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**MASTER THESIS**

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# The Gabriel-Roiter measure in representation theory

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Author's signature

I dedicate this thesis to my partner for his support and help during the last six years.

I want to thank my family for supporting my decision to study mathematics in Prague. I also want to thank my supervisor, Jan Šťovíček, for his valuable comments on my work and for deepening my understanding of the representation theory of artin algebras.

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Abstract: The Gabriel-Roiter measure is a module-theoretic invariant, defined in 1972 by P. Gabriel. It is an order-preserving function that refines a composition length of a module by also taking lengths of indecomposable submodules into account. We calculate all Gabriel-Roiter measures for finite-length representations of an orientation of a Dynkin graph $D_4$ and an orientation of a Euclidean graph $\tilde{A}_3$.

In 2007, H. Krause proposed a combinatorial definition of the Gabriel-Roiter measure based on other length functions instead of composition length. We study these alternative Gabriel-Roiter measures on thin representations of quivers whose underlying graph is a tree.

Keywords: Gabriel-Roiter measure, representations of quivers, thin representations

# Obsah

# Introduction

In 1968, Roiter proved the first Brauer-Thrall conjecture, [22]. He showed that a finite-dimensional algebra $A$ over an arbitrary field has an infinite number of pairwise non-isomorphic indecomposable representations (i.e., $A$ is representation-infinite) if and only if it has indecomposable representations of arbitrarily high composition length (i.e., $A$ is of unbounded representation type). Other, better-known proofs of the conjecture exist today. See, for example, [3, Chapter VI.2.].

The idea behind Roiter's proof was to show that for each finite-dimensional algebra $A$ of bounded representation type, there exists an order-preserving function $f$ assigning a natural number to indecomposable modules of finite length, refining the composition length. Moreover, the value on the image of an epimorphism is bouned by the values on direct summands of the domain (dual to the first part of the main property).

Gabriel, in his 1972 report, [5], inspired by the Roiter's proof, defines *the Roiter measure* for modules of finite length. In 1979, Auslander and Smaløwrote that their "*effort to explain* [Roiter's result] ... *in terms of the technics and ideas developed by Auslander and Reiten in connection with almost split sequences and irreducible morphisms*" was *"the original impetus"* for their work presented in [4], establishing a general definition of preprojective and preinjective modules over any artin algebra. The relationship between the Roiter measure and Auslander-Reiten theory was extensively studied in the late 2000s by Bo Chen, e.g., [11], [7] and [6]. It was also studied in the work of Ringel, [19], and Krause, [14]. Some of these results are gathered in Section 1.6.

Gabriel's definition does not assume algebras to be of bounded representation type, but the main property is proved under this assumption. There is a footnote in [5] saying that Ringel showed that the assumption is unnecessary. In [19], Ringel speculates that this restriction to the representation-finite algebras was why the Roiter measure was ignored in the 20th century. Indeed, most literature from the last two decades discusses representation-infinite algebras. A notable exception is [11].

The measure is now known under the name the *Gabriel-Roiter measure*, first coined by Ringel in his 2005's article [21]. Ringel extends Gabriel's definition to arbitrary modules. Krause utilised this in the study of the Ziegler spectrum, [12]. Other authors ignored this extension to infinitely-generated modules.

Ringel's article gives three new proofs of the first Brauer-Thrall conjecture. However, it should be noted that the Gabriel-Roiter measure does not satisfy the properties of the function $f$ from Roiter's proof, see Example 14. Ringel also defines the *Gabriel-Roiter comeasure*. This comeasure does not appear in later literature, but unlike the standard Gabriel-Roiter measure, this comeasure can be used as a function $f$ for Roiter's proof.

Ringel also defines a Gabriel-Roiter filtration, a sequence of indecomposable submodules witnessing the Gabriel-Roiter measure of a given indecomposable module and a related concept of a Gabriel-Rotier submodule. Results in [11] suggest that Gabriel-Roiter filtrations and submodules can be promising objects of study. See also subsection 2.1.3.

Despite the historical connection with artin algebras, the idea behind the Gabriel-Roiter measure is purely combinatorial. Krause formalised this in his 2007 article [13]. The importance of Krause's approach for this thesis lies in the idea that one does not need to derive the Gabriel-Roiter measure from the composition length of a module but rather from any *length function*, see Section 1.3. Chapter 2 studies these *alternative* Gabriel-Roiter measures. They have been only scarcely investigated in the literature. Still, most of the early results about the *standard* Gabriel-Roiter measure hold for alternative Gabriel-Roiter measures with only minor modifications of proofs needed, see Sections 1.4 and 1.5, or [14] for more details. Section 2.2 shows cases when the generalisation fails.

This thesis studies the Gabriel-Roiter measure of indecomposable finite-length $K$-linear representations of finite acyclic quivers. Results in Chapter 2 hold over a general field. The third Chapter requires $K$ to be algebraically closed.

The first chapter gathers well-known results from the representation theory needed for the rest of the thesis. The chapter also serves as an introduction to the theory of the Gabriel-Roiter measure, formulating its basics, illustrating them on examples and discussing its relation to the Auslander-Reiten theory.

Chapter 2 consists of new theoretical results about alternative Gabriel-Roiter measures with a particular interest in thin representations. An algorithm for calculating GR measures of thin representation of quivers whose underlying graph is a tree is given in Subsection 2.1.2. The main theoretical result is Theorem 38. While this setting may seem restrictive, it is sufficient for examples and counterexamples in Chapter 1. Also, the main results about lengths of Gabriel-Roiter filtrations can be proved in this setting, see Theorem 43.

The last chapter calculates the standard Gabriel-Roiter measure for all finite-dimensional indecomposable representations of $\tilde{A}_3$ with one source and one non-adjacent sink. It also illustrates some known results about the Gabriel-Rotier measure in this example.

# 1. Preliminaries and examples

This chapter aims to define the Gabriel-Roiter measure and to discuss some of its basic properties. Some well-known results from the representation theory needed in the rest of the thesis are gathered. Most of the chapter is compilatory, with some results presented in a slightly greater generality and some examples provided to illustrate the theory.

The first section gathers some general results about modules over rings. We also define the concept of an abelian length category. We will not use abelian length categories in later sections. However, it is the original setting where the GR measure was defined [5], and many results of Krause from [13] and [14] are formulated in this setting. The second section discusses quivers, their representations and associated integral forms.

The third section gives a combinatorial definition of the Gabriel-Roiter measure based on [13]. The fourth section then defines further terminology, formulates some basic properties and provides examples.

In the remaining two sections, well-known results about representations, in particular parts of the Auslander-Reiten theory, are presented in the context of their relation to the Gabriel-Roiter measure.

## 1.1 Rings, modules and abelian categories

This section gathers well-known results about rings and their modules used throughout the next. The abelian length category is defined in Subsection 1.1.1. The primary source for this chapter is [1].

By a *ring*, we mean an associative unital ring. In later sections, we will only write *module* for short instead of *right (unital) module*. The composition of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is denoted by $gf$. The unique one-element right $R$-module, the *zero module*, is denoted by 0. For each right $R$-module $M$, its submodules, partially ordered by inclusion, form a complete lattice, [1, Prop. 2.5.], denoted by $(M, \leq)$.

A right $R$-module $M$ is *noetherian* if $(M, \leq)$ is noetherian, i.e., there is no infinite strictly *ascending* chain in $(M, \leq)$. Dually, $M$ is *artinian* if $(M, \leq)$ is artinian, i.e., there is no infinite strictly *descending* chain in $(M, \leq)$. A module is *finite-length* if it is both noetherian and artinian. We denote the category of the right finite-length $R$-modules by *mod-$R$*.

A ring $R$ is *right artinian* if it is artinian as a right $R$-module. By the Akizuki-Hopkins-Levitzki theorem, [1, Thm. 15.20.], a right artinian ring is necessarily noetherian as a right $R$-module.

A right $R$-module $M$ is *finitely generated* if there is a natural $n$ and an $R$-epimorphism $\phi\colon R^n \to M$. The category of finitely generated right modules is known to be essentially small. All finite-length modules are finitely generated. Hence also *mod-R* is essentially small. We denote the set of isomorphism classes of finitely-generated right $R$-modules by [*mod-R*].

The class of artinian and the class of noetherian right modules are closed under finite direct sums and factors [1, Cor. 10.13., Prop. 10.12.]. In particular, a right $R$-module, for $R$ artinian, is finite-length if and only if it is finitely generated.

The *center* of a ring $R$ is a subring consisting of elements commuting with all elements of $R$. A ring $A$ is an *artin algebra* if its center is artinian and $A$ is finitely generated as a module over its center. Artin algebras are artinian as rings.

A right $R$-module is finite-length if it has a *composition series* [1, Prop. 11.1.], i.e., there is a natural number $n$ and a sequence of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M,$$

such that for any $i < n$ the factor module $M_{i+1}/M_i$ is simple. By the Jordan-Hölder theorem, [1, Thm. 11.3.], all composition series are equivalent in the sense that if we have another composition series

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_m = M,$$

then $m = n$ and for each $i < n$ there is $j < m$ such that $M_{i+1}/M_i \cong N_{j+1}/N_j$. This $n$ is called the *composition length* of $M$, denoted by $|M|$ throughout the text.

A non-zero right $R$-module $M$ is *indecomposable* if for any two right modules $N, N'$ such that $N \oplus N' = M$ either $N = 0$ or $N' = 0$. Otherwise, $M$ is *decomposable.* An artin algebra $A$ is *indecomposable* if it is indecomposable as right $A$-module.

We denote the full subcategory of right indecomposable $R$-modules by *ind-R*. Some authors use *ind-R* for a set of representatives of indecomposable modules. We use [*ind-R*] instead.

Any artinian right $R$-module $M$ can be decomposed as a finite direct sum of indecomposable modules

$$M \cong M_1 \oplus \cdots \oplus M_n \quad M_i \in \textit{ind-R}.$$

If $M$ is also noetherian, thus finite-length, then, by the Krull-Schmidt theorem [1, Thm. 12.9.], this decomposition is unique up to permutation of summands and isomorphism. This means that if we have a decomposition

$$M \cong M'_1 \oplus \cdots \oplus M'_m \quad M'_i \in \textit{ind-R},$$

then $m = n$ and for each $i \leq n$ there is $j \leq m$ such that $M_i \cong M'_j$.

A right module $M$ is *simple* if $(M, \leq) = \{0, M\}$. If $R$ is a right artinian ring, there are up to an isomorphism only finitely many simple right modules, [3, Prop. I.3.1.]. A direct sum of all simple submodules of a module $M$ is called the *socle* of $M$, denoted by $soc\ M$. If $soc\ M$ is a simple module, $M$ is called *uniform*. Module $N \leq M$ is *maximal* if $M/N$ is simple. The intersection of all maximal modules is called the *radical* of $M$, denoted by $rad\ M$.

A sequence

$$\epsilon \quad 0 \to K \xrightarrow{i} L \xrightarrow{p} M \to 0$$

in *mod-R* is a *short exact sequence* if $i$ is a monomorphism, $p$ is an epimorphism and $Im(i) = Ker(p)$. The monomorphism $i$ *splits* if it admits a left inverse in *mod-R*. The monomorphism $i$ splits if $p$ has a right inverse. The exact sequence $\epsilon$ *splits* if $i$ splits. If $\epsilon$ splits, then $Im(i)$ is a direct summand of $L$.

A module $P \in$ *mod-R* is *projective* if any short exact sequence $0 \to K \to L \to P \to 0$ in *mod-R* splits. A module $I \in$ *mod-R* is *injective* if any short exact sequence $0 \to I \to L \to M \to 0$ in *mod-R* splits.

### 1.1.1 Abelian length categories

The treatment of the abelian category here is based mainly on [2, Appendix A.1.].

A category $\mathcal{C}$ is called *additive* if it has all finite direct products, the zero object, and all hom-sets are equipped with the structure of an abelian group.

An additive category $\mathcal{C}$ is *abelian* if each morphism $f\colon X \to Y$ admits a kernel $i\colon Ker\ f \to X$, cokernel $p\colon Y \to Coker\ f$ and the induced morphism $\bar{f}\colon Coker\ i \to Ker\ p$ is an isomorphism.

*Example.* For a ring $R$, the category right $R$-modules is abelian.

Category *ind-R* is not abelian. For $f\colon N \to M \in$ *ind-R*, neither the kernel nor coker needs to be indecomposable.

*Remark.* By Corollary 17, if $f$ is a GR-inclusion, then both kernel and cokernel are indecomposable.

In some sense, *mod-R* is a canonical example. By the Freyd-Mitchell Embedding theorem, for any essentially small abelian category $\mathcal{C}$, there exists a ring $R$ such that $\mathcal{C}$ is essentially equivalent to a full subcategory of *mod-R*. This equivalency preserves kernels and cokernels, see [17].

This theorem implies that the Jordan-Hölder theorem and the Krull-Schmidt theorem hold in essentially small abelian categories. This allows us to define a length of an object in an abelian category. An essentially small abelian category is an *abelian length category* if all objects have finite length.

*Example.* For a ring $R$, the category *mod-R* is an abelian length category.

## 1.2 Path algebras and representations of quivers

This section defines quivers, their representations and associated integral forms. We formulate some classical results that will allow us to identify the representations with modules over finite-dimensional algebras. This is the setting in which most of the thesis is set.

The first two parts are based on [2]. The treatment of integral forms is based mainly on lecture notes by Krause, [15], but many results presented here can also be found in [2, Chapter VII]. Results in [2] and [15] are formulated and proven over an algebraically closed field. Sometimes, this assumption is unnecessary, and the more general results are cited from [3, Chapter III.1].

Let $K$ be a field. A ring $A$ is a $K$-*algebra* if an isomorphic copy of $K$ is contained in the center of $A$. We say that $A$ is *finite-dimensional* if it has finite dimension as a vector space over $K$. All finite dimensional algebras are artin algebras.

Note that we can concern ourselves only with right modules. Any right $A$-module can be viewed as a left $A^{op}$-module, where $A^{op}$ denotes the *opposite algebra* of the algebra $A = (A, +, -, \cdot, 0, 1)$ defined by signature $A^{op} := (A, +, -, *, 0, 1)$ where $a * b := b \cdot a$ for any two elements $a, b$ from $A$, [2, Sections 1.1., 1.2].

From now on, we will always say $A$-*module* instead of a *right* $A$-*module.* As with the general rings, (*ind*-$A$) *mod*-$A$ denotes the category of (indecomposable) finite-length $A$-modules.

### 1.2.1 Path algebras

A quiver $Q$ is a quadruple $(Q_0, Q_1, s, t)$ where $Q_0$ is a set of *vertices*, $Q_1$ a set of *arrows* and $s, t$ are two maps $Q_1 \to Q_0$ mapping an arrow to its *source* and *target*, respectively. All quivers in this text are assumed to be finite, i.e., the sets $Q_0$ and $Q_1$ are finite. Vertices are always considered to be assigned numbers from 1 to $|Q_0|$.

By $\bar{Q}$, we denote the *underlying graph of* $Q$ obtained from $Q$ by forgetting the orientation of arrows. $\bar{Q}$ is an unoriented graph, possibly with loops and multiple edges. A quiver $Q$ is *connected* if the graph $\bar{Q}$ is connected.

A quiver $Q'$ is a *subquiver* of $Q$ if $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and the source and the target map of $Q'$ are restrictions of the source and the target map on $Q$ respectively. For a set of vertices $V \subseteq Q_0$ the *full subquiver* induced by $V$ is given by $Q'_0 := V$ and

$$Q'_1 := \{\alpha \mid \alpha \in Q_1;\ t(\alpha) \in V \wedge s(\alpha) \in V\}.$$

A vertex $i \in Q_0$ is called a *sink* (*source*) if there is no arrow $\alpha$ such that $s(\alpha) = i$ ($t(\alpha) = i$).

A seqence of arrows $\alpha_1, \ldots, \alpha_n$ is a *path of length* $n > 0$ if $t(\alpha_i) = s(\alpha_{i+1})$ for every $i < n$. The vertex $s(\alpha_1)$ is the *source of a path* and $t(\alpha_n)$ the *target of a path.* We also consider *trivial* paths $e_i$ for each $i \in Q_0$, also called *stanionary paths.* A non-trivial path with the target equal to the source is called *cycle.* A quiver without cycles is *acylic.*

For a field $K$, a *path algebra* $KQ$ is the $K$-vector space with a basis consisting of paths in $Q$. The product of two paths is their concatenation if the target of the first path is the source of the second path and zero otherwise. By [3, Prop. III.1.1], the path algebra $KQ$ is a well-defined $K$-algebra. It is finite-dimensional if and only if $Q$ is acylic. An algebra $KQ$ is connected (as an algebra) if and only if $Q$ is connected.

An algebra is *hereditary* if every submodule of a finitely-generated projective module is projective. There are numerous other equivalent characterisations of hereditary algebras. See, for example, [2, Thm. VII.1.4.]. A characterisation that is the most important for our purposes is given by the following theorem. The first part follows from [3, Prop. III.1.4], the second from [2, Thm. VII.1.7.].

