FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Factorization of quaternion and dual quaternion polynomials

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Study programme: Mathematics for Information Technologies

Prague 2024

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Abstract: In this thesis we provide basic algorithms for factorization of polynomials over quaternions and dual quaternions. In the case of quaternions, we translate the art of finding roots of given polynomial into factorization. In the case of dual quaternions, we analyse the case of motion polynomials. We show, when it is possible to find a factorization of such polynomial into linear terms and give a workaround if it is not. All of this is supplied with geometric interpretations. More specifically, we use the factorizations to construct mechanical linkages.

Keywords: quaternions dual quaternions polynomials factorization

Contents

Introduction

When talking about polynomial factorizations, one can not omit the fundamental theorem of algebra. It states, that any complex polynomial P has a complex root z and hence the associated linear polynomial factor $t - z$. The rest of the original polynomial will have another complex zero and another linear factor, giving way for a factorization of P into linear terms. The natural question to ask is whether we can generalize it.

Another extension of $\mathbb{R}$, standing right beyond the complex numbers, is the set of quaternions. One of its most notable differences to real and complex numbers is the fact, that quaternion multiplication is not commutative. Therefore we do not speak of the field of quaternions, but rather a division ring or a skewfield.

Quaternions have found their use in geometry. Namely, they represent the elements of $SO(3)$ – the group of rotations about an axis through origin. Dual quaternions, which are an extension of quaternions, yield $SE(3)$ – the group of rigid body displacements. Both representations are commonly used in computer graphics, with dual quaternions having further use in mechanism science and robotics.

From geometric point of view, polynomials over dual quaternions parametrize motions in space. In fact we do not need all of them, as we can restrict ourselves to "motion polynomials". The quaternion polynomials then correspond to the special case of spherical motion. Factorizing these into linear factors allows us to decompose a complex motion into a sequence of simpler ones.

In this thesis we will discuss how to factorize polynomials whose indeterminate commutes with the coefficients. There are also more strict conventions regarding the polynomials over non-commutative rings. Some works, including Gordon and Motzkin [1965], study those as well. However, for applications in geometry, it is more natural to think of the indeterminate as a real parameter for our motion.

In the first chapter, we look briefly into the algebra and geometry of quaternions and dual quaternions. Most of the material is well-known, but we will also introduce simple lemmas, which will be useful in later chapters. For a more thorough introduction, see for example Selig [2004].

The second chapter covers the factorization of quaternion polynomials. Here we exploit the relation between roots and polynomial factors. The theory of finding quaternion zeros was already developed by Niven [1941], while more modern work of Huang and So [2002] gives the explicit solutions to degree two polynomials. Since neither were interested in factorizations, we will have to translate the results ourselves.

In chapter three we proceed to the case of dual quaternion polynomials. We will look into the factorization of motion polynomials, which was not widely considered until the work of Hegedüs et al. [2013]. As we will see, obtaining linear motion polynomials is not always possible. Since the paper only covers a special class of motion polynomials, it remains to us to disclose which cases admit a factorization. The paper by Li et al. [2019] gives an algorithm that decomposes the motion of given polynomial, when a factorization is not viable. We will use the same approach to get a more versatile version of the algorithm.

Finally, in the fourth chapter, we will see the application of factorizations to the construction of mechanical linkages. The main reason for the study of Hegedüs et al. [2013] was the construction of movable closed linkages. We will show how to obtain those from the factorization.

1. Preliminaries

The first chapter serves as a basic introduction into the world of quaternions and dual quaternions. We talk about these two concepts briefly from both algebraic and geometric point of view in sections 1.1 and 1.3. The section 1.2 gives us the basics of quaternion polynomials.

1.1 Quaternions

In this section we talk about the ring of quaternions. The aim is to introduce the basics necessary for further work. For more thorough read, see for example Joly [1905].

Definition 1. *The set of* quaternions *is the set*

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}.$$

For quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we say a is the real part *(or* scalar part*) of q, denoted $\operatorname{Re} q$, and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is the* imaginary part *(or* vector part*) of q, denoted $\operatorname{Im} q$.*

Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and let $a \in \mathbb{R}$. We define the quaternion $p = a + \mathbf{v}$ to be

$$p = a + v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

and identify $\mathbf{v} = \operatorname{Im} p$.

Definition 2. *Let $p = a + \mathbf{v}$ and $q = b + \mathbf{w}$ be quaternions. We define the sum and the product of p and q as follows.*

$$\begin{aligned} p + q &= (a + b) + (\mathbf{v} + \mathbf{w}), \\ p \cdot q &= (a \cdot b - \mathbf{v} \cdot \mathbf{w}) + (a\mathbf{w} + b\mathbf{v} + \mathbf{v} \times \mathbf{w}). \end{aligned}$$

The set $\mathbb{H}$ with these operations forms the ring of quaternions.

We may see, that the ring $\mathbb{H}$ is not commutative. However, this does not mean two quaternions can not commute. This simple geometric characterization will help us later.

Lemma 1. *Two quaternions commute if and only if their imaginary parts (viewed as a vector) are linearly dependent.*

Proof. Set $p = a + \mathbf{p}$ and $q = b + \mathbf{q}$. We compute the difference $pq - qp$.

$$\begin{aligned} pq - qp &= (ab - \mathbf{p} \cdot \mathbf{q} + a\mathbf{q} + b\mathbf{p} + \mathbf{p} \times \mathbf{q}) - (ab - \mathbf{p} \cdot \mathbf{q} + a\mathbf{q} + b\mathbf{p} + \mathbf{q} \times \mathbf{p}), \\ &= \mathbf{p} \times \mathbf{q} - \mathbf{q} \times \mathbf{p}, \\ &= 2\mathbf{p} \times \mathbf{q}. \end{aligned}$$

We may see that the difference is zero if and only $\mathbf{p} \times \mathbf{q} = \mathbf{0}$, which is equivalent to $\mathbf{p}$ and $\mathbf{q}$ being linearly dependent. ☐

Definition 3. *Let $p = a + \mathbf{v} \in \mathbb{H}$. The quaternion* conjugate *to p is*

$$\overline{p} = a - \mathbf{v}.$$

The norm *of p is*

$$\|p\| = \sqrt{p\overline{p}} = \sqrt{a^2 + \|v\|^2}.$$

We say p is a unit quaternion *if $\|p\| = 1$ and denote the set of unit quaternions $\mathbb{H}_1$. The* inverse *of p is*

$$p^{-1} = \frac{\overline{p}}{p\overline{p}} = \frac{\overline{p}}{\|p\|^2}.$$

The main use of quaternions is in geometry. The group $\mathbb{H}_1$ is a $2:1$ cover of the group of rotations $SO(3)$. Namely, the rotation about a unit vector $\mathbf{n}$ by angle θ can be represented by $q = \pm(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{n})$. The rotation is then acted by quaternion conjugation.

Lemma 2. *Let $\mathbf{x}, \mathbf{n} \in \mathbb{R}^3$, $x = 0 + \mathbf{x}$, $\|\mathbf{n}\| = 1$, $\theta \in \mathbb{R}$ and $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{n}$. Then vector*

$$\mathbf{y} = \operatorname{Im} qx\overline{q}$$

is the result of rotation of vector $\mathbf{x}$ about $\mathbf{n}$ by angle θ.

If $p = aq$ for some $a \in \mathbb{R} \setminus \{0\}$, then also

$$\mathbf{y} = \operatorname{Im} pxp^{-1} = \operatorname{Im}(px\overline{p})/(p\overline{p}).$$

The proof of the first part can be found in section 2.3 of Selig [2004]. The second part follows immediately from the fact that $p\overline{p} = a^2$.

Lemma 3. *Rotating vector $\mathbf{x}$ by quaternion q and subsequently by quaternion p yields the same result as rotating by quaternion pq.*

Proof. Simply note that

$$p(q\mathbf{x}q^{-1})p^{-1} = (pq)\mathbf{x}(pq)^{-1}.$$

□

Now let us assume a rotation by angle $\theta \neq 2k\pi$ for $k \in \mathbb{Z}$. Since $\sin\frac{\theta}{2} \neq 0$, we can divide our unit quaternion by this value and obtain the quaternion

$$p = \operatorname{cotg}\frac{\theta}{2} + \mathbf{n},$$

which gives the same rotation about unit vector $\mathbf{n}$ by the same angle θ.

This allows for a nice parametrization. We can simply replace the cotangent by a (real) time parameter t. As t attains real values the angle $\theta = 2 \cdot \operatorname{arccotg} t$ spans the interval $(0, 2\pi)$. To be able to reach a trivial rotation, that is $\theta = 0$, we define $\operatorname{arccotg}\infty = 0$.

Lemma 4. *Let $h = a + \mathbf{v} \in \mathbb{H}$, $\mathbf{v} \neq 0$, $t \in \mathbb{R}$. Then the map given by*

$$\mathbf{x} \mapsto (t-h)\mathbf{x}(t-h)^{-1} = \frac{(t-h)\mathbf{x}(t-\overline{h})}{(t-h)(t-\overline{h})}$$

corresponds to rotation of vector $\mathbf{x}$ about axis $-\mathbf{v}$ by angle

$$\theta = 2 \cdot arccotg \frac{t-a}{\|\mathbf{v}\|}.$$

Proof. Any real multiple of quaternion yields the same rotation. Set

$$p = (t - h)/\left\|\mathbf{v}\right\|.$$

By separating the real and imaginary part we get

$$p = \frac{(t-a)}{\|\mathbf{v}\|} - \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Since $\mathbf{v}/\left\|\mathbf{v}\right\|$ is a unit vector, by the work above we see that $\operatorname{Re} p = \operatorname{cotg}\frac{\theta}{2}$, where θ is the angle of rotation around $-\mathbf{v}/\left\|\mathbf{v}\right\|$. It now holds that

$$\begin{aligned}\theta &= 2 \cdot \operatorname{arccotg} \operatorname{Re} p \\ &= 2 \cdot \operatorname{arccotg} \frac{t-a}{\|\mathbf{v}\|},\end{aligned}$$

which concludes the proof. □

1.2 Polynomials

In this section we formally introduce the polynomials over quaternions. We present the most important algebraic properties and show the geometric interpretation as a parametrized spherical motion.

Definition 4. *Let $n \in \mathbb{Z}$, $n \geq 0$ and $\forall i \in \{0, \ldots, n\}$ let $h_i \in \mathbb{H}$. The* quaternion polynomial P *is given as*

$$P(t) = \sum_{i=0}^{n} h_i t^i.$$

The number n is called the degree *of P, and the polynomial*

$$\operatorname{Re} P(t) = \sum_{i=0}^{n} \operatorname{Re}(h_i) t^i$$

is called the real part *of P.*

Example. Setting $P_1(t) = t - \mathbf{i}$ gives a simple rotation about the x axis. The polynomial $P_2(t) = t - \mathbf{j} - \mathbf{k}$ parametrizes a rotation about the vector $\mathbf{v} = (0, -1, -1)$. Since $\|\mathbf{v}\| = \sqrt{2} \neq 1$, P_2 parametrizes the rotation at different speed than P_1. We compose these two rotation into a motion given by the degree two polynomial

$$\begin{aligned}P(t) &= P_2(t) \cdot P_1(t), \\ &= (t - \mathbf{j} - \mathbf{k}) \cdot (t - \mathbf{i}), \\ &= t^2 - (\mathbf{i} + \mathbf{j} + \mathbf{k})t + \mathbf{j} - \mathbf{k}.\end{aligned}$$

Lemma 5. *Let $P \in \mathbb{H}[t]$ be a quaternion polynomial with no real zeros and $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Then the formula*

$$\mathbf{y}(t) = \frac{P(t)\mathbf{x}\overline{P}(t)}{P(t)\overline{P}(t)}$$

prescribes a spherical motion. That is $\|\mathbf{y}(t)\| = \|\mathbf{x}\|$ $\forall t \in \mathbb{R}$.

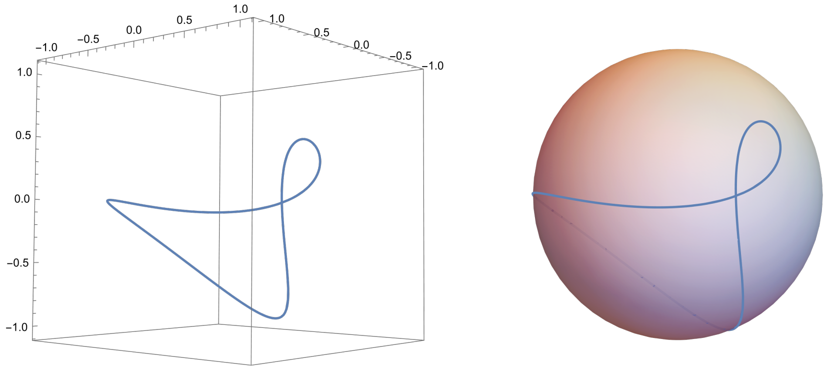

Figure 1.1: The trajectory of spherical motion of point $\mathbf{x} = (\frac{5}{13}, \frac{12}{13}, 0)$ given by the polynomial $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$ (left) and the same trajectory with added unit sphere (right).

Proof. For fixed $t \in \mathbb{R}$ the polynomial $P(t)$ becomes a non-zero quaternion p. Then $\overline{p}/(p\overline{p}) = p^{-1}$, which for our t means that $\mathbf{y}(t) = p\mathbf{x}p^{-1}$. The quaternion norm is multiplicative, therefore it holds that

$$\|\mathbf{y}(t)\| = \left\|p\mathbf{x}p^{-1}\right\| = \|p\| \, \|\mathbf{x}\| \, \|p\|^{-1} = \|\mathbf{x}\| \, .$$

□

Example. Let us now consider an arbitrary vector $\mathbf{x}$ and set $P(t) = P_2(t)P_1(t)$ as in previous example. The formula from lemma above gives the trajectory of $\mathbf{x}$ by the compound transformation given by $P_1(t)$ and then $P_2(t)$ with respect to t. We may see the trajectory of $\mathbf{x} = (5, 12, 0)/13$ in figure 1.1.

Now let us compare the polynomial $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$ with degree one polynomial from the lemma 4. It is not easy to see what motion P prescribes. For linear polynomials we only have to look at the numbers next to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to determine what axis are we rotating about. The real part (with t) will then describe the admissible angles. Readability gives us the motivation to factorize those higher-degree polynomials into linear terms.

In our example, we can imagine a rotation given by P_1, which transforms the axis $\operatorname{Im} P_2 = (0, -1, -1)$. This moving axis then rotates our given point, drawing its trajectory. We may see this in figure 1.2 with black segments (links) drawing the trajectories of both the axis and the point.

In order to get the polynomial $P = P_2P_1$ in previous section, we allowed the indeterminate t to commute with quaternion coefficients. This followed naturally from the assumption that t is a real time parameter. In doing the factorization we will need to occasionally evaluate given polynomials at quaternion values. This evaluation clearly depends on which side of coefficients is t written on.

Definition 5. *Let $M(t) = \sum_{i=0}^{n} h_i t^i$ be a quaternion polynomial. For $q \in \mathbb{H}$ we define the* right evaluation of M at q *to be*

$$M(q) = \sum_{i=0}^{n} h_i q^i.$$

Moreover q is called right zero of M *if $M(q) = 0$.*

Similarly, the left evaluation of M *is $\sum_{i=0}^{n} q^i h_i$ and q is called a* left zero of M, *if the left evaluation of M at q is equal to zero.*

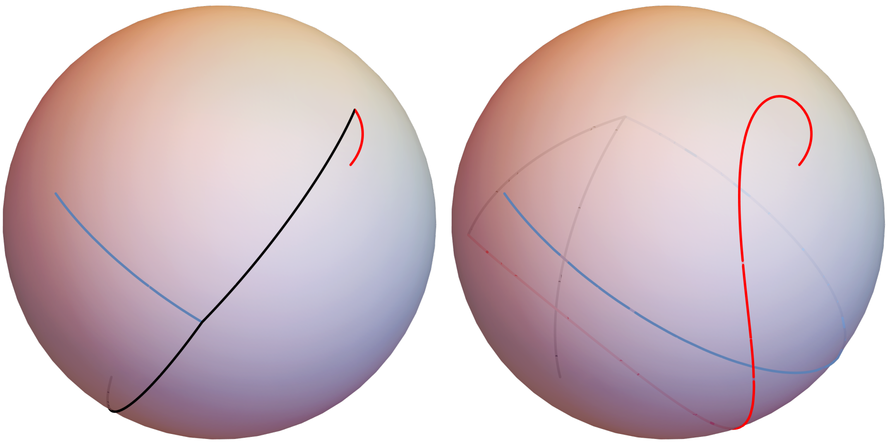

Figure 1.2: The trajectory of the motion of point $\mathbf{x} = (\frac{5}{13}, \frac{12}{13}, 0)$ given by the polynomial $P(t) = t^2 - (\mathbf{i} + \mathbf{j} + \mathbf{k})t + \mathbf{j} - \mathbf{k}$ (red) and trajectory of the axis $\operatorname{Im} P_2 = (0, -1, -1)$ (blue). Captured for $t \in (-\infty, -3)$ (left) and $t \in (-\infty, 1)$ (right).

From now on, we will be using only the right evaluation and refer to right zeros, unless we specify otherwise.

Let us now introduce more basic algebraic properties, to make the polynomials easier to talk about.

Lemma 6. *Let $M, N \in \mathbb{H}[t]$. Then there exist exactly one pair of polynomials $Q, R \in \mathbb{H}[t]$ such that* $\deg R < \deg N$ *and*

$$M(t) = Q(t) \cdot N(t) + R(t).$$

Note, that this is just a special case of lemma 1 in Hegedüs et al. [2013]. The proof can be found there.

Definition 6. *The process of computing $Q, R \in \mathbb{H}[t]$ in lemma 6 is called* right division of M by N. *Polynomial Q is the* right quotient, *denoted* $\operatorname{rquo}(M, N)$, *and R is the* right reminder, *denoted* $\operatorname{rrem}(M, N)$.

If $R = 0$, then we say that N is a right factor *of M. If M has no right factor of degree greater than* 0, *it is called* irreducible.

Note, that all of this can be defined for *left division* as well. We just have to write the formula in lemma 6 as

$$M(t) = N(t)Q(t) + R(t).$$

Then we can also get the analogous concept of *left quotient* ($Q = \operatorname{lquo}(M, N)$), *left reminder* ($R = \operatorname{lrem}(M, N)$) and *left factor*.

Lemma 7. *Let $M \in \mathbb{H}[t]$ and $h \in \mathbb{H}$. Then $M(h) = 0$ if and only if $(t - h)$ is a right factor of M.*

For the proof see proposition 16.2 in Lam [2001]. The book formulates it for more general setting of division rings.

