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Analysis of variations of stochastic integrals

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Abstract: In this thesis, we study the $1/H$-variations of stochastic integrals, where the integrators are the fractional Brownian motion and Rosenblatt process (with the Hurst parameter $H > 1/2$). The considered stochastic integrals are defined as the Skorokhod integrals within the framework of Malliavin calculus. We summarize the already established results about the $1/H$-variation of the integral with respect to the fractional Brownian motion and then apply the techniques used therein to obtain the form of the $1/H$-variation of the integral with respect to the Rosenblatt process.

Keywords: p-variation, stochastic integral, fractional Brownian motion, Rosenblatt process

Contents

Introduction

Self-similar processes are processes whose distribution is invariant under a suitable scaling of time and space. The most prominent class of self-similar processes is the family of fractional Brownian motions. The fractional Brownian motion, parameterized by the Hurst parameter $H \in (0,1)$, is a Gaussian self-similar process with stationary increments, which, for $H \in (\frac{1}{2}, 1)$, exhibits long-range dependence.

The Rosenblatt process, indexed by the Hurst parameter $H \in (\frac{1}{2}, 1)$, is a stochastic process that arises as a limit of normalized sums of long-range dependent random variables in the so-called Non-central limit theorem ([1, 2]). It is a self-similar, long-term memory process with stationary increments, but, contrary to the fractional Brownian motion, it is not Gaussian. As such, it poses a suitable alternative as a model to the fractional Brownian motion in cases where the system shows some clear signs of non-Gaussianity.

In [3], it has been established that for the p-variation of the fractional Brownian motion $Z^{H,1}$, it holds

$$\sum_{i=0}^{n-1} |Z^{H,1}_{t^n_{i+1}} - Z^{H,1}_{t^n_i}|^p \xrightarrow[n\to\infty]{\mathbb{P}} \begin{cases} 0, & p > \frac{1}{H}, \\ T\,\mathbb{E}|Z^{H,1}_1|^{\frac{1}{H}}, & p = \frac{1}{H}, \\ \infty, & p < \frac{1}{H}, \end{cases} \tag{1}$$

where $\{t^n_i\}_{i=0}^n$ is a suitable sequence of partitions of interval $[0,T]$ whose mesh tends to zero. Later on, for $H \in (\frac{1}{2}, 1)$, this result has been extended by Guerra and Nualart [4] to $1/H$-variation of stochastic integral with respect to the fractional Brownian motion, where the stochastic integral is defined as the divergence-type integral in the framework of the Malliavin calculus. In particular, they showed that the $1/H$-variation of the integral $\int_0^\cdot u_s \,\mathrm{d}Z^{H,1}_s$ is equal to $c_H \int_0^T |u_s|^{1/H}\,\mathrm{d}s$, where $c_H = \mathbb{E}|Z^{H,1}_1|^{1/H}$.

On the other hand, it has been shown (see [5, Proposition 2.3]) that for the $1/H$-variation of the Rosenblatt process $Z^{H,2}$, we have

$$\sum_{i=0}^{n-1} |Z^{H,2}_{t^n_{i+1}} - Z^{H,2}_{t^n_i}|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathbb{P}} T\,\mathbb{E}|Z^{H,2}_1|^{\frac{1}{H}}. \tag{2}$$

Similar to the fractional Brownian motion, one can use the Malliavin calculus to define a stochastic integral with respect to the Rosenblatt process (see e.g. [6]). In view of (1), (2), and the common properties of the fractional Brownian motion and Rosenblatt process, this brings up the question whether we can expect a result similar to the one by Guerra and Nualart also for stochastic integrals with respect to the Rosenblatt process.

The thesis consists of four chapters. Chapter 1 contains a summary of basic notions of Malliavin calculus such as the Malliavin derivative, divergence operator and multiple integral. In Chapter 2, we introduce the fractional Brownian motion and Rosenblatt process in the more general context of the Hermite processes and derive some of their properties. In Chapter 3, we use the Malliavin calculus to first construct an integral with respect to the Hermite process for deterministic integrands. Subsequently, we extend this definition and develop a stochastic integral with respect to the fractional Brownian motion and Rosenblatt process.

The majority of the original work lies in Chapter 4. Based on the techniques used by Guerra and Nualart in [4], we show that for a suitable class of integrands the stochastic integral with respect to the Rosenblatt process $\int_0^\cdot u_s \, \mathrm{d}Z_s^{H,2}$ has a finite $1/H$-variation of the form

$$C_H \int_0^T |u_s|^{\frac{1}{H}} \, \mathrm{d}s,$$

where $C_H = \mathbb{E}|Z_1^{H,2}|^{1/H}$.

1. Malliavin calculus

The Malliavin calculus (also known as the stochastic calculus of variations) is an infinite-dimensional differential calculus on a Gaussian space. Originally developed by Paul Malliavin [7] to obtain a probabilistic proof of the Hörmander theorem, the theory found an extensive application in other areas (e.g. in probabilistic approximations [8] or in anticipating stochastic calculus [9, Chapter 3]).

In this chapter, we summarize some basic notions and results of the Malliavin calculus which will be useful for the goals of this thesis. For an exhaustive explanation of the topic, we refer to [8, 9, 10]. The majority of the mentioned results will be stated without proofs; for the proofs, consult the reference accompanying each result.

1.1 Wiener-Itô chaos decomposition

We start by introducing the notion of an isonormal Gaussian process which plays the role of random noise in the presented theory. Throughout the whole text, we will be working on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$.

Definition 1.1. A family of random variables $W = \{W(h), h \in H\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an isonormal Gaussian process if W is a centered Gaussian family such that $\mathbb{E}(W(h)W(g)) = \langle h, g \rangle_H$ for all $h, g \in H$.

It is easy to show that the mapping $h \mapsto W(h)$ is linear. Also, given a real separable Hilbert space H, one can always construct a probability space and a Gaussian family $\{W(h), h \in H\}$ satisfying the conditions of Definition 1.1 (see [8, Proposition 2.1.1]).

The isonormal Gaussian process encodes a large class of random objects. For instance (see [8, Example 2.1.5]), any centered Gaussian process with covariance function R is an isonormal Gaussian process indexed by Hilbert space H, which is defined as the closed span of indicator functions with respect to the inner product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_H = R(t, s).$$

Another (and for the purposes of this thesis the most important) example is the isonormal Gaussian process which arises from the Wiener process.

Let $(W_t)_{t \in J}$ be a Wiener process (here, J can be an arbitrary bounded or unbounded interval, in most cases either $\mathbb{R}$ or $[0, T]$ for some $T > 0$), that is, $(W_t)_{t \in J}$ is a centered, continuous Gaussian process with the covariance function

$$R(s, t) = \frac{1}{2} \left(|s| + |t| - |s - t| \right), \quad s, t \in J.$$

Then $(W_t)_{t \in J}$ can be associated with an isonormal Gaussian process indexed by the Hilbert space $L^2(J)$ in the following way. For any $a, b \in J, a < b$, we set

$$W(\mathbf{1}_{[a,b)}) = W_b - W_a.$$

Denote $\mathcal{E}_1$ the space of functions of form

$$h(t) = \sum_{i=0}^{m-1} a_i \mathbf{1}_{[s_i, s_{i+1})}(t), \tag{1.1}$$

where $m \in \mathbb{N}, a_i \in \mathbb{R}$ and $s_i \in J$ such that $[s_i, s_{i+1})$ are disjoint intervals. For $h \in \mathcal{E}_1$ of the form (1.1), we set

$$W(h) = \sum_{i=0}^{m-1} a_i W(\mathbf{1}_{[s_i, s_{i+1})}). \tag{1.2}$$

Then we have

$$\mathbb{E}(W(h))^2 = \sum_{i=0}^{m-1} a_i^2 \mathbb{E}[W(\mathbf{1}_{[s_i,s_{i+1})})]^2 = \sum_{i=0}^{m-1} a_i^2 (s_{i+1} - s_i) = \int_J h^2(t)\, \mathrm{d}t = \|h\|^2_{L^2(J)}. \tag{1.3}$$

Since the space $\mathcal{E}_1$ is dense in $L^2(J)$, then, by (1.3), the mapping $h \mapsto W(h)$ can be extended to an isometry between $L^2(J)$ and a subspace of $L^2(\Omega)$ (see [5, Section 1.1.1]) which we will denote by $\mathcal{H}_1$. We will also denote this isometry by $W(h)$ or by

$$\int_J h(s)\, \mathrm{d}W_s.$$

This isometry is called *the Wiener integral with respect to the Brownian motion* (or simply the Wiener integral if there is no risk of confusion).

Consequently, the family $\{W(h), h \in L^2(J)\}$ is an isonormal Gaussian process, as it is clearly centered and Gaussian (since any $W(h)$ is a $L^2(\Omega)$-limit of jointly Gaussian random variables of form (1.2)) and for any $h, g \in L^2(J)$ it holds

$$\mathbb{E}W(h)W(g) = \langle f, g \rangle_{L^2(J)}.$$

In what follows, we will only consider the case of the isonormal Gaussian process given by the Wiener integral. Note, however, that all of the following theory can be generalized to the case of arbitrary isonormal Gaussian process.

Definition 1.2. For $n \in \mathbb{N}_0$, the n-th *Hermite polynomial* H_n is defined by $H_0(x) = 1$ and

$$H_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

For $n \in \mathbb{N}_0$, we will denote by $\mathcal{H}_n$ the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables $\{H_n(W(h)), h \in L^2(J), \|h\|_{L^2(J)} = 1\}$. The space $\mathcal{H}_n$ is called *the n-th Wiener chaos*. Clearly $\mathcal{H}_0 = \mathbb{R}$ is the subspace of constants and $\mathcal{H}_1 = \{W(h), h \in L^2(J)\}$ is the subspace of Gaussian random variables. The Wiener chaos provide an orthogonal decomposition of the space $L^2(\Omega, \mathcal{F}^W, \mathbb{P})$. Here, by $\mathcal{F}^W$, we denote the σ-algebra generated by the Wiener process $(W_t)_{t \in J}$.

Theorem 1.1 (Wiener-Itô chaos decomposition, [8, Theorem 2.2.4])**.** *The space $L^2(\Omega, \mathcal{F}^W, \mathbb{P})$ admits the following orthogonal decomposition:*

$$L^2(\Omega, \mathcal{F}^W, \mathbb{P}) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

In particular, any $F \in L^2(\Omega, \mathcal{F}^W, \mathbb{P})$ can be represented as

$$F = \mathbb{E}F + \sum_{n=1}^{\infty} F_n,$$

where $F_n \in \mathcal{H}_n$ and where the sum converges in $L^2(\Omega)$.

1.2 Malliavin derivative

In this section, the notion of derivative of random variables in $L^2(\Omega)$ is introduced. Since we do not impose any conditions on the topological structure of the probability space, the derivative is defined in a weak sense. Consider a space of "smooth" random variables

$$\mathcal{S} = \Big\{ F = f\Big(W(h_1), \dots, W(h_m)\Big), m \in \mathbb{N}, f \in \mathcal{C}_p^\infty(\mathbb{R}^m), h_i \in L^2(J) \Big\}, \tag{1.4}$$

where $C_p^\infty(\mathbb{R}^m)$ denotes the space of infinitely continuously differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$ such that f and all its partial derivatives have at most polynomial growth, that is,

$$\mathcal{C}_p^\infty(\mathbb{R}^m) = \Big\{ f \in C^\infty(\mathbb{R}^m) : \forall k \in \mathbb{N}_0 \, \exists \alpha, \beta \in (0, \infty) : \\ |f^{(k)}(x)| \leq \alpha(1 + |x|^\beta) \;\; \forall x \in \mathbb{R} \Big\}.$$

Note that the space $\mathcal{S}$ is dense in $L^p(\Omega)$ for any $p \in [1, \infty)$.

Let $F \in \mathcal{S}$ be of the form as in (1.4) and $n \in \mathbb{N}$. Then the *n-th Malliavin derivative of* F is defined as the element of $L^2(\Omega; L^2(J^n))$ given by

$$D^n_{x_1,\dots,x_n} F = \sum_{i_1,\dots,i_n=1}^{m} \left(\partial^n_{i_1,\dots,i_n} f \right) \Big(W(h_1), \dots, W(h_m) \Big) h_{i_1}(x_1) \dots h_{i_n}(x_n).$$

The n-th Malliavin derivative can also be viewed as an element of $L^2(\Omega \times J^n)$ or $L^2(J^n; L^2(\Omega))$, since by Fubini theorem we have

$$\mathbb{E} \int_{J^n} |g_s|^2 \, \mathrm{d}s = \int_{J^n \times \Omega} |g|^2 \, \mathrm{d}(\lambda^n \otimes \mathbb{P}) = \int_{J^n} \mathbb{E} |g_s|^2 \, \mathrm{d}s$$

and so the three spaces are isomorphic.

The following proposition allows us to extend the definition of the Malliavin derivative to a larger class of random variables.

Proposition 1.2 ([8, Proposition 2.3.4])**.** *Let $p \in [1, \infty)$ and $n \in \mathbb{N}$. Then the operator $D^n : L^p(\Omega) \supset \mathcal{S} \to L^p(\Omega; L^2(J^n))$ is closable; i.e., whenever there is a sequence $\{F_k\} \subseteq \mathcal{S}$ such that $F_k \to 0$ in $L^p(\Omega)$ and $D^n F_k \to \eta$ in $L^p(\Omega; L^2(J^n))$ for some η , then $\eta = 0$ a.s.*

Definition 1.3. For $p \in [1, \infty)$ and $n \in \mathbb{N}$, *the Sobolev-Watanabe space* $\mathbb{D}^{n,p}$ is defined as the closure of $\mathcal{S}$ with respect to the norm

$$\begin{aligned} \|F\|_{\mathbb{D}^{n,p}} &= \Big(\mathbb{E}|F|^p + \mathbb{E}\|DF\|^p_{L^2(J)} + \dots + \mathbb{E}\|D^n F\|^p_{L^2(J^n)} \Big)^{\frac{1}{p}} \\ &= \left(\mathbb{E}|F|^p + \mathbb{E} \left(\int_J |D_x F|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} + \dots + \mathbb{E} \left(\int_{J^n} |D^n_x F|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}. \end{aligned}$$

We additionally set $\mathbb{D}^{0,p} = L^p(\Omega)$ and $\|F\|_{\mathbb{D}^{0,p}} = \|F\|_{L^p(\Omega)}$.

Due to Proposition 1.2, the operator D^n can be consistently extended to the space $\mathbb{D}^{n,p}$. Moreover, from the form of the norm of Sobolev-Watanabe space and the usual embedding of $L^p(\Omega)$ spaces we have the following embedding

$$\mathbb{D}^{n,p} \hookrightarrow \mathbb{D}^{m,q}, \tag{1.5}$$

whenever $m \leq n, q \leq p$.

What follows is one of the chain rules for the Malliavin derivative.

Proposition 1.3 ([8, Proposition 2.3.7])**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with bounded derivative. Suppose that $F \in \mathbb{D}^{1,p}$ for some $p \in [1, \infty)$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and*

$$D\varphi(F) = \varphi'(F)DF.$$

The chain rule can also be applied to other classes of functions such as Lipschitz functions (see [8, Proposition 2.3.8]) or continuously differentiable functions with derivatives of polynomial growth (see [10, Proposition 3.3.2]).

The Malliavin derivative possesses a local property.

Proposition 1.4 ([9, Proposition 1.3.16])**.** *Let $F \in \mathbb{D}^{1,1}$ be a random variable such that $F = 0$ almost surely on some set $A \in \mathcal{F}$. Then $DF = 0$ almost surely on A.*

Remark 1.1. The notion of the Malliavin derivative can be extended to Hilbert space-valued random variables. Let $\mathcal{V}$ be a real separable Hilbert space and set

$$\mathcal{S}(\mathcal{V}) = \left\{ F = \sum_{j=1}^{m} F_j v_j, m \in \mathbb{N}, F_j \in \mathcal{S}, v_j \in \mathcal{V} \right\}. \tag{1.6}$$

For $F \in \mathcal{S}(\mathcal{V})$ of form (1.6), *the n-th Malliavin derivative* is defined by

$$D^n F = \sum_{j=1}^{m} (D^n F_j) \otimes v_j,$$

where $\otimes$ denotes the tensor product between two Hilbert spaces. Similarly as for the real-valued random variables the operator D^n is closable from $\mathcal{S}(\mathcal{V})$ to $L^p(\Omega; L^2(J^n) \otimes \mathcal{V})$ and consequently can be extended to the space $\mathbb{D}^{n,p}(\mathcal{V})$ which is defined as closure of $\mathcal{S}(\mathcal{V})$ with respect to the norm

$$\|F\|_{\mathbb{D}^{n,p}(\mathcal{V})} = \left(\mathbb{E}\|F\|_{\mathcal{V}}^p + \mathbb{E}\|DF\|_{L^2(J)\otimes\mathcal{V}}^p + \ldots + \mathbb{E}\|D^n F\|_{L^2(J^n)\otimes\mathcal{V}}^p \right)^{\frac{1}{p}}.$$

An embedding similar to (1.5) also holds for the spaces $\mathbb{D}^{n,p}(\mathcal{V})$.

