

We show this class admits only drawings of K_n with $n \leq 4$.

Lemma 26. *Let $\sigma \in \Sigma_n$ be a signature function such that is realizable by a (extended) $(d, 0)$-degree drawing. Then for any $i, j, k \in [n]$ with $i < k < j$, σ cannot change the sign between k and $k+1$, that is, $\sigma(i, k, j) = \xi$ and $\sigma(i, k+1, j) = \overline{\xi}$.*

Proof. The change of sign of $p_{i,j}$ between k and $k+1$ means that the portion of the curve $p_{i,j}$ to the left of the vertex k is on the opposite side of $p_{k,k+1}$ than its right portion to the right of $k+1$. Therefore, there is a crossing between two independent curves $p_{k,k+1}$ and $p_{i,j}$, which is not allowed since $d_i = 0$. $\square$

We can see that the drawing D of K_4 from Figure 4.1 is unique up to symmetries and a suitable homeomorphism. We can easily see that curve $p_{2,5}$ in K_n for $n \geq 5$ cannot be drawn to extend D. It suffices to consider only two cases as σ cannot change the sign due to Lemma 26. Hence, both the bounded and extended class of $(d, 0)$-degree drawings has only a finite set of allowed patterns.

4.3 Bounded (d_a, d_i)-degree drawings

Here, we will show that bounded (d_a, d_i)-degree drawings can always be characterized by finite forbidden patterns for any values of d_a and d_i.

4.3.1 $(1, 1)$-degree drawings and $(1, \infty)$-degree drawings

These classes are already described by finite forbidden patterns in Theorem 3 as simple and semisimple x-monotone drawings, respectively.

4.3.2 (d_a, d_i)-degree drawings

We now state Theorem 27 which generalizes Theorem 17. In particular, for the choices $d_a = 1, d_i = 1$ and $d_a = 1, d_i = \infty$ gives a characterization by forbidden patterns for the classes of simple and semisimple, respectively, drawings.

Theorem 27. *For integers $d_a \geq 1$ and $d_i \geq 1$, a signature function $\sigma \in \Sigma_n$ can be realized by a bounded (d_a, d_i)-degree drawing if and only if there is no ordered $(d_a + 3)$-tuple $(a_1, \ldots, a_{d+3})$ with $a_1 < \cdots < a_{d_a+3}$ that satisfies, for all feasible $i, j, k, l \in [n]$,*

$$\sigma(a_1, a_{2i}, a_{d_a+3}) = \overline{\sigma(a_1, a_{2j+1}, a_{d_a+3})} = \overline{\sigma(a_1, a_{2k}, a_{d_a+2})} = \sigma(a_1, a_{2l+1}, a_{d_a+3})$$

or

$$\sigma(a_1, a_{2i}, a_{d_a+3}) = \overline{\sigma(a_1, a_{2j+1}, a_{d_a+3})} = \overline{\sigma(a_2, a_{2k}, a_{d_a+3})} = \sigma(a_2, a_{2l+1}, a_{d_a+3})$$

and, additionally, there is no ordered (d_i+4)-tuple $(a_1, \ldots, a_{d_i+4})$ with $a_1 < \cdots < a_{d_i+4}$ that satisfies, for all feasible $i, j, k, l \in [n]$,

$$\sigma(a_1, a_{2i}, a_{d_i+3}) = \overline{\sigma(a_1, a_{2j+1}, a_{d_i+3})} = \overline{\sigma(a_2, a_{2k}, a_{d_i+4})} = \sigma(a_2, a_{2l+1}, a_{d_i+4})$$

or

$$\sigma(a_1, a_{2i}, a_{d_i+4}) = \overline{\sigma(a_1, a_{2j+1}, a_{d_i+4})} = \overline{\sigma(a_2, a_{2k}, a_{d_i+3})} = \sigma(a_2, a_{2l+1}, a_{d_i+3}).$$

We only sketch the proof of Theorem 27. Note that all lemmas and corollaries in the proof of Theorem 17 hold independently for adjacent and independent curves. Hence, it suffices to proceed in the same way, we only apply the lemmas that work with adjacent curves with parameter d_a and the ones that work with independent curves with parameter d_i.