**Theorem 1.** *Let $Q$ be an acylic quiver and $F$ a field.*

*(1) The path algebra $FQ$ is hereditary.*

*(2) Let $K$ be an algebraically closed field and $A$ a hereditary $K$-algebra.*

*Then there exists a quiver $Q_A$ such that mod-$A$ and mod-$KQ_A$ are equivalent as categories.*

### 1.2.2 Representations of quivers

For a quiver $Q$ and field $K$, the category of $K$-representations of $Q$ is defined, and its equivalence with *mod-$KQ$* is formulated. The subsection is concluded by several observations that will be useful throughout the text.

For a quiver $Q$ and a field $K$, a *$K$-linear representation $M$ of $Q$* consists of a collection of finite-dimensional $K$-vector spaces $M_a$ for each $a \in Q_0$ and a collection of $K$-linear *structural maps* $M_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)}$ for each $\alpha \in Q_1$. We refer to $M$ simply as a *($K$-)representation (of $Q$).* Because we assume $Q_0$ to be finite, $K$-representations are finite-dimensional as vector spaces over $K$.

Representations, together with morphisms, have a structure of an abelian length category. If we have two $K$-linear representations $M$ and $N$ of a quiver $Q$, a *morphism* $\phi \colon M \to N$ is a collection of maps $(\phi_a)_{a \in Q_0}$ such that for each arrow $\alpha \in Q_1$ the following diagram commutes

$$\begin{array}{ccc} N_{s(\alpha)} & \xrightarrow{N_\alpha} & N_{t(\alpha)} \\ \uparrow{\scriptstyle \phi_{s(\alpha)}} & & \uparrow{\scriptstyle \phi_{t(\alpha)}} \\ M_{s(\alpha)} & \xrightarrow{M_\alpha} & M_{t(\alpha)} \end{array}$$

For an acyclic quiver $Q$, the category of $K$-linear representations of a quiver $Q$ is equivalent to the category $mod\text{-}KQ$, [3, Thm. III.1.5.]. Furthermore, there is a canonical equivalence preserving simple, projective, injective and indecomposable objects and exact sequences, [3, Prop. III.1.8].

Whenever it is convenient, we shall view this equivalence as an identification. Note that a morphism $\phi\colon M \to N$ between two representations is a monomorphism (isomorphism) if and only if all $K$-linear maps $\phi_a\colon M_a \to N_a$ are monomorphisms (isomorphisms).

For a $K$-representation $M$ of a quiver $Q$ the vector

$$dim(M) = [dim_K(M_1), dim_K(M_2), \ldots, dim_K(M_{|Q_0|}]$$

is called a *dimension vector of $M$*. Two isomorphic representations have the same dimension vector. A representation is called *thin* if $dim(M) \in \{0,1\}^{|Q_0|}$.

A $K$-representation $M$ of a quiver $Q$ is *indecomposable* if the ring of endomorphisms, $End\ M$, is *local*. That means that for any $\phi \in End\ M$, either $\phi$ is invertible or $id - \phi$ is. By [2, Cor. 4.8.], this is equivalent with saying that $M$ is indecomposable as a $KQ$-module.

For simplicity, we will say that a representation $N$ is a *subrepresentation* of M, denoted by $N \leq M$, if there is a monomorphism from $N$ to $M$. If $\phi : N \to M$ is a monomorphism, then $\phi(N)$ is a subrepresentation of $M$ in the standard sense and $\phi(N) \cong N$.

**Definition 1.** *Let $Q$ be a quiver, $K$ field and $M \in mod\text{-}KQ$.*

*We define* support of M, *$supp(M)$ for short, as the subquiver $Q' \subseteq Q$ such that*

$$Q'_0 := \{a \mid a \in Q_0;\ M_a \neq 0\} \qquad Q'_1 := \{\alpha \mid \alpha \in Q_1;\ M_\alpha \neq 0\}$$

We end this subsection with several simple observations used extensively in the rest of the thesis.

Let $K$ be a field and $Q$ an acyclic quiver, then simple representations of $KQ$ are in bijection with vertices of $Q$ via $i \mapsto s(I)$, where $S(i)$ is defined by setting $S(i)_b := 0$ for $b \neq i$ and $S(i)_i := K$, [2, Lemma III.2.1.].

**Lemma 2.** *If $M$ is an indecomposable representation, then $supp(M)$ is connected.*

Unsurprisingly, the opposite is not true. Consider the one-vertex quiver without arrows. Then all non-zero representations have connected support, but there is (up to isomorphism) only one indecomposable representation.

**Lemma 3.** *Let $Q$ be a quiver, $K$ a field and $\phi\colon N \to M$ a monomorphism in $mod\text{-}KQ$.*

*Then for any arrow $\alpha$ such that $s(\alpha) \in supp(N)$, $t(\alpha) \notin supp(N)$ and $t(\alpha) \in supp(M)$, the linear map $M_\alpha$ is not injective.*

In the second chapter, Lemma 35 strengthens this lemma to characterise subrepresentations of indecomposable thin representations.

*Proof.* For such $\alpha \in Q_1$ and $M_\alpha$, consider the following commutative diagram

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{M_\alpha} & M_{t(\alpha)} \\ \uparrow{\scriptstyle \phi_{s\alpha}} & & \uparrow{\scriptstyle 0} \\ 0 \neq N_{s(\alpha)} & \xrightarrow{N_\alpha} & 0 \end{array}$$

with $N_{s(\alpha)}$ and $M_{s(\alpha)}$ non-zero. Then by the commutativity, $M_\alpha \circ \phi_{s(\alpha)} = 0$. Equivalently $Im(\phi_{s(\alpha)}) \subseteq Ker(M_\alpha)$. The map $M_\alpha$ cannot be injective. Otherwise, we would have $\phi_{s(\alpha)} = 0$, which contradicts $\phi$ being a monomorphism.

□

### 1.2.3 Integral forms and Gabriel's theorem

We define the Euler form of the quiver and the associated quadratic form. They will be helpful in calculations in the third chapter. We conclude with Gabriel's theorem, characterising representation-finite path-algebras in terms of the underlying graph of their quiver. The formulation here is not Gabriel's original version but an expanded version based on the lecture notes of Krause, [15].

For any natural $n$, there is the following partial order on $\mathbb{Z}^n$

$$(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n) \iff x_i \leq y_i \quad \forall i \leq n.$$

A vector $X \in \mathbb{Z}^n$ is said to be *positive* if $X > (0, 0, \dots, 0)$.

For a graph $G = (V, E)$ with $V = \{1, 2, \dots, n\}$, there is an associated integral quadratic form $q_G : Z^n \to Z^n$ defined as:

$$q_G(x_1, x_2, \dots, x_n) = \sum_{1 \leq i \leq n} x_i^2 - \sum_{1 \leq i \leq j \leq n} d_{ij} x_i x_j,$$

where $d_{ij}$ is the number of edges between vertices $i$ and $j$.

To a quiver $Q$ we assign a quadratic form $q_Q := q_{\bar{Q}}$. One can determine positive (semi)definiteness of $q_Q$ based on the type of $\bar{Q}$, [2, Prop. VII. 4.5.].

**Proposition 4.** *Let $Q$ be an acyclic, connected quiver.*

*(1) $q_Q$ is positive definite if and only if $\bar{Q}$ is a Dynkin graph.*

*(2) $q_Q$ is positive semidefinite but not positive definite if and only if $\bar{Q}$ is a Euclidean graph.*

*Example.* A path on $n$ vertices is a Dynkin graph $A_n$.

A cycle with $n+1$ vertices is a Euclidean graph $\tilde{A}_n$

The full list of Dynkin and Euclidean graphs can be found in [15, Chapter 4.2] (under the name *diagrams*) or in [20, Chapter 1.2]. If $\bar{Q}$ is Dynkin (Euclidean) graph, we say that $Q$ is a quiver of *Dynkin (Euclidean) type.* If there is $n$ such that $\bar{Q}$ is $(A_n)$ $\tilde{A}_n$ we say that $Q$ is of *type* $(A_n)$ $\tilde{A}_n$. If $Q$ is of Dynkin or Euclidean type, non-zero vectors $X \in \mathbb{Z}^n$ such that $q(X) \leq 1$ are *roots.*

For a quiver $Q$ with $Q_0 = \{1, 2, \ldots, n\}$ the integral bilinear *Euler form of the quiver* is defined as

$$\langle X, Y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_0} x_{s(\alpha)} y_{t(\alpha)}$$

For each quiver $Q$ of a Euclidean type, there exists a unique positive vector $\delta$ such that $\langle \delta, \delta \rangle = 0$. This vector is called the *minimal radical vector.* The value $\langle X, \delta \rangle$ is called the *defect of a vector* $X$. The *defect of a representation* is defined as the defect of its dimension vector.

*Example.* The minimal radical vector for $\tilde{A}_n$ is $[1, 1, \ldots, 1]$.

An indecomposable representation of a Euclidean quiver is *preprojective* if it has a negative defect, *regular* if it has zero defect and *preinjective* if it has a positive defect. In [3] and [2], decomposable preprojective, regular and preinjective representations are considered. We, like [15], use the terms more restrictively as most of the thesis is interested only in indecomposable representations.

An algebra $A$ is (*representation-finite*) *representation-infinite* if [*ind*-$A$] is (finite) infinite. We now state the classical Gabriel's theorem. The formulation used here is based on [15, Thm 5.1.1., Thm. 5.3.].

**Theorem 5** (Gabriel)**.** *Let $Q$ be an acyclic, connected quiver and $K$ an algebraically closed field.*
*(1) If $Q$ is of a Dynkin type, then the algebra $KQ$ is representation-finite and $X \mapsto dim(X)$ induces a bijection between isomorphism classes of indecomposable representations of $Q$ and positive roots of $q_Q$.*
*(2) If $Q$ is of a Euclidean type, then the algebra $KQ$ is representation-infinite and $X \mapsto dim(X)$ induces a bijection between isomorphism classes of preprojective and preinjective representations of $Q$ and positive roots of $q_Q$ with a non-zero defect.*

## 1.3 Definition of the Gabriel-Roiter measure

This section defines the Gabriel-Roiter measure. While the previous two sections focused only on the theory necessary for the rest of the thesis, this section gives a broader treatment of the subject. Despite being historically tied to the study of artin algebras and abelian length categories, the basic idea behind the Gabriel-Roiter measure is combinatorial. Shortly after Ringel reintroduced the Gabriel-Roiter measure in [21], Krause published a combinatorial treatment, [13], giving rise to alternative Gabriel-Roiter measures.

The combinatorial definitions are illustrated on quivers partially ordered by a subquiver relation. See Section 2.1 for their connection to the Gabriel-Roiter measure. Module-theoretic examples are used too, including the Gabriel-Roiter comeasure.

The primary source for this section [13] with aid from [14]. I found no published results about alternative Gabriel-Roiter measures except for these two articles. In Chapter 2, new results concerning alternative Gabriel-Roiter measures are presented.

### 1.3.1 Length functions

**Definition 2.** *Let* $(S, \leq)$ *and* $(T, \leq)$ *be two partially ordered sets.*

*Then* $l\colon S \to T$ *is a* length function *on* $(S, \leq)$ *if for any* $x, y \in S$ *it satisfies*

*(L1)* $x < y$ *implies* $l(x) < l(y)$

*(L2)* $l(x) \leq l(y)$ *or* $l(x) \geq l(y)$

*(L3)* $\{l(x') \mid x' \in S;\ x' \leq y\}$ *is finite.*

If $(T, \leq)$ is isomorphic to $(\mathbb{N}, \leq)$, then (2) and (3) are always satisfied.

*Example.* For a ring $R$ and $M \in mod-R$, recall $(M, \leq)$ from Subsection 1.1. The composition length is a length function on $(M, \leq)$.

*Example.* Let $R$ be a ring. We define a relation $\hookrightarrow$ on ($[ind\text{-}R]$) $[mod\text{-}R]$. If for two (indecomposable) modules $M$ and $N$ there is a monomorphism $N \to M$, then $[N] \hookrightarrow [M]$.

Then $([ind\text{-}R], \hookrightarrow)$ and $([mod\text{-}R], \hookrightarrow)$ are partially ordered sets, and the composition length is a length function.

*Example.* Consider the set $[\mathcal{Q}]$ of all isomorphism classes of quivers. We define relation $\leq_{\mathcal{Q}}$. For two quivers, $Q'$, $Q$, if $Q'$ is a full subquiver of $Q$, then $[Q'] \leq_{\mathcal{Q}} [Q]$.

Then the *number of vertices* is a length function on $([\mathcal{Q}], \leq_{\mathcal{Q}})$.

*Example.* For a ring $R$, we define a relation $\twoheadleftarrow$ on $[ind - R]$. If there is an epimorphism $M \to N$, then $[N] \twoheadleftarrow [M]$.

The composition length is a length function on $([ind - R], \twoheadleftarrow)$.

Definition 2 allows for numerous length functions on modules, but we only consider those respecting short exact sequences.

**Definition 3.** *Let $R$ be a ring. A function $l\colon$ mod-$R \to \mathbb{R}^+ \cup \{0\}$ is said to be a* length function *on mod-$R$ if the following two conditions hold:*

*(l1) $l(M) = 0$ if and only if $M = 0$.*

*(l2) If $0 \to K \to L \to M \to 0$ is a short exact sequence, then $l(L) = l(M) + l(K)$.*

*Example.* For a ring $R$, the composition length is a length function on *mod-$R$*.

*Example.* For a field $K$ and a $K$-algebra $A$, the $K$-dimension of a representation is a length function on *mod-$A$*.

**Lemma 6.** *Let $R$ be a ring and $l$ a length function on mod-$R$.*

*(1) The function $l$ is uniquely determined by its values on simple modules.*

*(2) The function $l$ induces a length function on $([\text{mod-}R], \hookrightarrow)$ and $([\text{ind-}R], \hookrightarrow)$.*

*Proof.* The first part follows from the Jordan-Hölder theorem. Let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

be a composition series for a module $M$. Then

$$l(M) = \sum_{0 \leq i < n} l(M_{i+1}/M_i).$$

As for the second part, we set $l([M]) = l(M)$. The axiom $(L3)$ holds because a finite-length module has only finitely many submodules. Axiom $(L2)$ holds because the codomain of $l$ is a totally ordered set.

If $[N] \hookrightarrow [M]$ and $[N] \neq [M]$, then there is a monomorphism $f\colon N \to M$ and the factor $M/f(N)$ is a non-zero module. By $(l2)$ we have

$$l(M) = l(N) + l(M/f(N)).$$

The axiom $(L1)$ then follows from $(l1)$.

□

### 1.3.2 Chains and lexicographical order

Let $(S, \leq)$ be a partially ordered set. A subset $X \subseteq S$ is a *chain in $S$* if the partial order $\leq$ restricted on $X$ is a total order.

*Example.* Let $R$ be a ring and consider $([mod\text{-}R], \hookrightarrow)$. Any sequence of submodules $M_1 \subsetneq \cdots \subsetneq M_{m-1} \subsetneq M_m = M$ induces a chain.

It also induces chain $[M/M_{m-1}] \twoheadleftarrow [M/M_{m-2}] \twoheadleftarrow \cdots \twoheadleftarrow [M/M_1]$ in $([ind\text{-}R], \twoheadleftarrow)$ if the factor modules are indecomposable.

*Example.* Any subset of a totally ordered set is a chain.

For a finite chain $X$, $\max(X)$ $(\min(X))$ denotes the maximum (minimum) of $X$. We use the convention that

$$\max(\emptyset) < x < \min(\emptyset) \quad \forall x \in S.$$

**Definition 4.** *Let $(S, \leq)$ be a partially ordered set.*

*We denote the set of finite chains in $S$ by $Ch(S)$.*

*For $x \in S$, all finite chains terminating with $x$ are denoted by $Ch(S, x)$.*

**Definition 5.** *Let $(S, \leq)$ be a partially ordered set.*

*We define the* lexicographical order *on $Ch(S)$ by*

$$X \leq Y \Leftrightarrow \min(Y \setminus X) \leq \min(X \setminus Y).$$

A lexicographical order is a total order.

**Example 7.** *Consider a partially ordered set $(\mathbb{N}, \leq)$.*

*A set $A \subseteq \mathbb{N}$ can be interpreted as an infinite countable word $w_A \in \{0,1\}^{\mathbb{N}}$, where the i-th coordinate of $w_A$ is 1 if and only if $i \in A$.*

*By [19, Lemma 1.], the set $Ch(\mathbb{N})$, ordered by lexicographical order, embeds into $\mathbb{R}^+ \cup \{0\}$ via the order-preserving map*

$$r(A) = \sum_{i \in A} \frac{1}{2^i}.$$

$\Delta$

*Example.* Let $R$ be a ring and $M \in mod\text{-}R$.

Then $Ch([mod\text{-}R], M)$ consists of all sequences of nested submodules of $M$ terminating with $M$. For a non-simple $M$, there are several maximal objects with respect to the inclusion of chains, namely all the composition series.

Maximal objects, with respect to inclusion, in $Ch([ind\text{-}R], M)$ are indecomposable filtrations, see Definition 8.

