FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

David Ryzák

Stochastic cooperative games

Department of Applied Mathematics

Supervisor of the master thesis: RNDr. Martin Černý
Consultant of the master thesis: doc. RNDr. Ing. Miloš Kopa, Ph.D.
Study programme: Probability, Mathematical Statistics and Econometrics

Prague 2024

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

In this part I would like to thank to all who helped me and supported in my studies and while writing this thesis. Mainly, I thank to my supervisor RNDr. Martin Černý for his consistent and great support in the whole thesis making process.

Title: Stochastic cooperative games

Author: David Ryzák

Department: Department of Applied Mathematics

Supervisor: RNDr. Martin Černý, Department of Applied Mathematics

Abstract: This thesis explores stochastic cooperative games, viewed here as cooperative games with a stochastic characteristic function, representing a generalization of the classical deterministic model by von Neumann and Morgenstern. To address the inherent randomness, it is essential to either access additional information about the game or understand its stochastic structure thoroughly. The main contribution of this thesis is the exploration of solution concepts within the stochastic context, defined by assuming the risk averse behaviors of the players. This is particularly achieved through the application of the second order stochastic dominance (SSD). We both define and examine the notion of the SSD-dominating core across various distributions of the characteristic function and apply it to the multiple newsvendors problem. Our findings concerning the nonemptiness of the SSD-dominating core offer a framework for addressing risk aversion in stochastic cooperative games without requiring specific assumptions about the levels of risk aversion among players.

Keywords: cooperative game, random characteristic function, newsvendor problem

Contents

Notation and Abbreviations

To avoid ambiguity and ensure clarity, we introduce some notations and abbreviations that require no further explanation.
The expressions below are presented in a more succinct form on the left-hand side to simplify the notation of coalitions and characteristic function:

- $v(i) = v(\{i\})$,
- $v(ijk) = v(\{i, j, k\})$,
- $x(S) = \sum_{i\in} x_i$,
- $S \setminus i$ instead of $S \setminus \{i\}$,
- $S \cup i$ instead of $S \cup \{i\}$.

Models of stochastic TU-games have the following abbreviations:

- SB model. . . Suijs and Borm model,
- CG model . . . Charnes and Granot model,
- SHS model . . . Sun et al. (optimization) model.

The following terms are listed in no specific order:

- The first derivative of a function f: $f'(x)$,
- The second derivative of a function f: $f''(x)$,
- Marginal contribution of i to S: $v(S \cup i) - v(S)$,
- If A is a matrix A_{ij} is entry in $i-$th row and j-th column,
- $x \in C_{\leq}(v)$ if $\forall S \subseteq N : x(S) \geq v(S)$,
- u_α^X is the α-quantile of X,
- Correlation between x_i and x_j: $\rho_{i,j} = \dfrac{\mathbf{cov}(x_i, x_j)}{\sqrt{\mathbf{Var}(x_i)\mathbf{Var}(x_j)}}$.

We use the following distributions with standard notation:

- Uniform distribution on $[a, b]$ denoted by $U[a, b]$,
- Normal distribution denoted by $N(\mu, \sigma^2)$,
- Gamma distribution denoted by $\Gamma(k, \theta)$.

Introduction

This thesis explores solution concepts within stochastic cooperative games, focusing on stochastic TU-games, i.e., when deterministic assumptions are inadequate. Specifically, the thesis surveys models and solution concepts for the stochastic TU-games and contributes to the theory with the solution concepts, namely those incorporating stability by using the second order stochastic dominance. The second order stochastic dominance allows for modeling a risk aversion in a general and structured manner, without needing specific assumptions about the players' levels of the risk aversion. As a practical application, the thesis extends the newsvendor problem to a multi newsvendors context, using solution concepts based on the second order stochastic dominance. A summarized content structure of the thesis is outlined below.

The first chapter provides a comprehensive survey of stochastic TU-games and introduces deterministic cooperative games, exploring their connections with noncooperative games and it defines stochastic dominance. It further surveys models of stochastic TU-games, spanning from 1973, the year the first model was introduced, to the present. The survey covers the most influential and studied models, namely those developed by Charnes and Granot [1], Suijs and Borm [2], Habis and Herings [3], and Sun et al. [4].

The second chapter provides an overview of the solution concepts developed for stochastic TU-games, as introduced in the first chapter, and discusses their properties and reasonableness. The solution concepts include generalization of both the core and the Shapley value. Subsequent chapters present the original results of this thesis.

The third chapter, which consists of the main contribution of this thesis, introduces a generalization of the core from deterministic TU-games to stochastic TU-games, assuming players are risk averse. This chapter focuses on the so called SSD-dominating core and SSD-undominated core. The distinction between the undominated and dominating aspects of the SSD-core arises because the SSD framework induces a partial ordering on random variables. We primarily examine the SSD-dominating core, deriving conditions for its nonemptiness across various distributions of the characteristic function v, including normal, uniform continuous, uniform discrete, and gamma distributions. These conditions are also explored through several types of payoffs.

The brief fourth chapter investigates the concepts of individual stability for coalition structures in stochastic TU-games. We present findings regarding the existence of an individually stable coalition structure for risk-averse players under a normally distributed characteristic function. Additionally, we explore the relationship between this stability and the stability induced by the SSD-undominated core.

The final chapter applies the concept of the SSD-dominating core to the multiple newsvendors game. Initially, the classical newsvendor problem is revisited, followed by an overview of its extension, as detailed in the literature, to the multiple newsvendors game, providing context for the application of the SSD-dominating core. Subsequently, the chapter derives the conditions for the nonemptiness of the SSD-dominating core when the demand is uniformly distributed.

1. Cooperative game theory

The main purpose of the first chapter of this thesis is to recall fundamental model of cooperative games under both deterministic payoffs and stochastic payoffs and to define terms needed further. In the survey of already existing results, we discuss a few ways to incorporate randomness of the characteristic function. Let us begin with preliminaries, formally introducing the notions of preferences and second-order stochastic dominance.

1.1 Preliminaries

Definition 1 (Preferences)**.** *A binary relation $\preceq$ over the set X is:*

- *a* partial order preference *if it is reflexive, antisymmetric and transitive,*
- *a* total order preference *if for any $a, b \in X$ holds $a \preceq b$ or $b \preceq a$.*

Definition 2 (Utility function)**.** Utility function *u is a function $u \colon \mathbb{R} \to \mathbb{R}$ satisfying the following two conditions:*

- *u is continuous,*
- *u is nondecreasing.*

The continuity of the utility function u serves as a technical assumption. The key assumption is that of monotonicity, which ensures that the utility u function does not decrease as the value increases. This property is essential for guaranteeing that higher values are always preferred or valued equally, reflecting rational preference behavior.

Definition 3 (Risk aversion, neutrality and lovingness)**.** *Let x be a random variable with a finite expected value, u_i be a utility function of a player i and $\preceq_i$ his preferences over the random variables with the finite expected value.*
The player i is:

- risk averse *if u_i is a concave function or $x \preceq_i \mathbb{E}x$,*
- risk neutral *if u_i is a linear function or $x \sim_i \mathbb{E}x$,*
- risk loving *if u_i is a convex function or $x \succeq_i \mathbb{E}x$.*

Remark. We use the symbol $\sim$ to denote indifference between two outcomes. In the context of a total order, this indifference implies that the two compared objects are equivalent in value or utility. In the case of a partial order, however, $\sim$ indicates that the objects are either incomparable due to the lack of a clear preference hierarchy or that they are equivalent.

The utility function directly specifies a preference relation; however, as we can see, for instance, in the notion of stochastic dominance, we may use more utility functions at once to describe a preference. Constructing a utility function for each player can sometimes be an impossible or too complex task. It is thus beneficial to have a method to order payoffs without the need for explicit utility functions. Stochastic dominance offers a framework for this purpose. In the book of Wolfstetter et al. [5] can be found the following definitions.

Definition 4 (First order stochastic dominance (FSD)). *Let X, Y be random variables and F_X, F_Y their cumulative distribution functions. We say X stochastically dominates Y if*

$$\forall u \in \mathbb{R} : F_X(u) \leq F_Y(u),$$

or equivalently if for all utility functions u:

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]. \tag{1.1}$$

Definition 5 (Second order stochastic dominance (SSD)). *Let X, Y be random variables and F_X, F_Y their cumulative distribution functions. We say X stochastically dominates Y in second order sense if*

$$\forall u \in \mathbb{R} : \int_{-\infty}^{u} (F_X(z) - F_Y(z))\, dz \leq 0,$$

or equivalently if for all concave utility functions u, i.e., utility functions for which $\forall x \in \mathbb{R} : u''(x) \leq 0$:

$$\mathbb{E}u(X) \geq \mathbb{E}u(Y). \tag{1.2}$$

Remark. We can similarly define the notion of the higher order stochastic dominance by restricting the set of utility functions in the definition of stochastic dominance even more.

For the derivation of our results, we use only the second order stochastic dominance. The first order stochastic dominance is defined for the sake of completeness. For the interpretation of the stochastic dominance notion, we use (1.1) and (1.2). First-order stochastic dominance occurs when all players, regardless of their individual utility functions, agree that one outcome is consistently better than or at least as good as another across all possible states. SSD is a slightly weaker condition since there are fewer utility functions that satisfy (1.2). Contrary to FSD, the condition for SSD considers only concave utility functions, hence SSD occurs when all risk averse players, i.e., utility maximizers with concave utility function, prefer one outcome over the other one. There is a useful property directly following from the definition of SSD and the fact that a linear function is also concave.

Claim 1. *If $X \succeq_{SSD} Y$ then $\mathbb{E}X \geq \mathbb{E}Y$.*

Both SSD and FSD induce partial orderings on random payoffs. Although partial orders present challenges, the insight provided by SSD into the preferences of risk averse players outweighs these issues.

1.2 Noncooperative vs cooperative games

This section serves as an introduction to cooperative game theory, primarily for those familiar with its noncooperative counterpart.

We can divide games into two main branches which are cooperative and noncooperative games. Most of the people familiar with mathematics, economics or computer science know the concept of noncooperative game (we can think of normal form games or more specifically matrix games), e.g., the game called

Rock-paper-scissors. One of the most famous solution concepts is the Nash equilibrium. The Nash equiblibrium can be vaguely described as a set of strategies of players, under which no player wants to deviate, e.g., in Rock-paper-scissors such a set of strategies is to uniformly randomize over all three possibilities. The Nash equilibrium describes a stable outcome of the game. Ideas of stability are also present in the cooperative game theory. In cooperative game theory, the solution concept known as the core is famous; it can also be viewed as a stable solution concept. However, let us focus on differences in the cooperative and noncooperative approach to games.

Cooperative and noncooperative games follow two different approaches to model strategic situations. Noncooperative games are mainly interested in actions that players can take. Such a framework enables us to model vast majority of situations and analyze sets of actions leading to adequate solutions in given terms (equilibria, efficient solutions, solutions maximizing social welfare). However, a cooperative game does not use explicitly sets of actions and strategies. The actions are rather inherently given in the game itself in the form of values of groups of players. We aim to understand the cooperation outcomes by values given to coalitions, i.e., to groups of players. From this perspective, we can model situations that are not addressable using solely noncooperative models. Given the differences in the model, the questions asked within the cooperative framework also differ from those in noncooperative models. In cooperative game theory, the focus shifts to questions about how to reasonably distribute value or determine which coalitions are likely to form, rather than concentrating on the strategies individual players might play.

The last thing we mention regarding the relationship is the Nash Program. It was initiated by John Nash in 1953, and it pursuits bridging of the disciplines of cooperative and noncooperative game theory for over 70 years. A survey of contributions made in this field over the last seven decades can be found in [6].

1.3 Cooperative games

This section seeks to provide fundamental definitions of cooperative games and should be used to get to know them together with the notation used further. It is not meant to be introductory text on general cooperative games. Thorough introduction with various solution concepts can be found in the book by Peters [7] together with all definitions from this section. Let us begin by recalling the fundamental deterministic model of transferable utility game.

Definition 6 (TU-game)**.** TU-game *is a pair* (N, v)*, where* $N = \{1, 2, \ldots, n\}$ *is a set of players and* v *is a function assigning a real value to each subset (coalition)* $S \subseteq N$ *and* $v(\emptyset) = 0$*. The function* v *is called characteristic function.*

TU-game and its characteristic function serves as a tool for modelling worths of groups of players. Apart from this, we can also define the cooperative coalitional model in other ways. One of these lead to nontransferable utility game or just NTU-game. The main difference is that in the NTU-game, we can not freely distribute payoff in a coalition but there are some constraints for the payoffs of the players. For more information on NTU-games, we refer to [8].

The following example illustrates a representation of a problem as a cooperative game, namely the *Bankruptcy problem*, a specific game whose study Aumann and Maschler initiated, as detailed in [9].

Example 1 (Bankruptcy problem)**.** *Suppose there is a value $E > 0$ which we want to distribute among n people (after a bankruptcy of a firm) who have their demands d_i on the value they should get. Then the characteristic function of TU-game (N, v) is defined for every nonempty coalition S as follows:*

$$v(S) = max\{0, E - \sum_{i \in N \setminus S} d_i\}.$$

For a specific example, let us have 3 players with the value $E = 30$ and demands $d_1 = 10$, $d_2 = 25$ and $d_3 = 25$. Values of coalitions are the following:

- $v(\{1, 2, 3\}) = 30$,
- $v(\{1, 2\}) = v(\{1, 3\}) = 5$,
- $v(\{2, 3\}) = 20$,
- $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$.

The bankruptcy problem is about how to distribute the value E among the players. To answer that, we need to get a bit more notation and definitions. We denote *payoff vector* or *allocation* by $(x_1, x_2, \ldots, x_n) = x \in \mathbb{R}^n$, which can be assigned to players. We can then say that *solution concept* is a set of payoffs $X \subseteq \mathbb{R}^n$ given by some condition or rule, e.g., $x_i \geq v(i)$. There are three useful terms relating to allocations:

- Payoff vector x is *individually rational* if $\forall i \in N : x_i \geq v(i)$.
- Payoff vector x is *efficient* if $x(N) = v(N)$.
- Payoff vector x is *imputation*, if it is individually rational and efficient.

Following is the definition of the *core*; the most studied solution concept, which is considered to be stable.

Definition 7 (Core)**.** *Let (N, v) be TU-game. The* core *of (N, v) denoted by $C(v)$ is the following set of payoffs:*

$$C(v) = \{x \in \mathbb{R}^n : x(S) \geq v(S) \ \ \& \ \ x(N) = v(N)\}.$$

The core is a set of efficient payoff vectors with additional property, so called *coalition rationality*, i.e. every coalition is guaranteed to receive at least the value of its worth. The following property of *balancedness* characterizes when is the core nonempty.

Definition 8 (Balancedness of TU-game)**.** *Let (N, v) be a TU-game. The game (N, v) is* balanced *if for every $\mu \colon 2^N \to [0, 1]$, where*

$$\forall i \in N : \sum_{S \in 2^N : i \in S} \mu(S) = 1,$$

the following condition holds:

$$\sum_{S \in 2^N} \mu(S) v(S) \leq v(N).$$

Remark. Function μ is called *balanced map.*

The following theorem, independently proved by Shapley in [10] and Bondareva in [11], characterizes games with nonempty core as precisely those that are balanced.

Theorem 2 (Bondareva-Shapley theorem)**.** *Let (N, v) be a TU-game. The game has nonempty core $C(v) \neq \emptyset$ if and only if (N, v) is balanced.*

Let us recall Example 1 with the bankruptcy game. In this game, the formation of the grand coalition is built into its structure. The primary objective is to distribute the total value $v(N) = E$ among all players. The demands of individual players are used exclusively to determine the allocation of this total value. If the core of the bankruptcy game is empty, it indicates that any proposed distribution of payoffs violates condition for at least one $S \subseteq N$, i.e. $x(S) < v(S)$. By the definition of $v(S)$, either $x(S) < 0$, in which case some of the players in S have to pay something, or $0 \leq x(S) < E - \sum_{i \in N \setminus S} d_i$, in which case it is more beneficial for players in S to pay the demands d_i to players $i \in N \setminus S$, and they are still left with more than $x(S)$. However, it is also worth noting that there are scenarios where dividing the set of players into disjoint coalitions and assigning each a specific value, $v(S)$, for redistribution is reasonable even though it may not be in the bankruptcy problem. These situations are explored further in this thesis when dealing with coalition structures.

The bankruptcy problem does not inherently adapt to such cooperative structures. Consequently, when the core of the bankruptcy game is empty, a different type of solution, allocating the value of the grand coalition, is required. The following solution concept addresses this challenge:

Definition 9 (Shapley value)**.** Shapley value ϕ *of TU-game (N, v) of i-th player is a solution concept given by the following formula:*

$$\phi_i = \sum_{S \subseteq N \setminus i} \frac{(|N| - |S| - 1)!(|S|!)}{|N|!} (v(S \cup i) - v(S)).$$

Remark. The Shapley value is efficient, i.e., $\sum_{i \in N} \phi_i = v(N)$.

Equivalently, the Shapley value can be characterized by a few axioms, which gives us means to understand its behaviour. The most known axiomatization is due to Shapley [12]. We can see that the value is given by the sum of weighted marginal contributions of a player to coalitions. The weight can be described as follows:

1. players are ordered according to a random permutation,
2. coalition S is created from the first $|S|$ players of the permutation,
3. the weight corresponds to the probability of player i being ($|S| + 1$)-th in the ordering of uniformly random permutation of the n players.

There is a clear distinction between the core and the Shapley value as one is single valued and the other might not be. A natural question to ask is whether the Shapley value lies within the core of a game? Before we answer this question, let us first define various families of games.

Definition 10 (Classes of games)**.** *Let* (N, v) *be a TU-game. We say* (N, v) *is*

- simple *if* $\forall S \subseteq N : v(S) \in \{0, 1\}$,[1]
- additive *if* $\forall S \subseteq N : v(S) = \sum_{i \in S} v(i)$,
- monotone *if* $\forall S \subseteq T \subseteq N : v(S) \leq v(T)$,
- superadditive *if* $\forall S, T \subseteq N, S \cap T = \emptyset : v(S) + v(T) \leq v(S \cup T)$,
- convex *if* $\forall S, T \subseteq N : v(S) + v(T) - v(S \cap T) \leq v(S \cup T)$.

These families are interesting enough to study on their own. Simple games form a well studied family of games. Its simple structure helps to establish interesting results and the family has its applications in social choice theory; it is studied in problems regarding establishing voting power, or preference aggregation. The family of monotone games has quite a straightforward interpretation. Every player in any coalition does have nonnegative impact on the worth of the coalition. There is no strict relation between superaditive and monotone or convex and monotone games contrary to superadditive and convex games, where the former is a subset of the latter. An important fact about the core and the Shapley value is that for convex games, the Shapley value always stays within the core. However, this might not be true for nonconvex games. There is another way to define convex games, which helps to understand them more easily: in a convex game, the marginal value that a player i brings to a coalition S, i.e., $v(S \cup i) - v(S)$, increases as more players join the coalition S.

Remark. The game (N, v) is convex if and only if $\forall i \in N,\ \forall S \subseteq T \subseteq N \setminus i$:

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T).$$

The following result by Shapley [13] resolves the question of core nonemptiness for convex games.

Theorem 3 (Core of a convex game)**.** *Let* (N, v) *be a convex game. Then* (N, v) *has nonempty core* $C(v) \neq \emptyset$.

Surprisingly, superadditivity is not sufficient condition for nonemptiness of the core. A counterexample can be obtained already for 3 players.