4.4 Extended (d_a, d_i)-degree drawings

In this section, we sketch the proof of the following result. For any $d_i \geq 1$, the class of extended $(1, d_i)$-degree drawings admits a characterization by finite forbidden patterns, while for any $d_a \geq 2$ and $d_i \geq 1$ the class of extended (d_a, d_i)-degree drawings does not admit any such characterization.

4.4.1 Extended $(1, 1)$-degree drawings

Similarly, as in Subsection 4.3.1, this class has already been described by finite forbidden patterns in Theorem 1.

4.4.2 Extended $(1, d)$-degree drawings with $d \geq 2$

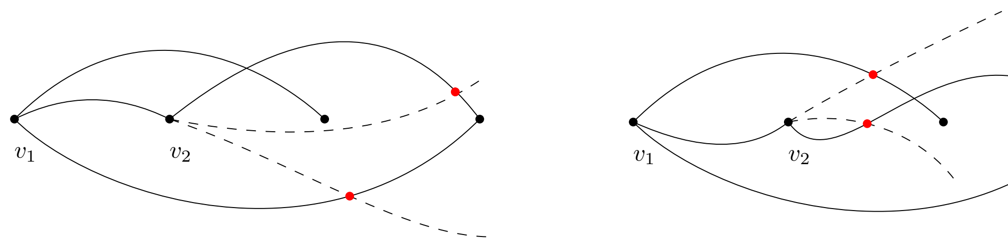


Figure 4.2: Up to symmetry, both cases of the drawings that can be obtained from the pattern $\xi\overline{\xi\xi}\xi$ depending on whether the right extension of the curve $p_{1,2}$ is below or above the curve $p_{2,4}$ at vertex v_2. Therefore, the pattern $\xi\overline{\xi\xi}\xi$ is not realizable by $(1, d)$-degree drawings of K_n.

We will show that the forbidden patterns of $(1, d)$-degree drawings are the same as for pseudoline drawings.

Theorem 28. *A signature function $\sigma \in \Sigma_n$ can be realized by a $(1, d)$-degree drawing if and only if every ordered 4-tuple of indices from $[n]$ is of one of the forms*

$$++++, ----, ++--, --++,$$
$$---+, +++-, +---, -+++$$

in σ.

By Theorem 28, the forbidden patterns in $(1, d)$-degree drawings are of the form

$$+-++, +-+-, -+--, -+-+,$$
$$++-+, --+-, +--+, -++-.$$

That is, those are exactly the forbidden patterns that appear in Theorem 1.

Proof. We can start with the forbidden patterns of form $\xi\bar{\xi}\xi\zeta$ and $\zeta\xi\bar{\xi}\xi$. Their realizability forces adjacent curves to cross and therefore are forbidden for this drawing class. The other forbidden patterns from the statement of Theorem 1 and also Theorem 28 are of form $\xi\overline{\xi\xi}\xi$. Here we need to show more carefully that it is not realizable by $(1, d)$-degree drawings; see Figure 4.2. Hence we have shown the forbidden patterns from the statement of Theorem 1 are also forbidden patterns for this class of drawings.

On the other hand, suppose we have a signature function σ without forbidden patterns from the statement of Theorem 28. As these are the same forbidden patterns as in the statement of Theorem 1 we can apply this theorem to prove that σ is realizable by pseudolinear drawing of K_n which is, by definition, also $(1, d)$-degree drawing for $d \geq 1$.

Hence, we have proved that the class of $(1, d)$-degree drawings has the same characterization by forbidden patterns as the class of pseudolinear drawings. $\square$

4.4.3 Extended $(d, 1)$-degree drawings with $d \geq 2$

Analogously to Theorem 15 we will formulate a similar Theorem 29, but the proof will be a little bit more technical.