**Definition 6.** *Let $l : (S, \leq) \to (T, \leq)$ be a length function.*

*To each chain, the* chain of lengths *is assigned via*

$$X = (x_1 \leq x_2 \leq \cdots \leq x_n) \;\longmapsto\; l(X) = (l(x_1) \leq l(x_2) \leq \cdots \leq l(x_n)).$$

*The length function $l$ also induces a* chain length function

$$S \to Ch(T): \quad x \longmapsto l^*(x) := max\{l(X) \mid X \in Ch(S, x)\}.$$

This means that for each $x \in S$, we consider the maximal chain of lengths (with respect to lexicographical order) of finite chains terminating in $x$. There might be multiple chains with the maximal chain of lengths.

*Example.* Let $R$ be a ring and $l$ the composition length. Then functions $l$ and $l^*$ are equivalent on $[mod\text{-}R]$, with $l^*(M) = \{1, 2, \ldots l(M)\}$.

*Example.* Similarly, for $([\mathcal{Q}], \leq_{\mathcal{Q}})$, functions $|Q_0|^*$ and $|Q_0|$ are equivalent.

Consider now a subset $C$ of $[\mathcal{Q}]$ induced by connected quivers. If $Q'$ is a subquiver of $Q$ such that all arrows with the source in $Q'$ have the target in $Q'$, we say $[Q'] \leq_C [Q]$.

Then $|Q_0|^*$ differs from $|Q_0|$. When $Q$ is a tree, there is a connection between $|Q_0|^*$ and the Gabriel-Roiter measure. See section 2.1.2.

**Definition 7** (The Gabriel-Roiter measure)**.** *For a ring $R$, consider the partially ordered set $([ind\text{-}R], \hookrightarrow)$. Let $l$ be a length function on mod-$R$ and let $l^*$ be the induced chain length function on $[ind\text{-}R]$.*

*If $M \in ind\text{-}R$ we define the* Gabriel-Roiter measure of $M$, with resepct to $l$ *as*

$$\mu_l(M) := l^*([M]).$$

*If a module $M \in mod\text{-}R$ is decomposable, then $\mu_l(M)$ is defined as the maximal Gabriel-Roiter measure of an indecomposable summand of $M$.*

*The Gabriel-Roiter measure will respect to the composition length will be called* standard Gabriel-Roiter measure*, denoted by $\mu(M)$*

We often write just *GR measure* to shorten the notation. By the Krull-Schmidt theorem, the GR measure of a decomposable module is well-defined. The GR measure of a finite direct power of a module is the same as that of the original module.

Ringel's definition of the standard GR measure motivates the extension to decomposable modules, [21, Cor. 1.]. The GR measure of decomposable modules is rarely discussed in the literature. Krause ignores them completely.

Krause found the following axiomatisation of chain length functions.

**Theorem 8.** *Let $l\colon (S, \leq) \to (T, \leq)$ be a length function.*

*Then there exists a map $\mu : S \to U$ such that for any $x, y \in S$*

*(1) If $x \leq y$ then $\mu(x) \leq \mu(y)$.*

*(2) If $\mu(x) = \mu(y)$ then $l(x) = l(y)$.*

*(3) If $\forall x' < x : \mu(x') < \mu(y)$ and $l(x) \geq l(y)$ then $\mu(x) \leq \mu(y)$.*

*Moreover, any function $\mu'\colon S \to U'$ satisfying the above condition, is equivalent to $\mu$, i.e., $\mu(x) \leq \mu(y)$ if and only if $\mu'(x) \leq \mu'(y)$.*

*Proof.* All the conditions are satisfied by $l^*$, [13, Prop. 1.5.]. A special case for a GR measure follows from Proposition 15. Krause proved the uniqueness of such a function, [13, Thm. 1.7.].

□

*Example.* For a ring $R$, consider the partially ordered set $([ind\text{-}R], \twoheadleftarrow)$. Let $l$ be the composition length.

Then $l^*$ corresponds to the *Gabriel-Roiter comeasure*, defined by Ringel in [21].

The comeasure defined in [19] differs, and it looks like it cannot be described as a chain length function.

We end with a recursive definition. It can be useful for calculations, see Example 29 and Chapter 2. It was first formulated by Krause, [14, Section 1.4.].

**Proposition 9.** *Let $R$ be a ring $R$ and $l$ a length function on mod-$R$.*

$$\mu_l(M) = max\{\mu_l(N) \mid N \subsetneq M;\ N \in ind\text{-}R\} \cup \{l(M)\} \quad for\ M \in ind\text{-}R$$

## 1.4 Basic properties of the GR measure

This section aims to define the terminology associated with the Gabriel-Roiter measure and formulate some basic properties. These concepts are illustrated on representations of quivers of type $A_n$. Proofs are omitted as they follow directly from the algorithm in Subsection 2.1.2. From now on, unless stated otherwise, all modules are considered to be right finite-length modules.

We start with some basic examples observed by Gabriel in [5].

**Example 10.** *Let $R$ be a ring and $M \in ind\text{-}R$.*
*(a) $\mu(M) = \{1\}$ if and only if $M$ is simple.*
*(b) $\mu(M) = \{1, 2\}$ if and only if $|M| = 2$.*
*(c) $\mu(M) = \{1, 2, 3\}$ if and only if $|M| = 3$ and $M$ has a simple socle.*
*(d) $\mu(M) = \{1, 3\}$ if and only if $|M| = 3$ and $|soc\ M| = 2$.*

△

Any non-zero finite-length module $M$ has a non-zero socle, thus $1 \in \mu(M)$. For the standard GR measure, we can generalise $(a)$ by saying that $\mu(M) = \{1\}$ if and only if $M$ is semisimple, i.e., if $M = soc\ M$. It is not true for the general GR measure. Take two indecomposable modules $M_1$ and $M_2$ with compositions lengths one and two, respectively, such that $\mu_l(M_1) = \{1\}$ and $\mu_l(M_2) = \{2, 3\}$. Then by the definition $\mu_l(M_1 \oplus M_2) = \mu_l(M_1) = \{1\}$ but $M_1 \oplus M_2$ is not a semisimple module.

We add two more examples by Ringel from [19]. They were given without proof. Bo Chen proves the second one in [10].

**Proposition 11.** *Let $R$ be a ring and $M \in$ ind-$R$ of length $n > 1$.*

*(1) $\mu(M) = \{1, n\}$ if and only if soc $M$ is its unique maximal ideal.*

*(2) $\mu(M) = \{1, 2, \ldots, n\}$ if and only if $M$ is uniform.*

The assumption that $M$ is indecomposable is indeed necessary. Take two indecomposable modules $N_t$ and $N_s$ with GR measures $\{1, t\}$ and $\{1, s\}$ where $1 < t \leq s$. Then $\mu(N_1 \oplus N_2) = \{1, t\}$, however $N_1 \oplus N_2$ has two maximal submodules.

*Proof.* Part (2) holds for general GR measure, see Proposition 42.

To prove (1), assume $\mu(M) = \{1, n\}$ and take $M'$, some maximal submodule of $M$. Then $|M'| < n$ and $1 \in \mu(M') \leq \{1, n\}$. Hence $\mu(M') = \{1\}$, so it is a semisimple module. If there is a simple submodule $S \leq M$ that is not included in $M'$, then $S \cap M' = 0$ hence by maximality of $M'$ we have $M' \oplus S \cong M$, but this is in contradiction with $M$ being indecomposable. So a maximal submodule of $M$ is necessarily its socle.

For the opposite direction, take an indecomposable submodule $I \subsetneq M$. Because $M$ is noetherian, there is a maximal submodule containing $N$, and by our assumptions on $M$, it is its socle. Hence $I$ is simple.

□

**Definition 8.** *Let $R$ be a ring, $M \in$ ind-$R$, and $l$ be a length function on mod-$R$.*

*By an* indecomposable filtration *of $M$, we mean a chain of indecomposable submodules*

$$\mathcal{I}: \; I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{m-1} \subsetneq I_m = M,$$

*such that $I_1$ is simple and if there is an indecomposable module $I$ and a number $j < m$ such that $I_j \subseteq I \subseteq I_{j+1}$ then either $I = I_j$ or $I = I_{j+1}$.*

*The indecomposable filtration is called a* Gabriel-Roiter filtration*, or a* GR filtration *for short, if*

$$\{l(I_1), \ldots, l(I_m)\} = \mu_l(M)$$

Any chain of indecomposable submodules can be refined into an indecomposable filtration. As shown in the next example, a module can generally have indecomposable filtrations of different lengths.

The term *Gabriel-Rotier filtration* was proposed by Ringel in [21]. He observed that a module has a GR filtration if and only if it is indecomposable. The following example shows that there might be more GR filtrations for a given module. For thin representations, any indecomposable filtration is a GR-filtration for some GR measure, see Theorem 38. Example 40 shows that this generally does not hold.

**Example 12.** *Let $K$ be a field and consider the following quiver*

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7.$$

*Let $M$ be the thin representation with $dim_K(M) = 7$ and all structural maps equal to identities. The standard GR measure of $M$ is $\{1, 2, 4, 7\}$ as witnessed by two GR filtrations*

$$S(1) \subsetneq K \xleftarrow{1} K \leftarrow 0 \rightarrow 0 \leftarrow 0 \rightarrow 0 \rightarrow 0 \subsetneq K \xleftarrow{1} K \xleftarrow{1} K \xrightarrow{1} K \leftarrow 0 \rightarrow 0 \rightarrow 0) \subsetneq M$$

$$S(7) \subsetneq 0 \leftarrow 0 \leftarrow 0 \rightarrow 0 \leftarrow 0 \rightarrow K \xrightarrow{1} K \subsetneq 0 \leftarrow 0 \leftarrow 0 \rightarrow K \xleftarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \subsetneq M.$$

*There are two more indecomposable filtrations of $M$.*

$$S(4) \subsetneq K \xleftarrow{1} K \xleftarrow{1} K \xrightarrow{1} K \leftarrow 0 \rightarrow 0 \rightarrow 0) \subsetneq M$$

$$S(4) \subsetneq 0 \leftarrow 0 \leftarrow 0 \rightarrow K \xleftarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \subsetneq M$$

$\Delta$

Note that a GR-filtration need not be longer than an indecomposable filtration. Suppose we define a length function $l$ by letting $l(S(4)) = 1$ and $l(S) = 2$ for any simple modules $S$ non-isomorphic with $S(4)$. In that case, the two indecomposable filtrations from the above example become GR filtrations.

**Definition 9.** *Let $R$ be a ring, $M \in$ ind-$R$, and $l$ be a length function on mod-$R$. An indecomposable submodule $N \subsetneq M$ is called a* GR submodule of M *if*

$$\mu_l(M) = \mu_l(N) \cup \{l(M)\}.$$

*A momomorphism $N' \xrightarrow{\phi} M$ is called a* GR-inclusion *if $\phi(N')$ is a GR submodule of $M$. The cokernel of a GR inclusion is called a* GR factor*.*

By Corollary 17, a GR factor is always an indecomposable module. The chain of submodules is a GR filtration if and only if any inclusion of two consecutive members is a GR inclusion. In particular, the factor of two consecutive members of GR-filtration is indecomposable.

**Example 13.** *The previous example shows that GR factors need not be isomorphic. The two GR factors corresponding to the last two members of GR filtrations are*

$$0 \leftarrow 0 \leftarrow 0 \rightarrow 0 \leftarrow K \xrightarrow{1} K \xrightarrow{1} K$$

$$K \xleftarrow{1} K \xleftarrow{1} K \rightarrow 0 \leftarrow 0 \rightarrow 0 \rightarrow 0$$

$\Delta$

**Example 14.** *The relationship between the GR measure of a module and its factor is complicated. In the previous example, GR factors have GR measure $\{1, 2, 3\}$. Strictly higher than the original module's GR measure $\{1, 2, 4, 7\}$.*

*On the other hand, there exist many simple GR factors, see Proposition 46.*

$\Delta$

**Proposition 15** (Basic properties of GR measure)**.** *Let $R$ be a ring, $l$ a length-function on mod-$R$, $M, N \in$ ind-$R$.*

*(GR1) If $N \subseteq M$ then $\mu_l(N) \leq \mu_l(M)$.*

*(GR2) If $\mu(M) = \mu(N)$ then $l(M) = l(N)$.*

*(GR3) If $l(N) \leq l(M)$ and $\mu_l(M') < \mu_l(N)$ for all $M' \subsetneq M$.*

*Then $\mu_l(M) \leq \mu_l(N)$*

This formulation of basic properties is based on [13], but similar claims appear in [5]. We will use (GR1) and (GR2) throughout the text without explicitly mentioning them. Note the similarity to the statement of Theorem 8.

*Proof.* (GR1) Any GR filtration of $N$ can be extended into an indecomposable filtration of $M$. (GR2) is trivial. To prove (GR3), consider a GR filtration

$$M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{m-1} \subsetneq M.$$

We define a new set

$$S := \{l(M_i) \mid i < m\} \cup \{l(N)\}.$$

Because $\mu_l(M_i) \leq \mu_l(N)$ for every $i < m$, we have $S \leq \mu_l(N)$. So it is enough to prove that $\mu_l(M) \leq S$. Because

$$\mu_l(M) = \mu_l(M_{m-1}) \cup l(M),$$

the conclusion follows from $l(N) \leq l(M)$.

□

We now formulate the *main property* of the GR measure. It was first proved by Gabriel, [5]. Our formulation is based on Krause's version, [13, Prop. 3.2.]. Both Gabriel's and Krause's versions discuss length categories.

**Proposition 16** (Main property of the GR measure)**.** *Let $R$ be a ring, $l$ a length function on mod-$R$. Let $X, Y_1, \ldots, Y_n \in$ ind-$R$ such that $X \leq \oplus Y_i$. Then*

$$\mu_l(X) \leq \max_{1 \leq i \leq n} \mu_l(Y_i).$$

*In the case of equality, $X$ is a direct summand of $\oplus Y_i$.*

We can now prove that the GR factors are indecomposable modules. It was first observed by Ringel, [21, Cor. 2., Cor. 3.], for the standard GR measure. The proof presented here is closely based on Ringel's.

**Corollary 17.** *Let $R$ be a ring, $l$ a length function on mod-$R$, $M \in$ ind-$R$ and let an indecomposable module $N$ be a GR submodule of $M$.*

*(1) Let $T$ be a proper submodule of $M$ containing $N$.*

*Then $\mu_l(T) = \mu_l(N)$ and $N$ is a direct summand of $T$.*

*(2) The GR factor $M/N$ is indecomposable.*

*Proof.* (1) Because $N$ is a GR submodule, $T$ has to be decomposable. If $\mu_l(N) < \mu_l(T)$, then, by definition, $T$ has an indecomposable direct summand with the GR measure strictly higher than $\mu_l(N)$. This contradicts $N$ being a GR submodule of $M$. The second part follows from the main property.

(2) Suppose there is a decomposition $M/N = Q_1 \oplus Q_2$. Let $T_1$ and $T_2$ be the preimages of $Q_1$ and $Q_2$ in the canonical projection $M \to M/N$. Clearly $T_1 \cup T_2 = M$ If $Q_1$ and $Q_2$ are non-zero modules we have $N \subsetneq T_1 \subsetneq M$ and $N \subsetneq T_2 \subsetneq M$. By (1), non-zero modules $N_1$ and $N_2$ exist such that $N \oplus N_1 = T_1$ and $N \oplus N_2 = T_2$. Because $T_1 \cap T_2 = N$ we see that $N_1 \cap N_2 = 0$, hence

$$M = N \oplus N_1 \oplus N_2.$$

This is in contradiction with the indecomposability of $M$. Thus either $Q_1$ or $Q_2$ have to be the zero module, and $M/N$ is indecomposable.

□

*Remark.* There are other results about the standard GR measure whose proofs can be almost verbatim used as proofs for general GR measures, e.g., Proposition 22 and Example 23. Also, Ringel's partition of *mod-R*, Theorem 21. Other statements cannot be generalised, e.g., Proposition 11, as explained by Theorem 43.

As another consequence of the main property, we get a family of rings with an upper bound for a GR measure.

**Corollary 18.** *Let $R$ be a ring such that there is a module $C$ such that for any $M \in$ mod-$R$ there exists a natural number $n$ and a monomorphism $\phi : M \hookrightarrow C^n$.*

*Then for any length function $l$ on mod-R, GR measures on mod-R are bounded by $\mu_l(C)$.*

In particular, artin algebras satisfy the assumptions of the above corollary.

*Proof.* Take $M \in$ *mod-R* and consider a monomorphism $\phi : M \hookrightarrow C^n$. By the definition, $\mu_l(C^n) = \mu_l(C)$, and by the main property $\mu_l(Im(\phi)) \leq \mu_l(C^n)$.

□

In general, such an upper bound need not exist.

**Example 19.** *Consider the Jordan quiver, i.e., the quiver given by one vertex and one arrow, and an algebraically closed field $K$. The indecomposable $K$-representations of $Q$ are given by pairs $(p, \lambda)$ and consist of a vector space $K^p$ and a multiplication by the Jordan block of dimension $p$ with a parameter $\lambda \in K$. They are pairwise non-isomorphic, [15, Thm. 9.2.1.].*

*The standard GR-measure of representation given by $(p, \lambda)$ is then $\{1, 2, \dots, p\}$.*

△

## 1.5 Artin algebras

This section aims to gather basic results about artin algebras used in the rest of the thesis and to comment on them from the perspective of the GR measure. However, some results often hold for an arbitrary artinian ring. We conclude this section with Ringel's partition of the category of representations of artin algebras. We omit definitions of *projective covers* and *injective envelopes* as they are used only in this section. Details can be found, for example, in [2, Chapter I]. This is, together with [21, Section 4.], the primary source for this section, but some results are cited from [3] and [16] for greater generality.

