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Mean variance optimalizace pro minimální entropickou míru

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Dedication. First of all I want to thank my supervisor, doc. RNDr. Jan Večeř, Ph. D., for all his valuable advice and insightful remarks. Another thanks belongs to my family who unconditionally supported me through the whole journey. Lastly thank you to my friends that I could always count on to play Dota with me when I needed (or not) distracting.

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Abstract: It can be shown that linear transformations of logarithm are the only utility functions whose optimal portfolios do not depend on numeraire. This thesis focuses on optimization of expected logarithmic utility of a portfolio. We show that, given our market opinion, the optimal expected payoff is the Kullback-Leibler divergence of the market state price density and the numeraire state price density. In an incomplete market however, the market density may not be replicable and we have to find the portfolio with the smallest K-L divergence to the market density. This problem does not have a general analytic solution but can be approximated, in two different ways, by a mean variance problem that possesses analytic solution and is not hard to calculate. Finally we compare these methods with hard numerical approach on simple portfolios of 2 assets and one or two maximal contracts, that are just shifted European options.

Keywords: state price density, Kullback-Leibler divergence, mean variance problem

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Introduction

In the utility maximization problem, generally the optimal portfolio depends on its reference asset, also called the numeraire. It can be shown that the only utility (up to additive and multiplicative constants) that is maximized for every numeraire all the same is the natural logarithm. Given our market opinion, more precisely state price density (SPD), the expected log-utility is maximized for the portfolio with the SPD corresponding to this opinion and is equal to Kullback-Leibler divergence of this opinion's and reference asset's SPDs. In incomplete markets however, it might be impossible to replicate this SPD exactly. From here the problem arises of finding a portfolio that is the closest to our opinion in Kullback-Leibler divergence. We can approximate the log-utility, in two different ways by its second order Taylor polynomial and we get a mean variance problem that has an analytic solution.

The thesis is organized in the following way: In Chapter 1 we give the necessary definitions and theorems used later on we show that prices are just scaled likelihood ratios of SPDs and the numeraire invariance of the logarithm. The relationship of log-utility maximization and K-L divergence is shown in Chapter 2 along with the conversion to a mean variance problem. Also the asymptotic behaviour of the price under the physical measure is discussed. In Chapter 3 we assume a geometric brownian motion model and approximately optimal portfolios of two assets and one or two maximal contracts (European options). Lastly in Chapter 4 we show some well known strategies that can be employed to profit on different market scenarios when we have one or two contracts in the portfolio.

1. State price densities and logarithmic utility

In the opening chapter we define the state price density of an asset and show that prices are scaled likelihood ratios of state price densities. We also show that the logarithmic utility is the only function whose optimal portfolio does not change with numeraire.

1.1 Prices as likelihood ratios

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis and X,Y two assets. We introduce the following notation.

Notation. By $X_Y(t)$ we mean the price of X in terms of Y at time $t \geq 0$

We will assume the stochastic process $\{X_Y(t), t \geq 0\}$ to be continuous, $(\mathcal{F}_t)$-adapted, $X_Y(0)$ to be deterministic and asset Y not worthless, so the price is well defined. This notation allows us to easily change the reference asset by the relationship

$$X_Z(t) = X_Y(t)Y_Z(t),$$

where Z is another arbitrary, not worthless asset.

Remark 1. *Assets by themselves do not have any precise mathematical meaning and are defined by their prices. We can look at them as equivalence classes where two assets X, Y are equivalent at time T if and only if $X_Y(T) = Y_X(T) = 1$.*

Definition 1. *Let Y be an asset and $T > 0$,* a contract (contingent claim) V settled in Y at time T *is a financial contract that delivers $V_Y(\omega, T)$ units of Y for market scenario $\omega \in \Omega$.*

Remark 2. *V_Y in the previous definition is a non-negative $(\mathcal{F}_T)$-random variable.*

Assets can however be characterized by their state price density. Let $T > 0$, for $A \in \mathcal{F}_T$ let

$$V(T) = \mathbb{1}_A X(T) \tag{1.1}$$

be a contract that pays a unit of no arbitrage asset X if the event A occurs at time T. These types of contracts are known as Arrow-Debreu securities. To calculate its price in terms of X at time 0 we are going to assume there exists measure $\mathbb{P}^X$ on $\mathcal{F}$ under which $\{V_X(t), t \geq 0\}$ is a martingale.

Remark 3. *By the symbol $\mathbb{E}^X$ we understand the expectation with respect to measure $\mathbb{P}^X$ or equivalently density $p(\omega|X)$.*

Then the martingale property implies

$$V_X(0) = \mathbb{E}^X V_X(T) = \mathbb{E}^X \mathbb{1}_A = \mathbb{P}^X(A).$$

Additionally if $\mathbb{P}^X \ll \mathbb{P}$, then by the Radon-Nikodým theorem there exists a density $p(\omega\, |X)$ on Ω such that

$$\mathbb{P}^X(A) = \int_A p(\omega|X) d\mathbb{P}(\omega). \tag{1.2}$$

Definition 2. *The function* $p(\omega|X)$ *in* (1.2) *is called the* state price density *of asset* X *at time* T.

The exsistence of $\mathbb{P}^X$, and consequently of $p(\omega|X)$, is not guaranteed, however we will assume it to exist, as we are treating state price densities as the main inputs of the model.

The following theorem states that prices can be expressed as scaled likelihood ratios of state price densities. Later on in Section 2 it will be useful to represent price as a state price density ratio of an asset and its numeraire.

Theorem 1. *Let* X, Y *be no arbitrage assets and* $\mathbb{P}^X \ll \mathbb{P}^Y$, *then for* $T > 0$ *it holds that*

$$X_Y(\omega, T) = X_Y(0) \cdot \frac{p(\omega|X)}{p(\omega|Y)}, \quad \mathbb{P}^Y\text{-a. s.} \tag{1.3}$$

Proof. Let $V(\omega, T)$ be an arbitrary contract with random payoff a time T. We can settle it either in asset X or Y, getting

$$V(0) = \mathbb{E}^X[V_X(T)]X(0) = \mathbb{E}^Y[V_Y(T)]Y(0).$$

The second equality above can be rewritten as

$$X(0) \int_\Omega V_X(\omega, T) \frac{p(\omega|X)}{p(\omega|Y)} p(\omega|Y) d\omega = Y(0) \int_\Omega V_Y(\omega, T) p(\omega|Y) d\omega$$

and since V is arbitrary, it must hold that

$$V_X(\omega, T) \frac{p(\omega|X)}{p(\omega|Y)} X(0) = V_Y(\omega, T) Y(0), \quad \mathbb{P}^Y\text{-a. s.}$$

and by dividing both sides by $V_X(\omega, T)$ and taking Y as reference asset we obtain

$$X_Y(\omega, T) = X_Y(0) \cdot \frac{p(\omega|X)}{p(\omega|Y)}, \quad \mathbb{P}^Y\text{-a. s.}$$

□

Remark 4. *The assumption* $\mathbb{P}^X \ll \mathbb{P}^Y$ *is usually satisfied in reality. The asset* Y *represents cash and if for some scenario it becomes worthless, then all other assets will as well.*

The relationship (1.3) has been first introduced in Geman and Rochet [1995], Theorem 1, in the form

$$p(\omega|X) = \frac{X_Y(\omega, T)}{X_Y(0)} \cdot p(\omega|Y)$$

and understood as a way to obtain the $\mathbb{P}^X$ from $\mathbb{P}^Y$ and the price process. We, on the other hand, and in the spirit of Theorem 1, are going to assume the state price densities to be known and use them to represent the price process at time T.

Now we define what we mean by a portfolio.

Definition 3. *Let $n \in \mathbb{N}, X^i, i = 1, ..., n$ be assets, $w_i \in \mathbb{R}, i = 1, , , , n,\ \sum w_i = 1$, then the asset*

$$P = \sum_{i=1}^{n} w_i X^i \tag{1.4}$$

is called the portfolio of assets $X^1, ..., X^n$ with weights $w_1, ..., w_n$.

In the previous definition we are allowing the weights to be any real numbers as long as they sum up to one. This is to include leverage ($|w_i| > 1$) and shorting ($w_i < 0$).

Remark 5. *The sum* (1.4) *is of course only symbolic, summing arbitrary assets is not defined. But the price of a portfolio in terms of an asset Y equals the sum of prices in Y of its assets with corresponding weights.*

The following Corollary of the Theorem 1 says that the state price density of a portfolio is a linear combination of the state price densities of the assets it contains. Since we allow the weights to be negative or absolutely greater than one, the resulting combination of densities does not need to be a density, more specifically it can attain negative values. Later when finding optimal weights, we are going to treat this by simply adding a condition on the corresponding state price density linear combination to be non-negative on its entire domain.

Corollary 1. *The state price density of a portfolio* (1.4) *is of the form*

$$p(\omega|P) = \sum_{i=1}^{n} [w_i X_P^i(0)] \cdot p(\omega|X^i), \tag{1.5}$$

if the right hand in non-negative for all $\omega \in \Omega$.

Proof. *Let Y be an arbitrary asset such that $\mathbb{P}^P \ll \mathbb{P}^Y$, then from Theorem 1 we have*

$$\begin{aligned} p(\omega|P) &= \frac{P_Y(\omega, T)}{P_Y(0)} \cdot p(\omega|Y) \\ &= \frac{p(\omega|Y)}{P_Y(0)} \cdot \sum_{i=1}^{n} w_i X_Y^i(\omega, T) \\ &= \frac{p(\omega|Y)}{P_Y(0)} \cdot \sum_{i=1}^{n} w_i X_Y^i(0) \frac{p(\omega|X^i)}{p(\omega|Y)} \\ &= \sum_{i=1}^{n} [w_i X_P^i(0)] p(\omega|X^i). \end{aligned} \tag{1.6}$$

Terms $[w_i X_P^i(0)]$ are the relative amounts of assets X_i in the portfolio and satisfy

$$\sum_{i=1}^{n} [w_i X_P^i(0)] = 1. \tag{1.7}$$

□

In order to show an application of the Theorem 1 in a geometric brownian motion model we are going to need the basic version of Itô formula.

Theorem 2 (Itô formula)**.** *Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}([0,T]\times\mathbb{R})$, $\{X_Y(t), t \in [0,T]\}$ a continuous martingale satisfying*

$$dX_Y(t) = a_t dt + b_t dW_t,$$

where $\{a_t\}, \{b_t\}$ are $\{\mathcal{F}_t\}$- predictable,

$$\int_0^T (a_t + b_t^2)dt < \infty \quad a.\ s.$$

and $\{W_t, t \geq 0\}$ is a Wiener process, then

$$\begin{aligned} df[X_Y(t),t] = &\left\{\frac{\partial f}{\partial t}[X_Y(t),t] + a_t\frac{\partial f}{\partial x}[X_Y(t),t] + \frac{b_t^2}{2}\frac{\partial^2 f}{\partial x^2}[X_Y(t),t]\right\}dt \\ &+ b_t\frac{\partial f}{\partial x}[X_Y(t),t]dW_t. \end{aligned} \tag{1.8}$$

Proof. Theorem holds in more general setting, i. e. Theorem 3.6, Karatzas and Shreve [1988].