Theorem 29. *Let d be an integer with $d \geq 2$. Then for extended $(d, 1)$-degree drawings, there does not exist a set of forbidden t-tuples, where $t \leq C$ for some fixed constant C.*

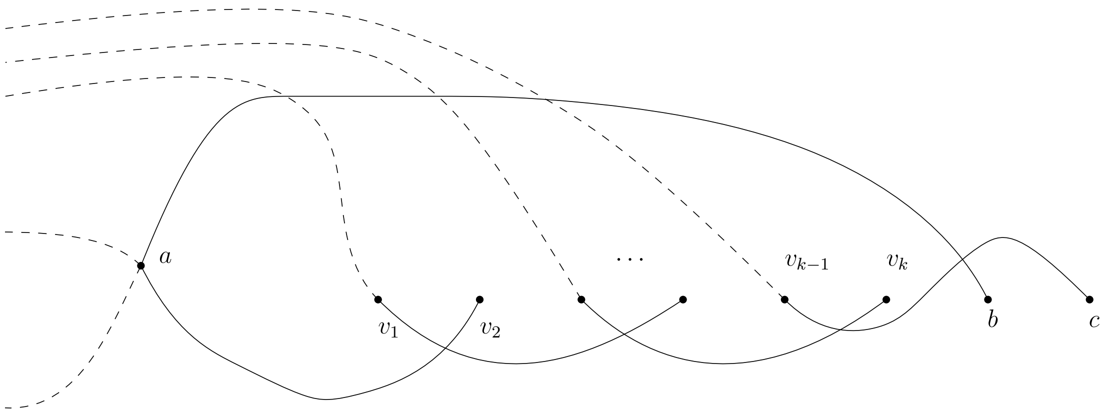


Figure 4.3: Full curves are given by signs of σ and dashed parts are extensions.

Proof. Similarly as in the proof of Theorem 15, suppose for contradiction that there is a positive integer C such that for every positive integer n all extended $(d, 1)$-degree drawings of K_n can be characterized by forbidden t-tuples with $t \leq C$. For some even positive integer k and a positive integer m with $m > k > t$, let D be an extended drawing of a graph on m vertices from Figure 4.3. We show that D is not an extended $(d, 1)$-degree drawing and contains k vertices $v_1, \ldots, v_k$ such that by deleting any of them, we obtain an extended $(d, 1)$-degree drawing. By case analysis, each of the smaller drawings on $m - 1$ vertices can be extended to an extended $(d, 1)$-degree drawing of K_{m-1}. Since D is not an extended $(d, 1)$-degree drawing, there is a t-tuple of its vertices that forms one of the forbidden patterns. Let v_c be a vertex of D that is not contained among these t vertices. The vertex v_c exists as $k > t$. Let D' be a drawing obtained from D by removing v_c.

We will show that D' is an extended $(d, 1)$-degree drawing. In this case, we cannot apply Lemma 16. We will do some casing to prove that D' can be extended to an extended $(d, 1)$-degree drawing D'' of K_{m-1}. Then, however, D'' contains the forbidden pattern T on t vertices which contradicts the assumption that these patterns characterize the extended $(d, 1)$-degree drawings of K_n.

Let σ be the signature function of D. We now show that σ is not realizable by an extended $(d, 1)$-degree drawing as extensions of its curves are forced to cross too many times.

We denote by $\gamma(v_i)$ the curve from D with left vertex v_i. We can see that the left extension of v_1v_4 needs to go up before curve av_2 as these independent curves already crossed each other. Analogously, left extension of each curve $\gamma(v_{2i+1})$, goes up before left extension of $\gamma(v_{2i-1})$ for all feasible integers i. Hence the left extension of the curve $v_{k-1}c$ and the curve ab are forced to cross for a second time, which is in contradiction with realizability by a $(d, 1)$-degree drawing as $v_{k-1}c$ and ab are independent. Therefore, D is not an extended $(d, 1)$-degree drawing.