### 1.5.1 Projective and injective modules

**Projective representations**

If $R$ is a right artinian ring, there are up to an isomorphism only finitely many projective right modules, corresponding to the projective covers of simple modules, [3, Cor. I.4.5.]. The projective cover of $S(i)$ is denoted by $P(i)$. See [2, Lemma III.2.4.] for a combinatorial description of $P(i)$ and $rad\ P(i)$ in the case where $R$ is a finite-dimensional algebra over an algebraically closed field.

If $A$ is artin algebra and $P$ is an indecomposable projective module, GR submodules are direct summands of $rad\ P$, see Example 23.

**Injective representations**

If $R$ is an artinian ring, there are up to an isomorphism only finitely many injective modules, corresponding to the injective envelopes of simple modules, [16, Thm. 3.61.]. The injective envelope of $S(i)$ is denoted by $I(i)$. See [2, Lemma III.2.6.] for a combinatorial description of $I(i)$ in the case when $R$ is a finite-dimensional algebra over an algebraically closed field.

An injective module over any ring $R$ is uniform if and only if it is indecomposable [16, Thm. 3.52.]. Proposition 11 shows that the GR measure of an injective indecomposable representation $I$ is $\{1, 2, \dots, |I|\}$. Because any direct summand of an injective module is injective, [1, Prop. 18.2], we conclude that the GR-measure of any injective module is of form $\{1, 2, \dots, m\}$, where $m$ is a maximal length of a direct summand.

Unsurprisingly, not all modules with this GR measure are injective. Take, for instance, non-injective simple modules. However, for an artin algebra $A$, if $n$ is maximal such that there exists a module $M \in ind - A$ with $\mu(M) = \{1, 2, \dots, n\}$, then $M$ is injective, see [21, Section 4].

It turns out that a similar claim holds for any GR measure, as shown by the following theorem by Krause, [13, Thm. 3.3.], originally stated for abelian length categories.

**Theorem 20.** *Let $R$ be a ring and let $I \in$ ind-$R$.*

*Then $I$ is injective if and only if there exists a length function on mod-$R$ such that $\mu_l(I)$ is the maximum of $\{\mu_l(M) \mid M \in$ mod-$R\}$.*

As seen in Example 19, the maximal GR measure might not exist for a given length function on *mod-$R$*. But for an injective $R$-module $I$, we may define $l$ as follows: let $S$ be the (simple) socle of $I$, then set $l(S) = 1$, and $l(S') = 2$ for any simple representations $S'$ not isomorphic with $I$.

### 1.5.2 Ringel's partition of mod-A

The following theorem is due to Ringel [21, Thm 2.]. Proof for a general GR measure is due to Krause, see [14, Section 4.]. The *take-off* part and *landing* part of the theorem give two proofs of the Brauer-Thrall conjecture. Example 19 shows that the theorem does not hold for general rings.

Let us fix a ring $R$ and a length function $l$ on *mod-$R$*. We say that a set $S \in \mathbb{R}^+$ is a *GR measure* on $A$ if there is $M \in ind-A$ such that $\mu_l(M) = S$. A GR measure $S$ on A is of *finite-type* if there exists, up to isomorphism, only finitely many indecomposable representations with such measure.

**Theorem 21.** *Let $A$ be a representation-infinite artin algebra.*

*Then for $i \in \mathbb{N}$, there exists finite-type GR measures $I^i$ and $I_i$ on $A$ such that*

$$I_1 < I_2 < I_3 < \quad \dots \quad < I^3 < I^2 < I_1.$$

*Furthermore, for any other GR-measure $I$ on $A$, inequality $I_t < I < I^t$ holds for every $t \in \mathbb{N}$.*

**Definition 10.** *Let $A$ be a representation-infinite artin algebra.*

*The measures $I_t$ for $t \in \mathbb{N}$ are called* take-off meausures*. A module with such measure is called a* take-off *module. The full subcategory of mod-A consisting of take-off modules is the* take-off *part.*

*The measures $I^t$ for $t \in \mathbb{N}$ are called* landing*. A module with such measure is called a* landing *module. The full subcategory of mod-A consisting of landing modules is the* landing *part.*

*The remaining nontrivial measures are called* central measures*. A module with such measure is called a* central module*, inducing the* central part *of mod-A.*

As shown in Section 3.6, some central measures are finite-type while others not.

## 1.6 The Auslander-Reiten Theory

The purpose of this section is two-fold. It prepares the ground for calculations in Chapter 3. It also comments on the relationship between the A-R theory and the GR measure. This has been a recurring theme in the study of the GR measure in the last two decades. Especially the cases of path algebras $KQ$, where $K$ is algebraically closed and $\bar{Q}$ is of either Dynkin or Euclidean type.

It is well-known that the A-R theory can be developed for general artin algebras, see [3]. We focus on path algebras over an algebraically closed field. The calculations of GR measures can often be based only on the dimension vectors of representations.

To avoid unnecessary repeating of well-known facts, the following treatment of the Auslander-Reiten theory takes several shortcuts. The Auslander-Reiten translate $\tau$ is not defined as a functor. The method to calculate $dim(\tau(M))$ for a given non-projective representation $M$ of a hereditary algebra is explained using *reflections*, [15, Chapter 3]. *Almost split sequences* are not defined, but the thesis uses that they exist and are exact.

Throughout this section, $Q$ is always a finite acyclic quiver, and $K$ is an algebraically closed field. Occasionally, some facts are formulated in a more general setting. The primary sources for A-R theory are [2] and [15]. The results about the GR measure are taken primarily from [6]. All three sources discuss algebras over an algebraically closed field.

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### 1.6.1 Irreducible morphisms

This section defines irreducible morphisms and shows how to find an irreducible morphism with a given codomain. The irreducible monomorphisms play a vital role in the calculations of the GR-measure. However, the relationship between these two concepts is more nuanced.

**Definition 11.** *Consider a morphism $\phi\colon X \to Y$ in mod-$KQ$.*

*We say that $\phi$ is an* irreducible morphism *if it has neither left nor right inverse and if there is a factorisation of $\phi$ as $X \xrightarrow{f} Z \xrightarrow{g} Y$ in mod-$A$, then either $g$ has a right inverse or $f$ has a left inverse.*

*Suppose further that $X, Y$ are indecomposable. By $Irr(X, Y)$, we denote the $K$-linear* space of irreducible morphisms *from $X$ to $Y$.*

By [15, Lemma 6.2.1.], an irreducible morphism is either a monomorphism or epimorphism but not an isomorphism. By [2, lemma IV.1.6.], $Irr(X, Y)$ has a structure of a finite-dimensional $K$-vector space for any $X, Y \in ind\text{-}A$.

We start with [11, Prop. 3.5.(5)].

**Proposition 22.** *Let $N \xrightarrow{i} M$ be a GR-inclusion in mod-$KQ$.*

*If all irreducible maps to $M$ are monomorphism, then $i$ is irreducible.*

*Remark.* The above observation holds for any GR measure on *mod-$A$*. The proof is the same as for the standard GR measure. In this text, the general version is used only in Examples 23 and 29.

In general, a GR inclusion is not given by an irreducible morphism. The validity of the following example will be clear from Subsection 1.6.3.

*Example.* Let $N \subsetneq M$ be a GR-inclusion in *mod-$KQ$* such that $M$ is a non-simple preinjective or non-simple quasi-simple module. Then the GR inclusion is not an irreducible morphism.

Applying the previous proposition, we get characterisations of GR-submodules of projective modules. It was first observed, without proof, for finite-dimensional algebras in [10]. This example also shows that an irreducible monomorphism need not be a GR inclusion.

**Example 23.** *Let $A$ be artin algebra and $P \in$ ind-$A$ a projective module.*

*If $X$ is a domain of an irreducible morphism $i \colon X \to P$, then $X$ is isomorphic to a direct summand of rad$(P)$, [3, Cor. V.1.6., Thm.V.5.3.].*

*In particular, $i$ is not an epimorphism. Because all irreducible morphisms to $P$ are monomorphisms, GR inclusions are irreducible monomorphisms, and GR submodule has to be a direct summand of rad$(P)$.*

*We see that $M \subseteq P$ is a GR-submodule of $P$ if and only if $M$ is a direct summand of rad$(P)$ with the maximal GR measure.*

$\Delta$

The dimension of the $K$-vector space $Irr(X, Y)$ for $X, Y \in$ *ind-$A$* can be calculated using the *Coxeter functor* $C^+$. From [2, Lemma VII.5.8.], follows that in the setting of this section, the Auslander-Reiten translate of a non-projective module $Y$ is isomorphic to $C^+(Y)$. The full definition of Coxeter functor can be found, for example, in [2, Chapter VII.5]. It is enough to calculate $dim(C^+(Y))$ for our purposes. The dimension vector can be calculated using *reflections* of dimension vectors. The following treatment is based on [15, Chapter 3].

Given a quiver $Q$ and a sink $i \in Q$, we get a new quiver $\sigma_i(Q)$ by reversing all arrows with the target $i$. An ordering of vertices $Q_0 = [n]$ is *admissible* if for any $j \leq n$, the vertex $j$ is a sink in quiver $\sigma_{j-1}\sigma_{j-2} \ldots \sigma_2\sigma_1 Q$. In particular, the quiver $\sigma_n\sigma_{n-1} \ldots \sigma_2\sigma_1 Q$ is well-defined and equal to the original quiver $Q$.

Recall the Euler form $\langle , \rangle$ on $Q$ from Subsection 1.2.3. It induces a symmetric integral bilinear form

$$\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n \qquad (x, y) = \langle x, y \rangle + \langle y, x \rangle.$$

For $i \in Q$ the *reflextion with respect to* $i$ is defined as

$$\sigma_i : \mathbb{Z}^n \to \mathbb{Z}^n \qquad \sigma_i(X) = X - \frac{2(X, e_i)}{(e_i, e_i)} e_i,$$

where $e_i$ is the $i$-th coordinate vector. With this terminology, we can now formulate how to calculate the dimension vector of $C^+(M)$. The proof follows from [15, Thm 3.3.5., Prop. 3.4.3.].

**Proposition 24.** *Let $M$ be a non-projective indecomposable $K$-representation of a quiver $Q$ with an admissible ordering $Q_0 = \{1, 2, \ldots, n\}$. Then*

$$dim(C^+(M)) = \sigma_n \ldots \sigma_2 \sigma_1(dim(M))$$

For $n \geq 1$ and $M \in$ *ind-A*, the module $C^n(M)$ is a module obtained from $M$ by aplying the functor $C^+$ $n$-times. If there exists $n$ such that $C^n(M)$ is projective, then $M$ is *preprojective*. If there exists an injective module $I$ and $n$ such that $C^n(I) \cong M$, then $M$ is preinjective. Note that these modules $C^+(M)$ and $I$ are necessarily indecomposable [2, Prop. 2.10.].

We can now formulate how to calculate $dim_K(Irr(X, M))$ for non-projective indecomposable modules, using [2, Thm IV.3.1. and prop IV.4.2.].

**Proposition 25.** *Let $M \in$ ind-$KQ$ be a non-projective module.*
*Then there exists exact* an almost split sequence

$$0 \to C^+(M) \to N \to M \to 0.$$

*For a module $X \in$ ind-$KQ$, an irreducible morphism from $X$ to $M$ exists if and only if $X$ is isomorphic to a direct summand of $N$. The dimension of $Irr(X, M)$ equals the multiplicity of isomorphic images of $X$ in an indecomposable decomposition of $N$.*

We have not defined *almost split sequences*. But as we will see later, when we need them for the calculations of GR measures, it is enough to know that they are exact. It follows from [2, Prop. IV.1.2.], that they are, up to isomorphism, uniquely determined by their end term.

### 1.6.2 The Auslander-Reiten quiver of an algebra

This subsection defines the Auslander-Reiter quiver of a path algebra $KQ$, where $K$ is algebraically closed and $Q$ is acyclic. Definition for an arbitrary artin algebra can be found in [3, Chapter VII]. Example 29 shows how to use an A-R quiver to calculate the GR measure of modules if $KQ$ is representation-finite.

**Definition 12.** *For an algebra $A = KQ$, the* Auslander-Reiten quiver $\Gamma(A)$*, or* A-R quiver *for short, is a quiver defined by the following data:*

*Vertices of* $\Gamma(A)$ *are isomorphism classes* $[X]$ *of modules in ind-A.*

*Let* $[N], [M]$ *be vertices in* $\Gamma(A)$ *corresponding to modules* $N, M \in$ *ind-A. The arrows* $[N] \to [M]$ *are in bijective correspondence with a basis of the $K$-vector space* $Irr(N, M)$.

Usually, we will identify vertices of $\Gamma(A)$ with indecomposable modules. Saying a *vertex* $N$ for $N \in$ *ind-A*, instead of *vertex* $[N]$. When the algebra is clear from the context, we will only write $\Gamma$.

Let us fix $X \in$ *ind-A*. By Example 23, for a projective indecomposable module $P$, the number of arrows between $X$ and $P$ in $\Gamma(A)$ equals the number of direct summands of $rad(P)$ isomorphic to $X$. By Theorem 25, for a non-projective $M \in$ *ind-A*, the number of arrows between $X$ and $M$ in $\Gamma(A)$ equals the number of direct summands of $N$ isomorphic to $X$, where $N$ is the module from the almost split sequence

$$0 \to C^+(M) \to N \to M \to 0.$$

The following proposition, [15, Prop. 7.3.4.], gives a formula for the number of arrows between preprojective vertices.

**Proposition 26.** *Let* $r, s \in \mathbb{N}$ *and* $P(i)$ *and* $P(j)$ *two projective KQ-modules.*

*Then the number of arrows from* $[C^r(P(i))]$ *to* $[C^s(P(j))]$ *is equal to*

*(1) Number of arrosws from $i$ to $j$ in $Q$ if* $r = s$.

*(2) Number of arrows from $j$ to $i$ in $Q$ if* $r = s + 1$.

*(3) zero otherwise.*

If $A$ is representation-finite indecomposable algebra, then the GR measures of indecomposable modules can be calculated directly from $\Gamma(A)$ thanks to the following theorem, [3, Thm. VI. 1.4.]. We present it in a greater generality than needed for this section because it implies the first Brauer-Thrall conjecture.

**Theorem 27.** *Let $A$ be an indecomposable artin algebra and $\mathcal{C}$ a component of* $\Gamma(A)$ *such that the lengths of objects in $\mathcal{C}$ are bounded.*

*Then $A$ is representation-finite and* $\Gamma(A) = \mathcal{C}$.

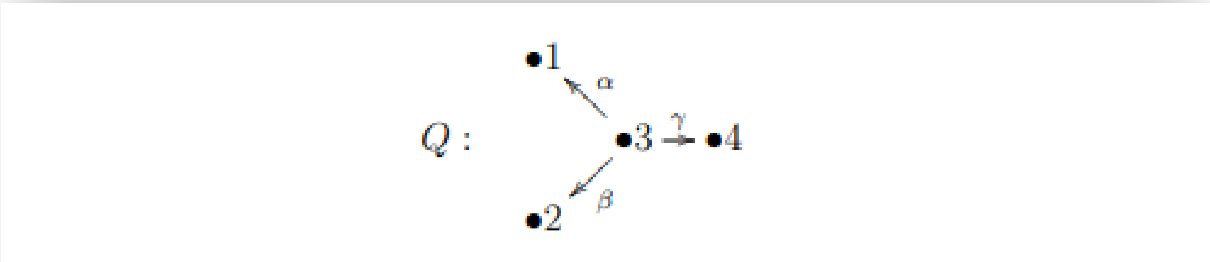
We immediately see that for an indecomposable artin algebra $A$, the quiver $\Gamma(A)$ is connected if and only if $A$ is representation-finite. In this case, any monomorphism $X \to Y$ in *ind-A* is a sum of compositions of irreducible morphisms, [2, Cor. IV. 5.6]. Furthermore, there are never multiple arrows between two vertices, [2, Prop. IV. 4.9.], and no loops. This gives us the means to calculate GR measures using the A-R quiver.

**Lemma 28.** *Assume $KQ$ tobe representation-finite and let $X \subsetneq Y$ for $X, Y \in$ ind-$A$ be a GR-inclusion.*

*Then there is a path from $X$ to $Y$ in $\Gamma(A)$ starting with a monomorphism.*

In principle, this gives a method to calculate all the GR measures for any indecomposable representation of a quiver of a Dynkin type.