□

Example 1. *Let $\sigma > 0$ and X, Y be assets and price $\{X_Y(t), t \geq 0\}$ be defined by dynamics*

$$dX_Y(t) = \sigma X_Y(t)dW^Y(t), \tag{1.9}$$

where $\{W^Y(t), t \geq 0\}$ is a Wiener process on $(\Omega, \{\mathcal{F}_t\}, \mathbb{P}^Y)$. The simple return process in Y is then

$$R(T) = \int_0^T \frac{dX_Y(t)}{X_Y(t)} = \sigma W^Y(t) \tag{1.10}$$

and has distribution $\mathcal{N}(0, \sigma^2 T)$. Clearly it holds $Y_X(t) = [X_Y(t)]^{-1}$, so the dynamics of $Y_X(t)$ is determined by Theorem 2 as

$$dY_X(t) = dX_Y(t)^{-1} = \sigma^2 Y_X(t)dt - \sigma Y_X(t)dW^Y(t).$$

It also holds that

$$dY_X(t) = \sigma Y_X(t)dW^X(t), \tag{1.11}$$

so we have a relationship

$$R(T) = \sigma W^Y(t) = -\sigma W^X(t) + \sigma^2 T \overset{d}{=} \sigma W^X(t) + \sigma^2 T \sim \mathcal{N}(\sigma^2 T, \sigma^2 T).$$

The simple return process is perceived to have a positive drift under $\mathbb{P}^X$. From Theorem 1 we can write the price $X_Y(t)$ as scaled likelihood ratio of $p(x|X)$ and $p(x|Y)$, where $x \sim \mathcal{N}(0, \sigma^2 T)$.

$$X_Y(T) = X_Y(0)\cdot\frac{p(x|X)}{p(x|Y)} = \frac{2\pi\sigma\sqrt{T}\exp\left(\frac{(x-\sigma^2 T)^2}{\sigma^2 T}\right)}{2\pi\sigma\sqrt{T}\exp\left(\frac{x^2}{\sigma^2 T}\right)} = X_Y(0)\exp\left(x - \frac{\sigma^2 T}{2}\right),$$

which is the same as a solution to the stochastic differential equation (1.9).

1.2 Numeraire invariance of logarithmic utility

Instead of maximizing directly the expected payoff of a portfolio, investor often transforms it by a so called *utility function*, that takes into account investor's greater aversion to some scenarios, typically going bankrupt.

Example 2. *One class of utility functions are isoelastic utilities*

$$U_b(x) = \begin{cases} \frac{x^{1-b}-1}{1-b} & x \geq 0, b \in [0,1) \\ \log(x) & x \geq 0, b = 1 \end{cases}$$

Now suppose we have an opinion of the market, i. e. state price density $p(\omega|M)$, $\omega \in \Omega$, and a driftless asset Y (cash).

Remark 6. *Except in Subsection 2.1.1, we are going to assume that $p(\omega|M)$ is the real state price density of the market, often called the physical measure. In reality this density is not directly observable and we can only estimate it which can be problematic. A nonparametric estimation is discussed for example in Aït-Sahalia and Lo [1998].*

By maximizing the expected utility with respect to Y we understand finding a random variable $P_Y(T)$ representing the price of a portfolio at time T that maximizes

$$\mathbb{E}^M U(P_Y(T))$$

under the condition that the expected price remains the same under $\mathbb{P}^Y$

$$\mathbb{E}^Y P_Y(T) = P_Y(0).$$

For well-behaved utilities this problem can be solved in general.

Theorem 3. *Let U be twice differentiable, increasing and concave on $[0, \infty)$. Define*

$$I(x) = [U'(x)]^{-1}$$

and let λ_Y satisfy

$$\mathbb{E}^Y I\left(\lambda_Y \cdot \frac{p(\omega|M)}{p(\omega|Y)}\right) = P_Y(0)$$

Then the $\mathbb{E}^M U(P_Y(T))$ is maximized for

$$P_Y(\omega, T) = I\left(\lambda_Y \cdot \frac{p(\omega|M)}{p(\omega|Y)}\right)$$

Proof. In a more general setting, Theorem 2.1. Kramkov and Schachermayer [1999].

□

Remark 7. *By taking* $U(x) = \log(x)$ *we can see that*

$$I(x) = \frac{1}{x}$$

and

$$\lambda_Y = \frac{1}{P_Y(0)}$$

giving the optimal

$$P_Y(\omega, T) = V_Y(0) \cdot \frac{p(\omega|M)}{p(\omega|Y)},$$

which is identical to the statement od Theorem 1.

This makes the logarithmic utility special, because prices are automatically log-optimal and is the reason a large section of this thesis focuses on this utility.

Theorem 3 can also be employed to prove that the logarithm is the only utility, up to additive and multiplicative constants, whose optimal expected payoff does not depend on the numeraire.

Theorem 4. *Let* U *be twice differentiable, increasing and concave on* $[0, \infty)$. *Then the optimal expected payoff* $\mathbb{E}^M U(P_Y(T))$ *is numeraire invariant if and only if* $\exists\, C_1, C_2 \in \mathbb{R}$:

$$U(x) = C_1 \log(x) + C_2 \tag{1.12}$$

Proof. The implication from right to left is a straight application of Theorem 3. By maximizing the expectation of 1.12 with reference assets X and Y we get

$$P_X(\omega, T) = \frac{p(\omega|M)}{p(\omega|X)} \cdot P_X(0),$$

$$P_Y(\omega, T) = \frac{p(\omega|M)}{p(\omega|Y)} \cdot P_Y(0)$$

and it holds that

$$P_Y \cdot Y_X = \frac{p(\omega|M)}{p(\omega|Y)} \cdot \frac{p(\omega|Y)}{p(\omega|X)} Y_X(0) P_Y(0) = \frac{p(\omega|M)}{p(\omega|X)} P_X(0) = P_X,$$

so the weights in the optimal portfolios with respect to X and Y are the same.

Conversely suppose that U is arbitrary, then for reference assets X, Y from Theorem 3 the optimal portfolios are

$$P_X(\omega, T) = I\left(\lambda_X \frac{p(\omega|X)}{p(\omega|M)}\right),$$

$$P_Y(\omega, T) = I\left(\lambda_Y \frac{p(\omega|Y)}{p(\omega|M)}\right).$$

From numeraire invariance it follows that

$$P_X(\omega, T) = P_Y(\omega, T) \cdot Y_X(\omega, T),$$

by Theorem 1 equivalently

$$I\left(\lambda_X \frac{p(\omega|X)}{p(\omega|M)}\right) \cdot p_X(\omega, T) = I\left(\lambda_Y \frac{p(\omega|Y)}{p(\omega|M)}\right) \cdot p(\omega|Y) Y_X(0).$$

Also, since the left hand side of the above equality does not depend on Y, the function I has to be of the form

$$I(x) = \frac{C_1}{x}, \ C_1 \in \mathbb{R}$$

and since $I = [U']^{-1}$, we get

$$U(x) = C_1 \log(x) + C_2.$$

□

Example 3. *Let us have a market opinion $p(\omega|M) \sim \mathcal{N}(\mu T, \sigma^2 T)$ and for assets X, Y assume the price dynamics as in Example 1. Asset Y in this case represents available cash that can be invested into asset X (for example a stock). In a little more general setting than this, Merton [1971] showed using stochastic control theory that the expected log-return is maximized if we invest $\frac{\mu}{\sigma^2}$ of our wealth in Y into asset X. Using Theorems 1 an 3 we can show this result directly. From Remark 7 we have*

$$P_Y(x, T) = P_Y(0) \cdot \frac{p(x|M)}{p(x|Y)} = P_Y(0) \exp\left(\frac{\mu}{\sigma^2} R(T) - \frac{\mu^2}{2\sigma^2} T\right), \tag{1.13}$$

the price $P_Y(T)$ is a martingale under $\mathbb{P}^Y$ so from Itô formula used on the simple return process $R(T)$ from (1.10) and

$$f(x, T) = P_Y(0) \exp\left(\frac{\mu}{\sigma^2} x - \frac{\mu^2}{2\sigma^2} T\right)$$

we obtain

$$dP_Y(R(T), T) = \frac{\mu}{\sigma^2} P_Y(R(T), T) dR(T),$$

equivalently

$$\frac{dP_Y(x, T)}{P_Y(x, T)} = \frac{\mu}{\sigma^2} \frac{dX_Y(x, T)}{X_Y(x, T)}.$$

The simple return process of the optimal portfolio is equal to $\frac{\mu}{\sigma^2}$ fraction of the simple return process $R(T)$, therefore we should invest $\frac{\mu}{\sigma^2}$ of our wealth into X.

2. Kullback-Leibler divergence and approximated portfolios

In this chapter we show the connection between expected log-utility maximization and Kullback-Leibler divergence. In an incomplete market we cannot construct the optimal portfolio exactly and have to settle for one that minimizes the relative entropy with the optimal portfolio. We are going to discuss three approaches to this problem: straightforward numerical maximization of the expectation, approximation by a mean variance problem similar to the one addressed by Markowitz [1952] and finally an approximation by a driftless problem, which is related to the Fischer information matrix, where it is enough to minimize a quadratic form defined by a covariance matrix of the state price densities. Appeals of the latter two methods are the existence of analytical solutions and their relative simplicity, where the inputs are the state price densities and outputs are the optimal weights.

2.1 Optimal portfolio and Kullback-Leibler divergence

Definition 4. *Let $p(x|X)$, $p(x|Y)$ be densities of two arbitrary real continuous distributions $\mathbb{P}^X$, $\mathbb{P}^Y$, the* Kullback-Leibler divergence *(relative entropy) of $\mathbb{P}^X, \mathbb{P}^Y$ is defined as*

$$D_{KL}(\mathbb{P}^X||\mathbb{P}^Y) = \int_{\mathbb{R}} \log\left(\frac{p(z|X)}{p(z|Y)}\right) p(z|X)dz = \mathbb{E}^X \log\left(\frac{p(Z|X)}{p(Z|Y)}\right)$$

It is well known that K-L divergence is non-negative and

$$D_{KL}(\mathbb{P}^X||\, \mathbb{P}^Y) = 0 \iff p(x|X) = p(x|Y),\ \lambda - a.\ e.,$$

where λ is the Lebesgue measure on the real line. On the other hand it is obviously not symmetric and does not satisfy the triangle inequality, so it is not a metric in a classical sense, but we can still interpret it as a notion of distance (divergence) between two distributions, in our case the state price densities. The following theorem shows the connection between the problem of maximizing log-utility and relative entropy.

Theorem 5. *Let Y be a reference asset, then the portfolio with state price density $p(x|M)$, assuming $\mathbb{P}^M \ll \mathbb{P}^Y$, has the maximal expected log-utility, given our market opinion is $p(x|M)$, i. e.*

$$D_{KL}(\mathbb{P}^M||\mathbb{P}^Y) = \mathbb{E}^M \log\left(\frac{M_Y(T)}{M_Y(0)}\right) = \max_{P_Y(T)} \mathbb{E}^M \log\left(\frac{P_Y(T)}{P_Y(0)}\right)$$

Proof. First equality follows from Theorem 1 and Definition 4. For an arbitrary portfolio P with state price density $p(x|P)$ we have

$$
\begin{aligned}
\mathbb{E}^M \log\left(\frac{P_Y(T)}{P_Y(0)}\right) &= \mathbb{E}^M \log\left(\frac{M_Y(T)P_M(T)}{M_Y(0)P_M(0)}\right) \\
&= \mathbb{E}^M \log\left(\frac{M_Y(T)}{M_Y(0)}\right) + \mathbb{E}^M \log\left(\frac{P_M(T)}{P_M(0)}\right) \\
&= D_{KL}(\mathbb{P}^M||\mathbb{P}^Y) - D_{KL}(\mathbb{P}^M||\mathbb{P}^P) \\
&\leq D_{KL}(\mathbb{P}^M||\mathbb{P}^Y).
\end{aligned} \tag{2.1}
$$

□

Definition 5. *A market is* complete *if any contract with payoff $V_Y(T)$ can be replicated by a portfolio of assets available on the market, i. e. there exist assets $X^1, ..., X^n$,*
$n \in \mathbb{N}$ and weights $w_1, ..., w_n$, $n \in \mathbb{N}$, $\sum w_i = 1$ such that

$$V_Y(T) = \sum_{i=1}^{n} w_i X_Y^i(T), \ \mathbb{P}^Y\text{-a. s.}$$

A market is incomplete *if it is not complete.*

If we are able to construct a portfolio with the same state price density as our opinion, from Theorem 5 we know it is optimal. However, in the case of incomplete markets, this may not be possible. The best we can do is to minimize the term $D_{KL}(\mathbb{P}^M||\mathbb{P}^P)$ in (2.1).