Example 2. *Let* (N, v) *be a TU-game, where* $N = \{1, 2, 3\}$ *and the values are as follows:*

- $v(\{1, 2, 3\}) = 10$,
- $v(\{1, 2\}) = v(\{1, 3\}) = 8$,
- $v(\{2, 3\}) = 5$,
- $v(\{1\}) = 5$,
- $v(\{2\}) = v(\{3\}) = 2$.

[1] In the literature, simple games are usually assumed to be also monotone.

This game is superadditive and has an empty core. To see this, we consider the following inequalities for x_1, x_2, and x_3, given by the core conditions:

1. *By combination of $x_i \geq v(\{i\})$ for $i \in N$, together with $v(N) = x(N)$, we get inequalities:*
   - $5 \geq x_2 + x_3$,
   - $8 \geq x_1 + x_3$,
   - $8 \geq x_1 + x_2$.
2. *From the worths of coalitions of two players and the already established inequalities, we can extract the following equalities:*
   - $5 = x_2 + x_3$
   - $8 = x_1 + x_3$,
   - $8 = x_1 + x_2$.
3. *It follows $x_2 = x_3 = 2.5$ and, together with the grand coalition value, we get $x_1 = 5.5$. This is a contradiction with the efficiency of the core because $2.5 + 2.5 + 5.5 \neq 10$. Therefore, the core is empty.*

A lot of research on cooperative game theory often assumes games to be superadditive, which can be usually justified by results connected to the *superadditive cover of a game* (see [14]). To every TU-game, one can assign a superadditive game, which is called the superadditive cover. This game has interesting properties, which connect it to the original one; its core is nonempty if and only if the core of the original game is nonempty. Therefore, if we are interested in games with nonempty cores, we can assume they are superadditive. As we do not need the superadditive cover later in our text, we omit its formal definition.

Except for the two main solution concepts we have already introduced, there are many others that are not in the scope of this thesis. Examples are the *nucleolus*, the *kernel*, the *Webber set*, to mention a few. We refer the reader to [8] to learn more about these. We shall say a few words about the nucleolus, though, since it is together with the core and the Shapley value the most studied solution concept. It is defined by a lexicographic ordering on *excesses*[2] of coalitions. The nucleolus resolves one of the issues of the core, and that is, how to choose a payoff vector from the core. Specifically, the nucleolus lies always within the core if the core is nonempty and might exist even if the core is empty. There are also other possibilities for choosing a vector from the core like the *center of gravity of the core*. However, this payoff vector does not exist when the core is empty.

So far, we interpreted value $v(S)$ as worth of coalition S, however, there might be scenarios, where it is more beneficial to view it as a *joint cost* of coalition. We refer to such games as *TU-cost games* and denote them (N, c), stressing the characteristic function c represents costs. For scenarios with cost games, modified variants of solution concepts are considered.

Definition 11 (Core of a cost game)**.** *Let (N, c) be TU-cost game. The* core of cost game (N, c)*, denoted by $C_{cost}(c)$, is the following set of payoffs:*

$$C_{cost}(c) = \{x \in \mathbb{R}^n : x(S) \leq c(S) \;\; \& \;\; x(N) = c(N)\}.$$

[2] *Excess* of coalition $S \subseteq N$ is given as $v(S) - x(S)$. It is formally introduced in Definition 24.

Before we conclude this section, we want to mention one last notion.

Definition 12 (Coalitional structure)**.** *Coalitional structure of a game* (N, v) *is a partition* $(S_1, S_2, \ldots, S_k)$ *of the set of players* N*, i.e., it is a set of disjoint coalitions* $S_l \cap S_p = \emptyset, l \neq p$*, where* $\bigcup_{i \in \{1,\ldots,k\}} S_i = N$*.*

The coalition structure is natural to consider in situations, when the core is empty, i.e., when it is not beneficial for the players to form the grand coalition but rather to split into smaller groups. We make use of this concept in Chapter 4 of this thesis.

Questions to ask in coalitional model. Depending on the structure of the game, there are two main key questions to consider:

- How should the total value, $v(N)$, be distributed among all players?
- Which coalition structures are likely to form?

1.4 Cooperative games with stochastic characteristic function

Most of this thesis focuses on games that incorporate elements of randomness. There are several methods for integrating randomness into these models, primarily driven by the need to create a payoff distribution that reflects the uncertainty of outcomes. We explore four most studied and established models from the literature. Each model introduces randomness through the characteristic function but approaches the problem from a unique angle. Additionally, the aim of this section is to present these stochastic models in a clear and structured manner. Following is the definition of *stochastic TU-game*, which is fundamental for the definitions of the rest of the models.

Definition 13 (Stochastic TU-game)**.** Stochastic TU-game *is a pair* (N, v)*, where* $N = \{1, 2, \ldots, n\}$ *is a set of players and* $v = (v(S)_{S \subseteq N})$ *is a multivariate random variable, i.e.,* $v : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}^{2^n}, \mathcal{B}^{2^n})$ *is a measurable map on the probability space* $(\Omega, \mathcal{F}, P)$*, where* Ω *is the sample space,* $\mathcal{F}$ *the sigma-algebra on* Ω*,* P *the probability measure on* $(\Omega, \mathcal{F})$ *and* $\mathcal{B}^{2^n}$ *is the Borel sigma-algebra on* $\mathbb{R}^{2^n}$*.*

Remark. We usually assume values $v(S)$ to be stochastically independent and $v(\emptyset) = 0$, if not specified otherwise.

Stochastic payoff x of players in N is a probability distribution over possible payoffs. A term we shall also use is a correlation structure.

Definition 14 (Correlation structure or covariance structure)**.** *The* correlation structure of the stochastic TU-game (N, v) *is the following set of covariances:*

$$\{cov(v(S), v(T)); T \subseteq N, S \subseteq N\}.$$

Specially, H*-correlation structure is correlation structure with respect to only one coalition* $H \subseteq N$*:*

$$\{cov(v(S), v(H)); S \subseteq N\}.$$

A way of defining a specific stochastic TU-game is to assign random variables to all the coalitions and a correlation structure, i.e., specify the multivariate distribution of the characteristic function v. For example, we might assign each coalition a distinct random variable such that all these variables are independent. Another example might be to create scenarios within the game to establish various possible outcomes and their implications. Regardless of the chosen approach, it is essential to either understand the structure of the problem or incorporate additional information about the situation and the players. This additional insight is crucial for determining how to appropriately redistribute payoffs under the randomness.

Interpretation of the stochastic model Let us discuss how the interpretation of a cooperative game changes, when $\forall S \subseteq N$, $v(S)$ is a nondegenerate random variable. In deterministic scenario, we usually distribute the value of the grand coalition, however, in the stochastic model, since $v(N)$ is a random variable, the value to distribute is not known until it is realized. This introduces a significant challenge: decisions about cooperation have to be made before the actual value of $v(N)$ is known, requiring an ex-ante approach to decision-making. In contrast, if the value of the grand coalition is deterministic and known as a real value, and at the same time $v(S)$ is a random variable for the rest of the coalition, the interpretative focus shifts. Under these conditions, we might analyze the situation through a model aimed at studying the robustness of solution concepts when there is noise in the values of smaller coalitions. This approach is explored in the recent work by Pantazis et al. [15], which examines the implications of such stochastic elements on strategic decision-making.

1.4.1 Scenario model

This model, as its title suggest, depend on a set of scenarios with given probabilities. We do not provide a formal definition, however, we illustrate it on an example, which is an adjustment of Example 1.

Example 3 (Stochastic bankruptcy game)**.** Stochastic bankruptcy problem *is a tuple* (S, E, d)*, where*

- $\mathcal{S} = \{1, 2, \ldots, k\}$ *is a finite set of states of nature or just scenarios,*
- $E = \{E_1, \ldots, E_k\}$ *is the set of values to be distributed among the players based on the scenario,*
- $d = \{d_{i,s}\}_{i \in N, s \in \mathcal{S}}$ *is the set of demands of players, where* $d_{i,s}$ *represents the demand of player* i *under scenario* s*.*

The stochastic bankruptcy game (N, v) *is stochastic TU-game, where* $\Omega = \mathcal{S}$ *and for every scenario* $s \in \mathcal{S}$ *which is equally probable to occur, we get deterministic game* (N, v_s) *defined as follows:*

$$v_s(T) = max\{0, E_s - \sum_{i \in N \setminus T} d_{i,s}\}, \ \forall T \subseteq N.$$

The goal of this game is to come up with a rule to distribute the payoffs before the scenario is realized. This example is motivated by [16], where stochastic *bankruptcy games are studied. The solution concept used in the paper is the* weak sequential core*, which is generally designed for stochastic games in the form of finite states. The weak sequential core for the stochastic games in the form of scenarios is introduced in [3].*

There is an obvious difference between the stochastic and the non-stochastic bankruptcy problem. In the latter, there are two stages of the game: one stage before the realization of the scenario and the second one after the realization.

For the scenario model, one can develop reasonable solution concepts as illustrated in the example. However, requiring uncertainties to fit into specific scenarios might be too limiting for some applications. Consequently, it might be necessary to assume less about the values of coalitions and at the same time assume additional information to be able to describe a broader range of situations. Models developed by Charnes and Granot, as well as by Suijs and Borm, extend the basic stochastic model of a TU-game (N, v) by incorporating extra details. These models either assume independence among the values of different coalitions or do not model the covariances at all. Hence, v is characterized by its marginal distributions $v(S)$. Nevertheless, these models require types of information beyond just the correlation structure such as preferences of players.

1.4.2 CG model

This is the oldest model, which incorporates randomness into the TU-game, developed by Charnes and Granot in the 1970s (the first paper mentioning the model dates back to 1973 [1]). Their approach is based mainly on chance-constrained optimization, i.e., constraints involving probability. This is a technique well established in the field of stochastic optimization. The model is based on two stage solutions:

1. Payoffs that are considered realizable are promised to the players.
2. After realization of the randomness, there are adjustments of the payoffs of players.

Let us summarize the model formally:

Definition 15 (CG model)**.** CG model *is an ordered triple* (N, v, α)*, where* $N = \{1, 2, \dots, n\}$*,* $v(S)$ *has an associated cumulative distribution function* F_S $\forall S \subseteq N$ *and* α *is the set of fixed assurances (confidence levels)* $\underline{\alpha}(S)$ *and* $\overline{\alpha}(S)$ *for* $\forall S \subseteq N$*.*

Remark. For the *prior core*, payoff $x \in \mathbb{R}^n$ is considered realizable, if it satisfies probability inequalities in the form of $P[x(S) \geq v(S)] \geq \alpha(S)$, however, after the realization of the randomness, it is further adjusted to satisfy efficiency. We discuss the prior core in detail in Chapter 2.

The CG model is relatively simple in terms of the number of components required: only a stochastic TU-game and levels of assurance are necessary. These assurance levels model the risk behavior of the coalitions in a sense. Solutions derived from the CG model, known as *prior solutions*, are based on the zero-order decision rule in chance-constrained programming. Consequently, such solutions do not incorporate the full spectrum of probabilistic information. They are designed primarily to guarantee the payoffs before the realization of the random variable $v(N)$, allocations are adjusted in specific ways, for example, to preserve ratios between pairs of players' payoffs.

1.4.3 SB model

The model presented by Suijs, Borm, Waegenaere, and Tijs [2] is similar to the CG model in that it models the values of coalitions as independent random variables; that is, the values are determined marginally. This model extends the stochastic TU-game by introducing preferences over random variables. We describe the model that incorporates preferences and, as a special case, the SB model as outlined in their paper, with one minor modification: we do not define the actions of coalitions within this model, as they are not essential for further discussion. Their inclusion would only complicate the model unnecessarily and introduce additional notation. We elaborate on potential adjustments to the model, specifically regarding the players' preferences and allocation types.

Definition 16 (Model with preferences)**.** *Model of a* stochastic TU-game with preferences *is a triple* $(N, v, (\succeq_i)_{i\in N})$ *where* (N, v) *is a stochastic TU-game and* $\succeq_i$ *is a preference of player* i *over set* X *of random variables.*

Remark. To be able to define preferences over X, we may restrict the set X for instance to variable with the finite expected value.

Definition 17 (SB model)**.** SB model *is a model with preferences* $(N, v, (\succeq_i)_{i\in N})$ *where the preference* $\succeq_i$ *is defined in the following way: For any random variables* X, Y *and given* $\alpha_i \in (0, 1)$, $\forall i \in N$ *it holds:*

$$X \succeq_i Y \iff F_X^{-1}(\alpha_i) \geq F_Y^{-1}(\alpha_i).$$

Remark. In [2], the authors implicitly assume a special type of stochastic payoff x associated with the SB model, where $x_i = d_i + r_i(v(N) - \mathbb{E}[v(N)])$ and:

- $\sum_{i\in S} d_i = \mathbb{E}[v(N)]$,
- $\sum_{i\in S} r_i = 1$ and $\forall i \in N,\ r_i \geq 0$.

We refer to these stochastic payoffs as *stochastic payoffs with transfer payments* and denote them by (d, r_+) (see Definition 26 in Chapter 3).

Preferences used in the SB model can be altered. Suijs and Borm also discussed preferences in the form of $\mathbb{E}[X] + b \cdot \mathbf{Var}(X)$ for $b \in \mathbb{R}$. The choice of stochastic payoffs with transfer payments improves the computational properties of solution concepts; however, it sacrifices generality due to this restriction. For this payoff type, in the case of forming the grand coalition, there are two possibilities based on the realization $z(N)$ of the $v(N)$):

$$z(N) \leq \mathbb{E}[v(N)] \tag{1.3}$$

$$z(N) \geq \mathbb{E}[v(N)] \tag{1.4}$$

According to the definition of an allocation, players receive a fixed amount d_i and a variable amount based on r_i. The greater the r_i the greater the potential profit or loss for the given player. If the realization of $v(N)$ corresponds to the scenario described in 1.3, the player receives less than d_i which is determined before the realization. If the realization of $v(N)$ corresponds to the second scenario in 1.4, the player receives more than d_i which is determined before realization.The greater the r_i of a given player, the higher his loss/gain corresponding to the realized value of $v(N)$.

Let us explore the motivation for the SB model using an example of an *insurance game*. More specifically, we shall illustrate the reason for a payoff structure in the form of (d, r_+). The application of the stochastic model in an insurance game, as introduced by Suijs et al. in [17], serves as an almost perfect example of a scenario where such types of payoffs are justified.

Example 4 (Non-life insurance games)**.** *Let N_I represent the set of insurers and N_P the set of individual persons. The objective for individuals in N_P is to obtain insurance coverage, which requires making a deterministic payment, specifically a premium, to an insurer. Insurers assume the risk of a potential insured event occurring. Although the occurrence of such events is uncertain, insurers are guaranteed to receive the premium payments from the individuals. Such a situation can be modeled using the model with random payoffs, more specifically by a slightly adjusted SB model with stochastic payoffs with transfer payments (d, r_+), however, its form is slightly modified. Rather than $x_i = d_i + r_i(v(N) - \mathbb{E}[v(N)])$, it is in the form of $x_i = d_i - r_i \cdot L$, where L is a random loss. We aim to determine from the model how much the players in N_P pay to the insurer, hence, $d_i < 0$ for $i \in N_P$. The insurers receive what the persons pay as premiums, thus $d_i > 0$ for $i \in N_I$. The random part $r_i \geq 0$ for $i \in N_P$ indicates the proportion of the realized loss that the insurer is required to pay in the event of its realization. Individual persons pay usually lower portion of the realized loss but there is also assigned $r_i \geq 0$ to all the individual persons.*

The original model used for an insurance game by Suijs et al. differs slightly and includes a few more components than those described in our example. The main idea motivating the type of payoff does not depend on the other components of the original model that are not described in the example. Hence, for the sake of clarity, we will not go into the technical details.

1.4.4 Optimization model

In the final model we discuss incorporating of a new idea and a slight generalization. The generalization lies in modelling the correlation structure of v and in the type of payoff which is inspired by the SB model. This model is based on the recent paper by Sun et al.[4], and we refer to it as the SHS model, named after the authors.

Definition 18 (SHS model)**.** *SHS model is a stochastic TU-game (N, v), where v has specified the N-correlation structure, i.e., the set $\{cov(v(S), v(N)); S \subseteq N\}$ is given.*

Remark. In [4], the authors implicitly assume a special type of stochastic payoff x associated with the generalization of the type from the SB model, where $x_i = d_i + r_i(v(N) - \mathbb{E}[v(N)])$ and:

- $\sum_{i \in S} d_i = \mathbb{E}[v(N)]$,

- $\sum_{i \in S} r_i = 1$ and $\forall i \in N,\ r_i \in \mathbb{R}$.

We refer to these stochastic payoffs as *stochastic payoffs with transfer payments and general risk part* and denote them by (d, r) (see Definition 26 in Chapter 3).

The main contribution of the SHS model lies in its approach to payoff distribution, which uses an optimization model with a specified objective function. We elaborate on this aspect in Chapter 2 about payoff distribution in stochastic TU-games.

Conclusion of models We have presented models from the literature that can be used to model stochastic TU-games. Each of the models is requiring additional information either directly (covariance structure, assurance levels or preferences) in the model definition or implicitly (types of payoff and objective functions) in the definition of its payoff distribution. The scenario model employs scenarios, thereby giving v a straightforward structure. In the CG model, it is the levels of assurances of coalitions. The SB model incorporates preferences over random variables and implicitly a given allocation type. The SHS model provides framework for general covariance structure of v.

2. Payoff distribution

This chapter provides a survey of solution concepts proposed in the literature and builds on the first chapter, where models were introduced. The purpose of this chapter is not only to conduct a survey but also to explore which questions might be posed about TU-games with stochastic payoffs in general. Additionally, this chapter aims to motivate some of the questions that are addressed later in the thesis.

2.1 Core and Shapley like solutions

Let us begin with solution concepts motivated by the core and Shapley value, which are adjusted for the stochastic setting. There are two models previously mentioned that introduce the core and the Shapley value in this context: the CG model and the SB model.

2.1.1 Prior core and Shapley value

So called *prior solution concepts* are defined in [1] for the CG model. Recall CG model is a triple (N, v, α) where (N, v) is a stochastic TU-game and α are measures of assurance for all coalitions. The characteristic function is given by its marginal distributions $v(S)$, i.e., for all $S \subseteq N$ there is a given cumulative distribution function F_S of a random variable $v(S)$. We present 2 types of core-like solution concepts and a Shapley-like value. In [1], the cores are called the *first* and the *second prior core*. To be more descriptive, we call them the *prior core* and the $\varepsilon-$*prior core*, respectively.

Definition 19 (Prior core)**.** *Let* (N, v, α) *be a CG model. The* prior core $C^1_\alpha(v)$ *is the following set:*

$$C^1_\alpha(v) = \{x \in \mathbb{R}^n : F_S(x(S)) \geq \underline{\alpha}(S),\ \forall S \subseteq N,\ \underline{\alpha}(N) \leq F_N(x(N)) \leq \overline{\alpha}(N)\}\,,$$

The parameters $\overline{\alpha}(S)$ for $S \neq N$ are not used in any of the solution concepts we introduce but could allow more general definitions of the solution concept where some characteristic of a coalition would be constrained not only from below but also from above. To summarize the definition, the prior core is well defined if the stochastic TU-game is well defined and levels of assurance are $\underline{\alpha}(S) \in (0,1),\ \forall S \subseteq N$ and $\overline{\alpha}(N) \in (0,1]$, $\underline{\alpha}(N) \leq \overline{\alpha}(N)$. The following claim, proved in [1], gives a characterization for the nonemptiness of the prior core $C^1_\alpha(v)$. It uses the following notation:

- $F_s^{-1}(\underline{\alpha}(S)) = \inf\{x(S) :\ F_S(x(S)) \geq \underline{\alpha}(S)\}$ for $S \subseteq N$,
- $\hat{F}_s^{-1}(\underline{\alpha}(S)) = \sup\{x(S) :\ F_S(x(S)) \leq \underline{\alpha}(S)\}$ for $S \subseteq N$.

