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Fine properties of functions and operators

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Abstract: We establish the equivalence between the boundedness of certain supremum operators and optimal spaces in Sobolev embeddings. We do this by exploiting known relations between higher-order Sobolev embeddings and isoperimetric inequalities. We provide an explicit way to compute both the optimal domain norm and the optimal target norm in a Sobolev embedding. Finally, we apply our results to higher-order Sobolev embeddings on John domains and on domains from the Maz'ya classes. Furthermore, our results are partially applicable to embeddings involving product probability spaces.

Keywords: isoperimetric function, optimal spaces, rearrangement-invariant spaces, Sobolev embeddings, supremum operators

Contents

Introduction **2**

1 Preliminaries **4**
1.1 Rearrangement invariant spaces 4
1.2 Sobolev spaces over r.i. spaces and isoperimetric function 9
1.3 Interpolation theory 11
1.4 Weighted inequalities involving suprema 14

2 Supremum operators **16**

3 Optimal spaces **24**
3.1 Optimal target space 24
3.2 Alternative norm in the optimal target space 32
3.3 Optimality and supremum operators 37

4 Examples **42**
4.1 Comparison of conditions 42
4.2 Higher-order Sobolev embeddings 44

Bibliography **47**

Introduction

Sobolev embeddings have been closely studied for several decades, ever since the period in which they were pioneered in the works of Gagliardo [6], Sobolev [19, 20] and Nirenberg [16]. Isoperimetric inequalities were explored by De Giorgi [4] and Federer and Fleming [5]. Some time later, Sobolev embeddings have been connected with isoperimetric inequalities in the works of Maz'ya [11, 12]. The study of the mentioned connection has been rapidly developing ever since, and in 2015 a comprehensive paper [2] presented a unified approach to the topic, and moreover, managed to find the optimal target spaces in the Sobolev embeddings on fairly general underlying measure spaces.

The aim of the thesis is to exploit the equivalence of Sobolev embeddings and isoperimetric inequalities in order to connect Sobolev embeddings with boundedness of two supremum operators S_I and T_I, which are formally introduced in Definition 2.2, in manner similar to that in [9] and [10], where I is the isoperimetric function of the domain. It turns out that the boundedness of S_I is related to the optimal domain spaces in the Sobolev embedding, and T_I corresponds to the optimal target spaces in the Sobolev embedding. We prove the intimate relation between Sobolev embeddings and the action of supremum operators under the assumptions of Theorem 3.13, which cover important Maz'ya class of domains $\mathcal{J}_\alpha$ for $\alpha \in \left[\frac{1}{n'}, 1\right)$. As the results rely on the boundedness of the Hardy-type operator $f \mapsto \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$, they are not generally applicable to $\mathcal{J}_1$, as the function $\frac{1}{I}$ is not integrable near zero. We in fact show that the norm of the optimal target space cannot be expressed in a manner stated in Theorem 3.13 for certain functions $I \in \mathcal{J}_1$ which is contained in a discussion after the mentioned theorem. Nonetheless, many results only require a certain condition regarding the function I, namely

$$\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \lesssim I(t), \quad t \in (0,1). \tag{1}$$

We show that this condition is always satisfied in the product probability spaces. Using this condition we show the boundedness of the operator T_I on the associate spaces of the optimal target spaces. However, in order to prove the analogous result for the operator S_I, we further require a condition similar to (1), namely

$$\int_0^t \frac{\mathrm{d}s}{I(s)} \lesssim \frac{t}{I(t)}, \quad t \in (0,1). \tag{2}$$

It is easy to see that this condition enforces integrability of $\frac{1}{I}$ and thus more restraining than (1) in this way.

The boundedness of T_I and S_I was previously studied for example in [8]. There, they are studied in the context of Orlicz L_A and Gamma $\Gamma_{p,\phi}$ spaces. Hence, once the equivalence of the Sobolev embeddings with boundedness of T_I and S_I is established, it allows us to partially recover and extend such results.

The work consists of four chapters. The first and preliminary chapter covers background results and is divided into four sections. First we recall the notion of the nonincreasing rearrangement and so-called rearrangement-invariant Banach function spaces, which will form our main framework. It will, however, be necessary to delve a bit deeper and work with quasi-Banach function spaces, such as the weak Lebesgue space $L^{1,\infty}$. The second section covers Sobolev spaces built upon rearrangement-invariant spaces and their connection to isoperimetric inequalities. Third section is devoted to the interpolation theory and, in particular, to the theory of the K-functional. The fourth and last section contains the characterization of boundedness of a general supremum operator on a weighted Lebesgue space.

In the second chapter we define supremum operators S_I and T_I and study their basic properties. The setting here will be that I is an increasing concave bijection of $(0, 1)$ onto itself. It is mainly due to operator S_I, and the Marcinkiewicz type space m_I, that we are forced to work with quasi-Banach function spaces. However, the condition (2) characterizes when the m_I is in fact a Banach space, and implies subaditivity of S_I. The end of the chapter then calls into play the condition (1) and we show its equivalence to two other statements, which will play a crucial role in the main, third chapter.

The third chapter finally connects optimal spaces with the boundedness of supremum operators. In its first section we present an alternative description of the associate optimal norm via a functional which admits boundedness of the operator S_I (Theorem 3.4). This is in turn used to describe the optimal target norm. Starting with the second section, we find an alternative description of the optimal target norm under the assumption of boundedness $f \mapsto \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$. The culmination of the chapter is then the third section which establishes equivalence between optimal spaces and boundedness of T_I or S_I on their associate spaces.

The fourth and final chapter summarises the conditions which have been used throughout the thesis. We show that the product probability spaces satisfy the main condition (1). We then translate Theorem 3.18 into examples.

1. Preliminaries

1.1 Rearrangement invariant spaces

Let (Ω, μ) be a nonatomic σ-finite measure space. We set

$$\mathcal{M}(\Omega, \mu) = \{f \colon \Omega \to [-\infty, \infty] \colon f \text{ is } \mu\text{-measurable in } \Omega\},$$
$$\mathcal{M}_+(\Omega, \mu) = \{f \in \mathcal{M}(\Omega, \mu) \colon f \geq 0\}$$

and

$$\mathcal{M}_0(\Omega, \mu) = \{f \in \mathcal{M}(\Omega, \mu) \colon f \text{ is finite } \mu\text{-a.e. in } \Omega\}.$$

We will often, for brevity, write only $\mathcal{M}(\Omega)$ if there is no risk of confusion, and similarly for the other two sets. When $\Omega \subset \mathbb{R}$ is measurable, unless stated otherwise, we will consider the Lebesgue measure which we will be denoted by λ. When considering a unit interval $(0, 1)$, which will be of particular interest to us, we will simply write $\mathcal{M}(0, 1)$.

Given $f \in \mathcal{M}(\Omega)$, we define the *distribution function* of f, denoted f_*, by

$$f_*(\lambda) = \mu(\{|f| > \lambda)\}), \quad \lambda \in [0, \infty). \tag{1.1}$$

Distribution function of a measurable function is a nonnegative, nonincreasing and right-continuous function on $[0, \infty)$ [1, Chapter 2, Proposition 1.3]. Given $f, g \in \mathcal{M}(\Omega)$, we say that they are *equimeasurable* if $f_* = g_*$ and write $f \sim g$.

Given $f \in \mathcal{M}(\Omega)$, we define its *nonincreasing rearrangement*, denoted f^*, by

$$f^*(t) = \inf\{\lambda \geq 0 \colon f_*(\lambda) \leq t\}, \quad t \in [0, \infty). \tag{1.2}$$

As the nonincreasing rearrangement plays a crucial role in the thesis, we list the basic properties of the nonincreasing rearrangement, the proof of which can be found in [1, Chapter 2, Proposition 1.7].

Fact 1.1. *Let $f, g, f_n \in \mathcal{M}(\Omega), n \in \mathbb{N}, a \in \mathbb{R}$ and $0 < p < \infty$. Then f^* is a nonnegative, nonincreasing and right-continuous function on $[0, \infty)$ and the following holds:*

(i) $(af)^* = |a|\, f^*$,

(ii) $|f| \leq |g|$ *μ-a.e.* $\implies f^* \leq g^*$.

(iii) $|f| \leq \liminf |f_n|$ *μ-a.e.* $\implies f^* \leq \liminf f_n^*$, *in particular,* $|f_n| \nearrow |f|$ *μ-a.e.* $\implies f_n^* \nearrow f^*$,

(iv) if f_* is decreasing and continuous, then $f^* = (f_*)^{-1}$,

(v) $f \sim f^*$,

(vi) $(|f|^p)^* = (f^*)^p$,

(vii) $(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2), \quad t_1, t_2 > 0.$

The nonincreasing rearrangement satisfies *Hardy-Littlewood inequality* [1, Chapter 2, Theorem 2.2]

$$\int_\Omega |f(x)g(x)| \,\mathrm{d}\mu(x) \le \int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t, \quad f,g \in \mathcal{M}(\Omega). \tag{1.3}$$

In particular,

$$\int_E |f(x)| \,\mathrm{d}\mu(x) \le \int_0^{\mu(E)} f^*(t)\,\mathrm{d}t, \quad f \in \mathcal{M}(\Omega), E \subset \Omega \text{ measurable.}$$

As Fact 1.1 suggests, $f \mapsto f^*$ need not be subaditive in the sense that $(f+g)^* \le f^* + g^*$ and only satisfies the weaker condition *(vii)*. It turns out that passing from f^* to its Hardy average, which we will call the *maximal nonincreasing rearrangement*, defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad f \in \mathcal{M}(\Omega), t \in (0,\infty), \tag{1.4}$$

we gain subaditivity. To be precise, it holds that $(f+g)^{**} \le f^{**} + g^{**}$ for $f,g \in \mathcal{M}(\Omega)$.

As the maximal nonincreasing rearrangement will, too, be of great importance, we list its properties. The proof can be found in [1, Chapter 2, Proposition 3.2] and in [1, Chapter 2, Theorem 3.4].

Fact 1.2. *Let $f, g, f_n \in \mathcal{M}(\Omega), n \in \mathbb{N}$, and $a \in \mathbb{R}$. Then f^{**} is nonnegative, nonincreasing and continuous on $(0,\infty)$, and the following holds:*

(i) $f^{**} \equiv 0 \iff f = 0$ *μ-a.e.,*

(ii) $f^* \le f^{**}$,

(iii) $|f| \le |g|$ *μ-a.e.* $\implies f^{**} \le g^{**}$,

(iv) $(af)^{**} = |a|\, f^{**}$,

(v) $|f_n| \nearrow |f|$ *μ-a.e.* $\implies f_n^{**} \nearrow f^{**}$,

(vi) $(f+g)^{**} \le f^{**} + g^{**}$.

We are finally ready to define the notion of a rearrangement-invariant (r.i.) Banach function norm.

Definition 1.3. A mapping $\rho\colon \mathcal{M}_+(0,1) \to [0,\infty]$ is called *rearrangement invariant Banach function norm*, or r.i. norm for short, if it satisfies the following conditions:

(P1) $\rho(f) = 0 \iff f = 0$ a.e.,
$\rho(af) = a\rho(f), \quad f \in \mathcal{M}_+(0,1), a \ge 0,$
$\rho(f+g) \le \rho(f) + \rho(g), \quad f,g \in \mathcal{M}_+(0,1),$

(P2) $f \le g$ a.e. $\implies \rho(f) \le \rho(g), \quad f,g \in \mathcal{M}_+(0,1),$

(P3) $f_n \nearrow f$ a.e. $\implies \rho(f_n) \nearrow \rho(f), \quad f, f_n \in \mathcal{M}_+(0,1), n \in \mathbb{N},$

(P4) $\rho(\chi_{(0,1)}) < \infty,$

(P5) $\int_0^1 f(t)\,\mathrm{d}t \lesssim \rho(f), \quad f \in \mathcal{M}_+(0,1)$,

(P6) $\rho(f) = \rho(f^*), \quad f \in \mathcal{M}_+(0,1)$.

Sometimes we will work with a functional which is not a norm, but still satisfies rearrangement invariance – so-called r.i. quasinorm.

Definition 1.4. A mapping $\rho\colon \mathcal{M}_+(0,1) \to [0,\infty]$ is called *rearrangement-invariant quasi-Banach function norm*, or r.i.q. norm for short, if it satisfies conditions (P2), (P3), (P4), (P6) and

(Q1) $\rho(f) = 0 \iff f = 0$ a.e.,
$\rho(af) = a\rho(f), \quad f \in \mathcal{M}_+(0,1), a \geq 0$,
$\exists C \geq 1\ \rho(f+g) \leq C\rho(f) + \rho(g), \quad f, g \in \mathcal{M}_+(0,1)$.

When ρ is an r.i. quasinorm, we define its *associate functional*, ρ', by

$$\rho'(f) = \sup_{\rho(g)\leq 1} \int_0^1 f(t)g(t)\,\mathrm{d}t, \quad f \in \mathcal{M}_+(0,1). \tag{1.5}$$

An immediate consequence of the definition of the associate functional is *Hölder's inequality*

$$\int_0^1 f(t)g(t)\,\mathrm{d}t \leq \rho(f)\rho'(g), \quad f, g \in \mathcal{M}_+(0,1), \tag{1.6}$$

under the convention $0 \cdot \infty = 0$ on the right-hand side.

By [1, Chapter 1, Theorem 2.7] and [1, Chapter 2, Proposition 4.2] if ρ is an r.i. norm, its associate norm ρ' is an r.i. norm as well and obeys the *principle of duality*, that is,

$$\rho'' := (\rho')' = \rho. \tag{1.7}$$

Given $f, g \in \mathcal{M}_+(0,1)$, *Hardy's lemma* [1, Chapter 2, Proposition 3.6] asserts that

$$f^{**}(t) \leq g^{**}(t),\ t \in (0,1) \implies \int_0^1 f^*(t)h(t)\,\mathrm{d}t \leq \int_0^1 g^*(t)h(t)\,\mathrm{d}t \tag{1.8}$$

for every $h \in \mathcal{M}_+(0,1)$ nonincreasing. An important consequence of Hardy's lemma and the principle of duality is the *Hardy-Littlewood-Pólya (HLP) principle* [1, Chapter 2, Theorem 4.6], which reads as follows:

$$f^{**}(t) \leq g^{**}(t),\ t \in (0,1) \implies \rho(f) \leq \rho(g) \tag{1.9}$$

whenever ρ is an r.i. norm.

For an r.i.q norm ρ we further define $X = X(\rho)$ as a collection of all $f \in \mathcal{M}(0,1)$ such that $\rho(|f|) < \infty$. Equipping X with a quasinorm defined by $\|f\|_X := \rho(|f|)$ for $f \in X$, we immediately see that $X = (X, \|\cdot\|_X)$ is a quasinormed linear space. By [1, Chapter 1, Theorem 1.6] and [15, Corollary 3.8], $(X, \|\cdot\|_X)$ is a complete metric space, and spaces defined in this manner are called *rearrangement-invariant quasi-Banach function spaces* or, as we will often say for brevity, *r.i.q. spaces*. If ρ is in fact an r.i. norm, the space $X = X(\rho)$ is called *rearrangement-invariant Banach function space* or briefly r.i. space. By X' we denote the space corresponding to ρ' and call it the *associate space* of X.

By X_b we denote the closure of simple functions in the space X.

The *fundamental function* corresponding to an r.i.q. space X, φ_X, is defined by

$$\varphi_X(t) = \|\chi_{(0,t)}\|_X, \quad t \in (0,1). \tag{1.10}$$

The fundamental function of r.i. space X satisfies [1, Chapter 2, Theorem 5.2]

$$\varphi_X(t) \cdot \varphi_{X'}(t) = t, \quad t \in (0,1).$$

A corollary to this is that if X is an r.i. space, then φ_X is a *quasiconcave* function, that is

$$t \mapsto \varphi_X(t) \text{ is nondecreasing}, \qquad t \mapsto \frac{\varphi_X(t)}{t} \text{ is nonincreasing}$$

and $\varphi_X(t) = 0 \iff t = 0$ [1, Chapter 2, Corollary 5.3].

Whenever φ is a quasiconcave function, by [1, Chapter 2, Proposition 5.10] there exists its *least concave majorant*, say $\tilde{\varphi}$, which satisfies

$$\frac{1}{2}\tilde{\varphi}(t) \le \varphi(t) \le \tilde{\varphi}(t), \quad t \in (0,1).$$

Furthermore, every r.i. space X can be equivalently renormed so that φ_X is a concave function [1, Chapter 2, Proposition 5.11] – we will from now on assume that every r.i. space has been renormed in this fashion.

We write $A \lesssim B$ if A is dominated by a constant multiple of B, independent of all quantities involved; these quantities will usually be evident from the context. By $A \approx B$ we mean that both $A \lesssim B$ and $A \gtrsim B$.

Let now X and Y be two r.i.q. spaces. We write $X \subset Y$ if $f \in X \Rightarrow f \in Y$. When T is an operator on $\mathcal{M}_+(0,1)$, we say that T is *bounded* from X to Y if

$$\|Tf\|_Y \lesssim \|f\|_X, \quad f \in X, \tag{1.11}$$

and denote this fact by $T\colon X \to Y$. If $X = Y$, we say that T is bounded on X. In the particular case when $T = Id$, an inclusion operator, we have [1, Chapter 1, Theorem 1.8] and [15, Corollary 3.10]

$$X \subset Y \iff Id\colon X \to Y. \tag{1.12}$$

In other words, inclusions between r.i.q. spaces are always continuous. The fact that $Id\colon X \to Y$ will be denoted as $X \hookrightarrow Y$.

We say that an operator T' on $\mathcal{M}_+(0,1)$ is an *associate operator* of T if

$$\int_0^1 (Tf)(t)g(t)\,\mathrm{d}t = \int_0^1 f(t)(T'g)(t)\,\mathrm{d}t, \quad f,g \in \mathcal{M}_+(0,1). \tag{1.13}$$

For two r.i. spaces X and Y one sees that

$$T\colon X \to Y \iff T'\colon Y' \to X' \tag{1.14}$$

and $\|T\| = \|T'\|$.

For $s > 0$ the dilation operator E_s defined for $f \in \mathcal{M}(0,1)$ by

$$(E_s f)(t) = f\left(\frac{t}{s}\right)\chi_{(0,\min\{s,1\})}(t) \quad t \in (0,1). \tag{1.15}$$

It is proved in [1, Chapter 3, Proposition 5.11] for r.i. spaces and, more generally, in [15, Theorem 3.23] for r.i.q. spaces, that E_s is bounded on every r.i.q. space.

Definition 1.5. Let T be an operator on $\mathcal{M}_+(0,1)$ and X and Y be r.i. spaces. We say that Y is an *optimal target space* for X under the mapping T, if $T\colon X \to Y$ and for every r.i. space Z the following implication holds:

$$T\colon X \to Z \implies Y \hookrightarrow Z. \tag{1.16}$$

Vice versa, we say that X is an *optimal domain space* for Y under the mapping T, if $T\colon X \to Y$ and for every r.i. space Z the following implication holds:

$$T\colon Z \to Y \implies Z \hookrightarrow X. \tag{1.17}$$

In the main chapter we will use the *level function* which is closely related to the nondecreasing rearrangement.