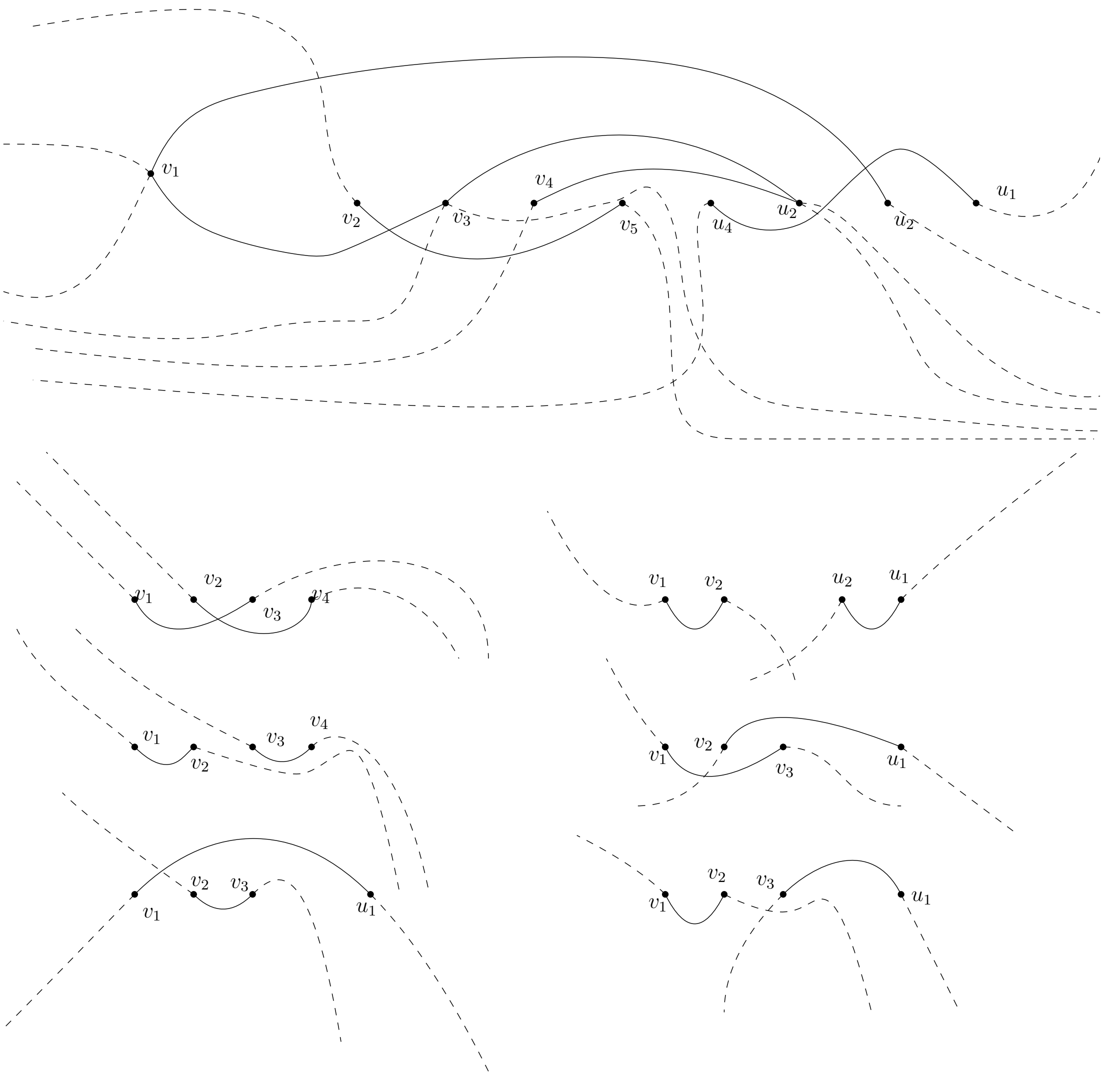


Figure 4.4: Redrawn Figure 4.3 so that left extension of curves v_iv_j go up trough inner curve segment v_1u_2 and all right extensions of v_iv_j go down the gap between last u and last v vertex. Analogously for u_iu_j curves, left extensions go down and the right go up. Below we can see a case analysis of possible crossings.