The preceding lemma could also be reformulated to left zeros and left factors. To avoid duplicating all the work, we can rely on the following relation between left zeros and right zeros.

Lemma 8. *Let $M \in \mathbb{H}[t]$ and $h \in \mathbb{H}$. Then h is a left zero of M if and only if $\overline{M}(\overline{h}) = 0$.*

Proof. Using a version of the preceding lemma, h is a left zero of M if and only if we may write $M(t) = (t - h)N(t)$ for some $N \in \mathbb{H}[t]$. This is then equivalent to $\overline{M}(t) = \overline{N}(t)(t - \overline{h})$, which by preceding lemma happens if and only if $\overline{M}(\overline{h}) = 0$. □

Definition 7. *Let $M \in \mathbb{H}[t]$ be a polynomial. The* factorization of M *is a sequence of irreducible polynomials $M_1, M_2, \ldots, M_k$ such that*

$$M(t) = M_1(t) \cdot M_2(t) \cdot \ldots \cdot M_k(t).$$

In this work we will only consider factorization of monic polynomials. Assuming $m \in \mathbb{H} \setminus \{0\}$ is a leading coefficient of $M(t)$, we may factorize $M'(t) = m^{-1}M(t)$ as $M_1(t) \cdot M_2(t) \cdot \ldots \cdot M_k(t)$. Then the factorization of $M(t)$ will be obtained by changing $M_1(t)$ into $mM_1(t)$.

While the factorization of polynomials over fields, such as $\mathbb{R}$ or $\mathbb{C}$, is unique up to ordering of factors, in the quaternion case this is not true. Since $\mathbb{H}$ is not commutative, different order typically leads to different product. Furthermore, one polynomial may have many different factorization, depending on which zeros we factor out first.

Example. Let us now return to the polynomial $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$. It can be easily verified that $P(\mathbf{i}) = 0$, which agrees with the fact that $P_1(t) = t - \mathbf{i}$ is a right divisor of P. But $\mathbf{i}$ is not the only right zero. In fact it holds that $P(\frac{4}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}) = 0$ as well. It gives us a new right factor and with it a brand new factorization

$$P(t) = \left(t - \frac{-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3}\right) \cdot \left(t - \frac{4\mathbf{i} + \mathbf{j} + \mathbf{k}}{3}\right).$$

The polynomial P is unchanged, so the trajectory of point $\mathbf{x}$ remains the same. However, the way we draw it changes. In figure 1.3 we may see segments drawing according to the new factorization.

1.3 Dual quaternions

In this section we introduce one of the possible generalizations of quaternions, the *dual quaternions*. Their geometric properties allow not only rotations about an axis through origin, but all rigid body displacements in space. However, we need the *dual numbers* to properly speak about them.

Definition 8. *We define the ring of* dual numbers *to be*

$$\mathbb{D} = \mathbb{R}[\varepsilon] / \left\langle \varepsilon^2 \right\rangle = \{a + \varepsilon b \mid a, b \in \mathbb{R}\}.$$

Definition 9. *The square root of $a + \varepsilon b \in \mathbb{D}$ is defined whenever $a > 0$ and it holds that*

$$\sqrt{a + \varepsilon b} = \sqrt{a} + \varepsilon \frac{b}{2\sqrt{a}}.$$

The inverse of $a + \varepsilon b \in \mathbb{D}$ is defined whenever $a \neq 0$ and it holds that

$$(a + \varepsilon b)^{-1} = \frac{1}{a + \varepsilon b} = \frac{1}{a} - \varepsilon \frac{b}{a^2}.$$

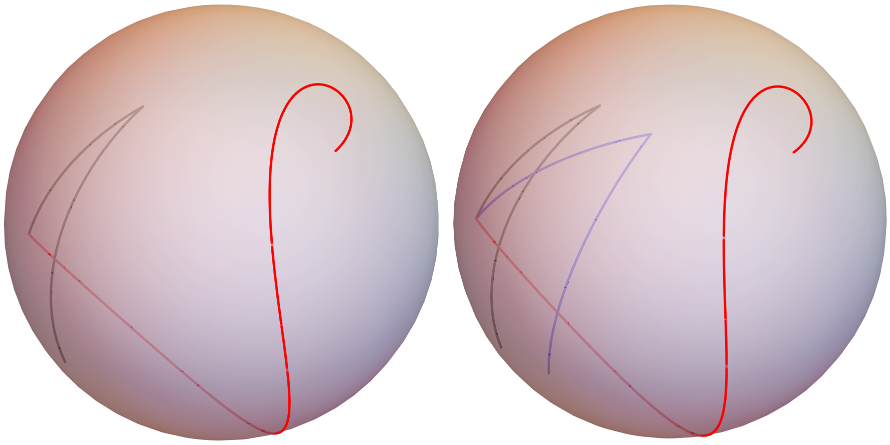

Figure 1.3: Trajectory of the motion of point $\mathbf{x} = (\frac{5}{13}, \frac{12}{13}, 0)$ given by polynomial $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$ (red). On the left, we may see the drawing segments for the new factorization. On the right, we see a comparison with the original factorization shown in blue.

We now may delve into the basics of dual quaternions. A more extensive text on the topic can be found in section 9.3 of Selig [2004].

Definition 10. *The set of* dual quaternions *is the set*

$$\mathbb{DH} = \{p + \varepsilon q \mid p, q \in \mathbb{H}\}.$$

For $h = p + \varepsilon q$ we call p the primal part *of h and q the* dual part *of h.*

Further we define the quaternion conjugate *(or just* conjugate*) of h to be $\overline{h} = \overline{p} + \varepsilon \overline{q}$ and the* dual conjugate *of h to be $h^\star = p - \varepsilon q$.*

Lastly we define the scalar part *of h as $\operatorname{Re} h = \operatorname{Re} p + \varepsilon \operatorname{Re} q \in \mathbb{D}$ and the* vector part *of h as $\operatorname{Im} h = \operatorname{Im} p + \varepsilon \operatorname{Im} q$.*

Definition 11. *Let $g = p + \varepsilon q$ and $h = r + \varepsilon s$ be dual quaternions. We define the sum and the product of g and h as follows.*

$$g + h = (p + r) + \varepsilon(q + s),$$
$$g \cdot h = (p \cdot r) + \varepsilon(p \cdot s + r \cdot q).$$

Definition 12. *We define the* norm *of $h \in \mathbb{DH}$ to be*

$$\|h\| = \sqrt{h \cdot \overline{h}} \in \mathbb{D}.$$

Lemma 9. *Let $h = p + \varepsilon q \in \mathbb{DH}$. Then*

$$h\overline{h} = p\overline{p} + \varepsilon(p\overline{q} + q\overline{p}).$$

Furthermore $\|h\|$ is well-defined for all $h \in \mathbb{DH}$.

Proof. First note, that $(p + \varepsilon q)(\overline{p} + \varepsilon \overline{q}) = p\overline{p} + \varepsilon(p\overline{q} + q\overline{p})$.

Now, if $p = 0$, then $h\overline{h} = 0$ and its square root is well-defined.

If $p \neq 0$, then $p\overline{p} > 0$ and again the square root of $h\overline{h}$ is well-defined. ☐

Lemma 10. *Dual quaternion $h = p + \varepsilon q$ is invertible if and only if $p \neq 0$ and it holds that*

$$h^{-1} = \frac{\overline{h}}{h\overline{h}}.$$

Proof. Since multiplying εq by any dual quaternion can only yield a multiple of ε, it is not possible to invert $h = p + \varepsilon q$ for $p = 0$.

If $p \neq 0$, then $h\overline{h} \in \mathbb{D}$ has a non-zero primal part and thus it is invertible. Hence $\overline{h}/(h\overline{h})$ is well defined. It holds that

$$\frac{\overline{h}}{h\overline{h}} \cdot h = h \cdot \frac{\overline{h}}{h\overline{h}} = \frac{h\overline{h}}{h\overline{h}} = 1.$$

Now if $hg = 1$ for some $g \in \mathbb{DH}$ then by multiplication from the left by h^{-1} we get $g = h^{-1}$. The same holds for $gh = 1$. □

Definition 13. *We call the elements of set*

$$\mathbb{DH}^{\times} = \{h \in \mathbb{DH} \mid \|h\| \in \mathbb{R} \setminus \{0\}\}$$

the Study quaternions*.*

Lemma 11. *Let $h = p+\varepsilon q \in \mathbb{DH}$, $p \neq 0$. Then $h \in \mathbb{DH}^{\times}$ if and only if it satisfies the* Study condition*:*

$$p\overline{q} + q\overline{p} = 0.$$

Proof. Follows straight from lemma 1.3. □

With basic algebraic knowledge of dual quaternions, we can move on to their geometric properties. In the case of quaternions, the vectorial part characterized a vector in $\mathbb{R}^3$. The vectorial Study quaternions characterize lines in $\mathbb{R}^3$.

Definition 14. *Let l be a line in $\mathbb{R}^3$ with direction vector $\mathbf{p}$ and let $\mathbf{q}$ be a point on it. Then the* Plücker coordinates *of line l is the vectorial dual quaternion h of the form*

$$h = \mathbf{p} - \varepsilon(\mathbf{p} \times \mathbf{q}).$$

If $\|\mathbf{p}\| = 1$, then we refer to h as normalized Plücker coordinates*.*

Lemma 12. *Plücker coordinates h of line l are uniquely determined by l up to a real multiple.*

For proof see section 6.2 of Selig [2004].

Lemma 13. *Let $h = p + \varepsilon q \in \mathbb{DH}^{\times}$, $\mathbf{x} \in \mathbb{R}^3$. Then the point $\mathbf{y}$ such that*

$$1 + \varepsilon\mathbf{y} = \frac{h^{\star}(1 + \varepsilon\mathbf{x})\overline{h}}{h\overline{h}}$$

is the result of rotating $\mathbf{x}$ by quaternion p and subsequent translation by vector $-2qp^{-1}$.[1]

The resulting motion is a pure rotation if and only if $-2qp^{-1} \perp \mathbf{p}$ and $\mathbf{p} \neq 0$.

The resulting motion is a pure translation if and only if $p \in \mathbb{R}$.

[1] Beware, that there are different conventions on how to define the transformation. Ours is in line with the work on polynomials by Li et al. [2019], as this is the main aim of this thesis. Other works, such as Selig [2004], use $h(1 + \varepsilon\mathbf{x})\overline{h}^{\star}/(h\overline{h})$.

Proof. Let us compute

$$\frac{h^\star(1+\varepsilon\mathbf{x})\overline{h}}{h\overline{h}} = \frac{(p-\varepsilon q)(1+\varepsilon\mathbf{x})(\overline{p}+\varepsilon\overline{q})}{p\overline{p}} = \frac{p\overline{p}+\varepsilon(p\mathbf{x}\overline{p}+p\overline{q}-q\overline{p})}{p\overline{p}} = 1+\varepsilon\frac{p\mathbf{x}\overline{p}-2q\overline{p}}{p\overline{p}}.$$

We used the Study condition $p\overline{q} = -q\overline{p}$. Also from the Study condition we have $0 = p\overline{q}+q\overline{p} = 2\operatorname{Re}(q\overline{p})$. Using the fact, that $p^{-1} = \overline{p}/(p\overline{p})$ we may write

$$\mathbf{y} = p\mathbf{x}p^{-1} - 2qp^{-1},$$

which is a rotation by p and subsequent translation by $2qp^{-1}$.

The transformation is a rotation about some axis if and only if the translation vector is perpendicular to the rotation vector, which gives the first statement.

To get a translation, we need the rotation part to be trivial, which is if and only if $p \in \mathbb{R}$, yielding the second statement. □

Note, that similarly to the case of quaternions, multiplying h by an arbitrary $a \in \mathbb{R}\setminus\{0\}$ gives the same motion, since

$$\frac{(ah)^\star(1+\varepsilon\mathbf{x})\overline{(ah)}}{ah\overline{(ah)}} = \frac{a^2h^\star(1+\varepsilon\mathbf{x})\overline{h}}{a^2h\overline{h}} = \frac{h^\star(1+\varepsilon\mathbf{x})\overline{h}}{h\overline{h}}.$$

Lemma 14. *Let l be a line in $\mathbb{R}^3$ with the Plücker coordinates $g = \mathbf{p}-\varepsilon(\mathbf{p}\times\mathbf{q}) \in \mathbb{DH}^\times$, $\|\mathbf{p}\| = 1$. The result of rotation of point $\mathbf{x}\in\mathbb{R}^3$ about line l by angle $\theta\in\mathbb{R}$ is the point $\mathbf{y}$ such that*

$$1+\varepsilon\mathbf{y} = h^\star(1+\varepsilon\mathbf{x})\overline{h},$$

where

$$h = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}g^\star.$$

Proof. Let us take $h = p+\varepsilon q$, so $p = \cos\frac{\theta}{2}+\sin\frac{\theta}{2}\mathbf{p}$ and $q = \sin\frac{\theta}{2}(\mathbf{p}\times\mathbf{q})$. Note, that $\|p\| = 1$. Now, let us compute qp^{-1}.

$$qp^{-1} = \left(\sin\frac{\theta}{2}(\mathbf{p}\times\mathbf{q})\right)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\mathbf{p}\right) \tag{1.1}$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}(\mathbf{p}\times\mathbf{q}) - \sin^2\frac{\theta}{2}(\mathbf{p}\times\mathbf{q})\times\mathbf{p}. \tag{1.2}$$

Since $qp^{-1}\perp\mathbf{p}$, the motion is a rotation. To determine it is a rotation about l, we need to prove all points of l remain unchanged by the motion.

So let us take an arbitrary point of l. Without loss of generality, we may choose $\mathbf{q}$. Then

$$\begin{aligned} p\mathbf{q}p^{-1} - qp^{-1} = {} & \cos^2\frac{\theta}{2}\mathbf{q} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\mathbf{p}\times\mathbf{q}) - \sin^2\frac{\theta}{2}(\mathbf{p}\times\mathbf{q})\times\mathbf{p} \\ & + \sin^2\frac{\theta}{2}(\mathbf{q}\cdot\mathbf{p})\mathbf{p} - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\mathbf{p}\times\mathbf{q}) + 2\sin^2\frac{\theta}{2}(\mathbf{p}\times\mathbf{q})\times\mathbf{p} \\ = {} & \cos^2\frac{\theta}{2}\mathbf{q} + \sin^2\frac{\theta}{2}((\mathbf{q}\cdot\mathbf{p})\mathbf{p} + (\mathbf{p}\times\mathbf{q})\times\mathbf{p}) \\ = {} & \cos^2\frac{\theta}{2}\mathbf{q} + \sin^2\frac{\theta}{2}\mathbf{q} = \mathbf{q}. \end{aligned}$$

□

Analogically to the case of quaternions, we may choose to parametrize this motion. Dividing the dual quaternion h by $\sin\frac{\theta}{2}$ we obtain

$$h' = \operatorname{cotg}\frac{\theta}{2} + g^{\star},$$

which represents the same transformation as h. By taking $t = \operatorname{cotg}\frac{\theta}{2}$ we get a polynomial parametrization

$$H(t) = t + g^{\star}.$$

2. Factorization of quaternion polynomials

In this chapter, we look into the factorization of polynomials over quaternions. We explore the number of such factorizations in section 2.1 following the steps of Gordon and Motzkin [1965] for division algebras, before giving explicit factorizations for any quaternion polynomial of degree two in section 2.2.

Some of the results from these sections will then be used in section 2.3. Here we derive the general factorization algorithm 1 combining the idea of Niven [1941] with the modern approach to polynomials over dual quaternions by Li et al. [2019].

2.1 Number of factorizations

We start with a section concerning the various factorization we may obtain. A classical work by Gordon and Motzkin [1965] studies the number of right zeros over division rings in general. This will be our stepping stone to determine the number of factorizations. We also look into the origin of those factorizations and introduce some concepts, that will be useful both now and further on.

The example at the end of section 1.2 showed us a polynomial of degree 2 with two different factorizations. This leads us to a question whether there could be more of those. Once we make it through the factorization algorithm, we will be able to see, that the answer is no for the polynomial $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$. In general a quaternion polynomial of degree n allows either at most $n!$ or infinite number of factorizations.

As we saw in lemma 7, to find a linear right factor of our polynomial, we need to find a right zero.

Theorem 15 (Gordon and Motzkin)**.** *Let $P(t) \in \mathbb{H}[t]$ have degree n. Then P has either at most n right zeros or infinitely many of them.*

This is just the theorem 5 in Gordon and Motzkin [1965], reformulated to quaternions. The proof can be found in the paper.

Corollary 16. *Let $P \in \mathbb{H}[t]$, $\deg P = n \geq 1$. Then there are either at most $n!$ or infinitely many factorizations of P.*

Proof. The proof is by induction. For $n = 1$ there is clearly exactly one factorization of P. For $n > 1$ we have two cases. By theorem 15 P has at most n zeros or infinitely many of them. If the latter is the case, then there are certainly infinitely many factorizations.

So let us assume P has at most n zeros. Then for any such zero $h \in \mathbb{H}$ we may find $Q^{(h)} \in \mathbb{H}[t]$ polynomial of degree $n-1$ such that $P(t) = Q^{(h)}(t)(t-h)$. If $Q^{(h)}$ has infinitely many factorizations for some h, so does P. Otherwise, by the inductive step, $Q^{(h)}$ has at most $(n-1)!$ factorizations for any zero h of P. The number of factorizations of P is then at most $n \cdot (n-1)! = n!$. □

Example. The polynomial $t^2 - (\mathbf{i} + \mathbf{j})t + \mathbf{k}$ has only one factorization, namely $(t - \mathbf{i})(t - \mathbf{j})$. Now note, that any $h = 0 + \mathbf{h} \in \mathbb{H}$ such that $\|\mathbf{h}\| = 1$ satisfies $h^2 = -1$. So, clearly, infinitely many such h are right zeros of $t^2 + 1$, hence the polynomial can be factored in infinitely many ways.

The aforementioned paper by Gordon and Motzkin [1965] studies a broader topic of polynomials over division algebras, rather than just quaternions. Their steps have interesting consequences in the world of quaternions, so we will apply them. For that we need a couple of definitions and a simple lemma, which introduce a useful concept of *conjugacy classes*.

Definition 15. *A* conjugacy class *of quaternion q is the set*

$$[q] = \{hqh^{-1} \mid h \in \mathbb{H} \setminus \{0\}\}.$$

Definition 16. *A* characteristic polynomial *of quaternion q is the real polynomial*

$$P_q(t) = (t - \overline{q})(t - q) = t^2 - 2 \cdot \mathrm{Re}(q)t + \|q\|^2 .$$

Lemma 17. *Quaternions p, q lie in the same conjugacy class if and only if it holds that $P_p = P_q$.*

Proof. We need to check that $p \in [q]$ if and only if $\mathrm{Re}\, p = \mathrm{Re}\, q$ and simultaneously $\|\mathrm{Im}\, p\| = \|\mathrm{Im}\, q\|$.