Obviously we can maximize the expectation numerically, solving a problem

$$
\begin{aligned}
&\max_{w_1,...,w_n} \mathbb{E}^M \log\left(\frac{P_Y(T)}{P_Y(0)}\right) \\
&\text{s. t.} \ \sum_{i=1}^{n} w_i = 1,
\end{aligned}
$$

but that does not have to be feasible for large portfolios. In the following sections we approximate the logarithm by its Taylor polynomial and get two types of mean variance problems that are generally much easier to solve.

2.1.1 Asymptotic price behaviour under the physical measure

Let $\hat{\mathbb{P}}$ be the physical measure of the market. If we assume the price increments over disjoint equidistant time intervals to be i.i.d., which is true for example in a geometric brownian motion model assumed in Chapter 3, then the asymptotic behaviour of the price in Y depends on whether the $\hat{\mathbb{P}}$ is closer to $\mathbb{P}^Y$ or $\mathbb{P}^M$ in the Kullback-Leibler divergence.

Proposition 1. *If $D_{KL}(\hat{\mathbb{P}}||\mathbb{P}^Y) > D_{KL}(\hat{\mathbb{P}}||\mathbb{P}^M)$, then*

$$M_Y(t) \xrightarrow[n\to\infty]{} +\infty, \ \hat{\mathbb{P}} - a. \ s.$$

and if the opposite inequality holds, then

$$M_Y(t) \xrightarrow[n\to\infty]{} 0, \ \hat{\mathbb{P}} - a. \ s.$$

Proof. Let $\hat{P}$ be a hypothetical asset with state price density $\hat{\mathbb{P}}$, then we have

$$\begin{aligned}\mathbb{E}^{\hat{\mathbb{P}}} \log\left(\frac{M_Y(T)}{M_Y(0)}\right) &= \mathbb{E}^{\hat{\mathbb{P}}} \log\left(\frac{M_{\hat{P}}(T)}{M_{\hat{P}}(0)}\right) + \mathbb{E}^{\hat{\mathbb{P}}} \log\left(\frac{\hat{P}_Y(T)}{\hat{P}_Y(0)}\right) \\ &= D_{KL}(\hat{\mathbb{P}}||\mathbb{P}^M) - D_{KL}(\hat{\mathbb{P}}||\mathbb{P}^Y).\end{aligned} \tag{2.2}$$

Now suppose the right hand side of (2.2) is strictly positive, i. e. there exists $\varepsilon > 0$ such that

$$\mathbb{E}^{\hat{\mathbb{P}}} \log\left(\frac{M_Y(T)}{M_Y(0)}\right) > \varepsilon.$$

Random variables

$$Y_n = \log\left(\frac{M_Y(nT)}{M_Y((n-1)T)}\right)$$

are i.i.d. and by the law of large numbers it holds

$$\frac{1}{n}\log\left(\frac{M_Y(nT)}{M_Y(0)}\right) = \frac{1}{n}\sum_{k=1}^{n} Y_k \xrightarrow[n\to\infty]{} \mathbb{E}^{\hat{\mathbb{P}}} \log\left(\frac{M_Y(T)}{M_Y(0)}\right) > \varepsilon,\ \hat{\mathbb{P}} - a.\ s.$$

Therefore from some $n_0 \in \mathbb{N}$ onward we have

$$\log\left(\frac{M_Y(nT)}{M_Y(0)}\right) > \frac{\varepsilon n}{2},\ \hat{\mathbb{P}} - a.\ s.$$

and the right hand side tends to infinity, so the left hands does as well. The argument for the second part of the statement is analogous. □

If we are able to make sure our opinion density has smaller relative entropy with the physical measure than the driftless measure does, the previous Proposition says we are going to profit in the longterm.

Example 4. *In Merton's setup and a complete market, from (1.13) we see that the best payoff we can get is*

$$D_{KL}(\mathbb{P}^M||\mathbb{P}^Y) = \frac{1}{2}\frac{\mu^2}{\sigma^2}T.$$

This will be used as a benchmark for approximated portfolios in Chapter 3, especially in the two asset case, i. e. Figure 3.1. Additionally, if the physical measure disagrees with our opinion only in drift parameter, i. e. $\hat{\mathbb{P}} \sim \mathcal{N}(\hat{\mu}T, \sigma^2 T)$*, then*

$$D_{KL}(\hat{\mathbb{P}}||\mathbb{P}^Y) = \frac{1}{2}\frac{\hat{\mu}^2}{\sigma^2}T$$

and

$$D_{KL}(\hat{\mathbb{P}}||\mathbb{P}^M) = \frac{1}{2}\frac{(\mu-\hat{\mu})^2}{\sigma^2}T$$

so, according to Proposition 1, we have to make sure that $|\mu - \hat{\mu}| < |\hat{\mu}|$*. If* $\hat{\mu} > 0$ *this translates to* $\mu \in (0, 2\hat{\mu})$*. In reality we do not know the value of* $\hat{\mu}$ *and in order to fulfill the condition in Proposition 1 it is safer to choose* μ *from the right neighbourhood of 0, even though that means smaller returns.*

2.2 Approximation by a mean variance problem

Suppose we have assets $X^1, ..., X^n$ to make a portfolio out of. We can approximate the problem by taking the second order Taylor polynomial of logarithm

$$\log(x) = (x-1) - \frac{1}{2}(x-1)^2 + o((x-1)^2).$$

First lets look at the difference

$$\begin{aligned}
\mathbb{E}^M \log\left(\frac{P_Y(T)}{P_Y(0)}\right) - \mathbb{E}^Y \log\left(\frac{P_Y(T)}{P_Y(0)}\right) &= \int_{\mathbb{R}} \log\left(\frac{P_Y(T)}{P_Y(0)}\right)[p(x|M) - p(x|Y)]dx \\
&\approx \int_{\mathbb{R}} \left(\frac{p(x|P)}{p(x|Y)} - 1\right)[p(x|M) - p(x|Y)]\,dx \\
&= \int_{\mathbb{R}} \left(\frac{p(x|P)}{p(x|Y)} - 1\right) p(x|M)dx \\
&= \mathbb{E}^M \left(\frac{p(x|P)}{p(x|Y)} - 1\right).
\end{aligned}$$

By this approximation and Theorem 1 we get

$$\begin{aligned}
\mathbb{E}^M \log\left(\frac{P_Y(T)}{P_Y(0)}\right) &\approx \mathbb{E}^M \left(\frac{p(x|P)}{p(x|Y)} - 1\right) + \mathbb{E}^Y \log\left(\frac{P_Y(T)}{P_Y(0)}\right) \\
&\approx \mathbb{E}^M \left(\sum_{i=1}^n w_i \frac{p(x|X^i)}{p(x|Y)} - 1\right) - \frac{1}{2}\mathbb{E}^Y \left(\sum_{i=1}^n w_i \frac{X_Y^i(T)}{X_Y^i(0)} - 1\right)^2 \\
&= \sum_{i=1}^n w_i \mathbb{E}^M \left(\frac{p(x|X^i)}{p(x|Y)} - 1\right) - \frac{1}{2}\mathbb{E}^Y \left(\sum_{i=1}^n w_i \frac{p(x|X^i)}{p(x|Y)} - 1\right)^2 \\
&= \boldsymbol{\mu}^\mathsf{T}\mathbf{w} - \frac{1}{2}\mathbf{w}^\mathsf{T}\Sigma_1\mathbf{w},
\end{aligned} \tag{2.3}$$

where

$$\boldsymbol{\mu} = \left(\mathbb{E}^M\left[\frac{p(x|X^1)}{p(x|Y)} - 1\right], ..., \mathbb{E}^M\left[\frac{p(x|X^n)}{p(x|Y)} - 1\right]\right)^\mathsf{T}$$

and

$$\Sigma_1 = \left[\mathbb{E}^Y\left(\frac{p(x|X^i)}{p(x|Y)} - 1\right)\left(\frac{p(x|X^j)}{p(x|Y)} - 1\right)\right]_{i,j=1}^n = \left[\operatorname{cov}^Y\left(\frac{X_Y^i(T)}{X_Y^i(0)}, \frac{X_Y^j(T)}{X_Y^i(0)}\right)\right]_{i,j},$$

which is similar to a mean variance problem addressed by Markowitz [1952], only the drift term is with respect to $\mathbb{P}^M$. We have a program

$$\max_{w_1,...,w_n} \boldsymbol{\mu}^\mathsf{T}\mathbf{w} - \frac{1}{2}\mathbf{w}^\mathsf{T}\Sigma_1\mathbf{w} \tag{2.4}$$

$$\text{s. t. } \sum_{i=1}^n w_i = 1.$$

If Σ is invertible, the solution can be found by using the Lagrange multipliers theorem and is of the form

$$\lambda = \frac{1 - \mathbf{1}^\intercal \Sigma_1^{-1} \boldsymbol{\mu}}{\mathbf{1}^\intercal \Sigma_1^{-1} \mathbf{1}},$$

$$\mathbf{w} = \Sigma_1^{-1} \left(\boldsymbol{\mu} + \frac{1 - \mathbf{1}^\intercal \Sigma_1^{-1} \boldsymbol{\mu}}{\mathbf{1}^\intercal \Sigma_1^{-1} \mathbf{1}} \cdot \mathbf{1} \right), \tag{2.5}$$

where λ is the Lagrange multiplier and $\mathbf{1}$ is a vector of ones and length n.

Remark 8. *From (2.3) it is apparent if the asset Y is part of the portfolio, and in Chapter 3 it will be, the row and column corresponding to Y in Σ_1 and $\boldsymbol{\mu}$ will both have only zero entries.*

In the classical Markowitz approach, the drift a volatility parameters are usually estimated from historical behaviour of the prices which brings more uncertainty. In our case this is not needed because they are determined by the state price densities of the assets.