1.3 Divergence operator

The divergence operator δ is defined as the adjoint of the Malliavin derivative. In the case of the isonormal Gaussian process indexed by $L^2(J)$, the operator can be interpreted as a stochastic integral. In fact, it can be shown that on processes adapted to the filtration generated by the Wiener process $(W_t)_{t \in J}$, the divergence operator coincides with the standard Itô integral.

Let $n \in \mathbb{N}$ and denote by $\operatorname{Dom} \delta^n$ the subset of $L^2(\Omega; L^2(J^n))$ consisting of processes u for which there is a constant $c \in (0, \infty)$ such that

$$|\mathbb{E} \langle D^n F, u \rangle_{L^2(J^n)}| = \left| \mathbb{E} \int_{J^n} (D_t^n F) u_t \, \mathrm{d}t \right| \le c \|F\|_{L^2(\Omega)}. \tag{1.7}$$

for every $F \in \mathcal{S}$. Then for any $u \in \operatorname{Dom} \delta^n$, by virtue of (1.7), the linear operator

$$F \mapsto \mathbb{E} \langle D^n F, u \rangle_{L^2(J^n)}$$

is continuous from $(\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$ to $\mathbb{R}$ and so it can be extended to a linear operator from $L^2(\Omega)$ to $\mathbb{R}$. According to the Riesz representation theorem, there is a unique element of $L^2(\Omega)$, which we will denote $\delta^n(u)$, satisfying the relation

$$\mathbb{E}F\delta^n(u) = \mathbb{E}\left\langle D^n F, u\right\rangle_{L^2(J^n)} \tag{1.8}$$

for every $F \in \mathcal{S}$. This allows us to state the following definition.

Definition 1.4. Let $u \in \operatorname{Dom}\delta^n$. Then $\delta(u)$ is the unique element of $L^2(\Omega)$ characterized by (1.8) for every $F \in \mathcal{S}$. The operator $\delta^n : \operatorname{Dom}\delta^n \to L^2(\Omega)$ is called *the divergence operator of order* n and we talk about the set $\operatorname{Dom}\delta^n$ as the *the domain of* δ^n.

Remark 1.2. The divergence operator is clearly, as the adjoint of closed linear operator D^n, also closed and linear. By taking $F = 1$ in (1.8), we obtain $\mathbb{E}\delta^n(u) = 0$ for any $u \in \operatorname{Dom}\delta^n$. Moreover, it can be shown (e.g. [8, Remark 2.5.3]) that $L^2(J^n) \subseteq \operatorname{Dom}\delta^n$ and $\delta(g) = W(g)$ for any $g \in L^2(J)$. Hence we see, that the divergence operator coincides with the Wiener integral for deterministic processes.

The following proposition allows us to factor out scalar random variables in the divergence operator.

Proposition 1.5 ([9, Proposition 1.3.3])**.** *Let $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta$ such that $Fu \in L^2(\Omega; L^2(J))$. Then $Fu \in \operatorname{Dom}\delta$ and*

$$\delta(Fu) = F\delta(u) - \left\langle DF, u\right\rangle_{L^2(J)},$$

provided that the right-hand side belongs to $L^2(\Omega)$.

The next proposition provides a large class of processes which belong to $\operatorname{Dom}\delta$ as well as a formula for the covariance of two divergences.

Proposition 1.6 ([9, Proposition 1.3.1])**.** *The space $\mathbb{D}^{1,2}(L^2(J))$ is included in $\operatorname{Dom}\delta$. Moreover, for any $u, v \in \mathbb{D}^{1,2}(L^2(J))$, we have the equality*

$$\mathbb{E}\Big(\delta(u)\delta(v)\Big) = \mathbb{E}\int_J u_s v_s \,\mathrm{d}s + \mathbb{E}\int_J\int_J D_x u_y D_y v_x \,\mathrm{d}x\,\mathrm{d}y. \tag{1.9}$$

Note that the space $\mathbb{D}^{1,2}(L^2(J))$ can be identified with the space $L^2(J; \mathbb{D}^{1,2})$ since

$$\begin{aligned}
\|g\|^2_{\mathbb{D}^{1,2}(L^2(J))} &= \mathbb{E}\|g\|^2_{L^2(J)} + \mathbb{E}\|Dg\|^2_{L^2(J^2)} \\
&= \mathbb{E}\int_J |g_s|^2 \,\mathrm{d}s + \mathbb{E}\int_J\int_J |D_x g_s|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&= \int_J \left(\mathbb{E}|g_s^2| + \mathbb{E}\int_J |D_x g_s|^2 \,\mathrm{d}x\right) \mathrm{d}s \\
&= \int_J \|g_s\|^2_{\mathbb{D}^{1,2}} \,\mathrm{d}s \\
&= \|g\|^2_{L^2(J;\mathbb{D}^{1,2})}.
\end{aligned}$$

Similar to the Malliavin derivative, the divergence also has a local property.

Proposition 1.7 ([9, Proposition 1.3.15])**.** *Let $u \in \mathbb{D}^{1,2}(L^2(J))$ be a stochastic process such that $u = 0$ almost surely on some set $A \in \mathcal{F}$. Then $\delta(u) = 0$ almost surely on A.*

Now we state the so-called Meyer inequalities which identify spaces between which the operator δ^n is bounded.

Theorem 1.8 (Meyer inequalities, [8, Theorem 2.5.5])**.** *For any $n, m \in \mathbb{N}_0, n \geq m$ and $p \in [1, \infty)$, the operator δ^m is bounded from $\mathbb{D}^{n,p}(L^2(J^m))$ to $\mathbb{D}^{n-m,p}$, that is, there is a finite positive constant $c_{m,n,p}$ such that*

$$\|\delta^m(u)\|_{\mathbb{D}^{n-m,p}} \leq c_{m,n,p} \|u\|_{\mathbb{D}^{n,p}(L^2(J^m))}, \quad u \in \mathbb{D}^{n,p}(L^2(J^m)).$$

Remark 1.3. Similar to the Malliavin derivative, the definition of the divergence δ^n can be extended to processes with values in a Hilbert space. Again, let $\mathcal{V}$ be a real separable Hilbert space and set $\operatorname{Dom} \delta^n$ to be the space of processes $u \in L^2(\Omega; \mathcal{V} \otimes L^2(J^n))$ for which there is a finite positive constant c such that

$$|\mathbb{E} \langle D^n F, u \rangle_{\mathcal{V} \otimes L^2(J^n)}| \leq c \|F\|_{L^2(\Omega; \mathcal{V})},$$

holds for all $F \in \mathcal{S}(\mathcal{V})$. Then, for $u \in \operatorname{Dom} \delta^n$, we define $\delta^n(u)$ as the unique element of $L^2(\Omega; \mathcal{V})$ satisfying the equality

$$\mathbb{E} \langle F, \delta^n(u) \rangle_{\mathcal{V}} = \mathbb{E} \langle D^n F, u \rangle_{\mathcal{V} \otimes L^2(J^n)}$$

for every $F \in \mathcal{S}(\mathcal{V})$.

The construction in Remark 1.3 allows us to interpret the symbol δ^n as the n-fold composition of δ. Choose $g \in L^2(J^n)$. Then, using the fact that the space $L^2(J^n)$ can written as $L^2(J^{n-m}) \otimes L^2(J^m)$ for any $0 < m < n$, we have that $\delta^m(g)$ is the unique element of $L^2(\Omega; L^2(J^{n-m}))$ chosen as in Remark 1.3 with $\mathcal{V} = L^2(J^{n-m})$. In addition, we can write

$$\delta^n(g) = \delta^{n-m}(\delta^m(g)), \quad g \in L^2(J^n). \tag{1.10}$$

In fact the relation (1.10) also holds for any $g \in \operatorname{Dom} \delta^n$.

Remark 1.4. As mentioned at the beginning of the section, the divergence operator can be viewed as an extension of the Itô integral as these two objects coincide for adapted square integrable process but the divergence is defined for a larger class of processes.

Indeed, let $L^2_{\mathcal{F}^W}([0,T] \times \Omega)$ be space of processes in $L^2([0,T] \times \Omega)$ that are adapted to the canonical filtration of the underlying Wiener process $(W_t)_{t \in [0,T]}$. Then we have (see [9, Proposition 1.3.11]) that $L^2_{\mathcal{F}^W}([0,T] \times \Omega) \subseteq \operatorname{Dom} \delta$ and for $u \in L^2_{\mathcal{F}^W}([0,T] \times \Omega)$ it holds

$$\delta(u) = (\text{Itô}) \int_0^T u_s \, \mathrm{d}W_s.$$

1.4 Multiple integrals

By $L^2_s(J^n)$ we will mean the elements of $L^2(J^n)$ which are symmetric functions.

Definition 1.5. Let $n \in \mathbb{N}$ and $f \in L^2_s(J^n)$. Then the *n-th multiple integral of f*, $I_n(f)$, is defined by $I_n(f) = \delta^n(f)$.

In view of Section 1.1 and Remark 1.3, it is clear that $I_1(f) = W(h)$, i.e., $I_1(f)$ is the Wiener integral of f. The next result shows the orthogonality of multiple integrals of different orders.

Proposition 1.9 ([8, Proposition 2.7.5]). *Let $m, n \in \mathbb{N}$. Then for $f \in L^2_s(J^m)$ and $g \in L^2_s(J^n)$, we have*

$$\mathbb{E} I_m(f) I_n(g) = \begin{cases} m! \left\langle f, g \right\rangle_{L^2(J^m)}, & m = n, \\ 0, & m \neq n. \end{cases}$$

The n-th multiple integral connects the space $L^2_s(J^n)$ to n-th Wiener chaos.

Theorem 1.10 ([8, Theorem 2.7.7]). *Let $f \in L^2(J)$ be such that $\|f\|_{L^2(J)} = 1$. Then for any $n \in \mathbb{N}$ we have*

$$H_n(W(f)) = I_n(f^{\otimes n}),$$

where $f^{\otimes n}(x_1, \ldots, x_n) = f(x_1) f(x_2) \ldots f(x_n)$. In particular, the linear operator I_n provides an isometry from $(L^2_s(J^n), \frac{1}{\sqrt{n!}} \|\cdot\|_{L^2(J^n)})$ to the n-th Wiener chaos $(\mathcal{H}_n, \|\cdot\|_{L^2(\Omega)})$.

According to Theorem 1.10, any random variable Y of the form $Y = I_n(f)$ belongs to the n-th Wiener chaos. Moreover, the following result shows that arbitrary moment of Y can be estimated by its second moment (or, in other words, within a fixed Wiener chaos, all $L^p(\Omega)$-norms are equivalent).

Theorem 1.11 ([8, Theorem 2.7.2]). *Let $p \in [1, \infty)$, $n \in \mathbb{N}$ and let Y be a random variable with the form $Y = I_n(f)$ for some $f \in L^2_s(J^n)$. Then there exists a finite positive constant $c_{n,p}$ such that*

$$\|Y\|_{L^p(\Omega)} \le c_{n,p} \|Y\|_{L^2(\Omega)}. \tag{1.11}$$

Proof. The case $p \in [1, 2]$ follows immediately with $c_{n,p} = 1$ from the standard embedding of $L^p(\Omega)$ spaces. Now let $p > 2$. According to the Meyer inequalities (Theorem 1.8), there is a finite positive constant $\hat{c}_{n,p}$ such that

$$\|I_n(f)\|_{L^p(\Omega)} = \|\delta^n(f)\|_{\mathbb{D}^{0,p}} \le \hat{c}_{n,p} \|f\|_{\mathbb{D}^{n,p}(L^2(J^n))}.$$

As f is deterministic, we have $\|f\|_{\mathbb{D}^{n,p}(L^2(J^n))} = \|f\|_{L^2(J^n)}$. In view of Proposition 1.9, we obtain

$$\|I_n(f)\|_{L^p(\Omega)} \le \hat{c}_{n,p} \|f\|_{L^2(J^n)} = \frac{\hat{c}_{n,p}}{\sqrt{n!}} \|I_n(f)\|_{L^2(\Omega)}.$$

□

An explicit form of the constant $c_{n,p}$ can be obtained by an alternative proof which appeals to the hypercontractivity of Ornstein-Uhlenbeck semigroup (see [8, Corollary 2.8.14]). As an example, for any random variable Y living in the second Wiener chaos, we have

$$\|Y\|_{L^p(\Omega)} \le (p-1) \|Y\|_{L^2(\Omega)}.$$

Remark 1.5. Given $f \in L_s^2(J^n)$, the n-th multiple integral $I_n(f)$ can be rewritten as an iterated Itô stochastic integral via a procedure similar to the construction of Wiener integral in Section 1.1. We summarize the main steps below, for a more detailed construction see [8, Exercise 2.7.6] and [5, Section 1.2.1] (or [11] for the original construction by Itô).

Let $\mathcal{E}_n$ be a set of elementary functions $f : J^n \to \mathbb{R}$, that is, a set of functions with form

$$h(t_1, \dots, t_n) = \sum_{i_1,\dots,i_n=1}^{m} a_{i_1,\dots,i_n} \mathbf{1}_{A_{i_1} \times \dots \times A_{i_n}}(t_1, \dots, t_n), \tag{1.12}$$

where $m \in \mathbb{N}, A_i \subseteq J$ are disjoint finite intervals, and $a_{i_1,\dots,i_n} = 0$ whenever two indices are the same. Recall that for $A_i = [a, b]$, we defined $W(A_i) = W_b - W_a$. For $h \in \mathcal{E}_n$ of form (1.12), we define

$$\hat{I}_n(h) = \sum_{i_1,\dots,i_n=1}^{m} a_{i_1,\dots,i_n} W(A_{i_1}) \cdot \ldots \cdot W(A_{i_n}).$$

Then it can be shown that $\hat{I}_n$ is a linear operator that satisfies the isometry relation

$$\mathbb{E}\hat{I}_n(h)\hat{I}_m(g) = \begin{cases} n! \left\langle h, g \right\rangle_{L^2(J^n)}, & n = m, \\ 0, & n \neq m, \end{cases}$$

for each $h \in \mathcal{E}_n$ and $g \in \mathcal{E}_m$. Since the space $\mathcal{E}_n$ is dense in $L_s^2(J^n)$, the operator $\hat{I}_n$ can be extended to a linear operator from $L_s^2(J^n)$ to $L^2(\Omega)$. This extension is usually denoted by

$$\int_{J^n} h(t_1, \dots, t_n) \,\mathrm{d}W_{t_1} \dots \mathrm{d}W_{t_n}$$

and called *the multiple Wiener-Itô integral of order* n. One can then show that the multiple integral I_n coincides with the multiple Wiener-Itô integral of order n. In case $J = [0, T]$, since the integrands are symmetric functions, we can also express the integral as an iterated integral in the classical Itô sense

$$I_n(h) = n! \,(\text{Itô}) \int_0^T \int_0^{t_1} \dots \int_0^{t_{n-1}} h(t_1, \dots, t_n) \,\mathrm{d}W_{t_n} \dots \mathrm{d}W_{t_2} \,\mathrm{d}W_{t_1}.$$

2. Fractional processes

As we mentioned in the introduction, we will be mainly interested in two particular cases of fractional processes, namely in the fractional Brownian motion and Rosenblatt process. However, for the purpose of deriving numerous common properties of these two processes, it will be advantageous to view them as two special examples of the class of processes called *Hermite processes*.

2.1 Hermite processes

In this section, the class of Hermite processes is introduced and we derive (or at least mention) some of their properties. The presentation of the topic in this section is based mainly on [5, 12, 13] and for more information on the theory of self-similar processes we refer to them.

The class of Hermite processes originally arose as a class of limiting processes in the so-called Non-central limit theorem. This theorem, extending the result of Rosenblatt [14], was independently proved by Taqqu [1], and Dobrushin and Majòr [2]. Let us briefly sketch the result.

Assume $g : \mathbb{R} \to \mathbb{R}$ to be a function such that $\mathbb{E}g(N) = 0$ and $\mathbb{E}[g(N)]^2 < \infty$, where $N \sim N(0,1)$. Any such function can be expanded into the basis of Hermite polynomials

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x),$$

where $c_j = \frac{1}{j!}\mathbb{E}(g(N)H_j(N))$ (see [8, Proposition 1.4.2], but it really is a consequence of Theorem 1.1 had we assumed the underlying isonormal Gaussian process to be indexed by the real line, see [9, Example 1.1.1]). Define

$$q = \min\{j : c_j \neq 0\}.$$

We usually refer to number q as *the Hermite rank of* g. Note that the Hermite rank of g is always at least 1 since we assumed $\mathbb{E}g(N) = 0$.

Let $\{\xi_n\}_{n\in\mathbb{N}_0}$ be a stationary Gaussian sequence such that $\xi_0 \sim N(0,1)$ and

$$\mathbb{E}[\xi_0\xi_n] = n^{\frac{2H-2}{q}} L(n),$$

where $H \in (\frac{1}{2}, 1)$, q is a Hermite rank of g and L is a slowly varying function, that is, a positive function satisfying

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1$$

for every $a > 0$. Then the Non-central limit theorem states that the sequence of partial sums

$$\frac{1}{n^H} \sum_{j=1}^{\lfloor nt \rfloor} g(\xi_j)$$

converges in distribution, as $n \to \infty$, to a stochastic process living in the q-th Wiener chaos. This limiting process is called the Hermite process of order q with the Hurst parameter (or self-similarity index) H.