Claim 4 (Nonemptiness of $C^1_\alpha(v)$)**.** *Let* (N, v, α) *be a CG model. The prior core is nonempty if and only if* $\mu > \hat{F}_N^{-1}(\overline{\alpha}(N))$*, where* μ *is the optimal solution of the* *following program:*

$$\begin{aligned} &min\ x(N)\\ &s.t.\ F_s^{-1}(\underline{\alpha}(S)) \leq x(S),\ \forall S \subseteq N,\\ &\quad\ \hat{F}_N^{-1}(\overline{\alpha}(N)) \leq x(N). \end{aligned}$$

What might ask what to do when the prior core is empty. The ε-prior core is always nonempty and at the same time, its definition is motivated by the characterization given in Claim 4.

Definition 20 (The ε-prior core)**.** *Let* (N, v, α) *be a CG model. The* ε-prior core $C_\alpha^\varepsilon(v)$ *is a set of optimal solutions of the following optimization program:*

$$\begin{aligned} &min\ \varepsilon\\ &s.t.\ P[x(S) \geq v(S)] \geq \underline{\alpha}(S)\ \forall S \subseteq N,\\ &\quad\ P[x(N) \geq v(N) - \varepsilon] \geq \underline{\alpha}(N),\\ &\quad\ P[x(N) \geq v(N)] < \overline{\alpha}(N),\\ &\quad\ \varepsilon \geq 0. \end{aligned}$$

It was shown in [1] that $C_\alpha^\varepsilon(v)$ is always nonempty when levels of assurance are well defined, i.e., $\underline{\alpha}(S) \in (0, 1),\ \forall S \subseteq N$ and $\underline{\alpha}(N) \leq \overline{\alpha}(N)$. They also showed that ε-prior core can be rewritten equivalently as the following program:

$$\begin{aligned} &\min\ \delta\\ &\text{s.t. } x(S) \geq F_S^{-1}(\underline{\alpha}(S))\ \forall S \subseteq N,\\ &\quad -\delta \leq x(N) - \delta_0 \leq \delta,\\ &\quad \delta \geq 0,\\ &\text{where } \delta_0 = \frac{1}{2}\left(F_N^{-1}(\underline{\alpha}(N)) + \hat{F}_N^{-1}(\overline{\alpha}(N))\right). \end{aligned}$$

The advantage of the ε-prior core over the prior core is that it is always nonempty. We can also see that for a fixed stochastic TU-game in the CG model (N, v, α), the prior core coincide with the ε-prior core if the minimum of the optimization problem in Definition 20 is 0. If the minimum is not 0, but something greater then 0 the prior core is empty and $\varepsilon-$prior core is nonempty.

Let us now look at a straightforward generalization of the Shapley value for stochastic TU-games, also introduced in [1].

Definition 21 (Prior Shapley value)**.** *Let* (N, v) *be a stochastic TU-game. The* prior Shapley value $\phi^{\mathbb{E}} \in \mathbb{R}^n$ *for such a game is defined as follows:*

$$\forall i \in N : \phi_i^{\mathbb{E}}(v) = \sum_{S \subseteq N, S \ni i} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \left[\mathbb{E}[v(S)] - \mathbb{E}[v(S \setminus i)]\right].$$

The prior Shapley value is expressed through the weighted marginal contributions of players to the expected value; therefore, it can be also applied to models other than the CG model. There are multiple reasons why this solution concept, the prior Shapley value, is well defined. One of these reasons, discussed in [1], is that the value satisfies the axioms that define the deterministic Shapley value, such as:

- (Permutation invariance): For any permutation π on N, and any game such that $v(\pi(S)) = v(S)$ for all $S \subseteq N$: $\phi^{\mathbb{E}}_{\pi(i)}(v) = \phi^{\mathbb{E}}_i(v)$,
- (Dummy player): if $v(S \setminus i) + v(i) = v(S)$, $\forall S \subseteq N, i \notin S$ then $\phi^{\mathbb{E}}_i(v) = v(i)$,
- (Additivity): for any two games (N, v) and (N, w) holds the following: $\phi^{\mathbb{E}}(v + w) = \phi^{\mathbb{E}}(v) + \phi^{\mathbb{E}}(W)$.

Another reason the prior Shapley value is well-defined is that it can be derived from a *weighted L_2 optimization problem* (see [1] for details).

Comment on prior solutions The prior solutions provide only prior payoffs x. Even though these payoffs are essentially promises, there is a possibility that the realization of v rule these out to be unfeasible since they might not be efficient. In the prior core and the ε-prior core, the allocation is chosen in such a way that it is highly probable that players will receive what is specified in the solution concept. The prior Shapley value almost always requires adjustment since it represents a single point in $\mathbb{R}^n$ and it is unlikely that the realization of $v(N)$ will exactly match $\mathbb{E}[v(N)]$. If this happens, adjustments have to be made according to rules such as keeping the payoffs *relatively the same* (see [1] for a discussion of adjustments).

2.1.2 Preferential core

This section provides an overview of the solution concepts within the model with preferences, as detailed in Definition 16. Specifically, we discuss the SB model from Definition 17, which incorporates specific preferences and payoff types. In the SB model, preferences are based on quantile preferences; for each player, there is an $\alpha_i \in (0, 1)$ representing the quantile according to which the player makes decisions. The α-quantile of a random variable X is denoted by u^X_α. The payoff considered are stochastic payoffs with transfer payments (d, r_+), (see Definition 17). We explore the solution concept known as the *preferential core*, which was presented in [2].

Definition 22 (Preferential core). *Let $(N, v, (\succeq_i)_{i\in N})$ be model with preferences. Preferential core is a set of stochastic payoffs x satisfying:*

- $\forall S \subseteq N : x(S) \succeq_i v(S),\ \forall i \in S,$
- $x(N) = v(N).$

The following lemma, proved in [2], enables us to calculate the preferential core of a game.

Lemma 5 (Conditions for preferential core). *Let $\Gamma_\alpha = (N, v, (\succeq_i)_{i\in N})$ be SB model. The preferential core is nonempty if and only if there is a stochastic payoff with transfer payments x, which satisfies $\forall S \subseteq N$,*

$$\sum_{i\in S}(d_i + r_i(u^{v(N)}_{\alpha_i} - \mathbb{E}[v(N)])) \geq \max_{i\in S} u^{v(S)}_{\alpha_i}.$$

Suijs et al. [2] also characterized nonemptiness of the preferential core using conditions which are based only on the values of the α-quantiles. The characterization makes use of generalization of *balancedness of SB model.*

Definition 23 (Balancedness in SB model)**.** *Let* $\Gamma_\alpha = (N, v, (\succeq_i)_{i\in N})$ *be SB model. A game* Γ_α *is called* balanced *if for each balanced map,*

$$\max_{i\in N} u_{\alpha_i}^{v_N} \geq \sum_{S\subseteq N} \mu(S) \max_{i\in N} u_{\alpha_i}^{v_S}.$$

This definition is just a generalization of balancedness to the SB model because $\max_{i\in N} u_{\alpha_i}^{v_S}$ is equal to $v(S)$ for deterministic games.

Theorem 6 (Nonemptiness of the preferential core)**.** *Let* $\Gamma_\alpha = (N, v, (\succeq_i)_{i\in N})$ *be SB model. The preferential core of* Γ_α *is nonempty if and only if the game* Γ_α *is balanced.*

Comment on core solutions in SB model. The preferential core in the SB model can be generalized to accommodate other types of preferences, such as ordering according to the $\mathbb{E}[X] + b \cdot \mathbf{Var}(X)$, where $b \in \mathbb{R}$ is constant. This type of preference relation, like the quantile preference, induces a total ordering. Suijs et al. [2] also discussed usage of first-order stochastic dominance as a preference relation but they do not explore it extensively. They argue that using the first-order dominance as a preference could result in an overly large core. They discuss its form, which would correspond to our notion of the *undominated core* in Chapter 3. A major contribution of the SB model is not only the incorporation of specific preferences but also the selection of a stochastic allocations with tranfer payments (d, r_+), which restricts feasible payoffs. This approach simplifies the problem by reducing a stochastic problem to finding numerical values for the vectors d and r, making it computationally more tractable but at the same time less general, as it does not account for all random allocations.

2.2 Program solutions

Another solution concept, recently proposed in the literature, is based on the SHS model, which is detailed in Definition 18. This model extends the payoff structure introduced for the SB model, generalizing it stochastic payoffs with transfer payments and general risk part (d, r); this generalization involves relaxing the conditions on the r_i part of the allocation; in this model, can be any real number. Negative r_i indicates that the player i benefits more with the realization of $v(N)$ being below its expected value $\mathbb{E}[v(N)]$ (see Definition 26). Unlike the SB model, where covariances between different coalitions are set to zero because the model assumes independent marginal distribution of v, the N-correlation structure of the game is more significant for solutions in this section.

The main idea of the paper [4] is to describe the solution concepts as solutions to optimization problems with a given objective, which captures the desired properties of the solution. A term heavily used in cooperative game theory, and particularly in the objective function for the subsequent solution concept, is the *excess* and *marginal excess.*

Definition 24 (Excess and average excess)**.** *Let* (N, v) *be a stochastic TU-game and* x *a stochastic payoff. The* excess $e(S, x, v)$ *of a coalition* $S \subseteq N$ *and stochastic payoff* x *in game* (N, v) *is defined as*

$$e(S, x, v) = v(S) - x(S),$$

and the average excess $\overline{e}(x, v)$ *of stochastic payoff* x *in game* (N, v) *is defined as*

$$\overline{e}(x, v) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S, x, v).$$

Definition 25 (Marginal excess and avarage marginal excess)**.** *Let* (N, v) *be a stochastic TU-game and* x *a stochastic payoff. The* marginal excess $m_i(S, x, v)$ *of a player* i*, coalition* S*, and stochastic payoff* x *is defined as*

$$m_i(S, x, v) = [v(S) - v(S \setminus i)] - x_i,$$

and the average marginal excess $\overline{m}(x, v)$ *of stochastic payoff* x *in game* (N, v) *is defined as*

$$\overline{m}(x, v) = \frac{1}{n \cdot 2^{n-1}} \sum_{i \in N} \sum_{S \ni i} m_i(S, x, v).$$

We provide a list of functions used as objective functions by Sun et al. [4] and we show solution concepts for 2 of them to see the approach.

$$\min \sum_{S \subseteq N} \mathbf{Var}[e(S, x, v)] \tag{2.1}$$

$$\min \sum_{S \subseteq N} \mathbb{E}[(e(S, x, v) - \overline{e}(x, v))^2] \tag{2.2}$$

$$\min \sum_{i \in N} \sum_{S \ni i} \mathbb{E}[(m_i(S, x, v) - \mathbb{E}[m_i(S, x, v)])^2] \tag{2.3}$$

$$\min \sum_{i \in N} \sum_{S \ni i} \mathbb{E}[(m_i(S, x, v) - \overline{m}(x, v))^2] \tag{2.4}$$

$$\min \sum_{S \subseteq N} w_S \mathbf{Var}[e(S, x, v)] \tag{2.5}$$

The idea of the last function is using weights w_S for coalitions. Let us present a results about the first function, as shown in [4].

Theorem 7. *Let* (N, v) *be SHS model and* X *be the set of stochastic payoffs with transfer payments and general risk part and finite expectation for* (N, v)*. Then the optimal solution of the following program:*

$$\begin{aligned} \min_{x \in X} \sum_{S \subseteq N} & \boldsymbol{Var}[e(S, x, v)] \\ s.t.\ & d(N) = \mathbb{E}[v(N)], \\ & r(N) = 1, \\ & r, d \in \mathbb{R}^n \end{aligned}$$

is a stochastic payoff with transfer payments and general risk part x^* *described by corresponding vectors* d^* *and* r^* *as follows:*

$$x_i^* = d_i^* + \left(\frac{1}{n} + \frac{n a_i(v) - \sum_{j \in N} a_j(v)}{n 2^{n-2} \boldsymbol{Var}[v(N)]} \right) (v(N) - \mathbb{E}[v(N)]), \tag{2.6}$$

where $a_i(v) = \sum_{S : i \in S} cov(v(S), v(N))$ *and* $\sum_{i \in N} d_i^* = \mathbb{E}[v(N)]$.

As we can see in (2.6), the term r_i^* of the payoff x_i is explicitly expressed as a function of the covariances between the grand coalition and the coalitions that include the given player, as well as a function of the number of players and coalitions. Consequently, it is not immediately clear how to distribute the value of $\mathbb{E}[v(N)]$ among the d_i^* terms for each player in N. Nevertheless, specifying d_i^* for all players may be feasible through some external constraints that are not explicitly stated within the objective funtion.

Idea of the proof. In Sun et al. [4], the theorem is proved by correctly rewriting the objective function into the following form:

$$\min \sum_{S\subseteq N} (r^2(S)\mathbf{Var}[v(N)] - 2r(S)cov(v(S), v(N))),\ s.t.\ r(N) = 1.$$

Then, a typical approach to the constrained optimization of a nonlinear function is applied, namely, the construction of the Lagrangian function; subsequently, the Karush-Kuhn-Tucker conditions are used to derive the optimal solution. $\square$

Let us show a similar result for (2.2). Averaging over all coalitions yields explicit results for the entire payoff, i.e., for both the $d \in \mathbb{R}^n$ and $r \in \mathbb{R}^n$ parts. This is achieved by substituting $\mathbb{E}[e(S, x, v)]$, the expected value of the excess for a given coalition S, in (2.1), with the average excess. This value $\mathbb{E}[e(S, x, v)]$ is implicitly represented in the expression $\mathbf{Var}[e(S, x, v)]$.

Theorem 8. *Let (N, v) be SHS model and X be the set of stochastic payoffs with transfer payments and general risk part for (N, v). Then the optimal solution of the following program:*

$$\begin{aligned} \min_{x\in X} &\sum_{S\subseteq N} \mathbb{E}[(e(S, x, v) - \overline{e}(v, x))^2] \\ s.t.\ & d(N) = \mathbb{E}[v(N)] \\ & r(N) = 1 \\ & r, d \in \mathbb{R}^n, \end{aligned}$$

is a stochastic payoff with transfer payments and general risk part x^* described by corresponding vectors d^* and r^* as follows:

$$\begin{aligned} x_i^* &= d_i^* + r_i^*(v(N) - \mathbb{E}[v(N)])), \\ d_i^* &= \frac{1}{n}\mathbb{E}[v(N)] + \frac{ne_i(v) - \sum_{j\in N} e_j(v)}{n2^{n-2}}, \\ r_i^* &= \frac{1}{n} + \frac{na_i(v) - \sum_{j\in N} a_j(v)}{n2^{n-2}\,\boldsymbol{Var}[v(N)]}, \end{aligned}$$

where $a_i(v) = \sum_{S:i\in S} cov(v(S), v(N))$ and $e_i(v) = \sum_{S:i\in S} \mathbb{E}[v(S)]$.

This solution represents a modified egalitarian approach[1]. Under vector d^*, each player receives almost $\frac{1}{n}$ fraction of $\mathbb{E}[v(N)]$ as this value is modified for every player i based on the sum of expected values of coalitions, $e_i = \sum_{S:i\in S} \mathbb{E}[v(S)]$. Specifically, if e_i is smaller than the average over all players, i.e., $e_i < (\sum_{j\in N} e_j)/n$,

[1] An egalitarian solution is a payoff where all players receive the same amount.

he receives less than $\frac{1}{n}$ fraction and if e_i is larger than the average, he receives more. Similarly, under r^*, each player receives almost $\frac{1}{n}$ part of $v(N) - \mathbb{E}[v(N)]$ as this value is modified for every player i, this time based on the sum of covariances of coalitions, $a_i = \sum_{S:i \in S} cov(v(S), v(N))$. If a_i is smaller for i than the average sum, he receives less than the fraction $\frac{1}{n}$ and if a_i is larger, he receives more.

Comments on optimization solutions Solution concepts proposed by Sun et al. in [4] can be particularly useful in certain situations. These solution concepts are applicable whenever SHS model (N, v) is well defined, which includes a correlation structure among components. Understanding the covariances within this structure can be the most challenging aspect of formulating the model. In the absence of a correlation structure, the random component r_i^* would be uniformly $\frac{1}{n}$ for any player i, making it an egalitarian payoff. Hence, the correlation structure is crucial if the egalitarian solution for the random part of the payoff is not desirable. When this correlation structure is present, these solution concepts are well justified, as discussed below Theorem 8, although optimizing functions such as the average excess may initially appear to be an ad hoc approach. However, in scenarios where this structure is absent or independence is assumed, i.e., covariances are set to zero, the optimization method results in r_i^* equal for all players $\forall i \in N$.

Discussion of solution concepts In this chapter, we discussed solution concepts for models introduced in Chapter 1. We did not define the solution concept of the *weak sequential core* mentioned in Chapter 1 as it arises from sequence game setting, which do not contribute to a better understanding of our results within cooperative game theory. Solution concepts in 2.1.1 represent pioneering work in stochastic cooperative games, employing a chance constraint programming approach. However, a major limitation of these concepts is in deterministic payoffs, which are initially promised and later adjusted post-realization. It is important to note that the probability of these promised payoffs can be very low, even when assurance measures are near one.

The SB model provides a framework to incorporate preferences over random variables. While it employ stochastic payoffs with transfer payments, which may seem restrictive, in Chapter 3, we justify the suitability of this type of payoff. Approach taken in the SB model significantly influences our discussion about the SSD-dominating core in the same chapter.

Furthermore, we examined the SHS model, whose method of optimizing a well-defined objective function is particularly insightful. This model offers explicitly given single-point payoffs (at least for some of the objective functions) for players based solely on the value of $v(N)$. Nevertheless, the assumptions of SHS model, especially the expected N-correlation structure $cov(V(S), V(N))$, $\forall S \subseteq N$, might be an overly strict assumption. If these assumptions can be satisfied, then the solution concepts of SHS model are viable.

In conclusion, each solution concept presents its own set of advantages and drawbacks, which are essential to consider when applying them to real-world problems.

3. Stochastic dominance core

In this chapter, we propose solution concepts motivated by the core, specifically for players with risk averse behaviour. We mainly develop later defined SSD-dominating core for various distributions such as uniform, normal or discrete uniform. Motivated by the SB model we propose a solution concept which incorporates second order stochastic dominance to model players preferences. This approach is inspired by the preferential core in the SB model, where player's preferences are quantiles of random variables. By integrating SSD we aim to address decision-making process of risk averse players to provide a robust framework for such players. In the SB model, the quantile preferences can be exchanged for preferences of type $\mathbb{E}[X] + b \cdot \mathrm{Var}(X)$, where $b \in \mathbb{R}$ represents a risk parameter. This modification preserves the result about preferential core in the SB model, allowing us to model different risk attitudes. Specifically, for a given player a value of $b = 0$ model risk neutrality, $b < 0$ risk aversion and $b > 0$ risk lovingness . Despite usefulness of this approach, determining an appropriate b for every player may be impractical or resource intensive. Moreover, the analysis and outcomes depend on the choice of b for each player. In scenarios where it is challenging to specify or uncover the exact utility function of each player, especially when wanting to understand broadly the risk averse behaviour without being too specific, adopting SSD as a preference relation proves to be advantageous. A notable limitation of SSD is that it is only partial ordering of random payoffs contrary to "quantile" preferences or "expected value plus variance" preferences which are total orders. Despite this limitation, SSD offers a robust framework that effectively uses the risk aversion characteristic of players. This strength can become useful, particularly in scenarios where robustness is required over the ability to rank all possible probability distributions completely.