To finish the proof, it remains to find the extended $(d, 1)$-degree drawing D'

with its signature function σ'. There is at least one vertex v_i which is not included in T as $k > C \geq t$. Therefore, there exists c for which the vertex v_c is not in T. We can therefore omit one of the curves of four consecutive vertices whose endpoint is v_c which splits v_is into two groups. We will label the vertices to the left to the gap as v_is numbered from the left and the vertices to the right to the gap as u_is numbered from the right. Hence, we can redraw D so that the left extension of $v_{k-1}c$ does not cross in the inner part of the curve ab; see Figure 4.4.

We will now extend the drawing D' so that it is a drawing of K_{m-1}. The remaining curves with both vertices of type v will have all signs $-$ in the inner part and the left extension will go up so that we cause a minimal number of crossings. The right extension will go below each of the vertices until it reaches the gap and then go down. Similarly, curves with both u type vertices will do the same but with left and right extensions having the opposite behavior. Lastly, curves with one u and one v vertices will have both extensions going down and the inner part will have all signs equal to $+$. We can see that none of the independent curves cross more than once; see the case analysis at the bottom of Figure 4.4.

A similar case analysis can be done for adjacent curves where we merge two out of the four vertices. Then one new crossing can occur because the extensions could go immediately to the opposite side than their inner part. However, only one such a crossing can occur, as if there were two, could swap the order at the joined vertex and reduce it to zero. Nevertheless $d \geq 2$, hence that means σ' is realizable with an extended $(d, 1)$-degree drawing D'. $\square$

4.4.4 Extended (d_a, d_i)-degree drawings with $d_a \geq 2$ and $d_i \geq 2$

One can easily see that in the proof of Lemma 16 it is only necessary to have $d_a, d_i \geq 2$. Then it only suffices to realize that the construction of σ in the proof of Theorem 25 is based only on independent edges; see Figures 3.1 and 3.2. We will therefore generalize Theorem 25.

Theorem 30 (Generalized Theorem 25)**.** *Let d_a, d_i be integers with $d_a \geq 2$ and $d_i \geq 2$. For extended (d_a, d_i)-degree drawing, there does not exist a set of forbidden t-tuples, where $t \leq C$ for some fixed constant C.*

We can apply Theorem 25 for $d = d_i$ to obtain that extended (d_a, d_i)-degree drawings with $d_a \geq 2$ and $d_i \geq 2$ cannot be characterized by finite forbidden patterns.

5. Conclusion

In this thesis, we have generalized the characterization by forbidden patterns. Bounded (d_a, d_i)-degree drawings can always be characterized by finite forbidden patterns for any values of d_a and d_i. On the other hand, extended (d_a, d_i)-degree drawings can be characterized by finite forbidden patterns only for $d_i = 0$ or $d_a \leq 1$. In other cases the characterization by finite forbidden patterns is impossible.

For classes where characterization by finite forbidden patterns is possible, there is an algorithm that checks the realizability of a signature function by (d_a, d_i)-degree drawing that is based on checking forbidden patterns of fixed size. For classes that do not allow such characterization, such an algorithm does not exist.

In this thesis, we were looking for forbidden patterns to characterize the signature function of many classes of x-monotone drawings. Another interesting question appears when one would like to do it from the opposite point of view.

We can consider, for example, generalized signotopes. It would be really interesting to have a geometric representation for generalized signotopes (see Subsection 1.3.1 for the definition) on 4-tuples. We can also proceed to an even more general version. For positive integers r, k with $k \leq r$, we can consider a class of (r, k)-*generalized signotopes* where on each $(r+1)$-tuple at most k changes of the sign are allowed. Such a class of signature functions deserves known geometric characterization.

Following a similar direction as Miyata [5], one can also generalize k-intersecting pseudoconfiguration of points and relax the third condition in its definition to allow intersecting at most k' times with $k' > k$. It would be interesting to know if this class of drawings can be combinatorially characterized.

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