**Example 29.** *The quiver*

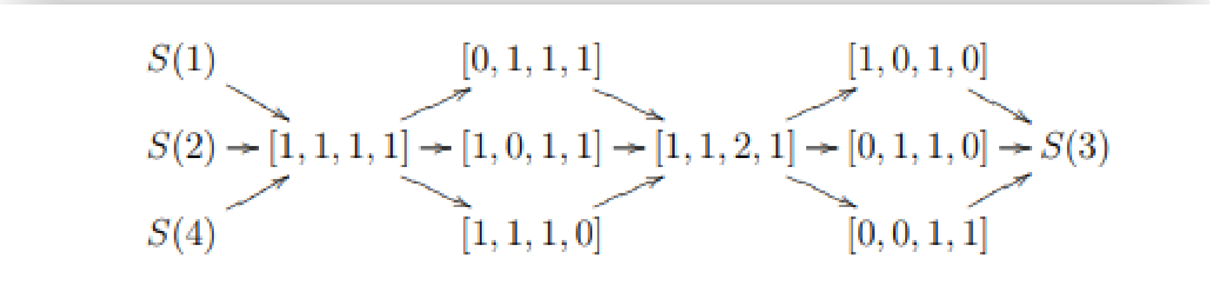



*is an orientation of a Dynkin graph $D_4$. By Gabriel's theorem, each indecomposable representation is determined by its dimension vector. We denote the indecomposable representation with the dimension vector $[a, b, c, d]$ by $R(a, b, c, d)$.*

*The associated integral quadratic form of $Q$ is*

$$q_Q(x_1, x_2, x_3, x_4) = (x_1 - 1/2x_3)^2 + (x_2 - 1/2x_3)^2 + (x_4 - 1/2x_3)^2 + 1/4x_3^2 = 1.$$

*There are twelve indecomposable representations (up to isomorphism). Eleven thin representations and the representation $R(1, 1, 2, 1)$. We now calculate $\Gamma(KQ)$. Recall that $S(i)$ is the simple representation with dimension vector $e_i$. In the non-simple vertices, we write the dimension vector of a corresponding representation:*




*We see from $\Gamma(KQ)$ that $S(3)$ does not inject into any other module. On the other hand, there is a path, starting with a monomorphism, from the module $S(1)$ to $R(0, 1, 1, 0)$, but $S(1)$ is not a submodule of $R(0, 1, 1, 0)$. We see that such a path is only a necessary but not sufficient condition. In total, irregardless of the chosen length function, the modules of length two have only one GR submodule.*

*The quiver $\Gamma(KQ)$ shows that only modules $S(1)$, $S(2)$ and $S(4)$ can be submodules of a module of length three or four. All three inject into $R(1, 1, 1, 1)$. For the* *standard GR measure, they are all GR submodules. For other measures, it depends on their length.*

*For each representation $R$ of length three, $S(i)$, $i \in \{1,2,4\}$ injects into $R$ if and only if $dim(S(i)) \leq dim(R)$. This gives two possible GR submodules. Note that here, GR inclusions are not irreducible morphisms.*

*Finally, all irreducible morphisms with codomain $R(1,1,2,1)$ are monomorphisms. By Proposition 22, the GR inclusion is an irreducible monomorphism. All three modules of length three are GR submodules for the standard GR measure.*

$\Delta$

Other methods for the calculation of GR measures exist. In Section 2.1.2, we present an algorithm to calculate GR measures of thin representations for quivers whose underlying graph is a tree.

### 1.6.3 A-R quiver and Ringel's partition of mod-A

For an artin algebra $A$, there are two partitions of the category *ind-A*. Take-off, central and landing modules following Theorem 21 by Ringel. The A-R quiver $\Gamma(A)$ gives partition in preprojective, regular and projective modules.

We compare these two partitions. Bo Chen studied the case when $Q$ is of Euclidean type in [6]. The description of $\Gamma(A)$ is based on [20].

Throughout this section, a quiver $Q$ is always of an Euclidean type with an acyclic orientation, $K$ is algebraically closed, and $A$ denotes the path algebra $KQ$. By Gabriel's theorem, $A$ is representation-infinite.

If the algebra $A$ is connected, then the full subquiver of $\Gamma(A)$ consisting of preprojective (preinjective) modules is connected, [3, Prop VII.1.11.]. A non-zero morphism with a preinjective domain (preprojective codomain) has a preinjective codomain (preprojective domain).

For general artin algebras, Ringel proved that landing modules are preinjective, [21, Thm.4.]. Section 3.6 shows that a preinjective module can also be central. There is a stronger result for quivers of type $\tilde{A}_n$, [9, Thm 5.7., Thm. 5.8.]

**Theorem 30.** *Assume $Q$ to be of type $\tilde{A}_n$ and consider $KQ$.*

*(1) All preinjective modules are landing modules if and only if $n$ is odd and $Q$ has a source-sink orientation, i.e., any vertex of $Q$ is either a source or a sink.*

*(2) If a preinjective central module exists, then there are infinitely many isomorphism classes of preinjective central modules.*

Bo Chen proved dual for quivers of Euclidean type. Any preprojective module is then a take-off module ([6, Thm. 4.4.]. Section 3.6 shows that regular take-off modules exist for some algebras.

Components of $\Gamma(A)$ containing regular modules are called *tubes*. Let $\mathcal{T}$ be a tube. For each module $X$ in $\mathcal{T}$, a unique irreducible monomorphism with domain $X$ exists. A module $X \in \mathcal{T}$ is *quasi-simple* if there is no irreducible monomorphism with codomain $X$. The number of quasi-simple modules is the *rank of the tube*, denoted $R$. For quivers of Euclidean type, then $r$ is a finite number. If for a tube $\mathcal{T}$ its rank is equal to one, the tube is called *homogenous*, the unique quasi-simple module $H \in \mathcal{T}$ is called *homogenous simple*, and there is a unique sequence of irreducible monomorphisms

$$H = H_1 \to H_2 \to H_3 \to H_4 \to \dots$$

The modules $H_i$ are called *homogenous (regular)*. If $Q$ is of Euclidean type, there are infinitely many homogenous tubes. The GR measure does not distinguish between homogenous tubes, [6, Cor. 4.5.].

**Proposition 31.** *Let $\delta$ be the minimal radical vector for $Q$.*

*Then for each $i > 1$, the module $H_i$ contains $H_{i-1}$ as the unique GR-submodule. In particular*

$$\mu(H_i) = \mu(H_1) \cup \{2\delta, 3\delta, \dots, i\delta\}.$$

If the rank $r$ of a tube $\mathcal{T}$ is greater than one, then the tube is called *exceptional*. We will also call its modules *exceptional*. Note that the word *exceptional module* can have different meanings in literature.

For a fixed exceptional tube $\mathcal{T}$, of rank $r$, we denote the quasi-simple exceptional modules by $X_1, X_2, \dots, X_r$. Quasi-simple modules are determined by their dimension vector. It is possible to choose the ordering of quasi-simple modules in $\mathcal{T}$ such that that

$$C^+(X_i) \cong X_{i+1} \ (mod\ r).$$

For each exceptional quasi-simple module $X_i$ there exists a unique seqence of irreducible monomorphisms

$$X_i = X_i[1] \to X_i[2] \to X_i[3] \to \dots$$

Section 3.6 shows that a GR submodule of an exceptional module can be preprojective. The following lemma gives a criterion when the above monomorphism is a GR inclusion, [6, Lemma 4.8. (2)].

**Lemma 32.** *Let $H_1$ be a homogenous simple module, and $X$ be a quasi-simple module of rank $r$ such that $\mu(X[r]) \geq \mu(H_1)$.*

*Then for $i \geq r$, the irreducible monomorphism $X[i] \hookrightarrow X[i+1]$ is a GR inclusion*

# 2. Alternative GR measures

This chapter gathers original results about alternative Gabriel-Roiter measures. Thin $K$-representations of quivers whose underlying graphs are trees are of particular interest. For simplicity, we will call these quivers *trees*. This includes, among others, all quivers of a Dynkin or a Euclidean type except for the type $\tilde{A}_n$. The presented results also apply in cases where the quiver is not a tree, but the *support* of a representation is. Results in this chapter do not assume any properties of the field $K$, except for Example 40.

In this setting, thin representations are, up to isomorphism, determined by their dimension vector, irregardless of the representation type of a quiver. This allows us to describe representations and their subrepresentations in terms of quivers and subquivers, see Subsection 2.1.1 for details. The assumption that a given quiver is a tree is necessary. Sections 3.3 and 3.4 illustrate that there are numerous non-isomorphic thin representations for quivers of type $\tilde{A}_n$

This transition from representations to subquivers enables a simple combinatorial procedure for finding GR-filtrations of such indecomposable thin representation. The procedure works for any GR measure, see Subsection 2.1.2. Subsection 2.1.3 shows that any indecomposable filtration is a GR filtration for some GR measure. This result is not true for general indecomposable representations as demonstrated by Example 40.

The last section is concerned with changes in the length of a GR filtration when the length function is changed. Even in the case of representations of quivers of type $A_n$, one can get various lengths as shown by Proposition 43.

## 2.1 Thin representations

Recall that a $K$-representation $M$ of a quiver $Q$ is *thin* if $dim_K(M_a) \leq 1$ for every $a \in Q_0$. For quivers of type $A_n$, all indecomposable representations are thin. Bo Chen observed that for any indecomposable representation of a Dynkin quiver, the first two terms of any GR filtration are thin, with the second one having a length at most four, [10, Prop 2.4.4.]. Also if $N \hookrightarrow M$ is a GR-inclusion and $N, M$ are indecomposable representations of $\tilde{A}_n$, the GR-factor $M/N$ is thin, [7, Thm 4.1.].

### 2.1.1 Basic properties of thin representations

This subsection prepares grounds for the study of GR measures of thin representations. Presented results allow us to focus on subquivers rather than representations. I assume that the results in this subsection are not novel. However, in the literature, I have consulted, they were not given in sufficient generality. A special case of Lemma 34 for quivers of type $A_n$ follows from [18, Thm. 1.2.].

**Definition 13.** *Let $Q$ be a quiver, $Q'$ its subquiver and $M \in$ mod-$KQ$.*
*We say that an arrow $\alpha \in Q_1$ is* incident with *$Q'$ if $s(\alpha) \in Q'_0$ or $t(\alpha) \in Q'_0$.*

Let $M$ be a thin representation of a quiver $Q$ and consider the structural map $M_\alpha$ for $\alpha \in Q_1$. If its domain or codomain is trivial, it is the zero map. If not, it is an isomorphism by Shur's lemma [1, Lemma 13.3.]. Lemma 34 shows that we can assume all those isomorphisms to be identities. Given this, we denote by $R(Q')$ the thin representation with the support $Q'$, such that all structural maps are either zero morphisms or identities.

**Lemma 33.** *Let $M$ be a thin representation of a quiver $Q$ such that $supp(M)$ is a tree.*
*Then $M$ is indecomposable.*

*Proof.* Without the loss of generality, we may assume that $supp(M) = Q$. For a contradiction, suppose we have a decomposition $M = N \oplus N'$ such that $N \neq 0$ and $N' \neq 0$.

Because $Q$ is a tree, there is a unique $\alpha \in Q_1$ such that $Q_1$ is a disjoint union of $supp(N)_1$, $supp(N')_1$ and $\{\alpha\}$. Furthermore, $Q_0$ is a disjoing union of $supp(N)_0$ and $supp(N')_0$.

Suppose that $s(\alpha) \in supp(N)$. Because $M$ is thin and $supp(M)$ is connected support, $M_\alpha$ is injective. By Lemma 3, we have a contradiction with $N$ being a subrepresentation of $N$

□

The following lemma allows us always to assume that maps $M_\alpha$ are either zero maps or identities. The case for representations of quivers of Dynkin type follows from Gabriel's theorem.

**Lemma 34.** *Let $M$ be a thin indecomposable representation of a quiver $Q$. Suppose that $Q$ is a tree and let $N$ be the thin representation $R(supp(M))$.*
*Then $N \cong M$.*

*Proof.* Without the loss of generality, we may assume that $supp(M) = Q$. We work by induction on $n := |Q_0|$. The case $n = 1$ is trivial.

For $n > 1$, we assume that the vertex $n$ is a leaf, i.e., only one arrow is incident with $n$. We denote this arrow by $\alpha$. We define a new indecomposable thin representation $M'$. The support of $M'$ is the full subquiver of $Q$ given by vertices $Q_0 \setminus \{n\}$. For each $\alpha \in supp(M')$ we set $M'_\alpha := M_\alpha$.

By induction, there is an isomorphism $\phi' \colon R(supp(M') \cong M'$.

We now construct an isomorphism $\phi \colon N \to M$. We consider two cases.

$$\text{Either } n = t(\alpha): \quad \begin{array}{ccc} \cdots K & \xrightarrow{a} & K \\ \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle \phi_n} \\ \cdots K & \xrightarrow{1} & K \end{array} \quad \text{or } n = s(\alpha): \quad \begin{array}{ccc} \cdots K & \xleftarrow{a} & K \\ \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle \phi_n} \\ \cdots K & \xleftarrow{1} & K \end{array}$$

Becuase $\phi'$ is isomorphism, $b \neq 0$ and because $M$ is indecomposable, $a \neq 0$. In the case $n = t(\alpha)$, we choose $\phi_n$ to be the multiplication by $ab$. In the second case, the multiplication by $b/a$.

□

A subrepresentation of a thin representation is also thin. The following lemma allows us to describe subrepresentations of a given representation only in terms of their support graphs.

**Lemma 35.** *Let $Q$ be a quiver and let $M, M'$ be thin representations of $Q$ such that $supp(M')$ is a subquiver of $supp(M)$. Further, assume that $supp(M)$ a tree.*

*Then $M'$ is a subrepresentation of $M$ if and only if all arrows with the source in $supp(M')$ have the target also in $supp(M')$.*

In particular, the simple representation $S(i)$ is a subrepresentation of $M$ if and only if $i$ is a sink in $supp(M)$.

*Proof.* All non-zero maps are injective. The forward implication then follows from Lemma 3. In the other direction, let us define $\phi \colon M' \to M$ by setting $\phi_a$ as the identity map if $a \in supp(M')_0$ and the zero map otherwise. This is a well-defined morphism. There are three possible cases to check.

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{M_\alpha} & M_{t(\alpha)} \\ \uparrow{\scriptstyle \phi_{s(\alpha)}} & & \uparrow{\scriptstyle \phi_{t(\alpha)}} \\ M'_{s(\alpha)} & \xrightarrow{M'_\alpha} & M'_{t(\alpha)} \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{id} & K \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} \\ K & \xrightarrow{id} & K \end{array} \quad \begin{array}{ccc} K & \xrightarrow{id} & K \\ \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle id} \\ 0 & \xrightarrow{0} & K \end{array} \quad \begin{array}{ccc} 0 & \xrightarrow{0} & 0 \\ \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\ 0 & \xrightarrow{0} & 0 \end{array}$$

□

By Corollary 17, if we have a GR-inclusion $N \subsetneq M$, then the factor is indecomposable. We can easily calculate the factor $M/N$ for thin representations. Becuase $dim(N) + dim(M/N) = dim(M)$, the support of $M/N$ is the full subquiver of $supp(M)$ induced by the set of vertices $supp(M)_0 \setminus supp(N)_0$.

### 2.1.2 Calculating GR-measure for thin representations

In this subsection, $Q$ denotes a tree quiver, and $K$ denotes an arbitrary field. Further, let $l$ be a length function on *mod*-$KQ$ and $M$ an indecomposable thin $K$-representation of $Q$. Without the loss of generality, we assume that $supp(M) = Q$.

We define a *weighted quiver* $Q_l$ by assigning a *weight* $l(S_i)$ to every vertex $i \in Q_0$. For a subquiver $Q'$ of $Q$, the *l-weight* of $Q'$ with respect to $l$ is defined as $\sum_{i \in Q'_0} l(S_i)$. Recall that a length function $l$ in *mod*-$A$ is determined by its values on simple modules. So we can also define a length function on *mod*-$KQ$ by a weighted quiver.

In this subsection, we present an algorithm for how to calculate $\mu_l(M)$ working only with the weighted quiver $Q_l$.

Let $\mathcal{S} = (M_1, \dots, M_m)$ be a chain of subrepresentations of a representation $M$. Following Definition 6, $l(\mathcal{S}) := (l(M_1), \dots, l(M_m)$ is the *chain of lenghts*.

Recall that a GR-filtration is an indecomposable filtration with a maximal chain of lengths (with respect to the lexicographical order). For each simple subrepresentation, we find an indecomposable filtration starting with the given simple subrepresentation with the maximal chain of lengths. From these filtrations, those with a maximal chain of lengths are GR-filtrations.

If the function $l$ is not constant on all simple subrepresentations, we can consider only filtrations starting with simple representations with minimal $l$-length.

**Initial step**

Let us start with a fixed simple subrepresentation $S(s)$ where $s$ is some sink in $supp(M) = Q$. If $S(s) = M$, we are done.

Otherwise, we consider the set $\mathcal{Q}(s)$ containing minimal (with respect to the inclusion) subquivers of $Q$ from the following set

$$\{Q' \mid s \in Q'_0;\ R(Q') \leq M\} \setminus \{s\}.$$

For each arrow incident with $s$, there is one minimal element of this set.

*Example.* If $s$ is a sink in a quiver of type $A_n$ and not one of its endpoints, we search to the left until we hit the first right-pointing arrow (not incident with $s$) or until we reach the endpoint. And we also search to the right until we find the first left-pointing arrow.

Then we find a subset $\mathcal{Q}(s)^+ \subseteq \mathcal{Q}(s)$ consisting of subquivers with minimal $l$-weight. And for each $Q' \in \mathcal{Q}(s)^+$ we build a two-element sequence $(\{s\}, Q')$.

If there is more than one minimal subquiver, we need to run the algorithm for all of them, see Example 37.