Definition 26 (Type of stochastic payoff of a player and coalition). *Let (N, v) be a stochastic TU-game and x be a stochastic vector. We say x is a stochastic payoff:*

- *without transfer payments if $\forall i \in N: \; x_i = r_i \cdot v(N)$, where $r_i \geq 0$,*
- *with transfer payments if $\forall i \in N : x_i = d_i + r_i(v(N) - \mathbb{E}[v(N)])$, where $d_i \in \mathbb{R}$ and $r_i \geq 0$,*
- *with transfer payments and general risk part if $\forall i \in N : x_i = d_i + r_i(v(N) - \mathbb{E}[v(N)])$, where $d_i \in \mathbb{R}$ and $r_i \in \mathbb{R}$,*
- *without structure (unstructured) if x_i is a random variable.*

A stochastic payoff of a coalition S is a sum of payoffs of individual players $x(S) = \sum_{i \in S} x_i$.

Remark. In the following text we use the term the *type of allocation* which just refers to a type of stochastic payoff. Different types of payoffs are denoted as follows:

- with transfer payments which is denoted by (d, r_+),

- with transfer payments and general risk part which is denoted by (d, r),
- without transfer payments which is denoted by r_+ [1].

We also refer to the allocated numbers vectors d, r or r as allocations of players.

Remark. We usually refer to the type of payoff with respect to a grand coalition. If we want to refer to a type of stochastic payoff with respect to a given coalitions S, then we write for instance for payoff without transfer payments $(d, r_+)_S$. Such a notion is useful when we talk about coalition structure.

Definition 27 (Feasible payoff of a coalition S)**.** *Let (N, v) be a stochastic TU-game. The stochastic payoff x with transfer payments is feasible for a coalition S if:*

- $d(S) = \mathbb{E}[v(S)]$,
- $r(S) = 1$.

The stochastic payoff x without transfer payments is feasible for a coalition S if:

- $r(S) = 1$.

The unstructured stochastic payoff x is always feasible.

Remark. The definition also defines feasible payoff for payoffs with transfer payments and general risk part even though it is not explicitly stated.

All of the four payoff types are random variables. An allocation with and without transfer payments determine what exactly the player gets after the realization. The unstructured payoff does not specify straightforwadly what should the player get after the realization. In the section about the unstructured type of allocation we discuss why the types of allocation (d, r) and (d, r_+) are reasonable. Assuming players preferences being modelled by SSD, there is a straightforward generalization of the core to stochastic TU-games.

Definition 28 (SSD-dominating core)**.** *Let (N, v) be a stochastic TU-game. The SSD-dominating core is a set of feasible stochastic payoffs x of players in N denoted by $\boldsymbol{DC}(v)$ for which it holds that*

$$\forall S \subseteq N : x(S) \succeq_{SSD} v(S) \ \ \& \ \ x(N) \text{ has the same distribution as } v(N).$$

Remark. If we want to stress that the partial order is SSD we can write $\mathbf{DC}_{SSD}(v)$. Since no other partial order is assumed in this thesis we write just $\mathbf{DC}(v)$ for the SSD-dominating core as it was defined.

Remark. To distinguish between types of allocations we denote the SSD dominating core by:

- $\mathbf{DC}^{(r+)}(v)$ if the payoff is without transfer payments,
- $\mathbf{DC}^{(d,r+)}(v)$ if the payoff is with transfer payments,
- $\mathbf{DC}^{(d,r)}(v)$ if the payoff is with transfer payments and general risk part.

[1] this type of payoff is motivated by the work of Timmer et al. [18]

For partial orderings we can also define slightly weaker concept called undominated core.

Definition 29 (SSD-undominated core)**.** *Let (N, v) be a stochastic TU-game. The SSD-undominated core is a set of feasible stochastic payoffs x denoted by $\boldsymbol{UDC}(v)$ for which it holds that*

$$\nexists S \subseteq N : x(S) \prec_{SSD} v(S) \;\;\&\;\; x(N) \text{ has the same distribution as } v(N).$$

Remark. Similarly to the dominating core we can distinguish multiple types of allocations or various partial orders.

Remark. The solution concepts of the dominating and the undominated core do not work at all with covariances between coalitions even if they can be present. That is a drawback of the core-like solution concepts in general, however, it also simplifies the analysis.

There is no need to differentiate between the concepts of the undominated and the dominating core when preference order is complete since they are the same. However, for the partial order preferences like SSD, differentiation is neccesary. We shall discuss the properties of both. The interpretation of the dominating core is straightforward as it aligns with what we are used to in the deterministic games or games with total order preferences. Each coalition gets at least what its worth. However, the interpretation of the undominated core is as follows: no coalition can receive a strictly better payoff. We shall discuss the undominated core in more details. We present an example demonstrating that the concept is far from perfect and then we present interpretation under special circumstances.

Example 5. *We consider the following example of a game involving 2 players. Let $(N = \{1, 2\}, v)$ be a stochastic TU-game. The characteristic function is normally distributed with independent marginals, i.e., $v(S) \sim N(\mu_S, \sigma_S^2)$. We prescribe random variables to each coalition:*

- $v(1) \sim N(10, 1)$,
- $v(2) \sim N(10, 1)$,
- $v(N) \sim N(2, 10)$.

It is apparent that the SSD-dominating core with the type (d, r_+) is empty since there are no pairs allocations (d_1, r_1) and (d_2, r_2) satisfying

$$d_1 + d_2 = 2 \;\;\&\;\; d_1 \geq 10 \;\;\&\;\; d_2 \geq 10.$$

However, the undominated SSD core is not empty. The problem is symmetric, so it is possible to swap roles of player 1 and 2. The following set of payoffs lies in the SSD-undominated core:

$$(d_1, d_2, r_1, r_2) = (x, 2 - x, t, 1 - t), \;\; \text{where } x > 10, t > \frac{9}{10}.$$

Thus, the first player receives payoff from normal distribution $N(d_1, r_1^2 \cdot 10)$ with a much greater expected value than the second player and much higher variance than that of the second player. The solution of this particular example could be coalitional structure, i.e., no cooperation, where each of the players receives their payoff according to their singleton value $v(i)$.

Remark. The undominated core is not necessarily nonempty in general. Suppose the same example as above, with $v(i) \sim N(10, 20)$. In this case the undominated core is empty.

In conclusion, we would not advise to use the SSD-undominated core every time when the SSD-dominating core is empty. However, there is a situation in which the undominated core seems reasonable. We illustrate it with the following example.

Example 6 (Undominated core interpretation)**.** *Let (N, v) be a stochastic TU-game and we question the undominated core $\boldsymbol{UDC}^{(d,r_+)}(v)$. For simplicity, we assume $v(S) \sim N(\mu_S, \sigma_S^2)$ is normally distributed. Suppose we already allocated d_i part of the allocation, i.e., d_i part of the payoff is already established for all the players. In such cases, we do not even need to know the whole distribution. It suffices to specify σ_S for all coalitions to be able to determine the undominated core. Knowing μ_S would suggest that the dominating core is nonempty at best, which directly follows from the type of allocation and Claim 10 which is proved further in the thesis, since $d(S) \geq \mu_S$ is necessary for the dominating core to be nonempty. However, without any information about μ_S and with no possibility of changing payments d_i we do not really need to model the entire distribution, the variance suffices at least for the case of normal distribution. Thus, we find r_i part of an allocation for each player in such a way that:*

$$\frac{\sigma_S}{\sigma_N} > r(S), \ \forall S \subseteq N.$$

Then such a pair of vectors d, r is at least in the undominated core.

The following lemma present the conditions for various distributions under which stochastic dominance occurs. These conditions to compare two distributions differing only in parameters helps with the analysis of SSD-dominating core. This lemma is crucial since two sufficiently different distributions can be often incomparable by SSD, e.g., if one distribution is normal with positive variance and the second is uniform. The following conditions can be either found in [5] or can be easily straightforwardly derived.

Lemma 9 (SSD conditions)**.** *Let X, Y be random variables. The following conditions are characterization of the relation $X \succeq_{SSD} Y$ for various distributions:*

- $\mu_X \geq \mu_Y$ *and* $\sigma_X^2 \leq \sigma_Y^2$ *if X and Y are normally distributed as* $X \sim N(\mu_X, \sigma_X^2)$ *and* $Y \sim N(\mu_Y, \sigma_Y^2)$.
- $a_X \geq a_Y$ *and* $b_Y \leq b_X + (a_X - a_Y)$, *or the latter equivalently* $\mathbb{E}[X] \geq \mathbb{E}[Y]$, *if X and Y are uniformly distributed as* $X \sim U[a_X, b_X]$ *and* $Y \sim U[a_Y, b_Y]$.
- $k_X \cdot \theta_X \geq k_Y \cdot \theta_Y$ *and* $\theta_X \geq \theta_Y$ *if X and Y are gamma distributed as* $X \sim \Gamma(k_X, \theta_X)$ *and* $Y \sim \Gamma(k_Y, \theta_Y)$, *where k is the shape parameter and θ is the scale parameter.*
- $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ *if X and Y are discretely uniformly distributed with realizations* $x_1 \leq x_2 \leq \ldots, x_K$ *and* $y_1 \leq y_2 \leq \ldots, y_K$ *and each of the realization having probability* $\frac{1}{K}$.

The following definitions are used in the following sections to consider TU-games which are given as a function of v.

Definition 30 (Mean, deviation and variance games)**.** *Let (N, v) be a stochastic TU-game. We denote $\mathbb{E}[v(S)]$ by μ_S, $\sqrt{\boldsymbol{Var}([v(S)])}$ by σ_S and $\boldsymbol{Var}([v(S)])$ by σ_S^2. We say:*

- *(N, μ) is called the mean game with characteristic function given by expected values of $v(S)$, i.e., $\mu(S) = \mu_S$.*
- *$(N, \sigma_{||\cdot||})$ is called the normalized standard deviation game with cost function given by $\sigma_{||\cdot||}(S) = \frac{\sigma_S}{\sigma_N}$.*
- *(N, σ^2) is called the variance game with cost function given by $\sigma^2(S) = \sigma_S^2$.*

To clarify, the characteristic function in the standard deviation game is denoted by $\sigma_{||\cdot||}$, with normalization performed relative to σ_N. In the mean game, there is no ambiguity in the notation of characteristic function, as μ_S represents $\mathbb{E}[v(S)]$. Consequently, the symbols $\mu(S)$ and μ_S can be used interchangeably. A similar notation approach applies to other games where characteristic function is some function of v.

3.1 Allocations with transfer payments

In this section, we focus on payoffs with transfer payments, i.e., where payoff of a player i is given as $x_i = d_i + r_i(v(N) - \mathbb{E}[v(N)]$. Primarily, we provide necessary and sufficient conditions for the SSD-dominating core to be nonempty for normal and continuous uniform distribution. Further, for the discrete distribution we only provide a way how to find a payoff from the SSD-dominating core. The section is divided into two parts, one dealing with stochastic payoffs with transfer payments and the other one dealing with stochastic payoffs with transfer payments and a general risk part.

We restrict our analysis to distributions from the scale-location family, where linear transformations change only parameters of the distributions. This makes the comparison much simpler, specifically for distributions from Lemma 9. Comparing two different distributions, even from scale-location family, can lead to useless results, for example, if $x(S)$ is normally distributed with a positive variance and $v(S)$ uniformly distributed within the finite bounds, then neither $x(S)$ dominates $v(S)$ nor does $v(S)$ dominate $x(S)$ in the terms of SSD.

Let us begin with a general observation about the relationship between the stochastic dominance and the expected value.

Claim 10 (Stochastic dominance and expected value)**.** *Let (N, v) be a stochastic TU-game. Let x be a stochastic payoff without transfer payments, i.e., (d, r_+). If $x(S) \succeq_{SSD} v(S)$, then $d(S) \geq \mathbb{E}[v(S)]$.*

Proof. The proof straightforwardly follows from Claim 1. □

This observation provides one of the conditions for the SSD-dominating core. Moreover, primarily it serves as a neccesary condition for the nonemptiness of the dominating core.

3.1.1 Nonnegative risk part

Here, we derive conditions describing the SSD-dominating core for stochastic payoffs with transfer payments, specifically when the risk part is positive, i.e., type of allocation is (d, r_+).

Theorem 11 (SSD dominating core under normal distribution)**.** *Let (N, v) be a stochastic TU-game. Suppose $v(S) \sim N(\mu_S, \sigma_S^2)$, $\forall S \subseteq N$ be normally distributed, where parameters $\mu_S \in \mathbb{R}, \sigma_S^2 \in \mathbb{R}_+$ are known. Then $\boldsymbol{DC}^{(d,r_+)}(v)$ is nonempty if and only if $C(\mu) \neq \emptyset$ and $C_{cost}(\sigma_{||\cdot||}) \neq \emptyset$.*

Proof. A payoff $x \in \mathbf{DC}^{(d,r_+)}(v)$ needs to satisfy $x(S) \succeq_{SSD} v(S),\ \forall S \subseteq N$. To be able to use Lemma 9, we need to calculate moments of $x(S)$ at first for any $S \subseteq N$. The first moment

$\mathbb{E}[x(S)] = d(S) + r(S)(\mathbb{E}[v(N)] - \mathbb{E}[v(N)]) = d(S),$
and the second central moment, i.e., variance
$\mathbf{Var}[x(S)] = \mathbf{Var}[d(S) + r(S)(v(N) - \mathbb{E}v(N)] = \mathbf{Var}[r(S)v(N)] = (r(S))^2\sigma_N^2.$

According to Lemma 9 and the calculations of moments such a payoff needs to satisfy the following inequalities:

$$d(S) \geq \mu_S, \tag{3.1}$$

$$\sigma_S^2 \geq \sigma_N^2(r(S))^2 \iff \frac{\sigma_S}{\sigma_N} \geq r(S), \text{ if } \sigma_N^2 > 0. \tag{3.2}$$

Inequalities of the core payoffs of the game (N, μ) and the core of the cost game $(N, \sigma_{||\cdot||})$ correspond precisely to those in (3.1) and (3.2) respectively. The same applies to the equality constraints $\mu_N = d(N)$ and $1 = r(N) = \frac{\sigma_N}{\sigma_N}$ (efficiency of the core payoffs). ☐

Claim 12 (Nonemptiness of the standard deviation game)**.** *Let $(N, \sigma_{||\cdot||})$ be a standard deviation game. The core $C_{cost}(\sigma_{||\cdot||})$ is nonempty if and only if for any balanced map $\mu : 2^N \longrightarrow [0, 1]$, where*

$$\forall i \in N : \sum_{S \in 2^N : i \in S} \mu(S) = 1$$

the following condition holds:

$$\sum_{S \in 2^N} \mu(S)v(S) \leq 1, \text{ where } v(S) = 1 - \sigma_{||\cdot||}(N \setminus S).$$

Proof. The statement follows from Theorem 2, after transforming the cost core inequalities of the standard deviation game to a classical profit game (N, v) as follows: $v(S) = 1 - \sigma_{||\cdot||}(N \setminus S)$. The conditions of the cost core are $r(S) \leq \sigma_{||\cdot||}(S)$ leading to $1 - r(N \setminus S) \leq \sigma_{||\cdot||}(S)$, which yields $1 - \sigma_{||\cdot||}(S)$. By substituting $v(N \setminus S) = 1 - \sigma_{||\cdot||}(S)$, we formulate a profit game (N, v) for which we can already use Theorem 2. ☐

Remark. In the broader setting of cooperative games, (N, v) is a so called *dual game* of $(N, \sigma_{||\cdot||})$ and the property characterizing nonemptiness of $C_{cost}(\sigma_{||\cdot||}$ is referred to as *balancedness*.

We proceed with uniform distribution and the definition of lower bound game and statement of a lemma. Similarly to mean value game, we arbitrarily switch between $a(S)$ and a_S.

Definition 31 (Lower bound game)**.** *Let (N, v) be a stochastic TU-game and $v(S) \sim U[a_S, b_S]$, $\forall S \subseteq N$ be uniformly distributed. Then lower bound game is a TU-game (N, a), defined as $a(S) = a_S$. We denote the core of such a game by $C(a)$.*

Lemma 13. *Let (N, a) be a TU-game, $r \in \mathbb{R}^n$, $K \in \mathbb{R}$ and (N, v_r) additive TU-game, i.e., $v_r(S) = \sum_{i \in S} r(i)$. Then $C(a + K \cdot v_r) = C(a) + K \cdot r$, i.e., payoffs in the core of the game $(N, a + K \cdot v_r)$ are just shifted vectors by $K \cdot r$ of the core of the game (N, a).*

Proof. Let us begin by noting that $C(K \cdot v_r) = K \cdot r$ for an additive game (N, v_r) and $K \in \mathbb{R}$. For the clarity of the notation, we denote a payoff in the core of game (N, a) by x, in the core of game (N, v) by y and in the core of game $(N, a+K \cdot v_r)$ by z . We show both inclusions. The first one $C(a + K \cdot v_r) \supseteq C(a) + K \cdot r$ holds directly since $a(S) \leq x(S)$, $r(S) \leq y(S)$ imply $a(S)+r(S) \leq x(S)+y(S) = z(S)$. The second inclusion is more intricate. Suppose $z \in C(a + K \cdot v_r)$, then

$$z(S) \geq a(S) + K \cdot r(S) \;\; \& \;\; z(N) = \mu_N.$$

The core of the additive game is a single point, i.e., $C(K \cdot v_r) = K \cdot r$. We define y as $z(S) - K \cdot r(S) = y(S)$. It is left to show that $y \in C(a)$. The inequality $z(S) \geq a(S) + K \cdot r(S)$ directly yields $y(S) \geq a(S)$, thus, confirming the second inclusion. □

Theorem 14 (SSD-dominating core under uniform distribution)**.** *Let (N, v) be a stochastic TU-game. Suppose $\forall S \subseteq N, v(S) \sim U[a_S, b_S]$ is uniformly distributed, where parameters $a_S, b_S \in \mathbb{R}$ are known and distributions are not degenerated, i.e., $a_S < b_S$. Then the following implications hold:*

$$\boldsymbol{DC}^{(d,r+)}(v) \neq \emptyset \implies C(\mu) \neq \emptyset \;\; \& \;\; C(a) \neq \emptyset,$$
$$(N, a) \textit{ is a convex game} \quad \& \;\; C(\mu) \neq \emptyset \implies \boldsymbol{DC}^{(d,r+)}(v) \neq \emptyset,$$

where (N, μ) is the mean game and (N, a) is the lower bound game of (N, v).

Idea of the proof. We reformulate the problem of finding a payoff in the dominating core to a problem of finding a payoff vector in an intersection of two deterministic TU-games. We can further restate this as a problem of finding vectors $x \in C(a)$ and $y \in C(\mu)$ such that the vector $y - x$ has nonnegative entries. The sufficient condition tells us that finding such x is possible for an arbitrary $y \in C(\mu)$ if the convexity of the game (N, a) is assumed. The idea of the proof of the sufficient condition is as follows: We take a payoff $y \in C(\mu)$ and we iterate the following process: We decrease the value of one arbitrary entry of y in such a way that the new vector does not violate inequalities given by $C_{\leq}(a)$. Then we iterate by choosing another entry till we end up with a vector from the core $C(a)$.