We will, however, define the Hermite process by means of a more explicit representation. By $(x)_+ = \max\{0, x\}$, we denote the non-negative part of x.

Definition 2.1. Let $(W_t)_{t\in\mathbb{R}}$ be a two-sided Wiener process. *The Hermite process* $(Z_t^{H,q})_{t\geq 0}$ *of order* q *with Hurst parameter* $H \in (\frac{1}{2}, 1)$ is defined by

$$Z_t^{H,q} = c(H,q) \int_{\mathbb{R}^q} \left(\int_0^t \prod_{i=1}^q (u-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \,\mathrm{d}u \right) \mathrm{d}W_{y_1} \dots \mathrm{d}W_{y_q}, \quad t \geq 0,$$

where $c(H,q)$ is a normalizing constant such that $\mathbb{E}\left(Z_1^{H,q}\right)^2 = 1$.

From the fact that the Hermite process of order q is defined as the q-th multiple integral, we immediately obtain that the process is centered and (by Theorem 1.10) that it lives in the q-th Wiener chaos.

In case $q = 1$, the process is called *the fractional Brownian motion*, in case $q = 2$, the resulting process is called *the Rosenblatt process*.

Let us now compute a covariance function of $Z_t^{H,q}$ (and, as a by-product, an explicit formula for $c(H,q)$). First, for $t \geq 0$ and $y_1, \dots, y_q \in \mathbb{R}$, denote by $L_t^{H,q}$ the kernel

$$L_t^{H,q}(y_1, \dots, y_q) = c(H,q) \int_0^t \prod_{i=1}^q (u-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \,\mathrm{d}u.$$

Then clearly $Z_t^{H,q} = I_q(L_t^{H,q})$ and by virtue of Proposition 1.9, we can write

$$\mathbb{E} Z_t^{H,q} Z_s^{H_q} = \mathbb{E} I_q(L_t^{H,q}) I_q(L_s^{H,q}) = q! \left\langle L_t^{H,q}, L_s^{H,q} \right\rangle_{L^2(\mathbb{R}^q)},$$

provided that $L_t^{H,q} \in L^2(\mathbb{R}^q)$ for every $t \geq 0$. Making use of the Fubini theorem and identity (A.1) with $a = -\left(\frac{1}{2} + \frac{1-H}{q}\right)$, we can write

$$\begin{aligned}
\mathbb{E} Z_t^{H,q} Z_s^{H_q} &= q! \int_{\mathbb{R}^q} L_t^{H,q}(y_1, \dots, y_q) L_s^{H,q}(y_1, \dots, y_q) \,\mathrm{d}y_1 \dots \mathrm{d}y_q \\
&= q!\, c(H,q)^2 \int_{\mathbb{R}^q} \Bigg(\int_0^t \int_0^s \prod_{i=1}^q (u-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \\
&\qquad\qquad \times (w-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \,\mathrm{d}u \,\mathrm{d}w \Bigg) \mathrm{d}y_1 \dots \mathrm{d}y_q \\
&= q!\, c(H,q)^2 \int_0^t \int_0^s \Bigg(\int_{\mathbb{R}^q} \prod_{i=1}^q (u-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \\
&\qquad\qquad \times (w-y_i)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \,\mathrm{d}y_1 \dots \mathrm{d}y_q \Bigg) \mathrm{d}u \,\mathrm{d}w \\
&= q!\, c(H,q)^2 \int_0^t \int_0^s \left(\int_{\mathbb{R}} (u-y)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} (w-y)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \,\mathrm{d}y \right)^q \mathrm{d}u \,\mathrm{d}w \\
&= q!\, c(H,q)^2 \int_0^t \int_0^s \left(\int_{-\infty}^{u\wedge w} (u-y)^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} (w-y)^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \,\mathrm{d}y \right)^q \mathrm{d}u \,\mathrm{d}w \\
&= q!\, c(H,q)^2 B\left(\frac{1}{2} - \frac{1-H}{q}, \frac{2-2H}{q}\right)^q \int_0^t \int_0^s \left(|u-w|^{\frac{2H-2}{q}} \right)^q \mathrm{d}u \,\mathrm{d}w \\
&= q!\, c(H,q)^2 B\left(\frac{1}{2} - \frac{1-H}{q}, \frac{2-2H}{q}\right)^q \int_0^t \int_0^s |u-w|^{2H-2} \,\mathrm{d}u \,\mathrm{d}w,
\end{aligned}$$

where B denotes the beta function (see Definition A.2). As there is the equality

$$H(2H-1)\int_0^t\int_0^s |u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w = \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right), \tag{2.1}$$

we obtain

$$\mathbb{E}Z_t^{H,q}Z_s^{H_q} = q!\,c(H,q)^2\frac{B\left(\frac{1}{2}-\frac{1-H}{q},\frac{2-2H}{q}\right)^q}{H(2H-1)}\cdot\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

In particular, $L_t^{H,q}\in L^2(\mathbb{R}^q)$ for every $t\geq 0$ and at time $t=s=1$, we have

$$\mathbb{E}(Z_1^{H,q})^2 = q!\,c(H,q)^2\frac{B\left(\frac{1}{2}-\frac{1-H}{q},\frac{2-2H}{q}\right)^q}{H(2H-1)},$$

so, in order for the normalization condition to hold, we must set

$$c(H,q)^2 = \frac{H(2H-1)}{q!B\left(\frac{1}{2}-\frac{1-H}{q},\frac{2-2H}{q}\right)^q}.$$

2.1.1 Basic properties of Hermite processes

Recall that stochastic process $(X_t)_{t\geq 0}$ is H-self-similar if there is $H\in(0,1)$ such that the processes $(X_{at})_{t\geq 0}$ and $(a^H X_t)_{t\geq 0}$ have the same finite-dimensional distributions for every $a>0$. We say that process $(X_t)_{t\geq 0}$ has stationary increments if for every $t\geq s\geq 0$

$$X_t - X_s \stackrel{d}{=} X_{t-s},$$

where $\stackrel{d}{=}$ denotes equality in distribution.

It can be shown that Hermite process $(Z_t^{H,q})_{t\geq 0}$ of an arbitrary order q is H-self-similar and has stationary increments. This result is mainly the consequence of the form of kernel $L_t^{H,q}$ and the fact that the Wiener process is a $\frac{1}{2}$-self-similar process (see e.g. [13, Theorem 1.2.1]) with stationary increments.

Proposition 2.1 ([5, Proposition 2.2]). *The Hermite process $(Z_t^{H,q})_{t\geq 0}$ is H-self-similar and has stationary increments.*

In fact, every non-trivial finite-variance H-self-similar process X with stationary increments must necessarily have covariance function of the form

$$R(s,t) = \frac{\mathbb{E}X_1^2}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

Indeed, assuming $t\geq s$, we can write

$$X_tX_s = \frac{1}{2}\left(X_t^2+X_s^2-(X_t-X_s)^2\right)$$

and thus, in view of stationarity of increments and H-self-similarity, we have

$$\begin{aligned}\mathbb{E}X_tX_s &= \frac{1}{2}\left(\mathbb{E}X_t^2+\mathbb{E}X_s^2-\mathbb{E}(X_t-X_s)^2\right)\\ &= \frac{1}{2}\left(\mathbb{E}X_t^2+\mathbb{E}X_s^2-\mathbb{E}X_{t-s}^2\right)\\ &= \frac{\mathbb{E}X_1^2}{2}\left(t^{2H}+s^{2H}-(t-s)^{2H}\right).\end{aligned}$$

Then it suffices to note that every finite-variance H-self-similar process with stationary increments is necessarily centered (see [13, Theorem 3.1.1]).

As an immediate consequence, the fractional Brownian motion is the only (up to a multiplicative constant) Gaussian H-self-similar process with stationary increments.

Another aftereffect of self-similarity and stationary increments is the regularity of sample paths of the Hermite process. By these properties, for any $p \geq 1$ and $t \geq s \geq 0$ we obtain

$$\mathbb{E}|Z_t^{H,q} - Z_s^{H,q}|^p = \mathbb{E}|Z_{t-s}^{H,q}|^p = |t-s|^{Hp}\mathbb{E}|Z_1^{H,q}|^p \tag{2.2}$$

and so, according to the Kolmogorov-Čentsov theorem (e.g. [15, Theorem 2.8]), the process $Z_t^{H,q}$ admits a modification with Hölder continuous sample paths of order δ for every $\delta < H$. Note that the p-th moment $\mathbb{E}|Z_1^{H,q}|^p$ is indeed finite thanks to Theorem 1.11.

As a final property, let us mention that the Hermite process $(Z_t^{H,q})_{t\geq 0}$ exhibits long-range dependence. What we mean by long-range dependence is the following. Consider the sequence of increments

$$\xi(n) = Z_{n+1}^{H,q} - Z_n^{H,q}, \quad n \in \mathbb{N}_0.$$

Then from the form of the covariance function of Hermite process, we obtain

$$\begin{aligned} r(n) = \mathbb{E}\xi(n)\xi(0) &= \mathbb{E}[Z_1^{H,q}(Z_{n+1}^{H,q} - Z_n^{H,q})] \\ &= \mathbb{E}Z_1^{H,q}Z_{n+1}^{H,q} - \mathbb{E}Z_1^{H,q}Z_n^{H,q} \\ &= \frac{1}{2}\left(1 + (n+1)^{2H} - n^{2H}\right) - \frac{1}{2}\left(1 + n^{2H} - (n-1)^{2H}\right) \\ &= \frac{1}{2}\left((n-1)^{2H} + (n+1)^{2H} - 2n^{2H}\right). \end{aligned}$$

Hence for large n, $r(n)$ behaves as n^{2H-2} and therefore (since $2H-2 > -1$)

$$\sum_{n=1}^{\infty} r(n) = \infty. \tag{2.3}$$

Property (2.3) is what we usually call long-range dependence (or long-term memory) of process $Z^{H,q}$.

2.1.2 P-variation of Hermite process

Fix $T > 0$. Consider a dyadic partition $\{t_i^n\}_{i=0}^{2^n}$ of the interval $[0,T]$, that is $t_i^n = \frac{iT}{2^n}$. For a stochastic process $X = (X_t)_{t\in[0,T]}$ and $p > 0$, we define random variables

$$V_n^p(X) = \sum_{i=0}^{2^n-1} |X_{t_{i+1}^n} - X_{t_i^n}|^p.$$

By *p-variation of* X, we mean the $L^1(\Omega)$-limit of $V_n^p(X)$ as $n \to \infty$, provided that this limit exists.

In view of (2.2), we can write

$$\mathbb{E}V_n^2(Z^{H,q}) = \mathbb{E}\sum_{i=0}^{2^n-1} |Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}|^2 = \mathbb{E}|Z^{H,q}_1|^2 \sum_{i=0}^{2^n-1} |t^n_{i+1} - t^n_i|^{2H}$$
$$= \sum_{i=0}^{2^n-1} \left|\frac{(i+1)T}{2^n} - \frac{iT}{2^n}\right|^{2H} = T^{2H} 2^{-n(2H-1)} \xrightarrow[n\to\infty]{} 0.$$

We have shown that the Hermite process is a process of zero quadratic variation. As a consequence the Hermite process is not a semimartingale. In general, we have the following result about the p-variation of Hermite processes.

Proposition 2.2 ([5, Proposition 2.3]). *Let $p > 0$. Then*

$$V_n^p(Z^{H,q}) \xrightarrow[n\to\infty]{\mathbb{P}} \begin{cases} 0, & p > \frac{1}{H}, \\ T\,\mathbb{E}|Z^{H,q}_1|^{\frac{1}{H}}, & p = \frac{1}{H}, \\ \infty, & p < \frac{1}{H}. \end{cases}$$

For the special case of the fractional Brownian motion, Proposition 2.2 was originally proved by Rodgers [3].

Note, however, that the convergence in Proposition 2.2 is only in probability. For our purposes we need the convergence in $L^1(\Omega)$ at least for the case $p = \frac{1}{H}$.

Proposition 2.3. *Let $Z^{H,q}$ be a Hermite process of order q. Then*

$$V_n^{1/H}(Z^{H,q}) \xrightarrow[n\to\infty]{L^1(\Omega)} T\,\mathbb{E}|Z^{H,q}_1|^{\frac{1}{H}}.$$

Proof. In view of Proposition 2.2, to ensure the convergence in $L^1(\Omega)$ it suffices to show that the sequence $\{V_n^{1/H}(Z^{H,q})\}_{n\in\mathbb{N}}$ is uniformly integrable.

Let $K > 0$. By Hölder's inequality, we obtain

$$\mathbb{E}V_n^{1/H}(Z^{H,q})\mathbf{1}_{\{V_n^{1/H}(Z^{H,q})\ge K\}} = \mathbb{E}\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}} \mathbf{1}_{\left\{\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}\ge K\right\}}$$
$$\le \left[\mathbb{E}\left(\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}\right)^2\right]^{\frac{1}{2}} \left[\mathbb{P}\left(\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}} \ge K\right)\right]^{\frac{1}{2}}$$

Successively using the generalized Minkowski inequality (see [16, Theorem 202]), self-similarity, and stationarity of increments yields

$$\left[\mathbb{E}\left(\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}\right)^2\right]^{\frac{1}{2}} \le \sum_{i=0}^{2^n-1}\left(\mathbb{E}\left|Z^{H,q}_{t^n_{i+1}} - Z^{H,q}_{t^n_i}\right|^{\frac{2}{H}}\right)^{\frac{1}{2}}$$
$$= \sum_{i=0}^{2^n-1}\left(\mathbb{E}\left|Z^{H,q}_{\frac{(i+1)T}{2^n}} - Z^{H,q}_{\frac{iT}{2^n}}\right|^{\frac{2}{H}}\right)^{\frac{1}{2}}$$
$$= \sum_{i=0}^{2^n-1}\left(\mathbb{E}\left(T^H 2^{-nH}\left|Z^{H,q}_{i+1} - Z^{H,q}_i\right|\right)^{\frac{2}{H}}\right)^{\frac{1}{2}}$$
$$= \sum_{i=0}^{2^n-1} T\,2^{-n}\left(\mathbb{E}\left|Z^{H,q}_1\right|^{\frac{2}{H}}\right)^{\frac{1}{2}}$$
$$= \sum_{i=0}^{2^n-1} T\,2^{-n}\left(\mathbb{E}\,|Z_1|^{\frac{2}{H}}\right)^{\frac{1}{2}} = T\left(\mathbb{E}\,|Z_1|^{\frac{2}{H}}\right)^{\frac{1}{2}}.$$

As for the second term, by Markov's inequality and same arguments as above, we obtain

$$\left[\mathbb{P}\left(\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}}-Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}\geq K\right)\right]^{\frac{1}{2}}\leq\left(\frac{\sum_{i=0}^{2^n-1}\mathbb{E}\left|Z^{H,q}_{t^n_{i+1}}-Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}}{K}\right)^{\frac{1}{2}}$$
$$=\frac{\sqrt{T}\left(\mathbb{E}\left|Z_1\right|^{\frac{1}{H}}\right)^{\frac{1}{2}}}{\sqrt{K}}.$$

Altogether,

$$\lim_{K\to\infty}\sup_{n\in\mathbb{N}}\mathbb{E}\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}}-Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}\mathbf{1}_{\left\{\sum_{i=0}^{2^n-1}\left|Z^{H,q}_{t^n_{i+1}}-Z^{H,q}_{t^n_i}\right|^{\frac{1}{H}}\geq K\right\}}$$
$$\leq\lim_{K\to\infty}\sup_{n\in\mathbb{N}}T^{\frac{3}{2}}\left(\mathbb{E}\left|Z_1\right|^{\frac{2}{H}}\right)^{\frac{1}{2}}\frac{\left(\mathbb{E}\left|Z_1\right|^{\frac{1}{H}}\right)^{\frac{1}{2}}}{\sqrt{K}}=0.$$

□

2.2 Fractional Brownian motion

The fractional Brownian motion is defined as the Hermite process of order 1. In words of Definition 2.1, the fractional Brownian motion $Z^{H,1}$ of Hurst parameter $H\in(\frac{1}{2},1)$ is given by

$$Z^{H,1}_t=c(H,1)\int_{\mathbb{R}}\left(\int_0^t(u-y)_+^{H-\frac{3}{2}}\,\mathrm{d}u\right)\mathrm{d}W_y,\quad t\geq 0.$$

In view of Section 2.1.1, the fractional Brownian motion is H-self-similar, long-range dependent process with stationary increments that admits a continuous modification. Additionally, the fractional Brownian motion lives in the first Wiener chaos and therefore must be a Gaussian process.

Thanks to the Gaussianity of the process, the fractional Brownian motion could have been alternatively defined as a centered, Gaussian process with covariance function given by

$$R(t,s)=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

It is well-known that, in this way, one can define the fractional Brownian motion also for $0<H<\frac{1}{2}$.

