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Complexity of classification problems in topology

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I would like to express my gratitude to my supervisor, Benjamin Vejnar, for his continuous support not only while I was working on this thesis, but throughout my entire studies at MFF UK.

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Abstract: This thesis consists of three articles. The first article focuses on compact metrizable spaces homeomorphic to their respective squares, the main result being that there exists a family of size continuum of pairwise non-homeomorphic compact metrizable zero-dimensional spaces homeomorphic to their respective squares. This result answers a question of W. J. Charatonik. In the second article we prove that there exists a Borel measurable mapping assigning to each Peano continuum X a continuous function from $[0, 1]$ onto X. We also show that there exists a Borel measurable mapping assigning to each triple (X, x, y), where X is a Peano continuum and x, y are distinct points in X, an arc in X with endpoints x, y. In the third article we prove that the homeomorphism relation for absolute retracts in $\mathbb{R}^2$ is Borel bireducible with the isomorphism relation for countable graphs. Moreover, we prove that neither the homeomorphism relation for Peano continua in $\mathbb{R}^2$ nor the homeomorphism relation for absolute retracts in $\mathbb{R}^3$ is classifiable by countable structures. We also show that the homeomorphism relation (as well as the ambient homeomorphism relation) for compacta in $[0, 1]^n$ is Borel reducible to the homeomorphism relation for continua in $[0, 1]^{n+1}$.

Keywords: Borel reduction, homeomorphism relation, metrizable compact space, Peano continuum

Contents

Introduction

Classification problems in general are a very important part of virtually any mathematical discipline. For any class of mathematical structures (e.g. graphs, groups, topological spaces, normed vector spaces, etc.), one naturally wants to be able classify members of that class (or of a specific subclass of interest) up to a suitable equivalence relation. For example, a classification up to isomorphism for the class of vector spaces over a given field $\mathbb{F}$ is given by the dimension: vector spaces X and Y over $\mathbb{F}$ are isomorphic if and only if $\dim X = \dim Y$. Another example from linear algebra is that of square matrices of a given order: two matrices of order $n \in \mathbb{N}$ are similar if and only if they have the same Jordan canonical form (up to the order of Jordan blocks). This type of classification is particularly useful since it provides for each equivalence class a specific representative (the Jordan canonical form) which is easy to work with.

In general, whenever we deal with a new classification problem, one of the first questions we should ask is how many equivalence classes there are for the equivalence relation describing the classification problem. The answer to that question may be interesting in itself and it can significantly affect our decision on how to further study the problem. For example, since there are (up to homeomorphism) only two continua contained in $\mathbb{R}$ (the degenerate continuum and the arc), there is not much to study regarding the homeomorphism classification problem for continua in $\mathbb{R}$. On the other hand, it is easy to show that there are (up to homeomorphism) continuum many continua contained in $\mathbb{R}^2$, which indicates that more advanced methods may be needed to study the homeomorphism classification problem for continua in $\mathbb{R}^2$. Let us note that it may be of interest that L. C. Hoehn and L. G. Oversteegen have obtained in [HO16] the following intriguing result: every non-degenerate homogeneous continuum in $\mathbb{R}^2$ is homeomorphic either to the unit circle, the pseudo-arc, or the circle of pseudo-arcs.

Let us turn our attention to the topic of complexity. Intuitively, a very rough measure of complexity of a classification problem is the number of equivalence classes of the associated equivalence relation (as discussed in the previous paragraph). In practice, this viewpoint is often not completely satisfactory. Up to isomorphism, there are continuum many countable graphs. Yet, this fact itself is hardly a reason to consider the classification of real numbers (up to equality) to be as complex as the classification of countable graphs up to isomorphism. A more subtle approach to the problem can be provided by methods of descriptive set theory, namely the notion of a Borel reduction. If E and F are equivalence relations on Polish (or standard Borel) spaces X and Y, respectively, then a Borel measurable mapping $f\colon X \to Y$ is said to be a Borel reduction from E to F if for all $x_1, x_2 \in X$ we have $x_1 E x_2 \iff f(x_1) F f(x_2)$. If there exists a Borel reduction from E to F, we say that E is Borel reducible to F and write $E \leq_B F$. If it is the case that $E \leq_B F$ and $F \leq_B E$, we say that E is Borel bireducible with F. Intuitively, for any two points $x_1, x_2 \in X$, if $f\colon X \to Y$ is a Borel reduction from E to F, then the question whether $x_1 E x_2$ can be reduced (in a "sufficiently definable" manner, as f is Borel measurable) to the question whether $f(x_1) F f(x_2)$. Because of this, if $E \leq_B F$, then E is considered to be at most as complex as F. Clearly, the relation $\leq_B$ is a quasiorder (i.e. it is reflexive and transitive) on the class of all equivalence relations on Polish (or standard Borel) spaces. It is worth noting, however, that not every two members of this class are comparable with respect to $\leq_B$. Also, it is clear that if $E \leq_B F$ and F is Borel (or $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$, etc.), then so is E. This shows that the descriptive quality of E as a subset of $X \times X$ sheds some light on the possible complexity of E in terms of Borel reducibility.

In order to study the complexity of a given classification problem using the tool of Borel reductions, one has to first represent the classification problem in a natural way by an equivalence relation on a Polish or standard Borel space. Such a representation is sometimes referred to as coding. Whenever we have two different codings – say equivalence relations E and F on standard Borel spaces X and Y, respectively – representing the same classification problem, we want these codings to be equivalent in the sense that E is Borel bireducible with F. Ideally, they should be equivalent in an even stronger sense: there should exist a Borel reduction f from E to F such that f is a Borel bijection (isomorphism) between X and Y (in that case, f^{-1} is a Borel reduction from F to E). As an example, let us mention the isometry classification problem for separable complete metric spaces (i.e. the problem of classifying separable complete metric spaces up to isometry). Since the Urysohn universal space $\mathbb{U}$ contains an isometric copy of every separable complete metric space as its closed subset, one coding we can naturally consider for this problem is the space $\mathcal{F}(\mathbb{U})$ of closed subsets of $\mathbb{U}$ (equipped with the Effros Borel structure) together with the equivalence relation F on $\mathcal{F}(\mathbb{U})$ identifying those members of $\mathcal{F}(\mathbb{U})$ that are isometric. Another natural coding is based on the fact that every separable complete metric space can be reconstructed (up to isometry) from any of its countable dense subsets (by taking a completion of the dense set). Let $\mathcal{D}$ be the space of all metrics on $\mathbb{N}$ considered as a subspace of the Polish space $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$. Then $\mathcal{D}$ is G_δ in $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, hence it is Polish. Define an equivalence relation E on $\mathcal{D}$ by $d_1 E d_2 \iff$ the completions of $(\mathbb{N}, d_1)$ and $(\mathbb{N}, d_2)$ are isometric. Then $\mathcal{D}$ together with E naturally constitute an alternative coding (note that for every infinite separable complete metric space X there is $d \in \mathcal{D}$ such that the completion of $(\mathbb{N}, d)$ is isometric to X). Fortunately, by [Gao09, 14.1], the two codings are equivalent (even in the stronger sense). It is generally believed that any two natural codings for a given classification problem are equivalent. This, however, is a philosophical statement rather than a mathematical one as there is no precise definition of what makes a coding natural. Thus, in practice, this belief has to be put under a test every time a new natural coding emerges.

Let us briefly mention a topic which may not seem related to classification problems at first glance. It is the topic of qualitative versions of existence theorems: For some theorems of the form $\forall\, a \in A\ \exists\, b \in B : T(a, b)$ it is possible to prove that there exists a “well-behaved” mapping $f : A \to B$ such that $T(a, f(a))$ holds for every $a \in A$. If A, B are topological spaces, “well-behaved” can mean continuous or Borel measurable. Results of this type are not only interesting in themselves, they are actually very relevant to the problematics of codings. For example, consider the homeomorphism classification problem for metrizable compact spaces. A natural coding for this problem is provided by the hyperspace of the Hilbert cube (with the Vietoris topology), since any compact metrizable space is homeomorphic to a compact subset of the Hilbert cube. On the other hand, since every (nonempty) compact metrizable space is a continuous image of the Cantor space, an alternative coding is provided by the space of all continuous mappings from the Cantor space into the Hilbert cube (with the topology of uniform convergence) together with the equivalence relation identifying those mappings whose images are homeomorphic. If one manages to show that there exists a Borel measurable mapping assigning to each nonempty compact subset X of the Hilbert cube a continuous mapping from the Cantor space onto X, then it becomes easy to prove that the two coding are equivalent.

This thesis consists of three chapters corresponding to the following three articles written by the author in collaboration with Benjamin Vejnar:

1. *Compact spaces homeomorphic to their respective squares*, [DV24]
2. *Borel Measurable Hahn-Mazurkiewicz theorem*, [DV23a]
3. *The complexity of homeomorphism relations on some classes of compacta with bounded topological dimension*, [DV23b].

These articles are presented here in a modified form (with changes aimed at making the text look better in the format of this thesis). The first article revolves around the homeomorphism classification problem for compact metrizable spaces homeomorphic to their respective squares. The main result of the article is a theorem stating that there are continuum many pairwise non-homeomorphic compact metrizable zero-dimenzional spaces homeomorphic to their respective squares. The second article contains qualitative versions of three well-know existence theorems (most importantly, the Hahn-Mazurkiewicz theorem). In the third article, Borel reductions are used to study homeomorphism classification problems for absolute retracts in $\mathbb{R}^2$ and $\mathbb{R}^3$, for 1-dimensional Peano continua in $\mathbb{R}^2$, and for compacta in $[0, 1]^n$.

1. Compact spaces homeomorphic to their respective squares

Abstract: We deal with topological spaces homeomorphic to their respective squares. Primarily, we investigate the existence of large families of such spaces in some subclasses of compact metrizable spaces. As our main result we show that there is a family of size continuum of pairwise non-homeomorphic compact metrizable zero-dimensional spaces homeomorphic to their respective squares. This answers a question of W. J. Charatonik. We also discuss the situation in the classes of continua, Peano continua and absolute retracts.

1.1 Introduction

There are many (separable metrizable) topological spaces which are known to be homeomorphic to their respective squares. Examples include the rationals, the irrationals, the Cantor space, the Erdös complete or rational space, an infinite discrete space, an infinite power of any space, mutual products of these spaces, etc. Some of these spaces have well-known topological characterizations which can be used to prove that the spaces are homeomorphic to their respective squares.

If we restrict ourselves to the case of compact metrizable spaces, the topological dimension becomes a natural obstruction since if $\dim X \geq 1$ for a metrizable compact space X, then $\dim X^n \geq n$ for every $n \in \mathbb{N}$ [Eng95, 1.8.K (b)]. Hence, a compact metrizable space homeomorphic to its square is necessarily either zero-dimensional, or infinite-dimensional.

On May 14 2020, W. J. Charatonik presented during the Wroclaw Set Theory seminar a joint result with S. Sahan about the existence of uncountably many pairwise non-homeomorphic zero-dimensional compact metrizable spaces homeomorphic to their respective squares [CS19]. In his presentation, he asked whether such a collection can be found of cardinality $\mathfrak{c} = |\mathbb{R}|$. Here we answer the question positively, that is, we prove the following theorem.

Theorem 1.1.1. *There exists a family $\mathcal{F}$ of size continuum of pairwise non-homeomorphic compact metrizable zero-dimensional spaces such that $X \times X$ is homeomorphic to X for each $X \in \mathcal{F}$.*

A naive attempt to prove this theorem by searching for spaces of the form $X = Y^{\mathbb{N}}$ is doomed to fail since if $Y^{\mathbb{N}}$ is a compact metrizable zero-dimensional space with more than one point, then it is homeomorphic to the Cantor space. It should also be noted that a countable compact space is homeomorphic to its square if and only if it has at most one point, since the Cantor-Bendixson rank of any infinite countable metrizable compact space X is strictly less than that of $X \times X$.

Obviously, if a topological space X is homeomorphic to $X \times X$, then it is homeomorphic to X^n for each $n \in \mathbb{N}$. On the other hand, if X is homeomorphic to X^n for some $n > 2$, it may not be the case that X is homeomorphic to X^2. A class $\mathcal{C}$ of topological spaces is said to have the *Tarski cube property* if every $X \in \mathcal{C}$ which is homeomorphic to its cube is homeomorphic to its square. Trnková proved in [Trn80a] that the class of countable metrizable spaces has the Tarski cube property while the class of connected metrizable spaces does not. Going further, it was shown in [Trn80b] that every compact zero-dimensional metrizable space X homeomorphic to X^n for some $n \in \mathbb{N} \setminus \{1\}$ is actually homeomorphic to X^2. It is worth noting that the original statement of the latter result was formulated in the language of Boolean algebras, their free products and isomorphisms, and can be translated via the Stone duality to the language of compact zero-dimensional spaces, their products and homeomorphisms.

It was shown by van Douwen in [vD81, Theorem 17.1] that every locally compact space homeomorphic to its square has a compactification homeomorphic to its square. The fundamental idea behind the construction can be generalized into a universal method of obtaining spaces homeomorphic to their respective square as follows. Starting with any topological space X and any continuous mapping $f : X^2 \to X$, consider the inverse sequence

$$X \xleftarrow{\quad f \quad} X^2 \xleftarrow{\quad f\times f \quad} (X^2)^2 \xleftarrow{f\times f\times f\times f} (X^2)^4 \xleftarrow{\qquad\qquad} \cdots .$$

Then the inverse limit is a space homeomorphic to its square. Conversely, every space X homeomorphic to its square can be realized using this method in a trivial way (take any homeomorphism $f\colon X^2 \to X$, then the inverse limit of the above inverse sequence is a space homeomorphic to X). The Pełczyński compactum is a nice example of a space which can be easily obtained with this method in a non-trivial way: If $X = \mathbb{N} \cup \{\infty\}$ is the one-point compactification of $\mathbb{N}$ and $f : X^2 \to X$ maps $\mathbb{N}^2$ onto $\mathbb{N}$ injectively and $X^2 \setminus \mathbb{N}^2$ onto $\{\infty\}$, then the inverse limit of the above inverse sequence is a space homeomorphic to the Pełczyński compactum.

There exists a Peano continuum which is homeomorphic to its square but not to its infinite powers [vM01, p.87-89]. The proof relies, among other things, on the idea behind van Douwen's result.

Let us mention one more result somewhat related to our topic. Recently, Medini and Zdomskyy proved that every filter in $P(\omega)/fin$ (considered as a subspace of $P(\omega) \sim 2^\omega$) is homeomorphic to its square [MZ16].

Let us briefly outline the content of this chapter: In Section 1.2 we define preliminary notions. In Section 1.3 we discuss various ways to produce continuum many pairwise non-homeomorphic infinite dimensional compact metrizable spaces (continua) which are homeomorphic to their respective squares. We also discuss the cases of Peano continua, absolute retracts and some other classes. In Section 1.4 we prove Theorem 1.1.1 as the main result of this chapter. In Section 1.5 we provide a correction of the proof of [CS19, Theorem 3.3], which is a theorem very important for us since its corollary, [CS19, Theorem 3.4], is used in the proof of our Theorem 1.1.1 in Section 1.4.

1.2 Preliminaries

We use the symbol $\mathfrak{c}$ for the cardinality of the continuum, that is, $\mathfrak{c} = |\mathbb{R}|$. We denote by ω the least infinite ordinal and by ω_1 the least uncountable ordinal. Ordinal numbers and ordinal arithmetic operations will play a role in section 1.4.

For any topological space X and any subset S of X, we denote by S' the set of all the points in X which are limit points of S. A subset P of a topological space is said to be *perfect* if it is closed and has no isolated points. Hence, P is perfect if and only if $P' = P$.

Lemma 1.2.1. *Let X be a metrizable space with infinitely many isolated points and let $z \in X \setminus X'$. Then $X \setminus \{z\}$ is homeomorphic to X.*

Proof. If X' is not open in X, fix an infinite set $A \subseteq X \setminus X'$ containing z such that A has exactly one limit point. If X' is open in X, let $A := X \setminus X'$. Either way, fix any bijection $\varphi\colon A \to A \setminus \{z\}$ and define a mapping $h\colon X \to X \setminus \{z\}$ by $h(x) = \varphi(x)$ for $x \in A$ and by $h(x) = x$ for $x \in X \setminus A$. It is easy to see that h is a homeomorphism. □

A topological space is said to be *Polish* if it is separable and completely metrizable. A subspace Y of a Polish space X is Polish if and only if Y is G_δ in X (see, e.g., [Kec95, Theorem 3.11]).

By [Kec95, Theorem 6.2], every nonempty perfect subset of a Polish space contains a homeomorphic copy of the Cantor space. In particular, every nonempty countable Polish space has an isolated point. By the Cantor-Bendixson theorem (see, e.g., [Kec95, Theorem 6.4]), for every Polish space X, there exists a unique perfect set $P \subseteq X$ such that $X \setminus P$ is countable. We call this set the *perfect kernel* of X and we denote it by $\mathrm{PK}(X)$. The complement of $\mathrm{PK}(X)$ can be characterized as the maximal countable open subset of X. In other words, $X \setminus \mathrm{PK}(X)$ is the set of all the points in X which have a countable neighbourhood. Note that $\mathrm{PK}(X)$ is nonempty (and thus of size continuum) if and only if X is uncountable. These facts and observations imply the following proposition.

Proposition 1.2.2. *For every Polish space X, the set $X \setminus \mathrm{PK}(X)$ is contained in the closure of the set $X \setminus X'$. In particular, if X is countable, then $X \setminus X'$ is dense in X.*

For a metrizable compact space X, the *Cantor-Bendixson derivative* of order α of X, denoted by $X^{(\alpha)}$, is defined for every ordinal number α inductively as follows:

(1) $X^{(0)} = X$,
(2) $X^{(\alpha)} = \left(X^{(\beta)}\right)'$ if $\alpha = \beta + 1$,
(3) $X^{(\alpha)} = \bigcap\{X^{(\gamma)}\,;\ \gamma < \alpha\}$ if α is a limit ordinal.

If X is countable, there exists an ordinal $\alpha < \omega_1$ with $X^{(\alpha+1)} = \emptyset$. The least such α is called the *Cantor-Bendixson rank* of X and we denote it by $\mathrm{CB}(X)$. If X is, in addition, nonempty, $\mathrm{CB}(X)$ can be characterized as the unique ordinal α for which $X^{(\alpha)}$ is nonempty and finite.

For every metrizable compact space X and every $x \in X \setminus \mathrm{PK}(X)$, we let

$$\mathrm{CB}(x, X) := \min\left\{\mathrm{CB}(K)\,;\ K \subseteq X \text{ is a countable compact neighborhood of } x\right\}.$$

It is easy to show that $\mathrm{CB}(x, X)$ is the least ordinal α for which $x \notin X^{(\alpha+1)}$.