Definition 1.6. Let $f \in M_+(0,1)$. Then the level function of f, denoted f°, is the derivative of the least concave majorant of $t \mapsto \int_0^t f(s)\,\mathrm{d}s, t \in (0,1)$.

Take note that, as f^* is nonincreasing, $t \mapsto \int_0^t f^*(s)\,\mathrm{d}s$ is a concave function, and so

$$\int_0^t f^\circ(s)\,\mathrm{d}s \le \int_0^t f^*(s)\,\mathrm{d}s, \quad f \in \mathcal{M}_+(0,1), t \in (0,1). \tag{1.18}$$

G. Sinnamon proved in [18, Corollary 2.4] that

$$\|f^\circ\|_{X'} = \|f\|_{X'_d}, \quad f \in \mathcal{M}_+(0,1). \tag{1.19}$$

Here, $\|\cdot\|_{X'_d}$ refers to the *down dual associate norm* of an r.i. space X, which is defined by

$$\|f\|_{X'_d} = \sup_{\|g\|_X \le 1} \int_0^1 f(t) g^*(t)\,\mathrm{d}t, \quad f \in \mathcal{M}_+(0,1). \tag{1.20}$$

Evidently $\|f\|_{X'_d} \le \|f\|_{X'}$ for every $f \in \mathcal{M}_+(0,1)$. Observe, however, that $\|f\|_{X'_d} = \|f\|_{X'}$ whenever f is nonincreasing.

Definition 1.7. Let $I\colon (0,1) \to (0,\infty)$ be a function. We say that I satisfies Δ_2 condition if

$$I(2t) \approx I(t), \quad t \in \left(0, \frac{1}{2}\right),$$

and denote this fact as $I \in \Delta_2$.

Classical examples of r.i. spaces would be *Lebesgue's* $L^p(0,1)$ spaces, where $1 \le p \le \infty$, whose norm is defined by

$$\|f\|_p = \left(\int_0^1 |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \tag{1.21}$$

if $1 \le p < \infty$ and

$$\|f\|_\infty = \operatorname{ess\,sup} |f|\,. \tag{1.22}$$

We use the convention that $\frac{1}{\infty} = 0 \cdot \infty = 0$. Defining $p' = \frac{p}{p-1}$ for $p \in [1,\infty]$, one has $(L^p)' = L^{p'}$.

Classical examples of r.i.q. spaces, which are not normed nor embedded in L^1, are Lebesgue's L^p spaces with $p \in (0,1)$, whose quasinorm is defined as in (1.21), or the weak Lebesgue space $L^{1,\infty}$ with a quasinorm defined as

$$\|f\|_{1,\infty} = \sup_{0<t<1} t f^*(t). \tag{1.23}$$

There is the largest and the smallest r.i. space. To be precise, by [1, Chapter 2, Corollary 6.7], it holds true that

$$L^\infty \hookrightarrow X \hookrightarrow L^1 \tag{1.24}$$

for every r.i. space X.

One possible generalization of Lebesgue's spaces are *Lorentz' $L^{p,q}$ spaces*, where $1 \le p, q \le \infty$, whose quasinorm is defined by

$$\|f\|_{p,q} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_q . \tag{1.25}$$

It is known that $\|\cdot\|_{p,q}$ is equivalent to an r.i. norm if and only if one of the following conditions is satisfied:

$$\begin{aligned}&1 < p < \infty, \quad 1 \le q \le \infty,\\ &p = q = 1,\\ &p = q = \infty.\end{aligned}$$

1.2 Sobolev spaces over r.i. spaces and isoperimetric function

Let $\Omega \subset \mathbb{R}^n$ be a domain, that is, a connected open set. We equip Ω with a finite measure μ which is absolutely continuous with respect to the Lebesgue measure with density ω. More precisely,

$$\mathrm{d}\mu(x) = \omega(x)\mathrm{d}x,$$

where ω is a Borel measurable function satisfying $\omega(x) > 0$ for a.e. $x \in \Omega$. Thus, the measure of an arbitrary measurable set $E \subset \Omega$ is given by

$$\mu(E) = \int_E \omega(x)\mathrm{d}x.$$

Throughout the thesis we will assume, for simplicity, that μ is normalized in such a way that $\mu(\Omega) = 1$. We now recall the definition of the perimeter of a set with respect to our space (Ω, μ) and the isoperimetric function.

Definition 1.8. Let $E \subset \mathbb{R}^n$ be measurable. We define the *perimeter* of E in (Ω, μ) by

$$P_\mu(E, \Omega) = \int_{\Omega \cap \partial^M E} \omega(x)\, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Here, $\mathcal{H}^{n-1}$ stands for the $n-1$ dimensional Hausdorff measure on $\mathbb{R}^n$ and $\partial^M E$ denotes the essential boundary of E in the sense of the geometric measure theory [13, 21].

Definition 1.9. The *isoperimetric function* of (Ω, μ) is a mapping $I_{\Omega,\mu} \colon [0,1] \to [0, \infty]$ defined by

$$I_{\Omega,\mu}(t) = \inf \left\{ P_\mu(E, \Omega) \colon E \subset \Omega, t \le \mu(E) \le \frac{1}{2} \right\} \quad \text{for } t \in \left[0, \frac{1}{2}\right]$$

and $I_{\Omega,\mu}(t) = I_{\Omega,\mu}(1-t)$ for $t \in (\frac{1}{2}, 1]$.

An easy consequence of this definition is the *isoperimetric inequality*

$$I_{\Omega,\mu}(\mu(E)) \leq P_\mu(E,\Omega), \quad E \subset \Omega \text{ is measurable.}$$

It is evident from the definition that $I_{\Omega,\mu}$ is a nondecreasing function on $[0, \frac{1}{2}]$. Further, by [2, Proposition 4.1], we know that $I_{\Omega,\mu}(t) \lesssim t^{\frac{1}{n'}}$ for t sufficiently small.

Given an r.i. space X, we define $X(\Omega) = X(\Omega,\mu)$ as the collection of all $u \in \mathcal{M}(\Omega)$ such that

$$\|u\|_{X(\Omega)} \coloneqq \|u^*\|_X$$

is finite. The functional $\|\cdot\|_{X(\Omega)}$ defines a norm on $X(\Omega)$. The space $X(\Omega)$ endowed with this norm is also called rearrangement-invariant space, and the space X is called its *representation space*.

The space $X'(\Omega)$ is then defined analogously via $\|\cdot\|_{X'}$.

Throughout the thesis we will, for the most part, not distinguish between $X(\Omega)$ and its representation space, as it will be evident whether we work in $X(\Omega)$ or in X.

Let $m \in \mathbb{N}$ and $X(\Omega,\mu)$ be an r.i. space. We define the *m-th order Sobolev space* $V^m X(\Omega,\mu)$ as

$$\begin{aligned} V^m X(\Omega,\mu) = \{u \colon u \text{ is } m\text{-times weakly differentiable in } \Omega \\ \text{and } |\nabla^m u| \in X(\Omega,\mu)\}. \end{aligned} \tag{1.26}$$

The results of [2] do not require one to work exactly with $I_{\Omega,\mu}$. It suffices to have a lower bound in terms of a nondecreasing function. To be precise, we work with a nondecreasing function $I \colon [0,1] \to [0,\infty)$ satisfying $I_{\Omega,\mu}(t) \geq cI(ct), t \in [0, \frac{1}{2}]$ for some $c > 0$. In view of [2, Proposition 4.2], it is natural to assume that $I(t) \gtrsim t, t \in (0,1)$, as this guarantees that $V^1L^1(\Omega) \subset L^1(\Omega)$ and, consequently, that $V^1X(\Omega) \subset L^1(\Omega)$ for every r.i. space X.

We continue by introducing a pair of integral operators, R_I and H_I, on $\mathcal{M}_+(0,1)$ which are defined by

$$R_I f(t) = \frac{1}{I(t)} \int_0^t f(s)\,\mathrm{d}s, \quad t \in (0,1), \tag{1.27}$$

and

$$H_I f(t) = \int_t^1 \frac{f(s)}{I(s)}\,\mathrm{d}s, \quad t \in (0,1). \tag{1.28}$$

Further, for $m \in \mathbb{N}$ we set

$$R_I^m = \underbrace{R_I \circ \ldots \circ R_I}_{m\text{-times}} \quad \text{and} \quad H_I^m = \underbrace{H_I \circ \ldots \circ H_I}_{m\text{-times}}. \tag{1.29}$$

Fubini's theorem reveals that operators R_I and H_I are mutually associate. Hence, R_I^m and H_I^m are also mutually associate for every $m \in \mathbb{N}$.

The operator G_I is then defined by

$$G_I f(t) = \sup_{t \leq s < 1} R_I f^*(s), \quad f \in \mathcal{M}_+(0,1), t \in (0,1).$$

Therefore, for every $f \in \mathcal{M}_+(0,1)$, $G_I f$ is a nonincreasing function and $R_I f \leq R_I f^* \leq G_I f$ and so $(R_I f)^* \leq G_I f$.

It holds true that

$$\|G_I f\|_X \approx \|R_I f^*\|_X, \quad f \in \mathcal{M}_+(0,1), \tag{1.30}$$

whenever X is an r.i. space [2, Theorem 9.5].

The core result of [2, Theorem 5.1] reads as follows:

Theorem 1.10 (Reduction principle)**.** *Let* (Ω, μ) *be such that* $I_\Omega(t) \gtrsim t$*. Let* $m \in \mathbb{N}$*, and let* X *and* Y *be r.i. spaces. Then*

$$\|H_I^m f\|_Y \lesssim \|f\|_X, \quad f \in \mathcal{M}_+(0,1), \tag{1.31}$$

implies

$$V^m X(\Omega) \to Y(\Omega). \tag{1.32}$$

Now, if we consider

$$\|f\|_{X'_m} = \|R_I^m f^*\|_{X'}, \tag{1.33}$$

by [2, Theorem 5.4] we know the following.

Theorem 1.11 (Optimal target)**.** *The functional* $\|\cdot\|_{X'_m}$ *defined in* (1.33) *is an r.i. norm, whose associate norm,* $\|\cdot\|_{X_m}$*, satisfies*

$$V^m X(\Omega) \to X_m(\Omega). \tag{1.34}$$

Moreover, if (1.32) *implies* (1.31) *(and so they are equivalent), then the space* $X_m(\Omega)$ *is the optimal target space in* (1.34) *among all r.i. spaces.*

Unless stated otherwise, by Y_X we will mean the optimal target space of X under the mapping H_I, and by X_Y we mean the optimal domain space of Y under the mapping H_I (if it exists). By the symbol Y'_X we understand $(Y_X)'$ and the symbol Y_{X_Z} stands for $Y_{(X_Z)}$.

The existence of the optimal target space Y_X is justified in [2, Proposition 8.3]. We will leave the question of the optimal domains to the beginning of the Chapter 3.

1.3 Interpolation theory

Definition 1.12. Let X_0 and X_1 be quasi-Banach spaces. We say that (X_0, X_1) is a *compatible couple* of quasi-Banach spaces if there exists a Hausdorff topological vector space H such that $X_0 \hookrightarrow H$ and $X_1 \hookrightarrow H$.

Let us recall the definition of the *K-functional.* In [1, Chapter 5] the K-functional is defined only for Banach spaces. However, it is not hard to see that extending this notion over quasi-Banach spaces does not invalidate any theorems that we will need, and their proofs would only need minor, if any, modifications. Let us also note that, by [1, Theorem 1.4] and [15, Theorem 3.4], for every r.i.q. space X we have $X \hookrightarrow \mathcal{M}_0$, where $\mathcal{M}_0$ is equipped with the (metrizable) topology of convergence in measure on the sets of finite measure. Consequently, any two r.i.q. spaces form a compatible couple.

Definition 1.13. Let (X_0, X_1) be a compatible couple quasi-Banach spaces. We define the K-functional on $X_0 + X_1$ by

$$K(f, t, X_0, X_1) = \inf\{\|g\|_{X_0} + t\|h\|_{X_1} \colon f = g + h, g \in X_0, h \in X_1\}, \quad t \in (0, \infty).$$

The next theorem will be of use to us, especially when combined with Theorem 1.15. The proof can be found in [1, Chapter 5, Proposition 5.2].

Theorem 1.14. *Let (X_0, X_1) be a compatible couple of quasi-Banach spaces. Then for every $f \in X_0 + X_1$ the map $t \mapsto K(f, t, X_0, X_1)$ is nonnegative, nondecreasing and concave on $(0, \infty)$. Consequently,*

$$K(f, t, X_0, X_1) = K(f, 0+, X_0, X_1) + \int_0^t k(f, s, X_0, X_1)\, \mathrm{d}s, \tag{1.35}$$

where $t \mapsto k(f, t, X_0, X_1)$ is the uniquely determined nonincreasing and right-continuous function.

There is a nice characterization in [1, Chapter 5, Proposition 1.15], stating when the first term of the righthand side of (1.35) can be omitted. Note that since the spaces involved need not be normed, one should be familiar with a *generalised Riesz–Fischer theorem* [15, Theorem 3.3].

Theorem 1.15. *Let (X_0, X_1) be a compatible couple of quasi-Banach spaces. Then*

$$K(f, 0+, X_0, X_1) = 0, \quad f \in X_0 + X_1$$

if and only if $X_0 \cap X_1$ is dense in X_0.

The proof of the next theorem can be found in [1, Chapter 5, Theorem 1.11].

Theorem 1.16. *Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of quasi-Banach spaces. Let T be a sublinear operator such that*

$$T \colon X_0 \to Y_0 \quad \textit{and} \quad T \colon X_1 \to Y_1.$$

Then there is $c > 0$ such that

$$K(Tf, t, Y_0, Y_1) \lesssim K(f, ct, X_0, X_1), \quad f \in X_0 + X_1, t > 0. \tag{1.36}$$

In many theorems, we will use a certain elementary decomposition of f, to which we will refer as our *favourite decomposition.*

Definition 1.17 (Favourite decomposition)**.** Let $f \in \mathcal{M}(0, 1)$ and $t \in (0, 1)$ be given. We define the favourite decomposition of f at point t by

$$f_0(s) = \min\{|f(s)|, f^*(t)\} \operatorname{sgn} f(s),$$

and

$$f_1(s) = \max\{|f(s)| - f^*(t), 0\} \operatorname{sgn} f(s).$$

Then $f = f_0 + f_1$ and it further satisfies

$$\begin{aligned} f_0^*(s) &= \min\{f^*(s), f^*(t)\}, \\ f_1^*(s) &= (f^*(s) - f^*(t))\chi_{(0,t)}(s), \end{aligned} \tag{1.37}$$

and $f^* = f_0^* + f_1^*$.

Next we state and prove two inequalities concerning the K-functional for (X, L^∞) and (L^1, X).

Proposition 1.18. *Let X be a r.i.q. space and assume its fundamental function, φ_X, is an increasing bijection on $(0,1)$. Then*

$$\|f^*\chi_{(0,\varphi_X^{-1}(t))}\|_X \lesssim K(f,t,X,L^\infty), \quad f \in \mathcal{M}_+(0,1), t \in (0,1). \tag{1.38}$$

Proof. Let $f \in X + L^\infty$ and $t \in (0,1)$ be given. Write $f = f_0 + f_1$, where $f_0 \in X$ and $f_1 \in L^\infty$. Using the boundedness of the dilation operator and the monotonicity of $\|\cdot\|_X$, we estimate

$$\begin{aligned}
\|f^*\chi_{(0,\varphi_X^{-1}(t))}\|_X &\lesssim \left\|f^*(2s)\chi_{(0,\varphi_X^{-1}(t))}(2s)\right\|_X \le \left\|f^*(2s)\chi_{(0,\varphi_X^{-1}(t))}(s)\right\|_X \\
&\lesssim \left\|f_0^*\chi_{(0,\varphi_X^{-1}(t))}\right\|_X + \left\|f_1^*\chi_{(0,\varphi_X^{-1}(t))}\right\|_X \\
&\le \left\|f_0^*\chi_{(0,\varphi_X^{-1}(t))}\right\|_X + \|f_1^*\|_\infty \cdot \left\|\chi_{(0,\varphi_X^{-1}(t))}\right\|_X \\
&= \left\|f_0^*\chi_{(0,\varphi_X^{-1}(t))}\right\|_X + \|f_1^*\|_\infty \cdot t \le \|f_0\|_X + t\|f_1\|_\infty.
\end{aligned}$$

On taking infimum over all such decompositions we obtain

$$\|f^*\chi_{(0,\varphi_X^{-1}(t))}\|_X \lesssim K(f,t,X,L^\infty).$$

□

Proposition 1.19. *Let X be an r.i.q. space such that $\varphi(t) := \frac{t}{\varphi_X(t)}$ is an increasing bijection on $(0,1)$. Then*

$$K(f,t,L^1,X) \lesssim \left\|f^*\chi_{(0,\varphi^{-1}(t))}\right\|_1 + t\left\|f^*\chi_{(\varphi^{-1}(t),1)}\right\|_X, \quad f \in \mathcal{M}_+(0,1), t \in (0,1). \tag{1.39}$$

Proof. Let $f \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$ be given. Let f_0 and f_1 be our favourite decomposition of f at point $\varphi^{-1}(t)$ in place of t. Using the rearrangement invariance of both L^1 and X and (1.37), we estimate

$$\begin{aligned}
K(f,t,L^1,X) &\le \|f_1\|_1 + t\|f_0\|_X = \|f_1^*\chi_{(0,\varphi^{-1}(t))}\|_1 + t\|f_0^*\|_X \\
&= \|f^*\chi_{(0,\varphi^{-1}(t))}\|_1 - \varphi^{-1}(t)f^*(\varphi^{-1}(t)) \\
&\quad + t\|f^*(\varphi^{-1}(t))\chi_{(0,\varphi^{-1}(t))} + f^*\chi_{(\varphi^{-1}(t),1)}\|_X \\
&\lesssim \|f^*\chi_{(0,\varphi^{-1}(t))}\|_1 - \varphi^{-1}(t)f^*(\varphi^{-1}(t)) \\
&\quad + t\varphi_X(\varphi^{-1}(t))f^*(\varphi^{-1}(t)) + t\|f^*\chi_{(\varphi^{-1}(t),1)}\|_X \\
&= \|f^*\chi_{(0,\varphi^{-1}(t))}\|_1 + t\|f^*\chi_{(\varphi^{-1}(t),1)}\|_X.
\end{aligned}$$

□

Definition 1.20. Let X_0, X_1 and X be quasi-Banach spaces which all embed to a Hausdorff topological vector space $\mathcal{H}$ and satisfy $X_0 \subset X \subset X_1$. We say that X is an *interpolation space* between X_0 and X_1, the fact being denoted $X \in \operatorname{Int}(X_0, X_1)$, if for any linear operator T the following holds:

$$T\colon X_0 \to X_0 \quad \text{and} \quad T\colon X_1 \to X_1 \quad \implies \quad T\colon X \to X.$$

The next theorem [1, Chapter 5, Theorem 1.19] of an interpolation nature appears to be indispensable in the proof of Theorem 3.4.