First let $h \in \mathbb{H} \setminus \{0\}$ be such that $p = hqh^{-1}$. Then

$$\mathrm{Re}\, p = h\, \mathrm{Re}(q)h^{-1} = \mathrm{Re}\, q.$$

Since $h \neq 0$, then $\mathrm{Im}\, p = h\, \mathrm{Im}(q)h^{-1}$ is a rotation of vector $\mathrm{Im}\, q$ onto vector $\mathrm{Im}\, p$. Therefore $\|\mathrm{Im}\, p\| = \|\mathrm{Im}\, q\|$.

For the converse assume $\mathrm{Re}\, p = \mathrm{Re}\, q$ and $\|\mathrm{Im}\, p\| = \|\mathrm{Im}\, q\|$. Any two vectors of the same length can be rotated one onto the other. So there is $h \in \mathbb{H} \setminus \{0\}$, such that

$$\mathrm{Im}\, p = h\, \mathrm{Im}(q)h^{-1}.$$

Then clearly $p = hqh^{-1}$, so $p \in [q]$. □

The concept of characteristic polynomials is not a part of Gordon and Motzkin [1965]. It is however a useful approach to conjugacy classes of quaternions, and can be found for example in the work of Kalantari [2013] on finding zeros of quaternion polynomials. We use them in section 2.3, in particular to prove lemma 19.

Lemma 18. *Let $M \in \mathbb{H}[t]$ be a polynomial of degree n. Then M has zeros in at most n different conjugacy classes.*

The lemma is just a restating of theorem 2 in Gordon and Motzkin [1965] specified for quaternions. The proof can be found there.

Lemma 19. *Let $M \in \mathbb{H}[t]$ be a polynomial and $x \in \mathbb{H}$, $M(x) = 0$. If M has another zero in conjugacy class $[x]$, then $\forall y \in [x]$ it holds that $M(y) = 0$.*

This time, the lemma 19 is a bit stronger then just a restatement of theorem 4 in Gordon and Motzkin [1965]. The theorem in the paper states, that there are infinitely many zeros in the conjugacy class $[x]$. However, if we apply its constructive proof on quaternions, we may see that for any $q \in \mathbb{H}$ such that

$$M(qxq^{-1}) = 0,$$

any $t \in \mathbb{R}$ yields a new zero of the form

$$(t+q)x(t+q)^{-1}.$$

This does not give us the entirety of $[x]$ yet. However, given $y = qxq^{-1}$, there are infinitely many other axes about which one can rotate $\mathbf{x} = \operatorname{Im} x$ onto $\mathbf{y} = \operatorname{Im} y$. Parametrizing all of them similarly to q above would give an entirety of $[x]$.

However, this approach is not the most straightforward. Therefore, we will give a formal proof of lemma 19 once we have the general factorization algorithm in section 2.3.

2.2 Factorization of degree two polynomials

The core of this section is the proof of theorem 20, that gives us a way to factorize degree two polynomials of various forms over quaternions. The results, in particular for real coefficients, will be useful for the general algorithm in section 2.3 and will give us other valuable information.

The theorem is based on the work of Huang and So [2002], who derive formulas for quaternion zeros. We adjust the formulation to talk about factorizations and take a different route to some of the cases, which will hopefully provide some new insights. Since we have different preliminaries and different end products to the paper, we will do our own proofs. The corollaries in this section are also our own work.

Theorem 20. *Let $P(t) = t^2 + bt + c \in \mathbb{H}[t]$ be a polynomial. Then one of the following holds:*

a) If $b, c \in \mathbb{R}$ and $b^2 - 4c \geq 0$, then there are at most two different factorizations given as

$$\begin{aligned} P(t) &= \left(t + \frac{b+\sqrt{b^2-4c}}{2}\right) \cdot \left(t + \frac{b-\sqrt{b^2-4c}}{2}\right) \\ &= \left(t + \frac{b-\sqrt{b^2-4c}}{2}\right) \cdot \left(t + \frac{b+\sqrt{b^2-4c}}{2}\right) \end{aligned}$$

b) If $b, c \in \mathbb{R}$ and $b^2 - 4c < 0$, then there are infinitely many factorizations of the form

$$P(t) = (t - \overline{q})(t - q),$$

where $q \in \mathbb{H}$ is of the form

$$\frac{-b + \sqrt{4c - b^2}\mathbf{v}}{2},$$

for any $\mathbf{v} \in \mathbb{R}^3$, $\|\mathbf{v}\| = 1$.

c) If $P \notin \mathbb{R}[t]$ and $\mathbf{b} = \operatorname{Im} b$ and $\mathbf{c} = \operatorname{Im} c$ are linearly dependent, then there are one or two factorizations of the form

$$P(t) = (t - p)(t - q) = (t - q)(t - p)$$

for $p, q \in \mathbb{H}$.

d) If $P \notin \mathbb{R}[t]$ and $\mathbf{b} = \operatorname{Im} b$ and $\mathbf{c} = \operatorname{Im} c$ are linearly independent, then there are exactly two factorizations

$$P(t) = (t - p_1)(t - q_1) = (t - p_2)(t - q_2).$$

The exact formulas for parts c) and d) are rather complicated. They can be found in full in corollaries 26 and 27.

Proof of parts a) and b). The formula from a) is well known to depict the real roots of $P(t) = t^2 + bt + c$, given the condition $b^2 - 4c \geq 0$. There are no roots in $\mathbb{C}$, we however suggest, there are no zeros in $\mathbb{H}$ either.

So let us assume $P(t) = (t - p)(t - q)$. Then $q = \rho + \mathbf{q}$ satisfies $P(q) = 0$. In order for $b \in \mathbb{R}$, it holds that $\mathbf{p} = -\mathbf{q}$. Since $c \in \mathbb{R}$ as well, the real parts of p and q are the same. So necessarily $p = \overline{q}$.

Then

$$\begin{aligned} P(t) = t^2 + bt + c &= (t - \overline{q})(t - q), \\ &= t^2 - 2\operatorname{Re}(q)t + \|q\|^2 . \end{aligned}$$

We may write $\rho = -\frac{b}{2}$. Since $\|q\|^2 = \rho^2 + \|\mathbf{q}\|^2$, it holds that

$$c = \|q\|^2 \geq \rho^2 = \frac{b^2}{4}.$$

Observing, that equality only holds for $q \in \mathbb{R}$ we get $4c > b^2$, which is the desired condition for b). We are now left to compute the imaginary part of q.

Return to the relation $\|q\|^2 = \rho^2 + \|\mathbf{q}\|^2$. Since $\|q\|^2 = c$ and $\rho = -\frac{b}{2}$, we may write $\|\mathbf{q}\|^2 = c - \frac{b^2}{4}$. We may see that it is actually sufficient condition for $\mathbf{q}$, since the coefficients of P only hold information about real part of q and its norm. For any unit vector $\mathbf{v}$ we may write the solution as

$$\frac{-b + \sqrt{4c - b^2}\mathbf{v}}{2}.$$

□

Corollary 21. *Quaternions p, q lie in the same conjugacy class if and only if $P_p(q) = 0$.*

Proof. First recall that by lemma 17 p, q lie in the same conjugacy class only if $P_p = P_q$. Then

$$P_p(q) = P_q(q) = 0,$$

from the definition of characteristic polynomial.

For the converse see that

$$P_p(t) = t^2 - 2\operatorname{Re}(p)t + \|p\|^2$$

has right zeros p and q. By the proof above, both norm and real part of such zero are uniquely determined. Hence $\operatorname{Re} p = \operatorname{Re} q$, $\|p\| = \|q\|$ and $P_p(t) = P_q(t)$. □

Corollary 22. *Let $M \in \mathbb{R}[t]$ be degree two polynomial with a quaternion zero $h \notin \mathbb{R}$. Then $M(t) = P_h(t)$.*

Next, the case c) gives us a look into when the two factors of P commute. As we will find out, this case is analogical to factorizing complex quadratic polynomials. First we need to expand on the commutativity.

Lemma 23. *Let $P(t) = (t-p)(t-q) = t^2 + bt + c \in \mathbb{H}[t]$ and let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{b}$, $\mathbf{c}$ be the imaginary parts of p, q, b, c, respectively. Then $\mathbf{p}$, $\mathbf{q}$, $\mathbf{b}$, $\mathbf{c}$ are either pairwise linearly dependent or pairwise linearly independent.*

Proof. Assume the latter is not the case, i. e. there is a pair of vectors among $\mathbf{p}$, $\mathbf{q}$, $\mathbf{b}$, $\mathbf{c}$, which are linearly dependent. These two vectors are therefore a multiple of some unit vector $\mathbf{n}$. By definition $\mathbf{b} = -\mathbf{p} - \mathbf{q}$, so if two of those vectors are a multiple of $\mathbf{n}$, so is the third. With $\mathbf{p}$, $\mathbf{q}$ linearly dependent, it now holds that

$$\mathbf{c} = \mathrm{Re}(p)\mathbf{q} + \mathrm{Re}(q)\mathbf{p} + \mathbf{p} \times \mathbf{q} = \mathrm{Re}(p)\mathbf{q} + \mathrm{Re}(q)\mathbf{p}$$

is a multiple of $\mathbf{n}$ as well.

The other case is when $\mathbf{c}$ is one of the two linearly dependent vectors. Since $\mathbf{p}$, $\mathbf{q}$, $\mathbf{b}$ all lie in the span of $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{c}$ lies there as well. However

$$\mathrm{Re}(p)\mathbf{q} + \mathrm{Re}(q)\mathbf{p} \in \mathit{span}(\mathbf{p}, \mathbf{q}) \text{ and } \mathbf{p} \times \mathbf{q} \perp \mathit{span}(\mathbf{p}, \mathbf{q}),$$

which means, that the vector product $\mathbf{p} \times \mathbf{q}$ must be zero. As such $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent and from the work above, all four vectors are multiple of some unit vector $\mathbf{n}$. □

Corollary 24. *Let $P(t) = (t-p)(t-q) = t^2 + bt + c \in \mathbb{H}[t]$. Then p, q, b, c are either pairwise commutative or no pair of them commutes.*

Proof. Follows from straightforward application of commutativity characterization in lemma 1 on the statement from preceding lemma. □

Corollary 25. *Let $P(t) = (t-p)(t-q) = t^2 + bt + c \in \mathbb{H}[t]$. If b, c commute, then also $P(t) = (t-q)(t-p)$ and both p and q are right zeros of P.*

We now have all we need for the proof of the next part.

Proof of theorem 20, part c). Let $P(t) = (t-p)(t-q)$ and $\mathbf{q} = \mathrm{Im}\, q$ and $\mathbf{p} = \mathrm{Im}\, p$. As we saw in lemma 23, the condition for $\mathbf{b}$ and $\mathbf{c}$ is equivalent to all four vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{b}$, $\mathbf{c}$ being a real multiple of some unit vector $\mathbf{n}$. We will show, that the choice of orientation is arbitrary, but for now we define $\mathbf{n} = \mathbf{b}/\|\mathbf{b}\|$, if $\mathbf{b} \neq 0$ and $\mathbf{c}/\|\mathbf{c}\|$ otherwise. Let us also define $\zeta \in \mathbb{R}$ such that $\mathbf{c} = \zeta\mathbf{n}$. Clearly $|\zeta| = \|\mathbf{c}\|$.

The key to the formula is an observation, that the set $\{a_1 + a_2\mathbf{n} \mid a_1, a_2 \in \mathbb{R}\}$ behaves exactly as the field of complex numbers. Rather than taking the imaginary unit $\mathbf{i}$, we consider the imaginary unit $\mathbf{n}$. It can be easily verified, that both addition and multiplication act the same on both sets.

With this in mind, we can solve the equation just like in complex case. First let us see that $(t + b/2)^2 = (4c - b^2)/4$. We know the complex roots of

$$x^2 = a_1 + a_2\mathbf{n}$$

to be

$$x = \pm\left(\sqrt{\frac{\sqrt{a_1^2+a_2^2}+a_1}{2}} + \operatorname{sgn}(a_2)\sqrt{\frac{\sqrt{a_1^2+a_2^2}-a_1}{2}}\mathbf{n}\right),$$

where sgn is a modified sign function, giving $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and $\operatorname{sgn}(x) = -1$ otherwise.

In our case $a_1 = (4\operatorname{Re} c - \operatorname{Re}^2 b + \|\mathbf{b}\|^2)/4$ and $a_2 = (4\zeta - 2\operatorname{Re} b\,\|\mathbf{b}\|)/4$. Setting $\xi_1 = 4^2(a_1^2 + a_2^2)$ and $\xi_2 = 4a_1$, we get

$$t + \frac{b}{2} = \pm\left(\frac{1}{2}\sqrt{\frac{\sqrt{\xi_1}+\xi_2}{2}} + \frac{1}{2}\sqrt{\frac{\sqrt{\xi_1}-\xi_2}{2}}\mathbf{n}\right).$$

Subtracting $b/2$ gives us right zeros of P, which by corollary above are exactly our p and q.

Note that choosing $-\mathbf{n}$ in place of $\mathbf{n}$ changes the sign in a_2. If $a_2 = 0$ than either $\sqrt{a_1^2} = a_1$ or $-a_1$, so at most one of the coefficients is non-zero. The $\pm$ in front of parenthesis then makes the choice of sign arbitrary. If $a_2 \neq 0$, then $\operatorname{sgn}(-a_2) = -\operatorname{sgn}(a_2)$, so $\operatorname{sgn}(-a_2)\cdot(-\mathbf{n}) = \operatorname{sgn}(a_2)\mathbf{n}$. □

Corollary 26. *For $P \notin \mathbb{R}[t]$ and $\mathbf{b} = \operatorname{Im} b$, $\mathbf{c} = \operatorname{Im} c$ linearly dependent, the two factorizations*

$$P(t) = (t-p)(t-q) = (t-q)(t-p)$$

satisfy

$$p = -\frac{b}{2} + \frac{1}{2}\left(\sqrt{\frac{\sqrt{\xi_1}+\xi_2}{2}} + \sqrt{\frac{\sqrt{\xi_1}-\xi_2}{2}}\mathbf{n}\right),$$
$$q = -\frac{b}{2} - \frac{1}{2}\left(\sqrt{\frac{\sqrt{\xi_1}+\xi_2}{2}} + \sqrt{\frac{\sqrt{\xi_1}-\xi_2}{2}}\mathbf{n}\right),$$

for $\mathbf{n} = \mathbf{b}/\|\mathbf{b}\|$ or $\mathbf{n} = \mathbf{c}/\|\mathbf{c}\|$, whichever is defined, and

$$\xi_1 = (4\operatorname{Re} c + \operatorname{Re}^2 b - \|\mathbf{b}\|^2)^2 + (4\|\mathbf{c}\| + 2\operatorname{Re} b\,\|\mathbf{b}\|)^2,$$
$$\xi_2 = 4\operatorname{Re} c + \operatorname{Re}^2 b - \|\mathbf{b}\|^2.$$

For the final part of the proof, we will actually use the work by Huang and So [2002]. While it is possible to verify that the two factorizations provided below are correct with our current knowledge, we still need to prove there are no other viable factorizations. Here we will use the right zeros provided by the paper.

Lemma 27. *Let*

$$c' = c - \frac{1}{4}(\operatorname{Re} b)^2 - \frac{1}{2}\operatorname{Re} b\mathbf{b},$$
$$\mu = \|\mathbf{b}\|^2 + 2\operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)^2 = \|\mathbf{b}\|^2 + 2\operatorname{Re} c',$$
$$\nu = 2\mathbf{b}\cdot\mathbf{c} - \operatorname{Re} b\,\|\mathbf{b}\|^2 = 2\operatorname{Re}(\mathbf{b}c'),$$
$$\xi = \left(\operatorname{Re} c - \frac{1}{4}(\operatorname{Re} b)^2\right)^2 + \left\|\mathbf{c} - \frac{1}{2}\operatorname{Re} b\mathbf{b}\right\|^2 = \|c'\|^2.$$

Then the two factorizations in part d) of theorem 20 satisfy

$$\begin{aligned} p_1 &= -\operatorname{Re} b/2 + (y - c')(\sqrt{z} + \mathbf{b})^{-1}, \\ q_1 &= -\operatorname{Re} b/2 + (-\sqrt{z} + \mathbf{b})^{-1}(x - c'), \\ p_2 &= -\operatorname{Re} b/2 + (x - c')(-\sqrt{z} + \mathbf{b})^{-1}, \\ q_2 &= -\operatorname{Re} b/2 + (\sqrt{z} + \mathbf{b})^{-1}(y - c'), \end{aligned}$$

where one of the following holds:

(1) There is a unique positive $z \in \mathbb{R}$ such that

$$z^3 + 2\mu z^2 + (\mu^2 - 4\xi)z - \nu^2 = 0.$$

Then

$$x = \frac{\sqrt{z}^3 + \mu\sqrt{z} - \nu}{2\sqrt{z}}, \; y = \frac{\sqrt{z}^3 + \mu\sqrt{z} + \nu}{2\sqrt{z}}.$$

(2) Otherwise $z = 0$ and

$$x = \frac{\mu - \sqrt{\mu^2 - 4\xi}}{2}, \; y = \frac{\mu + \sqrt{\mu^2 - 4\xi}}{2}.$$

We start by restating the case 4 of theorem 2.3 in Huang and So [2002] using our own notation, to make it easier to use.

Theorem 28 (Huang and So, case 4)**.** *Using the notation of lemma 27, if $b \notin \mathbb{R}$, then the solutions q such that $P(q) = 0$ can be obtained by*

$$q = -\frac{1}{2}\operatorname{Re} b - (\mathbf{b} + Z)^{-1}(c' - X),$$

where the pair (Z, X) is chosen as follows.

1. *$Z = 0$, $X = (\mu \pm \sqrt{\mu^2 - 4\xi})/2$ provided that $\nu = 0$ and $\mu^2 \geq 4\xi$.*
2. *$Z = \pm\sqrt{2\sqrt{\xi} - \mu}$, $X = \sqrt{\xi}$ provided that $\nu = 0$ and $\mu^2 < 4\xi$.*
3. *$Z = \pm\sqrt{z}$, $X = (Z^3 + \mu Z + \nu)/(2Z)$ provided that $\nu \neq 0$ and z is the unique positive root of real polynomial*

$$z^3 + 2\mu z^2 + (\mu^2 - 4\xi)z - \nu^2.$$

As this is just a restatement, the proof can be found in Huang and So [2002].

Before delving into the proof of lemma 27 and consequently the proof of part d) of theorem 20, we need just one more technical lemma.