For any metrizable topological space X, we denote by $\mathcal{K}(X)$ the hyperspace of nonempty compact subsets of X and we endow it with the Vietoris topology. If (X, d) is a metric space, we denote by d_H the Hausdorff metric on $\mathcal{K}(X)$ induced by d. That is, for all $K, L \in \mathcal{K}(X)$,

$$d_H(K, L) = \max\left\{\max_{x\in K} \mathrm{dist}_d(x, L)\,,\ \max_{y\in L} \mathrm{dist}_d(y, K)\right\}.$$

Note that the topology induced by d_H is exactly the Vietoris topology on $\mathcal{K}(X)$. Also, recall that a metrizable topological space X is compact if and only if $\mathcal{K}(X)$ is compact.

For any topological space X and subsets Y and Z of X with $Z \subseteq Y$, a partition $\mathcal{A}$ of Z is said to be Y*-clopen* provided that every set $A \in \mathcal{A}$ is relatively clopen in Y.

A family $\mathcal{A}$ of subsets of a metric space X is said to be a *null family* if the set $\{A \in \mathcal{A}\,;\ \mathrm{diam}(A) \geq \varepsilon\}$ is finite for every $\varepsilon > 0$. Also, for any family $\mathcal{A}$ of subsets of a metric space X, we denote

$$\mathrm{mesh}(\mathcal{A}) := \sup\left\{\mathrm{diam}(A)\,;\ A \in \mathcal{A}\right\}.$$

1.3 Infinite-dimensional case

In this section we discuss several ways to construct a large family of pairwise non-homeomorphic continua homeomorphic to their respective squares.

In 1934 Waraszkiewicz constructed an uncountable family of continua none of which can be continuously mapped onto any other. This was improved e.g. in [MT84, 20.3, 20.4, 20.9] or [KP07, page 2] into a result stating that there is a family $\mathcal{Z}$ of size continuum of pairwise non-homeomorphic continua such that for every continuum X there are at most countably many members of $\mathcal{Z}$ onto which X can be continuously mapped.

Lemma 1.3.1. *Let G be a directed graph with $\mathfrak{c}$-many vertices and with a countable outdegree at every vertex. Then there is a set of vertices T of size $\mathfrak{c}$, such that for no pair of distinct vertices $u, v \in T$ both (u, v) and (v, u) form an edge in the graph.*

Proof. Using transfinite recursion, we construct a transfinite sequence v_α, $\alpha < \mathfrak{c}$, of pairwise distinct vertices such that (v_α, v_β) is not an edge when $\alpha < \beta < \mathfrak{c}$. Given $\alpha < \mathfrak{c}$, assume we have already constructed v_γ for each $\gamma < \alpha$. Let M be the set of vertices v in G for which there is $\gamma < \alpha$ such that (v_γ, v) is an edge. Clearly, $|M| \leq |\alpha| \cdot \omega < \mathfrak{c}$. Consequently, there is a vertex v_α in G which is not in $M \cup \{v_\gamma\,;\ \gamma < \alpha\}$. It follows that (v_γ, v_α) is not an edge in G for any $\gamma < \alpha$. $\square$

Theorem 1.3.2. *There exists a family $\mathcal{C}$ of pairwise non-homeomorphic continua such that $|\mathcal{C}| = \mathfrak{c}$ and $X \times X$ is homeomorphic to X for each $X \in \mathcal{C}$.*

Proof. Consider the family $\mathcal{Z}$ described in the first paragraph of this section. Let G be a directed graph with $\mathcal{Z}$ as the set of vertices such that $(X, Y) \in \mathcal{Z} \times \mathcal{Z}$ forms an edge in G if and only if there is a continuous mapping from $X^{\mathbb{N}}$ onto $Y^{\mathbb{N}}$.

Then G has $\mathfrak{c}$-many vertices and it has a countable outdegree at every vertex. By Lemma 1.3.1, there is a family $\mathcal{Z}_0 \subseteq \mathcal{Z}$ of size $\mathfrak{c}$ such that for any two distinct continua $X, Y \in \mathcal{Z}_0$ either $X^{\mathbb{N}}$ can not be continuously mapped onto $Y^{\mathbb{N}}$ or $Y^{\mathbb{N}}$ can not be continuously mapped onto $X^{\mathbb{N}}$ (either way, $X^{\mathbb{N}}$ is not homeomorphic to $Y^{\mathbb{N}}$). Letting $\mathcal{C} := \{X^{\mathbb{N}};\ X \in \mathcal{Z}_0\}$ completes the proof. □

Remark 1.3.3. An alternative proof of Theorem 1.3.2 can be obtained using Cook continua. Let us recall that Cook constructed a non-degenerate (even hereditarily indecomposable) continuum with no continuous mappings between its subcontinua except for constant and identity mappings [Coo67]. Any such continuum is usually called a Cook continuum. Later, Mackowiak constructed a hereditarily decomposable arc-like Cook continuum C with no cut-points [Mac86, Theorem 6.1]. By the theory of tranches (see e.g. [Kur68, p. 200]), there exists a monotone surjective mapping $f : C \to [0, 1]$ (possessing a decomposition into tranches). Then, letting $C_s := f^{-1}(\{s\})$ for every $s \in (0, 1)$, the family $\{C_s\,;\ s \in (0, 1)\}$ is of cardinality $\mathfrak{c}$ and it consists of pairwise disjoint nondegenerate (since C has no cut-points) Cook continua. Moreover, for all $s, t \in (0, 1)$, any continuous mapping between subcontinua of C_s and C_t is either constant, or it is the identity (and $s = t$ in that case). Finally, given any two distinct numbers $s, t \in (0, 1)$, assume towards contradiction that there is a homeomorphism $h\colon C_s^{\mathbb{N}} \to C_t^{\mathbb{N}}$. Then, letting $\Delta\colon C_s \to C_s^{\mathbb{N}}$ be the diagonal mapping, it follows that $\pi_i \circ h \circ \Delta : C_s \to C_t$ is a constant mapping for every $i \in \mathbb{N}$. Thus, $h \circ \Delta$ is a constant mapping, which is a contradiction.

Remark 1.3.4. Yet another proof of Theorem 1.3.2 can be obtained from the following general result by Orsatti and Rodinò [OR86]. They have shown that for every $r \in \mathbb{N}$ and every infinite cardinal number λ, there is a class $\mathcal{C}$ of size 2^{λ} of pairwise non-homeomorphic compact connected Hausdorff topological Abelian groups of weight λ with the property that for all $m, n \in \mathbb{N}$ and $X \in \mathcal{C}$, X^m is homeomorphic to X^n if and only if $m = n \pmod r$. Hence, as every Hausdorff compact space of weight ω is metrizable, it suffices to take $\lambda = \omega$ and $r = 1$.

A question arises as to whether Theorem 1.3.2 remains true even if we additionally require each of the continua to be Peano. It turns out that such a strengthening of Theorem 1.3.2 can be realized using algebraic topology invariants.

Theorem 1.3.5. *There exists a family $\mathcal{P}$ of pairwise non-homeomorphic Peano continua such that $|\mathcal{P}| = \mathfrak{c}$ and $X \times X$ is homeomorphic to X for each $X \in \mathcal{P}$.*

Proof. Let $\mathbb{P}$ denote the set of primes. For every $p \in \mathbb{P}$ fix a Peano continuum Y_p whose fundamental group is isomorphic to $\mathbb{Z}_p$ (it is a folklore result that every finitely presented group can be realized as the fundamental group of a compact, connected, smooth manifold of dimension 4). For every nonempty set $A \subseteq \mathbb{P}$, let $X_A := \prod_{p\in A} Y_p$. Then the fundamental group of X_A is isomorphic to $G_A := \prod_{p\in A} \mathbb{Z}_p$ since by [Hat02, Proposition 4.2] the fundamental group of a product (even infinite) is isomorphic to the corresponding product of fundamental groups. Consequently, X_A is not homeomorphic to X_B if $A \neq B$ since the groups G_A and G_B are not isomorphic (if $p \in A \setminus B$, then G_A contains a point of order p, whereas G_B does not contain a point of order p). □

In the preceding proof we used the tool of infinite powers to obtain spaces homeomorphic to their respective squares and the algebraic tool of fundamental groups to prove that the spaces are pairwise non-homeomorphic. However, there are natural classes of (Peano) continua where these tools can not be used, e.g. countable-dimensional continua (see [Eng95, 5.1]) or continua with trivial shape (see [DS78]).

Question 1.3.6. Is there a non-degenerate countable-dimensional (Peano) continuum X homeomorphic to its square? If so, how many such continua are there (up to homeomorphism)?

Note that the non-existence of a non-degenerate countable-dimensional continuum homeomorphic to its square would immediately follow if we knew that $\operatorname{trind}(X) < \operatorname{trind}(X^2)$ for any infinite-dimensional countable-dimensional metrizable compact space X (see [Eng95, Corollary 7.1.32]).

Question 1.3.7. How many (Peano) continua with trivial shape homeomorphic to their respective squares are there (up to homeomorphism)?

Going further, an absolute retract homeomorphic to its square is either trivial or it is homeomorphic to the Hilbert cube [vM80]. Note that, as was shown by Borsuk [Bor67, Corollary 11.2], there is a locally contractible and contractible (hence Peano) continuum which is not an absolute retract.

Question 1.3.8. How many contractible (Peano) continua homeomorphic to their respective squares are there (up to homeomorphism)?

1.4 Zero-dimensional case

Countable compact spaces will play an important role in this section. The techniques used to work with such spaces date back to Mazurkiewicz and Sierpiński. In the first volume of Fundamenta Mathematicae [MS20] they showed that every infinite countable compact metrizable space X is homeomorphic to a countable ordinal of the form $\omega^\alpha \cdot k + 1$, where $\alpha > 0$ is equal to $\operatorname{CB}(X)$ and $k \in \omega \setminus \{0\}$ is the cardinality of $X^{(\alpha)}$.

For every $\alpha < \omega_1$, we consider the space $Z(\alpha)$ from [CS19]. That is, $Z(\alpha)$ is uncountable, metrizable, compact, zero-dimensional and it satisfies the following two conditions:

- $\operatorname{CB}(x, Z(\alpha)) < \alpha$ for every $x \in Z(\alpha) \setminus \operatorname{PK}(Z(\alpha))$;
- $\operatorname{PK}(Z(\alpha)) \subseteq \overline{\left\{x \in Z(\alpha) \setminus \operatorname{PK}(Z(\alpha))\,;\ \operatorname{CB}(x, Z(\alpha)) = \beta\right\}}$ for every $\beta < \alpha$.

The following two lemmata follow from Proposition 1.2.2 and [CS19, Theorem 3.4], respectively.

Lemma 1.4.1. *For every $\alpha < \omega_1$ with $\alpha \neq 0$, the set $Z(\alpha) \setminus (Z(\alpha))'$ is dense in $Z(\alpha)$.*

Lemma 1.4.2. *Let $\alpha < \omega_1$. Then every uncountable clopen subset of $Z(\alpha)$ is homeomorphic to $Z(\alpha)$.*

Let $\mathcal{O} := \{1\} \cup \{\omega^n + 1\, ;\ n \in \omega\, ,\ n \neq 0\}$. It is clear that every ordinal in $\mathcal{O}$ is (with the order topology) a countable metrizable compact space with finite Cantor-Bendixson rank.

Now we are ready to describe the basic strategy behind the proof of Theorem 1.1.1. For each infinite subset M of $\omega \setminus \{0\}$, we will consider the family $\mathscr{S}(M)$ of all finite products of members of $\mathcal{O} \cup \{Z(m)\, ;\, m \in M\}$. We will prove that every $S \in \mathscr{S}(M)$ admits a finite S-clopen partition into arbitrarily small sets such that each member of the partition is homeomorphic to a member of $\mathscr{S}(M)$. This will allow us to prove the uniqueness of a specific compactification $X(M)$ of the topological sum of the family $\mathscr{S}(M)$, where each $S \in \mathscr{S}(M)$ is taken infinitely (countably) many times in the sum. The space $X(M)$ will be constructed in such a way that its topological characterization will make it possible to verify (with the help of Lemma 1.4.2) that $X(M) \times X(M)$ is homeomorphic to $X(M)$ and that $X(M)$ is not homeomorphic to $X(L)$ for any other infinite set $L \subseteq \omega \setminus \{0\}$.

Lemma 1.4.3. *Let X be a nonempty countable compact metric space with finite Cantor-Bendixson rank and let $\varepsilon > 0$. Then there is a finite X-clopen partition $\mathcal{F}$ of X with* $\operatorname{mesh}(\mathcal{F}) < \varepsilon$ *such that every $F \in \mathcal{F}$ is homeomorphic to a member of $\mathcal{O}$. If, moreover, X is homeomorphic to a member of $\mathcal{O}$, then at least one element of $\mathcal{F}$ is homeomorphic to X.*

Proof. Since X is compact and zero-dimensional, there is a finite X-clopen partition $\mathcal{A}$ of X with $\operatorname{mesh}(\mathcal{A}) < \varepsilon$. We claim that, for every $A \in \mathcal{A}$, there is a finite X-clopen partition $\mathcal{F}_A$ of A such that every element of $\mathcal{F}_A$ is homeomorphic to a member of $\mathcal{O}$. Let $A \in \mathcal{A}$ be given. If A is finite, we define $\mathcal{F}_A := \{\{x\};\, x \in A\}$. If A is infinite, then, as $\mathrm{CB(A)} \leq \mathrm{CB}(X) < \omega$, there are $k, n \in \omega \setminus \{0\}$ such that A is homeomorphic to $\omega^n \cdot k + 1$. In that case, however, there is an A-clopen (and thus also X-clopen) partition $\mathcal{F}_A$ of A with exactly k elements each of which is homeomorphic to $\omega^n + 1$. Let $\mathcal{F} := \bigcup\{\mathcal{F}_A\, ;\, A \in \mathcal{A}\}$.

Assume X is homeomorphic to $\omega^n + 1$ for some $n \in \omega \setminus \{0\}$. Then $X^{(n)} = \{x\}$ for some $x \in X$. Consequently, for any X-clopen partition $\mathcal{F}$ of X, the member F of $\mathcal{F}$ containing x satisfies $F^{(n)} = \{x\}$ and thus is homeomorphic to $\omega^n + 1$. □

Lemma 1.4.4. *Let $n \in \omega$ and let X be a metric space homeomorphic to $Z(n)$. For every $\varepsilon > 0$, there is a finite X-clopen partition $\mathcal{F}$ of X with* $\operatorname{mesh}(\mathcal{F}) < \varepsilon$ *such that every $F \in \mathcal{F}$ is homeomorphic either to $Z(n)$, or to a member of $\mathcal{O}$. In particular, at least one member of $\mathcal{F}$ is homeomorphic to $Z(n)$.*

Proof. Since X is compact and zero-dimensional, there is a finite X-clopen partition $\mathcal{A}$ of X such that $\operatorname{mesh}(\mathcal{A}) < \varepsilon$. Denote $\mathcal{A}_0 := \{A \in \mathcal{A}\, ;\ A \text{ is countable}\}$ and $\mathcal{A}_1 := \mathcal{A} \setminus \mathcal{A}_0$. By Lemma 1.4.2, every member of $\mathcal{A}_1$ is homeomorphic to $Z(n)$. For every $A \in \mathcal{A}_0$, since $n < \omega$, we have $\mathrm{CB}(A) < \omega$. Hence, by Lemma 1.4.3, there is a finite A-clopen (and thus also X-clopen) partition $\mathcal{F}_A$ of A with $\operatorname{mesh}(\mathcal{F}_A) < \varepsilon$ such that every $F \in \mathcal{F}_A$ is homeomorphic to a member of $\mathcal{O}$. Let $\mathcal{F} := \mathcal{A}_1 \cup \bigcup\{\mathcal{F}_A\, ;\, A \in \mathcal{A}_0\}$. □

For every infinite set $M \subseteq \omega$, let $\mathscr{S}(M)$ be the family of topological spaces defined by $\mathscr{S}(M) :=$

$$\Big\{\alpha_0 \times \cdots \times \alpha_k \times Z(m_1) \times \cdots \times Z(m_n)\, ;\ k, n \in \omega,\ \alpha_0, \ldots, \alpha_k \in \mathcal{O},\ m_1, \ldots, m_n \in M\Big\}.$$

Clearly, $\mathscr{S}(M)$ is countable and every member of $\mathscr{S}(M)$ is a nonempty compact metrizable zero-dimensional space. Note that $\alpha_0 \times \dots \times \alpha_k$ is in $\mathscr{S}(M)$ for any $k \in \omega$ and $\alpha_0, \dots, \alpha_k \in \mathcal{O}$. In particular, $1 \in \mathscr{S}(M)$. Also note that the product of finitely many members of $\mathscr{S}(M)$ is again a space homeomorphic to a member of $\mathscr{S}(M)$.

Using Lemmata 1.4.3 and 1.4.4 we easily deduce the following.

Lemma 1.4.5. *Assume $M \subseteq \omega$ is an infinite set, let Y be a metric space homeomorphic to a member of $\mathscr{S}(M)$ and let $\varepsilon > 0$. Then there is a finite Y-clopen partition $\mathcal{F}$ of Y with* $\operatorname{mesh}(\mathcal{F}) < \varepsilon$ *such that every $F \in \mathcal{F}$ is homeomorphic to a member of $\mathscr{S}(M)$ and at least one element of $\mathcal{F}$ is homeomorphic to Y.*

As a consequence we get the following lemma.

Lemma 1.4.6. *Let $M \subseteq \omega$ be an infinite set, let X be a compact metric space and let $\mathcal{A}$ be a family of pairwise disjoint clopen subsets of X such that every member of $\mathcal{A}$ is homeomorphic to a member of $\mathscr{S}(M)$. Then there is a null family $\mathcal{A}^*$ of pairwise disjoint clopen subsets of X such that:*

(i) $\bigcup \mathcal{A}^* = \bigcup \mathcal{A}$;
(ii) *for every $B \in \mathcal{A}^*$, there is $A \in \mathcal{A}$ with $B \subseteq A$;*
(iii) *for every $A \in \mathcal{A}$, there is $B \in \mathcal{A}^*$ with $B \subseteq A$ such that A is homeomorphic to B;*
(iv) *every member of $\mathcal{A}^*$ is homeomorphic to a member of $\mathscr{S}(M)$.*

Proof. If $\mathcal{A}$ is finite, let $\mathcal{A}^* := \mathcal{A}$. Assume $\mathcal{A}$ is infinite. Since X is separable, $\mathcal{A}$ is countable. Thus, we can write $\mathcal{A} = \{Y_n ;\ n \in \omega\}$, where $Y_k \neq Y_n$ (and hence $Y_k \cap Y_n = \emptyset$) when $k \neq n$. For every $n \in \omega$, there is (by Lemma 1.4.5) a finite Y_n-clopen (and thus X-clopen) partition $\mathcal{F}_n$ of Y_n with $\operatorname{mesh}(\mathcal{F}_n) < 2^{-n}$ such that every $F \in \mathcal{F}_n$ is homeomorphic to a member of $\mathscr{S}(M)$ and at least one element of $\mathcal{F}_n$ is homeomorphic to Y_n. Let $\mathcal{A}^* := \bigcup\{\mathcal{F}_n ;\ n \in \omega\}$. □

Several variants of the next proposition are well known and were reproved many times for different purposes (e.g. [KR53,Peł65,Lor81,Ter97,Zie16], or [IN99, Proposition 8.8]). For the sake of completeness we include our proof.