**Example 36.** *Consider the quiver*

$$1 \to 2 \leftarrow 3 \leftarrow 4 \to 5 \leftarrow 6 \to 7 \to 8$$

*There are two minimal quivers containing sink 5.*

$$2 \leftarrow 3 \leftarrow 4 \rightarrow 5 \qquad 5 \longleftarrow\!\!\longrightarrow 7 \rightarrow 8$$

*They both have the same weight with respect to the composition length but may have different weights if we change the length function.*

$\Delta$

**Inductive step**

Suppose we already have a sequence of subquivers $\mathcal{S} = (\{s\} \leq Q^1 \leq \cdots \leq Q^k)$. We create new sequences, starting with $\mathcal{S}$. By $\mathcal{Q}(\mathcal{S})$ we denote set of minimal subquivers $Q'$ strictly containing $Q^k$ such that $R(Q') \leq M$. Then choosing the subset $\mathcal{Q}(\mathcal{S})^+$ consisting of those minimal with respect to their $l$-weight.

**Correctness**

Observe that if

$$R(\{s\}) \subsetneq R(Q^1) \subsetneq \cdots \subsetneq R(Q^k)$$

was an indecomposable filtration of $R(Q^k)$ and $Q' \in \mathcal{Q}(\mathcal{S})^+$ then

$$R(\{s\}) \subsetneq R(Q^1) \subsetneq \cdots \subsetneq R(Q^k) \subsetneq R(Q')$$

is an indecomposable filtration of $R(Q')$.

The algorithm gives several indecomposable filtrations of the input representation $M$, and we choose one with the maximal chain of lengths. The correctness of the algorithm follows from the recursive definition of the GR measure, Proposition 9.

**Example 37.** *Consider the algebra $KQ$ where*

$$Q: \ 1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \leftarrow 7 \leftarrow 8 \leftarrow 9 \rightarrow 10$$

*and let $M := R(Q)$.*

*We will show the run of an algorithm starting with sink 6. Notice that there are possible branching in the first few steps, but they give the same outcome.*

$$M_1: \ 0 \xrightarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} K \xleftarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} 0$$

$$M_3: \ 0 \xrightarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} 0$$

$$M_5: \ 0 \xrightarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} K \xrightarrow{1} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xleftarrow{0} 0 \xrightarrow{0} 0$$

*At this point, the algorithm branches once again. This time it gives different outcomes. We first go to the left.*

$$M_6: \ 0 \xrightarrow{0} K \xleftarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xleftarrow{0} 0 \xrightarrow{0} 0$$

$$M_7: \ K \xrightarrow{1} K \xleftarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xleftarrow{0} 0 \xrightarrow{0} 0$$

$$M_8: \ K \xrightarrow{1} K \xleftarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xrightarrow{1} K$$

*Resulting in the sequence* $\mathcal{S}_6 = \{1,2,3,4,5,7,8,10\}$*. If we go the right, we get*

$$M_6' : \; 0 \xrightarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} K \xrightarrow{1} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xrightarrow{1} K$$
$$M_7' : \; 0 \xrightarrow{0} K \xleftarrow{1} K \xrightarrow{K} K \xrightarrow{1} K \xrightarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xleftarrow{1} K \xrightarrow{1} K$$
$$M_8' = M_8$$

*Resulting in the sequence* $\mathcal{S}_6' = \{1,2,3,4,5,7,9,10\} < \mathcal{S}_6$.

*This difference is indeed important to the outcome of the algorithm. There are two more sinks, vertices 2 and 10, giving sequences* $\mathcal{S}_2 = \{1,2,6,7,8,10\}$ *and* $\mathcal{S}_{10} = \{1,5,6,7,9,10\}$*. Both* $\mathcal{S}_2$ *and* $\mathcal{S}_{10}$ *are strictly lesser than* $\mathcal{S}_6$*.*

$\triangle$

### 2.1.3 Indecomposable filtrations

**Theorem 38.** *Let $K$ be a field, $M$ a thin indecomposable representation of a tree quiver $Q$. And let*

$$\mathcal{R} : \; R_1 \subsetneq R_2 \subsetneq \cdots \subsetneq R_m = M$$

*be an indecomposable filtration of $M$.*

*Then there exists a length function $l$ on mod-$KQ$ such that $\mathcal{R}$ is a GR-filtration for $\mu_l(M)$.*

Thanks to the above theorem, propositions about GR filtrations and GR factors of indecomposable thin representations can often be easily generalised into propositions about indecomposable filtrations. Example 40 shows that the claim does not hold for general representation.

The idea of the proof is that we run a modified version of the algorithm from the previous section. In each step, we adjust the length function so that algorithm chooses $\mathcal{R}$ as a GR-filtration.

*Proof.* We build a series of length functions $l_1, \dots, l_l$ such that for each $i \le m$, the sequence $R_1 \subsetneq \cdots \subsetneq R_i$ is a GR-filtration for $\mu_{l_i}(R_i)$ and for each vertex $j \in Q_0$ not contained in *supp*$(R_i)$, the weight of $j$ is 1. Then we set $l := l_m$.

For $l_1$, we set the length of the simple representation $R_1$ to be 1/2 and all other simple representations to have length 1.

Suppose we have already constructed $l_1, \dots, l_i$. Consider the set $\mathcal{Q}(R_i)$ of all indecomposable subrepresentations of $M$ strictly containing $R_i$ minimal with respect to inclusion.

Observe that elements of $\mathcal{Q}(R_i)$ are in bijection with arrows incident with *supp*$(R_i)$ and the graphs *supp*$(R/R_i)$ for each $R \in \mathcal{Q}(R_i)$ are pairwise disjoint. This follows from the fact that $Q$ is a tree, i.e., $\bar{Q}$ contains no cycles.

Choose $R$ minimal with respect to $l_i(R/R_i) = |R/R_i|$.

If $R_{i+1} = R$ then set $l_{i+1} := l_i$.

If $R_{i+1} \neq R$ then $supp(R_{i+1}/R_i)$ and $supp(R/R_i)$ are disjoint. For each simple representation $S(a)$ corresponding to a vertex $a \in supp(R_{i+1}/R_i)_0$ we assign

$$l_{i+1}(S(a)) := \frac{|R/R_i|}{|R_{i+1}/R_i| + 1}.$$

For $a \notin supp(R_{i+1})_0$ we set $l_{i+1}(S(a)) := l_i(S(a)) = 1$.

For simple representations corresponding to vertices of $supp(R_i)$, we set their new length to be their $l_i$-length divided by some fixed constant, chosen to be big enough such that $R_1 \subsetneq R_2 \subsetneq \cdots \subsetneq R_i$ is a GR-filtration for $\mu_{l_{i+1}}(R_i)$.

□

**Example 39.** *Consider a quiver*

$$Q: \; 1 \leftarrow 2 \rightarrow 3 \leftarrow 4$$

*and the representation $M := R(Q)$ with the following indecomposable filtration*

$$S(3) \subsetneq (K \xleftarrow{1} K \xrightarrow{1} K \xleftarrow{0} 0) \subsetneq M.$$

*This is not a GR-filtration for the standard GR-measure $\{1, 2, 4\}$. We define a length function*

$$l_1: \; 1 \leftarrow 1 \rightarrow 1/2 \leftarrow 1.$$

*A GR-filtration for $\mu_{l_1}(M)$ is same as for $\mu(M)$, that is*

$$S(3) \subsetneq (0 \xleftarrow{0} 0 \xrightarrow{0} K \xleftarrow{1} K) \subsetneq M.$$

*We define*

$$l_1': \; 1/3 \leftarrow 1/3 \rightarrow 1/2 \leftarrow 1.$$

*Both $\mu(M)$ and $\mu_{l_1}$ have only one GR filtration, namely*

$$S(1) \subsetneq (K \xleftarrow{1} K \xrightarrow{1} K \xleftarrow{0} 0) \subsetneq M.$$

*We have ensured that our desired representation is indeed the second element of some GR-filtration, but we have changed the first member.*

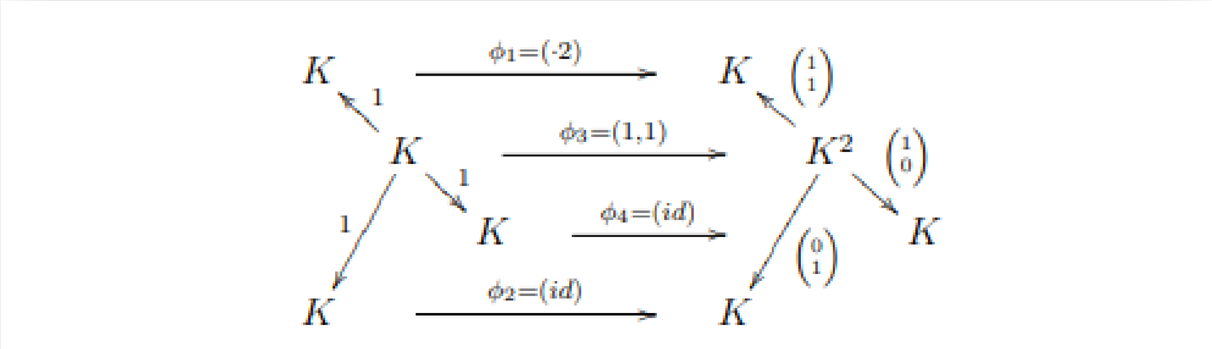
*Finally, we define*

$$l_2: \; 1/3 \leftarrow 1/3 \rightarrow 1/4 \leftarrow 1.$$

Δ

The assumption that the representation $M$ is thin is necessary.

**Example 40.** *Recall the quiver from Example 29. There is a following inclusion*




*of the representation* $R(1,1,1,1)$ *into* $R(1,1,2,1)$*. This gives us the following indecomposable filtrations*

$$S(i) \hookrightarrow R[1,1,1,1] \hookrightarrow R[1,1,2,1],$$

*for* $i \in \{1,2,4\}$*. But neither of these filtrations can be a GR filtration. As seen in Example 29, a GR subrepresentation of* $R[1,1,2,1]$ *has to be of length three.*

$\Delta$

In Example 29, we noticed that all GR inclusions for any GR measure are also GR inclusion for the standard GR measure. This is not true in general.

**Example 41.** *Recall Example 37. The thin indecomposable representation of length ten has only one GR filtration with respect to the composition length. But there are two indecomposable filtrations whose penultimate member differs from the one in the said GR filtration. By Theorem 38, these indecomposable filtrations are GR filtrations for some length functions.*

## 2.2 Lenght of a GR-filtration

In [19], Ringel formulates two dual examples. Suppose $M$ an indecomposable non-simple module. We might ask how many indecomposable modules are in $(M, \leq)$. These are two extreme cases described in Proposition 11. Either all non-zero submodules are indecomposable, then $|\mu(M)| = |M|$. Or the opposite case, $|\mu(M)| = 2$, when only $M$ itself and simple submodules are indecomposable. Despite this duality, the first case holds for any GR measure as shown by Proposition 42, while the second case does not, as will follow from Theorem 43.

Recall that a module is uniform if it has a simple socle. This condition characterises modules whose all non-zero submodules are indecomposable.

**Proposition 42.** *Let $R$ be a ring and $M \in$ ind-$R$.*

*The module $M$ is uniform if and only if $|\mu_l(M)| = |M|$ for any length function $l$ on mod-$R$.*

*Proof.* First, observe that a submodule $N \subseteq M$ has to be indecomposable. Suppose that we have a decomposition $N \cong N_1 \oplus N_2$. If $N_1 \neq 0 \neq N_2$, they have a nontrivial socle. But then we have found at least two distinct (though possibly isomorphic) simple submodules of $M$. The forward implication then follows from the Jordan-Hölder theorem.

Set $n := |M|$ and suppose $n = |\mu_l(M)|$. Let

$$M_1 \subsetneq M_2 \subset \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

be a GR filtration of $M$. This filtration is also a composition series.

We prove by induction that all submodules $M_i$ are uniform. All simple modules are uniform, and so is $M_1$. Now assume $M_{i-1}$ is uniform. If $M_i$ does not have a simple socle, it means that there is a simple submodule of $M_i$, let us call it $S$, different from $M_1$. Because *soc* $M_i = M_1$, we have $S \cap M_{i-1} = 0$ and tgus $S \oplus M_{i-1} = M_i$ hence a contradiction with $M_i$ being indecomposable.

□

The following proposition explains that we can't add similar examples to this list. In particular, why part (1) of Proposition 11 is not true for a general measure.

**Theorem 43.** *Let $t$ be a positive integer such that $t \geq 4$ and $K$ be a field.*

*Then there exists a $K$-algebra $A$ such that for any $1 < a < b < t$ there is $M \in$ ind-$A$ and two length function $k, l$ on mod-$A$ such that $|\mu_k(M)| = a$ and $|\mu_l(M)| = b$.*

*Proof.* For given $t$, we find a quiver $Q$ of type $A_n$ such that for any $1 < a < b < t$, a representation $M$ of $Q$ exists with the above properties.

We partition $A_n$, ensuring that for any such pair $(a, b)$, there is a subpath $P_{a,b}$ of length $a + b - 1$ and that two such subpaths, their intersection is either empty or consists only of one vertex. This is possible for a high enough $n$.

We choose an orientation on $A_n$ such that the endpoints of the subpaths are the sinks. There is exactly one source in each subpath $P_{a,b}$—namely, the $a$-th vertex of the path. The remaining vertices are neither sinks nor sources.

For a pair $(a, b)$, we choose a thin representation $M := R(P_{a,b})$ There are two simple subrepresentations $S_a$ and $S_b$ of $M$ corresponding to two sinks. If we choose $l$ such that all simple representations have length 1 and $l(S_b) = 2$ then

$$\mu_k(M) = \{1, 2, \ldots, a - 1, a + b\}.$$

For $l$ such that all simple representations have length 1 and $l(S_a) = 2$ we have

$$\mu_l(M) = \{1, 2, \ldots, b-1, a+b\}.$$

□

**Example 44.** *For $n = 5$ the possibe pairs $(a, b)$ are $(2, 3)$, $(3, 4)$ and $(2, 4)$.*

*The proof of Proposition 42 suggests a quiver*

$$1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \leftarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \leftarrow 10 \rightarrow 11 \rightarrow 12 \rightarrow 13.$$

*Consider the representation*

$$M: \ 0 \leftarrow 0 \rightarrow 0 \rightarrow K \leftarrow K \leftarrow K \rightarrow K \rightarrow K \rightarrow K \leftarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0.$$

*Recalling lenght function $l$ and $k$ from the proof of the proposition, we get $\mu_k(M) = \{1, 2, 7\}$ and $\mu_l(M) = \{1, 2, 3, 7\}$.*

△

Fix a ring $R$. We might ask whether two length functions $l, k$ on *mod*-$R$ give equivalent (in the sense of Theorem 8) GR measures. If $l$ and $k$ induce different ordering of simple representations, the GR measures $\mu_l$ and $\mu_k$ are not equivalent. The following example shows that even if $l$ and $k$ give the same ordering on simple representations, they still can give rise to different GR measures.

**Example 45.** *Consider the following orientation of $A_n$*

$$Q: \ 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6.$$

*We define two different length functions on mod-$KQ$*

$$l: \ 1 \leftarrow 2 \rightarrow 2 \rightarrow 2 \leftarrow 3 \rightarrow 1$$

$$k: \ 1 \leftarrow 2 \rightarrow 2 \rightarrow 2 \leftarrow 5 \rightarrow 1.$$

*The ordering of the lengths of simple representations is the same for both length functions. Set $M := R(Q)$. Then*

$$\mu_l(M) = \{1, 6, 8, 11\}$$

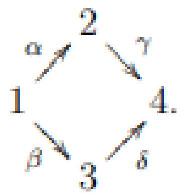
$$\mu_k(M) = \{1, 7, 13\}.$$

*Not only are GR filtrations different, but also their lengths differ.*

△

# 3. The GR measure for $\tilde{A}_3$

This chapter describes all standard GR-measures for a representation-infinite connected hereditary algebra $A := KQ$, where $K$ is an algebraically closed field, and $Q$ is the following orientation of




GR measures for acyclic orientations of $\tilde{A}_1$ and $\tilde{A}_2$ were already calculated by Ringel, [21]. Bo Chen calculated the GR measures for representations of $\tilde{A}_3$ with the source-sink orientation, [8].

The approach in this chapter is based mainly on findings in [8]. Some results were later generalised for any quiver of a Euclidean type, [6], and are gathered in Subsection 1.6.3.

Recall that in this thesis, by *preprojective*, *regular* or *preinjective* module, we always mean an indecomposable module, see Subsections 1.2.3 and 1.6.3.

## 3.1 Dimension vectors of indecomposables

We begin by calculating dimension vectors of all indecomposable representations of $Q$ using Gabriel's theorem. Used notation indicates which vector corresponds to preprojective ($P$), regular ($R$) or preinjective ($I$) modules.