Lemma 29. *Using the notation of lemma 27, it holds that*

$$\mu + 2\sqrt{\xi} \geq 0.$$

Proof. First note, that by definition

$$\mu = \|\mathbf{b}\|^2 + 2\operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)^2 \geq 2\operatorname{Re} c - \frac{1}{2}(\operatorname{Re} b)^2 = 2\operatorname{Re} c'.$$

Hence

$$\begin{aligned}\mu + 2\sqrt{\xi} &\geq 2\operatorname{Re} c' + 2\|c'\| = 2\operatorname{Re} c' + 2\sqrt{(\operatorname{Re} c')^2 + \|\mathbf{c}'\|^2} \\ &\geq 2\operatorname{Re} c' + 2\sqrt{(\operatorname{Re} c')^2} = 2(\operatorname{Re} c' + |\operatorname{Re} c'|) \\ &\geq 0.\end{aligned}$$

□

Proof of lemma 27. First we need to match our q_1 and q_2 to the zeros from Huang and So [2002], which we see in the restated theorem 28. Then we will verify, that the p_1 and p_2 yield the corresponding right factors.

First let us assume that there is $z > 0$ such that

$$z^3 + 2\mu z^2 + (\mu^2 - 4\xi)z - \nu^2 = 0.$$

If $\nu \neq 0$ we are in the third case of theorem 28. The choice of $Z = \sqrt{z}$ gives

$$X = \frac{\sqrt{z}^3 + \mu\sqrt{z} + \nu}{2\sqrt{z}} = y$$

and hence the root $q = q_2$. On the other hand choosing $Z = -\sqrt{z}$ gives us

$$X = \frac{-\sqrt{z}^3 - \mu\sqrt{z} + \nu}{-2\sqrt{z}} = \frac{\sqrt{z}^3 + \mu\sqrt{z} - \nu}{2\sqrt{z}} = x,$$

yielding the root $q = q_2$.

Now if $\nu = 0$ the positive root z is clearly a root of $z^2 + 2\mu z + (\mu^2 - 4\xi)$. Hence it is of the form

$$z = \frac{-2\mu \pm \sqrt{4\mu^2 - 4\mu^2 + 4\xi}}{2} = -\mu \pm 2\sqrt{\xi}.$$

Note, that $\xi = \|c'\|^2 \geq 0$. Since $z > 0$, then by lemma 29 necessarily $z = 2\sqrt{\xi} - \mu$ and $\mu < 2\sqrt{\xi}$, which is equivalent to the condition of case 2 in theorem 28. Now clearly

$$x = y = \frac{\sqrt{z}^3 + 2\mu\sqrt{z}}{2\sqrt{z}} = \frac{1}{2}(z + \mu) = \sqrt{\xi} = X.$$

The choice of $Z = \sqrt{z}$ yields $q = q_2$ and $Z = -\sqrt{z}$ yields $q = q_1$.

For the final part let us assume, the polynomial $z^3 + 2\mu z^2 + (\mu^2 - 4\xi)z - \nu^2$ has no positive root. By theorem 28 that happens only if $\nu = 0$ and by the work above necessarily $\mu^2 \geq 4\xi$. Hence we are in the first case of the theorem and $Z = \sqrt{z} = 0$. The choice of sign in X gives us either

$$X = \frac{\mu + \sqrt{\mu^2 - 4\xi}}{2} = y,$$

or

$$X = \frac{\mu - \sqrt{\mu^2 - 4\xi}}{2} = x.$$

The first case corresponds to $q = q_1$, the second case corresponds to $q = q_2$.

Now we only need to verify, that $p_1 q_1 = c = p_2 q_2$ and $p_1 + q_1 = -b = p_2 + q_2$. For the sake of simplicity we only prove these identities for p_1 and q_1. The proof for p_2 and q_2 follows analogically.

First notice, we have the following identities

$$\begin{aligned} x + y &= \mu + z, \\ \sqrt{z}(y - x) &= \nu, \\ xy &= \xi. \end{aligned}$$

Then see that for $\tilde{p} = p_1 + \operatorname{Re} b/2$ and $\tilde{q} = q_1 + \operatorname{Re} b/2$ we have

$$\begin{aligned} (t - \tilde{p})(t - \tilde{q}) &= t^2 - (\tilde{p} + \tilde{q}) + \tilde{p}\tilde{q} \\ &= t^2 - (\operatorname{Re} b + p_1 + q_1)t + (\operatorname{Re} b)^2 + \operatorname{Re} b/2(p_1 + q_1) + p_1 q_1. \end{aligned}$$

Hence, the pair of identities $p_1 + q_1 = -b$ and $p_1 q_1 = c$ is equivalent to the pair $\tilde{p} + \tilde{q} = -\mathbf{b}$ and $\tilde{p}\tilde{q} = c'$. We will prove those instead.

Let us start with

$$\begin{aligned} \tilde{p} + \tilde{q} &= (y - c')(\sqrt{z} + \mathbf{b})^{-1} + (-\sqrt{z} + \mathbf{b})^{-1}(x - c') \\ &= \frac{1}{\|\sqrt{z} + \mathbf{b}\|^2}(y - c')(\sqrt{z} - \mathbf{b}) + \frac{1}{\|-\sqrt{z} + \mathbf{b}\|^2}(-\sqrt{z} - \mathbf{b})(x - c') \\ &= \frac{y\sqrt{z} - y\mathbf{b} - \sqrt{z}c' + c'\mathbf{b} - \sqrt{z}x + \sqrt{z}c' - x\mathbf{b} + \mathbf{b}c'}{z + \|\mathbf{b}\|^2} \\ &= \frac{\sqrt{z}(y - x) - (x + y)\mathbf{b} + c'\mathbf{b} + \mathbf{b}c'}{z + \|\mathbf{b}\|^2} \\ &= \frac{\nu - (\mu + z)\mathbf{b} + c'\mathbf{b} + \mathbf{b}c'}{z + \|\mathbf{b}\|^2} \\ &= \frac{\mathbf{b}\overline{c'} + c'\overline{\mathbf{b}} - (\|\mathbf{b}\|^2 + z)\mathbf{b} - 2\operatorname{Re}(c')\mathbf{b} + c'\mathbf{b} + \mathbf{b}c'}{z + \|\mathbf{b}\|^2} \\ &= \frac{2\mathbf{b}\operatorname{Re} c' - 2\operatorname{Re}(c')\mathbf{b} - (z + \|\mathbf{b}\|^2)\mathbf{b}}{z + \|\mathbf{b}\|^2} = -\mathbf{b}. \end{aligned}$$

We used the identities from above, definitions of μ, ν, ξ and the fact that $\overline{\mathbf{b}} = -\mathbf{b}$.

Now, for the product we have

$$\begin{aligned}
\tilde{p}\tilde{q} &= (y-c')(\sqrt{z}+\mathbf{b})^{-1}(-\sqrt{z}+\mathbf{b})^{-1}(x-c') \\
&= \frac{1}{(z+\|\mathbf{b}\|^2)^2}(y-c')(\sqrt{z}-\mathbf{b})(-\sqrt{z}-\mathbf{b})(x-c') \\
&= \frac{-\|\sqrt{z}+\mathbf{b}\|^2}{(z+\|\mathbf{b}\|^2)^2}(y-c')(x-c') \\
&= -\frac{(y-c')(x-c')}{z+\|\mathbf{b}\|^2} \\
&= -\frac{xy-(x+y)c'+c'^2}{z+\|\mathbf{b}\|^2} \\
&= -\frac{\xi-(\mu+z)c'+c'^2}{z+\|\mathbf{b}\|^2} \\
&= -\frac{c'\overline{c'}-(c'+\overline{c'}+\|\mathbf{b}\|^2+z)c'+c'^2}{z+\|\mathbf{b}\|^2} \\
&= -\frac{(c'-c')(c'-\overline{c'})-(z+\|\mathbf{b}\|^2)c'}{z+\|\mathbf{b}\|^2} = c'.
\end{aligned}$$

Once again we only used the identities for x, y, z and the definitions of μ, ν, ξ.

Now we have the desired properties for $\tilde{p}, \tilde{q}$ and therefore the second part of the proof. □

The proof of theorem 20, part d), follows from lemma 27.

2.3 General factorization

This section will give us the tools to factorize all quaternion polynomials. The basic idea from Niven [1941] is to use the norm and the real part of quaternion zeros to find right factors. We modernise and simplify this approach, taking inspiration from the case of dual quaternions in Li et al. [2019], to get the algorithm 1.

More precisely, we will obtain an algorithm, that given a factorization of a certain real polynomial, returns the factorization of the original. This entire process revolves around *norm polynomials*, which are the polynomial counterparts of quaternion norm.

Definition 17. *Given $P \in \mathbb{H}[t]$, we call the polynomial $P\overline{P}$ the* norm polynomial *of P.*

Lemma 30. *For any $P \in \mathbb{H}[t]$, it holds that $P\overline{P} \in \mathbb{R}[t]$.*

Proof. Assume $P(t) = \sum_{i=0}^{n} a_i t^i$. Then the j-th coefficient of $P\overline{P}$ is of the form $\sum_{i=0}^{j} a_i \overline{a_{j-i}}$. It is easy to see, that whenever $i \neq j-i$, then

$$a_i\overline{a_{j-i}} + a_{j-i}\overline{a_i} = a_i\overline{a_{j-i}} + \overline{a_i\overline{a_{j-i}}} = 2\operatorname{Re}(a_i\overline{a_{j-i}}).$$

On the other hand, if $i = j-i$, then

$$a_i\overline{a_{j-i}} = a_i\overline{a_i} = \|a_i\|^2 \in \mathbb{R}.$$

Therefore the coefficients are all real. □

Example. The norm polynomial of $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$ from the section 1.2 is

$$P\overline{P}(t) = t^4 + 3t^2 + 2.$$

Given a linear polynomial $t - h$ for some $h \in \mathbb{H}$, its norm polynomial is

$$(t-h)(t-\overline{h}) = (t-\overline{h})(t-h) = P_h(t),$$

which is the characteristic polynomial of h.

The main advantage of norm polynomial $P\overline{P}$ is, that it has all the right zeros that P does, while possibly adding new ones. This is the case for all polynomials P, not just the linear ones.

Lemma 31. *Let $P \in \mathbb{H}[t]$ and $h \in \mathbb{H}$ such that $P(h) = 0$. Then also $P\overline{P}(h) = 0$.*

Proof. The fact, that h is a right zero of P can be equivalently written as $P(t) = Q(t) \cdot (t-h)$ for some $Q \in \mathbb{H}[t]$. Then, we may write

$$\begin{aligned} P\overline{P}(t) &= P(t) \cdot \overline{P}(t), \\ &= Q(t) \cdot (t-h) \cdot \overline{Q(t) \cdot (t-h)}, \\ &= Q(t) \cdot (t-h) \cdot (t-\overline{h}) \cdot \overline{Q}(t), \\ &= Q(t)\overline{Q}(t) \cdot \left((t-h)(t-\overline{h})\right), \end{aligned}$$

where the final equation holds, as $((t-h)(t-\overline{h})) \in \mathbb{R}[t]$, so it commutes with the other polynomials. Furthermore $(t-h)(t-\overline{h}) = (t-\overline{h})(t-h)$, so we may write

$$P\overline{P}(t) = \tilde{Q}(t) \cdot (t-h)$$

for $\tilde{Q}(t) = Q(t)\overline{Q}(t)(t-\overline{h})$. Therefore h is a right zero of $P\overline{P}$. □

Corollary 32. *Let $P \in \mathbb{H}[t]$. If $h \in \mathbb{H}$ satisfies $P(h) = 0$, then $P_h(t)$ divides $P\overline{P}(t)$.*

While it may not be obvious, this lemma is crucial for the factorization. We know, that the factorization can be carried out by finding and factoring out right zeros of the given polynomial P. The proof tells us to look for the common zeros of the polynomial P and some real quadratic polynomial P_h. In fact, P_h is the characteristic polynomial of our desired root h, carrying the information about the norm of h and its real part.

If we find any such characteristic polynomial, it will be easier to find the root. So we need to extract P_h from $P\overline{P}$.

Since $P\overline{P} \in \mathbb{R}[t]$, it can be written as a product of real quadratic[1] polynomials. The question now is, which of these quadratic factors give us the desired zeros? The answer is all of them.

Lemma 33. *Let $P \in \mathbb{H}[t]$, $n = \deg P$, and let $R_1, R_2, \ldots, R_n \in \mathbb{R}[t]$ be quadratic polynomials, such that*

$$P\overline{P}(t) = R_1(t) \cdot R_2(t) \cdot \ldots \cdot R_n(t).$$

Then for any $i \in \{1, \ldots, n\}$ there exists $h \in \mathbb{H}$ such that $P(h) = R_i(h) = h$.

[1] At most quadratic. For $h \in \mathbb{R}$ we may factor $(t-h)(t-\overline{h}) = (t-h)^2$. But $\deg P\overline{P}$ is always even, so we will be able to couple those into real quadratic polynomials.

Proof. Let $R = R_i$ for arbitrary $i \in \{1, \ldots, n\}$. We start by dividing P by R from the right and let $Q, S \in \mathbb{H}[t]$ be such, that $P(t) = Q(t)R(t) + S(t)$, where $\deg S < \deg R = 2$. Now note that

$$\begin{aligned}
P\overline{P}(t) &= (Q(t)R(t) + S(t)) \cdot \overline{(Q(t)R(t) + S(t))}, \\
&= Q(t)R(t)R(t)\overline{Q}(t) + Q(t)R(t)\overline{S}(t) + S(t)R(t)\overline{Q}(t) + S(t)\overline{S}(t), \\
&= Q(t)\overline{Q}(t)R^2(t) + Q(t)\overline{S}(t)R(t) + S(t)\overline{Q}(t)R(t) + S(t)\overline{S}(t).
\end{aligned}$$

We used the fact, that R is real and commutes with other polynomials. Now we may see, that since R is a right factor of $P\overline{P}$, as well as three of our terms on the right, it must also divide $S(t)\overline{S}(t)$.

We split the proof into two cases, depending on degree of S. If $\deg S = 1$, then $\deg S\overline{S} = 2 = \deg R$ and therefore $S\overline{S} = \nu R$ for some $\nu \in \mathbb{R}$. Seeing as $S(t) = at + b$ for some $a, b \in \mathbb{H}$, the quaternion $h = -a^{-1}b$ is a zero of both S and R. Since $(t - h)$ is a right divisor of $R(t)$ and $S(t)$ it is also a right divisor of $Q(t)R(t) + S(t) = P(t)$, so h is a right zero of P as well.

Now, if $\deg S = 0$ and R is a right divisor of $S\overline{S} \in \mathbb{H}$, then necessarily S is a zero polynomial. Then clearly anytime $(t - h)$ is a right divisor of $R(t)$, it is also a right divisor of $P(t) = Q(t)R(t)$. □

This gives us a way of finding right zeros.

Corollary 34. *Let us have a real quadratic polynomial R dividing $P\overline{P}$ and set $S(t) = \operatorname{rrem}(P, R)$. Then $S(t) = at + b$. If $a \neq 0$, then P has a right zero $-a^{-1}b$. If $a = 0$, then $b = 0$ and the right zeros can be calculated from R by formula in part a) or part b) of theorem 20.*

Example. So let us consider $P(t) = t^2 - (\mathbf{i}+\mathbf{j}+\mathbf{k})t + \mathbf{j} - \mathbf{k}$, we saw at the beginning of section 1.2. We will use the newly found wisdom. The norm polynomial is

$$P\overline{P}(t) = t^4 + 3t^2 + 2 = (t^2 + 1)(t^2 + 2) = R_1(t) \cdot R_2(t).$$

Using our $R_1(t) = t^2 + 1$, we calculate

$$S(t) = \operatorname{rrem}(P(t), R_1(t)) = -(\mathbf{i} + \mathbf{j} + \mathbf{k})t - (1 - \mathbf{j} + \mathbf{k}),$$

which has the root $\mathbf{i}$. Dividing our P on the right by $(t - \mathbf{i})$ yields the factorization

$$P(t) = (t - \mathbf{j} - \mathbf{k})(t - \mathbf{i}).$$

Now, what if used R_2, the other quadratic factor, instead? We have

$$S'(t) = \operatorname{rrem}(P(t), R_2(t)) = -(\mathbf{i} + \mathbf{j} + \mathbf{k})t - (2 - \mathbf{j} + \mathbf{k}),$$

which has the right zero $(4\mathbf{i} + \mathbf{j} + \mathbf{k})/3$. This gives us the second factorization, we saw at the end of section 1.2:

$$P(t) = \left(t - \frac{-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3}\right) \cdot \left(t - \frac{4\mathbf{i} + \mathbf{j} + \mathbf{k}}{3}\right).$$

To get a right zero of P, we used the real quadratic factors of $P\overline{P}$. Different factors gave us different zeros. Once we find h such that $P(h) = 0$, we may continue our factorization with $Q(t)$, where $P(t) = Q(t)(t - h)$. For different h we get different factorizations. The proof of lemma 31 shows, that any right zero can be obtained by choosing the right factor of $P\overline{P}$. Therefore all possible factorizations of P are obtainable by algorithm 1.

Algorithm 1 Factorization of quaternion polynomials

Input: polynomial $P \in \mathbb{H}[t]$ of degree n

Output: polynomials $P_1, P_2, \ldots, P_n \in \mathbb{H}[t]$ of degree one, such that $P = P_n \cdot \ldots \cdot P_1$

```
N ← P · P̄
For i = 1, ..., n Do
    R ← real monic quadratic factor of N
    S ← rrem(P, R)
    a ← linear coefficient of S
    If a = 0 Do
        b ← linear coefficient of R
        c ← constant coefficient of R
        P_i ← t + (b + √(4c − b²) i)/2
    Else Do
        b ← constant coefficient of S
        P_i ← t + a^{-1} b
    End If
    N ← N/R
    P ← rquo(P, P_i)
End For
Return P_1, ..., P_n
```

We now return to lemma 19 to find all roots in given conjugacy class.

Proof of lemma 19. Let $x_1, x_2 \in \mathbb{H}$ be two distinct zeros of M from the same conjugacy class. From lemma 17, we have that $P_{x_1} = P_{x_2}$. Let $S(t) = \mathrm{rrem}(M, P_{x_1})$. From the corollary 34 we know that S either has only one root or $S = 0$. Since M and P_{x_1} have at least two common roots, then the latter is the case. Hence for some $Q \in \mathbb{H}[t]$ and any $y \in [x_1]$ we have

$$M(t) = Q(t)P_{x_1}(t) = Q(t)(t - \overline{y})(t - y).$$

We see that y is a right zero of M. $\square$

3. Dual quaternion polynomials

This chapter gives us some insight into the factorization of polynomials over dual quaternions. In section 3.1 we start with the basics of how such polynomials work. The section 3.2 will begin the work towards factorization, drawing similarities and differences to quaternion polynomials. We propose the theorem 43 summing up our options and give an example in the form of algorithm by Hegedüs et al. [2013], which factorizes the *generic* polynomials.