Theorem 1.21. *Let* (X_0, X_1) *and* (Y_0, Y_1) *be two compatible couples of quasi-Banach spaces and* λ *be an r.i. norm. Suppose* $X_0 \cap X_1$ *is dense in* X_0 *and that* $Y_0 \cap Y_1$ *is dense in* Y_0*. Set* $\alpha(f) = \lambda(k(f, t, X_0, X_1))$ *and* $\beta(f) = \lambda(k(f, t, Y_0, Y_1))$ *for* $f \in \mathcal{M}_+(0, 1)$*. Then for any linear operator* T *satisfying*

$$T\colon X_0 \to Y_0 \quad \textit{and} \quad T\colon X_1 \to Y_1,$$

we have

$$\beta(Tf) \lesssim \alpha(f), \quad f \in \mathcal{M}_+(0, 1).$$

Additionally, if X_0 *and* X_1 *are r.i. spaces, then the functional* α *is an r.i. norm.*

Remark 1.22. It is important that the functional λ in the theorem above is an r.i. norm, so that we have the HLP principle at our disposal.

1.4 Weighted inequalities involving suprema

We will also use [7, Theorem 3.2] and its reduction to our setting. By weights we understand positive measurable functions on $(0, \infty)$.

Theorem 1.23. *Let* u, v, w *be weights on* $(0, \infty)$ *such that* $0 < \int_0^x v(t)\,\mathrm{d}t < \infty$ *and* $0 < \int_0^x w(t)\,\mathrm{d}t < \infty$ *for every* $x \in (0, \infty)$*. Then*

$$\int_0^\infty \sup_{t \le \tau < \infty} u(\tau)\varphi(\tau)w(t)\,\mathrm{d}t \lesssim \int_0^\infty \varphi(t)v(t)\,\mathrm{d}t \tag{1.40}$$

holds for all $\varphi \in \mathcal{M}_+(0, \infty)$ *nonincreasing if and only if*

$$\int_0^x \sup_{t \le \tau \le x} u(\tau)w(t)\,\mathrm{d}t \lesssim \int_0^x v(t)\,\mathrm{d}t \textit{ for every } x \in (0, \infty). \tag{1.41}$$

Corollary 1.24. *Let* u, v, w *be weights on* $(0, 1)$ *for each of which there exists the limit at* 1 *from the left that is nonzero and finite. Then*

$$\int_0^1 \sup_{t \le \tau < 1} u(\tau)\varphi(\tau)w(t)\,\mathrm{d}t \lesssim \int_0^1 \varphi(t)v(t)\,\mathrm{d}t \tag{1.42}$$

holds for all $\varphi \in \mathcal{M}_+(0, 1)$ *nonincreasing if and only if*

$$\int_0^x \sup_{t \le \tau \le x} u(\tau)w(t)\,\mathrm{d}t \lesssim \int_0^x v(t)\,\mathrm{d}t \textit{ for every } x \in (0, 1). \tag{1.43}$$

Proof. It suffices to show that (1.40) is equivalent to (1.42) and (1.41) is equivalent to (1.43). We begin by extending all u, v, w constantly on $[1, \infty)$ by their respective left limits at 1. We will not distinguish between u, v, w and their extensions.

(1.42) $\Rightarrow$ (1.40): Let $\varphi \in \mathcal{M}_+(0,\infty)$ be nonincreasing. Then

$$\begin{aligned}
&\int_0^\infty \sup_{t\le\tau<\infty} u(\tau)\varphi(\tau)w(t)\,\mathrm{d}t \\
&= \int_0^1 \sup_{t\le\tau<1} u(\tau)\varphi(\tau)w(t)\,\mathrm{d}t + \int_1^\infty \sup_{t\le\tau<\infty} u(\tau)\varphi(\tau)w(t)\,\mathrm{d}t \\
&\lesssim \int_0^1 \varphi(t)v(t)\,\mathrm{d}t + \int_1^\infty u(1)\varphi(t)w(1)\,\mathrm{d}t \\
&\lesssim \int_0^1 \varphi(t)v(t)\,\mathrm{d}t + \int_1^\infty \varphi(t)v(1)\,\mathrm{d}t \\
&= \int_0^\infty \varphi(t)v(t)\,\mathrm{d}t.
\end{aligned}$$

(1.43) $\Rightarrow$ (1.41): Let $x \in [1,\infty)$. Then

$$\begin{aligned}
\int_0^x \sup_{t\le\tau\le x} u(\tau)w(t)\,\mathrm{d}t &= \int_0^1 \sup_{t\le\tau\le x} u(\tau)w(t)\,\mathrm{d}t + \int_1^x \sup_{t\le\tau\le x} u(\tau)w(t)\,\mathrm{d}t \\
&\lesssim \int_0^1 v(t)\,\mathrm{d}t + \int_1^x u(1)w(1)\,\mathrm{d}t \lesssim \int_0^x v(t)\,\mathrm{d}t.
\end{aligned}$$

For $x \in (0,1)$ there is nothing to be proved. As implications (1.40) $\Rightarrow$ (1.42) and (1.41) $\Rightarrow$ (1.43) are trivial, the proof is complete. $\square$

2. Supremum operators

In this chapter, we will introduce two supremum operators S_I and T_I and explore their boundedness and interpolation properties. These two supremum operators, as their name suggest, will be defined in terms of a nondecreasing function $I\colon (0,1) \to (0,1)$. However, for our purposes, we restrict ourselves to concave functions, even though many theorems would hold in a more general setting. There are two things that lead us to this.

First, we will work with the Marcinkiewicz type space m_I given by the functional $\|f\|_{m_I} = \sup_{0<t<1} I(t) f^*(t)$ for $f \in \mathcal{M}_+(0,1)$. From here, we impose a Δ_2 condition on I, because:

Fact 2.1. *Let $I\colon (0,1) \to (0,1)$ be a nondecreasing function. Then m_I is an r.i.q. space if and only if $I \in \Delta_2$.*

Secondly, we will want I to be such that $t \mapsto \frac{I(t)}{t}, t \in (0,1)$, is nonincreasing. In other words, we want I to be quasiconcave. However, since there exists $\widetilde{I}$ concave such that $I \approx \widetilde{I}$ on $(0,1)$, and we will not particularly care about constants, there is no real loss of generality if we assume I to be concave.

For the remainder of the thesis, whenever we mention a concave function I, we implicitely assume that $I\colon (0,1) \to (0,1)$ is a bijection with $I(0+) = 0$ and $I(1-) = 1$. Note that this implies that $I(t) \geq t, t \in (0,1)$.

We proceed by defining two supremum operators.

Definition 2.2. Let I be a concave function. We define supremum operators S_I and T_I on $\mathcal{M}_+(0,1)$ by

$$(S_I f)(t) \coloneqq \frac{1}{I(t)} \sup_{0<s\leq t} I(s) f^*(s), \quad f \in \mathcal{M}_+(0,1), t \in (0,1),$$

and

$$(T_I f)(t) \coloneqq \frac{I(t)}{t} \sup_{t\leq s<1} \frac{s}{I(s)} f^*(s), \quad f \in \mathcal{M}_+(0,1), t \in (0,1).$$

Observe that $f^* \leq T_I f$ and $f^* \leq S_I f$ for every $f \in \mathcal{M}_+(0,1)$. We also see that both of these operators are monotone – if $f, g \in \mathcal{M}_+(0,1)$ are such that $f \leq g$, then $S_I f \leq S_I g$ and $T_I f \leq T_I g$.

Moreover, $t \mapsto T_I f(t)$ is a nonincreasing function for every $f \in \mathcal{M}_+(0,1)$.

Deploying Fact 1.1 and Δ_2 condition of I, we see that

$$(S_I(f+g))(t) \lesssim (S_I f)\left(\frac{t}{2}\right) + (S_I g)\left(\frac{t}{2}\right), \quad f \in \mathcal{M}_+(0,1), t \in (0,1), \tag{2.1}$$

and

$$(T_I(f+g))(t) \lesssim (T_I f)\left(\frac{t}{2}\right) + (T_I g)\left(\frac{t}{2}\right), \quad f \in \mathcal{M}_+(0,1), t \in (0,1). \tag{2.2}$$

We continue by defining three function spaces. To simplify notation, by symbol $\widetilde{I}$ we will denote a function $\widetilde{I}(t) = \frac{t}{I(t)}, t \in (0,1)$.

Definition 2.3. Let I be a concave function. We introduce three functionals defined on $\mathcal{M}_+(0,1)$ with values in $[0,\infty]$ by

$$\|f\|_{m_I} := \sup_{0<t<1} I(s)f^*(s),$$
$$\|f\|_{m_{\widetilde{I}}} := \sup_{0<t<1} \frac{s}{I(s)} f^*(s),$$
$$\|f\|_{\Lambda_I} := \int_0^1 \frac{I(s)}{s} f^*(t)\,\mathrm{d}s.$$

We further denote $m_I := \{f \in \mathcal{M}_+(0,1) \colon \|f\|_{m_I} < \infty\}$. Analogously we define spaces $m_{\widetilde{I}}$ and Λ_I.

Before we begin exploring the mapping properties of the operator S_I, let us observe that $t \mapsto S_I f(t)$ is a nonincreasing function for every $f \in \mathcal{M}_+(0,1)$.

Lemma 2.4. *Let I be a concave function and $f \in \mathcal{M}_+(0,1)$. Then $S_I f$ is a nonincreasing function on $(0,1)$.*

Proof. We put $Rf(t) = \sup_{0<s\le t} I(s)f^*(s)$ for $t \in (0,1)$. Let $0 < t_1 < t_2 < 1$ be given. We consider two cases: If $Rf(t_1) = Rf(t_2)$, then $(S_I f)(t_2) \le (S_I f)(t_1)$ because $t \mapsto \frac{1}{I(t)}$ is nonincreasing. If $Rf(t_1) < Rf(t_2)$, we consider a function

$$f_1(s) := \begin{cases} f^*(s), & s \le t_1, \\ f^*(t_1), & t_1 < s < 1. \end{cases}$$

Then $f_1^* = f_1$ and $f^* \le f_1^*$. Hence, as $Rf(t_1) < Rf(t_2)$, we have

$$Rf(t_2) = \sup_{t_1<s\le t_2} I(s)f^*(s)$$

and, consequently,

$$Rf_1(t_2) = \sup_{t_1<s\le t_2} I(s)f_1^*(s).$$

We estimate

$$\begin{aligned}(S_I f)(t_2) \le (S_I f_1)(t_2) &= \frac{1}{I(t_2)} \sup_{t_1<s\le t_2} I(s)f_1^*(s) \\ &= \frac{1}{I(t_2)} \cdot I(t_2)f_1^*(t_2) = f_1^*(t_1) = f^*(t_1) \le (S_I f)(t_1).\end{aligned}$$

□

Theorem 2.5. *Let I be a concave function. Then the operator S_I has the following endpoint mapping properties:*

(i) $S_I \colon L^\infty \to L^\infty$,

(ii) $S_I \colon m_I \to m_I$.

Proof. (i) Given $f \in L^\infty$ we estimate

$$\begin{aligned}\|S_I f\|_\infty = \sup_{0<t<1} \frac{1}{I(t)} \sup_{0<s\le t} I(s)f^*(s) &\le \|f\|_\infty \sup_{0<t<1} \frac{1}{I(t)} \sup_{0<s\le t} I(s) \\ &= \|f\|_\infty \sup_{0<t<1} \frac{1}{I(t)} \cdot I(t) = \|f\|_\infty.\end{aligned}$$

(ii) Let $f \in m_I$ be given. By Lemma 2.4 we have

$$\begin{aligned}\|S_I f\|_{m_I} &= \sup_{0<t<1} I(t) \left(\tau \mapsto \frac{1}{I(\tau)} \sup_{0<s\le\tau} I(s) f^*(s) \right)^* (t) \\ &= \sup_{0<t<1} I(t) \cdot \frac{1}{I(t)} \sup_{0<s\le t} I(s) f^*(s) = \|f\|_{m_I}.\end{aligned}$$

□

Theorem 2.6. *Let I be a concave function. Then the following holds:*

(i) $T_I \colon m_{\widetilde{I}} \to m_{\widetilde{I}}$,

(ii) $T_I \colon L^1 \to L^1$ if and only if $\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \lesssim I(t)$ for $t \in (0,1)$.

Proof. (i) Let $f \in m_{\widetilde{I}}$ be given. As $T_I f$ is nonincreasing, we estimate

$$\|T_I f\|_{m_{\widetilde{I}}} = \sup_{0<t<1} \frac{t}{I(t)} \cdot \frac{I(t)}{t} \sup_{t\le s<1} \frac{s}{I(s)} f^*(s) = \sup_{0<s<1} \frac{s}{I(s)} f^*(s) = \|f\|_{m_{\widetilde{I}}}.$$

(ii) We use Corollary 1.24 with weights $u(t) = \frac{t}{I(t)}$, $w = \frac{1}{u}$ and $v = 1$. Hence, T_I is bounded on L^1 if and only if

$$\int_0^t \sup_{s\le\tau\le t} \frac{\tau}{I(\tau)} \cdot \frac{I(s)}{s}\,\mathrm{d}s \lesssim t, \quad t \in (0,1).$$

As $t \mapsto \frac{t}{I(t)}$ is nondecreasing, $\sup_{s\le\tau\le t} \frac{\tau}{I(\tau)} = \frac{t}{I(t)}$. Thus, multiplying through by $\frac{I(t)}{t}$, we equivalently rewrite this as

$$\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \lesssim I(t), \quad t \in (0,1).$$

□

We proceed by defining a certain *average condition* for a concave function I, which says that the reciprocal of I is approximately the average integral of itself.

Definition 2.7. Let I be a concave function. We say that I has the average property, if

$$\int_0^t \frac{\mathrm{d}s}{I(s)} \approx \frac{t}{I(t)}, \quad t \in (0,1).$$

This condition, among other things, also appeared in [2], and allows one to simplify the description of the optimal target norm of the m-th order Sobolev embedding. Note that this condition implies integrability of $\frac{1}{I}$. Classical examples of functions satisfying the average property are the polynomials $t \mapsto t^\alpha, t \in (0,1)$ for $\alpha \in (0,1)$. Functions which do not possess this property include for example $t \mapsto t$ or $t \mapsto t\sqrt{\log \frac{2}{t}}$ for $t \in (0,1)$.

Lemma 2.8. *Assume that I satisfies the average property. Then*

$$\sup_{0<s\le t} I(s) f^*(s) \approx \sup_{0<s\le t} I(s) f^{**}(s), \quad f \in \mathcal{M}_+(0,1), t \in (0,1).$$

In particular, $m_I = M_I$ with equivalent norms, where M_I is the Marcinkiewicz space with norm given by

$$\|f\|_{M_I} = \sup_{0<t<1} I(t) f^{**}(t).$$

Proof. Fix $f \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$. Denoting

$$M = \sup_{0<s\le t} I(s) f^*(s),$$

which we can without loss of generality assume to be finite, we have for every $s \in (0,t)$ that

$$f^*(s) \le M \frac{1}{I(s)}.$$

Thus

$$\sup_{0<s\le t} I(s) f^{**}(s) \le M \sup_{0<s\le t} \frac{I(s)}{s} \int_0^s \frac{\mathrm{d}r}{I(r)} \lesssim M,$$

where the second inequality is exactly the average property. The converse inequality holds trivially, as $f^* \le f^{**}$. □

Remark 2.9. One easily sees from the proof that, in fact, $m_I = M_I$ is equivalent to I enjoying the average property. Indeed, one simply tests the inequality on $\frac{1}{I(t)}$. Even more is true – m_I is an r.i. space if and only if I has the average property. We have already proved the sufficiency of this claim. Assuming $\|\cdot\|_{m_I}$ to be equivalent to an r.i. norm, say $\|\cdot\|$, their fundamental functions coincide. As m_I and M_I have the same fundamental function, $m_I \hookrightarrow M_I$ by [1, Chapter 2, Theorem 5.13] and so $m_I = M_I$, since $M_I \hookrightarrow m_I$ holds regardless of what the function I satisfies.

One could also wonder whether S_I is bounded on M_I. We now claim that this happens if and only if $M_I = m_I$. Indeed, sufficiency was proved in Theorem 2.5. As for necessity, assume that $S_I \colon M_I \to M_I$. This means that

$$\sup_{0<t<1} \frac{I(t)}{t} \int_0^t \frac{1}{I(s)} \sup_{0<r\le s} I(r) f^*(r)\,\mathrm{d}s \lesssim \sup_{0<t<1} \frac{I(t)}{t} \int_0^t f^*(s)\,\mathrm{d}s, \quad f \in \mathcal{M}_+(0,1). \tag{2.3}$$

Let $r \in (0,1)$ be given and consider $f = f^* = \chi_{(0,r)}$. Then the right-hand side of (2.3) can be written as

$$\sup_{0<t<1} \frac{I(t)}{t} \int_0^t f^*(s)\,\mathrm{d}s = \max\left\{ \sup_{0<t\le r} I(t), \sup_{r\le t<1} \frac{I(t)}{t} r \right\} = I(r).$$

Similarly, the left-hand side can be rewritten as

$$\begin{aligned} &\sup_{0<t<1} \frac{I(t)}{t} \int_0^t \frac{1}{I(s)} \sup_{0<r\le s} I(r) f^*(r)\,\mathrm{d}s \\ &= \max\left\{ \sup_{0<t\le r} I(t), \sup_{r<t<1} \frac{I(t)}{t} \left(r + \int_r^t \frac{I(r)}{I(s)}\,\mathrm{d}s \right) \right\}. \end{aligned}$$

Therefore, necessarily,

$$\sup_{r<t<1} \frac{I(t)}{t} \int_r^t \frac{I(r)}{I(s)}\,\mathrm{d}s \lesssim I(r), \quad r \in (0,1).$$

Dividing by $I(r)$ we deduce that

$$\sup_{0<r<1} \sup_{r<t<1} \frac{I(t)}{t} \int_r^t \frac{\mathrm{d}s}{I(s)} < \infty.$$

In particular, should we fix $t \in (0,1)$ and take limit for $r \to 0^+$, we obtain that I has to satisfy the average property. Consequently, by Remark 2.9, $M_I = m_I$.

It is evident that the operator S_I and the space m_I are intertwined in a sense that $S_I f$ is finite if and only if $f \in m_I$. Next theorem provides us with a result, which essentially says that S_I is the greatest operator which is bounded simultaneously on L^∞ and m_I. To prove this, we will need a K-functional related lemma first.