□

Proof. Let us begin the proof with a preparatory part before proving the implications. A stochastic payoff in $x \in \mathbf{DC}^{(d,r+)}(v)$ needs to satisfy $x(S) \succeq_{SSD} v(S),\ \forall S \subseteq N$. To be able to use Lemma 9, we calculate the distribution of every $x(S)$ as a linear transformation of $v(S)$. The distribution of $x(S)$ is uniform:

$$x(S) \sim U[d(S) + r(S)(a_N - \mu_N), d(S) + r(S)(b_N - \mu_N)],$$

where $\mu_S = \mathbb{E}[v(S)]$. Then by using Lemma 9, we get the following conditions for a stochastic payoff with transfer payments to be in the $\mathbf{DC}^{(d,r+)}(v)$:

$$d(S) \geq \mu_S, \tag{3.3}$$
$$d(S) \geq a_S + r(S)(\mu_N - a_N). \tag{3.4}$$

A stochastic payoff satisfies (3.3) if the game (N, μ) has a nonempty core, i.e., $C(\mu) \neq \emptyset$. Both (3.3) and (3.4) have $d(S)$ on the left hand side. Thus, we can view these conditions for d as the conditions for cores of two games (N, μ) and $(N, a + v_r(\mu_N - a_N))$, where (N, v_r) represents an additive game given as $v_r(S) = r(S)$. Then the vectors d, r satisfy both inequalities if and only if the corresponding stochastic payoff vector $x \in \mathbf{DC}^{(d,r+)}(v)$ and that is equivalent to vector d being in the intersection of two cores $d \in C(\mu) \cap C(a + v_r(\mu_N - a_N))$. By Lemma 13, we can reformulate the problem of the nonemptiness of the intersection of the two cores to the problem of finding $d \in C(\mu) \cap (C(a) + r(\mu_N - a_N))$, i.e., finding d in the intersection of $C(\mu)$ and $C(a)$ shifted by $r \cdot (\mu_N - a_N)$. We can observe that for any $d \in C(\mu)$ it also holds $d \in C_{\leq}(a)$ from the fact that $\mu_S \geq a_S$ for any uniform distribution.

Now that we established the preparatory part we can move to the neccesary and sufficient condition of the nonemptiness of $\mathbf{DC}^{(d,r+)}(v)$. Let us prove the first implication. Since $\mathbf{DC}^{(d,r+)}(v) \neq \emptyset$, therefore, there $\exists d$, and $\exists r \geq 0$ such that $d \in C(\mu) \cap (C(a) + r(\mu_N - a_N))$. We can see that such a vector d can exists only if there is a vector in both $C(a)$ and $C(\mu)$. Therefore, we proved the first implication.

Let us prove the first implication, i.e., the sufficient condition for the dominating core to be nonempty. Suppose $C(\mu) \neq \emptyset$ and (N, a) is a convex game. Since the game (N, a) is convex, it holds $C(a) \neq \emptyset$ from Theorem 3. We want to show that then $C(\mu) \cap (C(a) + r(\mu_N - a_N)) \neq \emptyset$ for some non-negative r satisfying $r(N) = 1$. We show that for $y \in C(\mu)$ there exists an $x \in C(a)$ and non-negative vector $r:\ r_i \geq 0,\ r(N) = 1$ such that $x + r(\mu_N - a_N) = y$. Any such r satisfies the condition of the dominating core for the grand coalition, i.e,

$$x(N) + r(N)(\mu_N - a_N) = a_N + (\mu_N - a_N) = \mu_N.$$

Let us now prove that we can find such r by finding a vector x such that $x_i \leq y_i,\ \forall i \in N$. For that we use the following process $\mathcal{P}$:

1. Set $x^0 = y$.

2. At step m, select player k_m defined as:

$$k_m = \min\{k \in N : a_S < x^{m-1}(S),\ \forall S \subseteq N,\ k \in S\}.$$

   The player k_m is only included in the coalitions where $x^{m-1}(S)$ exceeds the bound a_S, i.e., $a_S < x^{m-1}(S),\ \forall S \subseteq N,\ S \ni k_m$. If multiple players meet this condition, choose the one with the smallest index $k_m \in N$.

3. Set $x^m = x^{m-1} - e_{k_m} t_m$, where e_k is a vector of canonical basis in $\mathbb{R}^n$ and t_m is a length of step m given by:

$$t_m = \min_{S \subseteq N, S \ni k_m} [x^{m-1}(S) - a_S].$$

We denote $\mathcal{S}_m = \arg\min_{S \subseteq N, S \ni k_m} [x^{m-1}(S) - a_S]$.

4. If $x^m(N) \neq a_N$ then go to the step $m+1$. If $x^m(N) = a_N$, we are done. We denote the final step as m_{end}.

Notice that at each step m, x^m is in $C_{\leq}(a)$. However, we still need to verify that $x^{m_{end}}(N) = a_N$, i.e., the process $\mathcal{P}$ provides a vector within the core $C(a)$. A key observation for the verification is that $\mathcal{S}_m$ is closed under union, or equivalently, that $\bigcup_{S \in \mathcal{S}_m} S$ is equal to the inclusion-wise maximal element in $\mathcal{S}_m$. Suppose that S, $T \in \mathcal{S}_m$. Thus, $a_S = x^m(S)$ and $a_T = x^m(T)$. Given the convexity of the game (N, a), it follows that $a_S + a_T \leq a_{S \cup T} + a_{S \cap T}$. Since $x \in C_{\leq}(a)$, we also have $a_{S \cup T} \leq x(S \cup T)$ and $a_{S \cap T} \leq x(S \cap T)$. From these inequalities, it straightforwardly follows that $a_{S \cup T} = x(S \cup T)$. Let us follow the process $\mathcal{P}$ assuming the convexity of (N, a).

- At step $m = 1$: We can easily see $x^1(S_1) = a_{S_1}$ from the definition of the process $\mathcal{P}$.

- At step $m = 2$: Let S_2 be the inclusion-wise maximal coalition from $\mathcal{S}_2$. We have $x^2(S_2) = a_{S_2}$. Possibly the coalition S_1 and S_2 have nonempty intersection, i.e., $S_1 \cap S_2 \neq \emptyset$. Thus, due to the convexity of (N, a),

$$x^2(S_1 \cup S_2) + x^2(S_1 \cap S_2) = a_{S_1} + a_{S_2} \leq a_{S_1 \cup S_2} + a_{S_1 \cap S_2}.$$

Since $x^2(S)$ is always in $C_{\leq}(a)$, we know that $a_S \leq x^2(S)$, $\forall S \subseteq N$. Thus,

$$x^2(S_1 \cup S_2) + x^2(S_1 \cap S_2) = a_{S_1 \cup S_2} + a_{S_1 \cap S_2},$$

and specifically $x^2(S_1 \cup S_2) = a_{S_1 \cup S_2}$ and $x^2(S_1 \cap S_2) = a_{S_1 \cap S_2}$.

- At step $m > 2$: Let S_m be the inclusion-wise maximal coalition from $\mathcal{S}_m$. We have $x^m(S_m) = a_{S_m}$. The general step follows the argument of $m = 2$. It holds

$$x^m(S_1 \cup \ldots \cup S_m) + x^m((S_1 \cup \ldots \cup S_{m-1}) \cap S_m) = a_{(S_1 \cup \ldots \cup S_{m-1})} + a_{S_m},$$

and at the same time from the convexity of (N, a),

$$a_{(S_1 \cup \ldots \cup S_{m-1})} + a_{S_m} \leq a_{((S_1 \cup \ldots S_{m-1}) \cap S_m)} + a_{(S_1 \cup \ldots \cup S_m)}.$$

Similar to $m = 2$, we get $x^m(S_1 \cup \ldots \cup S_m) = a_{(S_1 \cup \ldots \cup S_m)}$.

- If $(S_1 \cup \ldots \cup S_m) \neq N$, we can continue with the process by choosing a player $i \in N \setminus \bigcup_{i=1}^m S_i$, because if there exists a player i which is not in any of the coalitions S_i, i.e., $i \notin \bigcup_{i=1}^m S_i = \bigcup_{i=1}^m \mathcal{S}_i$, then for such a player and every coalition $S, i \in S$, we still obtain $x^m(S) \geq \mu_S > a_S$ since the distribution of $v(S)$ is not degenerated, i.e., $a_S < b_S$. On the other hand, if $(S_1 \cup \ldots \cup S_m) = N$ then $x^m(N) = a_N$ and since $x^m \in C_{\leq}(a)$, we also get $x^m \in C(a)$.

The payoff $x^{m_{end}}$ lies in $C(a)$. By setting

$$r = \sum_{m=1}^{m_{end}} t_m e_{k_m},$$

we get feasible r, because $r_i \geq 0$ and the construction of r and the choice of t_m yield $r(N) = 1$ which follows from the fact that $x^{m_{end}}(N) - y(N) = (\mu_N - a_N)$. Hence, $x^{m_{end}} + r(\mu_N - a_N) = y$.

□

From the theorem, we obtain a sufficient condition for the nonemptiness of the SSD-dominating core. The following example shows that superadditivity of (N, a) together with nonemptiness of $C(\mu)$ is not a sufficient condition.

Example 7 (Process failure for a superadditivity)**.** *In the example, we show that superadditivity of (N, a) is not sufficient together with the nonemptiness of $C(\mu)$ for the nonemptiness of the SSD-dominating core. Let (N, a) be the following superadditive game:*

- $a_{12} = 3$,
- $a_{23} = 3$,
- $a_N = 3$,
- $a_S = 0$, *otherwise.*

Hence, $C(a) = \{(0, 3, 0)\}$. Let (N, μ) be the mean game, which together with (N, a) define stochastic TU-game where $v(S) \sim U[a_S, b_S]$:

- $\mu_1 = \mu_3 = 5$,
- $\mu_{12} = 3$,
- $\mu_{23} = 3$,
- $\mu_N = 12$,
- $\mu_S = 1$, *otherwise.*

Notice that for all $S \subseteq N$, the random variable $v(S)$ is not degenerated. For these values of μ, the core $C(\mu)$ is one point in $\mathbb{R}^3$, specifically $C(\mu) = \{(5, 2, 5)\}$. Thus, according to the proof of Theorem 14, we need to find $d \in C(\mu) \cap (C(a) + r(\mu_N - a_N))$, however, such r with nonnegative entries does not exist since $(0, 3, 0) \in C(a)$ has the second entry greater than any entry in the $C(\mu)$. Therefore, superadditivity of (N, a) does not suffice.

In the previous example, the convexity of (N, a) is violated on the pair of coalitions $\{1, 2\}$ and $\{2, 3\}$. Specifically, the following inequality $a_{12} + a_{23} > a_{123} + a_2$ is violating the convexity of (N, a). We can extend this idea by showing that when the convexity of (N, a) is violated for a pair of coalitions with a singleton intersection, then there exists a game (N, μ) such that the SSD-dominating core is empty. In the following claim, we investigate this idea while using the standard property of the core of convex games:

$$\text{If a game } (N, a) \text{ is convex then } \forall S \subseteq N,\ \exists x \in C(a) : a(S) = x(S).$$

Derivation of this result is a straightforward exercise based on results connecting the core of a convex game with the *Weber set* (see [13] for details on the relation). This further points out the difference between convexity and superadditivity of (N, a) in the sufficient condition of Theorem 14. The following claim determines a situation which yields the empty SSD-dominating core.

Claim 15. *Let (N, v) be a stochastic TU-game, where $v(S) \sim U[a_S, b_S],\ \forall S \subseteq N$ follows uniform distribution $\forall S \subseteq N$ with $a_S, b_S \in \mathbb{R}$ and $a_S < b_S$. If there exists a player $i \in N$ such that for any payoff $x \in C(a)$ in the lower bound game (N, a) holds $x_i > a_i$, then there exist values $b_S,\ \forall S \subseteq N$ such that $\boldsymbol{DC}^{(d,r_+)}(v) = \emptyset$.*

Proof. Instead of constructing the values b_S, we construct values of the mean game (N, μ), from which the values b_S follow. We set $\mu_i = a_i + \varepsilon, \varepsilon > 0$, where ε is chosen small enough to ensure $\mu_i < x_i$ for every $x \in C(a)$. Then by defining $\mu_{N\setminus i} = \mu_N - \mu_i$, we ensure that for every $y \in C(\mu)$, we have $y_i = \mu_i$. We set the rests of the values of (N, μ) in such a way that for every remaining coalition S, $a_S < \mu_S$. It is immediate by Theorem 14, that $\mathbf{DC}^{(d,r_+)}(v) = \emptyset$. □

Let us now consider uniform discrete distribution, where realizations are equiprobable. We denote realizations by ω_i and for the analysis of the SSD, we also assume the realizations are ordered $\omega_1 \leq \omega_2 \leq \ldots \leq \omega_T$. Each of the realizations has the assigned probability $\frac{1}{T}$. We also note that this distribution is able to describe situation where some realizations are identical, i.e., $\omega_i = \omega_j$. In what follows, we have a uniform discrete distribution $v(S),\ \forall S \subseteq N$. We thus use ω_i^S to denote the i-th realization of coalition S.

Let us derive conditions for the nonemptiness of the SSD-dominating core under discrete distribution with equiprobable realizations. Since the distribution is part of scale-location family, the distribution of $x(S) = d(S) + r(S)(v(N) - \mathbb{E}[v(N)])$ is also discrete with equiprobable realizations. Let us describe the relation $x(S) \succeq_{SSD} v(S)$. For such a discrete distribution, the criteria for SSD are based on partial sums thanks to Lemma 9 of the ordered realizations of the random variable. Hence, $x(S) \succeq_{SSD} v(S)$ if

$$k \cdot d(S) + r(S)\Big(\sum_{i=1}^{k} \omega_i^N - \mathbb{E}[v(N)]\Big) \geq \sum_{i=1}^{k} \omega_i^S,\ k = 1, 2, \ldots, T. \tag{3.5}$$

For such conditions, we can find vectors d and r by solving corresponding linear program. The number of conditions for establishing the SSD-dominating core under the discrete uniform distribution is exactly $(2^n - 1) \cdot T$. Solving a linear program with this number of conditions can become untractable even for a small number of players and a moderate number of realization. In the case of a larger number of realizations, the approximation by continuous distribution is more suitable, however, if we insist on using the discrete distribution, we can implement heuristics to improve the computational complexity. Both of the heuristics use compute one of d, r first and then compute the other vector. Let us start with the heuristic computing vector r first:

1. **Identify feasible vector** r: By setting $d(S) = \mu_S$ in (3.5), compute a feasible r, i.e., vector r satisfying $r(N) = 1$ and (3.5).

2. **Compute corresponding** d: Compute a feasible vector d, i.e., $d(N) = \mu_N$ and $d(S) \geq \mu_S$.

Such an approach yields only a subset of the dominating core. However, each step of the heuristic can be solved by a linear program with half of the inequalities and half of the variables, which might lead to a significant reduction of computational complexity, at least in some instances. Specifically, to determine the vector r in the first step, the following set of conditions needs to be satisfied:

$$\forall S \subset N : r(S) \geq \max_{k \in \{1,\ldots,T\}} \frac{\sum_{i=1}^{k} (\omega_i^S - \mathbb{E}[v(S)])}{\sum_{i=1}^{k} (\omega_i^N - \mathbb{E}[v(N)])}.$$

This vector r, together with any feasible d, corresponds to a random payoff from the SSD-dominating core. It is possible that the heuristic fails to find a solution even though the SSD-dominating core is nonempty.

The second heuristic finds a feasible vector d first and subsequently finds all feasible r corresponding to that given vector d:

1. **Identify feasible vector** d: Find one feasible vector d, i.e., $d(N) = \mathbb{E}[v(N)]$, satisfying $d(S) \geq \mathbb{E}[v(S)]$.

2. **Compute corresponding** r: Find all feasible r for a given vector d. It results in pairs (d, r) corresponding to feasible payoffs within SSD-dominating core.

Once again, we divide the problem into two subproblems, each of half the size, which might lead to reduction of computational complexity. It is also crucial to note that the vectors r are guaranteed to be feasible only with respect to the initially set feasible vector d and not necessarily for any other feasible d.

3.1.2 General risk part

The stochastic payoff with transfer payments and general risk parts enhances the ability to model a wider range of scenarios. This generalization allows to capture situations where players have conflicting expectations regarding the outcome. For instance, consider a player predicting that the realization of the random variable will be lower than its expectation. Such a player would prefer higher payoff if the actual realization is less than expected, effectively desiring negative correlation, $corr(x_i, v(N)) = -1$, between their payoff and the value of the grand coalition. This approach enables to model conflicting interest among players, reflecting a negative linear relationship between the random payoff and value of the grand coalition.

We present a few results concerning this type of stochastic payoff. Proofs of the propositions are straightforward or at least similar to the proofs of already proved theorems for the stochastic payoff with transfer payments and nonnegative risk part, i.e., type of payoff (d, r_+).

Claim 16. *Let (N, v) be a stochastic TU-game, where $v(S) \sim N(\mu_S, \sigma_S^2)$, $\forall S \subseteq N$ follows normal distribution with parameters $\mu_S \in \mathbb{R}, \sigma_S^2 \in \mathbb{R}_+$. It holds $\boldsymbol{DC}^{(d,r)}(v)$ is nonempty if and only if there are $d \in \mathbb{R}^n, r \in \mathbb{R}_+^n$:*

- *$d(S) \geq \mu_S$, $\forall S \subseteq N$ and $d(N) = \mu_N$,*
- *$|r(S)| \leq \frac{\sigma_S}{\sigma_N}$, $\forall S \subseteq N$ and $r(N) = 1$.*

Proof. This follows from Lemma 9 and the proof can proceed in a similar manner to the proof of Theorem 11 with the difference that vector r does not necessarily have all nonnegative entries. □

Claim 17. *Let (N, v) be a stochastic TU-game. Suppose $\forall S \subseteq N, v(S) \sim U[a_S, b_S]$ is uniformly distributed, where parameters $a_S, b_S \in \mathbb{R}\ \forall S \subseteq N$ are known and $a_S < b_S$. Then $\boldsymbol{DC}^{(d,r)}(v)$ is nonempty if and only if $C(a) \neq \emptyset$ & $C(\mu) \neq \emptyset$.*

Proof. Similarly to the proof of Theorem 14, we can derive the conditions for nonemptiness of the SSD-dominating core under uniform distribution of $v(S)$:

$$d(S) \geq \mu_S, \tag{3.6}$$
$$d(S) \geq a_S + r(S)(\mu_N - a_N). \tag{3.7}$$

The crucial part is again using Lemma 13 to reformulate the problem of the non-emptiness of the SSD-dominating core to the non-emptiness of the intersection of cores of two deterministic TU-games. Again, we only need to find $d \in (C(a) + r(\mu_N - a_N)) \cap C(\mu)$. This is equivalent to showing that for some $x \in C(a)$ and $y \in C(\mu)$, we can find $r \in \mathbb{R}^n, r(N) = 1$ such that the intersection is nonempty. Notice, this is always possible since r has to satisfy $y = x + r(\mu_N - a_N)$, thus $r = \frac{1}{\mu_N - a_N}(y - x)$. For such r the condition $r(N) = 1$ is satisfied. □

3.2 Allocations without transfer payments

In this section, we analyze games for a given distribution of v where the allocation type for each player is $x_i = r_i \cdot v(N)$, where $r_i \geq 0\ \forall i \in N$. To be able to directly apply Lemma 9 and simplify the analysis of the stochastic dominance, we assume that $v(S)$ belongs to a scale family of distributions. In such family, multiplication by a constant only changes the parameters of the probability distribution and does not change the underlying distribution. Given that this type of allocation can be used well for scale family of distributions, we show conditions for the dominating core not only for the distributions already assumed before, however, we also derive conditions for the gamma distribution, which is outside the scale-location family.