These results will then help us in section 3.3. Here we factorize a motion into rotations rather than just factorizing the polynomial itself. We follow the footsteps by Li et al. [2019], while also improving the variety of achievable factorizations with our own algorithm.

3.1 Polynomials

This section formally introduces the polynomials over dual quaternions. Most of the work here is analogical to the case of quaternion polynomials in section 1.2. However, we use the concept of motion polynomials, which makes our effort easier, with no loss of generality on the geometric side.

Definition 18. *Let $P, Q \in \mathbb{H}[t]$. A* dual quaternion polynomial *is given as*

$$H(t) = P(t) + \varepsilon Q(t).$$

The set of such polynomial is denoted by $\mathbb{DH}[t]$.

The polynomial P is called the primal part *of H, polynomial Q is called the* dual part *of H.*

Lemma 35. *Let $H(t)$ be as above. Then for $n = \max\{\deg P, \deg Q\}$ and $\forall i \in \{0, \ldots, n\}$ there exist $a_i \in \mathbb{DH}$ such that*

$$H(t) = \sum_{i=0}^{n} a_i t^i.$$

Proof. Let $P(t) = \sum_{i=0}^{n} p_i t^i$ and $Q(t) = \sum_{i=0}^{n} q_i t^i$. Then $\forall i \in \{0, \ldots, n\}$ we may set $a_i = p_i + \varepsilon q_i$ to obtain the desired form. □

Definition 19. *Let $H = P + \varepsilon Q \in \mathbb{DH}[t]$. The number $n = \max\{\deg P, \deg Q\}$ is called the* degree *of H.*

Definition 20. *Let $M(t) = \sum_{i=0}^{n} a_i t^i$ be a dual quaternion polynomial. For $h \in \mathbb{DH}$ we define the* right evaluation of M at h *to be*

$$M(h) = \sum_{i=0}^{n} a_i h^i.$$

Moreover h is called right zero of M *if $M(q) = 0$.*

Lemma 36. *Let $H, G \in \mathbb{DH}[t]$. Then there exist exactly one pair of polynomials $Q, R \in \mathbb{DH}[t]$ such that $\deg R < \deg G$ and*

$$H(t) = Q(t) \cdot G(t) + R(t).$$

This is just a restatement of lemma 1 in Hegedüs et al. [2013], the proof can be found there.

Definition 21. *The process of computing* $Q, R \in \mathbb{DH}[t]$ *in lemma 36 is called the* right division *of* H *by* G*. Polynomial* Q *is the* right quotient*, denoted* $\mathrm{rquo}(H, G)$*, and* R *is the* right reminder*, denoted* $\mathrm{rrem}(H, G)$*.*

If $R = 0$*, then we say that* G *is a* right factor *of* H*. If* H *has no right factor of degree greater than* 0*, it is called* irreducible*.*

Similarly to the case of quaternions, we may talk of *left division* by writing

$$H(t) = G(t)Q(t) + R(t)$$

in lemma 36. This would then define the *left quotient* ($Q = \mathrm{lquo}(H, G)$), *left reminder* ($R = \mathrm{lrem}(H, G)$) and *left factor*.

Lemma 37. *Let* $H \in \mathbb{DH}[t]$ *and* $h \in \mathbb{DH}$*. Then* $H(h) = 0$ *if and only if* $(t - h)$ *is a right factor of* H*.*

The statement with proof can be found as lemma 2 in Hegedüs et al. [2013].

We would now like to prepare for similar approach we had in the case of quaternion polynomials. We used the concept of norm polynomial to help us find the right zeros. We can use the very same definition here, however, we will have to restrict us further, to make sure such polynomial is real.

Definition 22. *We call the polynomial* $H\overline{H}$ *the* norm polynomial *of* H*.*

Definition 23. *Let* $P, Q \in \mathbb{H}[t]$*. A polynomial* $P + \epsilon Q \in \mathbb{DH}[t]$ *is* motion polynomial *if* $P\overline{Q} + Q\overline{P} = 0$ *and its leading coefficient is invertible. The equation* $P\overline{Q} + Q\overline{P} = 0$ *is called the* Study condition*.*

The condition on leading coefficient allows us to consider only monic polynomials. The Study condition guarantees the motion polynomial to be real.

Lemma 38. *Let* $H \in \mathbb{DH}[t]$ *be a polynomial with invertible leading coefficient. Then* H *is a motion polynomial if and only if* $H\overline{H} \in \mathbb{R}[t]$*.*

Proof. Let $H = P + \varepsilon Q$ for $P, Q \in \mathbb{H}[t]$. Then

$$H\overline{H} = (P + \varepsilon Q)(\overline{P} + \varepsilon\overline{Q}) = P\overline{P} + \varepsilon(P\overline{Q} + Q\overline{P}),$$

which is real if and only if H satisfies the Study condition. □

Corollary 39. *Let* $H \in \mathbb{DH}[t]$ *be a motion polynomial. If* H *has no real zero, then for any* $t_0 \in \mathbb{R}$ *it holds that* $H(t_0) \in \mathbb{DH}^\times$*.*

Theorem 40 (Jüttler)**.** *Let* $H \in \mathbb{DH}[t]$ *be a motion polynomial with no real zeros and* $\mathbf{x} \in \mathbb{R}^3$*. Then the curve* $\mathbf{y}(t)$ *such that*

$$1 + \varepsilon\mathbf{y}(t) = \frac{H^\star(t)(1 + \varepsilon\mathbf{x})\overline{H}(t)}{H(t)\overline{H}(t)}$$

is rational.

Moreover, for any rational curve $\mathbf{y}(t)$ *there exists motion polynomial* H *and* $\mathbf{x} \in \mathbb{R}^3$ *such that the expression above holds.*

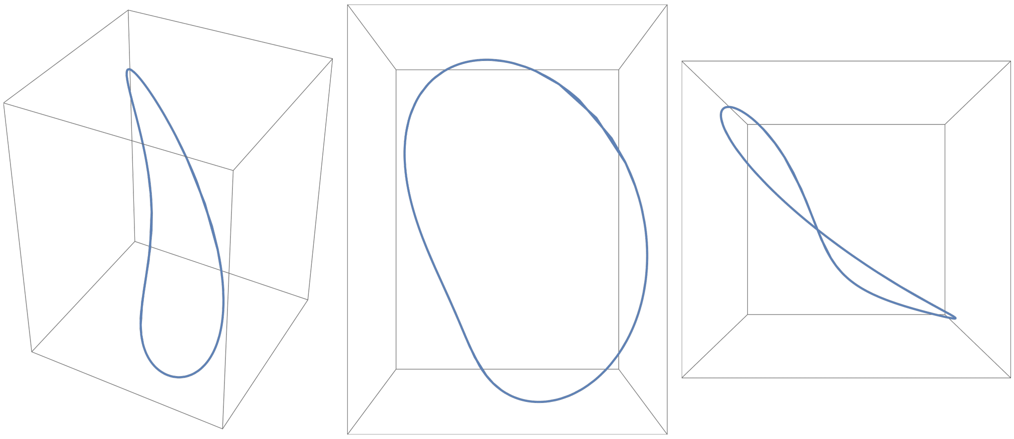

Figure 3.1: The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ given by polynomial $H(t) = t^2 + (-\mathbf{i} - \mathbf{k})t + (1 + \mathbf{i} - \mathbf{k}) + \varepsilon\left((\mathbf{j} + 2\mathbf{k})t + (2 - 2\mathbf{i} + \mathbf{j})\right)$. The view on the left picture is from front and top right, the middle picture is a front view, the right picture is a top view of the trajectory.

This is a famous result by Jüttler [1993]. The proof can be found in the paper, although with different terminology.

Note, that the first part also holds for any dual quaternion polynomial, since the given form guarantees, that the trajectories of all points are rational curves. The other direction is more important. It tells us, that by restricting ourselves to motion polynomials, we do not lose any variability in obtainable motions.

3.2 Factorizations in $\mathbb{DH}[t]$

In this section we begin our journey towards factorization of motion polynomials. Our aim is to use what we learned from the case of quaternions and apply the factorization of the norm polynomial. The main result is the theorem 43 and subsequently corollary 44, which sum up how and when we can obtain the desired right factors.

Example. Let us consider the polynomial

$$
\begin{aligned}
H(t) &= t^2 + (-\mathbf{i} - \mathbf{k})t + (1 + \mathbf{i} - \mathbf{k}) + \varepsilon\left((\mathbf{j} + 2\mathbf{k})t + (2 - 2\mathbf{i} + \mathbf{j})\right) \\
&= (t - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j})) \cdot (t - \mathbf{j} + \varepsilon(-\mathbf{i} - 2\mathbf{k})).
\end{aligned}
$$

Its norm polynomial is $H\overline{H}(t) = t^4 + 4t^2 + 3$, so H is a motion polynomial. The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ can be seen in figure 3.1.

It is hard to say just from the polynomial, what motion does H prescribe. The factorization shows us, that the motion is a composition of two rotations. One is about the axis $(1, 1, 1) - \varepsilon(1, -1, 0)$, the other is about $(0, 1, 0) - \varepsilon(-1, 0, -2)$. We may see the two axis in figure 3.2, in several steps of drawing the trajectory.

Definition 24. *Let $H \in \mathbb{DH}[t]$ be a polynomial. The* factorization of H *is a sequence of irreducible polynomials H_1, H_2, …, H_k such that*

$$H(t) = H_1(t) \cdot H_2(t) \cdot \ldots \cdot H_k(t).$$

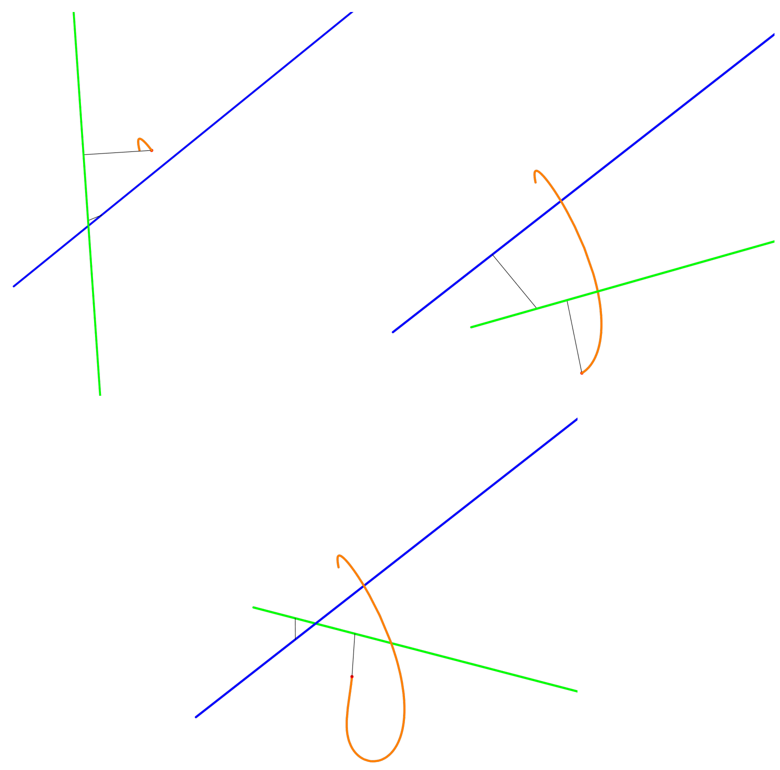

Figure 3.2: The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ being drawn (orange). The axis $(1,1,1) - \varepsilon(1,-1,0)$ is shown in blue, the rotating axis (originally $(0,1,0) - \varepsilon(-1,0,-2)$) is shown in green. The black lines are the links connecting the axes and the moving point. The pictures depict the state at $t = -2$, $t = 0$ and $t = 2$, respectively

We proceed by giving a definition of characteristic polynomials for dual quaternions. The main difference to quaternion case is that these are in general polynomials over dual numbers.

Definition 25. *Let $h \in \mathbb{DH}$. We define the* characteristic polynomial *of h as*

$$P_h(t) = (t - h)(t - \overline{h}) \in \mathbb{D}[t].$$

Since any motion polynomial H has a real norm polynomial, we would like to restrict ourselves to $\mathbb{R}[t]$. Fortunately, such restriction still allows for a full access to pure rotations and translations.

Lemma 41. *Let $h = p + \varepsilon q \in \mathbb{DH}$, $\mathbf{p} = \operatorname{Im} p$ and $\mathbf{q} = \operatorname{Im} q$. Then the following are equivalent:*

(1) $h \in \mathbb{DH}^\times$ and it gives a pure rotation or pure translation.

(2) $P_h(t) \in \mathbb{R}[t]$.

(3) $p \neq 0$, $\operatorname{Re} q = 0$ and $\mathbf{p} \perp \mathbf{q}$.

Proof. (1) $\Leftrightarrow$ (3): The first statement says, that $p \neq 0$, the Study condition holds, and that either $p \in \mathbb{R}$ or $qp^{-1} \perp \mathbf{p}$. Note, that the latter holds even in the case $p \in \mathbb{R}$ as $\mathbf{p} = \mathbf{0}$. Multiplying qp^{-1} by non-zero real number $\|p\|^2$ gives us an equivalent relation $q\overline{p} \perp \mathbf{p}$. We compute

$$\begin{aligned}(q\overline{p}) \cdot \mathbf{p} &= (-\operatorname{Re} q\mathbf{p} + \operatorname{Re} p\mathbf{q} - \mathbf{q} \times \mathbf{p}) \cdot \mathbf{p} \\ &= -\operatorname{Re} q \, \|\mathbf{p}\|^2 + \operatorname{Re} p(\mathbf{q} \cdot \mathbf{p}).\end{aligned}$$

We also rewrite the Study condition as

$$0 = p\overline{q} + q\overline{p} = 2\operatorname{Re}(p\overline{q}) = 2 \cdot (\operatorname{Re} p \cdot \operatorname{Re} q + \mathbf{p} \cdot \mathbf{q}).$$

Now, if (3) holds, then $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p} = 0$, irregardless of whether or not $p \in \mathbb{R}$. Since also $\operatorname{Re} q = 0$, we may see that both conditions of (1) are satisfied.

If (1) holds, then $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p} = -\operatorname{Re} p \cdot \operatorname{Re} q$ from Study condition. This transforms the other condition into

$$0 = -\operatorname{Re} q \, \|p\|^2 - (\operatorname{Re} p)^2 \cdot \operatorname{Re} q = -\operatorname{Re} q \cdot \|p\|^2 .$$

Since $p \neq 0$, then necessarily $\operatorname{Re} q = 0$. From Study condition we get $0 = \mathbf{p} \cdot \mathbf{q}$, which gives $\mathbf{p} \perp \mathbf{q}$.

(2) $\Leftrightarrow$ (3): Let us compute $P_h(t)$:

$$\begin{aligned}P_h(t) &= (t - p - \varepsilon q)(t - \overline{p} - \varepsilon \overline{q}) \\ &= t^2 - (p + \overline{p} + \varepsilon(q + \overline{q}))t + p\overline{p} + \varepsilon(p\overline{q} + q\overline{p}).\end{aligned}$$

Clearly $P_h(t) \in \mathbb{R}[t]$ if and only if $\operatorname{Re} q = (q + \overline{q})/2 = 0$ and

$$\operatorname{Re} p \cdot \operatorname{Re} q + \mathbf{p} \cdot \mathbf{q} = (p\overline{q} + q\overline{p})/2 = 0.$$

This is equivalent to $\operatorname{Re} q = 0$ and $\mathbf{p} \perp \mathbf{q}$, which concludes the proof. □

Corollary 42. *Let $p \in \mathbb{H}$ and set $M(t) = P_p(t) \in \mathbb{R}[t]$. Then for any $q \in \mathbb{H}$, the following are equivalent:*

(1) $M(p + \varepsilon q) = 0$.

(2) $p + \varepsilon q \in \mathbb{DH}^\times$ *and* $\operatorname{Re} q = 0$.

Now we have an entire class of Study quaternions given by a real polynomial M. In the case of quaternions, finding such M among the factors of the norm polynomial would guarantee us a right zero. Adding dual part to a polynomial makes things more complicated.

Example. Let us consider the motion polynomial

$$H(t) = t^2 + 1 + \varepsilon\left((\mathbf{i} + \mathbf{j})t + \mathbf{k}\right).$$

The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ is an ellipse, as we can see in figure 3.3. It can be factorized in infinitely many ways. For example, for any $\mu \in \mathbb{R}$ we have

$$H(t) = \left(t + \mathbf{i} + \varepsilon\frac{\mathbf{i}}{2} + \varepsilon\mu\mathbf{j}\right) \cdot \left(t - \mathbf{i} + \varepsilon\frac{\mathbf{i}}{2} + \varepsilon(1 - \mu)\mathbf{j}\right).$$

However, the norm polynomial of left factor is $t^2 + 1 + \varepsilon$ and the norm polynomial of right factor is $t^2 + 1 - \varepsilon$. By lemma 41 these do not give us rotation nor a translation.

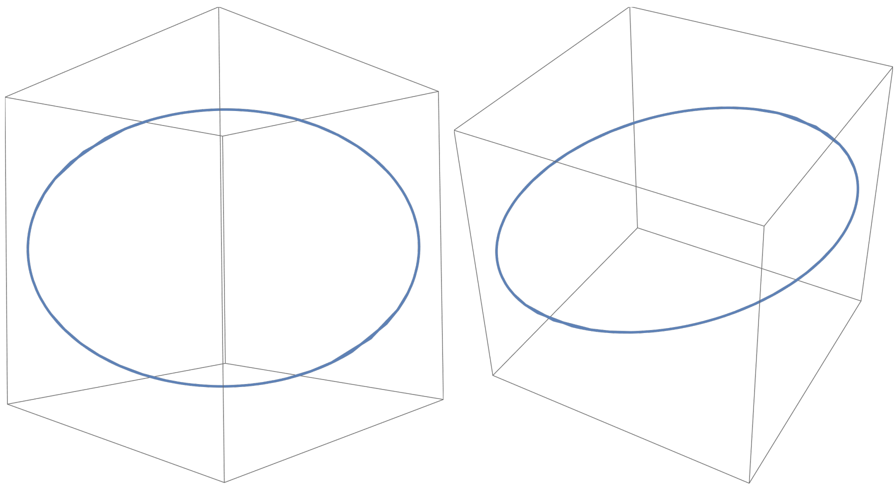

Figure 3.3: The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ given by motion polynomial $H(t) = t^2 + 1 + \varepsilon\left((\mathbf{i}+\mathbf{j})t + \mathbf{k}\right)$. The picture on the left is a front right view, the picture on the right is a view from front top right corner.