In addition to properties from Section 2.1.1, we state a finite time interval representation of the fractional Brownian motion. For $H\in(\frac{1}{2},1)$, consider the function $P^H(t,s)$ given by

$$P^H(t,s)=\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)_+^{H-\frac{3}{2}},\quad t,s\geq 0. \tag{2.4}$$

Then the fractional Brownian motion can be represented on a finite time interval in terms of the function P^H.

Theorem 2.4 ([17, Corollary 3.1]). *Let P^H be a function given by* (2.4) *and let* $(Z_t^{H,1})_{t\in[0,T]}$ *be a fractional Brownian motion. Then the process*

$$Y_t^{H,1} = c(H,1)\int_0^T \left(\int_0^t P^H(u,y)\,\mathrm{d}u\right)\mathrm{d}W_y, \quad t\in[0,T]$$

has the same distribution as $Z^{H,1}$.

2.3 Rosenblatt process

The Rosenblatt process $Z^{H,2}$ was defined as the Hermite process of order 2, which was given by

$$Z_t^{H,2} = c(H,2)\int_{\mathbb{R}^2}\left(\int_0^t (u-y_1)_+^{\frac{H}{2}-1}(u-y_2)_+^{\frac{H}{2}-1}\,\mathrm{d}u\right)\mathrm{d}W_{y_1}\,\mathrm{d}W_{y_2}, \quad t\geq 0.$$

From Section 2.1.1 it follows that it is a H-self-similar long-range dependent process with stationary increments that admits a continuous modification. Contrary to the fractional Brownian motion, the Rosenblatt process is not Gaussian.

The Rosenblatt process (or rather its distribution) first appeared in the paper by Rosenblatt [14] as a counterexample to one of the central limit theorems. The name "Rosenblatt process" was subsequently used by Taqqu in [18]. For more detailed history and additional properties of the Rosenblatt process we refer to [19].

Similar to the fractional Brownian motion, the Rosenblatt process can also be represented on a finite time interval in terms of function P^H.

Theorem 2.5 ([6, Proposition 1]). *Let P^H be a function given by* (2.4) *and let* $(Z_t^{H,2})_{t\in[0,T]}$ *be a Rosenblatt process. Then the process*

$$Y_t^{H,2} = c(H,2)\int_{[0,T]^2}\left(\int_0^t P^{\frac{H+1}{2}}(u,y_1)P^{\frac{H+1}{2}}(u,y_2)\,\mathrm{d}u\right)\mathrm{d}W_{y_1}\,\mathrm{d}W_{y_2}, \quad t\in[0,T]$$

has the same distribution as $Z^{H,2}$.

Remark 2.1. The original proofs of the representations in Theorems 2.4 and 2.5 exploit the properties of the Wiener chaos the processes live in. In particular, the distribution of a random variable in the first Wiener chaos is uniquely determined by its variance and the distribution of a random variable living in the second Wiener chaos is uniquely determined by its cumulants (see [5, Proposition 1.9]).

Alternatively, by using a regularization technique, one can obtain a finite time interval representation of Hermite process of general order (see [20]) of the form

$$Y_t^{H,q} = c(H,q)\int_{[0,T]^q}\left(\int_0^t P^{H'}(u,y_1)\dots P^{H'}(u,y_q)\,\mathrm{d}u\right)\mathrm{d}W_{y_1}\dots\mathrm{d}W_{y_q}, \qquad (2.5)$$

where

$$H' = 1+\frac{H-1}{q}.$$

3. Stochastic integration with respect to fractional Brownian motion and Rosenblatt process

We now turn to the task of constructing stochastic integrals with respect to the fractional Brownian motion and Rosenblatt process. As we have seen in Section 2.1.2, neither of these processes are semimartingales, hence a different approach to stochastic integration is required.

Let us first introduce some notation. By $A \propto B$ we mean that there is a finite positive constant c such that $A = cB$. Similarly, by $A \preceq B$ we mean that there is a finite positive constant c such that $A \leq cB$.

3.1 Integration with respect to Hermite process

In order to motivate the construction of stochastic integral, we will first consider the case of deterministic integrands. Since the construction is somewhat similar for both the fractional Brownian motion and the Rosenblatt process we will again consider the case of general Hermite process. The construction will closely follow [21] but it will be translated to the setting of a finite time interval using the representation from Remark 2.1.

Let $H \in (\frac{1}{2}, 1), q \in \mathbb{N}$, and consider the operator $\mathcal{K}^{H,q}$ which is defined on set of functions $f : [0,T] \to \mathbb{R}$, takes values in the set of functions $g : [0,T]^q \to \mathbb{R}$ and is given by

$$\begin{aligned}\mathcal{K}^{H,q}(f)(y_1,\dots,y_q) &= c(H,q) \int_0^T f(u) \prod_{k=1}^{q} P^{H'}(u, y_k)\,\mathrm{d}u \\ &= c(H,q) \int_0^T f(u) \prod_{k=1}^{q} \left(\frac{y_k}{u}\right)^{-\left(\frac{1}{2}-\frac{1-H}{q}\right)} (u-y_k)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}\,\mathrm{d}u.\end{aligned}$$

Clearly, from the representation (2.5), we have that

$$Y_t^{H,q} = \int_{[0,T]^q} \mathcal{K}^{H,q}(\mathbf{1}_{[0,t]})(y_1,\dots,y_q)\,\mathrm{d}W_{y_1}\dots\mathrm{d}W_{t_q} \tag{3.1}$$

is a Hermite process of order q on $[0,T]$. In the rest of the thesis, by $Z^{H,q}$, we will mean the Hermite process of order q given by (3.1). Let $\mathcal{A}$ be a space of elementary functions with form

$$f(s) = \sum_{j=0}^{m-1} a_j \mathbf{1}_{[t_j,t_{j+1})}(s), \tag{3.2}$$

where $m \in \mathbb{N}, a_j \in \mathbb{R}$ and $\{t_j\}_{j=0}^m$ is a partition of interval $[0,T]$. For $f \in \mathcal{A}$ of form (3.2) the most natural choice for the integral is to set

$$\int_0^T f(s)\,\mathrm{d}Z_s^{H,q} = \sum_{j=0}^{m-1} a_j \left(Z_{t_{j+1}}^{H,q} - Z_{t_j}^{H,q}\right).$$

Then, in view of (3.1) and the linearity of operator $\mathcal{K}^{H,q}$, we have

$$\begin{aligned}\int_0^T f(s)\,\mathrm{d}Z_s^{H,q} &= \sum_{j=0}^{m-1} a_j \left(Z_{t_{j+1}}^{H,q} - Z_{t_j}^{H,q}\right)\\ &= \sum_{j=0}^{m-1} a_j \int_{[0,T]^q} \mathcal{K}^{H,q}(\mathbf{1}_{[t_j,t_{j+1})})(y_1,\ldots,y_q)\,\mathrm{d}W_{y_1}\ldots\mathrm{d}W_{t_q}\\ &= \int_{[0,T]^q} \mathcal{K}^{H,q}\left(\sum_{j=0}^{m-1} a_j \mathbf{1}_{[t_j,t_{j+1})}\right)(y_1,\ldots,y_q)\,\mathrm{d}W_{y_1}\ldots\mathrm{d}W_{t_q}\\ &= \int_{[0,T]^q} \mathcal{K}^{H,q}(f)(y_1,\ldots,y_q)\,\mathrm{d}W_{y_1}\ldots\mathrm{d}W_{t_q}.\end{aligned}$$

Therefore, a natural extension to a larger class of functions f would be to define the integral by

$$\int_0^T f(s)\,\mathrm{d}Z_s^{H,q} = \int_{[0,T]^q} \mathcal{K}^{H,q}(f)(y_1,\ldots,y_q)\,\mathrm{d}W_{y_1}\ldots\mathrm{d}W_{t_q}.$$

However, in order to do so, we need to know for which f the right-hand side of the definition above makes sense. Recall that the q-th multiple integral was defined for symmetric functions from $L^2([0,T]^q)$. Thus we have to investigate for which functions $f:[0,T]\to\mathbb{R}$ it holds $\mathcal{K}^{H,q}(f)\in L_s^2([0,T]^q)$. Introduce the space

$$\mathfrak{H}_H = \left\{f:[0,T]\to\mathbb{R} : \mathcal{K}^{H,q}(f)\in L_s^2([0,T]^q)\right\}.$$

The space $\mathfrak{H}_H$ can be endowed with the norm

$$\|f\|_{\mathfrak{H}_H} = \sqrt{q!}\,\|\mathcal{K}^{H,q}(f)\|_{L^2([0,T]^q)}, \quad f\in\mathfrak{H}_H.$$

By using the Fubini theorem and identity (A.2) with $a = -\left(\frac{1}{2}+\frac{1-H}{q}\right)$, the norm $\|\cdot\|_{\mathfrak{H}_H}$ can be expressed as follows:

$$\begin{aligned}\|f\|_{\mathfrak{H}_H}^2 &= q! \int_{[0,T]^q} |\mathcal{K}^{H,q}(f)(y_1,\ldots,y_q)|^2\,\mathrm{d}y_1\ldots\mathrm{d}y_q\\ &= q!\,c(H,q)^2 \int_{[0,T]^q} \left(\int_0^T f(u) \prod_{k=1}^q \left(\frac{y_k}{u}\right)^{-\left(\frac{1}{2}-\frac{1-H}{q}\right)} (u-y_k)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}\,\mathrm{d}u\right)\\ &\qquad\times \left(\int_0^T f(w) \prod_{k=1}^q \left(\frac{y_k}{w}\right)^{-\left(\frac{1}{2}-\frac{1-H}{q}\right)} (w-y_k)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}\,\mathrm{d}w\right)\mathrm{d}y_1\ldots\mathrm{d}y_q\\ &= q!\,c(H,q)^2 \int_0^T\int_0^T f(u)f(w)(uw)^{q\left(\frac{1}{2}-\frac{1-H}{q}\right)}\\ &\qquad\times \left(\int_0^{u\wedge w} y^{-2\left(\frac{1}{2}-\frac{1-H}{q}\right)}(u-y)^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}(w-y)^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)}\,\mathrm{d}y\right)^q \mathrm{d}u\,\mathrm{d}w\\ &= q!\,c(H,q)^2 \int_0^T\int_0^T f(u)f(w)(uw)^{q\left(\frac{1}{2}-\frac{1-H}{q}\right)}\\ &\qquad\times \left(B\left(\frac{1}{2}-\frac{1-H}{q},\frac{2-2H}{q}\right)(uw)^{-\left(\frac{1}{2}-\frac{1-H}{q}\right)}|u-w|^{\frac{2H-2}{q}}\right)^q \mathrm{d}u\,\mathrm{d}w\\ &= q!\,c(H,q)^2 B\left(\frac{1}{2}-\frac{1-H}{q},\frac{2-2H}{q}\right)^q \int_0^T\int_0^T f(u)f(w)|u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w\\ &= H(2H-1)\int_0^T\int_0^T f(u)f(w)|u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w.\end{aligned}$$

Hence, we showed that the space $\mathfrak{H}_H$ coincides with the space of functions $f : [0,T] \to \mathbb{R}$ that satisfy

$$H(2H-1)\int_0^T\int_0^T f(u)f(w)|u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w < \infty. \tag{3.3}$$

The representation of the space $\mathfrak{H}_H$ via (3.3) allows us to endow it with the inner product

$$\langle f, g\rangle_{\mathfrak{H}_H} = H(2H-1)\int_0^T\int_0^T f(u)g(w)|u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w, \quad f, g \in \mathfrak{H}_H.$$

Denote

$$C(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T].$$

We derived in Section 2.1 that C is the covariance function of Hermite process $Z^{H,q}$. For any $f \in \mathcal{A}$ of form (3.2), we have

$$\begin{aligned}
&\left\|\int_0^T f(s)\,\mathrm{d}Z_s^{H,q}\right\|_{L^2(\Omega)}^2 = \left\|\sum_{j=0}^{m-1} a_j\left(Z_{t_{j+1}}^{H,q} - Z_{t_j}^{H,q}\right)\right\|_{L^2(\Omega)}^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_i a_j \mathbb{E}\left((Z_{t_{i+1}}^{H,q} - Z_{t_i}^{H,q})(Z_{t_{j+1}}^{H,q} - Z_{t_j}^{H,q})\right) \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_i a_j\left(C(t_{i+1},t_{j+1}) - C(t_{i+1},t_j) - C(t_{j+1},t_i) + C(t_i,t_j)\right) \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_i a_j H(2H-1)\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} |u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_i a_j\left\langle \mathbf{1}_{[t_i,t_{i+1})}, \mathbf{1}_{[t_j,t_{j+1})}\right\rangle_{\mathfrak{H}_H} \\
&= \left\langle \sum_{i=0}^{m-1} a_i\mathbf{1}_{[t_i,t_{i+1})}, \sum_{j=0}^{m-1} a_j\mathbf{1}_{[t_j,t_{j+1})}\right\rangle_{\mathfrak{H}_H} = \|f\|_{\mathfrak{H}_H}^2,
\end{aligned}$$

where we used the identity (2.1). In particular, the space $\mathcal{A}$ is included in $\mathfrak{H}_H$ and the mapping

$$f \mapsto \int_0^T f(s)\,\mathrm{d}Z_s^{H,q}$$

is an isometry from $\mathcal{A}$ to $L^2(\Omega)$. One can show that the space of elementary functions $\mathcal{A}$ is dense in $\mathfrak{H}_H$ (see [22]) and so the mapping

$$f \mapsto \int_0^T f(s)\,\mathrm{d}Z_s^{H,q}$$

can be extended to an isometry from $\mathfrak{H}_H$ to $L^2(\Omega)$. We call this isometry *the Wiener integral with respect to the Hermite process* $Z^{H,q}$.

In what follows, we will not be working with the space $\mathfrak{H}_H$. Instead we restrict the integrands to a smaller class of functions. Namely, we restrict to the space $L^{\frac{1}{H}}([0,T])$. The following proposition is based on [23, Lemma 1].

Proposition 3.1. *We have the following inclusion*

$$L^{\frac{1}{H}}([0,T]) \subseteq \mathfrak{H}_H.$$

In particular, the linear operator $\mathcal{K}^{H,q}$ is bounded from $L^{\frac{1}{H}}([0,T])$ to $L^2([0,T]^q)$.

Proof. We have

$$\begin{aligned}\|\mathcal{K}^{H,q}(g)\|^2_{L^2([0,T]^q)} \propto \|g\|^2_{\mathfrak{H}_H} \preceq \int_0^T \int_0^T |g(u)||g(w)||u-w|^{2H-2}\,\mathrm{d}u\,\mathrm{d}w \\ \propto \int_0^T |g(u)| \int_0^u |g(w)|(u-w)^{2H-2}\,\mathrm{d}w\,\mathrm{d}u.\end{aligned}$$

By Hölder's inequality with $p = \frac{1}{H}$,

$$\begin{aligned}&\int_0^T |g(u)| \int_0^u |g(w)|(u-w)^{2H-2}\,\mathrm{d}w\,\mathrm{d}u \\ &\quad\le \left(\int_0^T |g(u)|^{\frac{1}{H}}\,\mathrm{d}u\right)^H \left(\int_0^T \left(\int_0^u |g(w)|(u-w)^{2H-2}\,\mathrm{d}w\right)^{\frac{1}{1-H}}\,\mathrm{d}u\right)^{1-H}.\end{aligned}$$

The second factor on the right-hand side of the above inequality is, up to a multiplicative constant, equal to $\|\mathcal{I}^{2H-1}_{0+}|g|\|_{L^{\frac{1}{1-H}}([0,T])}$, where $\mathcal{I}^{2H-1}_{0+}$ is the left-sided fractional integral of order $2H-1$ (see Definition A.3). According to Theorem A.2 with $\alpha = 2H-1$, $p = \frac{1}{H}$, and $q = \frac{1}{1-H}$ we have

$$\|\mathcal{I}^{2H-1}_{0+}|g|\|_{L^{\frac{1}{1-H}}([0,T])} \preceq \|g\|_{L^{\frac{1}{H}}([0,T])}.$$

Altogether, we showed

$$\|\mathcal{K}^{H,q}(g)\|^2_{L^2([0,T]^q)} \propto \|g\|^2_{\mathfrak{H}_H} \preceq \|g\|^2_{L^{\frac{1}{H}}([0,T])}.$$

□

3.2 Stochastic integration with respect to the fractional Brownian motion

We now turn to the definition of a stochastic integral with respect to the fractional Brownian motion. In the previous section, we saw that for any function $f \in \mathfrak{H}_H$, we can define the integral by

$$\begin{aligned}\int_0^T f(s)\,\mathrm{d}Z^{H,1}_s &= \int_0^T \mathcal{K}^{H,1}(f)(y)\,\mathrm{d}W_y \\ &= c(H,1)\int_0^T \left(\int_0^T f(u)\left(\frac{y}{u}\right)^{\frac{1}{2}-H}(u-y)_+^{H-\frac{3}{2}}\,\mathrm{d}u\right)\mathrm{d}W_y.\end{aligned}$$

Recall that the multiple integral I_1 was defined as the divergence operator δ, which, in general, acted on stochastic processes $u \in \operatorname{Dom}\delta$. The stochastic integral with respect to the fractional Brownian motion can then be defined in terms of δ and the transfer operator $\mathcal{K}^{H,1}$.