Proposition 1.4.7. *Let X_1 and X_2 be infinite compact metrizable spaces and M a nonempty countable set. For each $i \in \{1,2\}$, let $\mu_i \colon X_i \setminus X_i' \to M$ be a mapping such that, for every $m \in M$,*

$$X_i' \subseteq \overline{\left\{x \in X_i \setminus X_i' ;\ \mu_i(x) = m\right\}}. \tag{$*$}$$

Let $h\colon X_1' \to X_2'$ be a homeomorphism. There is a homeomorphism $\overline{h}\colon X_1 \to X_2$ extending h such that $\mu_1(x) = \mu_2(\overline{h}(x))$ for every $x \in X_1 \setminus X_1'$.

Proof. Since X_1 and X_2 are infinite and compact, the sets X_1', X_2' are nonempty. Hence, it follows from $(*)$ that $X_1 \setminus X_1'$ and $X_2 \setminus X_2'$ are infinite sets. On the other hand, these two sets are countable as X_1, X_2 are separable. Thus, for each $i \in \{1,2\}$, we can fix a bijection $\varphi_i \colon \omega \to X_i \setminus X_i'$. Let ϱ and σ be compatible metrics on X_1 and X_2, respectively. Using $(*)$, it is easy to construct (inductively) sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ of elements of ω in such a way that the following holds for each $n \in \omega$:

- $\mu_1(\varphi_1(a_n)) = \mu_2(\varphi_2(b_n))$.
- If n is even, then $b_n \notin \{b_i\,;\ i < n\}$, $a_n = \min(\omega \setminus \{a_i\,;\ i < n\})$ and there is $p \in X_1'$ with $\varrho(\varphi_1(a_n), p) = \operatorname{dist}_\varrho(\varphi_1(a_n), X_1')$ and $\sigma(\varphi_2(b_n), h(p)) < 2^{-n}$.
- If n is odd, then $a_n \notin \{a_i\,;\ i < n\}$, $b_n = \min(\omega \setminus \{b_i\,;\ i < n\})$ and there is $q \in X_2'$ with $\sigma(\varphi_2(b_n), q) = \operatorname{dist}_\sigma(\varphi_2(b_n), X_2')$ and $\varrho(\varphi_1(a_n), h^{-1}(q)) < 2^{-n}$.

Clearly, $\{a_n\,;\ n \in \omega\} = \{b_n\,;\ n \in \omega\} = \omega$ and $a_i \neq a_j$, $b_i \neq b_j$ for all $i, j \in \omega$ with $i \neq j$. Therefore, the mapping $g\colon X_1 \setminus X_1' \to X_2 \setminus X_2'$ given by $g(\varphi_1(a_n)) = \varphi_2(b_n)$, $n \in \omega$, is a well-defined bijection. Define a mapping $\overline{h}\colon X_1 \to X_2$ by $\overline{h}(x) = h(x)$ for $x \in X_1'$ and by $\overline{h}(x) = g(x)$ for $x \in X_1 \setminus X_1'$. It is clear that $\mu_1(x) = \mu_2(\overline{h}(x))$ for every $x \in X_1 \setminus X_1'$, it remains to show that $\overline{h}$ is a homeomorphism. Since $\overline{h}$ is a bijection and X_1 is compact, it suffices to prove that $\overline{h}$ is continuous. Trivially, $\overline{h}$ is continuous at every point of the set $X_1 \setminus X_1'$. Given $z \in X_1'$ and $\varepsilon > 0$, let us find $\delta > 0$ such that $\sigma(\overline{h}(z), \overline{h}(x)) < 2\varepsilon$ for every $x \in X_1$ with $\varrho(z, x) < \delta$. By the continuity of h, there is $\delta_1 > 0$ such that $\sigma(h(z), h(x)) < \varepsilon$ for every $x \in X_1'$ with $\varrho(z, x) < 2\delta_1$. Also, there is $\delta_2 > 0$ such that $2^{-n} < \min\{\varepsilon, \delta_1\}$ for every $n \in \omega$ with $\varrho(z, \varphi_1(a_n)) < \delta_2$. By the compactness of X_2, there is $\delta_3 > 0$ such that $\operatorname{dist}_\sigma(\varphi_2(b_n), X_2') < \varepsilon$ for every $n \in \omega$ with $\varrho(z, \varphi_1(a_n)) < \delta_3$. Let $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ and let $x \in X_1$ satisfy $\varrho(z, x) < \delta$. If $x \in X_1'$, then (as $\delta \le \delta_1$) we immediately receive $\sigma(\overline{h}(z), \overline{h}(x)) = \sigma(h(z), h(x)) < \varepsilon$. Assume $x \in X_1 \setminus X_1'$ and let $n \in \omega$ satisfy $\varphi_1(a_n) = x$. Since $\delta \le \min\{\delta_2, \delta_3\}$, we have $2^{-n} < \min\{\varepsilon, \delta_1\}$ and $\operatorname{dist}_\sigma(\varphi_2(b_n), X_2') < \varepsilon$. If n is even, there is $p \in X_1'$ such that $\sigma(\varphi_2(b_n), h(p)) < 2^{-n} < \varepsilon$ and

$$\varrho(x, p) = \varrho(\varphi_1(a_n), p) = \operatorname{dist}_\varrho(\varphi_1(a_n), X_1') \le \varrho(\varphi_1(a_n), z) = \varrho(z, x).$$

Then $\varrho(z, p) \le \varrho(z, x) + \varrho(x, p) \le 2\varrho(z, x) < 2\delta \le 2\delta_1$. Hence, $\sigma(h(z), h(p)) < \varepsilon$, implying that

$$\begin{aligned}\sigma(\overline{h}(z), \overline{h}(x)) &\le \sigma(\overline{h}(z), \overline{h}(p)) + \sigma(\overline{h}(p), \overline{h}(x)) = \sigma(h(z), h(p)) + \sigma(h(p), \varphi_2(b_n)) \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

If n is odd, there is $q \in X_2'$ such that $\varrho(x, h^{-1}(q)) = \varrho(\varphi_1(a_n), h^{-1}(q)) < 2^{-n} < \delta_1$ and

$$\sigma(\overline{h}(x), q) = \sigma(\varphi_2(b_n), q) = \operatorname{dist}_\sigma(\varphi_2(b_n), X_2') < \varepsilon.$$

Then, denoting $p := h^{-1}(q)$, we have $\varrho(z, p) \le \varrho(z, x) + \varrho(x, p) < \delta + \delta_1 \le 2\delta_1$. Therefore, we obtain $\sigma(\overline{h}(z), q) = \sigma(h(z), h(p)) < \varepsilon$. Thus,

$$\sigma(\overline{h}(z), \overline{h}(x)) \le \sigma(\overline{h}(z), q) + \sigma(q, \overline{h}(x)) < 2\varepsilon.$$

□

Proposition 1.4.8. *Let $\mathscr{S}$ be a countable family of nonempty metrizable compact topological spaces and let (X_1, d^1), (X_2, d^2) be compact metric spaces. For each $i \in \{1, 2\}$, let $\mathcal{A}_i$ be a null family of pairwise disjoint clopen subsets of X_i such that:*

(1) *$C_i := X_i \setminus \bigcup \mathcal{A}_i$ is a perfect set;*

(2) *every member of $\mathcal{A}_i$ is homeomorphic to a member of $\mathscr{S}$;*

(3) *for every $S \in \mathscr{S}$ and every open set $V \subseteq X_i$ with $V \cap C_i \neq \emptyset$, there is $A \in \mathcal{A}_i$ such that A is homeomorphic to S and $A \subseteq V$.*

Then every homeomorphism $h\colon C_1 \to C_2$ can be extended to a homeomorphism $\overline{h}\colon X_1 \to X_2$.

Proof. Let $h\colon C_1 \to C_2$ be a homeomorphism. For each $i \in \{1, 2\}$, define

$$\mathcal{D}_i := \mathcal{A}_i \cup \Big\{\{x\};\ x \in C_i\Big\}.$$

It is clear that $\mathcal{D}_i \subseteq \mathcal{K}(X_i)$ for $i \in \{1, 2\}$.

Claim 1.4.8.1. For each $i \in \{1, 2\}$, the set $\mathcal{D}_i$ is closed in $\mathcal{K}(X_i)$.

Proof. Given $i \in \{1, 2\}$ and $K \in \mathcal{K}(X_i) \setminus \mathcal{D}_i$, let us find an open set $\mathcal{U} \subseteq \mathcal{K}(X_i)$ such that $K \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{D}_i = \emptyset$. If $K \cap A \neq \emptyset$ for some $A \in \mathcal{A}_i$, we just let $\mathcal{U} := \{L \in \mathcal{K}(X_i)\,;\ L \cap A \neq \emptyset\} \setminus \{A\}$. Assume $K \subseteq C_i$. As $K \notin \mathcal{D}_i$, the set K is not a singleton. Hence, the number $r := \operatorname{diam}_{d^i}(K)$ is positive. Since $\mathcal{A}_i$ is a null family, the set $\mathcal{F} := \{A \in \mathcal{A}_i\,;\ \operatorname{diam}_{d^i}(A) > r/2\}$ is finite and thus closed in $\mathcal{K}(X_i)$. Let $\mathcal{U} := \{L \in \mathcal{K}(X_i)\,;\ \operatorname{diam}_{d^i}(L) > r/2\} \setminus \mathcal{F}$. Then $\mathcal{U}$ is open in $\mathcal{K}(X_i)$, $K \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{D}_i = \emptyset$. ■

As X_1, X_2 are compact, so are $\mathcal{K}(X_1)$, $\mathcal{K}(X_2)$. Hence, $\mathcal{D}_1$ and $\mathcal{D}_2$ are compact by Claim 1.4.8.1. For each $i \in \{1, 2\}$, since every member of $\mathcal{A}_i$ is an isolated point of $\mathcal{D}_i$ and since $\{\{x\}; x \in C_i\}$ is homeomorphic to the perfect set C_i, we have $\mathcal{D}_i' = \{\{x\}; x \in C_i\}$. Let $\mathscr{S}_0$ be a subfamily of $\mathscr{S}$ such that for every $S \in \mathscr{S}$ there is exactly one $T \in \mathscr{S}_0$ homeomorphic to S. Then we have by (2) that, for each $i \in \{1, 2\}$, there is a unique mapping $\mu_i\colon \mathcal{A}_i \to \mathscr{S}_0$ such that $\mu_i(A)$ is homeomorphic to A for every $A \in \mathcal{A}_i$. For each $i \in \{1, 2\}$ and $S \in \mathscr{S}_0$, it follows from (3) that

$$\mathcal{D}_i' \subseteq \overline{\Big\{A \in \mathcal{A}_i\,;\ A \text{ is homeomorphic to } S\Big\}} = \overline{\Big\{A \in \mathcal{D}_i \setminus \mathcal{D}_i'\,;\ \mu_i(A) = S\Big\}}$$

in $\mathcal{K}(X_i)$. Define a mapping $\varphi\colon \mathcal{D}_1' \to \mathcal{D}_2'$ by $\varphi(\{x\}) = \{h(x)\}$, $x \in C_1$. Since h is a homeomorphism, so is φ. By Proposition 1.4.7, there is a homeomorphism $\overline{\varphi}\colon \mathcal{D}_1 \to \mathcal{D}_2$ which extends φ and satisfies $\mu_1(A) = \mu_2(\overline{\varphi}(A))$ for every $A \in \mathcal{A}_1$. Fix a homeomorphism $h_A\colon A \to \overline{\varphi}(A)$ for every $A \in \mathcal{A}_1$, and define a mapping $\overline{h}\colon X_1 \to X_2$ by $h(x) = x$ for $x \in C_1$ and by $h(x) = h_A(x)$ for $x \in A \in \mathcal{A}_1$. Then $\overline{h}$ is a well-defined bijection and, since $\mathcal{A}_1$ is an X_1-clopen partition of $X_1 \setminus C_1$, it is continuous at every point of $X_1 \setminus C_1$. Given $x \in C_1$ and $\varepsilon > 0$, let us find $\delta > 0$ such that $d^2(\overline{h}(x), \overline{h}(y)) < \varepsilon$ for every $y \in X_1$ with $d^1(x, y) < \delta$. By the continuity of $\overline{\varphi}$, there is $\delta_1 > 0$ such that $d^2_H(\overline{\varphi}(D), \overline{\varphi}(\{x\})) < \varepsilon$ for every $D \in \mathcal{D}_1$ with $d^1_H(D, \{x\}) < 2\delta_1$. As $\mathcal{A}_1$ is a null family, so is $\mathcal{D}_1$. Thus, as $\mathcal{D}_1$ consists of closed sets and $\{x\}$ is the only member of $\mathcal{D}_1$ containing x, there is $\delta_2 > 0$ such that $\operatorname{diam}_{d^1}(D) < \delta_1$ for every $D \in \mathcal{D}_1$ with $\operatorname{dist}_{d^1}(x, D) < \delta_2$. Let $\delta := \min\{\delta_1, \delta_2\}$ and let $y \in X_1$ be any point satisfying $d^1(x, y) < \delta$. There is $D \in \mathcal{D}_1$ with $y \in D$. Clearly, $\operatorname{dist}_{d^1}(x, D) \leq d^1(x, y) < \delta \leq \delta_2$, hence $\operatorname{diam}_{d^1}(D) < \delta_1$. Consequently,

$$d^1_H(D, \{x\}) = \max\{d^1(x, z)\,;\ z \in D\} \leq d^1(x, y) + \operatorname{diam}_{d^1}(D) < \delta + \delta_1 \leq 2\delta_1,$$

which gives us $d_H^2(\overline{\varphi}(D), \overline{\varphi}(\{x\})) < \varepsilon$. However, by the definition of $\overline{h}$, we have $\overline{h}(y) \in \overline{\varphi}(D)$ and $\{\overline{h}(x)\} = \overline{\varphi}(\{x\})$, hence

$$d^2(\overline{h}(x), \overline{h}(y)) \leq d_H^2\Big(\{\overline{h}(x)\}, \overline{\varphi}(D)\Big) = d_H^2\Big(\overline{\varphi}(D), \overline{\varphi}(\{x\})\Big) < \varepsilon.$$

Having shown that $\overline{h}$ is continuous, it follows from the compactness of X_1 that $\overline{h}$ is a homeomorphism. □

Proposition 1.4.9. *Let $M \subseteq \omega$ be an infinite set. Then there exists a compact metrizable space X with the property that there is a family $\mathcal{A}$ of pairwise disjoint clopen subsets of X such that:*

(1) *$C := X \setminus \bigcup \mathcal{A}$ is homeomorphic to the Cantor space;*

(2) *every member of $\mathcal{A}$ is homeomorphic to a member of $\mathscr{S}(M)$;*

(3) *for every $S \in \mathscr{S}(M)$ and every open set $V \subseteq X$ with $V \cap C \neq \emptyset$, there is $A \in \mathcal{A}$ such that A is homeomorphic to S and $A \subseteq V$.*

Moreover, the space X is unique up to homeomorphism.

Proof. Take a sequence $(S_n)_{n\in\omega}$ of members of $\mathscr{S}(M)$ such that $\{n \in \omega\,;\, S_n = S\}$ is an infinite set for every $S \in \mathscr{S}(M)$. Fix a set $C_0 \subseteq \mathbb{R}$ homeomorphic to the Cantor space and let $\{q_k\,;\, k \in \omega\}$ be a countable dense subset of C_0. Let $(t_n)_{n\in\omega}$ be a strictly decreasing sequence of positive real numbers converging to 0. For every $n \in \omega$, since S_n is separable, metrizable and zero-dimensional, it can be embedded in $\mathbb{R}$. Thus, for each $n \in \omega$, there is a set $A_n \subseteq (t_{n+1}, t_n)$ homeomorphic to S_n. Define

$$\mathcal{A} := \Big\{\{q_k\} \times A_n\,;\; k, n \in \omega\,,\; k \leq n\Big\}$$

and $X := (C_0 \times \{0\}) \cup \bigcup \mathcal{A}$. It is easy to see that X is a compact subset of $\mathbb{R}^2$ and that $\mathcal{A}$ is a disjoint family consisting of relatively clopen subsets of X. Obviously, assertion (2) is satisfied and the set $C := X \setminus \bigcup \mathcal{A}$ is equal to $C_0 \times \{0\}$ and thus it is homeomorphic to the Cantor space. Now, given $S \in \mathscr{S}(M)$ and an open set $V \subseteq \mathbb{R}^2$ with $V \cap C \neq \emptyset$, let us show that there are $k, n \in \omega$ with $k \leq n$ such that $\{q_k\} \times A_n$ is a subset of V homeomorphic to S. Since $V \cap C \neq \emptyset$, there is $k \in \omega$ with $(q_k, 0) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $\{q_k\} \times (0, \varepsilon) \subseteq V$. Fix $m \in \omega$ with $t_m \leq \varepsilon$. Since $\{n \in \omega\,;\, S_n = S\}$ is infinite, there is $n \in \omega$ such that $n \geq \max\{k, m\}$ and $S_n = S$. Then $\{q_k\} \times A_n$ is homeomorphic to S and, as $A_n \subseteq (t_{n+1}, t_n) \subseteq (0, t_m) \subseteq (0, \varepsilon)$, it is a subset of V.

The uniqueness of X easily follows from Lemma 1.4.6 and Proposition 1.4.8. □

For every infinite set $M \subseteq \omega$, denote by $X(M)$ the corresponding space whose existence and uniqueness was proven in Proposition 1.4.9. Note that $X(M)$ is zero-dimensional since it is the union of countably many closed zero-dimensional subspaces (see [Eng95, Theorem 1.3.1]).