Lemma 2.10. *Let I be a concave function. Then*

$$K(f, t, m_I, L^\infty) \approx \sup_{0<s\leq I^{-1}(t)} I(s) f^*(s), \quad f \in m_I, t \in (0,1).$$

Proof. By virtue of Proposition 1.18, it remains to prove $K(f, t, m_I, L^\infty) \lesssim \sup_{0<s\leq I^{-1}(t)} I(s) f^*(s)$. To this end, let $f \in m_I$ and $t \in (0,1)$ be given. Let f_0 and f_1 be our favourite decomposition of f at point $I^{-1}(t)$. Then

$$\begin{aligned} K(f, t, m_I, L^\infty) &\leq \|f_1\|_{m_I} + t\|f_0\|_\infty = \sup_{0<s\leq I^{-1}(t)} I(s) f_1^*(s) + t f^*(I^{-1}(t)) \\ &\leq \sup_{0<s\leq I^{-1}(t)} I(s) f^*(s) + t f^*(I^{-1}(t)) \leq 2 \sup_{0<s\leq I^{-1}(t)} I(s) f^*(s). \end{aligned}$$

□

Theorem 2.11. *Let I be a concave function and let S be a sublinear operator defined on m_I. If S is bounded on L^∞ and on m_I, then*

$$(Sf)^*(t) \lesssim (S_I f)(t), \quad f \in \mathcal{M}_+(0,1), t \in (0,1).$$

If, in addition, I has the average property, then

$$(Sf)^{**}(t) \lesssim (S_I f)(t), \quad f \in \mathcal{M}_+(0,1), t \in (0,1), \tag{2.4}$$

and so

$$(S_I f)^{**}(t) \lesssim (S_I f)(t), \quad f \in \mathcal{M}_+(0,1), t \in (0,1). \tag{2.5}$$

In particular, for an r.i. space $X \subset m_I$ we have $X \in \mathrm{Int}(L^\infty, m_I)$ whenever S_I is bounded on X.

Proof. By Lemma 2.10, we have

$$K(f, t, m_I, L^\infty) \approx \sup_{0<s\leq I^{-1}(t)} I(s) f^*(s).$$

Fix $f \in m_I$ and $t \in (0, d)$, where $d = \min\{\frac{1}{c}, cI\left(\frac{1}{c}\right)\}$ and $c \geq 1$ is a constant from Theorem 1.16. We estimate

$$\sup_{0<s\leq I^{-1}(t)} I(s)(Sf)^*(s) \lesssim \sup_{0<s\leq I^{-1}(ct)} I(s) f^*(s). \tag{2.6}$$

Passing from t to $I(t)$ we arrive at

$$\sup_{0<s\leq t} I(s)(Sf)^*(s) \lesssim \sup_{0<s\leq I^{-1}(cI(t))} I(s) f^*(s).$$

Thus, choosing $s = t$ on the left-hand side and dividing by $I(t)$ gives us

$$(Sf)^*(t) \lesssim \frac{1}{I(t)} \sup_{0<s\le I^{-1}(cI(t))} I(s)f^*(s) \approx (S_I f)(I^{-1}(cI(t))) \le (S_I f)(t),$$

where the last inequality stems from Lemma 2.4 and c being no less than one.

Next, assuming S_I is bounded on X we estimate using the boundedness of the dilation operator:

$$\|(Sf)^*\|_X \approx \|(Sf)^*\chi_{(0,d)}\|_X \lesssim \|S_I f\chi_{(0,d)}\|_X \approx \|S_I f\|_X \lesssim \|f\|_X.$$

Now, regarding the "in addition" part of the theorem, we only need to show (2.4) and (2.5), as the rest follows from the previous part. First, (2.4) is a direct application of Lemma 2.8 on the left-hand side of (2.6). Second, (2.5) holds true, because of (2.1) and the boundedness of the dilation operator on r.i.q. spaces. □

Remark 2.12. Should we assume that I satisfies the average property, then $X \in \operatorname{Int}(L^\infty, M_I) \iff S_I\colon X \to X$. "$\Leftarrow$" is exactly Theorem 2.11. The other implication is proved in [3, Theorem 1].

Corollary 2.13. *Let X be an r.i. space. Define $\|f\|_Z := \|S_I f\|_X$ for $f \in \mathcal{M}(0,1)$. Then $\|\cdot\|_Z$ is an r.i.q. norm and, denoting by Z the r.i.q. space corresponding to $\|\cdot\|_Z$, we have that*

$$S_I\colon Z \to Z.$$

If I satisfies the average property, then $\|\cdot\|_Z$ is equivalent to an r.i. norm, and so Z is an r.i. space.

Proof. The only property of (Q1) that requires some comment is the quasi-triangle inequality. To this end, let $f, g \in \mathcal{M}(0,1)$ be given. Using (2.1) and the boundedness of the dilation operator on r.i. spaces we calculate

$$\begin{aligned}\|f+g\|_Z = \|S_I(f+g)\|_X &\lesssim \left\|S_I f\left(\frac{t}{2}\right) + S_I g\left(\frac{t}{2}\right)\right\|_X \\ &\le \left\|S_I f\left(\frac{t}{2}\right)\right\|_X + \left\|S_I g\left(\frac{t}{2}\right)\right\|_X \approx \|S_I f\|_X + \|S_I g\|_X = \|f\|_Z + \|g\|_Z.\end{aligned}$$

Property (P2) obviously holds.

Let $f_n, f \in \mathcal{M}_+(0,1)$ be such that $f_n \nearrow f$ a.e. Then $f_n^* \nearrow f^*$. Fix $t \in (0,1)$ and let $K < \sup_{0<s\le t} I(s)f^*(s)$. We find $t_0 \in (0,t]$ such that $I(t_0)f^*(t_0) > K$. Then

$$I(t_0)f_n^*(t_0) \nearrow I(t_0)f^*(t_0)$$

and so $(S_I f_n^*)(t) \nearrow (S_I f^*)(t)$. As X is an r.i. norm, we get that $\|f_n\|_Z \nearrow \|f\|_Z$ and (P3) holds.

Regarding (P4), we have $\|\chi_{(0,1)}\|_Z = \|S_I\chi_{(0,1)}\|_X = \|\chi_{(0,1)}\|_X < \infty$.

Thus, functional $\|\cdot\|_Z$ is an r.i.q. norm.

Since $\|\cdot\|_X$ possesses property (P5), we can say the same about $\|\cdot\|_Z$, because $\|\cdot\|_X \le \|\cdot\|_Z$.

Next, for every $f \in \mathcal{M}(0,1)$, we estimate

$$\begin{aligned}\|S_I f\|_Z = \|S_I(S_I f)\|_X &= \left\|\frac{1}{I(t)} \sup_{0<s\le t} I(s) \left(\tau \mapsto \frac{1}{I(\tau)} \sup_{0<r\le\tau} I(r)f^*(r)\right)^*(s)\right\|_X \\ &\le \left\|\frac{1}{I(t)} \sup_{0<s\le t} I(s) \cdot \frac{1}{I(s)} \sup_{0<r\le s} I(r)f^*(r)\right\|_X = \|S_I f\|_X = \|f\|_Z,\end{aligned}$$

where the inequality comes from Lemma 2.4.

If I satisfies the average property, Lemma 2.8 tells us that

$$\|f\|_Z = \|S_I f\|_X \approx \|S_I f^{**}\|_X =: \|f\|, \quad f \in \mathcal{M}_+(0,1).$$

It follows from the previous and subadditivity of both $f \mapsto f^{**}$ and supremum, that $\|\cdot\|$ is an r.i. norm and $\|\cdot\|_Z \approx \|\cdot\|$. □

In view of Theorem 2.6, we will assume that the function I satisfies

$$\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \lesssim I(t), \quad t \in (0,1), \tag{2.7}$$

when working with T_I.

Lemma 2.14. *Let I be a concave function satisfying* (2.7)*. Then*

$$(T_I f)^{**}(t) \lesssim (T_I f^{**})(t), \quad f \in \mathcal{M}_+(0,1), t \in (0,1). \tag{2.8}$$

Proof. Fix $f \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$. Let f_0 and f_1 be our favourite decomposition of f at point t.

Then, as $T_I f$ is nonincreasing and supremum is subadditive, we have

$$\begin{aligned}(T_I f)^{**}(t) &= \frac{1}{t}\int_0^t \frac{I(s)}{s} \sup_{s \le r < 1} \frac{r}{I(r)} f^*(r)\,\mathrm{d}s \\ &\le \underbrace{\frac{1}{t}\int_0^t \frac{I(s)}{s} \sup_{s \le r < 1} \frac{r}{I(r)} f_0^*(r)\,\mathrm{d}s}_{I} + \underbrace{\frac{1}{t}\int_0^t \frac{I(s)}{s} \sup_{s \le r < 1} \frac{r}{I(r)} f_1^*(r)\,\mathrm{d}s}_{II}.\end{aligned}$$

As for I, we estimate

$$\begin{aligned} I &= \frac{1}{t}\int_0^t \frac{I(s)}{s} \sup_{s \le r < 1} \frac{r}{I(r)} \min\{f^*(r), f^*(t)\}\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t \frac{I(s)}{s} \max\left\{\sup_{s \le r < t} \frac{r}{I(r)} f^*(t), \sup_{t \le r < 1} \frac{r}{I(r)} f^*(r)\right\}\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t \frac{I(s)}{s} \sup_{t \le r < 1} \frac{r}{I(r)} f^*(r)\,\mathrm{d}s = \frac{I(t)}{t} \cdot \frac{1}{I(t)}\int_0^t \frac{I(s)}{s} \sup_{t \le r < 1} \frac{r}{I(r)} f^*(r)\,\mathrm{d}s \\ &\lesssim \frac{I(t)}{t} \sup_{t \le r < 1} \frac{r}{I(r)} f^*(r) = (T_I f)(t).\end{aligned}$$

Next, using Theorem 2.6, *(ii)*, we estimate II:

$$\begin{aligned} II &= \frac{1}{t}\int_0^t \frac{I(s)}{s} \sup_{s \le r < 1} \frac{r}{I(r)} f_1^*(r)\,\mathrm{d}s \le \frac{1}{t}\int_0^1 \frac{I(s)}{s} \sup_{s \le r < 1} \frac{r}{I(r)} f_1^*(r)\,\mathrm{d}s \\ &\lesssim \frac{1}{t}\int_0^1 f_1^*(s)\,\mathrm{d}s = \frac{1}{t}\int_0^t (f^*(s) - f^*(t))\,\mathrm{d}s \le f^{**}(t).\end{aligned}$$

Putting everything together, we arrive at what we wanted:

$$(T_I f)^{**}(t) \lesssim (T_I f(t) + f^{**}(t)) \lesssim (T_I f^{**})(t).$$

□

Remark 2.15. Notice that in the proof we used both (2.7) and the boundedness of T_I on L^1, which we know are equivalent. Now, taking the limit $t \to 1^-$ in (2.8), we get

$$\int_0^1 (T_I f)(t)\,\mathrm{d}t \lesssim \lim_{t\to 1^-} \frac{I(t)}{t} \sup_{t\le s<1} (R_I f^*)(s) = \int_0^1 f^*(t)\,\mathrm{d}t.$$

In other words, (2.8) is equivalent to the boundedness of T_I on L^1.

Now, an important consequence of the previous lemma is the following corollary.

Corollary 2.16. *Let X be an r.i. space and let I be a concave function satisfying (2.7). Then*

$$\left\| \frac{t}{I(t)} (T_I f)^{**}(t) \right\|_X \lesssim \left\| \frac{t}{I(t)} f^{**}(t) \right\|_X, \quad f \in \mathcal{M}_+(0,1).$$

In other words, operator T_I is bounded on Y'_X whenever X is an r.i. space.

Proof. For $f \in \mathcal{M}_+(0,1)$ we estimate

$$\begin{aligned} \left\| \frac{t}{I(t)} (T_I f)^{**}(t) \right\|_X &\lesssim \left\| \frac{t}{I(t)} (T_I f^{**})(t) \right\|_X \\ &= \left\| \sup_{t\le s<1} \frac{s}{I(s)} f^{**}(s) \right\|_X \approx \left\| \frac{t}{I(t)} f^{**}(t) \right\|_X \end{aligned}$$

where the "$\approx$" is (1.30). □

3. Optimal spaces

3.1 Optimal target space

As we said in the preliminary chapter of the thesis, given an r.i. space X, its optimal target space Y_X under the map H_I always exists. Moreover, the following inclusions hold:

$$Y_{L^\infty} \subset Y_X \subset Y_{L^1}. \tag{3.1}$$

The situation with optimal domain spaces is a bit different. The proof of the following statement can be found in [14, Proposition 3.3].

Proposition 3.1. *Let I be a concave function and let Y be an r.i. space. Then the functional*

$$\|f\| := \sup_{h\sim f} \|H_I h\|_Y, \quad f \in \mathcal{M}_+(0,1),$$

is an r.i. norm if and only if $H_I 1 \in Y$. In that case $\|\cdot\|_{X_Y} := \|\cdot\|$ is the optimal domain norm for Y under the mapping H_I.

From this proposition it easily follows that if $\frac{1}{I}$ is integrable, then X_{L^∞} exists, and the inclusions

$$X_{L^\infty} \subset X_Y \subset X_{L^1} \tag{3.2}$$

hold for every r.i. space Y.

Lemma 3.2. *Let I be a concave function, $X \subset \Lambda_I$ be an r.i. space and assume that $T_I \colon X' \to X'$. Then $X \in \mathrm{Int}(L^\infty, \Lambda_I)$. If I in addition satisfies (2.7), then $Y_{X'} \in \mathrm{Int}(L^\infty, \Lambda_I)$.*

Proof. Let $f \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$ be given. Let f_0 and f_1 be our favourite decomposition of f at point t. Let T be a linear operator bounded on both L^∞ and Λ_I. Using the subadditivity of $f \mapsto f^{**}$ and Hardy's lemma (1.8), we estimate

$$\begin{aligned}
\int_0^t \frac{I(s)}{s}(Tf)^*(s)\,\mathrm{d}s &= \int_0^t \frac{I(s)}{s}(T(f_0+f_1))^*(s)\,\mathrm{d}s \\
&\le \int_0^t \frac{I(s)}{s}\left((Tf_0)^*(s) + (Tf_1)^*(s)\right)\mathrm{d}s \\
&= \int_0^t \frac{I(s)}{s}(Tf_0)^*(s)\,\mathrm{d}s + \int_0^t \frac{I(s)}{s}(Tf_1)^*(s)\,\mathrm{d}s =: I + II.
\end{aligned}$$

Now, for I, we estimate

$$\begin{aligned}
\int_0^t \frac{I(s)}{s}(Tf_0)^*(s)\,\mathrm{d}s &\le \int_0^t \frac{I(s)}{s}\|Tf_0\|_\infty\,\mathrm{d}s \\
&\lesssim \int_0^t \frac{I(s)}{s}\|f_0\|_\infty\,\mathrm{d}s = \int_0^t \frac{I(s)}{s} f_0^*(s)\,\mathrm{d}s.
\end{aligned}$$

As for the II, we compute

$$\begin{aligned}
\int_0^t \frac{I(s)}{s}(Tf_1)^*(s)\,\mathrm{d}s &\le \int_0^1 \frac{I(s)}{s}(Tf_1)^*(s)\,\mathrm{d}s \\
&\lesssim \int_0^1 \frac{I(s)}{s} f_1^*(s)\,\mathrm{d}s = \int_0^t \frac{I(s)}{s} f_1^*(s)\,\mathrm{d}s.
\end{aligned}$$

Combining the last two estimates yields

$$\int_0^t \frac{I(s)}{s}(Tf)^*(s)\,\mathrm{d}s \lesssim \int_0^t \frac{I(s)}{s} f^*(s)\,\mathrm{d}s.$$

Applying Hardy's lemma (1.8) to $h(s) = \sup_{s\le t<1} \frac{t}{I(t)} g^*(t), g \in \mathcal{M}_+(0,1), s \in (0,1)$, we obtain

$$\int_0^1 \frac{I(s)}{s} \sup_{s\le t<1} \frac{t}{I(t)} g^*(t)(Tf)^*(s)\,\mathrm{d}s \lesssim \int_0^1 \frac{I(s)}{s} \sup_{s\le t<1} \frac{t}{I(t)} g^*(t) f^*(s)\,\mathrm{d}s,$$

which is nothing else than

$$\int_0^1 (T_I g)(s)(Tf)^*(s)\,\mathrm{d}s \lesssim \int_0^1 (T_I g)(s) f^*(s)\,\mathrm{d}s.$$

Finally,

$$\begin{aligned}\int_0^1 (Tf)^*(t) g^*(t)\,\mathrm{d}t \le \int_0^1 (Tf)^*(s)(T_I g)(t)\,\mathrm{d}t \lesssim \int_0^1 f^*(t)(T_I g)(t)\,\mathrm{d}t \\ \le \|f\|_X \|T_I g\|_{X'} \lesssim \|f\|_X \|g\|_{X'}.\end{aligned}$$

Division by $\|g\|_{X'}, g \ne 0$, followed by taking supremum over $\|g\|_{X'} \le 1$ provides us with $\|Tf\|_X \lesssim \|f\|_X, f \in \mathcal{M}_+(0,1)$.

Assume now that I satisfies (2.7). Corollary 2.16 guarantees that

$$\|T_I f\|_{Y'_{X'}} \lesssim \|f\|_{Y'_{X'}}, \quad f \in \mathcal{M}_+(0,1).$$

It remains to show that $Y_{X'} \subset \Lambda_I$ for every r.i. space X. This is by (3.1) equivalent to showing that $Y_{L^1} \subset \Lambda_I$. We show that, in fact, $Y_{L^1} = \Lambda_I$. We know

$$\|f\|_{Y'_{L^1}} = \|R_I f^*\|_\infty = \sup_{0<t<1} \frac{t}{I(t)} f^{**}(t) = \|f\|_{M_{\widetilde{I}}}, \quad f \in \mathcal{M}_+(0,1).$$

Observe that the condition (2.7) simply states that $\widetilde{I}$ satisfies has the average property. Therefore, Remark 2.9 asserts that $M_{\widetilde{I}} = m_{\widetilde{I}}$, from whence it follows that $Y_{L^1} = \Lambda_I$. □

Our next objective is to describe the norm of the optimal target space Y_X. The first step in this direction is to describe the norm of Y'_X via a functional, $\|\cdot\|_Z$, such that S_I is bounded thereon. We begin by exploring the mapping properties of the operator R_I.

Lemma 3.3. *Let I be a concave function satisfying* (2.7)*. Then the operator R_I has the following mapping properties:*

(i) $R_I \colon m_{\widetilde{I}} \to L^\infty$,

(ii) $R_I \colon L^1 \to m_I$,

(iii) $R_I \colon L^1 \to (m_I)_b$.

Proof. Let $f \in m_{\widetilde{I}}$ and estimate

$$\|R_I f\|_\infty = \sup_{0<t<1} \frac{t}{I(t)} f^{**}(t) \approx \sup_{0<t<1} \frac{t}{I(t)} f^*(t) = \|f\|_{m_{\widetilde{I}}},$$

where the approximation follows from the assumption that $\widetilde{I}$ has the average property.