Before we proceed with results concerning allocations without transfer payments, we discuss the difference between (d, r_+) and r_+ types of allocation. It is observable that in the case of scale-location distribution, the allocation type (d, r_+) is slightly more general than r_+. Let us illustrate this on the following situation. The conditions for the SSD-dominating core for the stochastic TU-game under normally distributed values of coalitions and payoff with transfer payments are presented in Theorem 11 as follows:

$$d(S) \geq \mu_S, \tag{3.8}$$
$$\sigma_S^2 \geq \sigma_N^2 r^2(S) \iff \frac{\sigma_S}{\sigma_N} \geq r(S), \text{ if } \sigma_N^2 > 0. \tag{3.9}$$

Similarly, we can derive the conditions for the SSD-dominating core when the allocation type is r_+:

$$r(S) \geq \frac{\mu_S}{\mu_N}, \tag{3.10}$$

$$\sigma_S^2 \geq \sigma_N^2 r^2(S) \iff \frac{\sigma_S}{\sigma_N} \geq r(S), \text{ if } \sigma_N^2 > 0. \tag{3.11}$$

The difference lies in (3.8) and (3.10), while (3.9) and (3.11) remain the same. Condition (3.10) significantly restricts the feasible vector r in the case of the allocation type r_+. Thus, the type of allocation (d, r_+) shows more generality at least for the normal distribution. However, for the distribution characterized by the first 2 moments, the generality of (d, r_+) becomes apparent from the calculation of the moments of $x(S)$. A payoff x_i of player i with transfer payments has the moments as follows: $\mathbb{E}[x_i] = d_i$ and $\mathbf{Var}[x_i] = r_i^2 \cdot \mathbf{Var}[v(N)]$ as opposed to a payoff without transfer payments: $\mathbb{E}[x_i] = r_i \cdot \mathbb{E}[v(N)]$ and $\mathbf{Var}[x_i] = r_i^2 \cdot \mathbf{Var}[v(N)]$. The variance for both types of allocations is the same and it depends on r_i. However, the expected value is determined by d_i for the type (d, r_+) and by r_i for r_+ which suggests the allocation (d, r_+) is more general for distributions determined by the first two moments. We do not provide rigorous argument as well as an argument for general distributions.

Let us now present some of the conditions for distributions we already assumed. These were normal and uniform, both discrete and continuous, distributions. We also propose the conditions for a distribution outside scale-location family; the gamma distribution, which lies within scale family. Since the arguments are basically the same as for the allocation with transfer payments (d, r_+), we do not provide complete proofs but only the conditions for the SSD-dominating core. For a more detailed analysis and a derivation of the conditions, we refer to the previous section on the allocation with transfer payments.

Theorem 18 (SSD-dominating core conditions)**.** *Let (N, v) be a stochastic TU-game. A feasible stochastic payoff vector $x = r \cdot v(N)$ lies in the SSD-dominating core, $\boldsymbol{DC}^{r_+}(v)$, if and only if the following conditions are met:*

- *If $v(S) \sim N(\mu_S, \sigma_S^2)$, $\forall S \subseteq N$ is normally distributed, where parameters $\mu_S \in \mathbb{R}, \sigma_S^2 \in \mathbb{R}_+$ then*

$$r(S) \geq \frac{\mu_S}{\mu_N} \ \& \ r(S) \leq \frac{\sigma_S}{\sigma_N}, \ \forall S \subseteq N.$$

- *If $v(S) \sim U[a_S, b_S]$, $\forall S \subseteq N$ is uniformly distributed, where parameters $a_S, b_S \in \mathbb{R}$ are known and distributions are not degenerated, i.e., $a_S < b_S$ then*

$$r(S) \geq \max\{\frac{\mu_S}{\mu_N}, \frac{a_S}{a_N}\}, \ \forall S \subseteq N.$$

- *If $v(S)$ has a discrete uniform distribution with equiprobable realizations $\omega_1 \leq \omega_2 \leq \ldots \leq \omega_T$, $\forall S \subseteq N$, then*

$$\forall S \subset N : r(S) \geq \max_{k \in \{1,\ldots,T\}} \frac{\sum_{i=1}^k v(S, \omega_i)}{\sum_{i=1}^k v(N, \omega_i)}, \ \forall S \subseteq N.$$

Remark. The conditions in Theorem 18 are given in the form of fractions which needs the assumption of the denominator to be nonzero. If the denominator is zero than the condition simply says that the numerator of the fraction needs to be less than or equal to 0. The case of σ_S does not need this discussion since $\sigma_S > 0$.

Let us now derive conditions for gamma distributed $v(S)$.

Theorem 19 (SSD-dominating core under gamma distribution)**.** *Let (N, v) be a stochastic TU-game. Suppose $\forall S \subseteq N,\ v(S) \sim \Gamma(k_S, \theta_S)$ is gamma distributed, where parameters $k, \theta \in (0, \infty)$ are known and k is called the shape parameter and θ is called the scale parameter. Then $\boldsymbol{DC}^{r+}(v) \neq \emptyset$ if and only if*

$$r(S) \geq \frac{k_S \cdot \theta_S}{k_N \cdot \theta_N} \ \ \& \ \ r(S) \geq \frac{\theta_S}{\theta_N}.$$

Proof. The distribution of $x(S)$ when $v(S)$ is gamma distributed can be expressed as $x(S) \sim \Gamma(k_S, r(S) \cdot \theta_S)$. The conditions for the $\mathbf{DC}^{r+}(v) \neq \emptyset$ follow directly from Lemma 9. □

3.3 Unstructured allocation

In this section, we assume the most general setting, i.e., stochastic payoff x is a multivariate random variable. We call the allocation unstructured since we do not assume any covariance structure of x as we do for the other allocation types, where $|\rho_{i,j}| = 1$. In Theorem 20, we state a result concerning the nonemptiness of the SSD-dominating core for a normal distribution and unstructured allocation. In the corollary following this theorem, we highlight the connection between Theorem 20 and Theorem 11; namely that one can view Theorem 11 as a special case of Theorem 20, where the covariances are ± 1. This allows to view allocations with transfer payments and with or without general risk part (d, r_+) and (d, r) as a reasonable framework for a payoff allocation of players in a coalition, which is in detail discussed in the rest of this section. We specifically employ normal distribution due to its structural properties. Notably, linear transformation does not change the normal distribution (only parameters), and importantly, the sum of normally distributed random variables remains normally distributed, irrespective of the covariance relationship among these random variables.

Theorem 20 (Unstructured allocation under normal distribution)**.** *Let (N, v) be a stochastic TU-game, where $v(S) \sim N(\mu_S, \sigma_S^2)$, $\forall S \subseteq N$ is normally distributed, where parameters $\mu_S \in \mathbb{R}, \sigma_S^2 \in \mathbb{R}_+$. We assume that x has a multivariate normal distribution $x \sim N_n(\overline{\mu}, \Sigma)$, where $\overline{\mu} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Var}(x_i) = \Sigma_{ii} = \overline{\sigma}_i^2$. Then $\boldsymbol{DC}(v) \neq \emptyset$ if and only if $C(\mu) \neq \emptyset$ and $C(\sigma^2) \neq \emptyset$. Specifically, $x \in \boldsymbol{DC}(v)$ if and only if it satisfies the following conditions:*

$$\begin{aligned} \forall S \subseteq N \ &: \mu_S \leq \sum_{i \in S} \overline{\mu}_i, \\ \forall S \subseteq N \ &: \sigma_S^2 \geq \sum_{i,j \in S} \rho_{i,j} \overline{\sigma}_i \overline{\sigma}_j = \Sigma_{ij}. \end{aligned}$$

Proof. We can proceed similarly as in the proof of Theorem 11. We just need to work straight with the variance instead of standard deviation game. We compute only the variance of $x(S),\ S \subseteq N$ to get the second inequality:

$$\mathbf{Var}[x(S)] = \sum_{i,j \in S} \mathbf{cov}(x_i, x_j) = \sum_{i,j \in S} \rho_{i,j} \overline{\sigma}_i \overline{\sigma}_j.$$

□

Corollary (Transfer payments are special case). Consider two stochastic TU-games, the first denoted by (N, z) is the one defined in Theorem 20, i.e., payoff x is a multivariate random variable. The second game denoted by (N, v) , i.e., game follows the model assumed in Theorem 11. In this setup, when the correlation coefficient $\rho_{i,j} = 1,\ \forall i, j \in N$, i.e., x_i and x_j are perfectly correlated for any players $i, j \in N$, and the variance of each x_i is defined as $\mathbf{Var}(x_i) \equiv \overline{\sigma}_i^2 = r_i^2 \sigma_N^2$, where $r_i \geq 0\ \forall i \in N$ and $r(N) = 1$, then $\mathbf{DC}(z) = \mathbf{DC}^{(d,r_+)}(v)$.

The corollary demonstrates that under the assumption of fully correlated payoffs, i.e., $\rho_{i,j} = 1,\ \forall i, j \in N$, the model employing the general allocation type can be described by the model with the allocation with transfer payments. This assumption is intuitively aligned with the notion of coalition formation (in the core the formation of N), where cooperating players are likely to exhibit correlated payoffs. Such a reasoning can be extended to allocation with transfer payment and general risk part (d, r), where the pairwise correlations can be -1, which indicates perfectly inversely correlated payoffs. This corollary supports our focus on studying coalitions where all players within the group exhibit strongly correlated values, with the pairwise correlations absolute value 1. Such a setting helps in understanding the dynamics and payoff distributions within cooperating groups.

Further exploration into the stochastic payoffs in cooperative games might consider the implications of arbitrary correlations among the marginal distributions of the random payoffs. This approach, although potentially less tractable, raises interesting questions about the definition of coalition formation and the appropriate methods for distributing profits among various coalitions. The complexity of arbitrary correlations presents challenges in precisely defining what it means for a coalition to form and how its profits should be allocated.

4. Coalition structures under risk aversion

The previous chapters are centered around solution concepts for grand coalition. In this chapter, we change our focus to coalition structure formation. Specifically, we target questions of stability of coalition structures. We have already encountered important stability concept in the cooperative game theory, which is the coalition rationality of the core. We defined the core as a stability concept for the grand coalition, however, it can be also generalized for the coalition structures. There are a few concepts for studying questions regarding stability in the cooperative game theory, such as the *cost of stability* [19], which studies stability by adding external payment to the grand coalition, or concepts based on coalition structures and coalition formation which are studied broadly and research from last decades is well summarized from different perspectives in [20] and [14] or in one older publication from the previous century [21].

In this chapter, we aim to describe a stability concept where no player can change the coalition within the coalition structure to which he belongs in a manner that is profitable for himself, his current coalition, and the coalition he joins. We motivate when this notion of stability is more reasonable then the stability given by the SSD-undominated core from the previous chapter. Due to a great number of results in the area of coalition structures, and the existence of the aforementioned surveys, we do not provide an exhaustive overview.

We generalize the notion of *individually stable contractual equilibrium*, which was initially discussed in [22] in the setting of so called *hedonic games.* The definition is based upon a possibility of a player to deviate from a coalition, where the possibility is represented by an existence of feasible payoffs for a given coalition structure. We must, therefore, generalize our notion feasibility from Definition 32 to *feasible payoff for a coalition structure.*

Definition 32 (Feasible payoff for a coalition structure)**.** *Let* (N, v) *be a stochastic TU-game. Stochastic payoff* x *with transfer payments is feasible for a coalition structure* $\Pi(N) = (S_1, \dots, S_K)$ *if* x_{S_j} *is feasible* $\forall S_j \in \Pi(N)$*. A feasible payoff for a coalition* S_j *with respect to* $\Pi(N)$ *is denoted by* x_{S_j}*.*

Remark. A feasible payoff can be similarly defined for other types of allocations like allocations with transfer payments and with general risk part (d, r) or allocations without transfer payments r_+.

The idea behind this notion of stability follows the view of an individual player. It says that coalition structure is stable if no player can change coalitions in a way, which is feasible and at the same time profitable for him and both the coalition he leaves and the coalition he joins. To evaluate the profitability of the transfer under the stochastic setting, we employ SSD.

Definition 33 (Credible deviation of a player)**.** *Let* (N, v) *be a stochastic TU-game and* x *a stochastic payoff with transfer payments. Further, let* $\Pi(N) = (S_1, \dots, S_K)$ *be a coalition structure. A player* $i \in S_j$ *has a* credible contractual deviation *from* $\Pi(N)$ *if for all payoff vectors* y *feasible for* $\Pi^1(N) = (S_1, \dots, S_k \setminus$

$i, \ldots, S_p \cup i, \ldots, S_K)$ and for all payoff vectors x feasible for $\Pi(N)$ satisfying the following condition

$$x_{S_k}(S_k \setminus i) \preceq_{SSD} v(S_k \setminus i) \quad \& \\ y_{S_p \cup i}(S_p) \succeq_{SSD} v(S_p) \quad \&$$

holds the following condition for a player i: $y_{S_p \cup i}(i) \succeq_{SSD} x_{S_k}(i)$ and at least one of the relations is strict, i.e., $\prec$.

Definition 34 ($S1$ stability)**.** *Let (N, v) be a stochastic TU-game and x stochastic payoff with transfer payments. We say $\Pi(N)$ is $S1$ stable if no player has a credible deviation and there exists an individually rational payoff for $\Pi(N)$. Stochastic payoff x is individually rational for $\Pi(N)$ if $\forall S \in \Pi(N) : \exists x_S, \forall i \in S :$ $x_S(i) \succeq_i v(i)$*

Remark. We do restrict ourselves to a coalition structures where individually rational payoff exists. We use the term $S1$ *stability* to refer to this concept since it concerns only one player deviations from a given coalition structure.

Let us now state the theorem for $S1$ stability for a normally distributed values of coalitions $v(S)$, $\forall S \subseteq N$.

Theorem 21. *Let (N, v) be a stochastic TU-game and x be a stochastic payoff with transfer payments. Let $v(S) \sim N(\mu_S, \sigma^2)$ be normally distributed random variables with parameters $\mu_S \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$. The game (N, v) has $S1$ stable partition.*

Proof. In the first step, we prove that it is not possible to obtain a sequence of credible deviations which starts and finishes with the same coalition structure. The second step is to prove that by executing a credible deviation from the partition where individually rational payoffs exist, individually rational payoffs also exist in the new coalition structure. Let us begin by rewriting conditions for existence of the credible deviation for a normal distribution from Definition 33. As in the definition, we denote a feasible payoff in $\Pi(N)$ by x and it is represented by $(d, r) \in \mathbb{R}^{2n}$ (see Definition 32). A feasible payoff in $\Pi^1(N)$, which is the coalition structure created from $\Pi(N)$ by one player changing a coalition, is denoted by y and represented by $(c, q) \in \mathbb{R}^{2n}$.

$$x_{S_k}(S_k \setminus i) \preceq_{SSD} v(S_k \setminus i) \iff \\ d(S_k \setminus i) \leq \mu_{S_k \setminus i} \;\; \& \;\; \frac{\sigma_{S_k \setminus i}}{\sigma_{S_k}} \leq r(S_k \setminus i) \iff \\ \mu_{S_k} - \mu_{S_k \setminus i} \leq d(i) \;\; \& \;\; r(i) \leq 1 - \frac{\sigma_{S_k \setminus i}}{\sigma_{S_k}}.$$

To rewrite we used Lemma 9 and Definition 32. We continue analogically with the payoff y:

$$y_{S_p \cup i}(S_p) \succeq_{SSD} v(S_p) \iff \\ c(S_p) \geq \mu_{S_p} \;\; \& \;\; \frac{\sigma_{S_p}}{\sigma_{S_p \cup i}} \leq q(S_p) \iff \\ \mu_{S_p \cup i} - \mu_{S_p} \geq c(i) \;\; \& \;\; q(i) \geq 1 - \frac{\sigma_{S_p}}{\sigma_{S_p \cup i}}.$$

The last preference relation concerning only the player i is the following:

$$y_{S_p \cup i}(i) \succeq_{SSD} x_{S_k}(i) \iff \\ c(i) \geq d(i) \;\; \& \;\; r(i)\sigma_{S_k} \geq q(i)\sigma_{S_p \cup i}.$$

Let us now combine the resulting inequalities about $d(i), r(i), c(i)$ and $q(i)$:

$$\mu_{S_p \cup i} - \mu_{S_p} \geq c(i) \geq d(i) \geq \mu_{S_k} - \mu_{S_k \setminus i} \;\; \& \\ \sigma_{S_k} - \sigma_{S_k \setminus i} \geq r(i)\sigma_{S_k} \geq q(i)\sigma_{S_p \cup i} \geq \sigma_{S_p \cup i} - \sigma_{S_p}.$$

Hence, if the player i has the credible deviation then $\mu_{S_p \cup i} + \mu_{S_k \setminus i} \geq \mu_{S_k} + \mu_{S_p}$ and $\sigma_{S_k} + \sigma_{S_p} \geq \sigma_{S_p \cup i} + \sigma_{S_k \setminus i}$, where at least one the inequalities is strict. Let us now define functions which we call *stochastic social welfare of the coalition structure*. For a coalition structure $\Pi(N) = (S_1, \ldots, S_K)$:

- $U(\Pi(N)) = \sum_{i=1}^{K} \mu_{S_i}$,
- $\Sigma(\Pi(N)) = \sum_{i=1}^{K} \sigma_{S_i}$.

If there is a credible deviation from $\Pi(N)$ to $\Pi^1(N)$ then

- $\Sigma(\Pi^1(N)) \leq \Sigma(\Pi(N))$,
- $U(\Pi^1(N)) \geq U(\Pi(N))$,

where at least one of the inequalities is strict. Thus, reaching the initial coalition structure through any number of credible deviations is not possible, as $U - \Sigma$, increases strictly following the sequence of a credible deviations.

In the second step, we prove that an individually rational payoff exists in the coalition structure $\Pi^1(N)$ when it existed in $\Pi(N)$. The proof is trivial for the player i since $y_{S_p \cup i}(i) \succeq x_{S_k}(i) \succeq v(i)$. For the rest of the players, we distinguish two cases, one for the coalition $S_p \cup i$ and the second for the coalition $S_k \setminus i$. We show the proof only for the case of $S_p \cup i$ since the line of reasoning is the same for the other coalition. We know that $c(S_p) \geq \mu_{S_p}$, $q(S_p) \leq \frac{\sigma_{S_p}}{\sigma_{S_p \cup i}}$, $d(S_p) = \mu_{S_p}$, and $r(S_p) = 1$ since the payoff needs to be feasible. Therefore, if x is feasible and individually rational payoff represented by $(d, r) \in \mathbb{R}^{2n}$ for $\Pi(N)$ then the individually rational payoff for players of S_p in coalition structure $\Pi^1(N)$ can be expressed as follows:

$$c(i) = d(i) + \varepsilon, \;\; \forall i \in S_p \;\;\; \& \\ q(i)\sigma_{S_p \cup i} = r(i)\sigma_{S_p} - \delta, \;\; \forall i \in S_p, \\ \text{where } \varepsilon > 0 \text{ or } \delta > 0.$$

In other words, there is an payoff in the coalition structure $\Pi(N)$ in which no player is worse off then in an individually rational payoff from $\Pi(N)$. □

This result motivates several other questions concerning the $S1$ stability. Let us name a few:

- Does $S1$ stable coalition structure always exist for any distribution of v?
- How many credible deviations are required in the iterative process to find an $S1$ stable coalition structure?