Proposition 1.4.10. *Let $M \subseteq \omega \setminus \{0\}$ be an infinite set. Then $X(M) \times X(M)$ is homeomorphic to $X(M)$.*

Proof. Let us write X instead of $X(M)$ throughout this proof. Fixing a family $\mathcal{A}$ witnessing the defining property of the space X (see the statement of Proposition 1.4.9), denote by C the set $X \setminus \bigcup \mathcal{A}$. Since $\mathcal{A}$ is infinite and countable, we can write $\mathcal{A} = \{A_i\,;\, i \in \omega\}$, where $A_i \neq A_j$ (and therefore $A_i \cap A_j = \emptyset$) for $i \neq j$. Let $\Gamma := \{i \in \omega\,;\, A_i \text{ is not a singleton}\}$ and note that A_i is infinite for every $i \in \Gamma$. As $0 \notin M$, it follows from Lemma 1.4.1 and Proposition 1.2.2 that $A_i \setminus A_i'$ is dense in A_i for every $i \in \omega$. In particular, $A_i \setminus A_i'$ is infinite for every $i \in \Gamma$. Hence, for each $i \in \Gamma$, there is an injective sequence $(a_{i,k})_{k\in\omega}$ with $A_i \setminus A_i' = \{a_{i,k}\,;\, k \in \omega\}$. For all $i \in \Gamma$ and $n \in \omega$, let

$$\mathcal{D}_i(n) := \Big\{\{a_{i,k}\}\,;\, 0 \le k \le n\Big\} \cup \Big\{A_i \setminus \{a_{i,k}\,;\, 0 \le k \le n\}\Big\}.$$

In addition to that, let $\mathcal{D}_i(n) := \{A_i\}$ for all $i \in \omega \setminus \Gamma$ and $n \in \omega$. Then, for all $i, n \in \omega$, the family $\mathcal{D}_i(n)$ is a finite X-clopen partition of A_i and every member of $\mathcal{D}_i(n)$ is either a singleton or (by Lemma 1.2.1) it is homeomorphic to A_i. In particular, every member of $\mathcal{D}_i(n)$ is homeomorphic to a member of $\mathscr{S}(M)$. Let $C^* := (C \times X) \cup (X \times C)$ and

$$\mathcal{A}^* := \bigcup_{i,j\in\omega} \Big\{E \times F\,;\; E \in \mathcal{D}_i(i+j),\, F \in \mathcal{D}_j(i+j)\Big\}.$$

It easily follows from the classical topological characterization of the Cantor space due to Brouwer that C^* is homeomorphic to the Cantor space. Moreover, we have

$$\bigcup \mathcal{A}^* = \bigcup_{i,j\in\omega} (A_i \times A_j) = \Big(\bigcup \mathcal{A}\Big) \times \Big(\bigcup \mathcal{A}\Big) = (X \setminus C) \times (X \setminus C) = (X \times X) \setminus C^*$$

and it is clear that $\mathcal{A}^*$ is a family consisting of pairwise disjoint clopen subsets of $X \times X$. Since it is obvious that the product of finitely many members of $\mathscr{S}(M)$ is homeomorphic to a member $\mathscr{S}(M)$, every member of $\mathcal{A}^*$ is homeomorphic to a member $\mathscr{S}(M)$. Given any $S \in \mathscr{S}(M)$ and any open set $W \subseteq X \times X$ satisfying $W \cap C^* \neq \emptyset$, let us show that there is $K \in \mathcal{A}^*$ with $K \subseteq W$ such that K is homeomorphic to S. Let $U, V \subseteq X$ be open sets satisfying $U \times V \subseteq W$ and $(U \times V) \cap C^* \neq \emptyset$. Then $U \times V$ intersects at least one of the sets $C \times (X \setminus C)$, $(X \setminus C) \times C$ and $C \times C$. We will assume the second possibility is true (the first two possibilities are symmetric and the third one is easier to deal with), that is, $U \setminus C \neq \emptyset$ and $V \cap C \neq \emptyset$. Then there is $i \in \omega$ such that $U \cap A_i \neq \emptyset$. Let us assume that $i \in \Gamma$ (the situation is easier if $i \notin \Gamma$). Then, since $\{a_{i,k}\,;\, k \in \omega\}$ is dense in A_i, there is $k \in \omega$ with $a_{i,k} \in U$. Let

$$V_0 := V \setminus \bigcup_{n=0}^{k} A_n.$$

Then V_0 is an open subset of X intersecting C. Hence, as $\mathcal{A}$ witnesses the defining property of the space X, there is $j \in \omega$ such that A_j is homeomorphic to S and $A_j \subseteq V_0$. Obviously, $j > k$. Thus, $E := \{a_{i,k}\}$ is in $\mathcal{D}_i(i+j)$. If $j \in \Gamma$, let

$$F := A_j \setminus \{a_{j,l}\,;\, 0 \le l \le i+j\},$$

otherwise let $F := A_j$. Finally, let $K := E \times F$. Then $F \in \mathcal{D}_j(i+j)$, $K \in \mathcal{A}^*$ and $K \subseteq U \times A_j \subseteq U \times V_0 \subseteq W$. Moreover, since it is clear that K is homeomorphic to F, we conclude that K is homeomorphic to A_j, which is homeomorphic to S.

By the uniqueness part of Proposition 1.4.9, $X \times X$ is homeomorphic to X. $\square$

Lemma 1.4.11. *Assume $M \subseteq \omega \setminus \{0\}$ is an infinite set and let $S \in \mathscr{S}(M)$. For every uncountable open set $G \subseteq S$, there is a clopen set $H \subseteq S$ contained in G such that H is homeomorphic to $Z(m)$ for some $m \in M$.*

Proof. By the definition of $\mathscr{S}(M)$, we have $S = \alpha_0 \times \cdots \times \alpha_k \times Z(m_1) \times \cdots \times Z(m_n)$ for some $k, n \in \omega$, $\alpha_0, \dots, \alpha_k \in \mathcal{O}$ and $m_1, \dots, m_n \in M$. Since S is second-countable and G is uncountable, there is a point in G which does not have any countable neighbourhood. Thus, by the definition of the product topology and by the zero-dimensionality of the spaces $\alpha_0, \dots, \alpha_k, Z(m_1), \dots, Z(M_n)$, there are clopen sets $U_i \subseteq \alpha_i$, $0 \leq i \leq k$, and clopen sets $V_j \subseteq Z(m_j)$, $1 \leq j \leq n$, such that $W := U_0 \times \cdots \times U_k \times V_1 \times \cdots \times V_n$ is uncountable and $W \subseteq G$. Hence, as $\alpha_0 \times \cdots \times \alpha_k$ is countable, it follows that $n > 0$ and that there is $l \in \{1, \dots, n\}$ such that V_l uncountable. For every $i \in \{0, \dots, k\}$, fix an isolated point $x_i \in U_i$. Since $0 \notin M$, it follows from Lemma 1.4.1 that there is an isolated point y_j in V_j for every $j \in \{1, \dots, n\}$. Let $A_l := V_l$ and $A_j := \{y_j\}$ for $j \in \{1, \dots, n\} \setminus \{l\}$. Then $H := \{x_0\} \times \cdots \times \{x_k\} \times A_1 \times \cdots \times A_n$ is a clopen subset of S, it satisfies $H \subseteq W \subseteq G$ and it is homeomorphic to V_l. However, V_l is homeomorphic to $Z(m_l)$ by Lemma 1.4.2. □

Proposition 1.4.12. *Let $M \subseteq \omega \setminus \{0\}$ be an infinite set and let $k \in \omega$. Then $Z(k)$ is homeomorphic to an open subset of $X(M)$ if and only if $k \in M$.*

Proof. Again, we will write X in place of $X(M)$. Fix a family $\mathcal{A}$ witnessing the defining property of the space X and denote $C := X \setminus \bigcup \mathcal{A}$. Obviously, if $k \in M$, then $S := \{0\} \times Z(k)$ is in $\mathscr{S}(M)$. Hence, fixing a set $A \in \mathcal{A}$ homeomorphic to S, we have found a clopen subset of X homeomorphic to $Z(k)$.

Conversely, assume that V is an open subset of X homeomorphic to $Z(k)$. If $V \cap C \neq \emptyset$, we just fix arbitrary $m \in M$ and find a set $A \in \mathcal{A}$ with $A \subseteq V$ such that A is homeomorphic to $\{0\} \times Z(m)$, and thus to $Z(m)$. In that case, since A is uncountable and clopen, it follows from Lemma 1.4.2 that A is homeomorphic to $Z(k)$, hence $k = m$. Now assume $V \cap C = \emptyset$. Since V is uncountable and $\mathcal{A}$ is countable, there is $A \in \mathcal{A}$ such that $G := V \cap A$ is uncountable. Then, since A is homeomorphic to a member of $\mathscr{S}(M)$, we conclude by Lemma 1.4.11 that there is a clopen set H contained in G such that H is homeomorphic to $Z(m)$ for some $m \in M$. Then, however, H is an uncountable clopen subset of V, and thus it is homeomorphic to $Z(k)$ by Lemma 1.4.2. Thus, $k = m$. □

Proof of Theorem 1.1.1. Let

$$\mathcal{F} := \Big\{X(M)\,;\ M \text{ is an infinite subset of } \omega \setminus \{0\}\Big\}.$$

Then $|\mathcal{F}| = \mathfrak{c}$ and every member of $\mathcal{F}$ is a compact metrizable zero-dimensional space. By Proposition 1.4.10, $X \times X$ is homeomorphic to X for every $X \in \mathcal{F}$. Finally, it easily follows from Proposition 1.4.12 that $\mathcal{F}$ consists of pairwise non-homeomorphic spaces. □

Remark 1.4.13. The family constructed in the proof of Theorem 1.1.1 has the property that for every member X of the family and for every $k \in \omega$, there are (infinitely many) points $x \in X \setminus \mathrm{PK}(X)$ with $\mathrm{CB}(x, X) = k$. It is thus natural to ask, for each $n \in \omega$, how many compact metrizable zero-dimensional spaces X there are (up to homeomorphism) such that X is homeomorphic to $X \times X$ and

$\mathrm{CB}(x, X) \leq n$ for each $x \in X \setminus \mathrm{PK}(X)$. The situation is unclear even for $n = 0$, i.e. we have the following question.

Question 1.4.14. How many compact metrizable zero-dimensional spaces X are there (up to homeomorphism) such that X is homeomorphic to $X \times X$ and each point in X is either isolated or belongs to the perfect kernel of X?

1.5 A correction of [CS19, Theorem 3.3]

In this section we present our proof of [CS19, Theorem 3.3]. The original proof, which was largely based on the proof of [IN99, Proposition 8.8], is not entirely correct. Namely, the equality "$\inf\{d_1(p, S(X_1)) : p \in \cup_{k=1}^{\infty} P_k\} = 0$" in [CS19, p.607 line 4] is not justified. The corresponding part of the proof of [IN99, Proposition 8.8] is correct since any infinite subset of a compact metric space has a zero distance from the set of all limit points of the space. However, the set $S(X_1)$ in the proof of [CS19, Theorem 3.3] is the perfect kernel of X_1, and as such it does not necessarily contain every limit point of X_1.

We will use the following notation throughout this section: Let $\gamma_0 := 1$ and denote $\gamma_\beta := \omega^\beta + 1$ for every nonzero ordinal β. For every $\alpha < \omega_1$ let $\mathscr{S}_\alpha$ be the family of ordinals defined by $\mathscr{S}_\alpha := \{\gamma_\beta\,;\ \beta < \alpha\}$.

Lemma 1.5.1. *Let X be a compact metric space and $\alpha < \omega_1$. Assume $G \subseteq X$ is a countable open set such that $\mathrm{CB}(x, X) < \alpha$ for every $x \in G$. Then there is an X-clopen partition $\mathcal{A}$ of G such that $\mathcal{A}$ is a null family and every member of $\mathcal{A}$ is homeomorphic to a member of $\mathscr{S}_\alpha$.*

Proof. If G is finite, we just let $\mathcal{A} := \{\{x\}; x \in G\}$. Assume that G is infinite (in particular, $G \neq \emptyset$, hence $\alpha > 0$) and let $\{x_n\,;\ n \in \omega\}$ be an enumeration of G. For every $n \in \omega$, fix a compact neighbourhood K_n of x_n contained in G such that $\mathrm{CB}(K_n) = \mathrm{CB}(x_n, X)$. As G is countable, it is zero-dimensional. Hence, since G is open, it easily follows that every point in G admits a neighbourhood base consisting of clopen subsets of X. For each $n \in \omega$, let $U_n \subseteq X$ be a clopen set containing x_n such that $U_n \subseteq K_n$ and $\mathrm{diam}(U_n) < 2^{-n}$. Let

$$V_n := U_n \setminus \bigcup\{U_k\,;\ k < n\}$$

for every $n \in \omega$ and denote $N := \{n \in \omega\,;\ V_n \neq \emptyset\}$. Then $\{V_n\,; n \in N\}$ is an X-clopen partition of G and it is a null family. Moreover, for every $n \in N$,

$$\beta(n) := \mathrm{CB}(V_n) \leq \mathrm{CB}(U_n) \leq \mathrm{CB}(K_n) = \mathrm{CB}(x_n, X) < \alpha.$$

For each $n \in N$ with V_n infinite, since V_n is homeomorphic to $\omega^{\beta(n)} \cdot k(n) + 1$ for some $k(n) \in \omega \setminus \{0\}$, there is a finite X-clopen partition $\mathcal{A}_n$ of V_n such that every member of $\mathcal{A}_n$ is homeomorphic to $\omega^{\beta(n)} + 1 = \gamma_{\beta(n)} \in \mathscr{S}_\alpha$. For each $n \in N$ with V_n finite, let $\mathcal{A}_n := \{\{x\}; x \in V_n\}$ (then every member of $\mathcal{A}_n$ is homeomorphic to $\gamma_0 = \gamma_{\beta(n)} \in \mathscr{S}_\alpha$). Finally, define $\mathcal{A} := \bigcup\{\mathcal{A}_n\,; n \in N\}$. □

Proposition 1.5.2. *Let (X, d) be an uncountable compact metric space and α a countable ordinal. Assume that $\mathrm{CB}(x, X) < \alpha$ for every $x \in X \setminus \mathrm{PK}(X)$ and that*

$$\mathrm{PK}(X) \subseteq \overline{\{x \in X \setminus \mathrm{PK}(X)\,;\ \mathrm{CB}(x, X) = \beta\}}$$

for every $\beta < \alpha$. Then there is an X-clopen partition $\mathcal{A}$ of $X \setminus \mathrm{PK}(X)$ such that:

(i) *$\mathcal{A}$ is a null family in (X, d);*
(ii) *every member of $\mathcal{A}$ is homeomorphic to a member of $\mathscr{S}_\alpha$;*
(iii) *for every $S \in \mathscr{S}_\alpha$ and every open set $U \subseteq X$ with $U \cap \mathrm{PK}(X) \neq \emptyset$, there is $A \in \mathcal{A}$ such that A is homeomorphic to S and $A \subseteq U$.*

Proof. If $\alpha = 0$, then $X \setminus \mathrm{PK}(X) = \emptyset$ and $\mathscr{S}_\alpha = \emptyset$, therefore the choice $\mathcal{A} := \emptyset$ works. Assume $\alpha > 0$ and let $(\beta_n)_{n\in\omega}$ be a sequence of ordinals less than α such that the set $\{n \in \omega\,;\ \beta_n = \beta\}$ is infinite for each $\beta < \alpha$. Fix a sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers converging to zero. For every $n \in \omega$, let $F_n \subseteq \mathrm{PK}(X)$ be a finite ε_n-net for $\mathrm{PK}(X)$. We are going to construct (inductively) a sequence $(\mathcal{V}_n)_{n\in\omega}$ of finite families of pairwise disjoint nonempty clopen subsets of X such that the following conditions hold for every $n \in \omega$.

(1) $\forall\, k < n : (\bigcup \mathcal{V}_k) \cap (\bigcup \mathcal{V}_n) = \emptyset$;
(2) $\forall\, V \in \mathcal{V}_n : V \subseteq X \setminus \mathrm{PK}(X)$;
(3) $\forall\, V \in \mathcal{V}_n : \mathrm{CB}(V) = \beta_n$;
(4) $\forall\, V \in \mathcal{V}_n : \mathrm{diam}_d(V) < \varepsilon_n$;
(5) $\forall\, V \in \mathcal{V}_n : \mathrm{dist}_d(\mathrm{PK}(X), V) < \varepsilon_n$;
(6) $\forall\, z \in F_n\ \exists\, V \in \mathcal{V}_n : \mathrm{dist}_d(z, V) < \varepsilon_n$.

Before we start the construction, for every $x \in X \setminus \mathrm{PK}(X)$, let K_x be a compact neighbourhood of x such that $K_x \subseteq X \setminus \mathrm{PK}(X)$ and $\mathrm{CB}(K_x) = \mathrm{CB}(x, X)$. Now, let us construct $\mathcal{V}_0$. For every $z \in F_0$, since

$$\mathrm{PK}(X) \subseteq \overline{\{x \in X \setminus \mathrm{PK}(X)\,;\ \mathrm{CB}(x, X) = \beta_0\}},$$

there exists $x_0(z) \in X \setminus \mathrm{PK}(X)$ such that $\mathrm{CB}(x_0(z), X) = \beta_0$ and $d(x_0(z), z) < \varepsilon_0$. Clearly, we can assume that $x_0(z) \neq x_0(w)$ for any two distinct points $z, w \in F_0$. As $X \setminus \mathrm{PK}(X)$ is countable (and thus zero-dimensional) and open, every point in $X \setminus \mathrm{PK}(X)$ admits a neighbourhood base consisting of clopen subsets of X. Thus, for every $z \in F_0$, there is a clopen set $V_0(z) \subseteq X$ such that $x_0(z) \in V_0(z) \subseteq K_{x_0(z)}$ and $\mathrm{diam}_d(V_0(z)) < \varepsilon_0$. Again, we can assume that $V_0(z) \cap V_0(w) = \emptyset$ for any two distinct points $z, w \in F_0$. For every $z \in F_0$, we have

$$\mathrm{dist}_d\Big(\mathrm{PK}(X), V_0(z)\Big) \leq \mathrm{dist}_d(z, V_0(z)) \leq d(x_0(z), z) < \varepsilon_0$$

and

$$\beta_0 = \mathrm{CB}(x_0(z), X) \leq \mathrm{CB}(V_0(z)) \leq \mathrm{CB}(K_{x_0(z)}) = \beta_0.$$

Hence, letting $\mathcal{V}_0 := \{V_0(z)\,;\ z \in F_0\}$, the initial step is done. In the same fashion, given $n \in \omega$ and assuming the families $\mathcal{V}_0, \ldots, \mathcal{V}_n$ have already been constructed, it is possible to construct $\mathcal{V}_{n+1}$. Note that it is easy to make sure (1) is satisfied as $\bigcup\{\bigcup \mathcal{V}_i\,;\ i \leq n\}$ is a compact set disjoint from $\mathrm{PK}(X)$.