Definition 3.1. Let $M \subseteq [0,T]$ be an interval. A Borel measurable function $g : [0,T] \to L^2(\Omega)$ is said to be *Skorokhod integrable with respect to the fractional Brownian motion on* M if $\mathcal{K}^{H,1}(g\mathbf{1}_M) \in \operatorname{Dom}\delta$. In such case, the *Skorokhod integral* is defined by

$$\int_M g_s \, \mathrm{d}Z_s^{H,1} = \delta\left(\mathcal{K}^{H,1}(g\mathbf{1}_M)\right).$$

In general, the Skorokhod integral is defined for stochastic processes g such that $\mathcal{K}^{H,1}(g) \in \operatorname{Dom}\delta$, however, such class of processes is difficult to describe. Hence we will only consider a suitable subspace of these processes. Such suitable subspace is described by the following proposition which is based on [23].

Proposition 3.2. *The operator* $\int_0^T (\ldots)\,\mathrm{d}Z^{H,1}$ *is bounded from* $L^{\frac{1}{H}}([0,T];\mathbb{D}^{1,\frac{1}{H}})$ *to* $L^{\frac{1}{H}}(\Omega)$.

Proof. Let $g \in L^{\frac{1}{H}}([0,T];\mathbb{D}^{1,\frac{1}{H}})$. By Theorem 1.8, we have

$$\begin{aligned}
\left\| \int_0^T g_s \, \mathrm{d}Z_s^{H,1} \right\|_{L^{\frac{1}{H}}(\Omega)}^{\frac{1}{H}} &= \left\| \delta\left(\mathcal{K}^{H,1}(g)\right) \right\|_{L^{\frac{1}{H}}(\Omega)}^{\frac{1}{H}} \\
&\preceq \left\| \mathcal{K}^{H,1}(g) \right\|_{\mathbb{D}^{1,\frac{1}{H}}(L^2([0,T]))}^{\frac{1}{H}} \\
&= \mathbb{E}\left\| \mathcal{K}^{H,1}(g) \right\|_{L^2([0,T])}^{\frac{1}{H}} + \mathbb{E}\left\| |D(\mathcal{K}^{H,1}(g)) \right\|_{L^2([0,T]^2)}^{\frac{1}{H}}. \qquad (3.4)
\end{aligned}$$

Using Proposition 3.1 gives

$$\mathbb{E}\left\| \mathcal{K}^{H,1}(g) \right\|_{L^2([0,T])}^{\frac{1}{H}} \preceq \mathbb{E}\left\| g \right\|_{L^{\frac{1}{H}}([0,T])}^{\frac{1}{H}} = \int_0^T \mathbb{E}|g_s|^{\frac{1}{H}} \, \mathrm{d}s.$$

For the second term in (3.4), by linearity and closability of the Malliavin derivative, we can interchange $\mathcal{K}^{H,1}$ and D. Then from Proposition 3.1 and the generalized Minkowski inequality it follows that

$$\begin{aligned}
\mathbb{E}\left\| D(\mathcal{K}^{H,1}(g)) \right\|_{L^2([0,T]^2)}^{\frac{1}{H}} &= \mathbb{E}\left\| \mathcal{K}^{H,1}(Dg) \right\|_{L^2([0,T]^2)}^{\frac{1}{H}} \\
&= \mathbb{E}\left(\int_0^T \|\mathcal{K}^{H,1}(D_x g)\|_{L^2([0,T])}^2 \, \mathrm{d}x \right)^{\frac{1}{2H}} \\
&\preceq \mathbb{E}\left(\int_0^T \|D_x g\|_{L^{\frac{1}{H}}([0,T])}^2 \, \mathrm{d}x \right)^{\frac{1}{2H}} \\
&= \mathbb{E}\left(\int_0^T \left(\int_0^T |D_x g_s|^{\frac{1}{H}} \, \mathrm{d}s \right)^{2H} \mathrm{d}x \right)^{\frac{1}{2H}} \\
&\le \mathbb{E}\int_0^T \left(\int_0^T |D_x g_s|^2 \, \mathrm{d}x \right)^{\frac{1}{2H}} \mathrm{d}s \\
&= \int_0^T \mathbb{E}\|Dg_s\|_{L^2([0,T])}^{\frac{1}{H}} \, \mathrm{d}s.
\end{aligned}$$

Summarized, we have

$$\begin{aligned}\mathbb{E}\left\|\mathcal{K}^{H,1}(g)\right\|_{L^2([0,T])}^{\frac{1}{H}} &\preceq \mathbb{E}\left\|\mathcal{K}^{H,1}(g)\right\|_{L^2([0,T])}^{\frac{1}{H}} + \mathbb{E}\left\||D(\mathcal{K}^{H,1}(g))|\right\|_{L^2([0,T]^2)}^{\frac{1}{H}} \\ &\preceq \int_0^T \mathbb{E}|g_s|^{\frac{1}{H}}\,\mathrm{d}s + \int_0^T \mathbb{E}\|Dg_s\|_{L^2([0,T])}^{\frac{1}{H}}\,\mathrm{d}s \\ &= \int_0^T \|g_s\|_{\mathbb{D}^{1,\frac{1}{H}}}^{\frac{1}{H}}\,\mathrm{d}s = \|g\|_{L^{\frac{1}{H}}([0,T];\mathbb{D}^{1,\frac{1}{H}})}^{\frac{1}{H}}.\end{aligned}$$

□

A slightly more general approach to Skorokhod integrability with respect to the fractional Brownian motion can be found in [9, Section 5.2].

3.3 Stochastic integration with respect to the Rosenblatt process

A stochastic integral with respect to the Rosenblatt process can be defined similarly as the integral with respect to the fractional Brownian motion. In Section 3.1, we defined the integral for deterministic integrands $f \in \mathfrak{H}_H$ by

$$\int_0^T f(s)\,\mathrm{d}Z_s^{H,2} = \int_{[0,T]^2} \mathcal{K}^{H,2}(f)(y_1,y_2)\,\mathrm{d}W_{y_1}\,\mathrm{d}W_{y_2}.$$

Again, the stochastic integral with respect to $Z^{H,2}$ can be defined via the divergence δ^2 and the transfer operator $\mathcal{K}^{H,2}$.

Definition 3.2. Let $M \subseteq [0,T]$ be an interval. A Borel measurable function $g : [0,T] \to L^2(\Omega)$ is said to be *Skorokhod integrable with respect to the Rosenblatt process on* M if $\mathcal{K}^{H,2}(g\mathbf{1}_M) \in \operatorname{Dom}\delta^2$. In such case, the *Skorokhod integral* is defined by

$$\int_M g_s\,\mathrm{d}Z_s^{H,2} = \delta^2\left(\mathcal{K}^{H,2}(g\mathbf{1}_M)\right).$$

What follows is a Skorokhod integrability condition similar to Proposition 3.2.

Proposition 3.3. *The operator* $\int_0^T(\ldots)\,\mathrm{d}Z^{H,2}$ *is bounded from* $L^{\frac{1}{H}}([0,T];\mathbb{D}^{2,\frac{1}{H}})$ *to* $L^{\frac{1}{H}}(\Omega)$.

Proof. The proof follows similarly as the proof of Proposition 3.2. One uses Theorem 1.8 to obtain

$$\begin{aligned}&\left\|\int_0^T g_s\,\mathrm{d}Z_s^{H,2}\right\|_{L^{\frac{1}{H}}(\Omega)}^{\frac{1}{H}} = \left\|\delta^2\left(\mathcal{K}^{H,2}(g\mathbf{1}_M)\right)\right\|_{L^{\frac{1}{H}}(\Omega)}^{\frac{1}{H}} \\ &\preceq \mathbb{E}\left\|\mathcal{K}^{H,2}(g)\right\|_{L^2([0,T]^2)}^{\frac{1}{H}} + \mathbb{E}\left\||D(\mathcal{K}^{H,2}(g))|\right\|_{L^2([0,T]^3)}^{\frac{1}{H}} + \mathbb{E}\left\||D^2(\mathcal{K}^{H,2}(g))|\right\|_{L^2([0,T]^4)}^{\frac{1}{H}}.\end{aligned}$$

Then it is enough to use Proposition 3.1 and the generalized Minkowski inequality as in the proof of Proposition 3.2.

□

An alternative Skorokhod integrability condition for the Rosenblatt process can be found in [6, Lemma 1].

4. Variations of stochastic integrals

By Proposition 2.3, we have

$$V_n^{1/H}(Z^{H,q}) \xrightarrow[n\to\infty]{L^1(\Omega)} T\,\mathbb{E}|Z_1^{H,q}|^{\frac{1}{H}}. \tag{4.1}$$

It is natural to ask whether the result (4.1) can be extended to Skorokhod integrals introduced in Sections 3.2 and 3.3. For the fractional Brownian motion we have the following result which was proved by Guerra and Nualart [4]. For $n \in \mathbb{N}_0$ we denote $\mathbb{L}^{n,\frac{1}{H}} = L^{\frac{1}{H}}([0,T];\mathbb{D}^{n,\frac{1}{H}})$.

Theorem 4.1. *Let $g \in \mathbb{L}^{1,\frac{1}{H}}$. Set $X_t = \int_0^t g_s \,\mathrm{d}Z_s^{H,1}, t \in [0,T]$, then*

$$V_n^{1/H}(X) \xrightarrow[n\to\infty]{L^1(\Omega)} c_H \int_0^T |g_s|^{\frac{1}{H}} \,\mathrm{d}s,$$

where $c_H = \mathbb{E}|Z_1^{H,1}|^{\frac{1}{H}}$.

Proof. The proof follows in similar manner as the proof of Theorem 4.2 or see [4] for the original proof for a slightly larger class of integrands.

□

Remark 4.1. For the fractional Brownian motion, it is possible to develop stochastic integral even for the case $0 < H < \frac{1}{2}$. For such integrals one can prove a counterpart to Theorem 4.1 as was done by Essaky and Nualart [24].

In view of (4.1) and multiple common properties that the fractional Brownian motion and Rosenblatt process share, it is not unreasonable to expect that a statement similar to Theorem 4.1 should also hold for the Skorokhod integral with respect to the Rosenblatt process. Indeed, for the Rosenblatt process $Z^{H,2}$, we have the following

Theorem 4.2. *Let $g \in \mathbb{L}^{2,\frac{1}{H}}$. Set $X_t = \int_0^t g_s \,\mathrm{d}Z_s^{H,2}, t \in [0,T]$, then*

$$V_n^{1/H}(X) \xrightarrow[n\to\infty]{L^1(\Omega)} C_H \int_0^T |g_s|^{\frac{1}{H}} \,\mathrm{d}s,$$

where $C_H = \mathbb{E}|Z_1^{H,2}|^{\frac{1}{H}}$.

Our main goal in this chapter will be to prove Theorem 4.2. The proof will be based on the techniques which were used in [4] to prove Theorem 4.1.

We start with a general inequality for a difference of variations of two stochastic processes.

Lemma 4.3. *Let X and Y be two stochastic processes such that $\mathbb{E}V_n^{1/H}(X) < \infty$ and $\mathbb{E}V_n^{1/H}(Y) < \infty$ for all $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, it holds that*

$$\begin{aligned}&\mathbb{E}|V_n^{1/H}(X) - V_n^{1/H}(Y)| \\ &\qquad \le \left(\mathbb{E}V_n^{1/H}(X-Y)\right)^H \left(\left(\mathbb{E}V_n^{1/H}(X)\right)^{1-H} + \left(\mathbb{E}V_n^{1/H}(Y)\right)^{1-H}\right).\end{aligned}$$

Proof. Let $[a,b] \subset \mathbb{R}$ be an interval. We will show the inequality

$$\left||b|^{\frac{1}{H}} - |a|^{\frac{1}{H}}\right| \leq \frac{1}{H}|b-a|\left(|b|^{\frac{1}{H}-1} + |a|^{\frac{1}{H}-1}\right). \tag{4.2}$$

We will consider two cases:
Case 1. Assume that the interval (a,b) does not contain zero. Consider a function $f : [a,b] \to \mathbb{R}$ given by $f(x) = |x|^{\frac{1}{H}}$. Then, by the mean value theorem, there is a $\xi \in (a,b)$ such that

$$\frac{1}{H}|\xi|^{\frac{1}{H}-1}|b-a| = \left||b|^{\frac{1}{H}} - |a|^{\frac{1}{H}}\right|.$$

From here it follows that

$$\left||b|^{\frac{1}{H}} - |a|^{\frac{1}{H}}\right| = \frac{1}{H}|\xi|^{\frac{1}{H}-1}|b-a| \leq \frac{1}{H}|b-a|\left(|b|^{\frac{1}{H}-1} + |a|^{\frac{1}{H}-1}\right).$$

Case 2. Now assume that the interval (a,b) contains zero. By the mean value theorem applied to function $f_b : [0,b] \to \mathbb{R}$ given by $f_b(x) = |x|^{\frac{1}{H}}$, there is a $\xi_b \in (0,b)$ such that

$$|b|^{\frac{1}{H}} = \frac{1}{H}|\xi_b|^{\frac{1}{H}-1}b.$$

Similarly, there is a $\xi_a \in (a,0)$ such that

$$-|a|^{\frac{1}{H}} = \frac{1}{H}|\xi_a|^{\frac{1}{H}-1}a.$$

Altogether, we have

$$\begin{aligned}
\left||b|^{\frac{1}{H}} - |a|^{\frac{1}{H}}\right| &= \left|\frac{1}{H}|\xi_b|^{\frac{1}{H}-1}b + \frac{1}{H}|\xi_a|^{\frac{1}{H}-1}a\right| \\
&\leq \frac{1}{H}|\xi_b|^{\frac{1}{H}-1}|b| + \frac{1}{H}|\xi_a|^{\frac{1}{H}-1}|a| \\
&\leq \frac{1}{H}\left(|b|^{\frac{1}{H}-1} + |a|^{\frac{1}{H}-1}\right)(|b| + |a|) \\
&= \frac{1}{H}\left(|b|^{\frac{1}{H}-1} + |a|^{\frac{1}{H}-1}\right)(b-a) \\
&= \frac{1}{H}\left(|b|^{\frac{1}{H}-1} + |a|^{\frac{1}{H}-1}\right)|b-a|.
\end{aligned}$$

In view of (4.2), we obtain

$$
\begin{aligned}
\mathbb{E}|V_n^{1/H}(X) - V_n^{1/H}(Y)| &= \mathbb{E}\left|\sum_{i=0}^{2^n-1} |X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}} - \sum_{i=0}^{2^n-1} |Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}}\right| \\
&\le \mathbb{E}\sum_{i=0}^{2^n-1} \left||X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}} - |Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}}\right| \\
&\le \frac{1}{H}\mathbb{E}\Bigg[\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - X_{t_i^n}) - (Y_{t_{i+1}^n} - Y_{t_i^n})| \\
&\qquad\qquad \times \left(|X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}-1} + |Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}-1}\right)\Bigg] \\
&= \frac{1}{H}\mathbb{E}\Bigg[\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})| \\
&\qquad\qquad \times \left(|X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}-1} + |Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}-1}\right)\Bigg] \\
&= \frac{1}{H}\mathbb{E}\left[\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})||X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}-1}\right] \\
&\qquad + \frac{1}{H}\mathbb{E}\left[\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})||Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}-1}\right].
\end{aligned}
$$

Applying Hölder's inequality with $p = \frac{1}{H}$ on both of the terms yields

$$
\begin{aligned}
&\frac{1}{H}\mathbb{E}\left[\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})||X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}-1}\right] \\
&\qquad + \frac{1}{H}\mathbb{E}\left[\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})||Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}-1}\right] \\
&\le \frac{1}{H}\left(\mathbb{E}\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})|^{\frac{1}{H}}\right)^H \left(\mathbb{E}\sum_{i=0}^{2^n-1} |X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}}\right)^{1-H} \\
&\qquad + \frac{1}{H}\left(\mathbb{E}\sum_{i=0}^{2^n-1} |(X_{t_{i+1}^n} - Y_{t_{i+1}^n}) - (X_{t_i^n} - Y_{t_i^n})|^{\frac{1}{H}}\right)^H \left(\mathbb{E}\sum_{i=0}^{2^n-1} |Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}}\right)^{1-H} \\
&= \frac{1}{H}\left(\mathbb{E}V_n^{1/H}(X-Y)\right)^H \left(\left(\mathbb{E}V_n^{1/H}(X)\right)^{1-H} + \left(\mathbb{E}V_n^{1/H}(Y)\right)^{1-H}\right).
\end{aligned}
$$

□

A special case of Lemma 4.3 where the stochastic processes are Skorokhod integrals is described by the following lemma.