2.3 Fischer approximation by a variance problem

We can further simplify the problem by realizing

$$\mathbb{E}^M \left(\frac{P_M(T)}{P_M(0)} \right) = 1$$

to reformulate it as follows.

$$\begin{aligned} \mathbb{E}^M \log \left(\frac{P_Y(T)}{P_Y(0)} \right) &= \mathbb{E}^M \log \left(\frac{P_M(T)}{P_M(0)} \right) + \mathbb{E}^M \log \left(\frac{M_Y(T)}{M_Y(0)} \right) \\ &\approx D_{KL}(\mathbb{P}^M || \mathbb{P}^Y) - \frac{1}{2} \mathbb{E}^M \left(\frac{P_M(T)}{P_M(0)} - 1 \right)^2 \\ &= D_{KL}(\mathbb{P}^M || \mathbb{P}^Y) - \frac{1}{2} \mathbf{w}^\intercal \Sigma \mathbf{w}, \end{aligned} \tag{2.6}$$

for

$$\begin{aligned} \Sigma &= \left[\mathbb{E}^M \left(\frac{p(x|X^i)}{p(x|M)} - 1 \right) \left(\frac{p(x|X^j)}{p(x|M)} - 1 \right) \right]_{i,j=1}^n \\ &= \left[\operatorname{cov}^M \left(\frac{X_Y^i(T)}{X_Y^i(0)}, \frac{X_Y^j(T)}{X_Y^j(0)} \right) \right]_{i,j=1}^n . \end{aligned} \tag{2.7}$$

Considering the price with respect to M gets rid of the drift term in (2.3) and finding the optimal weights $\mathbf{w}$ that maximize the right hand side of (2.6) is computationally easier than maximizing the expectation. An insightful perspective is that the portfolio maximizing the log expected payoff under $\mathbb{P}^M$ has approximately the smallest variance under the same measure.

The problem of finding the minimum is very similar to (2.4),

$$\begin{aligned} \min_{w_1,\dots,w_n} \quad & \mathbf{w}^\mathsf{T}\Sigma\mathbf{w} \\ \text{s. t.} \quad & \sum_{i=1}^n w_i = 1, \end{aligned} \tag{2.8}$$

only with the drift vector $\boldsymbol{\mu}$ being zero and a different Σ matrix.

Therefore from (2.5) follows the solution

$$\begin{aligned} \lambda &= \frac{1}{\mathbf{1}^\mathsf{T}\Sigma_1^{-1}\mathbf{1}} \\ \mathbf{w} &= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\mathsf{T}\Sigma_1^{-1}\mathbf{1}}. \end{aligned} \tag{2.9}$$

2.3.1 Connection to Fischer information

The Σ matrix in (2.7) is closely related to the Fischer information matrix in statistics, hence the name of the method.

Definition 6. *Let $\{p(x|\mathbf{w});\ \mathbf{w} \in \Theta \subset \mathbb{R}^n,\ n \in \mathbb{N}\}$ be a family of probability densities, then the Fischer information matrix is*

$$[I(\mathbf{w})]_{i,k=1}^n = \mathbb{E}^{\mathbf{w}}\left[\left(\frac{\partial \log(p(X|\mathbf{w}))}{\partial w_i}\right)\left(\frac{\partial \log(p(X|\mathbf{w}))}{\partial w_k}\right)\right].$$

Under so called regularity conditions (Anděl [2007], Definition 7.8) we can rewrite it as

$$[I(\mathbf{w})]_{i,k=1}^n = -\mathbb{E}^{\mathbf{w}}\left[\frac{\partial^2 \log(p(X|\mathbf{w}))}{\partial w_i \partial w_k}\right].$$

Now consider family $\Phi = \{\sum_{i=1}^n w_i p(x|X^i);\ w_i \in \mathbb{R},\ \sum w_i = 1\}$, that represents the possible state price densities of our portfolio, then in the first line of (2.6) we have

$$\mathbb{E}^M \log\left(\frac{P_M(T)}{P_M(0)}\right) = -D_{KL}(\mathbb{P}^M||\mathbb{P}^P) = \int p(x|M)\log\left(\frac{\sum w_i p(x|X^i)}{p(x|M)}\right)dx.$$

The Fischer information matrix of this family has entries

$$[I(\mathbf{w})]_{i,k} = -\mathbb{E}^M\left[\frac{p(X|X^i)p(X|X^k)}{(\sum_{j=1}^n w_j p(X|X^j))^2}\right].$$

If $p(x|M) = \sum \hat{w}_i p(x|X^i) \in \Phi$, from the second order Taylor expansion around $\hat{\mathbf{w}}$ we obtain (under regularity conditions)

$$D_{KL}(\mathbb{P}^M||\mathbb{P}^P) \approx \frac{1}{2}(\mathbf{w}-\hat{\mathbf{w}})^T[I(\hat{\mathbf{w}})]_{i,k=1}^n(\mathbf{w}-\hat{\mathbf{w}}), \tag{2.10}$$

because

$$[I(\hat{\mathbf{w}})]_{i,k} = -\mathbb{E}^{\hat{\mathbf{w}}}\left[\frac{\partial^2 \log(p(X|\mathbf{w}))}{\partial w_i \partial w_k}\right] = \left(\frac{\partial^2 D_{KL}(\mathbb{P}^M||\mathbb{P}^P)}{\partial w_i \partial w_k}\right)_{\mathbf{w}=\hat{\mathbf{w}}}.$$

It is not hard to see that the right hand side of (2.10) is the same as the quadratic form defined by (2.7).

We have

$$
\begin{aligned}
-(\mathbf{w}-\hat{\mathbf{w}})^T[I(\hat{\mathbf{w}})](\mathbf{w}-\hat{\mathbf{w}}) &= \sum_i \sum_k (w_i - \hat{w}_i)(w_k - \hat{w}_k)[I(\hat{\mathbf{w}})]_{i,k} \\
&= \sum_i \sum_k w_i w_k [I(\hat{\mathbf{w}})]_{i,k} - \sum_i \sum_k \hat{w}_i w_k [I(\hat{\mathbf{w}})]_{i,k} \\
&\quad - \sum_i \sum_k w_i \hat{w}_k [I(\hat{\mathbf{w}})]_{i,k} + \sum_i \sum_k \hat{w}_i \hat{w}_k [I(\hat{\mathbf{w}})]_{i,k} \\
&= \sum_i \sum_k w_i w_k [I(\hat{\mathbf{w}})]_{i,k} - 1 - 1 + 1 \\
&= \mathbf{w}^{\mathsf{T}} \Sigma \mathbf{w}.
\end{aligned}
\tag{2.11}
$$

In the third equality we used the fact that

$$
\sum_i \sum_k \hat{w}_i w_k [I(\hat{\mathbf{w}})]_{i,k} = \mathbb{E}^M \left[\frac{p(X|M) \sum_k w_k p(X|X^k)}{(p(X|M))^2} \right] = 1,
$$

similarly for the other two terms. The last equality follows from

$$
1 = \sum_i \sum_k w_i w_k.
$$

Since we originally assume $p(x|M) \notin \Phi$, i. e. the market is incomplete, the initial problem can also be understood as finding a portfolio that contains the most Fischer information about $\mathbb{P}^M$.

3. Application in GBM model

In this chapter we are going compare the three approaches from Chapter 3 in a simple geometric brownian motion model of the prices, where

$$dX_Y(t) = \sigma_1 X_Y(T) dW^Y(t), \sigma_2 > 0,$$

or equivalently

$$X_Y(t) = X_Y(0) \exp\left\{\sigma_1 W^Y(t) - \frac{\sigma_1^2 t}{2}\right\}$$

$\mathbb{P}^Y$ almost surely, where W^Y is a Wiener process on $(x, \mathcal{F}, \mathbb{P}^Y)$. Let our market opinion be

$$\mathbb{P}^M \sim \mathcal{N}(\mu T, \sigma_2^2 T), \; \mu \in \mathbb{R}, \; \sigma_2 > 0. \tag{3.1}$$

We will consider three portfolios consisting of a risk-free asset Y and asset X with state price densities

$$\mathbb{P}^Y \sim \mathcal{N}(0, \sigma_1^2 T), \; \mathbb{P}^X \sim \mathcal{N}(\sigma_1^2 T, \sigma_1^2 T), \tag{3.2}$$

and one or two maximal contracts with payoff in an arbitrary asset Z at time T being

$$[O_K]_Z(T) = \max\left\{X_Z(T), K \cdot Y_Z(T)\right\},$$

where $K > 0$ is a so called *strike price* which serves purpose of scaling asset Y so the prices of $K \cdot Y$ and X are comparable. Maximal contracts by themselves are not traded in the real world markets because their payoff, by definition, dominates payoffs of both assets $K \cdot Y$ an X. What is actually traded are (among other contracts of course) the so called European options.

Definition 7. *For assets X, Y, Z the* European call option with payoff in Z based on X and Y *is a contract with payoff*

$$\max\{X_Z(T) - K \cdot Y_Z(T), 0\}.$$

A European put option *in the same setup has payoff*

$$\max\{K \cdot Y_Z(T) - X_Z(T), 0\}.$$

Remark 9. *Usually in the previous definition we take $Z = Y$ and the payoff becomes*

$$\max\{X_Y(T) - K, 0\}.$$

In cases of the first two portfolios we will assume $\sigma_1 = \sigma_2$, the general case is considered in Section 3.3. From the relationship

$$\max(X_Z(T), K \cdot Y_Z(T)) = \max\{X_Z(T) - K \cdot Y_Z(T), 0\} + K \cdot Y_Z(T) \tag{3.3}$$

we can see the maximal contract is just a shifted European call, or analogously put, option with the same strike price. From Theorem 1 we can calculate the state price density of the maximum contract

$$
\begin{aligned}
p(x|O_K) &= p(x|M) \cdot \frac{\max\{X_M(T), K \cdot Y_M(T)\}}{\max\{X_M(0), K \cdot Y_M(0)\}} \\
&= p(x|M) \cdot \frac{\max\{\frac{p(x|X)}{p(x|M)} X_M(0), K \cdot \frac{p(x|Y)}{p(x|M)} Y_M(0)\}}{\max\{X_M(0), K \cdot Y_M(0)\}} \\
&= \frac{\max\{p(x|X) X_M(0), K \cdot p(x|Y) Y_M(0))\}}{\max\{X_M(0), K \cdot Y_M(0)\}}
\end{aligned}
\tag{3.4}
$$

and by properly norming we obtain

$$
p(x|O_K) = \frac{\max\{p(x|X) X_M(0), K \cdot p(x|Y) Y_M(0))\}}{X_M(0)(1 - F_X(d)) + K \cdot Y_M(0) F_Y(d)},
$$

where

$$
d = \frac{\sigma_1^2 t}{2} - \log\left(\frac{X_M(0)}{K \cdot Y_M(0)}\right) \tag{3.5}
$$

and F_X and F_Y are distribution functions of $\mathcal{N}(\sigma_1^2 T, \sigma_1^2 T)$, $\mathcal{N}(0, \sigma_1^2 T)$ respectively.

In the following sections we are going to find the optimal weights and their respective portfolios payoffs for all three approaches given in Chapter 2 and compare them. All calculations and plots have been done in a Mathematica notebook attached (Wolfram Research, Inc. [2023]).

Notation. From now on, to simplify notation, we will be writing $X_M(0)$ as x_0 and $Y_M(0)$ as y_0.

3.1 Portfolio of 2 assets

First we will look at the simplest case, when no maximal contracts are added to the portfolio. For simplicity we will also assume that $\sigma = \sigma_1 = \sigma_2$. Thus we are finding log-optimal portfolio

$$
P^2 = wX + (1 - w)Y
$$

under market opinion (3.1), where X,Y have state price densities given by (3.2). For this case we chose the parameter values to be $\mu = 0.03$, $\sigma = 0.2$ and $T = 10$, which is not unreasonable in the real world.