We have the following quadratic form $q\colon \mathbb{Z}^4 \to \mathbb{Z}^4$ given by the quiver $Q$

$$q(x_1, x_2, x_3, x_4) = \sum_{i \in \{1,2,3,4\}} x_i^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4.$$

We rewrite it as a sum of squares

$$q(x_1, x_2, x_3, x_4) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}(x_1 - x_3)^2 + \frac{1}{2}(x_4 - x_2)^2 + \frac{1}{2}(x_4 - x_3)^2,$$

hence

$$q(x_1, x_2, x_3, x_4) = 0 \Longleftrightarrow x_1 = x_2 = x_3 = x_4.$$

We now investigate nonnegative vectors $X$ such that $q(X) = 1$. For any arrow $\psi \in Q_1$, we need $|x_{t(\psi)} - x_{s(\psi)}| \leq 1$ to ensure that $1/2(x_a - x_b)^2 \leq 1$. If equality $|x_{t(\psi)} - x_{s(\psi)}| = 1$ occurs, it happens for exactly two arrows.

We differentiate two cases. If these two arrows are not adjacent, we have the following vectors for $a \geq 0$

$$P_\alpha(a) = [a, a, a+1, a+1]; \; P_\beta(a) = [a, a+1, a, a+1]$$
$$I_\gamma(a) = [a+1, a, a+1, a]; \; I_\delta(a) = [a+1, a+1, a, a].$$

The other case is when three vector coordinates have the same value, and the fourth differs by one. This gives us eight families of vectors

$$I_1(a) = [a+1, a, a, a]; \; P_1(a) = [a, a+1, a+1, a+1]$$
$$P_4(a) = [a, a, a, a+1]; \; I_4(a) = [a+1, a+1, a+1, a]$$
$$R_2^+(a) = [a, a+1, a, a]; \; R_2^-(a) = [a+1, a, a+1, a+1]$$
$$R_3^+(a) = [a, a, a+1, a]; \; R_3^-(a) = [a+1, a+1, a, a+1],$$

defined for $a \geq 0$.

The minimal radical vector for $Q$ is $\delta = [1, 1, 1, 1]$, so the *defect* of a vector is

$$\delta(x_1, x_2, x_3, x_4) = \sum_{i \in \{1,2,3,4\}} x_i - x_2 - x_3 - 2x_4 = x_1 - x_4.$$

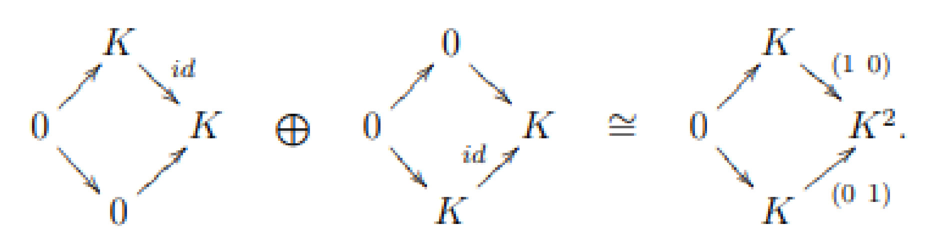
## 3.2 Preprojective component

An indecomposable module is preprojective if its defect is $-1$. In this case, it means that for a preprojective $P$ such that $dim(P) = [x_1, x_2, x_3, x_4]$ we have $x_4 > x_1$. This gives us families $P_1(a)$, $P_4(a)$, $P_\alpha(a)$ and $P_\beta(a)$ for non-negative $a$. By Gabriel's theorem, preprojectives are determined (up to isomorphism) by their dimension vector. From now on, we denote preprojective modules by their dimension vectors.

We start with the four indecomposable projective modules:

$$\mu(P_4(0)) = \{1\}, \; \mu(P_\beta(0)\} = \mu(P_\alpha(0)\} = \{1, 2\}; \; \mu(P_4(1)) = \{5, 2, 1\}$$

We calculate $\mu(P_4)(1) = P(1)$ using Example 23. We have the following decomposition of $radP(1)$, where the unspecified maps are zero.



For quivers of type $\tilde{A}_n$, GR inclusions between preprojective modules are given by irreducible morphisms, as illustrated in the following. Firstly, we calculate $C^+(P)$ for all non-projective preprojective modules.

$$C^+(P_1(a)) = P_4(a);\quad C^+(P_4(a)) = P_1(a-2)$$
$$C^+(P_\alpha(a)) = P_\beta(a-1);\quad C^+(\beta_4(a)) = P_\alpha(a-1)$$

The image is defined for all $a$ for which the domain is not projective.

By Proposition 26, we know that for any preprojective $P$, the vertex $[P]$ has two predecessors in $\Gamma(A)$. With this in mind, we can calculate all irreducible morphisms between preprojectives using Proposition 25, based on dimension vectors only.

For example, there is an AR-sequence

$$0 \to P_4(a) \to M \to P_1(a) \to 0,$$

where

$$dim(M) = [2a, 2a+1, 2a+1, 2a+2].$$

Recall that an irreducible morphism with a preprojective codomain has a preprojective domain, so $M$ is isomorphic to a direct sum of two preprojective modules. There are no loops in $\Gamma(A)$, so neither of these summands can be $P_1(a)$ nor $P_4(a)$. The only possibility then is

$$M \cong P_\alpha(a) \oplus P_\beta(a).$$

The complete list of irreducible morphisms is

$$P_\alpha(a) \to P_1(a);\ P_\beta(a) \to P_1(a);\ P_\alpha(a) \to P_4(a+1);\ P_\beta(a) \to P_4(a+1);$$

$$P_1(a) \to P_\beta(a+1);\ P_4(a) \to P_\beta(a);\ P_1(a) \to P_\alpha(a+1);\ P_4(a) \to P_\alpha(a),$$

for all non-negative $a$. Note that the list also includes irreducible morphisms with projective domains.

As witnessed by dimension vectors, none of the above irreducible morphisms can be epimorphism. Hence, all irreducible morphisms in the preprojective component are monomorphisms. This is true for any quiver $Q$ of type $\tilde{A}_n$. See, for example, [6, Cor. 2.3.].

Recall that any non-zero morphism with a preprojective codomain has to have a preprojective domain. Hence all irreducible morphisms with a preprojective codomain are monomorphisms. By Proposition 22, we conclude that all GR-inclusions in the preprojective component are given by irreducible morphisms.

Due to symmetry, the non-projective preprojectives of an even length have the same GR measure. Thus, the non-projective preprojectives of an odd length have two GR-submodules, namely $P_\beta(a)$ and $P_\alpha(a)$ where $a$ is maximal. Non-projective preprojectives of even lengths $4a+2$ have only one GR-submodule, namely $P_1(a-1)$. Measures $\mu(P_1(a-1))$ and $\mu(P_4(a))$ are the same except for the maximal element.

In total, for a preprojective module $P$ with length $4a+1$, $a \geq 2$, we have

$$\mu(P) = \{1,2,3,4+2,4+3,4\cdot 2+2,4\cdot 2+3,\dots,4(a-1)+2,4(a-1)+3,4a+1\},$$

where for $a=1$, it is the projective module $P(1)$ with GR measure $\{1,2,5\}$ and for $a=0$ it is the simple projective module $P(4)$.

For a preprojective module $P$ with an even length $4a+2$, $a \geq 0$, we have

$$\mu(P) = \{1,2,3,4+2,4+3,4\cdot 2+2,4\cdot 2+3,\dots,4(a-1)+2,4(a-1)+3,4a+2\},$$

and for an odd length $4a+3$, $a \geq 0$, we have

$$\mu(P) = \{1,2,3,4+2,4+3,4\cdot 2+2,4\cdot 2+3,\dots,4(a-1)+2,4(a-1)+3,4a+2,4a+3\}.$$

## 3.3 Homogenous tubes

We now calculate GR measures of homogenous modules. It depends only on their length. We start with an observation by Bo-Chen, [8, Prop. 3.2.].

**Proposition 46.** *Let $Q'$ be a quiver of type $\tilde{A}_n$, $H_1$ be a quasi-simple homogenous $KQ'$-module and $X$ its GR-submodule.*

*Then $H/X$ is a simple injective module.*

There is only one simple injective module, namely $I_1(0) = S(1)$. We see that $dim(X) = P_1(0)$, thus $\mu(H_1) = \{1,2,3,4\}$. By Proposition 31, $H_i$ is the unique GR submodule of $H_{i+1}$ for $i \geq 1$. We conclude that the GR measure of a homogenous module $H$ with composition length $4a$ is

$$\mu(H) = \{1,2,3,4,8,12,\dots,4a\}.$$

## 3.4 Exceptional tubes

An indecomposable module $R$ is regular if its defect is 0. In our case, the dimension vector of $R$ has the form $dim(X) = [x_1,x_2,x_3,x_1]$. This section calculates GR measures for regular modules from exceptional tubes. We will call these modules *exceptional* in this text.

Except for a few small cases, we do not need to specify the structure of modules. We, however, end this section by giving descriptions of all exceptional modules. It will be useful in the next section. It will become clear that the exceptional module $E$ with $q(dim(E)) = 1$ is determined (up to isomorphism) by its dimension vector. This is true for any quiver of a Euclidean type.

We use the following proposition. It is a special case of [20, Thm. 3.1.(3')].

**Proposition 47.** *Let $X \in$ mod-$A$ be a quasi-simple module from a tube of rank 2. Further suppose that $X$ is thin, i.e., $dim(X) \leq [1, 1, 1, 1] = \delta$. Then*

$$dim(X[j]) = \begin{cases} j\delta & j \text{ even}, \\ (j-1)\delta + dim(X) & j \text{ odd}. \end{cases}$$

Regular thin representations $X_1, X_2, Y_1, Y_2$ are given by dimension vectors

$$dim(X_1) = [0, 1, 0, 0];\ dim(X_2) = [1, 0, 1, 1]$$
$$dim(Y_1) = [0, 0, 1, 0];\ dim(Y_2) = [1, 1, 0, 1].$$

By simple calculation, we get

$$C^+(X_1) = X_2;\ C^+(X_2) = X_1;\ C^+(Y_1) = Y_1;\ C^+(Y_2) = Y_1.$$

There are at most two exceptional tubes in $\Gamma(A)$. Hence, $X_1, X_2, Y_1, Y_2$ is the complete list of exceptional quasi-simple modules.

Firstly, we calculate GR-measures of $X_2[i]$ and $Y_2[i]$. For $i = 1$, both modules are quasi-simple. There are sequences of monomorphisms

$$S(4) \cong P_4(0) \hookrightarrow P_\alpha(0) \hookrightarrow X_2 = X_2[1] \hookrightarrow X_2[2]$$

$$S(4) \cong P_4(0) \hookrightarrow P_\beta(0) \hookrightarrow Y_2 = Y_2[1] \hookrightarrow Y_2[2],$$

where the middle inclusions follow from Lemma 35. This gives us

$$\mu(X_2[2]) = \mu(Y_2[2]) = \{1, 2, 3, 4\} \geq \{1, 2, 3, 4\} = \mu(H_1).$$

By Lemma 32, for $r = 2$, the irredicble monomoprhisms $X_2[i] \hookrightarrow X_2[i+1]$ and $Y_2[i] \hookrightarrow Y_2[i+1]$ are GR inclusions for $i \geq 2$. For $i > 0$, we have

$$\mu(X_2[2i]) = \mu(Y_2[2i]) = \{1, 2, 3, 4, 4+3, 4\cdot 2, 4\cdot 2+3, \dots, 4(i-1), 4(i-1)+3, 4i\}$$

$$\mu(X_2[2i+1]) = \mu(Y_2[2i+1]) = \{1, 2, 3, 4, 4+3, \dots, 4(i-1), 4(i-1)+3, 4i, 4i+3\}.$$

We now calculate GR measures of modules $X_1[i]$. The case for $Y_1[i]$ is similar. For the rest of this section, we write $X := X_1$. We calculate the GR measures of $X[2]$ and $X[3]$. The rest follows from irreducible morphisms $X_1[i] \hookrightarrow X_1[i+1]$.

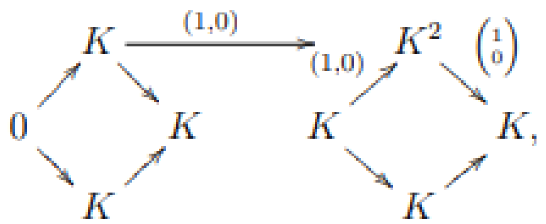
We show that $\mu(X[2]) = \{1,2,4\}$. There is an inclusion

$$X[1] \cong S(2) \hookrightarrow X[2],$$

where $dim(X[2]) = [1,1,1,1]$. By Lemma 3, we see that the structural map $X[2]_\gamma$ is the zero map. It is then easy to check that neither of the four modules of length three injects to $X[2]$. We get the following GR-filtration of $X[2]$

$$S(4) \hookrightarrow R_\alpha(0) \hookrightarrow X[2].$$

By [9, Prop. 2.2. (c)], a GR-submodule of $X[i+1]$ is either a preprojective module or $X[i]$. For module $X[3]$, there is a monomorphism $\phi\colon P_1(0) \hookrightarrow X[3]$




where $\phi_3$ and $\phi_4$ are identity maps and $\phi_1$ is the zero map. Vectors are considered to be row vectors. There is no preprojective module of lenght four and $\mu(X[2]) < \mu(P_1(0))$, so $\mu(X[3]) = \{1,2,3,5\}$.

Observe that for a preprojective module $X[2]$ of length strictly more then five, $\{1,2,3\} \in \mu(P)$ and $4 \notin \mu(P)$ and $5 \notin \mu(P)$. We see that $\mu(X[3])$ is bigger than any GR measure of a preprojective module. Futhermore, $X[3]$ injects into $X[i]$ for any $i > 3$. In summary, for $i > 3$ and a preprojective module $P$, we have

$$\mu(P) < \{1,2,3,5\} = \mu(X[3]) \leq \mu(X[i-1]),$$

showin that $P$ cannot be a GR-submodule of $X[i]$. Thus, $X[i-1]$ is the GR submodule of $X[i]$. For $i \geq 2$, we have

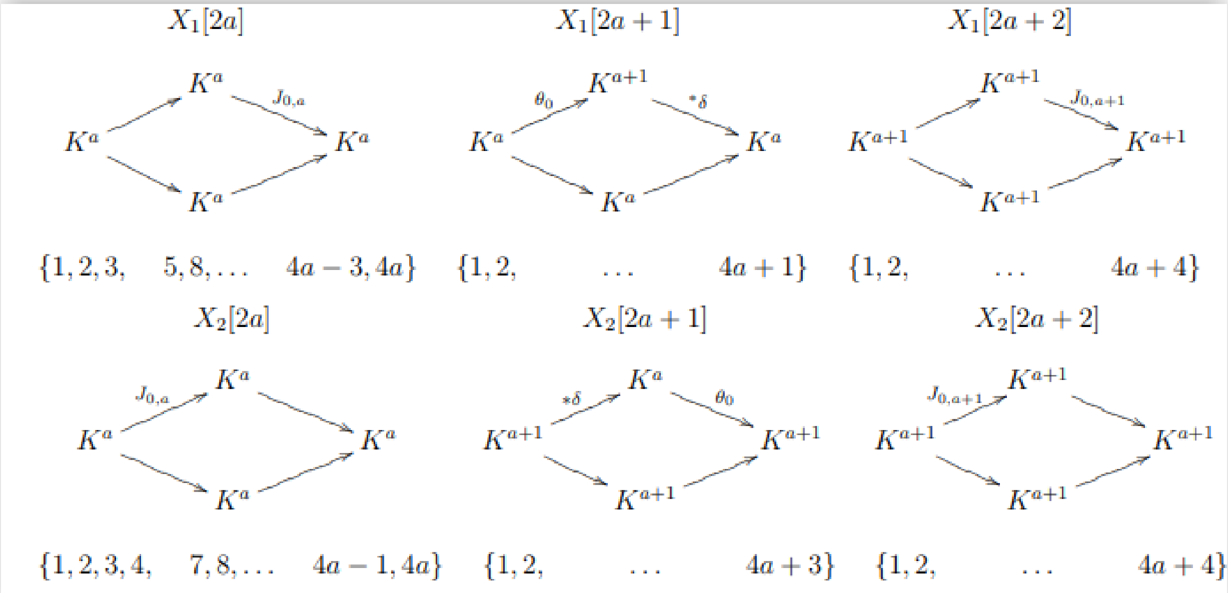
$$\mu(X[2i]) = \{1,2,3,5,8,9,12,\ldots,4(i-1),4(i-1)1,4i\}$$
$$\mu(X[2i+1]) = \{1,2,3,5,8,9,12,\ldots,4(i-1),4(i-1)1,4i,4i+1\}.$$

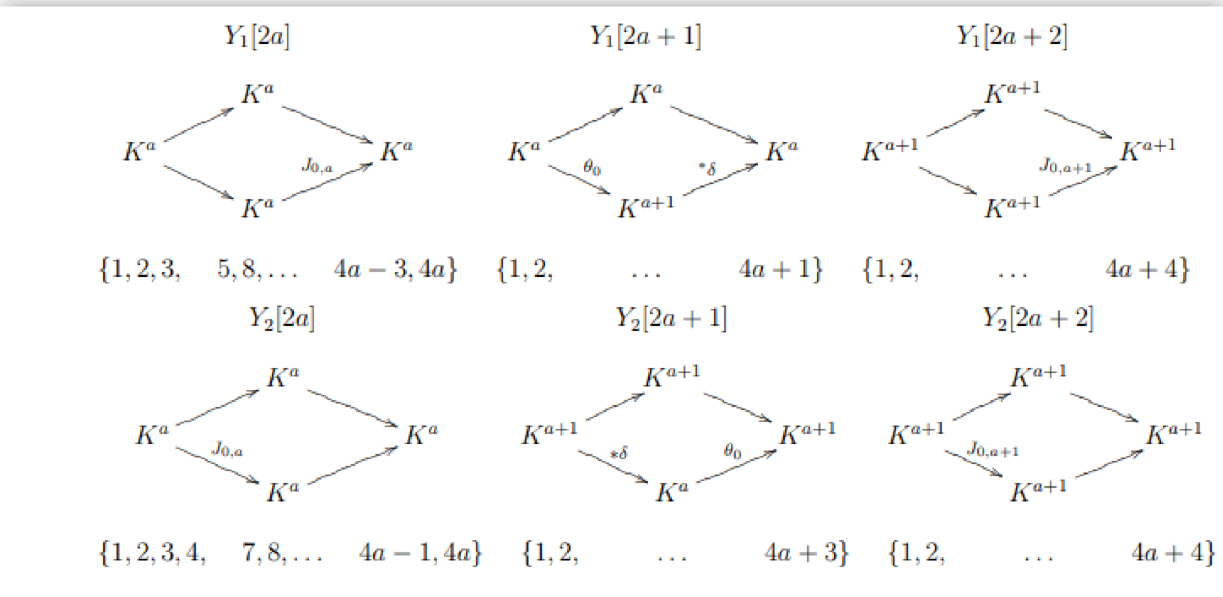
As we have seen, it was enough to specify finitely many regular modules to calculate all their GR measures. However, we need to specify the exceptional ones for calculating GR measures of preinjective modules. When classifying exceptional regular modules, we will use the GR measure to decide whether a module with dimension vector $i\delta$ is homogenous. Other methods exist.