Lemma 4.4. *Let $g, h \in \mathbb{L}^{2,\frac{1}{H}}$ and for $t \in [0,T]$ set*

$$X_t = \int_0^t g_s \,\mathrm{d}Z_s^{H,2}, \quad Y_t = \int_0^t h_s \,\mathrm{d}Z_s^{H,2}.$$

Then we have the following estimate

$$\mathbb{E}|V_n^{1/H}(X) - V_n^{1/H}(Y)| \le k_H \|g-h\|_{\mathbb{L}^{2,\frac{1}{H}}} \left(\|g\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1-H}{H}} + \|h\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1-H}{H}}\right),$$

where k_H is a finite positive constant.

Proof. Applying Lemma 4.3 gives

$$\mathbb{E}|V_n^{1/H}(X) - V_n^{1/H}(Y)| \le \frac{1}{H}\left(\mathbb{E}\sum_{i=0}^{2^n-1}\left|\int_{t_i^n}^{t_{i+1}^n}(g_s - h_s)\,\mathrm{d}Z_s^{H,2}\right|^{\frac{1}{H}}\right)^{H}$$
$$\times\left[\left(\mathbb{E}\sum_{i=0}^{n-1}\left|\int_{t_i^n}^{t_{i+1}^n} g_s\,\mathrm{d}Z_s^{H,2}\right|^{\frac{1}{H}}\right)^{1-H} + \left(\mathbb{E}\sum_{i=0}^{n-1}\left|\int_{t_i^n}^{t_{i+1}^n} h_s\,\mathrm{d}Z_s^{H,2}\right|^{\frac{1}{H}}\right)^{1-H}\right].$$

From the boundedness of the Skorokhod integral (Proposition 3.3) it follows that

$$\mathbb{E}\sum_{i=0}^{2^n-1}\left|\int_{t_i^n}^{t_{i+1}^n} g_s\,\mathrm{d}Z_s^{H,2}\right|^{\frac{1}{H}} = \sum_{i=0}^{2^n-1}\left\|\int_0^T g_s\mathbf{1}_{[t_i^n,t_{i+1}^n]}(s)\,\mathrm{d}Z_s^{H,2}\right\|_{L^{\frac{1}{H}}(\Omega)}^{\frac{1}{H}}$$
$$\preceq \sum_{i=0}^{2^n-1}\left\|g\mathbf{1}_{[t_i^n,t_{i+1}^n]}\right\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1}{H}}$$
$$= \sum_{i=0}^{2^n-1}\int_0^T\left\|g_s\mathbf{1}_{[t_i^n,t_{i+1}^n]}(s)\right\|_{\mathbb{D}^{2,\frac{1}{H}}}^{\frac{1}{H}}\,\mathrm{d}s$$
$$= \sum_{i=0}^{2^n-1}\int_0^T\mathbb{E}|g_s\mathbf{1}_{[t_i^n,t_{i+1}^n]}(s)|^{\frac{1}{H}}\,\mathrm{d}s + \sum_{i=0}^{2^n-1}\int_0^T\mathbb{E}\left(\int_0^T|D_x g_s\mathbf{1}_{[t_i^n,t_{i+1}^n]}(s)|^2\,\mathrm{d}x\right)^{\frac{1}{2H}}\mathrm{d}s$$
$$+ \sum_{i=0}^{2^n-1}\int_0^T\mathbb{E}\left(\int_0^T\int_0^T|D_{x,y}^2 g_s\mathbf{1}_{[t_i^n,t_{i+1}^n]}(s)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2H}}\mathrm{d}s$$
$$= \int_0^T\mathbb{E}|g_s|^{\frac{1}{H}}\,\mathrm{d}s + \int_0^T\mathbb{E}\left(\int_0^T|D_x g_s|^2\,\mathrm{d}x\right)^{\frac{1}{2H}}\mathrm{d}s$$
$$+ \int_0^T\mathbb{E}\left(\int_0^T\int_0^T|D_{x,y} g_s|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2H}}\mathrm{d}s$$
$$= \|g\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1}{H}},$$

where in the second to last equality we used the local property of Malliavin derivative (Proposition 1.4). This yields the desired inequality.

□

Let $g \in \mathbb{L}^{2,\frac{1}{H}}$ and set $X_t = \int_0^t g_s\,\mathrm{d}Z_s^{H,2}$, then from Lemma 4.4 (by choosing $h = 0$), we obtain the estimate

$$\mathbb{E}V_n^{1/H}(X) \le k_H\|g\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1}{H}}. \tag{4.3}$$

Consider the space

$$\mathbb{D}_b^{2,\frac{1}{H}} = \left\{F \in \mathbb{D}^{2,\frac{1}{H}} : F \text{ is bounded}\right\}.$$

Let $\mathfrak{U}_T$ be the space of bounded elementary processes, that is, the space of processes of the form

$$u = \sum_{j=0}^{m-1} F_j\mathbf{1}_{[s_j,s_{j+1})},$$

where $F_j \in \mathbb{D}_b^{2,\frac{1}{H}}$ and $\{s_j\}_{j=0}^m$ is a partition of interval $[0,T]$. We show that the space $\mathfrak{U}_T$ is dense in $\mathbb{L}^{2,\frac{1}{H}}$. We first prove the following auxiliary lemma.

Lemma 4.5. *Let $g \in \mathbb{L}^{2,\frac{1}{H}}$, then for any interval $[a,b] \subseteq [0,T]$, the random variable $\int_a^b g_s \,\mathrm{d}s$ belongs to $\mathbb{D}^{2,\frac{1}{H}}$.*

Proof. By the definition of the norm in $\mathbb{D}^{2,\frac{1}{H}}$, we have

$$\left\|\int_a^b g_s \,\mathrm{d}s\right\|_{\mathbb{D}^{2,\frac{1}{H}}}^{\frac{1}{H}} = \mathbb{E}\left|\int_a^b g_s \,\mathrm{d}s\right|^{\frac{1}{H}} + \mathbb{E}\left\|D\int_a^b g_s \,\mathrm{d}s\right\|_{L^2([0,T])}^{\frac{1}{H}} + \mathbb{E}\left\|D^2\int_a^b g_s \,\mathrm{d}s\right\|_{L^2([0,T]^2)}^{\frac{1}{H}}$$

For the first term, by Hölder's inequality with $p = \frac{1}{H}$, we obtain

$$\begin{aligned}
\mathbb{E}\left|\int_a^b g_s \,\mathrm{d}s\right|^{\frac{1}{H}} &\le \mathbb{E}\left(\int_a^b |g_s| \,\mathrm{d}s\right)^{\frac{1}{H}} \\
&\le \mathbb{E}\left(\int_0^T |g_s|^{\frac{1}{H}} \,\mathrm{d}s\right)\left(\int_0^T 1^{\frac{1}{1-H}} \,\mathrm{d}s\right)^{\frac{1-H}{H}} \\
&= T^{\frac{1-H}{H}}\mathbb{E}\int_a^b |g_s|^{\frac{1}{H}} \,\mathrm{d}s.
\end{aligned}$$

For the second term, by successively applying the generalized Minkowski and Hölder inequality, it is readily seen that the following holds:

$$\begin{aligned}
\mathbb{E}\left\|D\int_a^b g_s \,\mathrm{d}s\right\|_{L^2([0,T])}^{\frac{1}{H}} &= \mathbb{E}\left(\int_0^T \left|\int_0^T \mathbf{1}_{[a,b]}(s) D_x g_s \,\mathrm{d}s\right|^2 \mathrm{d}x\right)^{\frac{1}{2H}} \\
&\le \mathbb{E}\left(\int_0^T \left(\int_0^T |D_x g_s|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s\right)^{\frac{1}{H}} \\
&\le \mathbb{E}\left[\left(\int_0^T \left(\int_0^T |D_x g_s|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s\right)^{H}\left(\int_0^T 1^{\frac{1}{1-H}} \,\mathrm{d}s\right)^{1-H}\right]^{\frac{1}{H}} \\
&= T^{\frac{1-H}{H}}\mathbb{E}\int_0^T \left(\int_0^T |D_x g_s|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s.
\end{aligned}$$

For the last term, by the same arguments as for the second term, we have

$$\mathbb{E}\left\|D^2\int_a^b g_s \,\mathrm{d}s\right\|_{L^2([0,T]^2)}^{\frac{1}{H}} \le T^{\frac{1-H}{H}}\mathbb{E}\int_0^T \left(\int_0^T\int_0^T |D_{x,y} g_s|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2H}} \mathrm{d}s.$$

Altogether, we obtain

$$\begin{aligned}
\left\|\int_a^b g_s \,\mathrm{d}s\right\|_{\mathbb{D}^{2,\frac{1}{H}}}^{\frac{1}{H}} \preceq \mathbb{E}\int_a^b |g_s|^{\frac{1}{H}} \,\mathrm{d}s + \mathbb{E}\int_0^T \left(\int_0^T |D_x g_s|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s \\
+ \mathbb{E}\int_0^T \left(\int_0^T\int_0^T |D_{x,y} g_s|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2H}} \mathrm{d}s = \|g\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1}{H}}
\end{aligned}$$

and therefore $\int_a^b u_s \,\mathrm{d}s \in \mathbb{D}^{2,\frac{1}{H}}$.

□

We can now prove the following.

Lemma 4.6. *The set $\mathfrak{U}_T$ is dense in $\mathbb{L}^{2,\frac{1}{H}}$.*

Proof. For $g \in \mathbb{L}^{2,\frac{1}{H}}$, consider a sequence of processes $\{g^m\}_{m\in\mathbb{N}}$

$$g^m_t = \sum_{j=0}^{m-1} G^m_j \mathbf{1}_{[s^m_j, s^m_{j+1})}(t), \quad t \in [0,T],$$

where $\{s^m_j = \frac{jT}{m}\}_{j=0}^m$ is the uniform partition of interval $[0,T]$ and $\{G^m_j\}_{j=0}^{m-1}$ is defined by

$$G^m_j = \frac{1}{s^m_{j+1} - s^m_j} \int_{s^m_j}^{s^m_{j+1}} g_s \, \mathrm{d}s.$$

According to Lemma 4.5, we have that $G^m_j \in \mathbb{D}^{2,\frac{1}{H}}$. Furthermore, we will show that $g^m \in \mathbb{L}^{2,\frac{1}{H}}$ and that we have the uniform bound $\|g^m\|_{\mathbb{L}^{2,\frac{1}{H}}} \le \|g\|_{\mathbb{L}^{2,\frac{1}{H}}}$. We have

$$\|g^m\|^{\frac{1}{H}}_{\mathbb{L}^{2,\frac{1}{H}}} = \mathbb{E}\int_0^T |g^m_s|^{\frac{1}{H}} \, \mathrm{d}s + \int_0^T \mathbb{E}\|Dg^m_s\|^{\frac{1}{H}}_{L^2([0,T])} \, \mathrm{d}s + \int_0^T \mathbb{E}\|D^2 g^m_s\|^{\frac{1}{H}}_{L^2([0,T]^2)} \, \mathrm{d}s.$$

For the first term, using Hölder's inequality gives

$$\begin{aligned}
\mathbb{E}\int_0^T |g^m_s|^{\frac{1}{H}} \, \mathrm{d}s &= \mathbb{E}\int_0^T \left| \sum_{j=0}^{m-1} \mathbf{1}_{[s^m_j, s^m_{j+1})}(s) \frac{1}{s^m_{j+1} - s^m_j} \int_{s^m_j}^{s^m_{j+1}} g_t \, \mathrm{d}t \right|^{\frac{1}{H}} \mathrm{d}s \\
&= \sum_{j=0}^{m-1} \int_{s^m_j}^{s^m_{j+1}} \mathbb{E}\left| \frac{1}{s^m_{j+1} - s^m_j} \int_{s^m_j}^{s^m_{j+1}} g_t \, \mathrm{d}t \right|^{\frac{1}{H}} \mathrm{d}s \\
&\le \sum_{j=0}^{m-1} \int_{s^m_j}^{s^m_{j+1}} \mathbb{E}\left| \frac{1}{s^m_{j+1} - s^m_j} \left(\int_{s^m_j}^{s^m_{j+1}} |g_t|^{\frac{1}{H}} \, \mathrm{d}t \right)^H \left(\int_{s^m_j}^{s^m_{j+1}} 1^{\frac{1}{1-H}} \, \mathrm{d}t \right)^{1-H} \right|^{\frac{1}{H}} \\
&= \sum_{j=0}^{m-1} \int_{s^m_j}^{s^m_{j+1}} \frac{1}{s^m_{j+1} - s^m_j} \mathbb{E}\int_{s^m_j}^{s^m_{j+1}} |g_t|^{\frac{1}{H}} \, \mathrm{d}t \, \mathrm{d}s \\
&= \sum_{j=0}^{m-1} \mathbb{E}\int_{s^m_j}^{s^m_{j+1}} |g_t|^{\frac{1}{H}} \, \mathrm{d}t \\
&= \mathbb{E}\int_0^T |g_t|^{\frac{1}{H}} \, \mathrm{d}t
\end{aligned}$$

For the second term, successively using the generalized Minkowski inequality twice and Hölder's inequality yields

$$
\begin{aligned}
&\int_0^T \mathbb{E}\|Dg_s^m\|_{L^2([0,T])}^{\frac{1}{H}} \,\mathrm{d}s = \int_0^T \mathbb{E}\left(\int_0^T |D_x g_s^m|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \,\mathrm{d}s \\
&= \int_0^T \mathbb{E}\left(\int_0^T \left| D_x \left(\sum_{j=0}^{m-1} \mathbf{1}_{[s_j^m, s_{j+1}^m)}(s) \frac{1}{s_{j+1}^m - s_j^m} \int_{s_j^m}^{s_{j+1}^m} g_t \,\mathrm{d}t\right)\right|^2 \mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s \\
&= \sum_{j=0}^{m-1} \int_{s_j^m}^{s_{j+1}^m} \mathbb{E}\left(\int_0^T \left|\frac{1}{s_{j+1}^m - s_j^m} \int_{s_j^m}^{s_{j+1}^m} D_x g_t \,\mathrm{d}t\right|^2 \mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s \\
&= \sum_{j=0}^{m-1} (s_{j+1}^m - s_j^m)^{1-\frac{1}{H}} \mathbb{E}\left(\int_0^T \left|\int_{s_j^m}^{s_{j+1}^m} D_x g_t \,\mathrm{d}t\right|^2 \mathrm{d}x\right)^{\frac{1}{2H}} \\
&\le \sum_{j=0}^{m-1} (s_{j+1}^m - s_j^m)^{1-\frac{1}{H}} \mathbb{E}\left(\int_{s_j^m}^{s_{j+1}^m} \left(\int_0^T |D_x g_t|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}t\right)^{\frac{1}{H}} \\
&\le \sum_{j=0}^{m-1} (s_{j+1}^m - s_j^m)^{1-\frac{1}{H}} \left(\int_{s_j^m}^{s_{j+1}^m} \left(\mathbb{E}\left(\int_0^T |D_x g_t|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}}\right)^H \mathrm{d}t\right)^{\frac{1}{H}} \\
&\le \sum_{j=0}^{m-1} (s_{j+1}^m - s_j^m)^{1-\frac{1}{H}} \left(\left(\int_{s_j^m}^{s_{j+1}^m} \mathbb{E}\left(\int_0^T |D_x g_t|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}t\right)^H (s_{j+1}^m - s_j^m)^{1-H}\right)^{\frac{1}{H}} \\
&= \sum_{j=0}^{m-1} \int_{s_j^m}^{s_{j+1}^m} \mathbb{E}\left(\int_0^T |D_x g_t|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}t = \int_0^T \mathbb{E}\left(\int_0^T |D_x g_t|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}t.
\end{aligned}
$$