As for the Markowitz approach from Section 2.2, the Σ_1 matrix is very simple:

$$
\Sigma_1 = \begin{pmatrix} e^{\sigma^2 T} - 1 & 0 \\ 0 & 0 \end{pmatrix},
$$

the drift vector is

$$
\mu = \begin{pmatrix} e^{\mu T} - 1 \\ 0 \end{pmatrix},
$$

giving the weight

$$
w^1 = \frac{e^{\mu T} - 1}{e^{\sigma^2 T} - 1}.
$$

In the driftless case from Section 2.3, the Σ matrix is of the form

$$\Sigma = \begin{pmatrix} \exp\{\frac{(\mu-\sigma^2)T}{\sigma^2}\} & \exp\{(\frac{\mu^2}{\sigma^2}-\mu)T\} \\ \exp\{(\frac{\mu^2}{\sigma^2}-\mu)T\} & \exp\{\frac{\mu^2 T}{\sigma^2}\} \end{pmatrix}.$$

By solving (2.8) we obtain

$$w = \frac{\exp\{(\frac{\mu^2}{\sigma^2})T\} - \exp\{\frac{\mu(\mu+\sigma^2)T}{\sigma^2}\}}{2\exp\{(\frac{\mu^2}{\sigma^2})T\} - \exp\{\frac{\mu(\mu+\sigma^2)T}{\sigma^2}\} - \exp\{(\frac{\mu^2}{\sigma^2}+\sigma^2-\mu)T\}}.$$

The highest payoff achievable only in a complete market is

$$D_{KL}(\mathbb{P}^M||\mathbb{P}^Y) = \frac{1}{2}\frac{\mu^2}{\sigma^2}T = 0.1125.$$

Remark 10. *In both approaches it is not hard to calculate the limit*

$$\lim_{T\to 0+} w = \lim_{T\to 0+} w^1 = \frac{\mu}{\sigma^2}$$

that agrees with how much we should have invested in asset X according to Merton [1971].

The calculated weights and payoffs of all three approaches from Chapter 2 are laid out in Table 3.1. Here they are pretty similar and even for 10 year horizon not far off from the Merton's initial allocation from Remark 10. As expected, the Fischer approach outperforms Markowitz.

We know that Merton's portfolio is optimal so we can use it as a benchmark. If we plot the expected payoffs in Figure 3.1 as functions of time for our parameter values, the fall off of Markowitz starts to be significant after about 40 years.

The portfolio with no contracts also allows us to demonstrate how the approximate solutions fare against the exact one on a simple plot in Figure 3.2. Clearly the Fischer method does a better job at approximating the maximum of the real payoff, as we are expanding the Taylor polynomial 'around' $\mathbb{P}^M$ and Theorem 5 holds. The points in the plot represent portfolios with weights obtained above plugged into all three approaches with their respective colours. In the case of only two assets the difference is almost indistinguishable.

Weight and Payoff		
Method	**Weight**	**Payoff**
Numeric	0.767	0.111
Markowitz	0.711	0.1106
Fischer	0.769	0.111
Optimal		0.1125

Table 3.1: Calculated weights and payoffs for $\mu = 0.03,\ \sigma = 0.2,\ T = 10$.

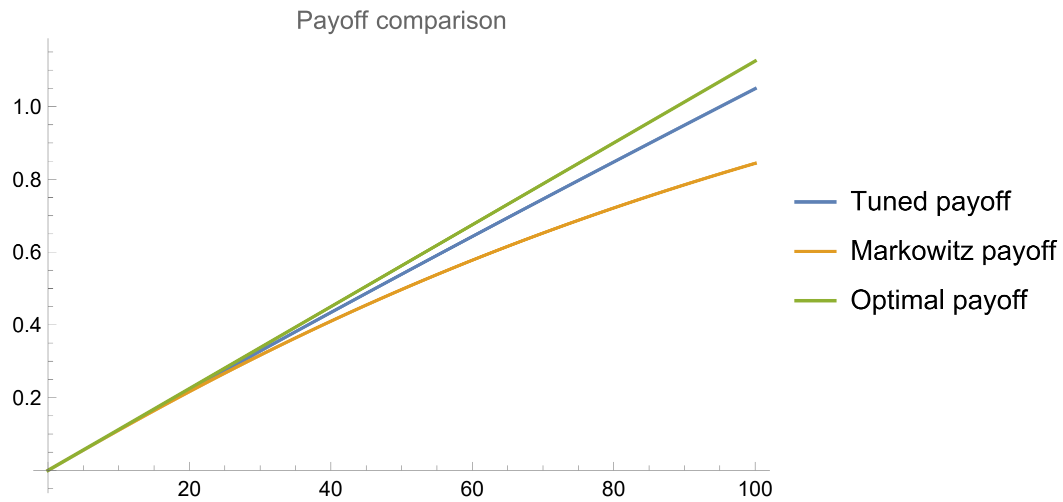

Figure 3.1: Expected payoff of 2 asset portfolio in time for $\mu = 0.03,\ \sigma = 0.2$.

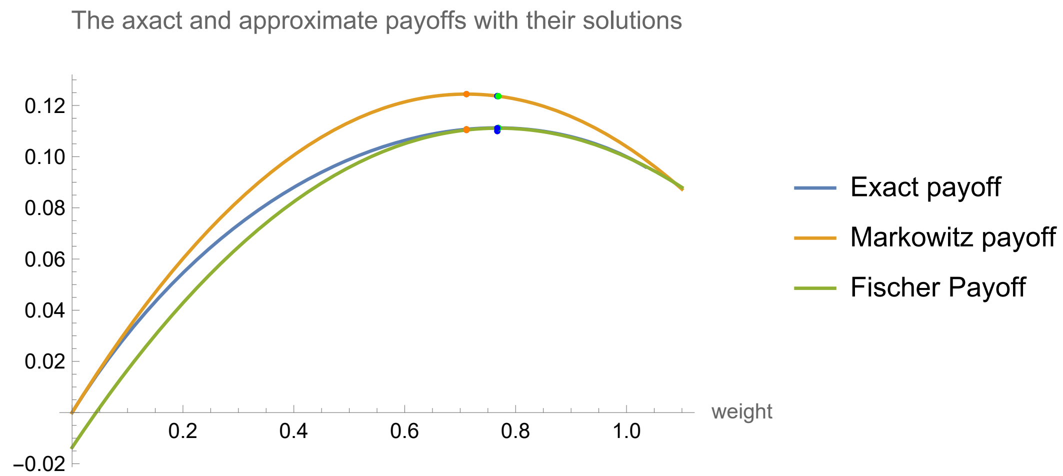

Figure 3.2: The exact and approximated expected logarithmic payoffs from Chapter 2 with their optimal portfolios. The colours of the points correspond to the methods used.

3.2 Portfolio of 2 assets and one maximal contract

By adding a maximal contract to the portfolio we can improve on the expected payoff. We are now optimizing the portfolio

$$P^3 = w_1X + w_2Y + (1 - w_1 - w_2)O_K.$$

Lets now consider $T = 10,\ \mu = 0.05,\ \sigma = 0.3,\ x_0 = y_0 = 1$. Unlike in the 2 asset portfolio, here we have one extra parameter to maximize the expected payoff, the strike price K of the maximal contract. Given an investing horizon T, we can find the optimal strike numerically by maximizing (2.4) with weights from (2.5) or as

$$K = \arg\min_{K>0.1} \hat{\mathbf{w}}^{\mathsf{T}}\Sigma\hat{\mathbf{w}},$$

where $\hat{\mathbf{w}}$ is the solution to (2.8).

Remark 11. *To prevent getting extreme values, we will be maximizing only over strikes greater that 0.1. This assumption is not vital and can be changed if needed.*

The entries of $\Sigma_{ij}, i, j \in \{1, 2\}$ (and also in the case of Σ_1) that do not depend on the strike K are of course the same as in the previous section so we only need to calculate the new right column and bottom row. The expressions representing the weights and new entries of Σ_1 and Σ matrices are very long so we leave them out, they can be found in the attached Mathematica notebook. The calculations are quite straightforward, but a little bit lengthy and tedious. One example of entry Σ_{13} is

$$\begin{aligned}
\Sigma_{13} &= \mathbb{E}^M\left(\frac{p(Z|X)p(Z|O_K))}{p(Z|M)^2}\right) \\
&= \int_{\mathbb{R}} \frac{p(z|X)p(z|O_K)}{p(z|M)}dz \\
&= C\left(\int_{-\infty}^{d} \frac{p(z|X)p(z|Y)}{p(z|M)}dz + \int_{d}^{\infty} \frac{p(z|X)p(z|X)}{p(z|M)}dz\right) \\
&= C\left[x_0\exp\left(\frac{(\mu-\sigma)^2T}{\sigma^2}\right)\left(1 + \operatorname{Erf}\left(\frac{(3\sigma^2 - 2\mu)t + 2\log(\frac{x_0}{Ky_0})}{\sigma^2T}\right)\right)\right. \\
&\quad + Ky_0\exp\left\{\left(\frac{\mu^2}{\sigma^2} - \mu\right)T\right\}\left(1 + \operatorname{Erf}\left(\frac{(\mu-\sigma^2)\sqrt{T}}{\sigma\sqrt{2}}\right)\right. \\
&\quad + \frac{\sigma^2-\mu}{|\sigma^2-\mu|}\operatorname{Erf}\left(\frac{|\sigma^2-\mu|\sqrt{T}}{\sigma\sqrt{2}}\right) \\
&\quad \left.\left. + \frac{(2\mu-\sigma^2)T - \log(\frac{x_0}{Ky_0})}{\left|(2\mu-\sigma^2)T - \log(\frac{x_0}{Ky_0})\right|}\operatorname{Erf}\left(\frac{\left|(2\mu-\sigma^2)T - \log(\frac{x_0}{Ky_0})\right|}{2\sqrt{2}\sigma\sqrt{T}}\right)\right)\right],
\end{aligned} \tag{3.6}$$

where

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt$$

is the Error function, d is from (3.5) and

$$C = \frac{1}{X_M(0)(1 - F_X(d)) + KY_M(0)F_Y(d)}.$$

The calculated optimal weights and strikes along with their payoffs are written out in Table 3.2. The optimal payoff derived in Example 4 is

$$D_{KL}(\mathbb{P}^M || \mathbb{P}^Y) = \frac{1}{2} \frac{\mu^2}{\sigma^2} T = 0.1389.$$

In this case the Markowitz method gives very different weights than the other two and its payoff is compensated by setting a significantly higher strike of the contract, but the hierarchy of payoffs remains the same. The Fischer method holds up well. The weight discrepancies in the Markowitz case hint that this approach might not approximate the original problem well enough.

Remark 12. *With the contract as part of the portfolio we cannot plot the payoff as in Figure 3.1 anymore because the optimal strike depends on the time horizon T.*

Weights, Strike, and Payoff				
Method	w_1	w_2	**Strike**	**Payoff**
Numeric	0.937	0.84	1.023	0.138
Markowitz	0.758	1.12	1.681	0.137
Fischer	0.949	0.851	1.019	0.1375
Optimal				0.1389

Table 3.2: Calcuated weights, strikes, and payoffs for $T = 10,\ \mu = 0.05,\ \sigma = 0.3,\ x_0 = y_0 = 1$

3.2.1 Mean variance frontier

Another useful perspective can be gained by plotting the approximate portfolios on a mean-variance (volatility) graph. For simplicity we are going to limit ourselves to the 1 contract case. If we leave all parameters the same, the result is in Figure 3.3.