Having finished the inductive construction, let $\mathcal{V} := \bigcup\{\mathcal{V}_n\,;\ n \in \omega\}$. Then $\mathcal{V}$ is a null family of pairwise disjoint nonempty clopen subsets of X contained in $X \setminus \mathrm{PK}(X)$. For every $V \in \mathcal{V}$, there is (by the same argument as in the proof of Lemma 1.5.1) a finite X-clopen partition $\mathcal{A}_V$ of V such that every member of $\mathcal{A}_V$ is homeomorphic to $\gamma_{\mathrm{CB}(V)} \in \mathscr{S}_\alpha$. It easily follows from (4) and (5) that the set $G := (X \setminus \mathrm{PK}(X)) \setminus \bigcup \mathcal{V}$ is open in X. Thus, by Lemma 1.5.1, there is an X-clopen partition $\mathcal{A}_G$ of G such that $\mathcal{A}_G$ is a null family and every member of $\mathcal{A}_G$ is homeomorphic to a member of $\mathscr{S}_\alpha$. Let $\mathcal{A} := \mathcal{A}_G \cup \bigcup\{\mathcal{A}_V\,;\ V \in \mathcal{V}\}$. Clearly,

$\mathcal{A}$ is an X-clopen partition of $X \setminus \mathrm{PK}(X)$ and it satisfies (i) and (ii). To verify assertion (iii), let $S \in \mathscr{S}_\alpha$ be given and let $U \subseteq X$ be an open set intersecting $\mathrm{PK}(X)$. Obviously, $S = \gamma_\beta$ for some $\beta < \alpha$. Fix a point $p \in U \cap \mathrm{PK}(X)$ and let r be a positive real number such that the open r-ball centered at p is contained in U. There is $m \in \omega$ such that $\varepsilon_n < r/3$ for each $n \in \omega$ with $n \geq m$. Since the set $\{n \in \omega\, ;\ \beta_n = \beta\}$ is infinite, there is $n \in \omega$ with $n \geq m$ such that $\beta_n = \beta$. As F_n is an ε_n-net for $\mathrm{PK}(X)$, there is $z \in F_n$ such that $d(p, z) < \varepsilon_n$. By (6), there is $V \in \mathcal{V}_n$ with $\mathrm{dist}_d(z, V) < \varepsilon_n$. Then $\mathrm{dist}_d(p, V) < 2\varepsilon_n < 2r/3$ and thus, since $\mathrm{diam}_d(V) < \varepsilon_n < r/3$ by (4), the set V is contained in the open r-ball centered at p. This shows that $V \subseteq U$. Moreover, $\mathrm{CB}(V) = \beta_n = \beta$ by (3). Hence, taking any $A \in \mathcal{A}_V$, we have found a member of $\mathcal{A}$ contained in U and homeomorphic to $\gamma_\beta = S$. □

Combining Proposition 1.4.8 with Proposition 1.5.2, we immediately obtain the following restatement of [CS19, Theorem 3.3].

Corollary 1.5.3. *Let α be a countable ordinal and let X_1, X_2 be uncountable metrizable compact spaces such that the following conditions are satisfied for each $i \in \{1, 2\}$.*

- $\mathrm{CB}(x, X_i) < \alpha$ *for every* $x \in X_i \setminus \mathrm{PK}(X_i)$*;*
- $\mathrm{PK}(X_i) \subseteq \overline{\left\{x \in X_i \setminus \mathrm{PK}(X_i)\, ;\ \mathrm{CB}(x, X_i) = \beta\right\}}$ *for every* $\beta < \alpha$*.*

Then every homeomorphism $h\colon \mathrm{PK}(X_1) \to \mathrm{PK}(X_2)$ extends to a homeomorphism $\overline{h}\colon X_1 \to X_2$.

2. Borel measurable Hahn-Mazurkiewicz theorem

Abstract: It is well known due to Hahn and Mazurkiewicz that every Peano continuum is a continuous image of the unit interval. We prove that an assignment, which takes as an input a Peano continuum and produces as an output a continuous mapping whose range is the Peano continuum, can be realized in a Borel measurable way. Similarly, we find a Borel measurable assignment which takes any nonempty compact metric space and assigns a continuous mapping from the Cantor set onto that space. To this end we use the Burgess selection theorem. Finally, a Borel measurable way of assigning an arc joining two selected points in a Peano continuum is found.

2.1 Introduction

A lot of results in mathematics are of the form $\forall\, a \in A\; \exists\, b \in B : T(a, b)$. In many cases the sets A and B can be equipped with natural topologies or standard Borel structures. Then it makes sense to ask whether there exists a continuous or Borel measurable mapping $b : A \to B$ satisfying $T(a, b(a))$ for every $a \in A$. A natural way to prove this kind of result is to use a suitable selection (resp. uniformization) theorem, applying it to the set $\{(a, b);\, T(a, b)\}$. Some of the most useful selection theorems for this matter include results by Kuratowski and Ryll-Nardzewski, Kunugui and Novikov, or Arsenin and Kunugui [Kec95, theorems 12.13, 28.7, 35.46]. However, it may happen that none of the above selection theorems can be directly applied.

The Dugundji extension theorem is a nice example of a result of the above form. Fixing a metric space Y and its closed subspace X, it follows from the Tietze extension theorem that every bounded continuous function $f : X \to \mathbb{R}$ can be extended to a bounded continuous function $F : Y \to \mathbb{R}$. Dugundji proved that this assignment $f \in C^*(X) \mapsto F \in C^*(Y)$ can be realized in a continuous way (with respect to the topology of uniform convergence) and linear at the same time (see [Dug51] or [vM01, Theorem 6.6.2 and Remark 6.6.3]).

In this chapter we obtain Borel measurable variants of the following classical results: (a) every nonempty compact metrizable space is a continuous image of the Cantor space, (b) every Peano continuum is a continuous image of $[0, 1]$, (c) any two distinct points in a Peano continuum are end-points of an arc.

Theorem A (with details in Theorem 2.3.4)**.** *There is a Borel measurable way of assigning to every nonempty compact metrizable space K a continuous surjective mapping $f : \mathcal{C} \to K$, where $\mathcal{C}$ is the Cantor space.*

Theorem B (with details in Theorem 2.4.15)**.** *There is a Borel measurable way of assigning to every Peano continuum K a continuous surjective mapping $f : [0, 1] \to K$.*

Theorem C (with details in Theorem 2.5.7)**.** *There is a Borel measurable way* *of assigning to every Peano continuum K and a pair of distinct points $x, y \in K$ an arc A in K with end-points x and y.*

Theorem A is obtained by an application of a selection theorem by Burgess [Bur79]. Surprisingly, we were not able to apply any selection principle to prove Theorem B. Thus we followed the proof of the Hahn-Mazurkiewicz theorem as written in [Nad92] and we verified that all the steps can be carried out in a Borel measurable way. Theorem C is a nontrivial consequence of Theorem B.

Theorems A and B imply some consequences in the context of invariant descriptive set theory (see [Gao09]). Namely it follows that the space of all continuous mappings from $[0, 1]$ (or from the Cantor space) into the Hilbert cube with the topology of uniform convergence provides a new yet equivalent coding for the collection of all Peano continua (or compact metrizable spaces) when considered naturally as a subspace of the hyperspace of the Hilbert cube with the Vietoris topology. Details are included in Corollary 2.4.16. This gives us a parallel result to that of Gao for separable complete metric spaces which can be represented either as closed subspaces of the Urysohn space with the Effros Borel structure or as metrics on $\mathbb{N}$ with the topology of pointwise convergence [Gao09, Theorem 14.1.3].

It was proved independently by Moise [Moi49] and Bing [Bin49], as an answer to a question by Menger [Men28], that every Peano continuum admits a convex metric. A natural question related to the main focus of this chapter follows.

Question 2.1.1. Is it possible to assign a convex metric to every Peano continuum in a Borel way?

2.2 Preliminaries

In this section we introduce the notation, terminology, definitions and basic facts which will be used throughout this chapter. By a natural number we mean a strictly positive integer. We denote the set of natural numbers by $\mathbb{N}$. We use the symbol $\mathbb{N}_0$ to denote the set of non-negative integers, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, we denote by $\mathbb{R}^+$ the set of strictly positive real numbers.

A subset of a topological space X is said to be *Borel* if it belongs to the smallest σ-algebra on X containing every open subset of X. For any two topological spaces X and Y, a mapping $f\colon X \to Y$ is said to be *Borel measurable* if $f^{-1}(U)$ is a Borel subset of X for every open subset U of Y. A *Polish space* is a separable completely metrizable topological space. It is a well-known fact that a subspace Y of a Polish space X is Polish if and only if Y is a G_δ set in X. A *Polish group* is a topological group which is also a Polish space. A *continuum* is a compact connected metrizable topological space. We do not consider the empty topological space to be connected. Therefore, in particular, every continuum is a nonempty space. A *Peano continuum* is a locally connected continuum.

We denote by $\mathcal{C}$ the *Cantor space*, i.e. the space $\{0, 1\}^{\mathbb{N}}$ equipped with the product topology. Recall that a topological space X is homeomorphic to $\mathcal{C}$ if and only if X is a nonempty, compact, metrizable, zero-dimensional space with no isolated points (this is a classical theorem due to Brouwer). We denote by I the compact interval $[0, 1]$. The *Hilbert cube*, denoted by Q, is the space $I^{\mathbb{N}}$ endowed with the product topology.

For a Polish space X, we denote by $\mathcal{K}(X)$ the space of all nonempty compact subsets of X equipped with the Vietoris topology (and the Hausdorff metric). It is well-known that $\mathcal{K}(X)$ is a Polish space. Moreover, $\mathcal{K}(X)$ is compact if and only if X is compact. Let us consider the following two subspaces of $\mathcal{K}(X)$: We denote by $\mathsf{C}(X)$ the space of all continua in X and by $\mathsf{LC}(X)$ the space of all Peano continua in X. It is easy to see that $\mathsf{C}(X)$ is a closed subset of $\mathcal{K}(X)$. In particular, it is Borel. The set $\mathsf{LC}(X)$ is Borel in $\mathcal{K}(X)$ as well (see [GvM93] for reference).

Recall that every separable metrizable space is homeomorphic to a subspace of Q. In particular, Q contains a homeomorphic copy of every metrizable compact space. So, in this sense, the space $\mathcal{K}(Q)$ represents the class of nonempty metrizable compact spaces. Similarly, the classes of continua and Peano continua are represented by $\mathsf{C}(Q)$ and $\mathsf{LC}(Q)$, respectively.

For any metrizable compact space X and any Polish space Y, we denote by $C(X,Y)$ the set of all continuous mappings from X to Y and we equip $C(X,Y)$ with the topology of uniform convergence (equivalently, the compact-open topology). It is well-known that $C(X,Y)$ is a Polish space. We denote by $\mathcal{E}(X,Y)$ the subspace of $C(X,Y)$ consisting of injective mappings. Note that by the compactness of X, we have

$$\mathcal{E}(X,Y)=\Big\{f\in C(X,Y)\,;\ f \text{ is a homeomorphic embedding of } X \text{ into } Y\Big\}.$$

It is not difficult to show that $\mathcal{E}(X,Y)$ is G_δ in $C(X,Y)$. Hence, $\mathcal{E}(X,Y)$ is a Polish space.

For any set X and any equivalence relation E on X, a subset M of X is said to be *E-invariant* if $[x]_E\subseteq M$ for every $x\in M$. A *transversal* for E is a subset of X containing exactly one element from each E-equivalence class.

Recall that for any group G and any set X, a mapping $\alpha\colon G\times X\to X$ is said to be an *action* of G on X if $\alpha(e,x)=x$ and $\alpha(g,\alpha(h,x))=\alpha(gh,x)$ for all $g,h\in G$ and $x\in X$, where e is the identity element of G. If α is an action of G on X, the equivalence relation E on X defined by

$$xEy \iff \exists\, g\in G:\alpha(g,x)=y$$

is called the *orbit equivalence relation* (induced by α).

An equivalence relation E on a Borel subset Y of a Polish space X is said to be *countably separated* if there is a sequence $(Z_n)_{n=1}^\infty$ of E-invariant Borel subsets of Y such that for all $x,y\in Y$, the points x and y are E-equivalent if and only if $\{n\in\mathbb{N};\ x\in Z_n\}=\{n\in\mathbb{N};\ y\in Z_n\}$.

In the remaining part of this section we present various lemmata related to spaces of compact sets and spaces of continuous mappings.

Lemma 2.2.1. *Let X be a topological space, Y a Polish space and $f\colon X\to\mathcal{K}(Y)$ any mapping. Then the following four assertions are equivalent:*

(i) *f is Borel measurable;*
(ii) *the set $\{x\in X;\ f(x)\subseteq V\}$ is Borel for every open set $V\subseteq Y$;*
(iii) *the set $\{x\in X;\ f(x)\cap V\neq\emptyset\}$ is Borel for every open set $V\subseteq Y$;*
(iv) *the set $\{x\in X;\ f(x)\cap F\neq\emptyset\}$ is Borel for every closed set $F\subseteq Y$.*

Proof. Let $\mathcal{A}_V := \{K \in \mathcal{K}(Y)\,;\ K \cap V \neq \emptyset\}$ and $\mathcal{B}_V := \{K \in \mathcal{K}(Y)\,;\ K \subseteq V\}$ for every open set $V \subseteq Y$. Let $\mathscr{S}_1 := \{\mathcal{A}_V;\ V \subseteq Y \text{ open}\}$, $\mathscr{S}_2 := \{\mathcal{B}_V;\ V \subseteq Y \text{ open}\}$. By the definition of the Vietoris topology, the family $\mathscr{S}_1 \cup \mathscr{S}_2$ is a subbase for $\mathcal{K}(Y)$. Therefore, since $\mathcal{K}(Y)$ is a separable metrizable space, (i) is equivalent to the conjunction of (ii) and (iii). This is a consequence of the general fact that a mapping from a topological space to a separable metrizable space is Borel measurable if and only if the preimage of every subbasic set is Borel. Moreover, it is clear that (ii) is equivalent to (iv).

It remains to show that (ii) is equivalent to (iii). Assume that (ii) holds and let $V \subseteq Y$ be an open set. We are going to prove that $\{x \in X;\ f(x) \cap V \neq \emptyset\}$ is a Borel set. Since Y is metrizable, there are closed sets $F_1, F_2, \ldots \subseteq Y$ such that $V = \bigcup\{F_n\,;\ n \in \mathbb{N}\}$. Denoting $V_n := Y \setminus F_n$ for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \{x \in X;\ f(x) \cap V \neq \emptyset\} &= \bigcup_{n\in\mathbb{N}} \{x \in X;\ f(x) \cap F_n \neq \emptyset\} \\ = \bigcup_{n\in\mathbb{N}} \Big(X \setminus \{x \in X;\ f(x) \subseteq V_n\}\Big) &= X \setminus \bigcap_{n\in\mathbb{N}} \{x \in X;\ f(x) \subseteq V_n\}. \end{aligned}$$

Since (ii) holds and each of the sets $V_1, V_2, \ldots$ is open, we are done.

The implication from (iii) to (ii) can be proven in a similar fashion. □

Lemma 2.2.2. *Let X be a topological space and Y a compact metrizable space. Let $f_n \colon X \to \mathcal{K}(Y)$, $n \in \mathbb{N}$, be Borel measurable mappings. Then the mapping $f \colon X \to \mathcal{K}(Y)$ defined by*

$$f(x) = \overline{\bigcup_{n\in\mathbb{N}} f_n(x)}$$

is Borel measurable.

Proof. Clearly, for every open set $V \subseteq Y$, we have

$$\begin{aligned} \{x \in X;\ f(x) \cap V \neq \emptyset\} &= \Big\{x \in X;\ V \cap \bigcup_{n\in\mathbb{N}} f_n(x) \neq \emptyset\Big\} \\ &= \bigcup_{n\in\mathbb{N}} \{x \in X;\ f_n(x) \cap V \neq \emptyset\}. \end{aligned}$$

An application of Lemma 2.2.1 finishes the proof. □

Lemma 2.2.3. *For any Polish space X, the mapping from*

$$\Big\{(K_1, K_2) \in \mathcal{K}(X) \times \mathcal{K}(X)\,;\ K_1 \cap K_2 \neq \emptyset\Big\}$$

to $\mathcal{K}(X)$ given by $(K_1, K_2) \mapsto K_1 \cap K_2$ is Borel measurable.

Proof. Denote by $\mathcal{A}$ the domain of the mapping in question and let $V \subseteq X$ be an open set. We are going to show that the set $\mathcal{G} := \{(K_1, K_2) \in \mathcal{A}\,;\ K_1 \cap K_2 \subseteq V\}$ is relatively open (and hence relatively Borel) in $\mathcal{A}$. Let $(K_1, K_2) \in \mathcal{G}$ be given. Then $K_1 \cap K_2 \subseteq V$, which implies that $K_1 \setminus V$ and $K_2 \setminus V$ are disjoint closed sets. Therefore, there exist disjoint open sets $V_1, V_2 \subseteq X$ such that $K_1 \setminus V \subseteq V_1$ and $K_2 \setminus V \subseteq V_2$. Let $\mathcal{U} := \{(L_1, L_2) \in \mathcal{A}\,;\ L_1 \subseteq V \cup V_1 \text{ and } L_2 \subseteq V \cup V_2\}$. Clearly, $\mathcal{U}$ is relatively open in $\mathcal{A}$ and it contains (K_1, K_2). Moreover, it is easy to see that $\mathcal{U} \subseteq \mathcal{G}$.

By Lemma 2.2.1, we are done. □

Lemma 2.2.4. *Let X be a topological space and let Y be a Polish space. Assume that $f_n\colon X \to \mathcal{K}(Y)$, $n \in \mathbb{N}$, are Borel measurable mappings such that $\bigcap\{f_n(x);\, n \in \mathbb{N}\} \neq \emptyset$ for every $x \in X$. Then the mapping $f\colon X \to \mathcal{K}(Y)$ given by $f(x) = \bigcap\{f_n(x);\, n \in \mathbb{N}\}$ is Borel measurable.*

Proof. For every $n \in \mathbb{N}$, define a mapping $g_n\colon X \to \mathcal{K}(Y)$ by

$$g_n(x) = f_1(x) \cap \cdots \cap f_n(x).$$

Then $g_1 = f_1$ and $g_{n+1}(x) = g_n(x) \cap f_{n+1}(x)$ for every $n \in \mathbb{N}$ and $x \in X$. Thus, it easily follows from Lemma 2.2.3 that each of the mappings $g_1, g_2, g_3, \ldots$ is Borel measurable. Moreover, f can be shown to be the pointwise limit of the sequence $(g_n)_{n=1}^{\infty}$. Therefore, f is Borel measurable. $\square$

Lemma 2.2.5. *Let X be a compact metrizable space and Y a Polish space. Then the mapping from $C(X,Y) \times \mathcal{K}(X)$ to $\mathcal{K}(Y)$ given by $(f,K) \mapsto f(K)$ is continuous.*

Proof. By the definition of the Vietoris topology, it suffices to show that for every open set $V \subseteq Y$, the sets

$$\begin{aligned} \mathcal{A}_V &:= \Big\{(f,K) \in C(X,Y) \times \mathcal{K}(X)\,;\ f(K) \subseteq V\Big\}, \\ \mathcal{B}_V &:= \Big\{(f,K) \in C(X,Y) \times \mathcal{K}(X)\,;\ f(K) \cap V \neq \emptyset\Big\} \end{aligned}$$

are open in $C(X,Y) \times \mathcal{K}(X)$. Let $V \subseteq Y$ be an open set. To show that $\mathcal{A}_V$ is open, let $(f,K) \in \mathcal{A}_V$. Then $f(K)$ is a closed set contained in V. Thus, there is an open set $U \subseteq Y$ with $f(K) \subseteq U \subseteq \overline{U} \subseteq V$. Since f is continuous, there is an open set $G \subseteq X$ such that $K \subseteq G$ and $f(G) \subseteq U$. Let

$$\mathcal{U} := \{g \in C(X,Y)\,;\ g(\overline{G}) \subseteq V\} \times \{L \in \mathcal{K}(X)\,;\ L \subseteq G\}.$$

Then $\mathcal{U}$ is open in $C(X,Y) \times \mathcal{K}(X)$ and $\mathcal{U} \subseteq \mathcal{A}_V$. Moreover, since $K \subseteq G$ and $f(\overline{G}) \subseteq \overline{f(G)} \subseteq \overline{U} \subseteq V$, we have $(f,K) \in \mathcal{U}$.