As for the last term, by exactly the same arguments as for the second term, we obtain

$$
\int_0^T \mathbb{E}\|D^2 g_s^m\|_{L^2([0,T]^2)}^{\frac{1}{H}} \,\mathrm{d}s \le \int_0^T \mathbb{E}\left(\int_0^T \int_0^T |D_{x,y}^2 g_t|^2 \,\mathrm{d}x \,\mathrm{d}y\right)^{\frac{1}{2H}} \mathrm{d}t,
$$

from which we can conclude that $\|g^m\|_{\mathbb{L}^{2,\frac{1}{H}}} \le \|g\|_{\mathbb{L}^{2,\frac{1}{H}}}$ and hence $g^m \in \mathbb{L}^{2,\frac{1}{H}}$. Now, consider processes $g \in \mathbb{L}^{2,\frac{1}{H}}$ such that

$$
t \mapsto g_t
$$

is a continuous mapping. We denote the class of such processes $\mathbb{L}_C^{2,\frac{1}{H}}$. Then for any $g \in \mathbb{L}_C^{2,\frac{1}{H}}$, we have

$$
\begin{aligned}
\int_0^T \mathbb{E}|g_s - g_s^m|^{\frac{1}{H}} \,\mathrm{d}s &= \int_0^T \mathbb{E}\left|g_s - \sum_{j=0}^{m-1} \mathbf{1}_{[s_j^m, s_{j+1}^m)}(s) \frac{1}{s_{j+1}^m - s_j^m} \int_{s_j^m}^{s_{j+1}^m} g_t \,\mathrm{d}t\right|^{\frac{1}{H}} \mathrm{d}s \\
&= \int_0^T \mathbb{E}\left|\sum_{j=0}^{m-1} \mathbf{1}_{[s_j^m, s_{j+1}^m)}(s) \frac{1}{s_{j+1}^m - s_j^m} \int_{s_j^m}^{s_{j+1}^m} (g_s - g_t) \,\mathrm{d}t\right|^{\frac{1}{H}} \mathrm{d}s \\
&\le T \sup_{|t-s| \le \frac{T}{m}} \mathbb{E}|g_t - g_s|^{\frac{1}{H}} \xrightarrow[m\to\infty]{} 0
\end{aligned}
$$

and

$$\int_0^T \mathbb{E}\|D(g_s - g_s^m)\|_{L^2([0,T])}^{\frac{1}{H}} \,\mathrm{d}s = \int_0^T \mathbb{E}\left(\int_0^T |D_x(g_s - g_s^m)|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s$$

$$= \int_0^T \mathbb{E}\left(\int_0^T \left|D_x g_s - \sum_{j=0}^{m-1} \mathbf{1}_{[s_j^m, s_{j+1}^m)}(s) \frac{1}{s_{j+1}^m - s_j^m} \int_{s_j^m}^{s_{j+1}^m} D_x g_t \,\mathrm{d}t\right|^2 \mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s$$

$$= \int_0^T \mathbb{E}\left(\int_0^T \left|\sum_{j=0}^{m-1} \mathbf{1}_{[s_j^m, s_{j+1}^m)}(s) \frac{1}{s_{j+1}^m - s_j^m} \int_{s_j^m}^{s_{j+1}^m} D_x(g_s - g_t) \,\mathrm{d}t\right|^2 \mathrm{d}x\right)^{\frac{1}{2H}} \mathrm{d}s$$

$$\leq T \sup_{|t-s|\leq \frac{T}{m}} \mathbb{E}\left(\int_0^T |D_x(g_t - g_s)|^2 \,\mathrm{d}x\right)^{\frac{1}{2H}} \xrightarrow[m\to\infty]{} 0.$$

By the same arguments as above

$$\int_0^T \mathbb{E}\|D^2(g_s - g_s^m)\|_{L^2([0,T]^2)}^{\frac{1}{H}} \,\mathrm{d}s \xrightarrow[m\to\infty]{} 0.$$

Since $\mathbb{L}_C^{2,\frac{1}{H}}$ is dense in $\mathbb{L}^{2,\frac{1}{H}}$, we get that

$$\|g - g^m\|_{\mathbb{L}^{2,\frac{1}{H}}} \xrightarrow[m\to\infty]{} 0$$

for any $g \in \mathbb{L}^{2,\frac{1}{H}}$.

It remains to show that any $F \in \mathbb{D}^{2,\frac{1}{H}}$ can be approximated by a sequence $\{F_k\} \subset \mathbb{D}_b^{2,\frac{1}{H}}$. For $k \in \mathbb{N}$, let $\varphi_k : \mathbb{R} \to \mathbb{R}$ be functions such that $\varphi_k \in \mathcal{C}^2(\mathbb{R})$,

$$\begin{aligned} \varphi_k(x) &= x, \quad |x| \leq k, \\ \varphi_k(x) &\leq k, \quad |x| > k+1, \\ |\varphi_k'(x)| &\leq 1, \quad x \in \mathbb{R}, \\ |\varphi_k''(x)| &\leq K, \quad x \in \mathbb{R}, \end{aligned}$$

where K is a finite positive constant. Assume, moreover, that

$$\sup_{x\in\mathbb{R}} |\varphi_k'(x) - 1| \xrightarrow[k\to\infty]{} 0, \qquad \sup_{x\in\mathbb{R}} |\varphi_k''(x)| \xrightarrow[k\to\infty]{} 0.$$

For $k \in \mathbb{N}$, set $F_k = \varphi_k(F)$. Then by Proposition 1.3, we have that $F_k \in \mathbb{D}^{2,\frac{1}{H}}$ and that F_k is clearly bounded for each $k \in \mathbb{N}$. We will show that

$$\|F-F_k\|_{\mathbb{D}^{2,\frac{1}{H}}}^{\frac{1}{H}} = \mathbb{E}|F-F_k|^{\frac{1}{H}} + \mathbb{E}\|D(F-F_k)|_{L^2([0,T])}^{\frac{1}{H}} + \mathbb{E}\|D^2(F-F_k)|_{L^2([0,T]^2)}^{\frac{1}{H}} \tag{4.4}$$

converges to zero as $k \to \infty$. Consider the first term of (4.4). By the mean value theorem, we obtain

$$\begin{aligned} \mathbb{E}|F - F_k|^{\frac{1}{H}} &= \mathbb{E}|F - \varphi_k(F)|^{\frac{1}{H}} \\ &\leq 2^{\frac{1}{H}-1}\left(\mathbb{E}|F|^{\frac{1}{H}} + \mathbb{E}|\varphi_k(F)|^{\frac{1}{H}}\right) \\ &= 2^{\frac{1}{H}-1}\left(\mathbb{E}|F|^{\frac{1}{H}} + \mathbb{E}|\varphi_k(F) - \varphi_k(0)|^{\frac{1}{H}}\right) \\ &= 2^{\frac{1}{H}-1}\left(\mathbb{E}|F|^{\frac{1}{H}} + \mathbb{E}\Big(|\varphi_k'(\xi)| \cdot |F - 0|\Big)^{\frac{1}{H}}\right) \\ &\leq 2^{\frac{1}{H}-1}\left(\mathbb{E}|F|^{\frac{1}{H}} + \mathbb{E}|F|^{\frac{1}{H}}\right) \\ &= 2^{\frac{1}{H}}\mathbb{E}|F|^{\frac{1}{H}} < \infty. \end{aligned}$$

And so, by the dominated convergence theorem, we have $\mathbb{E}|F - F_k|^{\frac{1}{H}} \to 0$ as $k \to \infty$. Now consider the second term in (4.4). By using Proposition 1.3, it follows that

$$\begin{aligned}\mathbb{E}\|D(F - F_k)\|^{\frac{1}{H}}_{L^2([0,T])} &= \mathbb{E}\|D(F - \varphi_k(F))\|^{\frac{1}{H}}_{L^2([0,T])} \\ &= \mathbb{E}\|DF - \varphi_k'(F)DF\|^{\frac{1}{H}}_{L^2([0,T])} \\ &\le \sup_{x\in\mathbb{R}} |1 - \varphi_k'(x)|^{\frac{1}{H}} \mathbb{E}\|DF\|^{\frac{1}{H}}_{L^2([0,T])} \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

As for the last term in (4.4), since in view of Proposition 1.3 we have

$$D^2\varphi_k(F) = D\Big(\varphi_k'(F)DF\Big) = \varphi_k'(F)D^2F + \varphi_k''(F)(DF \otimes DF),$$

we obtain

$$\begin{aligned}\mathbb{E}\|D^2(F - F_k)\|^{\frac{1}{H}}_{L^2([0,T]^2)} &= \mathbb{E}\|D^2(F - \varphi_k(F))\|^{\frac{1}{H}}_{L^2([0,T]^2)} \\ &= \mathbb{E}\|D^2F - \varphi_k'(F)D^2F - \varphi_k''(F)(DF \otimes DF)\|^{\frac{1}{H}}_{L^2([0,T]^2)},\end{aligned}$$

where again the right-hand side converges to zero, as $k \to \infty$, by the conditions imposed on φ_k.

□

Proof of Theorem 4.2. Since, by Lemma 4.6, the space $\mathfrak{U}_T$ is dense in $\mathbb{L}^{2,\frac{1}{H}}$, we can find a sequence of bounded elementary processes $\{g^m\} \subseteq \mathfrak{U}_T$ of the form

$$g^m = \sum_{j=0}^{m-1} F_j^m \mathbf{1}_{[s_j^m, s_{j+1}^m)},$$

where $F_j^m \in \mathbb{D}_b^{2,\frac{1}{H}}$ and $\{s_j^m\}_{j=0}^m$ is a partition of interval $[0,T]$, such that

$$\|g - g^m\|_{\mathbb{L}^{2,\frac{1}{H}}} \xrightarrow[m\to\infty]{} 0.$$

Set $X_t^m = \int_0^t g_s^m \, \mathrm{d}Z_s^{H,2}$. Then by the triangle inequality, we have

$$\begin{aligned}\mathbb{E}\left|V_n^{1/H}(X) - C_H \int_0^T |g_s|^{\frac{1}{H}} \,\mathrm{d}s\right| &\le \mathbb{E}|V_n^{1/H}(X) - V_n^{1/H}(X^m)| \\ &\quad + \mathbb{E}\left|V_n^{1/H}(X^m) - C_H \int_0^T |g_s^m|^{\frac{1}{H}} \,\mathrm{d}s\right| \\ &\quad + C_H\mathbb{E}\left|\int_0^T (|g_s^m|^{\frac{1}{H}} - |g_s|^{\frac{1}{H}}) \,\mathrm{d}s\right| \\ &= a_n^m + b_n^m + c^m.\end{aligned}$$

We want to show that all three terms converge to zero. We will prove this in multiple steps.

Step 1. Consider term a_n^m first. From Lemma 4.4, we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} a_n^m &= \sup_{n\in\mathbb{N}} \mathbb{E}|V_n^{1/H}(X) - V_n^{1/H}(X^m)| \\ &\le k_H \|g - g^m\|_{\mathbb{L}^{2,\frac{1}{H}}} \left(\|g\|^{\frac{1-H}{H}}_{\mathbb{L}^{2,\frac{1}{H}}} + \|g^m\|^{\frac{1-H}{H}}_{\mathbb{L}^{2,\frac{1}{H}}}\right) \xrightarrow[m\to\infty]{} 0.\end{aligned}$$

Step 2. Now, consider term b_n^m. For any $F \in \mathbb{D}_b^{2,\frac{1}{H}}$ and interval $[a,b] \subseteq [0,T]$ we have by virtue of Proposition 1.5 the equality

$$\begin{aligned}
&\int_0^T F\mathbf{1}_{[a,b]}(s)\,\mathrm{d}Z_s^{H,2} = \delta^2\Big(\mathcal{K}^{H,2}(F\mathbf{1}_{[a,b]})\Big)\\
&= \delta^2\Big(F\mathcal{K}^{H,2}(\mathbf{1}_{[a,b]})\Big)\\
&= \delta\left(F\delta(\mathcal{K}^{H,2}\mathbf{1}_{[a,b]}) - \Big\langle DF, \mathcal{K}^{H,2}\mathbf{1}_{[a,b]}\Big\rangle_{L^2([0,T])}\right)\\
&= F\delta^2(\mathcal{K}^{H,2}\mathbf{1}_{[a,b]}) - \Big\langle DF, \delta(\mathcal{K}^{H,2}\mathbf{1}_{[a,b]})\Big\rangle_{L^2([0,T])} - \delta\left(\Big\langle DF, \mathcal{K}^{H,2}\mathbf{1}_{[a,b]}\Big\rangle_{L^2([0,T])}\right)\\
&= F(Z_b^{H,2} - Z_a^{H,2}) - \Big\langle DF, \delta(\mathcal{K}^{H,2}\mathbf{1}_{[a,b]})\Big\rangle_{L^2([0,T])} - \delta\left(\Big\langle DF, \mathcal{K}^{H,2}\mathbf{1}_{[a,b]}\Big\rangle_{L^2([0,T])}\right).
\end{aligned}$$

By appealing to this computation, we can express X_t^m for $t \in [0,T]$ as

$$\begin{aligned}
X_t^m &= \int_0^t g_s^m\,\mathrm{d}Z_s^{H,2}\\
&= \Big(\delta^2 \circ \mathcal{K}^{H,2}\Big)\left(g^m\mathbf{1}_{[0,t]}\right)\\
&= \Big(\delta^2 \circ \mathcal{K}^{H,2}\Big)\left(\sum_{j=0}^{m-1} F_j^m \mathbf{1}_{[s_j^m, s_{j+1}^m)}\mathbf{1}_{[0,t]}\right)\\
&= \sum_{j=0}^{m-1} F_j^m(Z^{H,2}_{s_{j+1}^m\wedge t} - Z^{H,2}_{s_j^m\wedge t}) - \sum_{j=0}^{m-1}\Big\langle DF_j^m, \delta(\mathcal{K}^{H,2}\mathbf{1}_{[s_j^m, s_{j+1}^m)}\mathbf{1}_{[0,t]})\Big\rangle_{L^2([0,T])}\\
&\qquad - \sum_{j=0}^{m-1}\delta\left(\Big\langle DF_j^m, \mathcal{K}^{H,2}\mathbf{1}_{[s_j^m, s_{j+1}^m)}\mathbf{1}_{[0,t]}\Big\rangle_{L^2([0,T])}\right)\\
&= A_t^1 - A_t^2 - A_t^3.
\end{aligned}$$

We have

$$\begin{aligned}
A_t^2 &= \sum_{j=0}^{m-1}\int_0^T D_y F_j^m\,\delta(\mathcal{K}^{H,2}\mathbf{1}_{[s_j^m, s_{j+1}^m)}\mathbf{1}_{[0,t]})(y)\,\mathrm{d}y\\
&= c(H,2)\sum_{j=0}^{m-1}\int_0^T D_{y_1}F_j^m\delta\Bigg(\int_0^T \mathbf{1}_{[s_j^m, s_{j+1}^m)}(u)\mathbf{1}_{[0,t]}(u)\\
&\qquad\qquad \times\left(\frac{u}{y_1}\right)^{\frac{H}{2}}(u-y_1)_+^{\frac{H}{2}-1}\left(\frac{u}{y_2}\right)^{\frac{H}{2}}(u-y_2)_+^{\frac{H}{2}-1}\,\mathrm{d}u\Bigg)\,\mathrm{d}y_1,
\end{aligned}$$

where the divergence acts on the variable y_2. By linearity and closability of the divergence, one can interchange the Lebesgue integral and the divergence operator (see [9, Exercise 3.2.7]) and write

$$\begin{aligned}
A_t^3 &= \sum_{j=0}^{m-1} \delta\left(\left\langle DF_j^m, \mathcal{K}^{H,2}\mathbf{1}_{[s_j^m,s_{j+1}^m)}\mathbf{1}_{[0,t]}\right\rangle_{L^2([0,T])}\right) \\
&= \sum_{j=0}^{m-1} \delta\left(\int_0^T D_y F_j^m\, \mathcal{K}^{H,2}(\mathbf{1}_{[s_j^m,s_{j+1}^m)}\mathbf{1}_{[0,t]})(y,\cdot)\,\mathrm{d}y\right) \\
&= \sum_{j=0}^{m-1} \int_0^T \delta\left(D_y F_j^m\, \mathcal{K}^{H,2}(\mathbf{1}_{[s_j^m,s_{j+1}^m)}\mathbf{1}_{[0,t]})(y,\cdot)\right)\,\mathrm{d}y \\
&= \sum_{j=0}^{m-1} \int_0^T D_y F_j^m \delta(\mathcal{K}^{H,2}\mathbf{1}_{[s_j^m,s_{j+1}^m)}\mathbf{1}_{[0,t]})(y)\,\mathrm{d}y \\
&\qquad - \sum_{j=0}^{m-1} \int_0^T \left\langle D_{y,\cdot}^2 F_j^m, \mathcal{K}^{H,2}(\mathbf{1}_{[s_j^m,s_{j+1}^m)}\mathbf{1}_{[0,t]})(y,\cdot)\right\rangle_{L^2([0,T])}\,\mathrm{d}y \\
&= A_t^2 - A_t^4,
\end{aligned}$$

where in the second to last equality we used Proposition 1.5. Finally,

$$\begin{aligned}
A_t^4 &= \sum_{j=0}^{m-1} \int_0^T \int_0^T D_{y_1,y_2}^2 F_j^m \mathcal{K}^{H,2}(\mathbf{1}_{[s_j^m,s_{j+1}^m)}\mathbf{1}_{[0,t]})(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&= c(H,2)\sum_{j=0}^{m-1} \int_0^T \int_0^T D_{y_1,y_2}^2 F_j^m \int_0^T \mathbf{1}_{[s_j^m,s_{j+1}^m)}(u)\mathbf{1}_{[0,t]}(u) \\
&\qquad \times \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\,\mathrm{d}u\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&= c(H,2)\int_0^t \int_0^T \int_0^T D_{y_1,y_2}^2 (g^m(u)) \\
&\qquad \times \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}u.
\end{aligned}$$

Summarized, we expressed the process X_t^m as

$$X_t^m = \sum_{j=0}^{m-1} F_j^m (Z_{s_{j+1}^m \wedge t}^{H,2} - Z_{s_j^m \wedge t}^{H,2}) - 2A_t^2 + A_t^4 = Y_t^m + Z_t^m,$$

where we set $Y_t^m = A_t^1$ and $Z_t^m = -2A_t^2 + A_t^4$. Then we can estimate

$$b_n^m \le \mathbb{E}|V_n^{1/H}(X^m) - V_n^{1/H}(Y^m)| + \mathbb{E}\left|V_n^{1/H}(Y^m) - C_H \int_0^T |u_s^m|\,\mathrm{d}s\right| = d_n^m + e_n^m.$$

Step 2.1. Take the term d_n^m. By Lemma 4.3, we have the estimate

$$d_n^m \le \frac{1}{H}\left(\mathbb{E}V_n^{1/H}(Z^m)\right)^H \left((\mathbb{E}V_n^{1/H}(X^m))^{1-H} + (\mathbb{E}V_n^{1/H}(Y^m))^{1-H}\right)$$

Using estimate (4.3) yields

$$\mathbb{E}V_n^{1/H}(X^m) \le k_H \|g^m\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1}{H}}$$

and

$$\mathbb{E}V_n^{1/H}(Y^m) \leq \sum_{j=0}^{m-1} \sup_{\omega\in\Omega} |F_j^m(\omega)|^{\frac{1}{H}} \mathbb{E}V_n^{1/H}\left(\int_0^{\cdot} \mathbf{1}_{[s_j^m, s_{j+1}^m)}\, \mathrm{d}Z^{H,2}\right)$$
$$\leq k_H \sum_{j=0}^{m-1} \sup_{\omega\in\Omega} |F_j^m(\omega)|^{\frac{1}{H}} \|\mathbf{1}_{[s_j^m, s_{j+1}^m)}\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1}{H}} < \infty.$$

Hence, we found bounds for terms $\mathbb{E}V_n^{1/H}(X^m)$ and $\mathbb{E}V_n^{1/H}(Y^m)$ which are independent of $n \in \mathbb{N}$. If we assume that Z^m is a process of zero $1/H$-variation, then we obtain $d_n^m \xrightarrow[n\to\infty]{} 0$.