We take care of *(ii)* and *(iii)* in one fell swoop. Let $f_n \to f$ in L^1. Then, for every $t \in (0,1)$, we have

$$|(R_I f)(t)| \leq \|f\|_\infty \frac{1}{I(t)} \int_0^t \mathrm{d}s = \|f\|_\infty \frac{t}{I(t)} \lesssim \|f\|_\infty,$$

from whence it follows that if f_n is a sequence of bounded functions, then $R_I f_n$ is a sequence of bounded functions. Now,

$$|(R_I f_n)(t) - (R_I f)(t)| = \left| \frac{1}{I(t)} \int_0^t (f_n(s) - f(s))\, \mathrm{d}s \right| \leq \frac{1}{I(t)} \|f - f_n\|_1$$

and so,

$$\left(s \mapsto |(R_I f_n)(s) - (R_I f)(s)|\right)^*(t) \leq \frac{1}{I(t)} \|f - f_n\|_1.$$

Multiplying through by $I(t)$ and taking supremum over $t \in (0,1)$ finish the proof. $\square$

Theorem 3.4. *Let I be a concave function satisfying* (2.7) *and let X be an r.i. space. Define $\|f\|_Z = \|S_I f\|_{X'}$ for $f \in \mathcal{M}_+(0,1)$. Then*

$$\|f\|_{Y_X'} = \left\| \frac{1}{I(t)} \int_0^t f^*(s)\, \mathrm{d}s \right\|_{X'} \approx \left\| \frac{1}{I(t)} \int_0^t f^*(s)\, \mathrm{d}s \right\|_Z \tag{3.3}$$

and S_I is bounded on Z.

Proof. We define three functionals

$$\begin{aligned} &\lambda(f) \coloneqq \|f^{**}(I(t))\|_{X'}, \\ &\alpha(f) \coloneqq \lambda(k(f, t, L^1, m_{\widetilde{I}})) \\ &\text{and} \\ &\beta(f) \coloneqq \lambda(k(f, t, (m_I)_b, L^\infty)) \end{aligned}$$

for $f \in \mathcal{M}_+(0,1)$. We check that λ is an r.i. norm. The triangle inequality follows from the subadditivity of $f \mapsto f^{**}$ and the triangle inequality of $\|\cdot\|_{X'}$. The rest of (P1) obviously holds. The same goes for (P2). When $0 \leq f_n \nearrow f$, then $f_n^{**} \nearrow f^{**}$ and so $f_n^{**}(I(t)) \nearrow f^{**}(I(t))$ for every $t \in (0,1)$. Hence (P3) for λ follows from (P3) of X'. Regarding (P4), we note that the maximal nonincreasing rearrangement of a constant function is the function itself, and so the required property follows from its counterpart in X'. As for (P5), we estimate

$$\begin{aligned} \lambda(f) = \|f^{**}(I(t))\|_{X'} &\geq \left\| f^{**}(I(t)) \chi_{(0,\frac{1}{2})}(t) \right\|_{X'} \geq \left\| f^{**}\left(I\left(\frac{1}{2}\right)\right) \chi_{(0,\frac{1}{2})} \right\|_{X'} \\ &\approx f^{**}\left(I\left(\frac{1}{2}\right)\right) \approx f^{**}\left(\frac{1}{2}\right) \approx \|f\|_1, \quad f \in \mathcal{M}_+(0,1). \end{aligned}$$

As λ is obviously rearrangement invariant, depending only on the nonincreasing rearrangement of f, we conclude that λ is an r.i. norm.

In general, if we have a compatible couple of quasi-Banach spaces (X_0, X_1) such that $X_0 \cap X_1$ is dense in X_0, Theorem 1.15 combined with the definition of λ give us

$$\begin{aligned} \lambda(k(f,t,X_0,X_1)) &= \left\| \frac{1}{I(t)} \int_0^{I(t)} k(f,t,X_0,X_1) \right\|_{X'} \\ &= \left\| \frac{1}{I(t)} K(f,I(t),X_0,X_1) \right\|_{X'}. \end{aligned} \tag{3.4}$$

Since $L^1 \cap m_{\widetilde{I}}$ is dense in L^1 and $I(t)$ is an increasing bijection of $(0,1)$ onto itself, by Proposition 1.19 we have

$$\begin{aligned} \alpha(f) = \left\| \frac{1}{I(t)} K(f,I(t),L^1,m_{\widetilde{I}}) \right\|_{X'} &\le \left\| \frac{1}{I(t)} \left(\int_0^t f^*(s)\,\mathrm{d}s + I(t) \sup_{t\le s<1} \frac{s}{I(s)} f^*(s) \right) \right\|_{X'} \\ \le \|f\|_{Y'_X} + \left\| \frac{t}{I(t)} (T_I f) \right\|_{X'} \lesssim \|f\|_{Y'_X}, \end{aligned} \tag{3.5}$$

where the last estimate comes from Corollary 2.16.

Next, $(m_I)_b \cap L^\infty$ is dense in $(m_I)_b$. Using (3.4) once more we have

$$\begin{aligned} \beta(f) &= \left\| \frac{1}{I(t)} K(f,I(t),(m_I)_b,L^\infty) \right\|_{X'} \\ &\ge \left\| \frac{1}{I(t)} K(f,I(t),m_I,L^\infty) \right\|_{X'} \approx \left\| \frac{1}{I(t)} \sup_{0<s\le t} I(s) f^*(s) \right\|_{X'} = \|f\|_Z, \end{aligned} \tag{3.6}$$

where the first inequality follows from enlarging the space $(m_I)_b$ to m_I and hence allowing more decompositions. The approximation is an application of Lemma 2.10.

Adding Lemma 3.3 to the kettle we see that all the assumptions of Theorem 1.21 are satisfied. Therefore

$$\beta\left(R_I f^*\right) \lesssim \alpha(f^*). \tag{3.7}$$

Finally, chaining (3.6), (3.7) and (3.5) together, we arrive at

$$\|R_I f^*\|_Z \lesssim \beta(R_I f^*) \lesssim \alpha(f^*) \lesssim \|f\|_{Y'_X}. \tag{3.8}$$

This finishes the proof, as the reverse inequality holds trivially. □

We now introduce two functionals, λ and μ, and exhibit their equivalence in Proposition 3.6.

Lemma 3.5. *Let I be a concave function and let X be an r.i. space. Define a functional $\|f\|_Z = \|S_I f\|_X, f \in \mathcal{M}_+(0,1)$. Then*

$$\mu(f) := \sup_{\substack{\|g\|_Z \le 1 \\ \|g\|_\infty < \infty}} \int_0^1 f^*(t)\,\mathrm{d}\operatorname*{csup}_{0<s\le t} I(s) g^*(s) + \|f\|_1, \quad f \in \mathcal{M}_+(0,1), \tag{3.9}$$

is an r.i. norm. Here, $\operatorname*{csup}_{0<s\le t} \varphi(s)$ stands for the least concave majorant of $t \mapsto \sup_{0<s\le t} \varphi(s)$.

Proof. Denote $h_g(t) = \operatorname*{csup}_{0<s\le t} I(s)g^*(s)$ for $g \in \mathcal{M}_+(0,1)$. Let us observe that $t \mapsto \sup_{0<s\le t} I(s)g^*(s)$ is a quasiconcave map for every $g \in Z$ and so h_g is a finite concave function. Indeed, for such functions g the supremum is finite, as $Z \subset m_I$, and with increasing t the supremum is nondecreasing. It thus remains to check that for $0 < t_1 < t_2 < 1$ the inequality

$$\frac{\sup_{0<s\le t_2} I(s)g^*(s)}{t_2} \le \frac{\sup_{0<s\le t_1} I(s)g^*(s)}{t_1}$$

holds. We calculate

$$\begin{aligned}\frac{\sup_{t_1<s\le t_2} I(s)g^*(s)}{t_2} &\le g^*(t_1)\frac{\sup_{0<s\le t_1} I\left(\frac{t_2}{t_1}\cdot s\right)}{\frac{t_2}{t_1}\cdot t_1}\\ &\le \frac{\sup_{0<s\le t_1} I(s)g^*(t_1)}{t_1} \le \frac{\sup_{0<s\le t_1} I(s)g^*(s)}{t_1},\end{aligned}$$

where in the second to last inequality we used the concavity of I. For $\|g\|_Z \le 1$ we therefore have

$$\int_0^1 f^*(t)\,\mathrm{d}h_g(t) = \int_0^1 f^*(t)\frac{\mathrm{d}h_g(t)}{\mathrm{d}t}\,\mathrm{d}t.$$

As $t \mapsto \frac{\mathrm{d}h_g(t)}{\mathrm{d}t}$ is nonincreasing, Hardy's lemma (1.8) gives us a triangle inequality of the functional μ. Property (P2) is obvious. (P3) follows from the monotone convegence theorem. To check (P4), let $\|g\|_Z \le 1$ be given. Using the fact that $Z \hookrightarrow m_I$, we estimate

$$\int_0^1 \mathrm{d}\operatorname*{csup}_{0<s\le t} I(s)g^*(s) \le \operatorname*{csup}_{0<s<1} I(s)g^*(s) \approx \sup_{0<s<1} I(s)g^*(s) = \|g\|_{m_I} \lesssim \|g\|_Z \le 1.$$

Thus $\mu(\chi_{(0,1)}) < \infty$. Regarding (P5), $\|f\|_1 \le \mu(f)$, $f \in \mathcal{M}_+(0,1)$, follows from the definition of μ. Finally, μ is rearrangement invariant, as the first expression in its definition (3.9) depends only on the nonincreasing rearrangement of a function and the second term, $\|\cdot\|_1$, is rearrangement invariant. ☐

Proposition 3.6. *Let I be a concave function and let X be an r.i. space. Put $\|f\|_Z := \|S_I f\|_X$ for $f \in \mathcal{M}_+(0,1)$. Then the functional λ on $\mathcal{M}_+(0,1)$ defined by*

$$\lambda(f) = \sup_{\|g\|_Z\le 1} \int_0^1 -I(t)g^*(t)\,\mathrm{d}f^*(t) + \|f\|_1, \quad f \in \mathcal{M}_+(0,1),$$

is equivalent to an r.i. norm.

Proof. We show that $\lambda \approx \mu$, where μ is the functional from Lemma 3.5. First observe that, by the monotone convergence theorem, it suffices to consider only bounded functions over which we take the supremum in the definition of the functional λ.

We show their equivalence in three steps. Let first $f \in \mathcal{M}_+(0,1)$ be such that $f^*(0+) < \infty$ and $f^*(1-) = 0$ and pick $g \in \mathcal{M}_+(0,1)$ bounded such that $\|g\|_Z \le 1$. Then

$$\lim_{t\to 0^+} f^*(t)\operatorname*{csup}_{0<s\le t} I(s)g^*(s) = \lim_{t\to 1^-} f^*(t)\operatorname*{csup}_{0<s\le t} I(s)g^*(s) = 0.$$

Consequently, integration by parts yields

$$\int_0^1 f^*(t)\,\mathrm{d}\operatorname*{csup}_{0<s\le t} I(s)g^*(s) = \int_0^1 -\operatorname*{csup}_{0<s\le t} I(s)g^*(s)\,\mathrm{d}f^*(t).$$

We therefore have

$$\begin{aligned}
&\sup_{\substack{\|g\|_Z\le 1\\ \|g\|_\infty<\infty}} \int_0^1 f^*(t)\,\mathrm{d}\operatorname*{csup}_{0<s\le t} I(s)g^*(s) = \sup_{\substack{\|g\|_Z\le 1\\ \|g\|_\infty<\infty}} \int_0^1 -\operatorname*{csup}_{0<s\le t} I(s)g^*(s)\,\mathrm{d}f^*(t)\\
&\approx \sup_{\substack{\|g\|_Z\le 1\\ \|g\|_\infty<\infty}} \int_0^1 -\sup_{0<s\le t} I(s)g^*(s)\,\mathrm{d}f^*(t) = \sup_{\substack{\|g\|_Z\le 1\\ \|g\|_\infty<\infty}} \int_0^1 -I(t)(S_I g)(t)\,\mathrm{d}f^*(t)\\
&= \sup_{\substack{\|S_I g\|_Z\le 1\\ \|g\|_\infty<\infty}} \int_0^1 -I(t)(S_I g)(t)\,\mathrm{d}f^*(t) = \sup_{\substack{\|g\|_Z\le 1\\ \|g\|_\infty<\infty}} \int_0^1 -I(t)g^*(t)\,\mathrm{d}f^*(t),
\end{aligned}$$

and so $\mu(f) \approx \lambda(f)$. Here we used the fact that $\|g\|_Z = \|S_I g\|_Z$ for every $g \in \mathcal{M}_+(0,1)$ and Theorem 2.5, *(i)*, which says that if g is bounded, so is $S_I g$.

Second, let $f \in \mathcal{M}_+(0,1)$ be such that $f^*(1-) = 0$. We show that for $\min\{f^*, n\} =: f_n = f_n^*$ we have $\lambda(f_n) \nearrow \lambda(f)$. Indeed, by the monotone convergence theorem we have

$$\begin{aligned}
\lambda(f_n) &= \sup_{\|g\|_Z\le 1} \int_0^1 -I(t)g^*(t)\,\mathrm{d}f_n^*(t) + \|f_n\|_1\\
&= \sup_{\|g\|_Z\le 1} \int_{t_n}^1 -I(t)g^*(t)\,\mathrm{d}f^*(t) + \|f_n\|_1 \nearrow \lambda(f),
\end{aligned}$$

where $t_n = \inf\{t \in (0,1)\colon f^*(t) \le n\}$.

Finally, let $f \in \mathcal{M}_+(0,1)$ be such that $f^*(1-) < \infty$. Then, using the previous step, we have

$$\begin{aligned}
\lambda(f) &= \sup_{\|g\|_Z\le 1} \int_0^1 -I(t)g^*(t)\,\mathrm{d}f^*(t) + \|f\|_1\\
&= \sup_{\|g\|_Z\le 1} \int_0^1 -I(t)g^*(t)\,\mathrm{d}(f^* - f^*(1-))(t) + \|f^* - f^*(1-)\|_1 + f^*(1-)\\
&\approx \mu(f^* - f^*(1-)) + f^*(1-) \approx \mu(f),
\end{aligned}$$

since

$$\begin{aligned}
\mu(f^* - f^*(1-)) + f^*(1-) &\le \mu(f) + \mu(f^*(1-)) + f^*(1-)\\
&\approx \mu(f) + \|f^*(1-)\|_1 \le \mu(f) + \|f\|_1 \approx \mu(f)
\end{aligned}$$

and

$$\mu(f) \le \mu(f^* - f^*(1-)) + \mu(f^*(1-)) \approx \mu(f - f^*(1-)) + f^*(1-).$$

□

Remark 3.7. Both functionals μ and λ contain $\|\cdot\|_1$ in their definitions. For the functional μ it guarantees that $X(\mu) \hookrightarrow L^1$, while for λ it guarantees that $\lambda(f) = 0 \iff f = 0$ a.e.

Before we establish an alternative description of the optimal target space norm, we require a technical lemma which extends (1.30) to our setting.

Lemma 3.8. *Let I be a concave function and X be an r.i. space. Put $\|f\|_Z = \|S_I f\|_X, f \in \mathcal{M}_+(0,1)$. Then*

$$\|R_I f^*\|_Z \approx \|G_I f\|_Z, \quad f \in \mathcal{M}_+(0,1).$$

Proof. We show that $S_I G_I f = G_I f$ for every $f \in \mathcal{M}_+(0,1)$. To this end, fix $f \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$. We calculate

$$\begin{aligned}
(S_I G_I f)(t) &= \frac{1}{I(t)} \sup_{0<s\le t} I(s) \sup_{s\le r<1} \frac{1}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \\
&= \frac{1}{I(t)} \sup_{0<s\le t} \sup_{s\le r<1} I(s) \frac{1}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \\
&= \frac{1}{I(t)} \max\left\{ \sup_{0<s\le t} \sup_{s\le r\le t} \frac{I(s)}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z, \sup_{0<s\le t} \sup_{t\le r<1} \frac{I(s)}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \right\} \\
&= \frac{1}{I(t)} \max\left\{ \sup_{0<r\le t} \sup_{0<s\le r} \frac{I(s)}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z, \sup_{t\le r<1} \frac{I(t)}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \right\} \\
&= \frac{1}{I(t)} \max\left\{ \sup_{0<r\le t} \int_0^r f^*(z)\,\mathrm{d}z, \sup_{t\le r<1} \frac{I(t)}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \right\} \\
&= \frac{1}{I(t)} \max\left\{ \frac{I(t)}{I(t)} \int_0^t f^*(z)\,\mathrm{d}z, \sup_{t\le r<1} \frac{I(t)}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \right\} \\
&= \max\left\{ \frac{1}{I(t)} \int_0^t f^*(z)\,\mathrm{d}z, \sup_{t\le r<1} \frac{1}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z \right\} \\
&= \sup_{t\le r<1} \frac{1}{I(r)} \int_0^r f^*(z)\,\mathrm{d}z = G_I f(t).
\end{aligned}$$

Therefore, by (1.30),

$$\|G_I f\|_Z = \|S_I G_I f\|_X = \|G_I f\|_X \approx \|R_I f^*\|_X \le \|S_I R_I f^*\|_X = \|R_I f^*\|_Z.$$

We know $R_I f^* \le G_I f$ and so $S_I R_I f^* \le S_I G_I f$. Furnishing this last inequality with $\|\cdot\|_X$ finishes the proof. □

Theorem 3.9. *Let I be a concave function satisfying* (2.7) *and let X be an r.i. space. Then*

$$\|f\|_{Y_X} \approx \lambda(f), \quad f \in \mathcal{M}_+(0,1),$$

where λ is the functional from Proposition 3.6.

Proof. Put $\|f\|_Z = \|S_I f\|_{X'}$ for $f \in \mathcal{M}_+(0,1)$. Let $f, g \in \mathcal{M}_+(0,1)$ be given and assume $f^*(0+) < \infty$ and $f^*(1-) = 0$. Then

$$\begin{aligned}\int_0^1 f^*(s)g^*(s)\,\mathrm{d}s &= \int_0^1 -g^*(s)\int_s^1 \mathrm{d}f^*(t)\,\mathrm{d}s = \int_0^1 -\int_0^t g^*(s)\,\mathrm{d}s\,\mathrm{d}f^*(t)\\ &= \int_0^1 -\frac{I(t)}{I(t)}\int_0^t g^*(s)\,\mathrm{d}s\,\mathrm{d}f^*(t) \le \int_0^1 -I(t)(G_I g)(t)\,\mathrm{d}f^*(t)\\ &\approx \frac{\|g\|_{Y_X'}}{\left\|\frac{1}{I(t)}\int_0^t g^*\right\|_Z}\int_0^1 -I(t)(G_I g)(t)\,\mathrm{d}f^*(t)\\ &\approx \frac{\|g\|_{Y_X'}}{\|G_I g\|_Z}\int_0^1 -I(t)(G_I g)(t)\,\mathrm{d}f^*(t)\\ &\le \|g\|_{Y_X'}\cdot \sup_{\|g\|_Z\le 1}\int_0^1 -I(t)g^*(t)\,\mathrm{d}f^*(t) \le \|g\|_{Y_X'}\cdot\lambda(f),\end{aligned}$$

where the first approximation is Theorem 3.4 and in the second approximation we used Lemma 3.8. Dividing by $\|g\|_{Y_X'}$ and taking supremum over $g \in \mathcal{M}_+(0,1)$ with $\|g\|_{Y_X'} \le 1$ gives us $\|f\|_{Y_X} \lesssim \lambda(f)$.