Now, let us show that $\mathcal{B}_V$ is open. To that end, let $(f,K) \in \mathcal{B}_V$ be given and let $x \in K$ be a point for which $f(x) \in V$. Let $U \subseteq Y$ be an open set such that $f(x) \in U \subseteq \overline{U} \subseteq V$. By the continuity of f, there is an open set $G \subseteq X$ with $x \in G$ and $f(G) \subseteq U$. Let

$$\mathcal{U} := \{g \in C(X,Y)\,;\ g(\overline{G}) \subseteq V\} \times \{L \in \mathcal{K}(X)\,;\ L \cap G \neq \emptyset\}.$$

Again, it is easy to see that $\mathcal{U}$ is open, $\mathcal{U} \subseteq \mathcal{B}_V$ and $(f,K) \in \mathcal{U}$. $\square$

The following lemma was essentially proved in [Ken88]. We present our proof here for the sake of completeness.

Lemma 2.2.6. *Let X be a compact metrizable space and Y a Polish space. Then the mapping from $C(X,Y)$ to $\mathcal{K}(X \times Y)$ given by*

$$f \mapsto \operatorname{graph}(f) = \{(x,y) \in X \times Y;\ y = f(x)\}$$

is a homeomorphic embedding.

Proof. Denote the mapping in question by Γ. Since Γ is injective, it suffices to show that Γ and Γ^{-1} are continuous. Let ϱ and σ be arbitrary compatible metrics on X and Y, respectively. Define a metric d on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{\varrho(x_1, x_2), \sigma(y_1, y_2)\}.$$

Moreover, let d_H be the Hausdorff metric on $\mathcal{K}(X \times Y)$ induced by d and let m be the uniform metric on $C(X, Y)$ induced by σ. That is,

$$d_H(K, L) = \max\Big\{ \max_{(x,y)\in K} \operatorname{dist}_d\big((x, y), L\big),\ \max_{(x,y)\in L} \operatorname{dist}_d\big((x, y), K\big)\Big\},$$
$$m(f, g) = \max\big\{\sigma(f(x), g(x))\,;\ x \in X\big\}.$$

Then d, d_H and m are compatible metrics on $X \times Y$, $\mathcal{K}(X \times Y)$ and $C(X, Y)$, respectively.

Claim 2.2.6.1. For all $f, g \in C(X, Y)$, we have $d_H(\Gamma(f), \Gamma(g)) \leq m(f, g)$.

Proof. Let $f, g \in C(X, Y)$. For every $(x, y) \in \Gamma(f)$, we have $y = f(x)$ and

$$\operatorname{dist}_d((x, y), \Gamma(g)) \leq d((x, y), (x, g(x))) = \sigma(y, g(x)) = \sigma(f(x), g(x)) \leq m(f, g).$$

Symmetrically, we have $\operatorname{dist}_d((x, y), \Gamma(f)) \leq m(g, f)$ for all $(x, y) \in \Gamma(g)$. Hence, $d_H(\Gamma(f), \Gamma(g)) \leq m(f, g)$. ■

The continuity of Γ is an immediate consequence of Claim 2.2.6.1. To show that Γ^{-1} is continuous, we have to prove that for all $f \in C(X, Y)$ and $\varepsilon > 0$, there is $\delta > 0$ such that $m(f, g) < \varepsilon$ for every $g \in C(X, Y)$ with $d_H(\Gamma(f), \Gamma(g)) < \delta$. Let $f \in C(X, Y)$ and $\varepsilon > 0$ be given. By the (uniform) continuity of f, there is $\Delta > 0$ such that $\sigma(f(x_1), f(x_2)) < \varepsilon/2$ for any two points $x_1, x_2 \in X$ satisfying $\varrho(x_1, x_2) < \Delta$. Let $\delta := \min\{\Delta, \varepsilon/2\}$.

Claim 2.2.6.2. Let $g \in C(X, Y)$ satisfy $d_H(\Gamma(f), \Gamma(g)) < \delta$. Then $m(f, g) < \varepsilon$.

Proof. Fix a point $x \in X$ satisfying $m(f, g) = \sigma(f(x), g(x))$. As $(x, g(x)) \in \Gamma(g)$ and $d_H(\Gamma(f), \Gamma(g)) < \delta$, there is $(z, y) \in \Gamma(f)$ such that $d((x, g(x)), (z, y)) < \delta$. Then we have $\varrho(x, z) < \delta \leq \Delta$, $\sigma(y, g(x)) < \delta \leq \varepsilon/2$ and $y = f(z)$. Therefore, $m(f, g) = \sigma(f(x), g(x)) \leq \sigma(f(x), f(z)) + \sigma(y, g(x)) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. ■

□

Lemma 2.2.7. *Let X be a compact metrizable space and Y a Polish space. Then the mapping from $\{(f, K) \in C(X, Y) \times \mathcal{K}(Y)\,;\ f^{-1}(K) \neq \emptyset\}$ to $\mathcal{K}(X)$ given by $(f, K) \mapsto f^{-1}(K)$ is Borel measurable.*

Proof. Denote by $\mathcal{A}$ the domain of the mapping in question and define a mapping $\Psi\colon \mathcal{A} \to \mathcal{K}(X \times Y)$ by $\Psi(f, K) = \operatorname{graph}(f) \cap (X \times K)$. Then Ψ is Borel measurable by Lemmata 2.2.3 and 2.2.6. Let π_1 be the coordinate projection from $X \times Y$ to X. By Lemma 2.2.5, the mapping from $\mathcal{A}$ to $\mathcal{K}(X)$ given by $(f, K) \mapsto \pi_1(\Psi(f, K))$ is Borel measurable. However, $\pi_1(\Psi(f, K)) = f^{-1}(K)$ for all $(f, K) \in \mathcal{A}$. □

The following lemma is implied by [Kur68, Theorem 3, §43].

Lemma 2.2.8. *Let X be a Polish space, Y a compact metrizable space and F a closed subset of $X \times Y$. Then the mapping from $\{x \in X\,;\ \exists\, y \in Y : (x, y) \in F\}$ to $\mathcal{K}(Y)$ given by $x \mapsto \{y \in Y;\ (x, y) \in F\}$ is Borel measurable (in fact, it is upper semicontinuous).*

2.3 Compacta as continuous images of the Cantor space in a Borel measurable way

In this section we prove that there exists a Borel measurable mapping $T\colon \mathcal{K}(Q) \to C(\mathcal{C}, Q)$ such that for every $K \in \mathcal{K}(Q)$, the mapping $T(K)$ maps $\mathcal{C}$ onto K. This can be accomplished using the Kuratowski and Ryll-Nardzewski selection theorem. However, we shall present a more elegant approach. The basic idea is the following: We fix a suitable continuous surjection $\varphi\colon \mathcal{C} \to Q$. Then for any $K \in \mathcal{K}(Q)$, the restriction of φ to $\varphi^{-1}(K)$ is a continuous mapping whose range is equal to K. Moreover, the assignment $K \mapsto \varphi\restriction_{\varphi^{-1}(K)}$ seems to be constructive enough to ensure Borel measurability. However, the problem is that the domain of $\varphi\restriction_{\varphi^{-1}(K)}$, i.e. the set $\varphi^{-1}(K)$, depends on K and it is not equal to $\mathcal{C}$ (unless $K = Q$). The main tool allowing us to get rid of this problem is the following theorem due to Burgess (see [Bur79]). We will refer to this theorem as the Burgess selection theorem.

Theorem. *Let G be a Polish group, X a Polish space and let $\alpha\colon G \times X \to X$ be a continuous action of G on X. Denote by E the orbit equivalence relation induced by α and let Y be an E-invariant Borel subset of X. Let E_Y be the restriction of E to Y and assume that E_Y is countably separated. Then there is a Borel transversal for E_Y.*

Let us split the proof of the main theorem into three lemmata.

Lemma 2.3.1. *There exists a continuous surjection $\varphi\colon \mathcal{C} \to Q$ such that $\varphi^{-1}(K)$ is homeomorphic to $\mathcal{C}$ for every $K \in \mathcal{K}(Q)$.*

Proof. It is well-known that every nonempty compact metrizable space is a continuous image of $\mathcal{C}$. Therefore, there is a continuous surjection $\psi\colon \mathcal{C} \to Q$. Define a mapping $\Psi\colon \mathcal{C} \times \mathcal{C} \to Q$ by $\Psi(\alpha, \beta) = \psi(\alpha)$. Then Ψ is a continuous surjection and $\Psi^{-1}(A) = \psi^{-1}(A) \times \mathcal{C}$ for every subset A of Q. Hence, for every $K \in \mathcal{K}(Q)$, we can easily see that $\Psi^{-1}(K)$ is a nonempty, compact, zero-dimensional space with no isolated points. This shows that $\Psi^{-1}(K)$ is homeomorphic to $\mathcal{C}$ for every $K \in \mathcal{K}(Q)$. Finally, since $\mathcal{C} \times \mathcal{C}$ is homeomorphic to $\mathcal{C}$ as well, we can finish the proof by fixing a homeomorphism $h\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and letting $\varphi := \Psi \circ h$. □

Before stating the next lemma, let us denote

$$\mathcal{K}_{\mathcal{C}} := \Big\{K \in \mathcal{K}(\mathcal{C})\,;\ K \text{ is homeomorphic to } \mathcal{C}\Big\}.$$

Lemma 2.3.2. *There exists a Borel set $\mathcal{B} \subseteq \mathcal{E}(\mathcal{C}, \mathcal{C})$ such that for every $K \in \mathcal{K}_{\mathcal{C}}$, there is exactly one $f \in \mathcal{B}$ with $f(\mathcal{C}) = K$.*

Proof. Let $X := \mathcal{E}(\mathcal{C}, \mathcal{C})$ and let G be the group of self-homeomorphisms of $\mathcal{C}$, where the group operation is the composition. Equip G with the subspace topology inherited from $C(\mathcal{C}, \mathcal{C})$. This makes G a Polish group (this is well-known, see e.g. [Gao09, Example 2.2.4]). Define a mapping $\alpha\colon G \times X \to X$ by $\alpha(g, f) = f \circ g$. Then α is an action of G on X and it is easy to see that α is continuous. Let E be the the orbit equivalence relation induced by α. Clearly, for all $f_1, f_2 \in X$,

$$f_1 E f_2 \iff f_1(\mathcal{C}) = f_2(\mathcal{C}).$$

Thus, to prove this lemma, it suffices to show that there is a Borel transversal for E. To that end, we are going to use the Burgess selection theorem with $Y := X$. We have already verified most of the assumptions of the selection theorem, it remains to show that E is countably separated. Let $(U_n)_{n=1}^{\infty}$ be a sequence of open subsets of $\mathcal{C}$ forming a base for the topology of $\mathcal{C}$. For each $n \in \mathbb{N}$, let

$$Z_n := \Big\{ f \in X\,;\ U_n \cap f(\mathcal{C}) \neq \emptyset \Big\}.$$

Then Z_n is E-invariant and open in X for all $n \in \mathbb{N}$. Moreover, for all $f_1, f_2 \in X$,

$$f_1 E f_2 \iff f_1(\mathcal{C}) = f_2(\mathcal{C}) \iff \{n \in \mathbb{N}\,;\ f_1 \in Z_n\} = \{n \in \mathbb{N}\,;\ f_2 \in Z_n\}.$$

□

Lemma 2.3.3. *There is a Borel measurable mapping $T_0 \colon \mathcal{K}_{\mathcal{C}} \to \mathcal{E}(\mathcal{C},\mathcal{C})$ such that for every $K \in \mathcal{K}_{\mathcal{C}}$, the image of $\mathcal{C}$ under $T_0(K)$ is equal to K.*

Proof. Let $\mathcal{B} \subseteq \mathcal{E}(\mathcal{C},\mathcal{C})$ be the Borel set given by Lemma 2.3.2. Define a mapping $\Phi \colon \mathcal{B} \to \mathcal{K}(\mathcal{C})$ by $\Phi(f) = f(\mathcal{C})$. It follows from Lemma 2.2.5 that Φ is continuous. In particular, Φ is Borel measurable. Moreover, Φ is injective and $\Phi(\mathcal{B}) = \mathcal{K}_{\mathcal{C}}$. Hence, $\mathcal{K}_{\mathcal{C}}$ is Borel and the mapping $\Phi^{-1} \colon \mathcal{K}_{\mathcal{C}} \to \mathcal{E}(\mathcal{C},\mathcal{C})$ is Borel measurable (see, e.g., [Kec95, Corollary 15.2]). Let $T_0 := \Phi^{-1}$. □

Theorem 2.3.4. *There is a Borel measurable mapping $T \colon \mathcal{K}(Q) \to C(\mathcal{C}, Q)$ such that for every $K \in \mathcal{K}(Q)$, the image of $\mathcal{C}$ under $T(K)$ is equal to K.*

Proof. Let $\varphi \colon \mathcal{C} \to Q$ and $T_0 \colon \mathcal{K}_{\mathcal{C}} \to \mathcal{E}(\mathcal{C},\mathcal{C})$ be the mappings guaranteed by Lemmata 2.3.1 and 2.3.3, respectively. Define mappings $\Theta \colon \mathcal{E}(\mathcal{C},\mathcal{C}) \to C(\mathcal{C}, Q)$ and $\Psi \colon \mathcal{K}(Q) \to \mathcal{K}(\mathcal{C})$ by $\Theta(f) = \varphi \circ f$ and $\Psi(K) = \varphi^{-1}(K)$. It is clear that Θ is continuous and, by Lemma 2.2.7, Ψ is Borel measurable. Moreover, note that $\Psi(\mathcal{K}(Q)) \subseteq \mathcal{K}_{\mathcal{C}}$. Finally, define $T := \Theta \circ T_0 \circ \Psi$. Then T is Borel measurable and it is easy to see that for every $K \in \mathcal{K}(Q)$, the image of $\mathcal{C}$ under $T(K)$ is equal to $\varphi(\varphi^{-1}(K)) = K$. □

2.4 The Hahn-Mazurkiewicz theorem in a Borel measurable way

The aim of this section is to prove the existence of a Borel measurable mapping $\Phi \colon \mathsf{LC}(Q) \to C(I, Q)$ such that for every $K \in \mathsf{LC}(Q)$, the image of I under $\Phi(K)$ is equal to K. Our basic strategy is to go through the proof of the Hahn-Mazurkiewicz theorem and verify that every step can be carried out in a Borel fashion.

The following definition and theorem can be found in [Nad92, p. 120]

Definition 2.4.1. A nonempty subset X of a metric space Y is said to have *property S* provided that for every $\varepsilon > 0$ there are $n \in \mathbb{N}$ and connected sets $A_1, \ldots, A_n \subseteq Y$ such that $A_1 \cup \cdots \cup A_n = X$ and $\operatorname{diam}(A_i) < \varepsilon$ for $i = 1, \ldots, n$.

Theorem 2.4.2. *A nonempty compact subset of a metric space is locally connected if and only if it has property S.*

Corollary 2.4.3. *A nonempty compact subset X of a metric space Y is locally connected if and only if for each $\varepsilon > 0$ there are $n \in \mathbb{N}$ and continua $K_1, \dots, K_n$ in Y such that $K_1 \cup \dots \cup K_n = X$ and* $\operatorname{diam}(K_i) < \varepsilon$ *for each $i = 1, \dots, n$.*

Proof. The "if" part immediately follows from Theorem 2.4.2. To prove the "only if" part assume that X is locally connected. By Theorem 2.4.2, X has property S. Let $\varepsilon > 0$ be arbitrary and let $A_1, \dots, A_n \subseteq Y$ be the corresponding connected sets given by the property S. For each $i = 1, \dots, n$, let $K_i = \overline{A_i}$. □

Let us denote by $\mathcal{Z}$ the Polish space of all finite sequences of continua in Q. Formally speaking, $\mathcal{Z}$ is the topological sum

$$\bigoplus_{n \in \mathbb{N}} \big(\mathsf{C}(Q)\big)^n.$$

Since Q is compact, so is $\mathcal{K}(Q)$. Thus, being a closed subspace of $\mathcal{K}(Q)$, the space $\mathsf{C}(Q)$ is compact and so are its powers. Therefore, $\mathcal{Z}$ is σ-compact.

Let ϱ be a compatible metric on Q. This metric will be fixed from now on (in this section).