Step 2.1.1. We show that process Z^m has zero $1/H$-variation. It is enough to show that both processes A_t^2 and A_t^4 have zero $1/H$-variation since for two arbitrary stochastic processes X and Y it holds

$$\begin{aligned} V_n^{1/H}(X+Y) &= \sum_{i=0}^{2^n-1} \left|(X_{t_{i+1}^n} + Y_{t_{i+1}^n}) - (X_{t_i^n} + Y_{t_i^n})\right|^{\frac{1}{H}} \\ &= \sum_{i=0}^{2^n-1} \left|(X_{t_{i+1}^n} - X_{t_i^n}) + (Y_{t_{i+1}^n} - Y_{t_i^n})\right|^{\frac{1}{H}} \\ &\leq 2^{\frac{1}{H}-1} \sum_{i=0}^{2^n-1} \left(|X_{t_{i+1}^n} - X_{t_i^n}|^{\frac{1}{H}} + |Y_{t_{i+1}^n} - Y_{t_i^n}|^{\frac{1}{H}}\right) \\ &= 2^{\frac{1}{H}-1} \left(V_n^{1/H}(X) + V_n^{1/H}(Y)\right). \end{aligned}$$

Consider the process A_t^4 first. We will show that the process

$$B_t = \int_0^t \int_0^T \int_0^T D^2_{y_1,y_2} F \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}u,$$

where $F \in \mathbb{D}^{2,\frac{1}{H}}$, is a process of bounded variation. Using twice Hölder's inequality, it follows that

$$\mathbb{E}\left|\int_0^t \int_0^T \int_0^T D^2_{y_1,y_2} F \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}u\right|$$

$$\leq \mathbb{E}\int_0^T \int_0^T \int_0^T |D^2_{y_1,y_2} F| \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}u$$

$$\leq \mathbb{E}\left(\int_0^T \int_0^T |D^2_{y_1,y_2} F|^2\, \mathrm{d}y_1\, \mathrm{d}y_2\right)^{\frac{1}{2}}$$

$$\times \left(\int_0^T \int_0^T \left(\int_0^T \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u\right)^2 \mathrm{d}y_1\, \mathrm{d}y_2\right)^{\frac{1}{2}} \tag{4.5}$$

$$\preceq \left(\mathbb{E}\left(\int_0^T \int_0^T |D^2_{y_1,y_2} F|^2\, \mathrm{d}y_1\, \mathrm{d}y_2\right)^{\frac{1}{2H}}\right)^H \|1\|_{\mathfrak{H}_H} < \infty,$$

where the second factor in (4.5) can be computed as was done in Section 3.1. The $\mathfrak{H}_H$-norm of 1 is indeed finite. In fact, it is equal to T^H due to (2.1). Hence B_t (and consequently A_t^4) is of bounded variation and therefore of zero $1/H$-variation.

In similar manner, we show that the process A_t^2 is also of bounded variation. Consider the process

$$N_t = \int_0^T D_{y_1} F\, \delta\Bigg(\int_0^T \mathbf{1}_{[0,t]}(u) \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u \Bigg)\, \mathrm{d}y_1$$
$$= \int_0^t \int_0^T D_{y_1} F\, \delta\Bigg(\left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \Bigg)\, \mathrm{d}y_1\, \mathrm{d}u$$

where $F \in \mathbb{D}^{2,\frac{1}{H}}$ and where again the divergence acts on the variable y_2. Applying twice Hölder's inequality yields

$$\mathbb{E}\left| \int_0^t \int_0^T D_{y_1} F\, \delta\Bigg(\left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \Bigg)\, \mathrm{d}y_1\, \mathrm{d}u \right|$$
$$= \mathbb{E}\left| \int_0^T D_{y_1} F\, \delta\Bigg(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u \Bigg)\, \mathrm{d}y_1 \right|$$
$$\leq \mathbb{E}\left(\int_0^T |D_{y_1} F|^2\, \mathrm{d}y_1 \right)^{\frac{1}{2}}$$
$$\times \left(\int_0^T \left| \delta\Bigg(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u \Bigg) \right|^2 \mathrm{d}y_1 \right)^{\frac{1}{2}}$$

$$\leq \left[\mathbb{E}\left(\int_0^T |D_{y_1} F|^2\, \mathrm{d}y_1 \right)^{\frac{1}{2H}} \right]^H \times$$
$$\left[\mathbb{E}\left(\int_0^T \left| \delta\Bigg(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u \Bigg) \right|^2 \mathrm{d}y_1 \right)^{\frac{1}{2(1-H)}} \right]^{1-H}$$

where the first factor is obviously finite since we assumed $F \in \mathbb{D}^{2,\frac{1}{H}}$. By the generalized Minkowski inequality applied to the second factor, we obtain

$$\left[\mathbb{E}\left(\int_0^T \left| \delta\Bigg(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u \Bigg) \right|^2 \mathrm{d}y_1 \right)^{\frac{1}{2(1-H)}} \right]^{1-H}$$

$$\leq \Bigg[\int_0^T \Bigg(\mathbb{E}\Bigg| \delta\Bigg(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1}$$
$$\times \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1}\, \mathrm{d}u \Bigg) \Bigg|^{\frac{1}{1-H}} \Bigg)^{2(1-H)} \mathrm{d}y_1 \Bigg]^{\frac{1}{2}}.$$

By using Theorem 1.11 and Proposition 1.9, it follows that

$$
\begin{aligned}
&\left(\mathbb{E}\left|\delta\left(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)\right|^{\frac{1}{1-H}}\right)^{2(1-H)} \\
&= \left\|\delta\left(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)\right\|_{L^{\frac{1}{1-H}}(\Omega)}^2 \\
&\preceq \left\|\delta\left(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)\right\|_{L^2(\Omega)}^2 \\
&= \left\|\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right\|_{L^2([0,T])}^2 \\
&= \int_0^T \left(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)^2 \mathrm{d}y_2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left[\mathbb{E}\left(\int_0^T \left|\delta\left(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)\right|^2 \mathrm{d}y_1\right)^{\frac{1}{2(1-H)}}\right]^{1-H} \\
&\preceq \left[\int_0^T\int_0^T \left(\int_0^t \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)^2 \mathrm{d}y_2\,\mathrm{d}y_1\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^T\int_0^T \left(\int_0^T \left(\frac{u}{y_1}\right)^{\frac{H}{2}} (u-y_1)_+^{\frac{H}{2}-1} \left(\frac{u}{y_2}\right)^{\frac{H}{2}} (u-y_2)_+^{\frac{H}{2}-1} \,\mathrm{d}u\right)^2 \mathrm{d}y_2\,\mathrm{d}y_1\right]^{\frac{1}{2}} \\
&\propto \|1\|_{\mathfrak{H}_H} < \infty
\end{aligned}
$$

and so the process N_t (and therefore also the process A_t^2) is of bounded variation and thus also of zero $1/H$-variation.
Step 2.2. By taking $p = \frac{1}{H}$ and $q = 2$ in (2.2), we obtain

$$
\mathbb{E}|Z_{s_{j+1}^m}^{H,2} - Z_{s_j^m}^{H,2}|^{\frac{1}{H}} = C_H |s_{j+1}^m - s_j^m|.
$$

Then we have

$$
\begin{aligned}
e_n^m &= \mathbb{E}\left|V_n^{1/H}(Y^m) - C_H \int_0^T |g_s^m| \, \mathrm{d}s\right| \\
&= \mathbb{E}\left|V_n^{1/H}(Y^m) - C_H \sum_{j=0}^{m-1} |F_j^m|^{\frac{1}{H}} (s_{j+1}^m - s_j^m)\right| \\
&= \mathbb{E}\left|V_n^{1/H}(Y^m) - \sum_{j=0}^{m-1} |F_j^m|^{\frac{1}{H}} \mathbb{E}|Z^{H,2}_{s^m_{j+1}} - Z^{H,2}_{s^m_j}|^{\frac{1}{H}}\right| \\
&\le \sum_{j=0}^{m-1} \sup_{\omega\in\Omega} |F_j^m(\omega)|^{\frac{1}{H}} \\
&\qquad \times \mathbb{E}\left|\left(\sum_{i\in J_j^{m,n}} |Z^{H,2}_{t^n_{i+1}} - Z^{H,2}_{t^n_i}|^{\frac{1}{H}}\right) - \mathbb{E}|Z^{H,2}_{s^m_{j+1}} - Z^{H,2}_{s^m_j}|^{\frac{1}{H}}\right| \xrightarrow[n\to\infty]{} 0,
\end{aligned}
$$

where $J_j^{m,n} = \{i : t_i^n \in [s_j^m, s_{j+1}^m)\}$ and where the fact that the last term converges to zero follows by (4.1), self-similarity, and the stationarity of increments of the Rosenblatt process. Therefore, we have that $b_n^m \to 0$ as $n \to \infty$.

Step 3. Finally, consider the term c^m. Applying the mean value theorem in the same way as in proof of Lemma 4.3 and Hölder's inequality gives

$$
\begin{aligned}
c^m &= C_H \mathbb{E}\left|\int_0^T (|g_s^m|^{\frac{1}{H}} - |g_s|^{\frac{1}{H}}) \, \mathrm{d}s\right| \\
&\le \frac{C_H}{H} \mathbb{E}\int_0^T |g_s^m - g_s|(|g_s^m|^{\frac{1}{H}-1} + |g_s|^{\frac{1}{H}-1}) \, \mathrm{d}s \\
&= \frac{C_H}{H} \mathbb{E}\int_0^T |g_s^m - g_s||g_s^m|^{\frac{1}{H}-1} \, \mathrm{d}s + \frac{C_H}{H} \mathbb{E}\int_0^T |g_s^m - g_s||g_s|^{\frac{1}{H}-1} \, \mathrm{d}s \\
&\le \frac{C_H}{H} \left(\mathbb{E}\int_0^T |g_s^m - g_s|^{\frac{1}{H}} \, \mathrm{d}s\right)^H \left(\mathbb{E}\int_0^T |g_s^m|^{\frac{1}{H}} \, \mathrm{d}s\right)^{1-H} \\
&\qquad + \frac{C_H}{H} \left(\mathbb{E}\int_0^T |g_s^m - g_s|^{\frac{1}{H}} \, \mathrm{d}s\right)^H \left(\mathbb{E}\int_0^T |g_s|^{\frac{1}{H}} \, \mathrm{d}s\right)^{1-H} \\
&\le \frac{C_H}{H} \|g^m - g\|_{\mathbb{L}^{2,\frac{1}{H}}} (\|g^m\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1-H}{H}} + \|g\|_{\mathbb{L}^{2,\frac{1}{H}}}^{\frac{1-H}{H}}) \xrightarrow[m\to\infty]{} 0,
\end{aligned}
$$

where we used the embedding $\mathbb{D}^{2,\frac{1}{H}} \hookrightarrow L^{\frac{1}{H}}(\Omega)$. This concludes the proof.

□

Conclusion

As the main result of the thesis, we showed that for $g \in \mathbb{L}^{2,\frac{1}{H}}$, the Skorokhod integral with respect to the Rosenblatt process $\int_0^\cdot g_s \, \mathrm{d}Z_s^{H,2}$ has $1/H$-variation equal to

$$C_H \int_0^T |g_s|^{\frac{1}{H}} \, \mathrm{d}s,$$

where $C_H = \mathbb{E}|Z_1^{H,2}|^{\frac{1}{H}}$.

It is plausible that a result similar to Theorems 4.1 and 4.2 should also hold for the stochastic integral with respect to the Hermite process of a general order q. Such integral would be defined by

$$\int_0^t g_s \, \mathrm{d}Z_s^{H,q} = \delta^q \left(\mathcal{K}^{H,q}(g\mathbf{1}_{[0,t]}) \right),$$

for any Borel measurable $g : [0,T] \to L^2(\Omega)$, such that $\mathcal{K}^{H,q}(g\mathbf{1}_{[0,t]}) \in \operatorname{Dom} \delta^q$. From the analysis of the proof of Theorem 4.2, it seems that out of the properties of the integrator, the proof depends only on the self-similarity, stationarity of increments, and the form of $1/H$-variation of the integrator itself. Furthermore, the proof builds on the mapping property of the integral (Proposition 3.3).

However, we have all these properties for a general Hermite process as well. The self-similarity and stationarity of increments follows form Proposition 2.1 and we have the right form of $1/H$-variation by Proposition 2.3. A mapping property similar to Proposition 3.3 can be derived also for the integral $\int_0^T (\dots) \, \mathrm{d}Z^{H,q}$ as the proof of this proposition is mainly a consequence of the Meyer inequalities and the mapping property of the transfer operator $\mathcal{K}^{H,q}$ (which was proven in Proposition 3.1 for arbitrary $q \in \mathbb{N}$).

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A. Appendix

A.1 Special functions

Definition A.1. For $x > 0$, we define *the gamma function* $\Gamma(x)$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} \, \mathrm{e}^{-t} \, \mathrm{d}t.$$

Definition A.2. For $x > 0, y > 0$, we define *the beta function* $B(x, y)$ by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, \mathrm{d}t$$

Proposition A.1. *Let $u, w, a \in \mathbb{R}$ be such that $a < -\frac{1}{2}$ and $u \neq w$. Then*

(i)

$$\int_{-\infty}^{u \wedge w} (u-y)^a (w-y)^a \, \mathrm{d}y = B(a+1, -2a-1)|u-w|^{2a+1}, \tag{A.1}$$

(ii)

$$\int_0^{u \wedge w} y^{-2a-2}(u-y)^a (w-y)^a \, \mathrm{d}y = B(a+1, -2a-1)(uw)^{-a-1}|u-w|^{2a+1}. \tag{A.2}$$

Proof. We first show (ii). Considering the case $w \leq u$, then by substitution $z = \frac{u-y}{w-y}$

$$\int_0^w y^{-2a-2}(u-y)^a (w-y)^a \, \mathrm{d}y = (u-w)^{2a+1} \int_{\frac{u}{w}}^\infty (wz-u)^{-2a-2} z^a \, \mathrm{d}z.$$

By change of variables $x = \frac{u}{wz}$, we obtain

$$\begin{aligned}(u-w)^{2a+1} \int_{\frac{u}{w}}^\infty (wz-u)^{-2a-2} z^a \, \mathrm{d}z &= (u-w)^{2a+1}(uw)^{-a-1} \int_0^1 x^a (1-x)^{-2a-2} \, \mathrm{d}x \\ &= (u-w)^{2a+1}(uw)^{-a-1} B(a+1, -2a-1),\end{aligned}$$

which proves (ii). In order to show (i), it is enough to use the substitution $z = \frac{u-y}{w-y}$.

□

A.2 Fractional calculus

We state basic definitions and a useful result regarding the fractional calculus. For more information about the topic see [25].

Definition A.3. Let $a, b \in \mathbb{R}, \alpha > 0$ and $f \in L^1(a, b)$. We define *the left and right-sided fractional integrals of f of order α* for $x \in (a, b)$ by

$$(\mathcal{I}_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} \, \mathrm{d}t,$$

$$(\mathcal{I}_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} \, \mathrm{d}t.$$

The following result provides a mapping property of fractional integrals between L^p-spaces.

Theorem A.2 ([25, Theorem 3.5])**.** *If* $0 < \alpha < 1, 1 < p < \frac{1}{\alpha}$*, then the fractional integrals* $\mathcal{I}_{a+}^{\alpha}, \mathcal{I}_{b-}^{\alpha}$ *are bounded linear operators from* $L^p(a,b)$ *to* $L^q(a,b)$*, where* $q = p/(1-\alpha p)$*.*