We use the following notation: when the description of the map is omitted, it is an identity map. A map $*\delta$ ($\delta *$) erases the first (last) coordinate of the vector, and a map ${}_0\theta$ ($\theta_0$) adds zero at the beginning (end) of a vector. $J_{\lambda,a}$ is the Jordan block of dimension $a$ with eigenvalue $\lambda$. Recall that $J_{0,a}(k_1,\ldots,k_a) = (k_2,\ldots,k_a,0)$.

As for irreducible morphisms, if $\phi\colon N \to M$ is a monomorphism between two consecutive modules $N, M$, then $\phi_x$ is an identity map if $N_x = M_x$. If $N_x = K^a$ and $M_x = K^{a+1}$, then $phi_x$ is the map $\theta_0$.

For $a \leq 1$, the GR measure may differ, and not all modules are defined.



It is worth justifying that the presented modules with dimension vector of form $i\delta$ are indeed exceptional. Modules $X_1[2j]$ and $Y_1[2j]$, for $j \geq 1$, cannot be homogenous because they inject in the modules $X_1[2j+1]$ and $Y_1[2j+1]$. This would contradict the calculated GR measure of $X_1[2j+1]$ and $Y_1[2j+1]$.

Modules $X_2[2j]$ and $Y_2[2j]$, for $j \geq 1$, cannot be homogenous because modules $X_2[2j-1]$ and $Y_2[2j-1]$ inject into them. This would contradict the formula for the GR measure of homogenous modules.

It remains to prove that the presented modules are indeed indecomposable. We only verify the case when the dimension vector is a multiple of $\delta$. The remaining cases can be proven using the analogue of Lemma 49.

Recall, that if an endomorphism ring of a $K$-representation is local, then the representation is indecomposable. Let $E$ be one of the representations marked as $X_i[2a]$, $Y_i[2a]$, $i \in \{1,2\}$. Take $\phi = (\phi_1, \dots, \phi_4) \in End\ E$. Then all four maps $\phi_i$ are equal because three structural maps of $E$ are identity maps.

We can thus identify ring $End\ E$ with the ring of matrices commuting with the Jordan block matrix $J_{0,a}$. Hence, $End(R)$ is isomorphic to the ring $R$ of matrices polynomial in $J_{0,a}$. There are two possibilities for a matrix $A$ that is a power of $J_{0,a}$. It can be the zero matrix of dimension $a$. Or there is $m \leq a$ such that values $a_{ij}$ for $i - j = m$ are 1, and the remaining entries are zeroes. In particular, $R$ consists of upper triangular matrices, with all elements on the main diagonal equal. Such a matrix is invertible if and only if the value on the diagonal is non-zero. If the matrix $A$ is not invertible, then $Id - A$ is. Hence the ring $End\ E$ is local.

## 3.5 Preinjective modules

An indecomposable module is preinjective if its defect is 1. In this case, it means that for a preinjective $I$ such that $dim(I) = [x_1, x_2, x_3, x_4]$ we have $x_4 < x_1$. This gives us families $I_1(a)$, $I_4(a)$, $I_\gamma(a)$ and $I_\delta(a)$ for non-negative $a$. By Gabriel's theorem, preinjectives are determined (up to isomorphism) by their dimension vector. From now on, we denote preinjective modules by their dimension vectors.

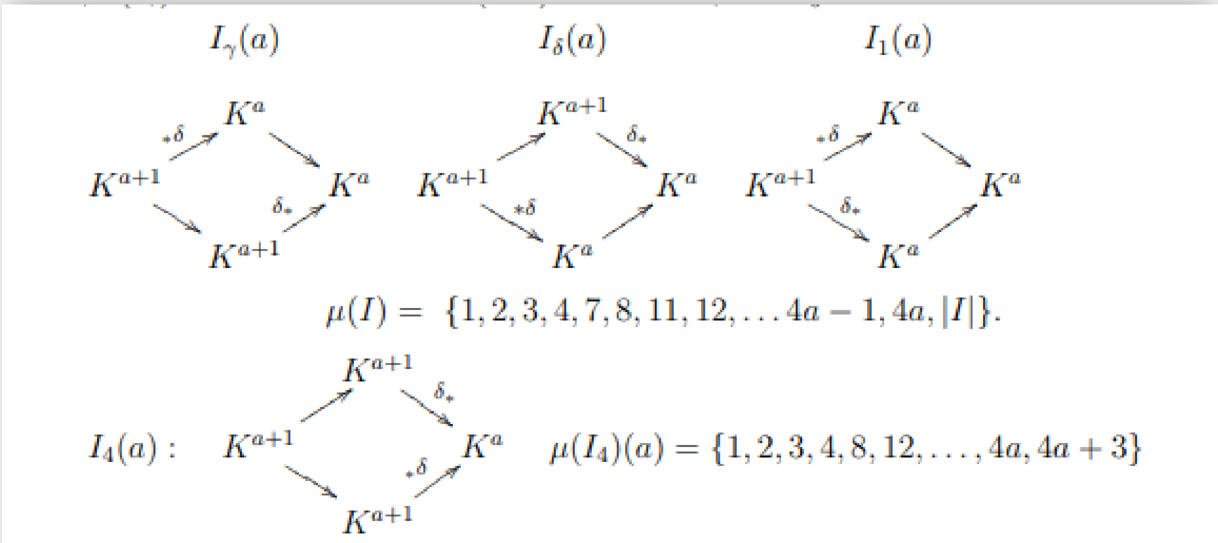
Recall from section 1.5 that the GR-measure of an indecomposable injective module $I$ is $\{1, 2, \dots, |I|\}$. There are four indecomposable injective modules $I_1(0)$ , $I_\delta(0)$, $I_\gamma(0)$ and $I_1(1)$ with lengths with lengths $1, 2, 2$ and $5$ respectively.

For any quiver $Q$ of type $\tilde{A}_n$, all irreducible morphisms between preinjective modules are epimorphisms, [6, Cor. 2.3.]. Recall that any map from a preprojective module to a preinjective one factors through a direct sum of regular modules. We see that all GR submodules of preinjective modules are regular modules. When investigating possible regular GR submodules, it is enough to concentrate on only a few thanks to the following theorem, [7, Thm. 4.1.].

**Theorem 48.** *Let $Q'$ be a quiver of type $\tilde{A}_n$. Assume that $N \subsetneq M$ is a GR inclusion in mod-$KQ'$.*

*Then $M/N$ is an indecomposable thin representation.*

We now list all indecomposable preinjective modules with their GR measures. The proof that given modules are indeed indecomposable is left for the end of the section. Whenever the structure map is not specified, it is an identity map. Maps ${}_*\delta$ ($\delta_*$) erase a vector's first (last) coordinate, and $\theta_0$ adds 0 at the end of a vector.




We start with $I_\delta(a)$ with dimension vector $[a+1, a+1, a, a]$ and $a > 0$. Consider regular modules $R$ with $dim(R) \leq dim(I_\delta(a))$. Maximal GR measure exists among these modules, namely $\{1, 2, 3, 4, \dots, 4a-1, 4a\}$. It is the GR measure for modules $X_2[2a]$ and $Y_2[2a]$. There exists a monomorphism

$$\phi\colon Y_2[2a] \to I_\delta(a) \qquad \phi_x = \begin{cases} \theta_0 & x \in \{1,2\}, \\ id & x \in \{3,4\}. \end{cases}$$

The case with $I_\gamma(a)$, for $a > 0$ is analogous. Exceptional module $X_2[2a]$ injects into $I_\gamma(a)$ via

$$\Phi\colon X_2[2a] \to I_\gamma(a) \qquad \phi_x = \begin{cases} \theta_0 & x \in \{1,3\}, \\ id & x \in \{2,4\}. \end{cases}$$

In the similar way, $X_2[2a]$ injects into $I_1(a)$. If we switch maps $\alpha$ and $\beta$ in $I_1(a)$, we get an isomorphic preinjective module. Using identities and $\theta_0$, can inject $Y_2[2a]$ into $I_1(a)$. We see that $I_1(a)$ has two GR submodules for $a > 1$.

It remains to investigate the case $I_4(a)$. Module $I_4(0)$ is thin. We can calculate that its GR measure is $\{1, 3\}$ using the algorithm from Section 2.1.2.

Let us assume $a \geq 1$ and let $M$ be a GR submodule of $I_1(a)$. We know that $dim(I_1(a)/M) \leq \delta$ hence $M$ is either a regular module with dimension vector $a\delta$ or one of the exceptional modules $X_1[2a+1]$ and $Y_1[2a+1]$.

The maps $I_4(a)_\alpha$ and $I_4(a)_\beta$ are injective. If $M$ injects into $I_4(a)$, then also $M_\alpha$ and $M_\beta$ have to be injective. We see that neither $X_2[2a]$ nor $Y_2[2a]$ can inject into $I_4(a)$.

Let $H$ be the thin homogenous module such that all structure maps are identities. Module $H$ injects into $I_4(a)$ via the morphism $\phi$, where $\phi_1 = \phi_2 = \phi_3$ send $k \in K$ to an $a$ copies of $k$ and $\phi_4$ sends $k$ to $a+1$ copies of $k$. Thus

$$\{1, 2, 3, 4\} = \mu(H) \leq \mu(I_4(a)) = \mu(M) \cup \{4a+3\}.$$

In particular, $\{1, 2, 3, 4\} \subseteq \mu(M)$. We see that $M$ cannot be $X_1[2a+1]$ or $Y_1[2a+1]$. We conclude that $M$ is a homogenous regular module of length $4a$.

We end this section by justifying that the presented modules are indeed indecomposable and thus preinjective. Using the following lemma, we can identify the ring of endomorphisms of given modules with the ring of diagonal matrices with all values on the diagonal equal. Such a ring is isomorphic to the field $K$, hence local.

**Lemma 49.** *Let $\phi$ and $\psi$ be two square matrices over a field $K$ with dimensions $a$ and $a+1$, respectively. Further, suppose that the following squares commute:*

$$\begin{array}{ccc} K^{a+1} & \xrightarrow{\delta_*} & K^a \\ \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\phi} \\ K^{a+1} & \xrightarrow{\delta_*} & K^a \end{array} \qquad \begin{array}{ccc} K^{a+1} & \xrightarrow{{}_*\delta} & K^a \\ \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\phi} \\ K^{a+1} & \xrightarrow{{}_*\delta} & K^a \end{array}.$$

*(a) $\phi$ is equal to the principal submatrix obtained from $\psi$ by removing the last column and last row.*

*(b) $\phi$ is equal to the principal submatrix obtained from $\psi$ by removing the first column and first row.*

*(c) $\psi$ is a diagonal matrix with all values on the diagonal equal.*

*Proof.* Parts (a) and (b) follow directly from the two commuting squares. In the last column of $\psi$, all but the last entry are zero (left square). Similarly, in the first column of $\psi$, all but the first entry are zeroes (right square).

□

## 3.6 Partition of mod-KQ

The take-off part of *mod-A* consists of preprojective modules, one simple injective, two simple regular modules and exceptional modules $X_1[2]$, $Y_1[2]$, $X_2[1]$, $Y_2[1]$. The take-off measures are

$$\begin{gathered}\{1\},\ \{1,2\},\ \{1,2,5\},\ \{1,2,4\},\\ \{1,2,3\},\ \{1,2,3,6\},\ \{1,2,3,6,9\},\\ \{1,2,3,6,7\},\ \{1,2,3,6,7,10\},\ \{1,2,3,5,6,7,10,13\},\\ \{1,2,3,6,7,10,11\},\ \{1,2,3,6,7,10,11,14\},\ \{1,2,3,6,7,10,11,14,17\},\\ \dots\\ \{1,2,3,6,7,\dots,4a-2,4a-1\},\ \{1,\dots,7,\dots,4a-2,4a-1,4a+2\},\\ \{1,\dots,4a-2,4a-1,4+2,4a+5\},\ \{1,\dots,4a-1,4a+2,4a+3\},\dots\end{gathered}$$

The landing part consists of preinjective modules $I_\gamma(a)$, $I_\delta(a)$ and $I_1(a)$ for $a > 0$. For two preinjective modules $I$, $J$ we have $\mu(I) \leq \mu(J)$ if and only if $|I| \geq |J|$. This holds for any acyclic quiver of type $\tilde{A}_n$, [9, Prop. 5.3.].

The central part consists of three GR segments. We list them in ascending order. The first two segments consist of the chain of subsets, so we only list the first element and then the sequence of new elements. The first segment corresponds to GR measures of modules $X_1[i]$ and $Y_1[i]$ for $i \geq 3$, and the second corresponds to $X_2[2]$ and $Y_2[2]$ and homogenous modules.

$$\begin{gathered}\{1,2,3,5\}\ 8\ 9\ 12\ \dots\ 4a-3\ 4a\ 4a+1\ 4(a+1)\ \dots\\ \{1,2,3,4\}\ 8\ 12\ \dots\ 4a\ 4(a+1)\ 4(a+2)\ \dots\end{gathered}$$

The third segment corresponds to the GR measures of the preinjective modules $I_4(a)$, for $a \geq 1$, and exceptional modules $X_2[i]$ and $Y_2[i]$ for $i \geq 3$. Note that central preinjective modules have smaller measures than infinitely many exceptional modules. The segment is unbounded from both sides.

$$\begin{gathered}\dots\{1,2,3,4,8,12,\dots,4a,4a+3\}\dots\{1,2,3,4,8,11\},\ \{1,2,3,4,7\}\\ \{1,2,3,4,7,8\}\ 11\ 12\ 15\ \dots\ 4(a-1)+3\ 4a\ 4a+3\ \dots\end{gathered}$$

Note that a central GR measure is not finite-type if and only if it corresponds to modules with dimension vector $i\delta$. There are three GR measures without a direct predecessor: the trivial case $\{1\}$, the minimal central measure $\{1,2,3,5\}$ and the measure $\{1,2,3,4\}$.

# Conclusion

The standard GR measure, as defined by Gabriel in [5], has a straightforward generalisation to modules of arbitrary length, as shown by Ringel in [21]. Krause utilised it in his study of *Ziegler closed* subsets of the set of isomorphism classes of indecomposable pure-injective $A$-modules for an artin algebra $A$. However, most existing literature on the subjects deals only with finite-length modules.

This thesis studied the Gabriel-Rotier measure, as defined by Krause in 8, of finite-dimensional representations of finite acyclic quivers. Particular attention was paid to results that can be obtained combinatorially, working only with subquivers and dimension vectors. This, among other considerations, motivated the choice of Krause's more general definition of the GR measure, despite staying in the setting of [21]. It was shown that for a thin representation of a tree, any GR measure can be calculated only by working with the quiver. There are some limitations to this approach. Section 3.5 suggests that GR measures of preinjective representations can generally only be calculated by referencing structural maps.

In Chapter 2, new theoretical results about alternative GR measures were presented. For a thin representation of a tree, any indecomposable filtration is a GR filtration for some suitable choice of a length function, Theorem 38. In Section 2.2, lengths of GR filtrations were studied. It was shown that while many results about standard GR inclusions can be easily transformed for general GR measures, the lengths of filtrations behave rather wildly, except for the trivial case of uniform modules.

GR measures for various representations of quivers of type $A_n$ were calculated in Sections 1.4 and 2.2. GR measure for all indecomposable representations of the orientation of $D_4$ with three sinks and one source was calculated in Example 29. In Chapter 3, the standard GR measure for all finite-dimensional representations of $\tilde{A}_3$ with one source and one non-adjacent sink was calculated. Similar methods can be used for any quiver of type $\tilde{A}_n$.

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