- What is the longest sequence of credible deviations yielding an S1 stable solution?
- How can we find an efficient[1] $S1$ stable solution in terms of stochastic social welfare functions U and Σ?

Now we want to motivate where it might be more reasonable to choose an individually rational payoff given by a $S1$ stable coalition structure instead of payoff from the undominated core. Take Example 5, where the undominated core leads to unreasonable payoffs and strongly asymmetric payoffs even though players are identical with respect to the values of v. The only $S1$ stable coalition structure in Example 5 is $\Pi(N) = \{\{1\}\{2\}\}$. In S1 stable coalition structure players can obtain individually rational payoffs contrary to the SSD-undominated core, where in Example 5 no payoff from the SSD-undominated core is individually rational. This discussion may motivate using some solution concept for some S1 stable coalition structure rather then choosing a payoff from the SSD-undominated core.

Although, we do not present many results in this chapter, understanding the concept of coalition structures remains crucial. This is particularly important when core-like solutions are absent, meaning no stability is achieved within the grand coalition. In such cases, exploring alternative concepts of stability tailored to specific coalition structures becomes invaluable. We discuss possible ideas for stability concepts in the conclusion of the thesis. We also need to emphasize that modeling the game solely with a characteristic function might be inadequate or impractical, and employing the partition function could be more advantageous. The partition function considers the value of coalitions within specific coalition structures, thus dealing with a larger set of values compared to the characteristic function. This method more effectively captures the interrelationships among different coalitions. However, the approach of using a stochastic partition function within the stochastic cooperative game theory has not been extensively explored.

[1] We mean the efficient solution in the context of multi-criteria optimization.

5. Multiple newsvendor problem

In this chapter, we apply the notion of the SSD-dominating core to the multiple newsvendors game. At first, we recall the classical newsvendor problem, which involves determining the optimal number of newspapers a newsboy should purchase at the beginning of the day to maximize the profit when the demand is random. Afterwards, we survey the generalization of the newsvendor problem to multiple players who can cooperate; by cooperation, we mean that players not only order newspapers together but also cover the demand of other newsvendors from their coalition. Lastly, we present conditions for the SSD-dominating core for the specific multiple newsvendors game under uniformly distributed demand.

5.1 Classical newsboy problem

Newsvendor or newsboy problem is a problem with a long tradition in the literature with variety of settings. Ideas of the newsvendor problem can be found in already more than a century old paper of Edgeworth [23] or in slightly younger paper of Arrow et al. [24]. In the basic setting, there is a newsboy selling newspapers in one place. Before the start of the day, he can buy a number of newspapers and for the rest of the day, he cannot buy more. We assume there is a random demand for the newspapers. The newsboy problem is then to decide how many newspapers to buy. Let us formulate this as an optimization problem where the newsboy wants to maximize the expected profit.

Definition 35 (Newsboy problem)**.** *Let c be the unit purchase price and p the unit selling price for newspapers. Further, denote by ω the random demand for newspapers. The quantity which the newsboy buys at the beginning of the day is denoted by q. Then the* newsboy problem *is the optimization program maximizing the profit and is defined as follows:*

$$\max_{q \in \mathbb{R}} \; p \cdot \mathbb{E}[\min\{\omega, q\}] - c \cdot q. \tag{5.1}$$

The optimal order q^* for a continuous demand can be derived in terms of the inverse cumulative distribution function F^{-1}. The optimal order quantity for the profit function in the example can be quite straightforwardly derived and is as follows:

$$q^* = F^{-1}(\frac{p-c}{p}).$$

The newsboy problem, even under a more general setting, is well studied. For more, we refer the reader to [25]. The cited book includes wide range of problems related to the newsboy problem from a multi-item newsboy problem to a multiple newsboys problems. We maintain our focus on the foundational concept illustrated by the classic newsboy problem, choosing not to complicate the scenario further. This approach allows us to concentrate on understanding the core of the problem rather than exploring its variants. Although these modified models might be more applicable to real-world scenarios, our priority is to grasp the essential principles that underlie the basic structure of the issue.

We can outline assumptions of the newsboy problem, which are typical for newsvendor-like situations:

1. Single period planning,
2. Random demand,
3. Products are delivered prior the demand,
4. Products do not have any value in the next period.

These assumptions can be adjusted or some other may be added to correspond to a given situation we want to model. To conclude, we presented the fundamentals of the newsboy problem, where the decision is made only for one newsboy.

5.2 Multiple newsvendors

In this section, we present a literature review of multiple newsvendor problems to get a grasp of what can be done and, at the same time, to put our approach to the context of already established results in the literature. It also helps us to motivate the construction of the problem in Section 5.3 using the cooperative game theoretic approach.

In multiple newsvendors problems, contrary to the newsboy problem, we define the optimization function for the whole coalition of players. Then the optimum of $v(S)$ of coalition S represents the worth that the given coalition is able to obtain on their own. This gives rise to modelling the problems as questions in cooperative games. Most of sources try to formulate the multiple newsvendors problems in this way. These games are special cases of a *cooperative inventory game* simply asking questions about how to work with inventory in multiple player or multiple location settings. We focus mostly on these results. In what follows, when we talk about multiple newsvendors game, we mean the respective cooperative game.

The paper of Özen et al. [26] studies convexity of a special construction of a multiple newsvendors game under several distributions of the demand. This model is an immediate generalization of the newsboy problem and the multiple newsvendor game (N, v) is defined in the following way. The demand of S is Y_S, p is the unit selling price, c is the unit purchasing price and q_S the ordered quantity for S. They construct the characteristic function in 3 steps. At first, the random variable describing the profit of a coalition S depending on the order quantity q is defined as $r^S(q, Y_S)$. In the second step, the function $\pi^S(q)$ is defined as the expected value of $r^S(q, Y_S)$. The last step is the actual construction of the characteristic function $v(S)$ as the maximum of $\pi^S(q)$ over all possible q. Formally:

$$\begin{aligned} r^S(q, Y_S) &= p \cdot \min\{q, Y_S\} - c \cdot q, \\ \pi^S(q) &= \mathbb{E}_{Y_S}[r^S(q, Y_S)], \\ v(S) &= \max_q \pi^S(q), \ \forall S \subseteq N. \end{aligned}$$

When we construct the characteristic function in Section 5.3, we do it in the similar manner. We provide the interpretation of this construction of (N, v) when discussing the next paper [27], where we compare different possibilities for modelling the multiple newsvendors games.

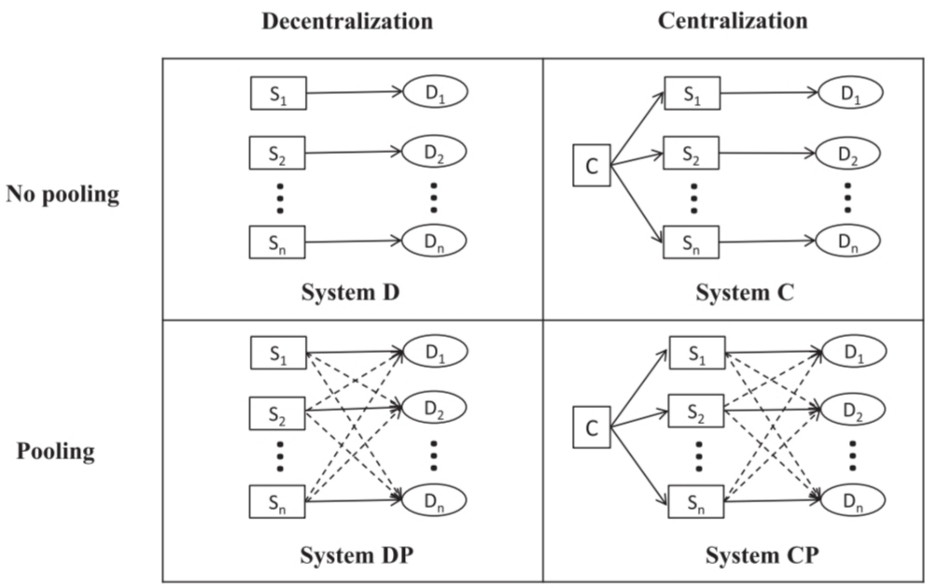

Figure 5.1: Systems of inventories and orders from [27].

The following paper of Yang et al. [27] assumes risk aversion in a special sense which we do not discuss deeply. More importantly, the paper examines various scenarios involving different strategies for *inventory pooling* or *centralization of orders* and proposes models for these configurations. The four systems, which they discuss, are based on a centralization of the order and a pooling of an inventory and understanding of them enhances the clarity of the approach one can take on in such problems:

- **System D (Independent newsvendors):** This is the basic model where each newsvendor independently places orders. In this system, newsvendors operate autonomously, meaning no newsvendor can fulfill the demand of another. Each newsvendor's orders are solved and optimized independently.
- **System C (Centralized ordering):** In this model, a single order is placed for all newsvendors collectively. Once the order is received, it is divided among the newsvendors. Despite the centralized ordering, each newsvendor cannot meet the excess demand of others; surplus or shortage at one newsvendor remains unaddressed by others.
- **System DP (Decentralized with pooling):** Each newsvendor orders independently, similar to System D. However, unlike System D, if a newsvendor's actual demand is lower than their ordered quantity, they can transfer the surplus to satisfy the demand of another newsvendor. This system allows for some level of flexibility and mutual support among newsvendors.
- **System CP (Centralized ordering with pooling):** Orders are made collectively for all newsvendors as in System C, but with a significant difference: newsvendors can satisfy each other's demands, thanks to a pooled inventory approach. The specific quantity each newsvendor initially receives after the order is less crucial because of the shared inventory dynamics.

This list is particularly useful for our setting as it provides a clearer framework for understanding the scenarios we intend to model. To articulate the differences even better, we present a figure of a scheme of these, which was taken directly from [27]. To conclude, the distinction between these systems is present and assuming various systems we need to adjust questions and models for that.

Among these models, those featuring pooled inventory are particularly relevant to the application of cooperative game theory. Systems such as DP (Decentralized with Pooling) and CP (Centralized Ordering with Pooling) naturally embody cooperative dynamics because they eliminate distinctions between individual players' inventories, encouraging a shared approach to demand satisfaction. Conversely, in the centralized ordering model (System C), cooperation looks different. Here, the cooperation is not through shared inventory, but through the decision-making process, where a central planner determines the collective order for all players, as opposed to each player ordering independently.

In conclusion, the significant differences between these systems require that we develop specific questions and models specialized to each one. When assuming different systems, it is crucial to adjust the analytical models accordingly to capture the unique elements of cooperation and competition inherent in each setup.

The next model of Zhang et al. [28] describes a way to incorporate risk averse behaviour in the multiple newsvendors problem, specifically, they use *exponential* and *power utilities* to deal with that. The survey of Dror and Hartman [29] presents several models for *joint-replenishment games*, *dynamic lot sizing games*, or multiple newsvendors problem and is useful for fuller picture about inventory games. In one of these models [30], Hartman et al. formulated the multiple newsvendors problem as a cost game, adding penalty to the profit formulation[1]. They showed a characterization of the nonemptiness of the core of the multiple newsvendors game, under several demand distributions such as symmetric, normal, etc. Mentioned papers [29] and [30] motivate our approach to use a specific distribution to deal with the multiple newsvendors problem. Finally, we mention the work of Slikker et al. [31], which extends the model originally presented in [30] to scenarios where the unit selling and unit purchasing prices among players vary and the costs of transshipment among players are included. These adjustments not only broadens the original model but also opens up potential for further generalization in our own research, leading to new research questions.

5.3 Multiple risk-averse newsvendors

In this section, we study risk-averse behavior of players in the multiple newsvendors problem. We restrict our analysis to a single-period setting, i.e., only one order is placed. We follow the system CP from the previous section, which involves a centralized order decision for the pooled inventory of players. To model risk-averse behaviour, we use the second-order stochastic dominance, which was not, to the best of our knowledge, considered in the literature. The main question we ask is under which conditions the SSD-dominating core is nonempty.

[1] Newsvendor-like problems are often formulated also like cost minimization problems.

Description of the parts of the model follows:

$c \ldots$ unit purchasing price for players,
$p \ldots$ unit selling price,
$q_S \ldots$ a quantity of ordered units for coalition S,
$Y_S \ldots$ a random demand of a coalition S for a single period.

These parts enable us to construct characteristic function for a simple model of multiple newsvendors situation with centralized order and pooled inventories. We define the characteristic function describing the profit of a coalition S for a given q_S and it is defined as follows:

$$v(S, q_S) = p \cdot \min(Y_S, q_S) - c \cdot q_S.$$

As we can see, it is slightly generalized objective function from the single newsvendor problem. At this point $v(S, q)$ is not only random but it also depends on a parameter q. Therefore, we define $v(S, q^*)$ with the optimal value of order q^* under expected value for a coalition S:

$$v(S) = v(S, q_S^*) \text{ , where } q_S^* = \arg\max_{q_S} \mathbb{E}[v(S, q_S)].$$

Value $v(S)$ describes the random profit of a coalition S under the optimal order quantity q_S^* which is derived under expectation.

Definition 36 (Stochastic multiple newsvendors game)**.** *Let $c \in \mathbb{R}$ and $p \in \mathbb{R}$, $0 < c < p$ be the unit purchasing price and the unit selling price. Further, let q_S be the order quantity of coalition S and $Y_S \sim U[a_S, b_S]$ the random demand of coalition S. Finally, let $v(S, q_S) = p \cdot \min(Y_S, q_S) - c \cdot q_S$. Stochastic TU-game (N, v) is a* stochastic multiple newsvendors game *if v is defined as follows:*

$$v(S) = v(S, q_S^*) = p \cdot \min(Y_S, q_S^*) - c \cdot q_S^*,$$

where q_S^* is defined as optimal value of the following function:

$$q_S^* = \arg\max_{q_S} \mathbb{E}[v(S, q_S)].$$

Remark. We already assume the random demand to be uniformly distributed. Other distributions can be also used in this definition.

Stochastic multiple newsvendors game from Definition 36 is almost identical to the one by Özen et al. [26] discussed in the previous section. The main difference between our and their approach is that we use *random* characteristic function with optimal value of the parameter q_S^*, $v(S) = v(S, q_S^*)$, while they use *deterministic* characteristic function with value of S being equal to $\max_{q_S} \mathbb{E}[v(S, q_S)]$.

Further, we remark on mathematical soundness of our construction. Under SSD, the choice of q_S^* for defining $v(S)$ is reasonable, since no $v(S, q_S)$ dominates $v(S, q_S^*)$ when $q_S \neq q_S^*$ if the maximum $v(S, q_S^*)$ is unique. This is due to Claim 1.

In practice, Definition 36 covers several scenarios. For instance, newsvendors order newspaper from a specified firm, and the unit purchasing price c already includes the company's transportation cost for transferring newspaper from one newsvendor to another. In another scenario, the transportation costs may be

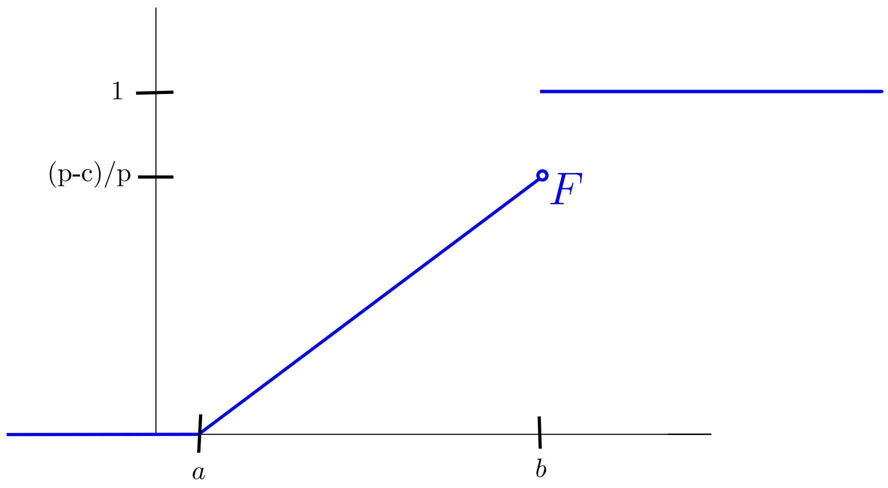

Figure 5.2: $(\frac{p-c}{p})$-cut uniform distributions.

negligible, or it might be assumed that there are none. In the last of the mentioned scenarios, it is assumed that if a newsvendor cannot meet the demand, he can refer the customer to another newsvendor with whom he cooperates.

Now that the stochastic multiple newsvendors game is defined, we are prepared to tackle the questions about the SSD-dominating core. Usually, the random demand is assumed to follow a given distribution and due to primary motivation of multiple newsvendors, these distributions should be discrete. However, for greater demand quantities, the problem might become intractable. Approximating the problem by continuous distributions has several advantages. First, as we already discussed in Chapter 3, the conditions for nonemptiness of the SSD-dominating core can be more easily derived for continuous distributions. Second, assuming demand to be continuous leads to a generalization of the problem to possibly not discrete commodities. In this section, we consider only continuous uniform distributions.

Derivation of the SSD dominating core conditions To derive conditions for the SSD-dominating core to be nonempty we can not directly use neither Theorem 14 for the payoffs with transfer payments nor Claim 18. However, we can derive the distribution of $v(S)$ in terms of the demand distribution for a given $S \subseteq N$ and then obtain the conditions for the SSD-dominating core as in the proof of Theorem 14. Recall (5.1), from which $q_S^* = F_{Y_S}^{-1}(\frac{p-c}{p})$. For a given uniform distribution $U[a_S, b_S]$, we are able to express precisely $q_S^* = a_S + \frac{p-c}{p}(b_S - a_S)$. The distribution of $v(S)$ is a combination of a continuous uniform and a discrete distribution and can be described using the outcomes ω of distribution Y_S as follows:

$$v(S,\omega) = \begin{cases} p \cdot \omega - c \cdot q_S^* & \text{if } \omega \in [a_S, q_S^*) \\ (p-c)q_S^* & \text{if } \omega \in [q_S^*, b_S] \end{cases}, \tag{5.2}$$

The cumulative distribution function is "uniform" up to $F_{Y_S}^{-1}(\frac{p-c}{p})$ and then there is a jump to 1. We provide Figure 5.2 to enhance the clarity of what a cumulative distribution function of this form looks like. In general, we call such distributions *α-cut uniform distribution.*

Definition 37 (α-cut uniform distribution)**.** *Let $a_Z, b_Z \in \mathbb{R}$ real parameters, where $a_Z < b_Z$ and $\alpha \in (0, 1)$. A random variable Z follows α-*cut uniform distribution *when it has the following cumulative distribution function:*

$$F_Z(x) = \begin{cases} 0 & \text{if } x < a_Z \\ \frac{x - a_Z}{b_Z - a_Z} \cdot \alpha & \text{if } x \in [a_Z, b_Z) \\ 1 & \text{if } x \geq b_Z \end{cases}, \tag{5.3}$$

To compare two α-cut uniform distributions for stochastic dominance, we cannot directly use Lemma 9. In the following lemma, we derive the conditions for SSD dominance for two α-cut uniform distributions with the same α.