The points labeled "H,V,F" represent portfolios obtained by numerical optimization, Markowitz approach and Fischer approach respectively. "M" is the ideal portfolio that is impossible to reach precisely and "MH" is the maximal contract with optimal strike. The blue line depicts returns and volatilities of solely contracts as strike changes. The green line are portfolios consisting of only the contract and asset X, taking out Y. By including Y we get the yellow line that is tangent to the frontier excluding Y. We also included, in red, the approximation of the logarithmic return by the Markowitz method as

$$\mu \approx \frac{\sigma^2}{2} + \frac{1}{2} \boldsymbol{\mu}^T \Sigma_1^{-1} \boldsymbol{\mu},$$

where $\boldsymbol{\mu}$, Σ_1 are from 2.3. The agreement on volatility and the longer time horizon makes the plot not really transparent but all the portfolios are pretty close, which is not surprising given the closeness of fit in Figure 3.6.

Much more illustrative is Figure 3.4 where we work with $\mu = 0.05$, $\sigma_1 = 0.3$, $\sigma_2 = 0.35$, $x_0 = y_0 = 1$, $T = 1$. Here the Markowitz portfolio has higher returns than the other two but it also has much higher volatility. At the same time the numerical and Fischer portfolios lie on a very comparable (purple) parabola which makes those preferable over Markowitz. Of course the optimal portfolio is "H" and "V" and "F" are only approximations.

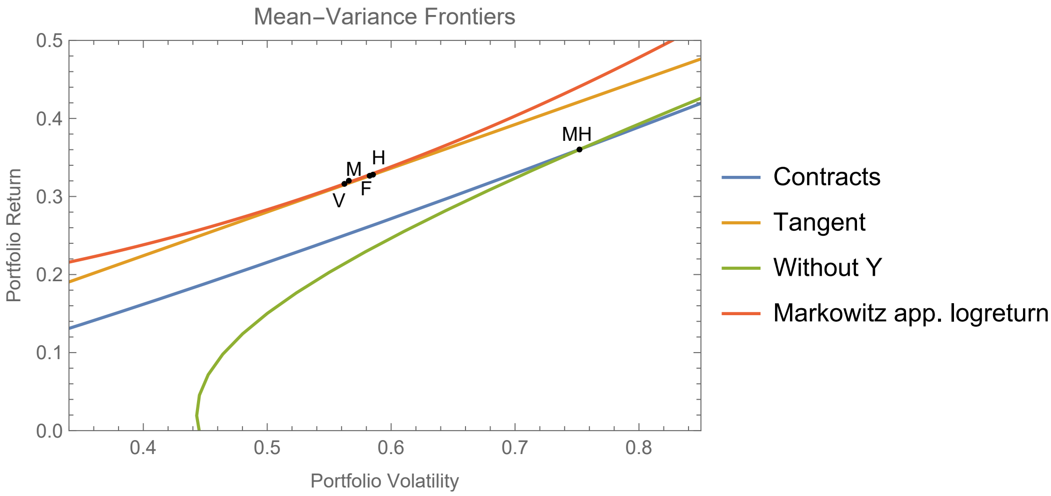

Figure 3.3: Approximate portfolios and particular mean variance frontiers for parameters $\mu = 0.05$, $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $x_0 = y_0 = 1$, $T = 10$.

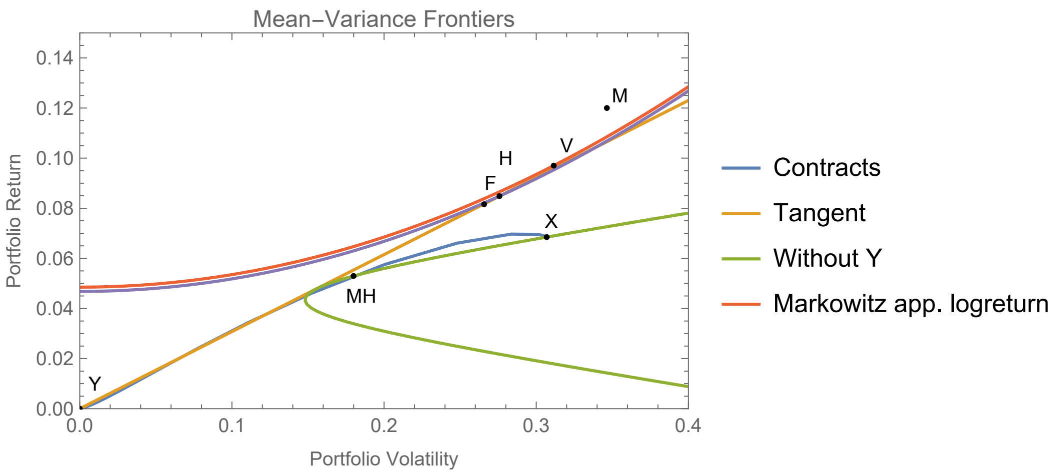

Figure 3.4: Approximate portfolios and particular mean variance frontiers for parameters $\mu = 0.05$, $\sigma_1 = 0.3$, $\sigma_2 = 0.35$, $x_0 = y_0 = 1$, $T = 1$.

3.3 Portfolio of 2 assets and two maximal contracts

Now we consider the general case where $\sigma_1, \sigma_2 > 0$ and a portfolio with two maximal contracts with two strike prices K_1, K_2:

$$P^4 = w_1 X + w_2 Y + w_3 O_{K_1} + (1 - w_1 - w_2 - w_3) O_{K_2}.$$

Without the loss of generality we are going to assume that $K_1 < K_2$ and take $\mu = 0.05,\ \sigma_1 = 0.3,\ \sigma_2 = 0.4,\ T = 10,\ x_0 = y_0 = 1.$

Again there is no point in typing out all the new entries and expressions of weights, see the attached notebook, but since we now do not assume $\sigma_1 = \sigma_2$, the calculation of the Σ matrix comes with constraints. For example the entry Σ_{11} is of the form

$$\Sigma_{11} = \mathbb{E}^M \left(\frac{p(x|X)}{p(x|M)} - 1 \right)^2 = \frac{\exp\left(-\frac{(\mu - \sigma_1^2)^2 T}{\sigma_1^2 - 2\sigma_2^2}\right)}{\sqrt{2\sigma_1^2 - \frac{\sigma_1^4}{\sigma_2^4}}}$$

and the expectation is real only for $0 < \sigma_1 < \sqrt{2}\sigma_2$. The same constraints are required for other entries as well.

The optimal payoff achievable only in a complete market is

$$D_{KL}(\mathbb{P}^M || \mathbb{P}^Y) \approx 0.24.$$

The resulting weights and strikes are summarized in Table 3.3. The numerical method surprisingly comes out to have lower return that Fischer. In the next Section we discuss that it has found a local maximum.

Note that assuming asset volatility to be lower than the market's allows for much higher expected return. This is intuitive because the portfolio has smaller volatility but is part of the market whose drift has not changed.

In the next section we will compare the approximated densities of the portfolios calculated in the last three sections with the desired market density.

Weights, Strikes, and Payoff						
Method	w_1	w_2	w_3	K_1	K_2	**Payoff**
Numeric	0.294	-1.478	2.184	2.411	4.498	0.221
Markowitz	0.405	-0.471	-0.059	0.1	1.388	0.21
Fischer	-2.121	-3.529	2.663	0.279	4.087	0.231
Optimal						0.24

Table 3.3: Calculated weights, strikes, and payoffs for $\mu = 0.05,\ \sigma_1 = 0.3,$ $\sigma_2 = 0.4,\ T = 10,\ x_0 = y_0 = 1.$

3.4 Approximated portfolios state price densities

Now lets compare state price densities of the three types of portfolios discussed in the previous sections. These densities (portfolios) should be, according to Theorem 5, close in the Kullback-Leibler divergence with the market density. Even though it is not the same as the relative entropy, the closeness of the fit is evident in Figures 3.5, 3.6 and 3.7.

The relative entropies of the portfolio densities are laid out in Table 3.4. We can see the Fischer approach outperforms Markowitz in every case and is even better than the standard optimization tools used in the two contracts case. This is probably because we are allowing shorting, i. e. negative weights and this makes the space of feasible solutions quite complicated, because not all combinations of weights result in non negative combination of densities on the whole real line. In order to find truly optimal attainable values, more sophisticated methods, beyond the scope of this thesis, would need to be employed.

From Table 3.4 of Figure 3.7 we observe that assuming different volatilities $\sigma_1,\ \sigma_2$, of $\mathbb{P}^X$ and $\mathbb{P}^M$ slightly increases the relative entropy. Intuitively it is clear, we are trying to 'match' both parameters $\mu,\ \sigma_2$ instead of just the drift as in the univolatility case.

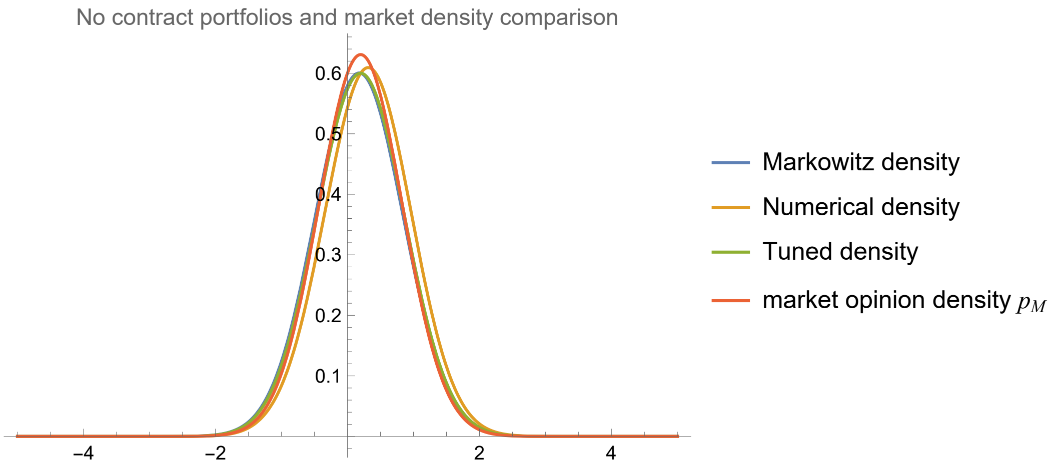

Figure 3.5: State price densities of 2 asset portfolios comparison with market opinion with parameters $\mu = 0.03,\ \sigma = 0.2,\ x_0 = y_0 = 1,\ T = 10$.

	No Contract	**1 Contract**	**2 Contracts**
Markowitz	0.001809	0.0021	0.0405
Fischer	0.001262	0.00137	0.008
Numeric	0.001261	0.00136	0.0222

Table 3.4: Kullback-Leibler divergences of calculated portfolios

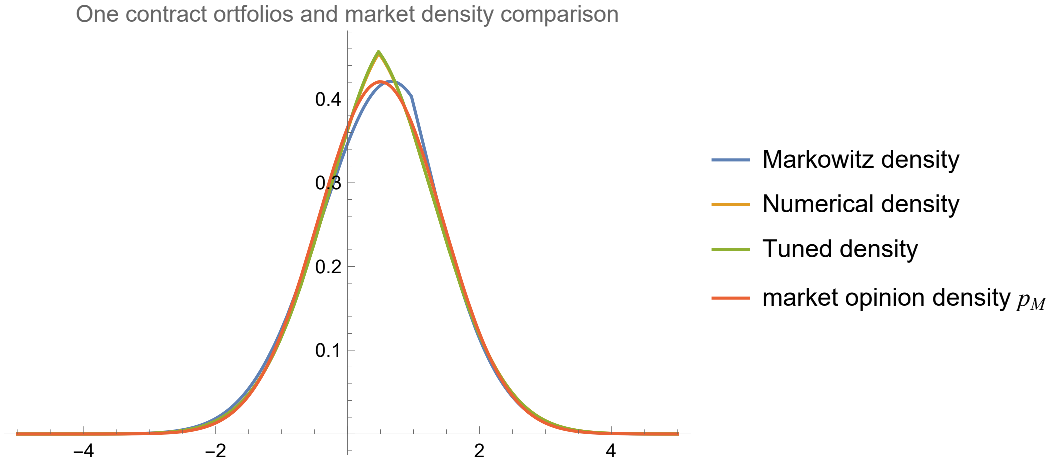

Figure 3.6: State price densities comparison of 2 asset and 1 maximal contract approximated portfolios against market opinion with parameters $\mu = 0.05,\ \sigma = 0.3,\ x_0 = y_0 = 1, T = 10$.