The motion described by these polynomials is referred to as *vertical Darboux motion*. More details on this can be found in Siegele et al. [2021]. We would like to restrict ourselves onto rotations and translations, hence we need the characteristic polynomial of our roots to be real.

Such an assumption limits our possibilities. In the following theorem, we explore what zeros can be found in different classes given by real characteristic polynomials.

Theorem 43. *Let $H = P + \varepsilon Q \in \mathbb{DH}[t]$ be a motion polynomial and let $M \in \mathbb{R}[t]$ be a monic degree two factor of $P\overline{P}$ with at most one distinct real zero.*

(1) If M is not a factor of P, then there is a unique $h \in \mathbb{DH}^\times$ such that $M(h) = H(h) = 0$.

(2) If M is a factor of both P and Q, then any $h \in \mathbb{DH}^\times$ satisfying $M(h) = 0$ also satisfies $H(h) = 0$.

(3) If M is a factor of P and $Q\overline{Q}$, but not a factor of Q, then the primal part of $h \in \mathbb{DH}^\times$ satisfying $M(h) = H(h) = 0$ is unique.

(4) If M is a factor of P, but not a factor of $Q\overline{Q}$, then $h \in \mathbb{DH}^\times$ satisfying $M(h) = H(h) = 0$ does not exist.

Proof. (1): Note that $\operatorname{rrem}(P+\varepsilon Q, M) = \operatorname{rrem}(P, M) + \varepsilon \operatorname{rrem}(Q, M)$. If M is not a factor of P, then by corollary after lemma 33 we have that $\deg \operatorname{rrem}(P, M) = 1$. Hence we may write $\operatorname{rrem}(P + \varepsilon Q, M) = at + b$ for some $a, b \in \mathbb{DH}$, where the primal part of a is non-zero, so a is invertible. We claim $h = -a^{-1}b$ is the unique quaternion satisfying $M(h) = H(h) = 0$ and $h \in \mathbb{DH}^\times$.

The uniqueness is clear since h is the only dual quaternion zero of $at + b$. We remain to prove $M(h) = 0$. Let $G \in \mathbb{DH}[t]$ be such, that $H = GM + (at + b)$. Since M is a factor of $P\overline{P} = H\overline{H}$, it is a factor of

$$(GM+(at+b))\overline{(GM+(at+b))} = (G\overline{G}M + G(\overline{a}t+\overline{b}) + (at+b)\overline{G})M + (at+b)(\overline{a}t+\overline{b}).$$

Therefore M is also a factor of

$$\begin{aligned}(at+b)(\overline{a}t+\overline{b}) &= a\overline{a}t^2 + (a\overline{b}+b\overline{a})t + b\overline{b}\\ &= (\overline{a}t+\overline{b})(at+b)\\ &= (t+\overline{b}\overline{a}^{-1})\overline{a}a(t+a^{-1}b)\\ &= a\overline{a}(t+\overline{b}\overline{a}^{-1})(t+a^{-1}b).\end{aligned}$$

Since a is invertible, so is $a\overline{a} \in \mathbb{D}$. Since M is monic, we get

$$M(t) = (t+\overline{b}\overline{a}^{-1})(t+a^{-1}b),$$

so $M(h) = 0$. To see that $h \in \mathbb{DH}^\times$, note that the absolute term of M is real and equal to

$$(a\overline{a})^{-1}(b\overline{b}) = (a^{-1}b)\overline{(a^{-1}b)},$$

which is the norm of h.
(2): This part is clear from the fact, that M is a factor of H.
(3) and (4): Similarly to the first case set $at+b = \operatorname{rrem}(P+\varepsilon Q, M)$. Now, since M is a factor of P, the primal part of both a and b is zero and hence they are not invertible. We may write $at+b = \varepsilon \operatorname{rrem}(Q, M)$. Hence $h \in \mathbb{DH}^\times$ satisfies $H(h) = M(h) = 0$ if and only if $\varepsilon Q(h) = 0$ and $\varepsilon M(h) = 0$. Since $h = p + \varepsilon q$ must be invertible, then $p \neq 0$, so the primal part of $M(h)$ and $Q(h)$ is equal to $M(p)$ and $Q(p)$ respectively. Hence

$$\varepsilon M(h) = \varepsilon M(p) \text{ and } \varepsilon Q(h) = \varepsilon Q(p).$$

By corollary 32, we know that a common zero p of Q and M exists only if M divides $Q\overline{Q}$. Since M is not a divisor of Q, by corollary 34 such p is unique.

We can then pick any q using lemma 42, for example $q = 0$. Then $M(h) = 0$ and $\varepsilon Q(h) = \varepsilon M(h) = 0$, hence also $H(h) = 0$. □

Corollary 44. *Let $H = P+\varepsilon Q \in \mathbb{DH}[t]$ be a motion polynomial and let $M \in \mathbb{R}[t]$ be a factor of $P\overline{P}$. Then there exists $h \in \mathbb{DH}^\times$ satisfying $M(h) = H(h) = 0$ if and only if (at least) one of the following holds:*

- *M is not a factor of P.*
- *M is a factor of $Q\overline{Q}$.*

Example. The polynomial $H(t) = t^2+1+\varepsilon\left((\mathbf{i}+\mathbf{j})t+\mathbf{k}\right)$ has a primal part equal to $P(t) = t^2+1 \in \mathbb{R}[t]$. The norm polynomial is

$$H\overline{H}(t) = t^4 + 2t^2 + 1 = (t^2+1)^2.$$

Since the norm polynomial of $Q(t) = (\mathbf{i}+\mathbf{j})t+\mathbf{k}$ is $2t^2+1$, it si not divisible by t^2+1, so by corollary 44 there is no $h \in \mathbb{DH}^\times$ satisfying $H(h) = h^2+1 = 0$. Hence the polynomial $H(t)$ can not be factorized into linear motion polynomials.

The wisdom of theorem 43 allows us to factorize at least some polynomials. The paper of Hegedüs et al. [2013] uses of the first part of the theorem. We present their algorithm as the simplest example of factorization over dual quaternions.

Definition 26. *Motion polynomial $H = P+\varepsilon Q \in \mathbb{DH}[t]$ is called* generic*, if its primal part P has no real factors.*

The idea is very simple. Since P has no real factors, the theorem 43 guarantees a unique right zero at every step. Moreover the proof also gives us a way of finding the factors, which is analogical to the factorization of quaternion polynomials.

Algorithm 2 Factorization of generic polynomials

Input: generic motion polynomial $H \in \mathbb{DH}[t]$ of degree n

Output: motion polynomials $H_1, H_2, \ldots, H_n \in \mathbb{DH}[t]$ of degree 1, such that $H = H_n \cdot \ldots \cdot H_2 \cdot H_1$

```
N ← H H̄
For i = 1, ..., n Do
    M ← real monic quadratic factor of N
    S ← rrem(H, M)
    a ← linear coefficient of S
    b ← constant coefficient of S
    H_i ← t + a^{-1} b
    N ← N/R
    H ← rquo(H, H_i)
End For
Return H_1, ..., H_n
```

Example. Let us now return to the motion polynomial

$$H(t) = t^2 + (-\mathbf{i} - \mathbf{k})t + (1 + \mathbf{i} - \mathbf{k}) + \varepsilon\left((\mathbf{j} + 2\mathbf{k})t + (2 - 2\mathbf{i} + \mathbf{j})\right)$$

from the beginning of this section. Its norm polynomial is

$$N(t) = t^4 + 4t^2 + 3 = (t^2 + 1)(t^2 + 3).$$

Clearly its primal part has no real zeros, otherwise the norm polynomial would have those zeros as well. Since its primal part is quadratic and non-real, it has no real factors either. So we may use algorithm 2 to factorize it.

First let $M(t) = t^2 + 3$. Then

$$\begin{aligned} S &= \operatorname{rrem}(H, M) \\ &= (-\mathbf{i} - \mathbf{k} + \varepsilon\mathbf{j} + 2\varepsilon\mathbf{k})t + (-2 + \mathbf{i} - \mathbf{k} + 2\varepsilon - 2\varepsilon\mathbf{i} + \varepsilon\mathbf{j}). \end{aligned}$$

Therefore the right factor is $H_1(t) = t - \mathbf{i} + \mathbf{j} - \mathbf{k} + \varepsilon(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. We can compute the other factor just by right division to get

$$H(t) = (t - \mathbf{j} + \varepsilon\mathbf{i})(t - \mathbf{i} + \mathbf{j} - \mathbf{k} + \varepsilon(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})).$$

If we used the other factor $t^2 + 1$ of $N(t)$, we would obtain the factorization from the beginning of this section:

$$H(t) = (t - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j})) \cdot (t - \mathbf{j} + \varepsilon(-\mathbf{i} - 2\mathbf{k})).$$

We may see the new factorization in action in figure 3.4. The axes are visibly different to the original factorization (see figure 3.2), but the trajectory remains the same.

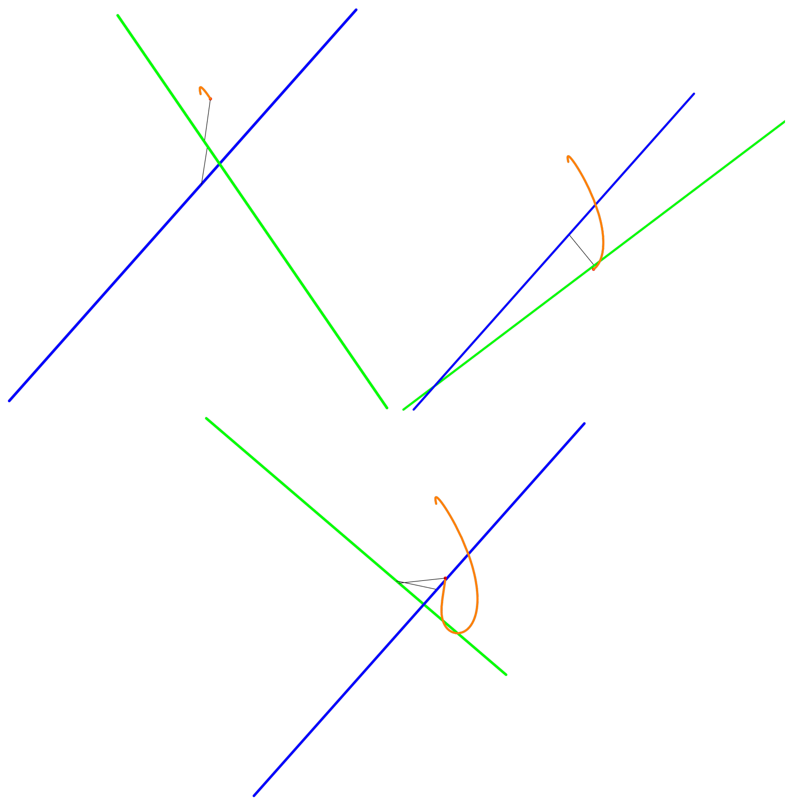

Figure 3.4: The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ drawn by a linkage from the computed factorization. The axis $(0,1,0) - \varepsilon(1,0,0)$ is shown in blue, the moving axis (originally $(1,-1,1) - \varepsilon(-1,1,2)$) is shown in green. The black lines are links connecting the axes and the moving point. The pictures depict the state at $t=-2$, $t=0$ and $t=2$, respectively.

3.3 Factorization into rotation polynomials

The next step is to decompose given motion into a sequence of rotations. The surprising result by Li et al. [2019] states that it can be done for any bounded motion, even if its motion polynomial itself can not be factorized into rotational linear terms.

In this section we look into how this can be done. The main difference to the aforementioned paper will be in the approach. Li et al. [2019] try to algorithmically change the given polynomial into a generic one, which we can factorize already. We deal with the problematic cases of theorem 43 as we get to them.

While the general tricks are the same, our algorithm 3 produces wider range of possible factorizations, while keeping all the properties of the one presented by Li et al. [2019].

Definition 27. *We say a motion polynomial $H = P + \varepsilon Q \in \mathbb{DH}[t]$ is* bounded*, if its primal part P has no real zeros.*

Theorem 45. *Given a bounded motion polynomial $H = P + \varepsilon Q \in \mathbb{DH}[t]$, there always exists $R \in \mathbb{R}[t]$ with no real zeros such that HR can be written as a product of linear rotation polynomials.*

If the polynomial P has a root $t_0 \in \mathbb{R}$, we get $H(t_0) = \varepsilon Q(t_0)$ which no longer lies in $\mathbb{DH}^\times$. Geometrically the closer the parameter t is to t_0, the closer is the trajectory of given point to infinity. Hence the requirement of bounded polynomials is not unreasonable. The multiplication by real polynomial does not change the motion since

$$\frac{(HR)^\star(1+\varepsilon\mathbf{x})\overline{(HR)}}{HR\overline{HR}} = \frac{R^2H^\star(1+\varepsilon\mathbf{x})\overline{H}}{R^2H\overline{H}} = \frac{H^\star(1+\varepsilon\mathbf{x})\overline{H}}{H\overline{H}}.$$

Example. The polynomial $H(t) = t^2 + 1 + \varepsilon\left((\mathbf{i}+\mathbf{j})t + \mathbf{k}\right)$ from previous section can not be factorized into linear motion polynomials. Since $t^2 + 1$ has no real roots, the polynomial is bounded. It holds that

$$H(t)\cdot(t^2+1) = \left(t + \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}\right)\cdot\left(t - \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} + \varepsilon\frac{3}{4}\mathbf{i} + \varepsilon\mathbf{j}\right)\cdot\left(t + \mathbf{j} + \varepsilon\frac{1}{4}\mathbf{i}\right)\cdot(t - \mathbf{j}).$$

One can easily verify, that every term in this product parametrizes a rotation.

Let us head towards the factorization. We would like to modify algorithm 2 so it works in all cases of theorem 43. By corollary 44 there is only one case, when the factor of the form $(t-h)$ for $h \in \mathbb{DH}^\times$ does not exist. We now work to deal with that.

Lemma 46. *Let $H = P+\varepsilon Q \in \mathbb{DH}[t]$ and let $M \in \mathbb{R}[t]$ be a degree two polynomial with no real roots that is a factor of P, but not $Q\overline{Q}$. Then for any $h_r \in \mathbb{H}$ such that $M(h_r) = 0$ there exists a unique $h_l \in \mathbb{H}$ such that the following holds:*

(1) $M(h_l) = 0$.

(2) Quaternion h_l is a left zero of $Q\cdot(t-\overline{h}_r)$.

(3) For $P' = (t-\overline{h}_l)(P/M)(t-\overline{h}_r)$ and $Q' = \operatorname{lquo}(Q\cdot(t-\overline{h}_r), (t-h_l))$ it holds that

$$(P+\varepsilon Q)M = (t-h_l)(P'+\varepsilon Q')(t-h_r).$$

Proof. For simplicity, let us denote $N = Q\cdot(t-\overline{h}_r)$. By lemma 8 finding a left zero of N is equivalent to finding a right zero of $\overline{N}$. Its norm polynomial satisfies

$$N\overline{N} = Q(t-\overline{h}_r)(t-h_r)\overline{Q} = Q\overline{Q}M.$$

Since M does not divide $Q\overline{Q}$, it can not divide $\overline{N}$ either. Then by corollary after lemma 33 there is a unique right quaternion zero h of $\overline{N}$. Setting $h_l = \overline{h}$ gives us the unique left zero of N.

We remain to prove that such h_l satisfies (3). Since h_l is a left factor of N, polynomial Q' is well-defined. Then we may write

$$\begin{aligned}(t-h_l)(P'+\varepsilon Q')(t-h_r) &= (t-h_l)P'(t-h_r) + \varepsilon(t-h_l)Q'(t-h_r)\\ &= M(P/M)M + \varepsilon Q(t-\overline{h}_r)(t-h_r)\\ &= PM + \varepsilon QM.\end{aligned}$$

□

Definition 28. *Let $M \in \mathbb{R}[t]$ be a degree 2 polynomial with no real zeros and let $Q \in \mathbb{H}[t]$ be such, that M is not a factor of $Q\overline{Q}$. Let us denote $O \subseteq \mathbb{H}$ the set of zeros of M. We define the function $f_{M,Q}\colon O \to O$ by assigning each $h \in O$ the unique left zero of both M and $Q\cdot(t-\overline{h})$.*

The idea is to modify the motion polynomial in such a way that does not change its degree nor the motion it defines. As we will see, the changes in part (3) of lemma 46 can be used to get rid of the real factors of P. We now show a way of computing $f_{M,Q}$. This is not included in Li et al. [2019].

Lemma 47. *For any $h \in O$ we have $f_{M,Q}(h) = a\overline{h}a^{-1}$, where $a = Q(h)$.*

Proof. Let us take $b, c \in \mathbb{H}$ such that $bt - c = \operatorname{rrem}(Q, M)$. Since M is not a factor of Q, b and c can not be both zero. Then we may write for some $R \in \mathbb{H}[t]$ that

$$Q(t) = R(t)M(t) + (bt - c) = R(t)(t - \overline{h})(t - h) + (bt - c).$$

Hence $a = Q(h) = 0 + bh - c$. Since M does not divide $Q\overline{Q}$, h is not a right zero of Q and $a \neq 0$.

Now let us compute a right zero of $(t - h)\overline{Q}$. Since M is a real polynomial, we have that $\operatorname{rrem}(\overline{Q}, M) = \overline{b}t - \overline{c}$. Furthermore $M(t) = P_h(t)$, so clearly

$$\begin{aligned}\operatorname{rrem}((t-h)\overline{Q}, M) &= (t-h)(\overline{b}t - \overline{c}) - M\overline{b} \\ &= \overline{b}t^2 - (h\overline{b} + \overline{c})t + h\overline{c} - (\overline{b}t^2 - (h + \overline{h})\overline{b}t + h\overline{h}\,\overline{b}) \\ &= (\overline{h}\,\overline{b} - \overline{c})t - h(\overline{h}\,\overline{b} - \overline{c}).\end{aligned}$$

The right zero is then

$$(\overline{h}\,\overline{b} - \overline{c})^{-1}h(\overline{h}\,\overline{b} - \overline{c}) = (bh - c)h(bh - c)^{-1} = aha^{-1}.$$

Then by lemma 8 the left zero of original is

$$f_{M,Q}(h) = \overline{(aha^{-1})} = a\overline{h}a^{-1}.$$

□

Corollary 48. *Let $P, S \in \mathbb{H}[t]$ and $h \in \mathbb{H}$ be such, that $P(t) = S(t) \cdot (t - h)$. Then $a\overline{h}a^{-1}$ for $a \in \mathbb{H}$ is a left zero of P if one of the following holds:*

(1) P_h is not a factor of $S\overline{S}$ and $a = S(\overline{h})$.

(2) $S(\overline{h}) = 0$ and $a \neq 0$.