Lemma 22 (SSD condition for α-cut uniform distribution)**.** *Let X and Y be random variables both possessing a α-cut uniform distribution for the same $\alpha \in (0, 1)$ and let a_X, a_Y, b_X, and b_Y be the corresponding parameters. Then $X \succeq_{SSD} Y$ if an only if*

$$a_X \geq a_Y \ \ \& \ \ (2 - \alpha) \cdot b_X + \alpha \cdot a_X \geq (2 - \alpha) \cdot b_Y + \alpha \cdot a_Y.$$

Proof. To derive the conditions, we use the formulation of SSD from Definition 5 concerning cumulative distribution function, i.e.,

$$\forall u \in \mathbb{R} : I(u) = \int_{-\infty}^{u} (F_X(z) - F_Y(z))\, dz \leq 0. \tag{5.4}$$

To abbreviate, we denote the integral by $I(u)$. We can easily see that for any $u \leq \min\{a_X, a_Y\}$, $I(u) = 0$, thus we only need to analyze situation, where $u > \min\{a_X, a_Y\}$. We can further see that if $a_X < u < a_Y$ then $I(u) > 0$, which means X cannot dominate Y. Hence, a necessary condition for X to dominate Y is $a_X \geq a_Y$. For $a_X \geq a_Y$, we derive further conditions for X to dominate Y. We can simply see that for $b_X \geq b_Y$, $F_X(u) \leq F_Y(u)$ for every $u \in \mathbb{R}$, thus, the variable X dominates Y (actually, in the first order stochastic dominance). Let the relation between b_X and b_Y be $b_X \leq b_Y$. Then the distribution function F_X and F_Y intersect on interval (a_Y, b_Y) at a point denoted by h. The crucial observation is that $I(u)$ is decreasing on interval (a_Y, h) and increasing on the interval (h, b_Y). This enables us to calculate the other condition for dominance just as $I(b_Y) \leq 0$ because if $I(b_Y) \leq 0$ then $I(u) \leq 0$, $\forall u \in \mathbb{R}$. We just need to calculate $I(b_Y)$:

$$\begin{aligned} I(b_Y) &= \int_{-\infty}^{b_Y} (F_X(z) - F_Y(z))\, dz = (b_Y - b_X) + \alpha \cdot \frac{b_X - a_X}{2} - \alpha \cdot \frac{b_Y - a_Y}{2} \\ &= -b_X \cdot (1 - \frac{\alpha}{2}) - \frac{\alpha}{2} \cdot a_X + (1 - \frac{\alpha}{2}) \cdot b_Y + \frac{\alpha}{2} \cdot a_Y \end{aligned}$$

Therefore, the second condition, which together with $a_X \geq a_Y$ make the conditions for $X \succeq_{SSD} Y$, is as follows

$$b_X \cdot (2 - \alpha) + \alpha \cdot a_X \geq (2 - \alpha) \cdot b_Y + \alpha \cdot a_Y.$$

Notice, that the condition works for $b_X > b_Y$. □

Remark. For $\alpha = 1$, the dominance conditions for the α-cut uniform distribution corresponds in terms of lower and upper bound a_X and b_X to the conditions for uniform distribution, see Lemma 9.

Now that we have conditions for two α-cut uniform distributions, we can derive conditions for the payoffs from the SSD-dominating core for the stochastic multiple newsvendors game.

Theorem 23. *Let $Y_S \sim U[a_S, b_S]$, $\forall S \subseteq N$ be a uniformly distributed demand with parameters $a_S, b_S \in [0, \infty)$, where $a_S < b_S$. Further, let (N, v) be a stochastic multiple newsvendors game with random demand Y_S. Then $\boldsymbol{DC}^{r+}(v) \neq \emptyset$ if and only if*

$$r(S)(a_N \cdot (p + c) + b_N(p - c) \geq a_S \cdot (p + c) + b_S \cdot (p - c) \;\;\&$$
$$r(S)(a_N \cdot p - (b_N - a_N) \cdot c) \geq (a_S \cdot p - (b_S - a_S) \cdot c).$$

Proof. To derive the SSD-dominating core conditions based on the distribution of the demand $Y_S \sim U[a_S, b_S]$, recall the distribution of $x(S)$ looks as follows:

$$x(S) = r(S) \cdot v(N) \sim U[r(S) \cdot a_{v(N)}, r(S) \cdot b_{v(N)}].$$

To distinguish between the bounds of the uniform distribution of Y_S and $\frac{p-c}{p}$-cut uniform distribution $v(S)$ we write $Y_S \sim U[a_S, b_S]$ and $v(S)$ has α-cut distribution with parameters $a_{v(S)}$ and $b_{v(S)}$, respectively. These are the bounds representing the $v(S)$ at points a_S and q_S^*. Let us begin with the derivation of the distribution of $v(S)$ to obtain its bound $a_{v(S)}$ and $b_{v(S)}$ to be in terms of the parameters a_S and b_S. We recall that the distribution is described in (5.2). We can easily see that the lower bound $a_{v(S)}$ is obtained when $\omega = a_S$, thus the lower bound $a_{v(S)} = p \cdot a_S - c \cdot q_S^*$. The upper bound is exactly the point where the cumulative distribution function is not continuous, i.e., for $\omega = q_S^*$. Thus, the upper bound $b_{v(S)}$ is $(p - c) \cdot q_S^*$. Together, the distribution of $v(S)$ is as follows:

$$v(S) \sim U[p \cdot a_S - c \cdot q_S^*, (p - c) \cdot q_S^*].$$

Let us now finally derive the nonemptiness conditions for the SSD-dominating core $\mathbf{DC}^{r+}(v)$. To do this, we use Lemma 22. We begin with the condition $a_N \cdot r(S) \geq a_S$. It can be equivalently rewritten as

$$r(S)(a_N \cdot p - q_N^* \cdot c) \geq a_S \cdot p - q_S^* \cdot c.$$

This can be rewritten using $q_S^* = a_S + (b_S - a_S)(\frac{p-c}{p})$ as follows:

$$r(S) \cdot \left(a_N(p - c + \frac{p - c}{p} \cdot c) - b_N \frac{p - c}{p} \cdot c\right) \geq a_S(p - c + \frac{p - c}{p} \cdot c) - b_S \frac{p - c}{p} \cdot c.$$

We can simplify this to:

$$r(S)(a_N \cdot p - (b_N - a_N) \cdot c) \geq (a_S \cdot p - (b_S - a_S) \cdot c).$$

Let us derive the second condition for the stochastic dominance of $x(S)$ over $v(S)$. It can be rewritten as follows:

$$r(S) \left(\frac{p + c}{p} \cdot (p - c) \cdot q_N^* + (a_N \cdot p - c \cdot q_N^*) \cdot \frac{p - c}{p}\right) \geq$$
$$\frac{p + c}{p} \cdot (p - c) \cdot q_S^* + (a_S \cdot p - c \cdot q_S^*) \cdot \frac{p - c}{p}.$$

This can be further simplified to:

$$q_S^* + a_S \geq r(S)(q_N^* + a_N).$$

By plugging in the optimal value $q_S^* = a_S + (b_S - a_S)(\frac{p-c}{p})$ we obtain:

$$r(S)(a_N \frac{p+c}{p} + b_N \frac{p-c}{p}) \geq a_S \frac{p+c}{p} + b_S \frac{p-c}{p}.$$

□

Theorem 23 provides conditions for the nonemptiness of the SSD-dominating core in the stochastic multiple newsvendors game with continuous uniform demand. These conditions can be interpreted as promoting cooperation among all players in the game. Since we use the SSD-dominating core, this cooperation occurs under the assumption that all players are risk averse.

Let us now move to the interpretation of the conditions to better understand the crucial points of interest for risk averse players in the stochastic multiple newsvendors game. The first and simpler condition to discuss is as follows:

$$r(S)(a_N \cdot p - (b_N - a_N) \cdot c) \geq (a_S \cdot p - (b_S - a_S) \cdot c).$$

In this case, players within coalition S want to cooperate within the grand coalition, if their guaranteed profit is less than or equal to the portion of the grand coalition's guaranteed profit that they can obtain. The second condition is a bit more intricate to interpret:

$$r(S)(a_N \cdot (p + c) + b_N(p - c) \geq a_S \cdot (p + c) + b_S \cdot (p - c).$$

We rewrite it to the following form:

$$r(S)\left(p \cdot \frac{a_N + b_N}{2} - c \cdot \frac{b_N - a_N}{2}\right) \geq p \cdot \frac{a_S + b_S}{2} - c \cdot \frac{b_S - a_S}{2}.$$

The first expression, $p \cdot (a_S + b_S)/2$, represents the expected net income of coalition S. The second term, $c \cdot (b_S - a_S)/2$, can be interpreted as a potential expected loss from choosing q_S^* instead of b_S. These two terms, the expected net income and the potential loss from choosing q_S^*, collectively represent the market quality within coalition S, i.e., players within S want to cooperate within the grand coalition, if their portion of market quality in N is better than or equal to the market quality within the coaltion S. The SSD-dominating condition suggests that such market quality is important for risk averse players when deciding on cooperation in the stochastic multiple newsvendors game.

To summarize the findings on the SSD-dominating core conditions, risk averse newsvendors consider the following questions when deciding whether to cooperate with other newsvendors in the already thoroughly explained situation:

- Is the portion of the guaranteed profit in N better or at least not worse than the guaranteed profit within S?
- Is the quality of the market within S, specifically, its expected net profit and expected loss from buying the optimal quantity q_S^* instead of the maximal quantity b_S, worse than a portion of the quality of the market within N?

Concluding remark to newsvendors problem We derived conditions for the nonemptiness of the SSD-dominating core for the given type of payoff. We modeled the problem of multiple newsvendors in an unconventional way by assuming a random characteristic function rather than one defined by the expected profit under optimal ordering. Conversely, we applied the concept of optimal ordering as is standard in other multiple newsvendors models, but in a mathematically sound manner appropriate for second order stochastic dominance. The results and insights have the potential to be generalized for bounded distributions of demand, such as discrete distributions with a finite number of realizations. However, they are likely not applicable to distributions with unbounded support, like the normal distribution.

The last remark concerns stochastic payoffs with tranfer payments (d, r_+), which, according to Theorem 14, seemed promising for use when the demand is continuous and uniform in the stochastic multiple newsvendors problem. However, the outcome was the opposite. We do not present the derivation of the condition for nonemptiness of $\mathbf{DC}^{(d,r_+)}(v)$ since it did not prove to be insightful, and also due to the extensive number of calculations that would not yield more usable results than those in Theorem 23, even though $\mathbf{DC}^{r_+}(v)$ should encompass fewer stochastic payoffs than $\mathbf{DC}^{(d,r_+)}(v)$ which was discussed in Chapter 3.

Conclusion

This thesis studying the stochastic TU-games contibutes to the game theory research in several aspects. It surveys already established models not only to provide context for its own results, but also to provide a unified overview of the known results. Then it defines solution concepts, motivated by the core of deterministic games, called the SSD-dominating core and the SSD-undominated core; the latter might be viewed as an alternative when the former is empty. The main focus is on the SSD-dominating core; however, examples of situations where the SSD-undominated core is more suitable, along with discussions on its drawbacks, are also provided. The SSD-dominating core is thoroughly studied for several distributions and types of stochastic payoff. The conditions for the SSD-dominating core to be nonempty often restate the problem as one of the nonemptiness of the cores of two deterministic games. Particularly interesting conditions can be derived for $v(S) \sim U[a_S, b_S]$, where the SSD-dominating core is nonempty if the game induced by the expected values $\mathbb{E}[v(S)]$ of $v(S)$ has a nonempty core and together with that the game induced by lower bounds a_S is convex. However, this is only a sufficient condition. Although we do not provide a characterization in this case in terms of games (N, a) and (N, μ), we discuss that superaddivity of (N, a) is not even sufficient, which suggests that a more complicated relationship between (N, a) and (N, μ) is needed for the characterization.

In exploring a broader range of payoffs, we also discuss the reasonableness of the type of payoffs used in the thesis by presenting results for normally distributed payoffs without assumption on covariance among payoffs of individual players. In this context, we justify the use of payoffs with transferable payments, specifically, those characterized by an absolute value of correlation being 1. This discussion on types of payoffs generates numerous new research questions. These questions may target completely different types of payoffs, which could be more suitable for preferences other than SSD, or they may focus on generalizing the payoff type to accommodate a general covariance matrix of the stochastic payoff x. Particularly interesting is the potential interpretation in such a context of a general covariance matrix of the stochastic payoff.

The results concerning the SSD-dominating core can be seen as strong because of the reformulation of the problem of the nonemptiness of the SSD-dominating core as a problem of the nonemptiness of cores in deterministic games. Although the results are satisfactory, the SSD-dominating core presents a significant drawback: deriving conditions for distributions beyond the scale-location or scale families proves highly impractical or unfeasible.

We also examine stability concepts for coalition structures, a brief yet significant part of the thesis deserving close attention. We introduce a stability concept that identifies individually stable coalition structures for risk averse players. Building on potential research questions highlighted in that chapter, our proposals primarily focus on computational aspects of stability, such as the maximum possible number of credible deviations and the possibility of evaluating efficiency in stochastic social welfare among different coalition structures. Besides studying individual stability, other stability concepts for risk-averse players could be developed using second-order stochastic dominance. This includes stability concerning union and division of coalitions, defined by the inability of coalitions to profitably merge or split. Such concepts could extend the SSD-undominated core to coalition structures, potentially offering greater utility than the SSD-undominated core itself.

Lastly, we apply the notion of the SSD-dominating core to the multiple newsvendors game, deriving conditions for its nonemptiness in a simple game with uniformly distributed demand among risk averse newsvendors. Future research could extend this to a wider range of demand distributions, although the limitations of stochastic dominance concerning distribution options remain. Another way of research could involve adapting the model to more realistic scenarios through the introduction of additional parameters.

Generally, the study of stochastic TU-games can also be approached through the development of single-point solution concepts. Research in this area holds the potential to yield significant insights and practical solutions for stochastic TU-games. This thesis did not explore this line of research due to differing objectives. Future research could build on the ideas from the SHS model and its solution concepts, which are based on the optimization of the objective function.

To conclude, this thesis primarily integrates stochastic dominance (SSD) into stochastic cooperative games, examining it across various distributions. Additionally, it contextualizes the results within cooperative game theory and cooperative inventory management.

Bibliography

[1] Abraham Charnes and Daniel Granot. *Prior solutions: Extensions of convex nucleus solutions to chance-constrained games.* Center for Cybernetic Studies, University of Texas, 1973.

[2] Jeroen Suijs, Peter Borm, Anja De Waegenaere, and Stef Tijs. Cooperative games with stochastic payoffs. *European Journal of Operational Research*, 113(1):193–205, 1999.

[3] Helga Habis and P. Jean-Jacques Herings. Transferable utility games with uncertainty. *Journal of Economic Theory*, 146(5):2126–2139, 2011.

[4] Panfei Sun, Dongshuang Hou, and Hao Sun. Optimization implementation of solution concepts for cooperative games with stochastic payoffs. *Theory and Decision*, 93(4):691–724, 2022.

[5] Elmar Wolfstetter, Uwe Dulleck, Roman Inderst, Peter Kuhbier, and M. Lands-Berger. *Stochastic dominance: theory and applications.* Humboldt-Univ., Wirtschaftswiss. Fak., 1993.

[6] Roberto Serrano. Sixty-seven years of the Nash program: time for retirement? *SERIEs*, 12(1):35–48, 2021.

[7] Hans Peters. *Game theory: A Multi-leveled approach.* Springer, 2015.

[8] Bezalel Peleg and Peter Sudhölter. *Introduction to the theory of cooperative games*, volume 34. Springer Science & Business Media, 2007.

[9] Robert J. Aumann and Michael Maschler. Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, 36(2):195–213, 1985.

[10] Lloyd S. Shapley. *On balanced sets and cores.* RAND Corporation, 1965.

[11] Olga N. Bondareva. Some applications of linear programming methods to the theory of cooperative games. *Problemy Kibernet*, 10:119, 1963.

[12] Lloyd S. Shapley. A value for n-person game. *Annals of Mathematical Studies*, 28:307–317, 1953.

[13] Lloyd S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1:11–26, 1971.

[14] Georgios Chalkiadakis, Edith Elkind, and Michael Wooldridge. *Computational aspects of cooperative game theory.* Springer Nature, 2022.

[15] George Pantazis, Filippo Fabiani, Filiberto Fele, and Kostas Margellos. Probabilistically robust stabilizing allocations in uncertain coalitional games. *IEEE Control Systems Letters*, 6:3128–3133, 2022.

[16] Helga Habis and P. Jean Jacques Herings. Stochastic bankruptcy games. *International Journal of Game Theory*, 42:973–988, 2013.

[17] Jeroen Suijs, Anja De Waegenaere, and Peter Borm. Stochastic cooperative games in insurance. *Insurance: Mathematics and Economics*, 22(3):209–228, 1998.

[18] Judith Timmer, Peter Borm, and Stef Tijs. On three Shapley-like solutions for cooperative games with random payoffs. *International Journal of Game Theory*, 32:595–613, 2004.

[19] Yoram Bachrach, Edith Elkind, Reshef Meir, Dmitrii Pasechnik, Michael Zuckerman, Jörg Rothe, and Jeffrey S. Rosenschein. The cost of stability in coalitional games. In *Algorithmic Game Theory: Second International Symposium, SAGT 2009, Paphos, Cyprus, October 18-20, 2009. Proceedings 2*, pages 122–134. Springer, 2009.

[20] Debraj Ray and Rajiv Vohra. Coalition formation. *Handbook of Game Theory with Economic Applications*, 4:239–326, 2015.

[21] Joseph Greenberg. Coalition structures. *Handbook of Game Theory with Economic Applications*, 2:1305–1337, 1994.

[22] Jacques H. Dreze and Joseph Greenberg. Hedonic coalitions: Optimality and stability. *Econometrica: Journal of the Econometric Society*, pages 987–1003, 1980.

[23] Francis Y. Edgeworth. The mathematical theory of banking. *Journal of the Royal Statistical Society*, 51(1):113–127, 1888.

[24] Kenneth J. Arrow, Theodore Harris, and Jacob Marschak. Optimal inventory policy. *Econometrica: Journal of the Econometric Society*, pages 250–272, 1951.

[25] Tsan-Ming Choi. *Handbook of Newsvendor problems: Models, extensions and applications*, volume 176. Springer, 2012.

[26] Ulaş Özen, Henk Norde, and Marco Slikker. On the convexity of newsvendor games. *International Journal of Production Economics*, 133(1):35–42, 2011.

[27] Chaolin Yang, Zhenyu Hu, and Sean X. Zhou. Multilocation newsvendor problem: Centralization and inventory pooling. *Management Science*, 67(1):185–200, 2021.

[28] Jiahua Zhang, Shu-Cherng Fang, and Yifan Xu. Inventory centralization with risk-averse newsvendors. *Annals of Operations Research*, 268:215–237, 2018.

[29] Moshe Dror and Bruce C. Hartman. Survey of cooperative inventory games and extensions. *Journal of the Operational Research Society*, 62:565–580, 2011.

[30] Bruce C. Hartman, Moshe Dror, and Moshe Shaked. Cores of inventory centralization games. *Games and Economic Behavior*, 31(1):26–49, 2000.

[31] Marco Slikker, Jan Fransoo, and Marc Wouters. Cooperation between multiple news-vendors with transshipments. *European Journal of Operational Research*, 167(2):370–380, 2005.