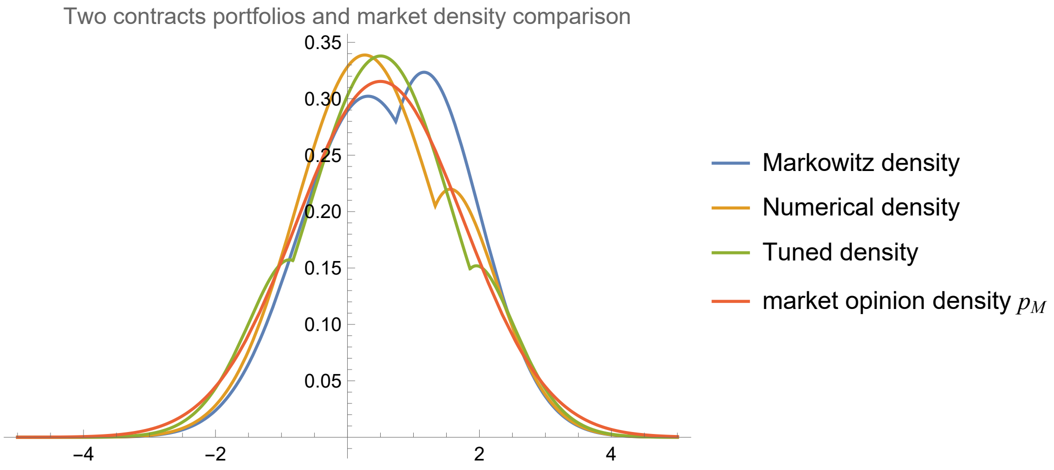

Figure 3.7: State price densities of 2 asset and 2 maximal contracts approximated portfolios against market opinion with parameters $\mu = 0.05,\ \sigma_1 = 0.2,$
$\sigma_2 = 0.3,\ x_0 = y_0 = 1,\ T = 10$.

4. Option strategies

By having options (or maximal contracts) in the portfolio, besides using them to maximize the expected log payoff, we can also utilize them in certain market scenarios, such as high or low volatility or negative return. These strategies are profitable given our opinion is more accurate than market's. In the previous chapter we calculated portfolios with approximately the highest expected logarithmic returns. Now we are going to showcase strategies involving one or two options in these (close to) optimal portfolios. Without the loss of generality we can work with portfolios with maximal contracts because these can be transformed into standard options by adding assets X or Y, as indicated in (3.3). The payoff function and option price (in Y) are then just shifted corresponding maximal contract payoff function and price. In the case of two options portfolio we will need to add a constraint that the options weights are the same for the strategies to work properly.

Of the two approximations from Chapter 3, the Fischer method seems to be the most accurate so we are going to use it exclusively to find the optimal strikes that affect the profit/loss of the option strategies.

As in the previous chapter we are going to assume the geometric brownian motion model with assets X and Y. A classical way to determine a price of a call option expiring at time T with strike price K that has payoff

$$V_Y(T) = \max\{0, X_Y(T) - K\} = \max\{X_Y(T), K\} - K$$

is the Black-Scholes formula (e. g. Večeř [2011], equation (1.77))

$$V_Y(0) = \mathbb{P}^X(X_Y(T) \geq K)X_Y(0) - K \cdot \mathbb{P}^Y(X_Y(T) \geq K) \tag{4.1}$$

and in the GBM model it is straight forward to calculate

$$V_Y(0) = X_Y(0)\Phi(d+) - K \cdot \Phi(d-)$$

where Φ is the distribution function of $\mathcal{N}(0,1)$ and

$$d\pm = \frac{1}{\sigma_1\sqrt{T}} \log\left(\frac{X_Y(0)}{K}\right) \pm \frac{1}{2}\sigma_1\sqrt{T}.$$

A put option has payoff

$$U_Y(T) = \max\{0, K - X_Y(T)\} = \max\{K, X_Y(T)\} - X_Y(T) = V_Y(T) + K - X_Y(T)$$

so its price at time 0 is

$$U_Y(0) = V_Y(0) + K - X_Y(0).$$

4.1 One option portfolio

With one option in the portfolio we can hedge against the market moving heavily into either direction. By buying call or put options we can speculate on the market movement in either direction, that would correspond to $\mu < 0$ or $\mu > 0$.

Lets look at the bearish case. Take $T = 10,\ \mu = -0.05,\ \sigma_1 = 0.3,\ \sigma_2 = 0.3$ and $X_Y(0) = 1$, then the optimal strike is $K = 0.677$ and the plot of Profit/Loss against the price $X_Y(T)$ is shown in Figure 4.1.

If we think that the market is more volatile, i. e. $\sigma_1 > \sigma_2$, we can buy the same amount of calls and puts with the same strike price and we will profit in case of high volatility. For $T = 10,\ \mu = 0.05,\ \sigma_1 = 0.3,\ \sigma_2 = 0.22$ and $X_Y(0) = 1$, by the same method as in previous chapter, we get optimal $K = 1.173$ and we can plot the Profit/Loss against the terminal price $X_Y(T)$ in Figure 4.2. This strategy is well known and fittingly called Straddle.

For simplicity the plots show the P/L in case of holding one put (Figure 4.1) or a put and call pair (Figure 4.2), in the context of our our portfolio it would just be scaled differently depending on the weight of the contract.

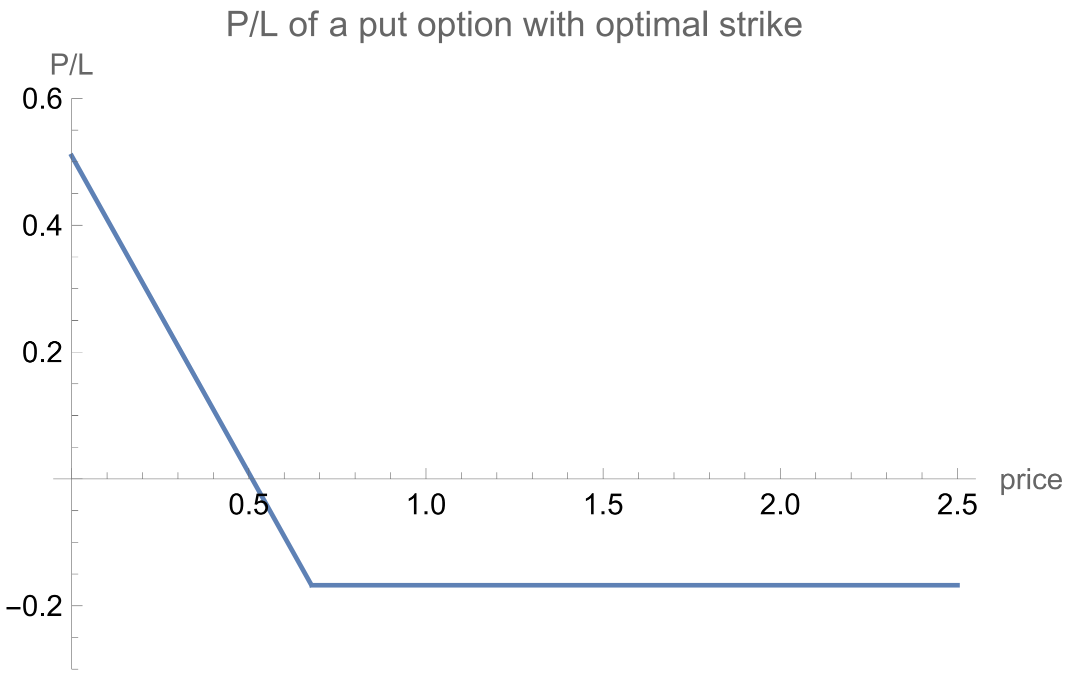

Figure 4.1: P/L of a put option with parameters $\mu = -0.05,\ \sigma_1 = 0.3,$ $\sigma_2 = 0.3,\ K = 0.677,\ T = 10.$

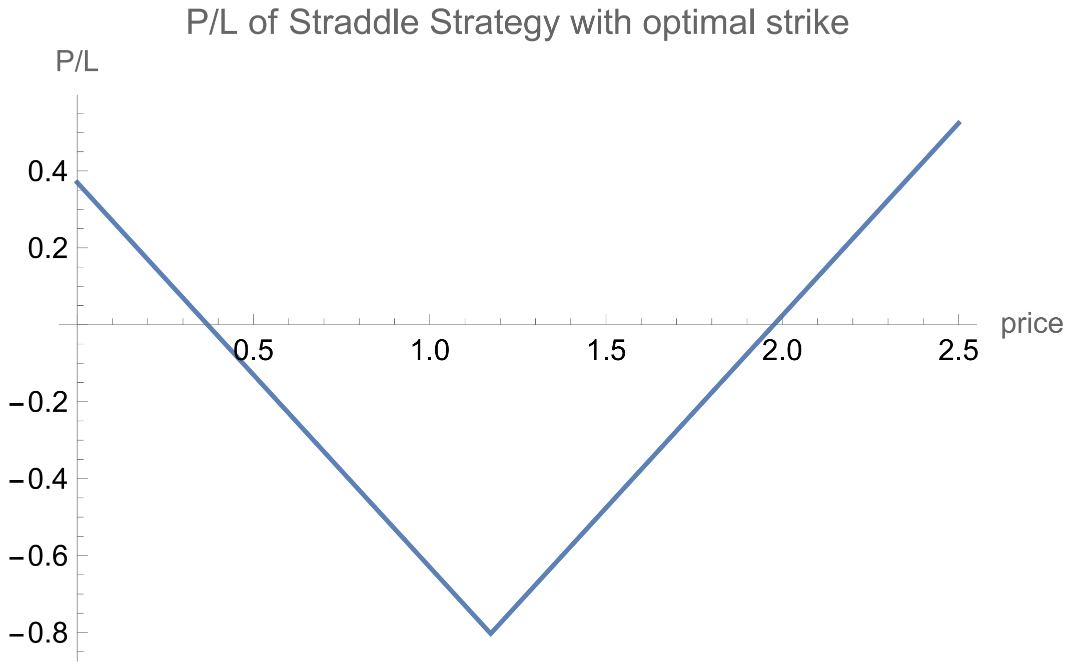

Figure 4.2: P/L of Straddle strategy with parameters $\mu = 0.05,\ \sigma_1 = 0.3,$ $\sigma_2 = 0.22,\ K = 1.173,\ T = 10.$

4.2 Two options portfolio

Adding a second option to the portfolio allows us to bet on the drift of the market while limiting the downside to the option premium. For the hedge against risk to work properly, we will add a constraint that both options in the portfolio have the same weight. More precisely, the portfolio is of the form

$$P^4 = w_1 X + (1 - w_1 - 2w_2)Y + w_2(O_{K_1} + O_{K_2}).$$

All the calculations as described in Chapter 2 remain the same.