Theorem 2.4.4. *There is a Borel measurable mapping $\Psi\colon \mathsf{LC}(Q) \times \mathbb{R}^+ \to \mathcal{Z}$ such that for every $X \in \mathsf{LC}(Q)$ and $\varepsilon \in \mathbb{R}^+$, if $\Psi(X, \varepsilon) = (K_1, \dots, K_n)$, then $K_1 \cup \dots \cup K_n = X$ and* $\operatorname{diam}_\varrho(K_i) < \varepsilon$ *for each $i = 1, \dots, n$.*

Proof. For every $\Gamma \in \mathcal{Z}$, there is unique $l(\Gamma) \in \mathbb{N}$ such that $\Gamma \in (\mathsf{C}(Q))^{l(\Gamma)}$. Using this notation, define a mapping $f\colon \mathcal{Z} \to \mathcal{K}(Q)$ by

$$f(\Gamma) = \bigcup_{i=1}^{l(\Gamma)} \Gamma_i.$$

Since $(\mathsf{C}(Q))^k$ is an open subset of $\mathcal{Z}$ for every $k \in \mathbb{N}$, it is easy to see that f is continuous. Define a function $g\colon \mathsf{C}(Q) \to \mathbb{R}$ by $g(K) = \operatorname{diam}_\varrho(K)$. Clearly, g is continuous as well. Finally, define

$$\mathcal{A} := \Big\{(X, \varepsilon, \Gamma) \in \mathsf{C}(Q) \times \mathbb{R}^+ \times \mathcal{Z}\,;\ f(\Gamma) = X,\ g(\Gamma_i) \le \varepsilon \text{ for } i = 1, \dots, l(\Gamma)\Big\}.$$

The continuity of f and g implies that $\mathcal{A}$ is a closed subset of the Polish space $\mathsf{C}(Q) \times \mathbb{R}^+ \times \mathcal{Z}$. In particular, $\mathcal{A}$ is a Borel set with σ-compact sections in $\mathcal{Z}$. By the Arsenin-Kunugui selection theorem [Kec95, Theorem 35.46], the set

$$\pi(\mathcal{A}) := \Big\{(X, \varepsilon) \in \mathsf{C}(Q) \times \mathbb{R}^+\,;\ \exists\, \Gamma \in \mathcal{Z} : (X, \varepsilon, \Gamma) \in \mathcal{A}\Big\}$$

is Borel in $\mathsf{C}(Q) \times \mathbb{R}^+$ and there exists a Borel measurable mapping $\widehat{\Psi}\colon \pi(\mathcal{A}) \to \mathcal{Z}$ such that $(X, \varepsilon, \widehat{\Psi}(X, \varepsilon)) \in \mathcal{A}$ for every $(X, \varepsilon) \in \pi(\mathcal{A})$. It follows from Corollary 2.4.3 that $\mathsf{LC}(Q) \times \mathbb{R}^+ \subseteq \pi(\mathcal{A})$. The desired mapping $\Psi\colon \mathsf{LC}(Q) \times \mathbb{R}^+ \to \mathcal{Z}$ can thus be defined by $\Psi(X, \varepsilon) = \widehat{\Psi}(X, \varepsilon/2)$. □

The very definition of property S implies the following proposition.

Proposition 2.4.5. *Let X be a nonempty subset of a metric space Y. If X has property S, then so does the closure of X.*

Let $\mathcal{R} := \Big\{(X, A, \varepsilon) \in \mathsf{LC}(Q) \times \mathsf{C}(Q) \times \mathbb{R}^+;\ A \subseteq X\Big\}$ and for all $(X, A, \varepsilon) \in \mathcal{R}$, define

$$\begin{aligned} S(X, A, \varepsilon) := \Big\{x \in X;\ & \exists\, n \in \mathbb{N}\ \exists\, D_1, \dots, D_n \subseteq X:\ x \in D_n\,,\ A \cap D_1 \neq \emptyset\,, \\ & \forall\, i \in \{2, \dots, n\} : D_{i-1} \cap D_i \neq \emptyset\,, \\ & \forall\, i \in \{1, \dots, n\} : D_i \text{ is connected}, \\ & \forall\, i \in \{1, \dots, n\} : \operatorname{diam}_\varrho(D_i) < \varepsilon \cdot 2^{-i}\Big\}. \end{aligned}$$

For every $(X, A, \varepsilon) \in \mathcal{R}$, it follows from the compactness of X that

$$\begin{aligned} S(X, A, \varepsilon) = \Big\{x \in X;\ & \exists\, n \in \mathbb{N}\ \exists\, K_1, \dots, K_n \in \mathsf{C}(Q): \\ & x \in K_n\,,\ A \cap K_1 \neq \emptyset\,, \\ & \forall\, i \in \{2, \dots, n\} : K_{i-1} \cap K_i \neq \emptyset\,, \\ & \forall\, i \in \{1, \dots, n\} : K_i \subseteq X, \\ & \forall\, i \in \{1, \dots, n\} : \operatorname{diam}_\varrho(K_i) < \varepsilon \cdot 2^{-i}\Big\}. \end{aligned}$$

The following proposition is a consequence of [Nad92, 8.7, 8.8] combined with Theorem 2.4.2 and Proposition 2.4.5.

Proposition 2.4.6. *For every $(X, A, \varepsilon) \in \mathcal{R}$, the set $\overline{S(X, A, \varepsilon)}$ is a Peano continuum and we have* $\operatorname{diam}_\varrho(S(X, A, \varepsilon)) \leq 2\varepsilon + \operatorname{diam}_\varrho(A)$.

Proposition 2.4.7. *The mapping $\Theta\colon \mathcal{R} \to \mathsf{LC}(Q)$ defined by*

$$\Theta(X, A, \varepsilon) = \overline{S(X, A, \varepsilon)}$$

is Borel measurable.

Proof. For every $n \in \mathbb{N}$ and $(X, A, \varepsilon) \in \mathcal{R}$, let

$$\begin{aligned} S_n(X, A, \varepsilon) = \Big\{x \in X;\ & \exists\, K^1, \dots, K^n \in \mathsf{C}(Q):\ x \in K^n\,,\ A \cap K^1 \neq \emptyset, \\ & \forall\, i \in \{2, \dots, n\} : K^{i-1} \cap K^i \neq \emptyset, \\ & \forall\, i \in \{1, \dots, n\} : K^i \subseteq X, \\ & \forall\, i \in \{1, \dots, n\} : \operatorname{diam}_\varrho(K^i) \leq \varepsilon \cdot 2^{-i}\Big\}. \end{aligned}$$

Claim 2.4.7.1. For any $n \in \mathbb{N}$ and $(X, A, \varepsilon) \in \mathcal{R}$, the set $S_n(X, A, \varepsilon)$ is compact.

Proof. Since X is metrizable and compact, it suffices to show that $S_n(X, A, \varepsilon)$ is sequentially closed in X. Let $(x_k)_{k=1}^\infty$ be a sequence in $S_n(X, A, \varepsilon)$ converging to some $x \in X$. For all $k \in \mathbb{N}$, let $K_k^1, \dots, K_k^n \in \mathsf{C}(Q)$ be continua witnessing that $x_k \in S_n(X, A, \varepsilon)$. As $\mathsf{C}(Q)$ is metrizable and compact, we can assume that there exist $K^1, \dots, K^n \in \mathsf{C}(Q)$ such that $\lim_{k\to\infty} K_k^i = K^i$ for $i = 1, \dots, n$. It is straightforward to verify that $K^1, \dots, K^n$ witness that $x \in S_n(X, A, \varepsilon)$. ∎

For every $n \in \mathbb{N}$, Claim 2.4.7.1 shows that S_n is a mapping from $\mathcal{R}$ to $\mathcal{K}(Q)$. It is easy to see that for each $(X, A, \varepsilon) \in \mathcal{R}$,

$$S(X, A, \varepsilon) = \bigcup_{k,n \in \mathbb{N}} S_n\Big(X, A, \tfrac{k}{k+1}\varepsilon\Big),$$

therefore,

$$\Theta(X,A,\varepsilon)=\overline{\bigcup_{k,n\in\mathbb{N}} S_n\Big(X,A,\tfrac{k}{k+1}\varepsilon\Big)}.$$

By Lemma 2.2.2, this proposition will be proved once we show that each of the mappings $S_1, S_2, S_3, \ldots$ is Borel measurable. Hence, let $n\in\mathbb{N}$ be fixed and let $F\subseteq Q$ be an arbitrary closed set. By Lemma 2.2.1, it suffices to show that the set $\mathcal{F} := \{(X,A,\varepsilon)\in\mathcal{R}\,;\ F\cap S_n(X,A,\varepsilon)\neq\emptyset\}$ is Borel in $\mathcal{R}$. We claim that this set is actually closed in $\mathcal{R}$. Let $((X_k,A_k,\varepsilon_k))_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ converging to some $(X,A,\varepsilon)\in\mathcal{R}$, we are going to show that $(X,A,\varepsilon)\in\mathcal{F}$. For every $k\in\mathbb{N}$, fix any point $x_k\in F\cap S_n(X_k,A_k,\varepsilon_k)$ and let $K_k^1,\ldots,K_k^n\in\mathsf{C}(Q)$ witness that $x_k\in S_n(X_k,A_k,\varepsilon_k)$. Since Q and $\mathsf{C}(Q)$ are metrizable and compact, we can assume that there exist $K^1,\ldots,K^n\in\mathsf{C}(Q)$ and $x\in Q$ such that $\lim_{k\to\infty}x_k=x$ and $\lim_{k\to\infty}K_k^i=K^i$ for every $i=1,\ldots,n$. Then $x\in F$ and it is straightforward to verify that $K^1,\ldots,K^n$ witness that $x\in S_n(X,A,\varepsilon)$. Thus, $(X,A,\varepsilon)\in\mathcal{F}$. □

Denote by $\mathcal{Z}_P$ the subspace of $\mathcal{Z}$ consisting of finite sequences of Peano continua. Formally,

$$\mathcal{Z}_P=\Big\{\Gamma\in\mathcal{Z}\,;\ \text{if } \Gamma=(K_1,\ldots,K_n),\text{ then } K_i\in\mathsf{LC}(Q)\text{ for } i=1,\ldots,n\Big\}.$$

The following theorem is a strengthening of Theorem 2.4.4 and it is a key ingredient for the proof of the Borel version of the Hahn-Mazurkiewicz theorem.

Theorem 2.4.8. *There is a Borel measurable mapping $\Psi_P\colon \mathsf{LC}(Q)\times\mathbb{R}^+\to\mathcal{Z}_P$ such that for every $X\in\mathsf{LC}(Q)$ and $\varepsilon\in\mathbb{R}^+$, if $\Psi_P(X,\varepsilon)=(K_1,\ldots,K_n)$, then $K_1\cup\cdots\cup K_n=X$ and $\operatorname{diam}_\varrho(K_i)<\varepsilon$ for each $i=1,\ldots,n$.*

Proof. Let $\Psi\colon\mathsf{LC}(Q)\times\mathbb{R}^+\to\mathcal{Z}$ and $\Theta\colon\mathcal{R}\to\mathsf{LC}(Q)$ be the mappings provided by Theorem 2.4.4 and Proposition 2.4.7, respectively. Define the desired mapping $\Psi_P\colon\mathsf{LC}(Q)\times\mathbb{R}^+\to\mathcal{Z}_P$ in the following manner: For any $(X,\varepsilon)\in\mathsf{LC}(Q)\times\mathbb{R}^+$, if $\Psi(X,\varepsilon/3)=(K_1,\ldots,K_n)$, let

$$\Psi_P(X,\varepsilon)=\Big(\Theta(X,K_1,\varepsilon/3),\ldots,\Theta(X,K_n,\varepsilon/3)\Big).$$

Since $\Psi(X,\varepsilon/3)$ equals $(K_1,\ldots,K_n)$, each of the sets $K_1,\ldots,K_n$ is a continuum contained in X with diameter less than $\varepsilon/3$. Thus, by Proposition 2.4.6, each of the sets $\Theta(X,K_1,\varepsilon/3),\ldots,\Theta(X,K_n,\varepsilon/3)$ is a Peano continuum with diameter less than ε. Moreover, it is clear that $K_i\subseteq S(X,K_i,\varepsilon/3)\subseteq X$ for every $i=1,\ldots,n$. Therefore, as $\Psi(X,\varepsilon/3)=(K_1,\ldots,K_n)$, we have

$$X=\bigcup_{i=1}^{n}K_i\subseteq\bigcup_{i=1}^{n}S(X,K_i,\varepsilon/3)\subseteq X,$$

which implies that

$$\bigcup_{i=1}^{n}\Theta(X,K_i,\varepsilon/3)=\bigcup_{i=1}^{n}\overline{S(X,K_i,\varepsilon/3)}=X.$$

Finally, as both of the mappings Ψ and Θ are Borel measurable, Ψ_P is Borel measurable too. □

The following lemma is a special case of [Nad92, 8.13].

Lemma 2.4.9. *Let $n \in \mathbb{N}$ and let $K_1, \dots, K_n \subseteq Q$ be continua such that the set $X := K_1 \cup \dots \cup K_n$ is connected. Then, for any two points $x, y \in X$, there is $m \in \mathbb{N}$ and continua $L_1, \dots, L_m \subseteq Q$ such that $\{K_1, \dots, K_n\} = \{L_1, \dots, L_m\}$, $x \in L_1$, $y \in L_m$ and $L_{i-1} \cap L_i \neq \emptyset$ for each $i \in \{2, \dots, m\}$.*

Now, let us present a Borel version of Lemma 2.4.9. Define

$$\mathcal{M} := \Big\{(\Gamma, x, y) \in \mathcal{Z} \times Q \times Q\,;\ \text{if } \Gamma = (K_i)_{i=1}^n, \text{ then } K_1 \cup \dots \cup K_n \\ \text{is a connected set containing both } x \text{ and } y\Big\}$$

and equip $\mathcal{M}$ with the subspace topology inherited from $\mathcal{Z} \times Q \times Q$.

Lemma 2.4.10. *There exists a Borel measurable mapping $\zeta\colon \mathcal{M} \to \mathcal{Z}$ such that for every $(\Gamma, x, y) \in \mathcal{M}$, if $\Gamma = (K_1, \dots, K_n)$ and $\zeta(\Gamma, x, y) = (L_1, \dots, L_m)$, then $\{K_1, \dots, K_n\} = \{L_1, \dots, L_m\}$, $x \in L_1$, $y \in L_m$ and $L_{i-1} \cap L_i \neq \emptyset$, $i = 2, \dots, m$.*

Proof. For any given $(\Gamma, x, y) \in \mathcal{M}$ with $\Gamma = (K_1, \dots, K_n)$, the task of finding the corresponding finite sequence $(L_1, \dots, L_m) \in \mathcal{Z}$ is equivalent to the task of finding a finite sequence $(k_1, \dots, k_m)$ of natural numbers such that $x \in K_{k_1}$, $y \in K_{k_m}$, $\{k_1, \dots, k_m\} = \{1, \dots, n\}$ and $K_{k_{i-1}} \cap K_{k_i} \neq \emptyset$ for $i = 2, \dots, m$. Indeed, once these numbers $k_1, \dots, k_m$ are found, we can simply define $L_i := K_{k_i}$ for each $i = 1, \dots, m$. Therefore, the task of finding the Borel measurable mapping ζ is closely related to the task of finding a suitable Borel measurable mapping from $\mathcal{M}$ to a suitable space of finite sequences of natural numbers. Formally, we are going to represent finite sequences of natural number by infinite sequences of nonnegative integers whose terms are equal to zero from some point on. Define

$$T := \Big\{\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}_0{}^{\mathbb{N}}\,;\ \exists\, m, n \in \mathbb{N}:\ \{\alpha_1, \dots, \alpha_m\} = \{1, \dots, n\}\,, \\ \alpha_{m+1} = \alpha_{m+2} = \dots = 0\Big\}$$

and equip this set with the discrete topology. For every $\alpha \in T$, let $m(\alpha), n(\alpha) \in \mathbb{N}$ be the numbers witnessing that $\alpha \in T$. For every $\Gamma \in \mathcal{Z}$, let $l(\Gamma) \in \mathbb{N}$ be the number satisfying $\Gamma \in (\mathsf{C}(Q))^{l(\Gamma)}$. For every $(\Gamma, x, y) \in \mathcal{M}$, let

$$A(\Gamma, x, y) := \Big\{\alpha \in T\,;\ l(\Gamma) = n(\alpha),\ x \in \Gamma_{\alpha_1},\ y \in \Gamma_{\alpha_{m(\alpha)}}, \\ \Gamma_{\alpha_{i-1}} \cap \Gamma_{\alpha_i} \neq \emptyset \ \text{ for } i = 2, \dots, m(\alpha)\Big\}.$$

By Lemma 2.4.9, the set $A(\Gamma, x, y)$ is nonempty for each $(\Gamma, x, y) \in \mathcal{M}$. Also, it is easy to see that for every $\alpha \in T$, the set $\mathcal{M}_\alpha := \{(\Gamma, x, y) \in \mathcal{M}\,;\ \alpha \in A(\Gamma, x, y)\}$ is closed in $\mathcal{M}$. As T is countable, we can write $T = \{\alpha^1, \alpha^2, \alpha^3, \dots\}$. Define mappings $\mu\colon \mathcal{M} \to \mathbb{N}$ and $\zeta_1\colon \mathcal{M} \to T$ by

$$\mu(\Gamma, x, y) = \min\Big\{k \in \mathbb{N}\,;\ \alpha^k \in A(\Gamma, x, y)\Big\}, \quad \zeta_1(\Gamma, x, y) = \alpha^{\mu(\Gamma, x, y)}.$$

Let us show that μ is Borel measurable. Since $\mathbb{N}$ is a discrete space, we have to show that $\mu^{-1}(\{j\})$ is Borel in $\mathcal{M}$ for each $j \in \mathbb{N}$. Clearly,

$$\mu^{-1}(\{1\}) = \Big\{(\Gamma, x, y) \in \mathcal{M}\,;\ \alpha^1 \in A(\Gamma, x, y)\Big\} = \mathcal{M}_{\alpha^1}$$

and for every $j \in \mathbb{N}$, we have $\mu^{-1}(\{j+1\}) = \mathcal{M}_{\alpha^{j+1}} \setminus (\mathcal{M}_{\alpha^1} \cup \cdots \cup \mathcal{M}_{\alpha^j})$. Hence, μ is Borel measurable and it follows that ζ_1 is Borel measurable too.

Finally, denote $\mathcal{N} := \{(\Gamma, \alpha) \in \mathcal{Z} \times T\,;\ n(\alpha) \leq l(\Gamma)\}$ and define a mapping $\zeta_2 \colon \mathcal{N} \to \mathcal{Z}$ by $\zeta_2(\Gamma, \alpha) = (\Gamma_{\alpha_1}, \ldots, \Gamma_{\alpha_{m(\alpha)}})$. Clearly, ζ_2 is continuous. Define the desired mapping $\zeta \colon \mathcal{M} \to \mathcal{Z}$ by $\zeta(\Gamma, x, y) = \zeta_2(\Gamma, \zeta_1(\Gamma, x, y))$. Since both of the mappings ζ_1, ζ_2 are Borel measurable, so is ζ. It is straightforward to verify that ζ is the mapping we are after. □

For any topological space Y, denote by $\mathcal{F}(Y)$ the family of all nonempty closed subsets of Y. Recall that for any two topological spaces X and Y, a mapping $f \colon X \to \mathcal{F}(Y)$ is said to be *upper semi-continuous* if the set $\{x \in X\,;\ f(x) \subseteq V\}$ is open in X for every open set $V \subseteq Y$. The following theorem can be found in [Nad92, 7.4].

Theorem 2.4.11. *Let X and Y be nonempty compact metric spaces and let $f_n \colon X \to \mathcal{F}(Y)$, $n \in \mathbb{N}$, be upper semi-continuous mappings such that:*

- $\forall\, n \in \mathbb{N}\ \forall\, x \in X : f_{n+1}(x) \subseteq f_n(x)$;
- $\forall\, n \in \mathbb{N} : \bigcup\{f_n(x)\,;\ x \in X\} = Y$;
- $\forall\, x \in X : \lim_{n\to\infty} \operatorname{diam}(f_n(x)) = 0$.

Then the unique mapping $f \colon X \to Y$ satisfying

$$\{f(x)\} = \bigcap_{n\in\mathbb{N}} f_n(x)$$

for each $x \in X$ is surjective and continuous.