Knowing the correspondence between left and right zeros within the same class is useful in analysis of polynomial modifications proposed in lemma 46. Our next aim is the lemma 53. It tells us which choices of h_l and h_r are to be avoided, if we want to get rid of real factors of P. First we will need some technical lemmas.

Lemma 49. *The function $f_{M,Q}$ is bijective.*

This is just the lemma 4 in Li et al. [2019]. The proof can be found there.

Lemma 50. *Let P, $Q \in \mathbb{H}[t]$ and let $R \in \mathbb{R}[t]$ be a degree 2 polynomial. If R is a factor of neither P or Q, but it is a factor of PQ, then there exist $h \in \mathbb{H}$ such that $R(t) = (t - h)(t - \overline{h})$ and $(t - h)$ is a right factor of $P(t)$ and $(t - \overline{h})$ is a left factor of $Q(t)$.*

The lemma is used quite commonly and it can be found with proof, for example in Li et al. [2016] as lemma 1.

Lemma 51. *Let $T \in \mathbb{H}[t]$ be a polynomial with no real factors and let $h \in \mathbb{H}$. Then the polynomial $T \cdot (t - \overline{h})$ has no real factor if and only if h is not a right zero of T.*

Proof. Clearly if $T(h) = 0$, the polynomial $P_h(t) = (t - h)(t - \overline{h})$ is a real factor of $T(t)(t - \overline{h})$.

In the other direction it is clear to see there is no real factor of degree one. So let R be a degree two factor of $T(t) \cdot (t - \overline{h})$. By applying lemma 50 on T and $(t - \overline{h})$ we get that $(t - h)$ must be a right factor of T, hence $T(h) = 0$. □

Corollary 52. *The polynomial $(t - h) \cdot T$ has no real factor if and only if $\overline{h}$ is not a left zero of T.*

Lemma 53. *Let $H = P + \varepsilon Q \in \mathbb{DH}[t]$ be a bounded motion polynomial and let $M \in \mathbb{R}[t]$ be a monic degree two polynomial with no real roots that is a factor of P, but not $Q\overline{Q}$. Furthermore let $h_r \in \mathbb{H}$, $M(h_r) = 0$, set $h_l = f_{M,Q}(h_r)$ and let us write $P = RT$, where $R \in \mathbb{R}[t]$ and $T \in \mathbb{H}[t]$ has no real factors. Then the following statements hold:*

(1) Polynomial $(t - \overline{h}_l) \cdot T \cdot (t - \overline{h}_r)$ has a real factor if and only if h_l is a left zero of $T \cdot (t - \overline{h}_r)$ or $T(h_r) = 0$.

(2) There exists finitely many h_r for which (1) holds.

Part (1) is just an application of lemma 51. The proof of part (2) uses the fact, that T has at most one right zero in the class defined by M. The rest of the proof can be found in Li et al. [2019] as lemma 6. The proof in the paper goes even further and shows, that there are at most two h_r for which $h_l = f_{M,Q}(h_r)$ is a left zero of $T(t) \cdot (t - \overline{h}_r)$.

Such a result allows us to just guess h_r, other than the possible zero of T. The verification of whether h_l is a left zero can be computed by left evaluation. In at most three guesses we will get a suitable pair of h_r, h_l.

Example. Recall the polynomial $H(t) = t^2 + 1 + \varepsilon\,((\mathbf{i} + \mathbf{j})t + \mathbf{k})$ with norm polynomial $(t^2 + 1)^2$. Let $M(t) = t^2 + 1$ and $Q(t) = (\mathbf{i} + \mathbf{j})t + \mathbf{k}$. Then we have

$$f_{M,Q}(h) = ((\mathbf{i} + \mathbf{j})h + \mathbf{k})\overline{h}((\mathbf{i} + \mathbf{j})h + \mathbf{k})^{-1}.$$

Note, that the primal part is $M(t) \in \mathbb{R}[t]$, so the polynomial T from lemma 53 is just constant 1. Hence it has no zeros and the only left zero of $1 \cdot (t - \overline{h}_r)$ is $\overline{h}_r$.

Let us note, that $f_{M,Q}(\mathbf{i}) = -\mathbf{i} = \overline{\mathbf{i}}$. Therefore $h_r = \mathbf{i}$ is not a valid choice for our procedure.

However,

$$f_{M,Q}(\mathbf{j}) = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$$

gives us a valid pair $h_r = \mathbf{j}$ and $h_l = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$. Then we may write

$$P'(t) = (t - \overline{h}_l)(t - \overline{h}_r) = t^2 - (\frac{4}{5}\mathbf{i} - \frac{8}{5}\mathbf{j})t - \frac{3}{5} - \frac{4}{5}\mathbf{k},$$
$$Q'(t) = (\mathbf{i} + \mathbf{j})t - \frac{4}{5} + \frac{3}{5}\mathbf{k}.$$

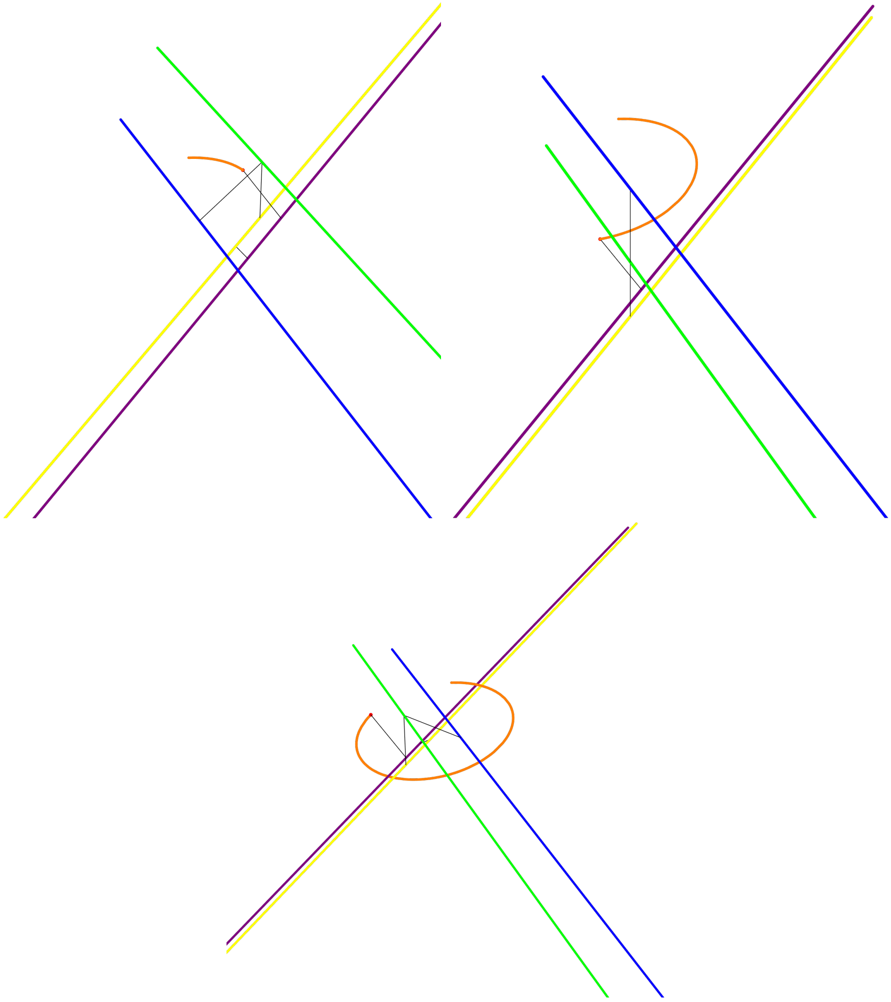

Figure 3.5: The trajectory of point $\mathbf{x} = (-2/3, 1/3, 2/3)$ drawn by a linkage from the computed factorization. The order of the axes is blue (fixed), green, yellow and purple (all moving). The black lines are links connecting the consecutive axes and the moving point. The pictures depict the state at $t = -2$, $t = 0$ and $t = 2$, respectively.

Since $P'(t)$ has no real factors, we can factorize $P'(t) + \varepsilon Q'(t)$ using the algorithm 2. We obtain

$$P'(t) + \varepsilon Q'(t) = \left(t - \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} + \varepsilon\frac{3}{4}\mathbf{i} + \varepsilon\mathbf{j}\right) \cdot \left(t + \mathbf{j} + \varepsilon\frac{1}{4}\mathbf{i}\right).$$

The factorization of the motion given by $H(t)$ is then

$$\begin{aligned} H(t) \cdot (t^2 + 1) &= \left(t + \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}\right) \cdot (P'(t) + \varepsilon Q'(t)) \cdot (t - \mathbf{j}) \\ &= \left(t + \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}\right)\left(t - \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} + \varepsilon\frac{3}{4}\mathbf{i} + \varepsilon\mathbf{j}\right)\left(t + \mathbf{j} + \varepsilon\frac{1}{4}\mathbf{i}\right)(t - \mathbf{j}). \end{aligned}$$

We may see how this factorization draws the trajectory of the point $\mathbf{x} = (-2/3, 1/3, 2/3)$ in figure 3.5.

We now remain to deal with the case when M is a factor of both P and $Q\overline{Q}$. By lemma 33 there is a quaternion p such that $M(p) = Q(p) = 0$. Since M is a factor of P, then also $P(p) = 0$ and hence $H(p) = 0$. Therefore finding the right factor in this case corresponds to finding the right factor of Q over quaternions.

Algorithm 3 Factorization of bounded polynomials

Input: bounded motion polynomial $H = P + \varepsilon Q$ of degree n

Output: real polynomial R of degree $2m$ and rotation polynomials $H_1, H_2, \ldots, H_{n+m}$, $G_1, G_2, \ldots, G_m$ of degree 1, such that $HR = G_1 \cdot G_2 \cdot \ldots \cdot G_m \cdot H_{m+n} \cdot \ldots \cdot H_2 \cdot H_1$

```
N ← H H̄
R ← 1
i ← 1
j ← 1
While deg H > 0 Do
    M ← real monic quadratic factor of N
    If rrem(P, M) ≠ 0 Do
        S ← rrem(H, M)
        a ← linear coefficient of S
        b ← constant coefficient of S
        H_i ← t + a^{-1} b
        N ← N/M
        H ← rquo(H, H_i)
        i ← i + 1
    Else If rrem(Q Q̄, M) = 0 Do
        S ← rrem(Q, M)
        a ← linear coefficient of S
        If a = 0 Do
            b ← linear coefficient of M
            c ← constant coefficient of M
            H_i ← t + (b + sqrt(4c − b^2) i)/2
        Else Do
            b ← constant coefficient of S
            H_i ← t + a^{-1} b
        N ← N/M
        H ← rquo(H, H_i)
        i ← i + 1
    Else Do
        compute h_r, h_l
        P' ← (t − h̄_l)(P/M)(t − h̄_r)
        Q' ← lquo(Q(t − h̄_r), (t − h_l))
        H_i ← (t − h̄_l)
        G_j ← (t − h̄_r)
        R ← R · M
        H ← P' + εQ'
        i ← i + 1
        j ← j + 1
    End If
Return H_1, H_2, ..., H_{n+m}, G_1, G_2, ..., G_m
```

Theorem 54. *The algorithm 3 is correct and always terminates.*

Proof. For correctness we need to verify, that all polynomials $H_1, \ldots, H_{n+m}$, $G_1, \ldots, G_m$ are rotation polynomials. Each of the polynomials is of the form $(t-h)$ for some $h \in \mathbb{DH}$, it is enough to prove h is purely rotational.

From the algorithm we know that for any h there is $M \in \mathbb{R}[t]$ such that M is a factor of N and $M(h) = 0$. Since H was originally a bounded polynomial, its primal part P had no real zeros. Therefore $P\overline{P} = H\overline{H} = N$ has no real zeros and neither does M, so $h \notin \mathbb{R}$. Then $M = P_h$ is its characteristic polynomial. Since $M \in \mathbb{R}[t]$, by lemma 41, h is either pure rotation or pure translation.

For contradiction, let us assume the latter is the case. Then $h = p + \varepsilon q$, where $p \in \mathbb{R}$ and $\operatorname{Re} q = 0$, again by lemma 41. We may write

$$M(t) = (t-h)(t-\overline{h}) = (t-p-\varepsilon q)(t-p+\varepsilon q) = (t-p)^2.$$

This contradicts the fact that M has no real factor. Hence we obtained only rotational polynomials.

For termination, let us write $P = UT$, where $U \in \mathbb{R}[t]$ and $T \in \mathbb{H}[t]$ has no real factors. We will watch the quantity $k = \deg H + \deg U$. Note, that $k = 0$ if and only if $\deg H = \deg U = 0$. Inside the loop, three different cases may occur. In first two, we find a factor by which we divide H, hence lowering $\deg H$ by one. Note, that this action may decrease $\deg U$, but it can not raise it.

In the third case we change $H = P + \varepsilon Q$ into $H' = P' + \varepsilon Q'$. Let $P' = U'T'$, with $U' \in \mathbb{R}[t]$ and $T' \in \mathbb{DH}[t]$ having no real factors. It holds that

$$\begin{aligned} P'(t) &= (t-\overline{h}_l)\frac{P(t)}{M(t)}(t-\overline{h}_r) \\ &= \frac{U(t)}{M(t)}(t-\overline{h}_l)T(t)(t-\overline{h}_r). \end{aligned}$$

By lemma 53 it is always possible to choose h_l and h_r in such a way, that $(t-\overline{h}_l)T(t-\overline{h}_r)$ has no real factors. Hence $U' = U/M$ and we lowered the degree of U by two. The degree of H remained unchanged.

So at every iteration of the loop we lower the value of k by at least one. Therefore the algorithm terminates after at most k iterations. □

Corollary 55. *The cofactor R from algorithm 3 has degree at most $\deg U$, where $U \in \mathbb{R}[t]$, $P = UT$ and $T \in \mathbb{DH}[t]$ has no real factors.*

Proof of theorem 45. We can use the algorithm 3 to compute the desired factorization, hence it exists. □

4. Application to the construction of mechanisms

A *mechanical linkage* is an assembly of rigid bodies called *links*, which are joined together by moveable *joints*. If the links are connected in series, we talk about *open linkage*. We saw examples of those in previous sections, see for example figures 1.2, 3.2 or 3.5. Recall, that we prescribed the motion of each joint and the endpoint of final link drew the trajectory of given point.

A more interesting example is a *closed linkage*, which we obtain by connecting the links in circle, see for example figure 4.3. Such linkage does not allow for an arbitrary movement. The compound motion given by all joints must be an identity in $\mathbb{R}^3$, since the initial link is rigid. For this to hold, the motion of joints must satisfy some constraints, called the *closure equations*. If such a linkage moves, the closure equations must have at least one-parametric set of solutions.

We will show, that one can use the different factorizations of quaternion and motion polynomials to create closed linkages. In section 4.1 the focus will be on spherical linkages. In section 4.2, we will create some over-constrained linkages – linkages, that move, even though it seems, they should not.

4.1 Closed spherical linkages

Spherical linkages are contained on a sphere. All of their joints allow a motion in $SO(3)$, so we can use quaternions to describe them.

Let us now look into what can a closed spherical 4-bar linkage look like. We will have four rigid links connected by four joints. The parametrization of each motion given by our joints is

$$(t_1 - h_1),\ (t_2 - h_2),\ (t_3 - h_3),\ (t_4 - h_4),$$

where t_1, t_2, t_3, t_4 are real parameters and $h_1, h_2, h_3, h_4 \in \mathbb{H}$.

Now, we would like to write the *closure equation* to see, whether such a link can exist and move. Without loss of generality, we may fix the link connecting the first and the last joint. The movement of said link can be described as the composed motion given by

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4).$$

Since the link is fixed in place, this polynomial has to be real. Conversely if this polynomial is real, the link will not move. This gives us the closure equation

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) \in \mathbb{R}[t].$$

We could equivalently write this a set of three real equations – each of the polynomial coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have to be zero. To obtain a motion, we need to get (at least) a one-parametric set of solutions.

Note the motion of the link connecting second and third node. Its movement is described by $(t_1 - h_1)(t_2 - h_2)$. Equivalently, since the fourth jointed is fixed, the same motion can be described by $(t_4 - \overline{h}_4)(t_3 - \overline{h}_3)$ – the inverse motion to $(t_3 - h_3)(t_4 - h_4)$.

We are able to obtain a particular solution to this using the factorization of polynomials. Let us take a quadratic polynomial $P \in \mathbb{H}[t]$ whose coefficients do not commute. By the theorem 20 this polynomial has two distinct factorizations

$$P(t) = (t - p_1)(t - q_1) = (t - p_2)(t - q_2).$$

We also know its norm polynomial is real and

$$\begin{aligned} P\overline{P}(t) &= P(t) \cdot (t - \overline{q}_2)(t - \overline{p}_2) \\ &= (t - p_1)(t - q_1)(t - \overline{q}_2)(t - \overline{p}_2). \end{aligned}$$

Hence for $h_1 = p_1$, $h_2 = q_1$, $h_3 = \overline{q}_2$, $h_4 = \overline{p}_2$ we have the solution $t_1 = t_2 = t_3 = t_4 = t \in \mathbb{R}$.

Example. Let us use the polynomial $P(t) = t^2 - (\mathbf{i} + \mathbf{j} + \mathbf{k})t + \mathbf{j} - \mathbf{k}$ from section 1.2. In section 2.3, we found out it has two factorizations

$$(t - \mathbf{j} - \mathbf{k})(t - \mathbf{i}) = \left(t - \frac{-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3}\right)\left(t - \frac{4\mathbf{i} + \mathbf{j} + \mathbf{k}}{3}\right).$$

Hence the compound motion

$$(t - \mathbf{j} - \mathbf{k})(t - \mathbf{i})\left(t - \frac{\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}}{3}\right)\left(t - \frac{-4\mathbf{i} - \mathbf{j} - \mathbf{k}}{3}\right)$$

describes an identity on the sphere as well as a closed spherical 4-bar linkage.

The resulting linkage can be seen in different positions in figure 4.1. The figure 4.2 visualises various states of the linkage on a single sphere.

4.2 Spatial linkages

In this section, we create a moveable 4-bar and 6-bar linkages. While the spherical 4-bar linkage had closure equations consisting of 4 variables and 3 equations, this time we will have 4 and 6 variables respectively, but only 6 equations. That is caused by the fact, that the vectorial part of a dual quaternion has 6 coefficients.

Having more equations than variables causes a problem, since it does not have a solution in general. Even in the case of 6 variables and 6 equations the solution should theoretically consist of isolated points, which does not give us a motion. This is why such linkages are called *over-constrained*.

However, the fact that a linkage is over-constrained in general does not mean, it can not move under special circumstances. We already saw a special case of a 4-bar linkage – the spherical 4-bar linkage. We will generalize the work from previous section to dual quaternions, to get over-constrained linkages. This approach was first done by Hegedüs et al. [2013].