Let us first assume the drift is positive, $\mu = 0.05 > 0$ and $\sigma_1 = \sigma_2 = 0.3$, $T = 10$, then the optimal strikes are

$$K_1 = 0.46,\ K_2 = 2.11.$$

If we buy a call with strike K_1 and sell a call with strike K_2, the P/L plot looks like in Figure 4.3. This strategy is known as the Bullish Spread.

After setting $\mu = -0.05 < 0$ and leaving the rest of the parameters the same, we calculate

$$K_1 = 0.34,\ K_2 = 0.84.$$

Buying a put with strike K_2 and selling a put with strike K_1 ensures profit given sufficient price decrease while risking ”only” the option price as is shown in Figure 4.4. This strategy is called the Bearish Spread.

More complex strategies are possible in portfolios with 3 or more options with different strikes. Principally it is not different than the two options case so we are not going to explore these cases further.

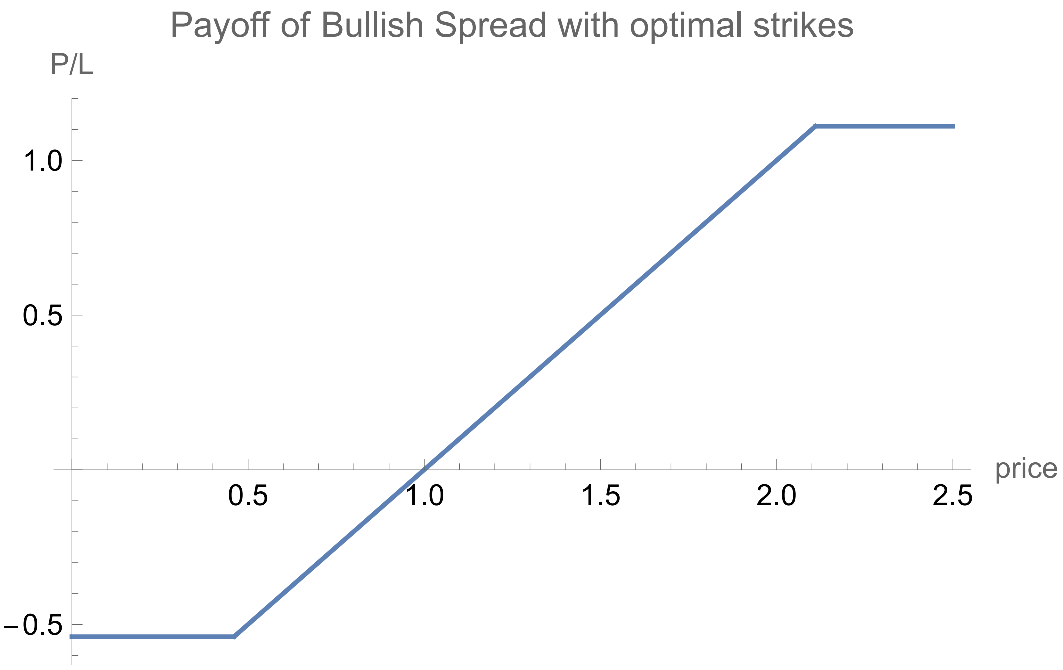

Figure 4.3: P/L of a Bullish Spread strategy with parameters $\mu = 0.05$, $\sigma_1 = \sigma_2 = 0.3,\ K_1 = 0.46,\ K_2 = 2.11,\ T = 10.$

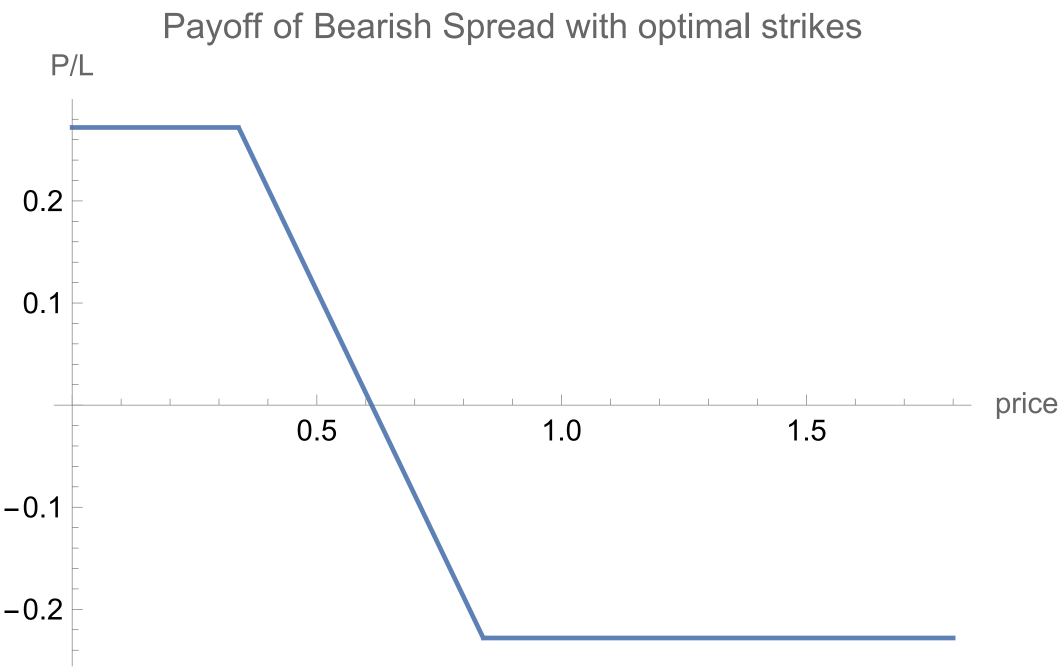

Figure 4.4: P/L of a Bearish Spread strategy with parameters $\mu = -0.05$, $\sigma_1 = \sigma_2 = 0.3,\ K_1 = 0.46,\ K_2 = 2.11,\ T = 10.$

4.2.1 Recovering the state price density from a P/L function

Given a P/L function of a combination of contracts, using Theorem 1 we can calculate the corresponding state price density. Let $p(\omega|S)$ be a state price density of the contract combination, then the profit/loss at time T is equal to

$$S_Y(T) - S_Y(0) = \frac{p(\omega|S)}{p(\omega|Y)} S_Y(0) - S_Y(0) = \frac{p(\omega|S) - p(\omega|Y)}{p(\omega|Y)} S_Y(0),$$

on the other hand the P/L is a linear combination of, sold or bought, put and call options with different strikes $K_1, ..., K_n, n \in \mathbb{N}$ and combined payoff

$$\sum_{i=1}^{n} a_i \max\{X_Y(T) - K_i, 0\}, \ a_i \in \{-1, 1\}.$$

Therefore

$$p(\omega|S) = \sum_{i=1}^{n} \max\{p(\omega|X)X_Y(0) - p(\omega|Y)K_i, 0\} + p(\omega|Y)S_Y(0).$$

$S_Y(0)$ is our initial cost of the contracts, i. e. sum of the option prices with their sign depending on whether we sold or bought them.

Example 5. *Consider the Bullish spread from the current section with the same parameters. Then we have*

$$\frac{p(\omega|S) - p(\omega|Y)}{p(\omega|Y)} S_Y(0) = \max\{X_Y(T) - K_1, 0\} + \min\{K_2 - X_Y(T), 0\},$$

therefore the state price density of the Bullish spread is

$$\begin{aligned} p(\omega|S) &= \max\{X_Y(0) \cdot p(\omega|X) - K_1 \cdot p(\omega|Y), 0\} \\ &+ \min\{K_2 \cdot p(\omega| \, Y) - X_Y(0) \cdot p(\omega|X), 0\} \\ &+ S_Y(0) \cdot p(\omega|Y) \end{aligned}$$

$S_Y(0)$ in this case is the difference between premiums of call options with optimal strikes 0.46 and 2.11, which, according to the Black-Scholes formula (4.1) comes out to 0.451. Plot of the density is in Figure 4.5.

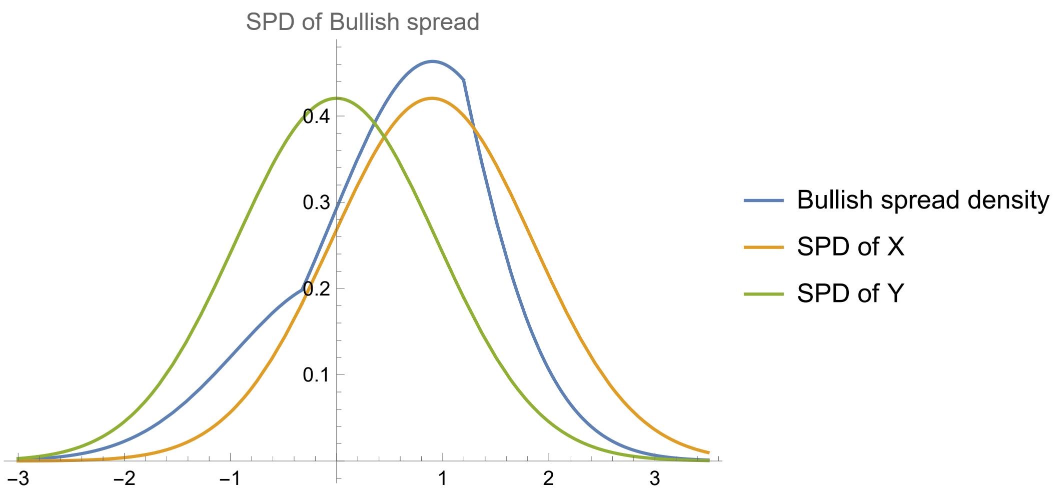

Figure 4.5: State price density of a bullish spread with $\mu = 0.05$, $\sigma = 0.3$, $T = 10$, $K_1 = 0.46$, $K_2 = 2.11$.

Conclusion

In the thesis we first showed that a price of an asset expressed in terms of another asset is just scaled likelihood ratio of their state price densities. From this fact and standard utility maximization techniques it follows that prices are naturally log-optimal and logarithm is the only utility (up to scaling and shifting) that is numeraire invariant, meaning the weights of the log-optimal portfolio do not change with the underlying asset.

Chapter 2 was dedicated to the connection between log utility maximization and Kullback-Leibler divergence. We showed that the maximal logarithmic payoff of a portfolio in terms of an asset $\mathbb{P}^Y$ in a market with physical measure $\mathbb{P}^M$ is exactly $D_{KL}(\mathbb{P}^M||\mathbb{P}^Y)$. This is however achievable only in a complete market, otherwise the best we can do is to find a portfolio with smallest relative entropy with the physical measure, which leads to a problem that does not have an analytical solution and may not be numerically feasible for large portfolios. For this purpose we presented two alternative approaches based on approximating the logarithm by its second order Taylor polynomial, resulting in mean variance type optimization problems, one similar to the one firstly addressed by Markowitz [1952] and the other related to Fischer information. Those have analytical solutions and are generally easier to calculate.

The accuracy of these methods was then demonstrated on a simple model with two assets with the same volatility, one of them driftless, when their prices follow the geometric brownian motion, and one or two maximal contracts (options) based on those assets. The physical measure was considered to have a different drift and also volatility than the other two, which is a generalization of the setup in Merton [1971], who considered all assets to have the same volatility. On the three different combinations of volatilities and drifts, we showed that the approximations, especially Fischer's, work quite well.

Finally the benefits, other than achieving smaller relative entropy with $\mathbb{P}^M$, of having the contracts in the portfolio when certain market scenarios occur were discussed.

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A. Attachments

All calculations and plots throughout the thesis have been done in the Mathematica notebook called ”calculations.nb”.