In the other direction, let $f \in \mathcal{M}_+(0,1)$ be arbitrary. As

$$\|f\|_{Y_X'} = \left\|\frac{1}{I(t)}\int_0^t f^*\right\|_{X'},$$

there exists $h \in \mathcal{M}_+(0,1)$ with $\|h\|_X \le 1$ and

$$\frac{1}{2}\|f\|_{Y_X'} \le \int_0^1 \frac{h(t)}{I(t)}\int_0^t f^*(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Put $g(t) = g^*(t) = \int_t^1 \frac{h(s)}{I(s)}\,\mathrm{d}s$. For $k \in \mathcal{M}_+(0,1)$ satisfying $\|k\|_Z \le 1$ we have

$$\begin{aligned}\int_0^1 -I(t)k^*(t)\,\mathrm{d}g^*(t) &= \int_0^1 I(t)k^*(t)\frac{h(t)}{I(t)}\,\mathrm{d}t\\ &= \int_0^1 k^*(t)h(t)\,\mathrm{d}t \le \|k\|_{X'}\|h\|_X \le \|k\|_Z\|h\|_X \le 1.\end{aligned}$$

By Fubini's theorem we estimate

$$\int_0^1 g^*(t)\,\mathrm{d}t = \int_0^1\int_t^1 \frac{h(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t = \int_0^1 h(s)\frac{s}{I(s)}\,\mathrm{d}s \lesssim \int_0^1 h(s)\,\mathrm{d}s \lesssim \|h\|_X \le 1.$$

The last two estimates tell us that $\lambda(g) \lesssim 1$.

We further have

$$\int_0^1 f^*(t)g^*(t)\,\mathrm{d}t = \int_0^1 f^*(t)\int_t^1 \frac{h(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t = \int_0^1 \frac{h(s)}{I(s)}\int_0^s f^*(t)\,\mathrm{d}t\,\mathrm{d}s \ge \frac{1}{2}\|f\|_{Y_X'}.$$

Finally, putting everything together yields

$$\frac{1}{2}\|f\|_{Y_X'} \le \int_0^1 f^*(t)g^*(t)\,\mathrm{d}t \le \lambda'(f)\lambda(g) \lesssim \lambda'(f)$$

or, equivalently, $\lambda(f) \lesssim \|f\|_{Y_X}$. □

3.2 Alternative norm in the optimal target space

Starting from Proposition 3.12, we will need boundedness of $f \mapsto f^{**}$ on Λ_I. The following characterization can be found in [17, Theorem 10.3.12].

Theorem 3.10. *The operator $f \mapsto f^{**}$ is bounded on Λ_I if and only if*

$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s \lesssim \frac{1}{t}\int_0^t \frac{I(s)}{s}\,\mathrm{d}s, \quad t \in (0,1).$$

Remark 3.11. In the proposition that follows, we will use a slighty stronger condition, namely

$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s \lesssim \frac{I(t)}{t}, \quad t \in (0,1). \tag{3.10}$$

This condition is indeed stronger, as

$$\frac{I(t)}{t} \le \frac{1}{t}\int_0^t \frac{I(s)}{s}\,\mathrm{d}s, \quad t \in (0,1),$$

is true due to $t \mapsto \frac{I(t)}{t}$ being nonincreasing. However, we will, as in many previous results, assume $\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \lesssim I(t), t \in (0,1)$. In other words, in our setting, (3.10) is equivalent to the condition in Theorem 3.10.

Proposition 3.12. *Let I be a concave function satisfying* (2.7) *and* (3.10)*. Let X be an r.i. space. Put $\|f\|_Z = \|S_I f\|_{X'}, f \in \mathcal{M}_+(0,1)$. Then*

$$\|f\|_{Y_X} \approx \sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t}(f^{**}(t) - f^*(t))g^*(t)\,\mathrm{d}t + \|f\|_1, \quad f \in \mathcal{M}_+(0,1).$$

Proof. By virtue of Theorem 3.10, the mapping $f \mapsto f^{**}$ is bounded on Λ_I which implies that the map $f \mapsto \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$ is, too, bounded on Λ_I. Lemma 3.2 tells us that $Y_X \in \mathrm{Int}(L^\infty, \Lambda_I)$. Therefore $f \mapsto \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$ and, consequently, $f \mapsto f^{**}$ are bounded on Y_X.

Now, for every $\varepsilon \in (0, \frac{1}{2})$ and $g \in \mathcal{M}_+(0,1)$ we have

$$\begin{aligned}
&\int_\varepsilon^{1-\varepsilon} -I(t)g^*(t)\,\mathrm{d}\left[\frac{1}{t}\int_0^t f^{**}(s)\,\mathrm{d}s\right] \\
&= \int_\varepsilon^{1-\varepsilon} -I(t)g^*(t)\left[-\frac{1}{t^2}\int_0^t f^{**}(s)\,\mathrm{d}s + \frac{1}{t}f^{**}(t)\right]\mathrm{d}t \\
&= \int_\varepsilon^{1-\varepsilon} I(t)\left[\frac{1}{t}\int_0^t (f^{**}(s) - f^*(s))\,\mathrm{d}s\right] g^*(t)\,\mathrm{d}t.
\end{aligned}$$

Thus, passing to the limit and using Fubini's Theorem, we have

$$\begin{aligned}
\int_0^1 I(t)g^*(t)\,\mathrm{d}\left[\frac{1}{t}\int_0^t f^{**}(s)\,\mathrm{d}s\right] &= \int_0^1 \frac{I(t)}{t^2}\int_0^t (f^{**}(s) - f^*(s))\,\mathrm{d}s\, g^*(t)\,\mathrm{d}t \\
&= \int_0^1 \frac{I(s)}{s}(f^{**}(s) - f^*(s))\frac{s}{I(s)}\int_s^1 \frac{I(t)}{t^2}g^*(t)\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^1 \frac{I(s)}{s}(f^{**}(s) - f^*(s))(Hg^*)(s).
\end{aligned}$$

Here, the operator H is defined as

$$Hg(s) = \frac{s}{I(s)} \int_s^1 \frac{I(t)}{t^2} g(t)\,\mathrm{d}t, \quad g \in \mathcal{M}_+(0,1), s \in (0,1). \tag{3.11}$$

Observe that, on one hand, we have

$$(Hg^*)(s) = \frac{s}{I(s)} \int_s^1 \frac{I(t)}{t^2} g^*(t)\,\mathrm{d}t \le g^*(s) \frac{s}{I(s)} \int_s^1 \frac{I(t)}{t^2}\,\mathrm{d}t \lesssim g^*(s), \tag{3.12}$$

and, on the other hand,

$$(Hg^*)\left(\frac{s}{2}\right) \gtrsim \frac{s}{I(s)} \int_{\frac{s}{2}}^s g^*(t) \frac{I(t)}{t^2}\,\mathrm{d}t \gtrsim \frac{s}{I(s)} g^*(s) \frac{I(s)}{s^2} \cdot s = g^*(s). \tag{3.13}$$

Finally, using Theorem 3.9, we get

$$\begin{aligned} \|f\|_{Y_X} &\approx \left\| \frac{1}{t} \int_0^t f^{**}(s)\,\mathrm{d}s \right\|_{Y_X} \\ &\approx \sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t} (f^{**}(t) - f^*(t))(Hg^*)(t)\,\mathrm{d}t + \|f\|_1. \end{aligned} \tag{3.14}$$

To finish the proof we need to show that

$$\sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t} (f^{**}(t) - f^*(t))(Hg^*)(t)\,\mathrm{d}t \approx \sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t} (f^{**}(t) - f^*(t)) g^*(t)\,\mathrm{d}t.$$

The inequality "$\lesssim$" immediately follows from (3.12).

The boundedness of the dilation operator followed by (3.14) and the fact that $(f^*(2\cdot))^{**}(t) = f^{**}(2t)$, substitution and (3.13) yield

$$\begin{aligned} \|f\|_{Y_X} \approx \|f^*(2t)\|_{Y_X} &\approx \sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t} (f^{**}(2t) - f^*(2t))(Hg^*)(t)\,\mathrm{d}t + \|f\|_1 \\ &\gtrsim \sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t} (f^{**}(t) - f^*(t))(Hg^*)\left(\frac{t}{2}\right)\mathrm{d}t + \|f\|_1 \\ &\gtrsim \sup_{\|g\|_Z \le 1} \int_0^1 \frac{I(t)}{t} (f^{**}(t) - f^*(t)) g^*(t)\,\mathrm{d}t + \|f\|_1. \end{aligned}$$

□

Theorem 3.13. *Let I be a concave function satisfying* (2.7), (3.10) *and the average property. Let X be an r.i. space such that S_I is bounded on X'. Assume*

$$c = \sup\left\{\lambda \ge 0 : \lambda\left(\frac{I(t)}{t^2} - 1\right) \le \int_t^1 \frac{I(s)}{s^3}\,\mathrm{d}s, \quad t \in (0,1)\right\} \in (0,\infty) \tag{3.15}$$

and let d denote the smallest positive number such that

$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s \le d \frac{I(t)}{t}.$$

If

$$\textit{either} \quad c - 1 \ge 0 \quad \textit{or} \quad (1-c)d \le c,$$

then

$$\|f\|_{Y_X} \approx \left\| \frac{I(t)}{t} (f^{**}(t) - f^*(t)) \right\|_X + \|f\|_1. \tag{3.16}$$

Proof. Since S_I is bounded on X', we have $\|S_I g\|_{X'} \approx \|g\|_{X'}, g \in \mathcal{M}_+(0,1)$. Therefore, by Proposition 3.12, (1.19) and the fact that $\|\cdot\|_X = \|\cdot\|_{X''}$, we obtain

$$\begin{aligned}
\|f\|_{Y_X} &\approx \sup_{\|S_I g\|_{X'} \le 1} \int_0^1 \frac{I(t)}{t}(f^{**}(t) - f^*(t))g^*(t)\,\mathrm{d}t + \|f\|_1 \\
&\approx \sup_{\|g\|_{X'} \le 1} \int_0^1 \frac{I(t)}{t}(f^{**}(t) - f^*(t))g^*(t)\,\mathrm{d}t + \|f\|_1 \\
&= \left\| \frac{I(t)}{t}(f^{**}(t) - f^*(t)) \right\|_{(X')'_d} + \|f\|_1 \\
&= \left\| \left(\frac{I(s)}{s}(f^{**}(s) - f^*(s)) \right)^{\circ} \right\|_X + \|f\|_1, \quad f \in \mathcal{M}_+(0,1).
\end{aligned} \tag{3.17}$$

It follows from (1.18) and the HLP principle (1.9) that $\|f^\circ\| \le \|f\|$ for every $f \in \mathcal{M}_+(0,1)$ and every r.i. norm $\|\cdot\|$. We have therefore obtained "$\lesssim$" in (3.16).

From the definition of the level function, for every $f \in \mathcal{M}_+(0,1)$, we have

$$\int_0^t \frac{I(s)}{s}(f^{**}(s) - f^*(s))\,\mathrm{d}s \le \int_0^t \left(\frac{I(y)}{y}(f^{**}(y) - f^*(y)) \right)^{\circ}(s)\,\mathrm{d}s, \quad t \in (0,1).$$

Using Hardy's lemma we get to

$$\int_0^t \frac{I(s)}{s}(f^{**}(s) - f^*(s))g^{**}(s)\,\mathrm{d}s \le \int_0^t \left(\frac{I(y)}{y}(f^{**}(y) - f^*(y)) \right)^{\circ}(s)g^{**}(s)\,\mathrm{d}s$$

for every $t \in (0,1), f, g \in \mathcal{M}_+(0,1)$. Using Fubini's theorem, we rewrite the last inequality as

$$\int_0^t g^*(s) \int_s^t \frac{I(y)}{y}(f^{**}(y) - f^*(y))\frac{\mathrm{d}y}{y}\,\mathrm{d}s \le \int_0^t g^*(s) \int_s^t \left(\frac{I(z)}{z}(f^{**}(z) - f^*(z)) \right)^{\circ}(y)\frac{\mathrm{d}y}{y}\,\mathrm{d}s. \tag{3.18}$$

For $f \in \mathcal{M}_+(0,1)$ we calculate

$$\begin{aligned}
&\int_t^1 \frac{I(s)}{s}(f^{**}(s) - f^*(s))\frac{\mathrm{d}s}{s} \\
&= \int_t^1 \frac{I(s)}{s^3} \int_0^1 \chi_{(0,s)}(y) f^*(y)\,\mathrm{d}y\,\mathrm{d}s - \int_t^1 \frac{I(s)}{s^2} f^*(s)\,\mathrm{d}s \\
&= \int_0^1 f^*(y) \int_t^1 \frac{I(s)}{s^3}\chi_{(y,1)}(s)\,\mathrm{d}s\,\mathrm{d}y - \int_t^1 \frac{I(s)}{s^2} f^*(s)\,\mathrm{d}s \\
&= \underbrace{\int_0^t f^*(y)\,\mathrm{d}y \int_t^1 \frac{I(s)}{s^3}\,\mathrm{d}s}_{I} + \underbrace{\int_t^1 f^*(y) \int_y^1 \frac{I(s)}{s^3}\,\mathrm{d}s\,\mathrm{d}y - \int_t^1 \frac{I(s)}{s^2} f^*(s)\,\mathrm{d}s}_{II}.
\end{aligned} \tag{3.19}$$

For I we estimate

$$I = t f^{**}(t) \int_t^1 \frac{I(s)}{s^3}\,\mathrm{d}s \ge c t f^{**}(t) \left(\frac{I(t)}{t^2} - 1 \right) = c\frac{I(t)}{t} f^{**}(t) - c t f^{**}(t). \tag{3.20}$$

For II we proceed similarly:

$$\begin{aligned}
II &\ge \int_t^1 f^*(y) c \left(\frac{I(y)}{y^2} - 1 \right)\mathrm{d}y - \int_t^1 \frac{I(y)}{y^2} f^*(y)\,\mathrm{d}y \\
&= \int_t^1 (c-1) f^*(y) \frac{I(y)}{y^2}\,\mathrm{d}y - c\int_t^1 f^*(y)\,\mathrm{d}y.
\end{aligned} \tag{3.21}$$

If $c-1\geq 0$, then we can omit the first term on the second line of (3.21), not making it greater. Therefore, putting (3.20) and (3.21) together and using (3.19), we arrive at

$$\begin{aligned} c\frac{I(t)}{t}(f^{**}(t)-f^*(t))-c\|f\|_1 &\leq c\frac{I(t)}{t}f^{**}(t)-ctf^{**}(t)-c\int_t^1 f^*(y)\,\mathrm{d}y \\ &\leq I+II=\int_t^1\frac{I(s)}{s}(f^{**}(s)-f^*(s))\frac{\mathrm{d}s}{s}. \end{aligned} \tag{3.22}$$

If $c-1<0$, we return to (3.21) and continue in our estimates:

$$\begin{aligned} \int_t^1(c-1)f^*(y)\frac{I(y)}{y^2}\,\mathrm{d}y-c\int_t^1 f^*(y)\,\mathrm{d}y &\geq (c-1)f^*(t)\int_t^1\frac{I(y)}{y^2}\,\mathrm{d}y-c\int_t^1 f^*(y)\,\mathrm{d}y \\ &\geq (c-1)d\frac{I(t)}{t}f^*(t)-c\int_t^1 f^*(y)\,\mathrm{d}y. \end{aligned}$$

Assuming therefore that $(1-c)d\leq c$, similarly as in (3.22), we get

$$\begin{aligned} (1-c)d\frac{I(t)}{t}(f^{**}(t)-f^*(t))-c\|f\|_1 &\leq c\frac{I(t)}{t}f^{**}(t)+(c-1)d\frac{I(t)}{t}f^*(t)-c\|f\|_1 \\ &\leq \int_t^1\frac{I(s)}{s}(f^{**}(s)-f^*(s))\frac{\mathrm{d}s}{s}. \end{aligned}$$

Either way, we have

$$\frac{I(t)}{t}(f^{**}(t)-f^*(t))\lesssim\int_t^1\frac{I(s)}{s}(f^{**}(s)-f^*(s))\frac{\mathrm{d}s}{s}+\|f\|_1,\quad f\in\mathcal{M}_+(0,1),t\in(0,1).$$

Hence, for every $f\in\mathcal{M}_+(0,1)$ we have

$$\begin{aligned} \left\|\frac{I(t)}{t}(f^{**}(t)-f^*(t))\right\|_X &\lesssim \left\|\int_t^1\frac{I(s)}{s}(f^{**}(s)-f^*(s))\frac{\mathrm{d}s}{s}\right\|_X+\|f\|_1 \\ &\lesssim \left\|\int_t^1\left(\frac{I(y)}{y}(f^{**}(y)-f^*(y))\right)^\circ(s)\frac{\mathrm{d}s}{s}\right\|_X+\|f\|_1, \end{aligned} \tag{3.23}$$

where the second inequality comes from (3.18) by taking $t=1$ and supremum over all $g\in\mathcal{M}_+(0,1)$ with $\|g\|_{X'}\leq 1$.

As I has the average property, by Remark 2.9 $f\mapsto\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$ is bounded simultaneously on L^∞ and m_I. Since the operator S_I is bounded on X', Theorem 2.11 asserts that $X'\in\operatorname{Int}(L^\infty,m_I)$. Therefore $f\mapsto\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$ is bounded on X' and so, by duality (1.14), its associate operator, $f\mapsto\int_t^1\frac{f(s)}{s}\,\mathrm{d}s$, is bounded on X. We therefore have

$$\left\|\int_t^1\left(\frac{I(y)}{y}(f^{**}(y)-f^*(y))\right)^\circ(s)\frac{\mathrm{d}s}{s}\right\|_X\lesssim\left\|\left(\frac{I(y)}{y}(f^{**}(y)-f^*(y))\right)^\circ\right\|_X.$$

Adding this estimate to (3.23) and using (3.17) we have

$$\left\|\frac{I(t)}{t}(f^{**}(t)-f^*(t))\right\|_X\lesssim\left\|\left(\frac{I(y)}{y}(f^{**}(y)-f^*(y))\right)^\circ\right\|_X+\|f\|_1\approx\|f\|_{Y_X},$$

which concludes the proof. □

Remark 3.14. Observe that the average property was only used in Theorem 2.11 to guarantee us the validity of the theorem for an arbitrary r.i. space. Therefore, if the map $f \mapsto \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$ is bounded on the associate space of a certain r.i. space X, we may drop this assumption and still obtain validity of (3.16).