Bennett's 4-bar linkage. The most general case[1] of 4-bar linkage with rotational joints is called the *Bennett linkage.* It was first described by Bennett [1903].

[1]That is: neither a spherical nor a planar linkage.

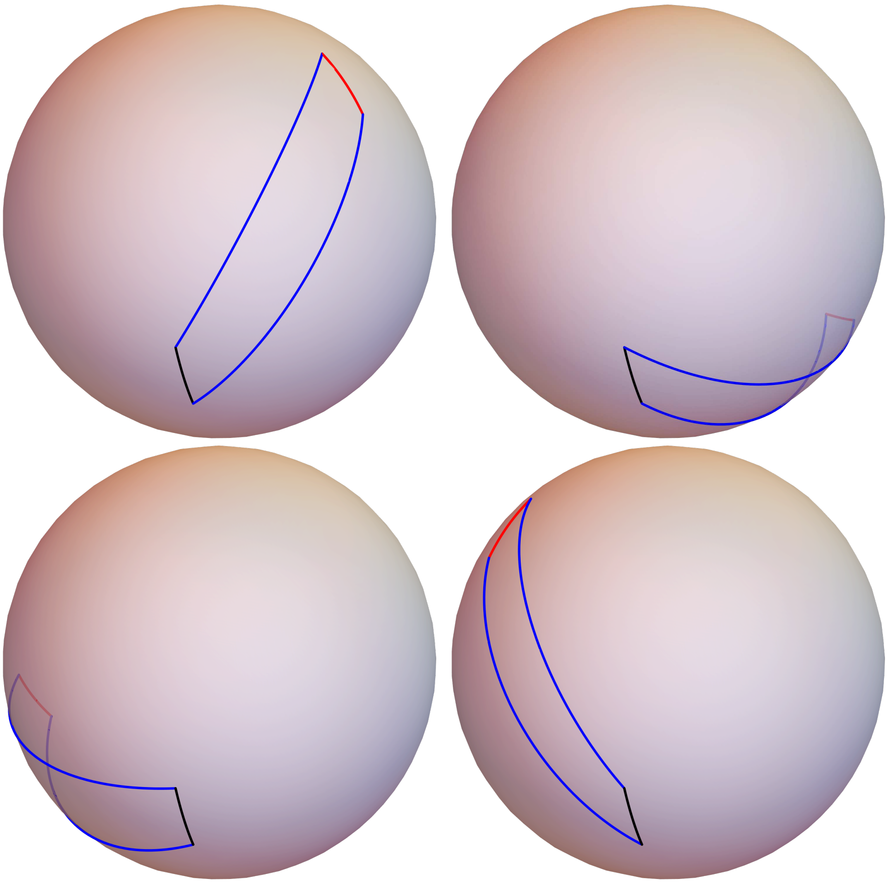

Figure 4.1: Spherical 4-bar linkage obtained by factorizations of $P(t) = t^2 - (\mathbf{i} + \mathbf{j} + \mathbf{k})t + \mathbf{j} - \mathbf{k}$. Captured at $t = -3$, $t = -0.5$, $t = 1$ and $t = 4$ (row-wise, each row left to right). The black link is fixed on the sphere, while the blue links are rotating the red one.

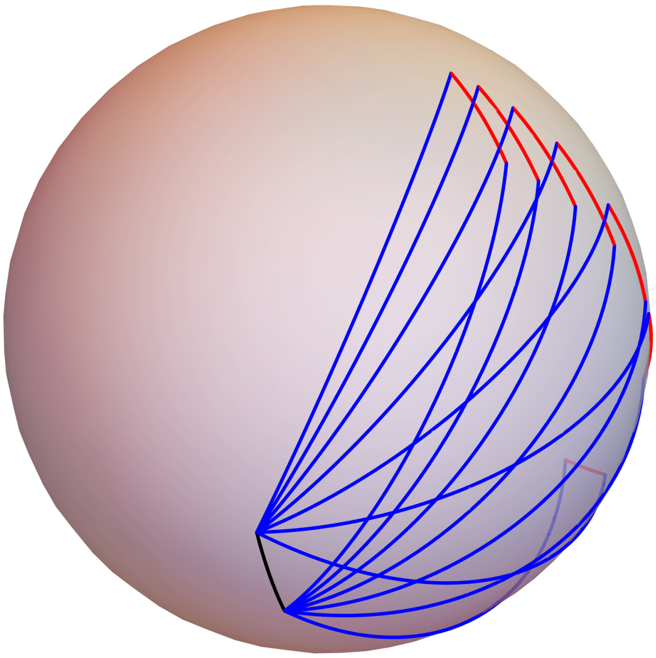

Figure 4.2: Different positions of the same spherical 4-bar linkage.

We are trying to construct a spatial 4-bar linkage. Once again we have four rigid links connected by four rotational joints. Using $h_1, h_2, h_3, h_4 \in \mathbb{DH}^\times$ which are purely rotational, we search for $t_1, t_2, t_3, t_4 \in \mathbb{R}$ so that

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) \in \mathbb{D}.$$

We will use the polynomial factorization to obtain such h_1, h_2, h_3, h_4 for which the linkage exists. First take a quadratic bounded motion polynomial $P \in \mathbb{DH}[t]$ with two different factorizations

$$P(t) = (t - g_1)(t - g_2) = (t - g_3)(t - g_4).$$

Then its norm polynomial is real, so

$$(t - g_1)(t - g_2)(t - \overline{g}_4)(t - \overline{g}_3) \in \mathbb{R}[t] \subseteq \mathbb{D}[t].$$

Hence, for $h_1 = g_1$, $h_2 = g_2$, $h_3 = \overline{g}_4$ and $h_4 = \overline{g}_3$ we get a solution $t_1 = t_2 = t_3 = t_4 = t \in \mathbb{R}$.

Example. Recall the polynomial

$$H(t) = t^2 + (-\mathbf{i} - \mathbf{k})t + (1 + \mathbf{i} - \mathbf{k}) + \varepsilon\left((\mathbf{j} + 2\mathbf{k})t + (2 - 2\mathbf{i} + \mathbf{j})\right)$$

from section 3.2. We have two different factorizations

$$\begin{aligned} H(t) &= (t - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j})) \cdot (t - \mathbf{j} + \varepsilon(-\mathbf{i} - 2\mathbf{k})) \\ &= (t - \mathbf{j} + \varepsilon\mathbf{i}) \cdot (t - \mathbf{i} + \mathbf{j} - \mathbf{k} + \varepsilon(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})). \end{aligned}$$

This allows us to create a closed linkage satisfying

$$(t - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j}))(t - \mathbf{j} + \varepsilon(-\mathbf{i} - 2\mathbf{k}))(t + \mathbf{i} - \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j} - 2\mathbf{k}))(t + \mathbf{j} - \varepsilon\mathbf{i}) \in \mathbb{R}[t].$$

We can see the resulting linkage in two different positions in figure 4.3. The figure 4.4 contains several states in one picture, to better show the movement of said linkage.

6R linkage. We will use the same approach to get a 6-bar linkage with rotational joints. The norm polynomial of a generic degree three motion polynomial H can be decomposed as

$$H\overline{H}(t) = M_1 M_2 M_3.$$

The different factorizations depend on the order in which we use M_1, M_2 and M_3. However not all choices will give us a 6-bar linkage.

Example. Let us take a motion polynomial

$$\begin{aligned} H(t) = {} & t^3 + (-1 + 2\mathbf{k})t^2 + (1 - 2\mathbf{j} - 2\mathbf{k})t + 1 - 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \\ & + \varepsilon\Big((\mathbf{i} - \mathbf{j} - \mathbf{k})t^2 + (2 - 3\mathbf{i} + 4\mathbf{j} + \mathbf{k})t - 4 + 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}\Big). \end{aligned}$$

Its norm polynomial is

$$t^6 - 2t^5 + 7t^4 - 8t^3 + 15t^2 - 6t + 9 = (t^2 - 2t + 3)(t^2 + 3)(t^2 + 1).$$

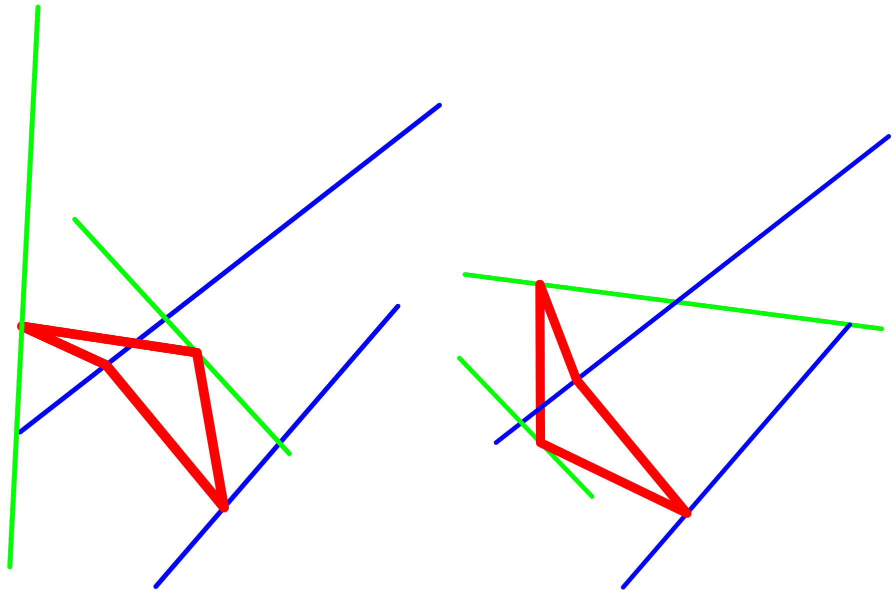

Figure 4.3: The Bennett linkage obtained from the polynomial $H(t) = t^2 + (-\mathbf{i} - \mathbf{k})t + (1 + \mathbf{i} - \mathbf{k}) + \varepsilon\left((\mathbf{j} + 2\mathbf{k})t + (2 - 2\mathbf{i} + \mathbf{j})\right)$. The red lines are links, the green and blue lines are axes of rotational joints.

So let $M_1(t) = t^2 - 2t + 3$, $M_2(t) = t^2 + 3$ and $M_3(t) = t^2 + 1$. Then taking M_1, M_2, M_3 in this order in algorithm 2 gives us

$$H(t) = (t + \mathbf{j} - \varepsilon\mathbf{i})(t + \mathbf{i} - \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j} - 2\mathbf{k}))(t - 1 - \mathbf{i} + \mathbf{k} + \varepsilon(\mathbf{i} + \mathbf{k})).$$

If we take M_1 then M_3 and finally M_2, we get

$$H(t) = (t + \mathbf{i} + \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j}))(t - \mathbf{j} + \varepsilon(-\mathbf{i} - 2\mathbf{k}))(t - 1 - \mathbf{i} + \mathbf{k} + \varepsilon(\mathbf{i} + \mathbf{k})).$$

Let us try to create a linkage from these two factorizations. The closure equation of such linkage is

$$\begin{aligned}&(t_1 + \mathbf{j} - \varepsilon\mathbf{i}) \cdot (t_2 + \mathbf{i} - \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j} - 2\mathbf{k}))\\ &\cdot (t_3 - 1 - \mathbf{i} + \mathbf{k} + \varepsilon(\mathbf{i} + \mathbf{k})) \cdot (t_4 - 1 + \mathbf{i} - \mathbf{k} + \varepsilon(-\mathbf{i} - \mathbf{k}))\\ &\cdot (t_5 + \mathbf{j} + \varepsilon(\mathbf{i} + 2\mathbf{k})) \cdot (t_6 - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(-\mathbf{i} + \mathbf{j})) \in \mathbb{D}[t].\end{aligned}$$

Note, that the third and fourth joints specify rotation about the same axis. The conjugation just changed the orientation of the line. In case of $t_3 = t = t_4$ we get two rotations that cancel out, which means, that the third link (between the joints parametrized by t_2 and t_3) and the fifth link (between t_4 and t_5) are moving in phase. The link between them is just spinning in place with no effect on its surroundings.

Alternatively, we may imagine the third and fifth link as the same rigid body. The fourth link in between them is just an appendage spinning irregardless of the remainder of linkage. Hence the rest of mechanism is just a 4-bar linkage.

Since the mechanism is circular, the situation is equivalent to first and last joints producing the opposite rotation. Therefore we have to make sure, that the first factors and the last factors in our factorizations differ.

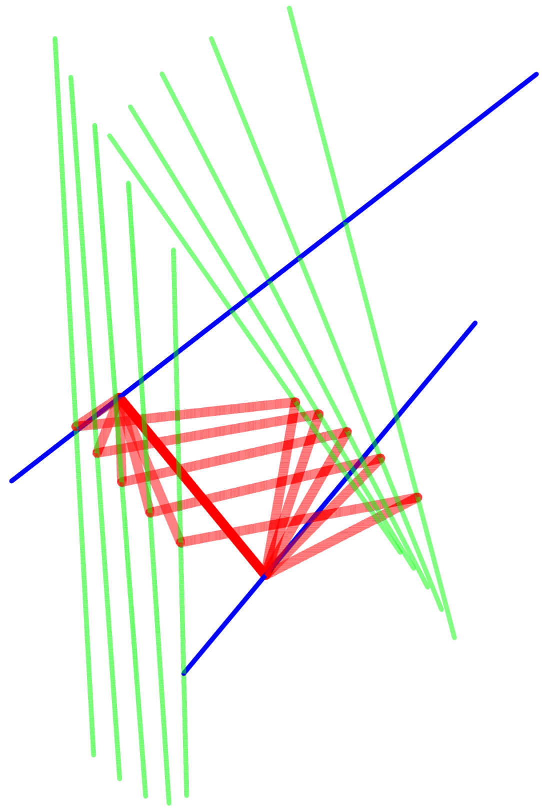

Figure 4.4: Various states of the same Bennett linkage obtained from the polynomial $H(t) = t^2 + (-\mathbf{i} - \mathbf{k})t + (1 + \mathbf{i} - \mathbf{k}) + \varepsilon\left((\mathbf{j} + 2\mathbf{k})t + (2 - 2\mathbf{i} + \mathbf{j})\right)$. The red lines are links, the green and blue lines are axes of rotational joints.

This can be done by picking the first and last polynomials from M_1, M_2, M_3 differently in each of our factorizations. If we have a factorization obtained by taking M_1 then M_2 and finally M_3, it will form a 6-bar linkage with another factorization, only if that other factorization started with something else than M_1 and ended with something else than M_3.

Example. Let us now return to the polynomial $H(t)$ above. We have the factorization

$$H(t) = (t + \mathbf{j} - \varepsilon\mathbf{i})(t + \mathbf{i} - \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j} - 2\mathbf{k}))(t - 1 - \mathbf{i} + \mathbf{k} + \varepsilon(\mathbf{i} + \mathbf{k})),$$

obtained from taking first M_1, then M_2 and then M_3. The second factorization for our linkage can start with M_2 or M_3 and end with M_1 or M_2. We get three possibilities: M_2, M_3, M_1 or M_3, M_1, M_2 or M_3, M_2, M_1.

We will use the second one. The factorization is

$$\begin{aligned} H(t) = {} & (t + \mathbf{i} + \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j})) \cdot \left(t - 1 - \mathbf{i} - \mathbf{j} + \varepsilon\frac{-3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}}{2}\right) \\ & \cdot \left(t + \mathbf{k} + \varepsilon\frac{3\mathbf{i} - 3\mathbf{j}}{2}\right). \end{aligned}$$

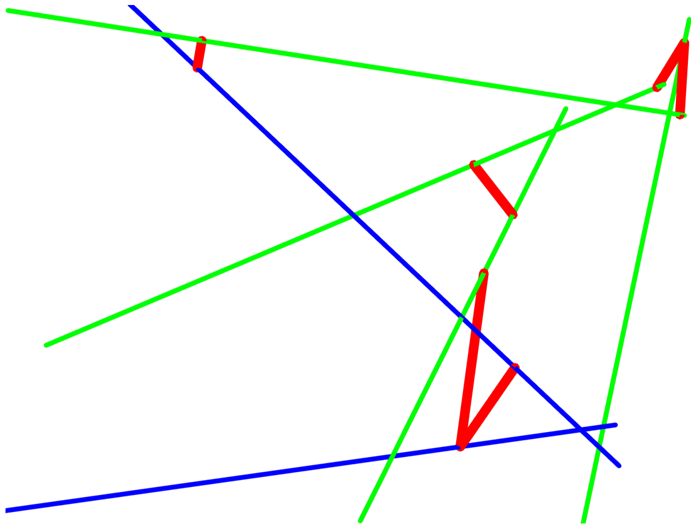

Figure 4.5: A 6-bar linkage with rotational joints. Red lines correspond to links. Blue lines are fixed axes, green axes move in space.

The resulting closure equation is

$$
\begin{aligned}
&(t_1 + \mathbf{j} - \varepsilon\mathbf{i}) \cdot (t_2 + \mathbf{i} - \mathbf{j} + \mathbf{k} + \varepsilon(\mathbf{i} - \mathbf{j} - 2\mathbf{k})) \\
&\cdot (t_3 - 1 - \mathbf{i} + \mathbf{k} + \varepsilon(\mathbf{i} + \mathbf{k})) \cdot \left(t_4 - \mathbf{k} + \varepsilon\frac{-3\mathbf{i} + 3\mathbf{j}}{2}\right) \\
&\cdot \left(t_5 - 1 + \mathbf{i} + \mathbf{j} + \varepsilon\frac{3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{2}\right) \cdot (t - \mathbf{i} - \mathbf{j} - \mathbf{k} + \varepsilon(-\mathbf{i} + \mathbf{j})) \in \mathbb{D}[t].
\end{aligned}
$$

We may see, that the consecutive joints rotate about different axes. The resulting linkage can be seen in figure 4.5.

Conclusion

In the thesis we have shown a complete factorization analysis for quaternion polynomials and given a factorization algorithm for generic polynomials in the case of dual quaternions. We have also characterized the obtainable factors. Finally, we have provided a workaround for decomposing a motion into rotations, even in the case when the given bounded polynomial can not be factorized directly. This allowed us to construct certain mechanical linkages.

The natural continuation would be to expand the set of motion polynomials, which one can factorize. Unbounded motion may allow for factorization into rotations and translations. If we permit other types of linear terms, such as Darboux motion, we no longer need to restrict ourselves to motion polynomials. The ongoing research pushes the bounds of what polynomials we can factorize, but there is still more to uncover.

Another, perhaps more practical direction would be the study of numerical behaviour of factorization algorithms. This would allow for further use of the theory in settings that are limited by finite precision arithmetic. There are already papers on numerical methods for finding quaternion zeros, so it could be only natural to attempt an application to factorizations.

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