Also, we can view the condition about I having the average property as the property of the operator H_I to push the optimal target space Y_{L^1} far enough from L^1, so that $f \mapsto f^{**}$ is bounded on Y_{L^1}.

Let us demonstrate that for a function I defined by $I(t) = t\log^{\alpha}\frac{2}{t}, \alpha \in \left[0, \frac{1}{2}\right]$, the conclusion of the theorem need not hold. To this end, let $\alpha \in \left[0, \frac{1}{2}\right]$ be given and set $X = L^1$. We first notice that $\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \lesssim I(t), t \in (0,1)$ holds for this particular choice of I (this assertion is contained as a special case in Proposition 4.2). At the end of the proof of Lemma 3.2, we showed that $Y_{L^1} = \Lambda_I$. Thus, in our setting, (3.16) reads as

$$\|f\|_{\Lambda_I} \approx \left\| \frac{I(t)}{t}(f^{**}(t) - f^*(t)) \right\|_1 + \|f\|_1.$$

We now show that the left-hand side does not majorize the right-hand side. To this end, it suffices to show that $f \mapsto f^{**}$ is not bounded on Λ_I. By Theorem 3.10 this is equivalent to showing that

$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s \lesssim \frac{I(t)}{t}, \quad t \in (0,1),$$

does not hold. Plugging in our choice of I, we get $\frac{I(t)}{t} = \log^{\alpha}\frac{2}{t}, t \in (0,1)$ and

$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s = \int_t^1 \frac{\log^{\alpha}\frac{2}{s}}{s}\,\mathrm{d}s \approx \log^{\alpha+1}\frac{2}{t}, t \in (0,1).$$

As $\log^{\alpha+1}\frac{2}{t}$ is not majorized by $\log^{\alpha}\frac{2}{t}$, the result follows.

The examples of functions which satisfy (3.10) are $I(t) = t^{\alpha}, \alpha \in (0,1)$.

As for the condition (3.15), it is satisfied by every concave function for $t \in \left(0, \frac{1}{2}\right)$. This follows from the following calculation:

$$\int_t^1 \frac{I(s)}{s^3}\,\mathrm{d}s \geq \int_t^{2t} \frac{I(s)}{s^3}\,\mathrm{d}s \approx \frac{I(t)}{t^2} \geq \frac{I(t)}{t^2} - 1.$$

Corollary 3.15. *Let X be an r.i. space and assume that I satisfies all the conditions we imposed in Theorem 3.13. Put $\|f\|_Z = \|S_I f^{**}\|_{X'}$. Then*

$$\|f\|_{Y_X} \approx \left\| \frac{I(t)}{t}(f^{**}(t) - f^*(t)) \right\|_{Z'} + \|f\|_1.$$

Proof. By Corollary 2.13 Z is an r.i. space which admits boundedness of the operator S_I. Thus, by Theorem 3.13, we have

$$\|f\|_{Y_{Z'}} \approx \left\| \frac{I(t)}{t}(f^{**}(t) - f^*(t)) \right\|_{Z'} + \|f\|_1.$$

However, by Proposition 3.12,

$$\begin{aligned}\|f\|_{Y_{Z'}} &\approx \sup_{\|S_I g\|_Z \leq 1} \int_0^1 \frac{I(t)}{t}(f^{**}(t) - f^*(t))g^*(t)\,\mathrm{d}t + \|f\|_1 \\ &\approx \sup_{\|g\|_Z \leq 1} \int_0^1 \frac{I(t)}{t}(f^{**}(t) - f^*(t))g^*(t)\,\mathrm{d}t + \|f\|_1 \approx \|f\|_{Y_X}.\end{aligned}$$

□

3.3 Optimality and supremum operators

Next proposition is an analogue of Corollary 2.16 for the operator S_I.

Proposition 3.16. *Let Y be an r.i. space and assume that a concave function I satisfies* (2.7) *and the average property. Then the operator S_I is bounded on X'_Y.*

Proof. By virtue of Proposition 3.1, the optimal domain space under the map H_I, X_Y, exists for every r.i. space Y and is optimal in

$$\|H_I f\|_Y \lesssim \|f\|_{X_Y}, \quad f \in \mathcal{M}_+(0,1).$$

This means, by duality, that X'_Y is optimal in

$$\|R_I f^*\|_{X'_Y} \lesssim \|f\|_{Y'}, \quad f \in \mathcal{M}_+(0,1). \tag{3.24}$$

From the definition of the norm of the space Y'_{X_Y}, it easily follows that Y'_{X_Y} is optimal in

$$\|R_I f^*\|_{X'_Y} \lesssim \|f\|_{Y'_{X_Y}}, \quad f \in \mathcal{M}_+(0,1). \tag{3.25}$$

Let us check that X'_Y, too, is optimal therein. Assume that

$$\|R_I f^*\|_Z \lesssim \|f\|_{Y'_{X_Y}}, \quad f \in \mathcal{M}_+(0,1),$$

for some r.i. space Z. We need to show that $X'_Y \hookrightarrow Z$. Optimality of Y'_{X_Y} in (3.25) and inequality (3.24) tell us that $Y' \hookrightarrow Y'_{X_Y}$, which in turn implies that

$$\|R_I f^*\|_Z \lesssim \|f\|_{Y'}.$$

Finally, optimality of space X'_Y in (3.24) yields $X'_Y \hookrightarrow Z$ as we wanted.

Define $\beta'(f) = \lambda(k(f, t, m_I, L^\infty))$, $f \in \mathcal{M}_+(0,1)$, where λ is as in Theorem 3.4 corresponding to the space X_Y – here the space X_Y plays the role of the space X. Since I satisfies the average property, by Remark 2.9, m_I is in fact an r.i. space and so, by Theorem 1.21, β' is an r.i. norm. Respecting the notation from Theorem 3.4, $\beta' \leq \beta$ and so

$$\beta'(R_I f^*) \leq \|f\|_{Y'_{X_Y}}.$$

Optimality of X'_Y in (3.25) therefore tells us that $\beta'(\cdot) \lesssim \|\cdot\|_{X'_Y}$. Further, using (3.6), we have

$$\beta'(f) \gtrsim \|S_I f\|_{X'_Y}, \quad f \in \mathcal{M}_+(0,1).$$

Connecting last two estimate, we arrive at $\|S_I f\|_{X'_Y} \lesssim \beta'(f) \lesssim \|f\|_{X'_Y}$. In other words, S_I is bounded on X'_Y. □

In the final theorem of the chapter, we will need an alternative description of the optimal domain norm. Boundedness of the operator T_I plays essential role in there.

Theorem 3.17. *Let I be a concave function satisfying* (2.7)*. Let Y be an r.i. space such that $H_I 1 \in Y$. If T_I is bounded on Y', then*

$$\sup_{h \sim f} \|H_I h\|_Y \approx \|H_I f^*\|_Y, \quad f \in \mathcal{M}_+(0,1).$$

In other words, for the optimal domain space we have

$$\|f\|_{X_Y} \approx \|H_I f^*\|_Y, \quad f \in \mathcal{M}_+(0,1).$$

Proof. First, as $H_I 1 \in Y$, $\|\cdot\|_{X_Y}$ is an r.i. norm by Proposition 3.1. Next, as T_I is bounded on Y', we have

$$\|g\|_{Y'} \approx \|T_I g\|_{Y'}, \quad g \in \mathcal{M}_+(0,1).$$

We need to check that

$$\sup_{h\sim f} \|H_I h\|_Y \lesssim \|H_I f^*\|_Y, \quad f \in \mathcal{M}_+(0,1).$$

To this end, fix $f \in \mathcal{M}_+(0,1)$. We estimate

$$\begin{aligned}
\left\| \int_t^1 \frac{f(s)}{I(s)}\,\mathrm{d}s \right\|_Y &= \sup_{\|g\|_{Y'}\le 1} \int_0^1 g^*(t) \int_t^1 \frac{f(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t \\
&= \sup_{\|g\|_{Y'}\le 1} \int_0^1 \frac{f(s)}{I(s)} \int_0^s g^*(t)\,\mathrm{d}t\,\mathrm{d}s \\
&\le \sup_{\|g\|_{Y'}\le 1} \int_0^1 \frac{f(s)}{I(s)} \int_0^s (T_I g)(t)\,\mathrm{d}t\,\mathrm{d}s \\
&\approx \sup_{\|g\|_{Y'}\le 1} \int_0^1 \frac{f(s)}{\int_0^s \frac{I(r)}{r}\,\mathrm{d}r} \int_0^s (T_I g)(t)\,\mathrm{d}t\,\mathrm{d}s \\
&\le \sup_{\|g\|_{Y'}\le 1} \int_0^1 \frac{f^*(s)}{\int_0^s \frac{I(r)}{r}\,\mathrm{d}r} \int_0^s (T_I g)(t)\,\mathrm{d}t\,\mathrm{d}s \\
&\approx \sup_{\|T_I g\|_{Y'}\le 1} \int_0^1 (T_I g)(t) \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t \\
&\approx \sup_{\|g\|_{Y'}\le 1} \int_0^1 g^*(t) \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t = \left\| \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s \right\|_Y.
\end{aligned}$$

Here, the first and second "$\approx$" is the assumption that $I(t) \approx \int_0^t \frac{I(s)}{s}\,\mathrm{d}s$. In the second inequality we used the Hardy-Littlewood inequality (1.3), and the fact that

$$s \mapsto \frac{1}{\int_0^s \frac{I(t)}{t}\,\mathrm{d}t} \int_0^s (T_I g)(t)\,\mathrm{d}t = \frac{1}{\int_0^s \frac{I(t)}{t}\,\mathrm{d}t} \int_0^s \sup_{t\le r<1} \frac{r}{I(r)} g^*(r) \frac{I(t)}{t}\,\mathrm{d}t$$

is nonincreasing, being the integral mean of a nonincreasing function with respect to the measure $\frac{I(t)}{t}\,\mathrm{d}t$. □

Theorem 3.18. *Let I be a concave function as in Theorem 3.13. Then an r.i. space X is the optimal domain space under the map H_I for some r.i. space Y if and only if S_I is bounded on X'. In that case,*

$$\|f\|_{Y_X} \approx \left\| \frac{I(t)}{t} (f^{**}(t) - f^*(t)) \right\|_X + \|f\|_1, \quad f \in \mathcal{M}_+(0,1). \tag{3.26}$$

Vice versa, an r.i. space Y is the optimal target space under the map H_I for some r.i. space X if and only if T_I is bounded on Y'. In that case,

$$\|f\|_{X_Y} \approx \left\| \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s \right\|_Y, \quad f \in \mathcal{M}_+(0,1). \tag{3.27}$$

Proof. Since $\frac{1}{I}$ is integrable, the optimal domain space X_Z exists for every r.i. space Z by Proposition 3.1. By Proposition 3.16, S_I is bounded on X_Y'. This, by Theorem 3.13, means that

$$\|f\|_{Y_{X_Y}} \approx \left\| \frac{I(t)}{t}(f^{**}(t) - f^*(t)) \right\|_{X_Y} + \|f\|_1, \quad f \in \mathcal{M}_+(0,1). \tag{3.28}$$

Now, given an r.i. space Z, operator T_I is bounded on Y_Z' by virtue of Corollary 2.16. Hence, by Theorem 3.17, we have

$$\|f\|_{X_{Y_Z}} \approx \left\| \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s \right\|_{Y_Z}, \quad f \in \mathcal{M}_+(0,1). \tag{3.29}$$

Assuming now that X is optimal for some Y, we have that $X = X_Y$. Consequently (3.28) turns into (3.26) and S_I is bounded X'.

On the contrary, assuming S_I is bounded on X', we obtain validity of (3.26) by Theorem 3.13. We show that $X_{Y_X} = X$. To this end, let $f \in \mathcal{M}_+(0,1)$ be given. Then

$$\begin{aligned}
\|f\|_{X_{Y_X}} &\approx \left\| \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s \right\|_{Y_X} \\
&\approx \left\| \frac{I(t)}{t}\left(\frac{1}{t}\int_0^t \int_s^1 \frac{f^*(r)}{I(r)}\,\mathrm{d}r\,\mathrm{d}s - \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s \right) \right\|_X + \int_0^1 \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t \\
&= \left\| \frac{I(t)}{t^2} \int_0^t \frac{r}{I(r)} f^*(r)\,\mathrm{d}r \right\|_X + \int_0^1 \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned} \tag{3.30}$$

because, by Fubini's theorem,

$$\frac{1}{t}\int_0^t \int_s^1 \frac{f^*(r)}{I(r)}\,\mathrm{d}r\,\mathrm{d}s - \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s = \frac{1}{t}\int_0^t \frac{s}{I(s)} f^*(s)\,\mathrm{d}s.$$

Further,

$$\int_0^1 \int_t^1 \frac{f^*(s)}{I(s)}\,\mathrm{d}s\,\mathrm{d}t \le \int_0^1 f^*(t) \int_t^1 \frac{\mathrm{d}s}{I(s)}\,\mathrm{d}t \le \int_0^1 f^*(t)\,\mathrm{d}t \int_0^1 \frac{\mathrm{d}s}{I(s)} \lesssim \|f\|_1 \lesssim \|f\|_X. \tag{3.31}$$

Observe that the operator R defined by

$$Rf(t) = \frac{I(t)}{t^2} \int_0^t \frac{s}{I(s)} f(s)\,\mathrm{d}s, \quad f \in \mathcal{M}_+(0,1), t \in (0,1),$$

is associate to the operator H defined in (3.11). Concavity of I implies that

$$\int_0^t \frac{s}{I(s)}\,\mathrm{d}s \approx \frac{t^2}{I(t)}, \quad t \in (0,1).$$

Therefore, for the operator R' defined by

$$R'f(t) = \frac{1}{\int_0^t \frac{s}{I(s)}\,\mathrm{d}s} \int_0^t \frac{s}{I(s)} f(s)\,\mathrm{d}s, \quad f \in \mathcal{M}_+(0,1), t \in (0,1),$$

we have

$$Rf(t) \approx R'f(t), \quad f \in \mathcal{M}_+(0,1), t \in (0,1).$$

The advantage of the operator R' is that $R'f^*$ is nonincreasing for every $f \in \mathcal{M}_+(0,1)$, being the integral mean of f^* with respect to the measure $\frac{t}{I(t)}\,\mathrm{d}t$. Hence, using (3.12), we have

$$\begin{aligned}\|Rf^*\|_X \approx \|R'f^*\|_X &= \sup_{\|g\|_{X'}\le 1} \int_0^1 R'f^*(t)g^*(t)\,\mathrm{d}t \approx \sup_{\|g\|_{X'}\le 1} \int_0^1 Rf^*(t)g^*(t)\,\mathrm{d}t \\ &= \sup_{\|g\|_{X'}\le 1} \int_0^1 f^*(t)Hg^*(t)\,\mathrm{d}t \lesssim \sup_{\|g\|_{X'}\le 1} \int_0^1 f^*(t)g^*(t)\,\mathrm{d}t = \|f^*\|_X,\end{aligned}$$

for every $f \in \mathcal{M}_+(0,1)$.

We also see that

$$\frac{I(t)}{t^2}\int_0^t \frac{r}{I(r)} f^*(r)\,\mathrm{d}r \ge f^*(t)\cdot \frac{I(t)}{t^2}\int_0^t \frac{r}{I(r)}\,\mathrm{d}r \ge f^*(t)\cdot\frac{I(t)}{t^2}\int_{\frac{t}{2}}^t \frac{r}{I(r)}\,\mathrm{d}r \approx f^*(t),$$

as $I \in \Delta_2$. Thus

$$\left\| \frac{I(t)}{t^2}\int_0^t \frac{r}{I(r)} f^*(r)\,\mathrm{d}r \right\|_X \approx \|f\|_X, \quad f \in \mathcal{M}_+(0,1).$$

Combining this with (3.30) and (3.31) yields

$$\|f\|_{X_{Y_X}} \approx \|f\|_X.$$

Turning our attention to the second part, the assumption on optimality of Y for some space X implies $Y = Y_X$. Hence T_I is bounded on Y' and (3.29) turns into (3.27).

In the other direction, let us assume that T_I is bounded on Y'. We show that $Y = Y_{X_Y}$, which shows optimality of Y for some space – X_Y in fact.

For $t \in (0, \frac{1}{2})$ we estimate

$$\begin{aligned}\int_t^1 \frac{I(s)}{s^2}(f^{**}(s)-f^*(s))\,\mathrm{d}s &= \int_t^1 \frac{I(s)}{s^3}\int_0^s f^*(r)\,\mathrm{d}r\,\mathrm{d}s - \int_t^1 \frac{I(s)}{s^2} f^*(s)\,\mathrm{d}s \\ &= \int_t^1 \frac{I(s)}{s^3}\int_0^t f^*(r)\,\mathrm{d}r\,\mathrm{d}s + \int_t^1 \frac{I(s)}{s^3}\int_t^s f^*(r)\,\mathrm{d}r\,\mathrm{d}s - \int_t^1 \frac{I(s)}{s^2} f^*(s)\,\mathrm{d}s \\ &\ge \int_t^1 \frac{I(s)}{s^3}\int_0^t f^*(r)\,\mathrm{d}r\,\mathrm{d}s + \int_t^1 \frac{I(s)}{s^3}(s-t) f^*(s)\,\mathrm{d}s - \int_t^1 \frac{I(s)}{s^2} f^*(s)\,\mathrm{d}s \\ &= \int_t^1 \frac{I(s)}{s^3}\int_0^t f^*(r)\,\mathrm{d}r\,\mathrm{d}s - t\int_t^1 \frac{I(s)}{s^3} f^*(s)\,\mathrm{d}s \\ &\ge t\int_t^1 \frac{I(s)}{s^3} f^{**}(t)\,\mathrm{d}s - tf^*(t)\int_t^1 \frac{I(s)}{s^3}\,\mathrm{d}s \\ &\ge t\int_t^{2t} \frac{I(s)}{s^3}\,\mathrm{d}s (f^{**}(t)-f^*(t)) \approx \frac{I(t)}{t}(f^{**}(t)-f^*(t)).\end{aligned}$$

Using this, (3.28) and Theorem 3.17, we have

$$\begin{aligned}\|f\|_{Y_{X_Y}} &\approx \left\|\frac{I(t)}{t}(f^{**}(t)-f^*(t))\right\|_{X_Y} + \|f\|_1 \\ &\lesssim \left\|\int_t^1 \frac{I(s)}{s^2}(f^{**}(s)-f^*(s))\,\mathrm{d}s\right\|_{X_Y} + \|f\|_1 \\ &\approx \left\|\int_t^1 \frac{1}{I(s)}\int_s^1 \frac{I(r)}{r^2}(f^{**}(r)-f^*(r))\,\mathrm{d}r\,\mathrm{d}s\right\|_Y + \|f\|_1.\end{aligned} \tag{3.32}$$

Now, for every $h \in \mathcal{M}_+(0,1)$, we have

$$\begin{aligned}\int_t^1 \frac{1}{I(s)}\int_s^1 h(r)\frac{\mathrm{d}r}{r}\,\mathrm{d}s &= \int_t^1 \frac{h(r)}{r}\int_t^r \frac{\mathrm{d}s}{I(s)}\,\mathrm{d}r \le \int_t^1 \frac{h(r)}{r}\int_0^r \frac{\mathrm{d}s}{I(s)}\,\mathrm{d}r \\ &\lesssim \int_t^1 \frac{h(r)}{r}\cdot\frac{r}{I(r)}\,\mathrm{d}r = \int_t^1 \frac{h(r)}{I(r)}\,\mathrm{d}r,\end{aligned} \tag{3.33}$$

where the first inequality is the average property of I. Further,

$$\begin{aligned}\int_t^1 \frac{1}{s^2}\int_0^s h(r)\,\mathrm{d}r\,\mathrm{d}s &= \int_t^1 \frac{1}{s^2}\int_0^t h(y)\,\mathrm{d}y\,\mathrm{d}s + \int_t^1 \frac{1}{s^2}\int_t^s h(r)\,\mathrm{d}r\,\mathrm{d}s \\ &\le \int_0^t h(y)\,\mathrm{d}y\int_t^1 \frac{\mathrm{d}s}{s^2} + \int_t^1 h(r)\int_r^1 \frac{\mathrm{d}s}{s^2}\,\mathrm{d}r \\ &\le \frac{1}{t}\int_0^t h(r)\,\mathrm{d}r + \int_t^1 \frac{h(r)}{r}\,\mathrm{d}r.\end{aligned} \tag{3.34}$$

Deploying (3.33) for $h(r) = \frac{I(r)}{r}(f^{**}(r)-f^*(r))$ and (3.34) for $h(r) = f^*(r)$ in this order in (3.32) yields

$$\begin{aligned}\|f\|_{Y_{X_Y}} &\lesssim \left\|\int_t^1 \frac{1}{s}(f^{**}(s)-f^*(s))\,\mathrm{d}s\right\|_Y + \|f\|_1 \\ &= \left\|\int_t^1 \frac{1}{s^2}\int_0^s f^*(r)\,\mathrm{d}r\,\mathrm{d}s - \int_t^1 f^*(s)\frac{\mathrm{d}s}{s}\right\|_Y + \|f\|_1 \\ &\le \|f^{**}\|_Y + \|f\|_1.\end{aligned} \tag{3.35}$$

We now claim that $T_I\colon Y' \to Y'$ implies $Y \subset \Lambda_I$. In the proof of Lemma 3.2 we showed that $\Lambda_I' = m_{\widetilde{I}}$ whenever I satisfies (2.7). Therefore, we can equivalently show that $Y' \supset m_{\widetilde{I}}$. Now, for every $f \in m_{\widetilde{I}}$ there exists $c \ge 0$ such that $f^*(t) \le c\frac{I(t)}{t}, t \in (0,1)$. Thus, from the lattice property of the space Y', it suffices to show that $t \mapsto \frac{I(t)}{t} \in Y'$. We observe that $(T_I 1)(t) = \frac{I(t)}{t}, t \in (0,1)$, and so, using the boundedness of the operator T_I on Y', we have

$$\left\|\frac{I(t)}{t}\right\|_{Y'} = \|T_I 1\|_{Y'} \lesssim \|1\|_{Y'} < \infty.$$

Hence, combining Lemma 3.2 with Theorem 3.10, we conclude that $f \mapsto f^{**}$ is bounded on Y. Adding this piece of information to (3.35), we obtain

$$\|f\|_{Y_{X_Y}} \lesssim \|f\|_Y.$$

As $Y_{X_Y} \subset Y$ holds trivially, the proof is complete. □

4. Examples

In this chapter, we first briefly discuss all the properties we have imposed on a function I so far and explore their relations. This will be followed by naming particular examples of functions I, for some of which, we will apply the results of Chapter 3.

4.1 Comparison of conditions

We have so far used these conditions:

(i)
$$\int_0^t \frac{\mathrm{d}s}{I(s)} \approx \frac{t}{I(t)}, \quad t \in (0,1),$$

(ii)
$$\int_0^t \frac{I(s)}{s}\,\mathrm{d}s \approx I(t), \quad t \in (0,1),$$

(iii)
$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s \lesssim \frac{I(t)}{t}, \quad t \in (0,1),$$

(iv)
$$\int_t^1 \frac{I(s)}{s^2}\,\mathrm{d}s \approx \frac{I(t)}{t} - 1, \quad t \in (0,1),$$

(v)
$$\int_t^1 \frac{I(s)}{s^3}\,\mathrm{d}s \approx \frac{I(t)}{t^2} - 1, \quad t \in (0,1).$$

We have always formulated all these properties on the whole unit interval $(0,1)$, but what is truly needed is its validity on some right neighbourhood of 0.

When checking validity of condition (ii), it is oft easier to check whether I satisfies $I'(t) \approx \frac{I(t)}{t}, t \in (0,1)$, the latter being stronger.

Similarly, if $\left(\frac{t}{I(t)}\right)' \approx \frac{1}{I(t)}, t \in (0,1)$, then

$$\left(\frac{I(t)}{t}\right)' = \left(\left(\frac{t}{I(t)}\right)^{-1}\right)' = -\frac{I(t)^2}{t^2} \cdot \left(\frac{t}{I(t)}\right)' \approx -\frac{I(t)}{t^2}, \quad t \in (0,1).$$

From here it follows that condition (iv), and consequently (iii), hold.

We will be interested in two types of domains. Firstly, those whose isoperimetric profile is related to polynomials $I(t) = t^\alpha, \alpha \in \left[\frac{1}{n'}, 1\right)$ and, secondly, product probability spaces. To this end, we check which conditions are satisfied.

The following proposition states, that strictly concave polynomials satisfy all the conditions that appeared throughout the thesis. The proof is omitted.

Proposition 4.1 (Polynomials t^α for $\alpha \in (0,1)$)**.** *Let $\alpha \in (0,1)$ be given. Then $I(t) = t^\alpha$ satisfies all the conditions (i)-(v). Moreover, the condition* (3.15) *is satisfied with $c = 2-\alpha \geq 1$ and so I satisfies all the assumptions in Theorem 3.13.*

Next, let us consider $\Phi\colon [0,\infty) \to [0,\infty)$. Assume Φ is twice differentiable, strictly increasing and convex in $(0,\infty)$, $\sqrt{\Phi}$ is concave and $\Phi(0) = 0$. Define further a measure on $\mathbb{R}$ by

$$\mathrm{d}\mu_\Phi(x) = c_\Phi e^{-\Phi(|x|)}\,\mathrm{d}x,$$

where c_Φ is such that $\mu_\Phi(\mathbb{R}) = 1$.

We then define its *product measure* $\mu_{\Phi,n}$ on $\mathbb{R}^n$ as

$$\mu_{\Phi,n} = \underbrace{\mu_\Phi \times \cdots \times \mu_\Phi}_{n\text{-times}}.$$

Then $(\mathbb{R}^n, \mu_{\Phi,n})$ is a probability space.

It is further known by [2, Chapter 7] that

$$I(t) = I_{\mathbb{R}^n,\mu_{\Phi,n}}(t) \approx t\Phi'\left(\Phi^{-1}\left(\log\frac{2}{t}\right)\right), \quad t \in \left(0, \frac{1}{2}\right). \tag{4.1}$$

Note that I is quasi-concave function – I being nondecreasing is proved in [2, Lemma 11.1], while $t \mapsto \frac{I(t)}{t}$ being nonincreasing follows from the fact that Φ is increasing and convex. Hence, the theory we built is applicable.

We will now check that $I'(t) \approx \frac{I(t)}{t}$ on some right neighbourhood of 0.

First, we observe that $\Phi \in \Delta_2$. Indeed, to see this, recall that $\sqrt{\Phi}$ is concave and so $\sqrt{\Phi} \in \Delta_2$. We estimate

$$\Phi(2t) = \left(\sqrt{\Phi(2t)}\right)^2 \le \left(c\sqrt{\Phi(t)}\right)^2 = c^2\Phi(t), \quad t \in (0,\infty).$$

As Φ is an increasing convex function with $\Phi(0) = 0$, we write

$$\Phi(t) = \int_0^t \Phi'(s)\,\mathrm{d}s, \quad t \in (0,\infty).$$

Convexity tells us that Φ' is nondecreasing and so, for every $t \in (0,\infty)$, we have

$$\Phi(t) = \int_0^t \Phi'(s)\,\mathrm{d}s \le t\Phi'(t) \le \int_t^{2t} \Phi'(s)\,\mathrm{d}s \le \Phi(2t).$$

Combining with the knowledge that $\Phi \in \Delta_2$, we obtain

$$\Phi'(t) \approx \frac{\Phi(t)}{t}, \quad t \in (0,\infty).$$

Plugging this new information into (4.1), we have

$$I(t) \approx \frac{t\log\frac{2}{t}}{\Phi^{-1}\left(\log\frac{2}{t}\right)}, \quad t \in \left(0, \frac{1}{2}\right), \tag{4.2}$$

and so we may as well consider the very last expression to be the representantive of I. Differentiating, we get

$$\begin{aligned} I'(t) &= \frac{\left(\log\frac{2}{t} - 1\right)\Phi^{-1}\left(\log\frac{2}{t}\right) - \frac{1}{\Phi'\left(\Phi^{-1}\left(\log\frac{2}{t}\right)\right)} \cdot \frac{-1}{t} \cdot t\log\frac{2}{t}}{\left(\Phi^{-1}\left(\log\frac{2}{t}\right)\right)^2} \\ &= \underbrace{\frac{\left(\log\frac{2}{t} - 1\right)}{\Phi^{-1}\left(\log\frac{2}{t}\right)}}_{A(t)} + \underbrace{\frac{\log\frac{2}{t}}{\Phi'\left(\Phi^{-1}\left(\log\frac{2}{t}\right)\right) \cdot \left(\Phi^{-1}\left(\log\frac{2}{t}\right)\right)^2}}_{B(t)}, \quad t \in \left(0, \frac{1}{2}\right). \end{aligned}$$

As for A, we simply note that $\log \frac{2}{t} - 1 \approx \log \frac{2}{t}$ on some right neighbourhood of 0, because $\lim_{t\to 0^+} \log \frac{2}{t} = \infty$. Therefore, it follows that $A(t) \approx \frac{I(t)}{t}$. Using both (4.1) and (4.2), we obtain

$$B(t) \approx \frac{\log \frac{2}{t}}{\frac{I(t)}{t}} \cdot \left(\frac{I(t)}{t \log \frac{2}{t}} \right)^2 = \frac{I(t)}{t \log \frac{2}{t}}.$$

Finally, observing that $\lim_{t\to 0^+} \frac{B(t)}{A(t)} = 0$, we conclude that B is negligible.

We have therefore proved the following proposition.

Proposition 4.2. *Let I be as in* (4.1)*. Then*

$$I'(t) \approx \frac{I(t)}{t}$$

on some right neighbourhood of 0*. Consequently, condition (ii) is satisfied.*

A corollary to this observation follows immediately from Corollary 2.16.

Corollary 4.3. *Let X be an r.i. space and let I as in* (4.1)*. Then T_I is bounded on Y'_X.*

4.2 Higher-order Sobolev embeddings

Recall that Maz'ya classes of domains $\mathcal{J}_\alpha$ for $\alpha \in \left[\frac{1}{n'}, 1\right)$ are defined as

$$\mathcal{J}_\alpha = \left\{ \Omega : I_\Omega(t) \gtrsim t^\alpha, t \in \left[0, \frac{1}{2}\right] \right\}.$$

Since $I(t) = t^\alpha$ enjoys the average property, we have, by virtue of [2, Proposition 8.6],

$$\|R_I^m f\|_X \approx \left\| \frac{t^{m-1}}{I(t)^m} \int_0^t f(s)\, \mathrm{d}s \right\|_X, \quad f \in \mathcal{M}_+(0,1),$$

and

$$\|H_I^m f\|_X \approx \left\| \int_t^1 \frac{s^{m-1}}{I(s)^m} f(s)\, \mathrm{d}s \right\|_X, \quad f \in \mathcal{M}_+(0,1),$$

for every r.i. space X. Define a function $J \colon (0,1) \to (0,\infty)$ by

$$J(t) = \frac{I(t)^m}{t^{m-1}} = t^{1-m(1-\alpha)}, \quad t \in (0,1).$$

From here, we see that whenever $1 - m(1-\alpha) > 0$, then $J(t)$ is an increasing, strictly concave bijection of $(0,1)$ onto itself. Proposition 4.1 therefore asserts that the function J satisfies all the conditions (i)-(v). Therefore, for this particular choice of I, Theorem 3.18 reads as follows:

Corollary 4.4. *Let $\alpha \in \left[\frac{1}{n'}, 1\right)$ be given and let $m \in \mathbb{N}$ be such that $1-m(1-\alpha) >$ 0. Put $J(t) = t^{1-m(1-\alpha)}$ for $t \in (0,1)$. Then an r.i. space X is the optimal domain* *space under the map H_I^m for some r.i. space Y if and only if S_J is bounded on X'. In that case,*

$$\|f\|_{Y_X} \approx \left\| \frac{J(t)}{t}(f^{**}(t) - f^*(t)) \right\|_X + \|f\|_1, \quad f \in \mathcal{M}_+(0,1).$$

Vice versa, an r.i. space Y is the optimal target space under the map H_I^m for some r.i. space X if and only if T_J is bounded on Y'. In that case,

$$\|f\|_{X_Y} \approx \left\| \int_t^1 \frac{f^*(s)}{J(s)} \,\mathrm{d}s \right\|_Y, \quad f \in \mathcal{M}_+(0,1).$$

Recall that a bounded open $\Omega \subset \mathbb{R}^n$ is said to be a *John domain*, if there exists $x_0 \in \Omega$ and $c > 0$ such that for every $x \in \Omega$ there exists a recitifiable curve $\gamma_x \colon [0,l] \to \Omega$, parametrized by its arclength, such that

$$\operatorname{dist}(\gamma_x(t), \partial\Omega) \geq ct, \quad t \in [0,l].$$

It is known that for every John domain Ω we have $I_\Omega(t) \approx t^{\frac{1}{n'}}$. Note that every Lipschitz domain is also a John domain.

Now, given $\Omega \subset \mathbb{R}^n$ a John domain, [2, Theorem 6.1] asserts that

$$H_{I_\Omega}^m \colon X \to Y \iff V^m X \to Y$$

whenever X and Y are r.i. spaces. Combining this with Corollary 4.4, we have:

Corollary 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a John domain. Let $m \in \mathbb{N}, m < n$ be given and put $J(t) = t^{1-\frac{m}{n}}$ for $t \in (0,1)$. Then an r.i. space $X(\Omega)$ is the optimal domain space in the m-th order Sobolev embedding for some r.i. space $Y(\Omega)$ if and only if S_J is bounded on X'. In that case,*

$$\|f\|_{Y_X} \approx \left\| t^{-\frac{m}{n}}(f^{**}(t) - f^*(t)) \right\|_X + \|f\|_1, \quad f \in \mathcal{M}_+(0,1).$$

Vice versa, an r.i. space $Y(\Omega)$ is the optimal target space in the m-th order Sobolev embedding for some r.i. space $X(\Omega)$, if and only if T_J is bounded on Y', in which case

$$\|f\|_{X_Y} \approx \left\| \int_t^1 s^{\frac{m}{n}-1} f^*(s) \,\mathrm{d}s \right\|_Y, \quad f \in \mathcal{M}_+(0,1).$$

The very last corollary recovers [10, Theorem A] for Lipschitz domains and extends the result to John domains.

The next proposition exhibits an Euclidean domain Ω_α which is extreme in the sense that $I_{\Omega_\alpha}(t) \approx t^\alpha$. The proof follows from a special case of [13, Section 5.3.3].

Proposition 4.6. *Let $\alpha \in \left[\frac{1}{n'}, 1\right)$ and define $\eta_\alpha \colon \left[0, \frac{1}{1-\alpha}\right] \to [0,\infty)$ by*

$$\eta_\alpha(r) = \omega_{n-1}^{-\frac{1}{n-1}} (1 - (1-\alpha)r)^{\frac{\alpha}{(1-\alpha)(n-1)}}, \quad r \in \left[0, \frac{1}{1-\alpha}\right],$$

where ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$. Define $\Omega_\alpha \subset \mathbb{R}^n$ as

$$\Omega_\alpha = \left\{ (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, 0 < x_n < \frac{1}{1-\alpha}, |x'| < \eta_\alpha(x_n) \right\}. \tag{4.3}$$

Then $|\Omega_\alpha| = 1$ and $I_{\Omega_\alpha}(t) \approx t^\alpha, t \in \left[0, \frac{1}{2}\right]$. In particular, $\Omega_\alpha \in \mathcal{J}_\alpha$.

The following theorem is a special case of [2, Theorem 6.4].

Theorem 4.7. *Let* $\alpha \in \left[\frac{1}{n'}, 1\right)$ *and* $m \in \mathbb{N}$ *be such that* $1 - m(1-\alpha) > 0$*. Put* $J(t) = t^{1-m(1-\alpha)}$*. Let* X *and* Y *be r.i. spaces and let* Ω_α *be as in* (4.3)*. Then the following statements are equivalent:*

(i)
$$H_J \colon X \to Y;$$

(ii)
$$\forall \Omega \in \mathcal{J}_\alpha \colon V^m X(\Omega) \to Y(\Omega);$$

(iii)
$$V^m X(\Omega_\alpha) \to Y(\Omega_\alpha).$$

Using this theorem, we formulate a result analogous to Corollary 4.4.

Corollary 4.8. *Let* $\alpha \in \left[\frac{1}{n'}, 1\right)$ *and* $m \in \mathbb{N}$ *be such that* $1 - m(1-\alpha) > 0$*. Put* $J(t) = t^{1-m(1-\alpha)}$ *and let* Ω_α *be as in* (4.3)*. Then an r.i. space* $X(\Omega_\alpha)$ *is the optimal domain space in the* m*-th order sobolev embedding for some r.i. space* $Y(\Omega_\alpha)$ *if and only if* S_J *is bounded on* X'*. In this case,* $V^m X(\Omega) \to Y(\Omega)$ *for every* $\Omega \in \mathcal{J}_\alpha$ *and*

$$\|f\|_{Y_X} \approx \|t^{-m(1-\alpha)}(f^{**}(t) - f^*(t))\|_X + \|f\|_1, \quad f \in \mathcal{M}_+(0,1).$$

Vice versa, an r.i. space $Y(\Omega_\alpha)$ *is the optimal target space in the* m*-th order Sobolev embedding for some r.i. space* $X(\Omega_\alpha)$ *if and only if* T_J *is bounded on* Y'*. In this case,* $V^m X(\Omega) \to Y(\Omega)$ *for every* $\Omega \in \mathcal{J}_\alpha$ *and*

$$\|f\|_{X_Y} \approx \left\| \int_t^1 s^{m(1-\alpha)-1} f^*(s) \, \mathrm{d}s \right\|_X, \quad f \in \mathcal{M}_+(0,1).$$

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