Denote by $\mathbb{N}^{<\mathbb{N}}$ the set of all (nonempty) finite sequences of natural numbers and equip $\mathbb{N}^{<\mathbb{N}}$ with the discrete topology. More formally, $\mathbb{N}^{<\mathbb{N}}$ is the topological sum

$$\bigoplus_{p\in\mathbb{N}} \mathbb{N}^p,$$

where $\mathbb{N}$ is considered as a discrete space.

Definition 2.4.12. Let $X \in \mathsf{C}(Q)$, $\varepsilon > 0$ and $\Gamma = (K_1, \ldots, K_k) \in \mathcal{Z}_P$. We say that Γ is a *weak ε-chain* covering X if the following three conditions hold:

(i) $K_1 \cup \cdots \cup K_k = X$;
(ii) $\operatorname{diam}_\varrho(K_i) < \varepsilon$ for $i = 1, \ldots, k$;
(iii) $K_{i-1} \cap K_i \neq \emptyset$ for $i = 2, \ldots, k$.

Definition 2.4.13. Let $\Gamma = (K_1, \ldots, K_k) \in \mathcal{Z}$, $\Gamma' = (L_1, \ldots, L_l) \in \mathcal{Z}$ and let $\gamma = (j_1, \ldots, j_m) \in \mathbb{N}^{<\mathbb{N}}$. We say that Γ' is a *refinement of* Γ *coded by* γ provided that $m = k$ and the following three assertions are satisfied:

(i) $1 = j_1 < j_2 < \cdots < j_k \leq l$;
(ii) $L_{j_k} \cup L_{j_k+1} \cup \cdots \cup L_l = K_k$;
(iii) $L_{j_{i-1}} \cup L_{j_{i-1}+1} \cup \cdots \cup L_{j_i-1} = K_{i-1}$ for $i = 2, \ldots, k$.

Lemma 2.4.14. *There exist Borel measurable mappings* $\mu_n\colon \mathsf{LC}(Q) \to \mathbb{N}^{<\mathbb{N}}$ *and* $\Psi_n\colon \mathsf{LC}(Q) \to \mathcal{Z}_P$, $n \in \mathbb{N}$, *such that for every* $X \in \mathsf{LC}(Q)$ *and every* $n \in \mathbb{N}$, *the following two assertions hold:*

(i) $\Psi_n(X)$ *is a weak* 2^{-n}*-chain covering* X.

(ii) *If* $n > 1$, *then* $\Psi_n(X)$ *is a refinement of* $\Psi_{n-1}(X)$ *coded by* $\mu_n(X)$.

Proof. Let $\Psi_P\colon \mathsf{LC}(Q) \times \mathbb{R}^+ \to \mathcal{Z}_P$ and $\zeta\colon \mathcal{M} \to \mathcal{Z}$ be the mappings provided by Theorem 2.4.8 and Lemma 2.4.10 respectively. Let $\sigma\colon \mathcal{K}(Q) \to Q$ be a Borel measurable mapping satisfying $\sigma(X) \in X$ for each $X \in \mathcal{K}(Q)$. It is well-known that such a mapping exists (it is a simple application of the Kuratowski and Ryll-Nardzewski selection theorem). Define $\Psi_1\colon \mathsf{LC}(Q) \to \mathcal{Z}_P$ by

$$\Psi_1(X) = \zeta\Big(\Psi_P(X, \tfrac{1}{2}), \sigma(X), \sigma(X)\Big)$$

and let $\mu_1\colon \mathsf{LC}(Q) \to \mathbb{N}^{<\mathbb{N}}$ be any Borel measurable mapping. It is easy to see that Ψ_1 is Borel measurable and that assertion (i) is satisfied for $n = 1$. Note that assertion (ii) does not say anything about $n = 1$. Let us proceed by induction. Assume that $n \in \mathbb{N}\backslash\{1\}$ is given and the mappings $\mu_1, \dots, \mu_{n-1}$ and $\Psi_1, \dots, \Psi_{n-1}$ have already been found. Let $X \in \mathsf{LC}(Q)$ be given, we are going to define $\Psi_n(X)$ and $\mu_n(X)$ in the following way: Assume that $\Psi_{n-1}(X) = (K_1, \dots, K_k)$. If $k = 1$, then (since $K_1 \cup \dots \cup K_k = X$) we have $K_1 = X$ and we can simply define

$$\mu_n(X) = (1)\,, \;\; \Psi_n(X) = \zeta\Big(\Psi_P(X, 2^{-n}), \sigma(X), \sigma(X)\Big).$$

Let us focus on the case when $k > 1$. Assume that

$$\zeta\Big(\Psi_P(K_1, 2^{-n}), \sigma(K_1), \sigma(K_1 \cap K_2)\Big) = (L^1_1, \dots, L^1_{l(1)}),$$
$$\zeta\Big(\Psi_P(K_k, 2^{-n}), \sigma(K_{k-1} \cap K_k), \sigma(K_k)\Big) = (L^k_1, \dots, L^k_{l(k)})$$

and

$$\zeta\Big(\Psi_P(K_i, 2^{-n}), \sigma(K_{i-1} \cap K_i), \sigma(K_i \cap K_{i+1})\Big) = (L^i_1, \dots, L^i_{l(i)})$$

for every $i \in \mathbb{N}$ with $1 < i < k$. Then we define

$$\Psi_n(X) = \Big(L^1_1, \dots, L^1_{l(1)}, L^2_1, \dots L^2_{l(2)}, \dots, L^k_1, \dots, L^k_{l(k)}\Big),$$
$$\mu_n(X) = \Big(1, 1 + l(1), \dots, 1 + l(1) + \dots + l(k-1)\Big) = \Big(1 + \sum_{j=1}^{i-1} l(j)\Big)_{i=1}^{k}.$$

It is straightforward (although quite tedious) to verify that Ψ_n and μ_n are Borel measurable (among other things, Lemma 2.2.3 is used here) and that both of the assertions (i) and (ii) are satisfied for n. □

Theorem 2.4.15. *There is a Borel measurable mapping* $\Phi\colon \mathsf{LC}(Q) \to C(I, Q)$ *such that for every* $X \in \mathsf{LC}(Q)$, *the image of* I *under* $\Phi(X)$ *is equal to* X.

Proof. Let $\mu_n\colon \mathsf{LC}(Q) \to \mathbb{N}^{<\mathbb{N}}$ and $\Psi_n\colon \mathsf{LC}(Q) \to \mathcal{Z}_P$, $n \in \mathbb{N}$, be the Borel measurable mappings given by Lemma 2.4.14. For every $\Gamma \in \mathcal{Z}$, let $l(\Gamma) \in \mathbb{N}$ be the number satisfying $\Gamma \in (\mathsf{C}(Q))^{l(\Gamma)}$. Let us define a function $\lambda\colon \mathsf{LC}(Q) \times \mathbb{N} \to \mathbb{N}$ by $\lambda(X, n) = l(\Psi_n(X))$. Clearly, λ is Borel measurable. For every $n \in \mathbb{N} \setminus \{1\}$, we define a mapping $\nu_n\colon \mathsf{LC}(Q) \to \mathbb{N}^{<\mathbb{N}}$ such that for every $X \in \mathsf{LC}(Q)$, roughly speaking, the i-th element of $\nu_n(X)$ is the length of the subsequence of $\Psi_n(X)$ corresponding to the i-th member of $\Psi_{n-1}(X)$. Formally, if $\mu_n(X) = (j_1, \ldots, j_m)$, we define $\nu_n(X)$ as follows: If $m = 1$, let $\nu_n(X) = (\lambda(X, n))$. If $m = 2$, define

$$\nu_n(X) = \Big(j_2 - 1,\ \lambda(X,n) + 1 - j_2\Big) = \Big(j_2 - j_1,\ \lambda(X,n) + 1 - j_m\Big).$$

Finally, if $m \geq 3$, let

$$\nu_n(X) = \Big(j_2 - j_1,\ j_3 - j_2,\ \ldots, j_m - j_{m-1},\ \lambda(X,n) + 1 - j_m\Big).$$

For each $n \in \mathbb{N} \setminus \{1\}$, since μ_n is Borel measurable, so is ν_n. Note that for every $n \in \mathbb{N} \setminus \{1\}$ and $X \in \mathsf{LC}(Q)$, if $\nu_n(X) = (i_1, \ldots, i_k)$ and $\mu_n(X) = (j_1, \ldots, j_m)$, then $k = m = \lambda(X, n-1)$ and $i_1 + \cdots + i_k = \lambda(X, n)$.

Let $W := \{(a, b, k) \in I \times I \times \mathbb{N};\ a < b\}$, $T := \bigoplus_{p \in \mathbb{N}} I^p$ and define a mapping $\phi\colon W \to T$ by

$$\phi(a, b, k) = \left(a + \frac{i}{k}(b - a)\right)_{i=0}^{k}.$$

In other words, for every $(a, b, k) \in W$, if $\phi(a, b, k) = (t_0, \ldots, t_m)$, then $m = k$, $a = t_0 < t_1 < \cdots < t_m = b$ and $t_i - t_{i-1} = (b-a)/k$ for $i = 1, \ldots, m$. Clearly, ϕ is continuous. Hence, ϕ is Borel measurable and so is the mapping $\tau_1\colon \mathsf{LC}(Q) \to T$ given by $\tau_1(X) = \phi(0, 1, \lambda(X, 1))$. Define a mapping $\tau_2\colon \mathsf{LC}(Q) \to T$ as follows: Let $X \in \mathsf{LC}(Q)$ be given and denote $m := \lambda(X, 1)$. If $\tau_1(X) = (t_0, \ldots, t_m)$, $\nu_2(X) = (k_1, \ldots, k_m)$ and $\phi(t_{i-1}, t_i, k_i) = (s^i_0, s^i_1, \ldots, s^i_{k_i})$ for every $i = 1, \ldots, m$, then we define

$$\tau_2(X) = \Big(0, s^1_1, \ldots, s^1_{k_1}, s^2_1, \ldots, s^2_{k_2}, \ldots, s^m_1, \ldots, s^m_{k_m}\Big).$$

Note that $s^m_{k_m} = t_m = 1$ and $s^i_0 = t_{i-1} = s^{i-1}_{k_{i-1}}$ for $i = 2, \ldots, m$. Moreover, if we relabel the elements of $\tau_2(X)$ so that $\tau_2(X) = (s_0, \ldots, s_l)$, then $l = k_1 + \cdots + k_m = \lambda(X, 2)$.

If we keep repeating the process used to construct τ_2, we obtain Borel measurable mappings $\tau_n\colon \mathsf{LC}(Q) \to T$, $n \in \mathbb{N}$, such that for every $X \in \mathsf{LC}(Q)$ and $n \in \mathbb{N}$, if $\tau_n(X) = (t_0, \ldots, t_m)$, then $m = \lambda(X, n)$ and

$$\tau_{n+1}(X) = \Big(0, s^1_1, \ldots, s^1_{k_1}, s^2_1, \ldots, s^2_{k_2}, \ldots, s^m_1, \ldots, s^m_{k_m}\Big),$$

where $(k_1, \ldots, k_m) = \nu_{n+1}(X)$ and $(s^i_0, s^i_1, \ldots, s^i_{k_i}) = \phi(t_{i-1}, t_i, k_i)$, $i = 1, \ldots, m$.

For every $n \in \mathbb{N}$, define a mapping $\Phi_n\colon \mathsf{LC}(Q) \to \mathcal{K}(I \times Q)$ as follows: For any $X \in \mathsf{LC}(Q)$, if $\Psi_n(X) = (K_1, \ldots, K_m)$ and $\tau_n(X) = (t_0, \ldots, t_m)$, let

$$\Phi_n(X) = \bigcup_{i=1}^{m} \Big([t_{i-1}, t_i] \times K_i\Big).$$

It is easy to see that Φ_n is Borel measurable for each $n \in \mathbb{N}$. For every $X \in \mathsf{LC}(Q)$ and $n \in \mathbb{N}$, define a mapping $\psi^X_n\colon I \to \mathcal{F}(X)$ by

$$\psi^X_n(t) = (\Phi_n(X))_t = \{x \in Q;\ (t, x) \in \Phi_n(X)\} = \{x \in X;\ (t, x) \in \Phi_n(X)\}.$$

Clearly, if $\Psi_n(X) = (K_1, \dots, K_m)$ and $\tau_n(X) = (t_0, \dots, t_m)$, then

$$\psi_n^X(t) = \begin{cases} K_1 & \text{if } t = 0 \\ K_m & \text{if } t = 1 \\ K_i & \text{if } t_{i-1} < t < t_i,\ i = 1, \dots, m \\ K_i \cup K_{i+1} & \text{if } t = t_i,\ i = 1, \dots, m-1. \end{cases}$$

For every $X \in \mathsf{LC}(Q)$, it is fairly straightforward to verify that the mappings $\psi_1^X, \psi_2^X, \psi_3^X, \dots$ satisfy the assumptions of Theorem 2.4.11. Hence, the mapping $\Phi\colon \mathsf{LC}(Q) \to C(I, Q)$ given by

$$\{\Phi(X)(t)\} = \bigcap_{n \in \mathbb{N}} \psi_n^X(t)$$

is well-defined and, for every $X \in \mathsf{LC}(Q)$, the image of I under $\Phi(X)$ is equal to X. It remains to show that Φ is Borel measurable. By Lemma 2.2.6, it suffices to prove that the mapping $\widehat{\Phi}\colon \mathsf{LC}(Q) \to \mathcal{K}(I \times Q)$ given by

$$\widehat{\Phi}(X) = \operatorname{graph}(\Phi(X))$$

is Borel measurable. However, we clearly have

$$\widehat{\Phi}(X) = \bigcap_{n \in \mathbb{N}} \Phi_n(X)$$

for every $X \in \mathsf{LC}(Q)$. Thus, by Lemma 2.2.4, $\widehat{\Phi}$ is Borel measurable. $\square$

Corollary 2.4.16. *There is a Borel measurable bijection $\phi : C(I, Q) \to \mathsf{LC}(Q)$ such that $\phi(f)$ is homeomorphic to the image of f for every $f \in C(I, Q)$.*

Proof. We can proceed similarly as Gao in [Gao09, 326-327]. Let $\Phi\colon \mathsf{LC}(Q) \to C(I, Q)$ be the mapping provided by Theorem 2.4.15. It is clear that Φ is a Borel measurable injection. On the other hand, we can construct a Borel measurable injection $\chi : C(I, Q) \to \mathsf{LC}(Q)$ such that $\chi(f)$ homeomorphic to $f(I)$ for every $f \in C(I, Q)$. That can be done the following way: By [Kec95, Theorem 15.6], there is a Borel measurable bijection $\theta : C(I, Q) \to I$, hence, identifying $\mathsf{LC}(Q)$ with $\mathsf{LC}(I \times Q)$, we can define χ by $\chi(f) = \{\theta(f)\} \times f(I)$ for every $f \in C(I, Q)$.

A standard Cantor Bernstein argument applied to Φ and χ gives us the desired Borel measurable bijection $\phi : C(I, Q) \to \mathsf{LC}(Q)$. $\square$

2.5 Peano continua are arcwise connected in a Borel measurable way

It is well-known that Peano continua are arcwise connected. In this section we prove that an arc connecting two points in a Peano continuum can be chosen in a Borel measurable way.

Lemma 2.5.1. *Let Y be a compact metrizable space and $\varphi\colon Y \to \mathbb{R}$ a continuous function. Then there exists a Borel measurable mapping $\nu\colon \mathcal{K}(Y) \to Y$ such that $\nu(K) \in K$ and $\varphi(\nu(K)) = \min\{\varphi(y)\,;\, y \in K\}$ for every $K \in \mathcal{K}(Y)$.*

Proof. Let $\mathcal{D} := \{(K,t) \in \mathcal{K}(Y) \times \mathbb{R}\,;\ \exists y \in K : \varphi(y) \le t\}$ and define a mapping $\xi\colon \mathcal{D} \to \mathcal{K}(Y)$ by $\xi(K,t) = \{y \in K\,;\ \varphi(y) \le t\}$. Fix $s \in \mathbb{R}$ such that $\varphi(y) > s$ for every $y \in Y$. Then we have $\xi(K,t) = K \cap \varphi^{-1}([s,t])$ for every $(K,t) \in \mathcal{D}$. By Lemmata 2.2.3 and 2.2.7, it easily follows that ξ is Borel measurable. Moreover, the function $\mu\colon \mathcal{K}(Y) \to \mathbb{R}$ given by $\mu(K) = \min\{\varphi(y);\ y \in K\}$ is continuous (this is a very easy exercise). Let $\sigma\colon \mathcal{K}(Y) \to Y$ be a Borel measurable mapping satisfying $\sigma(L) \in L$ for every $L \in \mathcal{K}(Y)$. Then we can define the desired mapping by $\nu(K) = \sigma(\xi(K, \mu(K)))$. □

By [Nad92, 4.33], for every compact metrizable space Z, there is a continuous function $\varphi\colon \mathcal{K}(Z) \to \mathbb{R}$ such that $\varphi(\{z\}) = 0$ for every $z \in Z$ and $\varphi(K) < \varphi(L)$ for all $K, L \in \mathcal{K}(Z)$ with $K \subsetneq L$. Such a function φ is called a *Whitney map*.

Lemma 2.5.2. *There exists a Borel measurable mapping $\Upsilon\colon \mathcal{K}(\mathcal{K}(I)) \to \mathcal{K}(I)$ such that for every $\mathcal{L} \in \mathcal{K}(\mathcal{K}(I))$, the set $\Upsilon(\mathcal{L})$ is a minimal element of the family $\mathcal{L}$ with respect to set inclusion.*

Proof. Let $\varphi\colon \mathcal{K}(I) \to \mathbb{R}$ be a Whitney map. It follows from Lemma 2.5.1 that there exists a Borel measurable mapping $\Upsilon\colon \mathcal{K}(\mathcal{K}(I)) \to \mathcal{K}(I)$ such that $\Upsilon(\mathcal{L}) \in \mathcal{L}$ and $\varphi(\Upsilon(\mathcal{L})) = \min\{\varphi(K)\,;\ K \in \mathcal{L}\}$ for all $\mathcal{L} \in \mathcal{K}(\mathcal{K}(I))$. Then, as φ is a Whitney map, it is easy to see that Υ is the desired mapping. □

Lemma 2.5.3. *For every continuous mapping $f\colon I \to Q$, the set*

$$\Lambda(f) := \Big\{K \in \mathcal{K}(I)\,;\ f(\min K) = f(0)\,,\ f(\max K) = f(1)\Big\}$$

is nonempty and compact. Moreover, the mapping $\Lambda\colon C(I,Q) \to \mathcal{K}(\mathcal{K}(I))$ defined by $f \mapsto \Lambda(f)$ is Borel measurable.

Proof. For every $f \in C(I,Q)$, the set $\Lambda(f)$ is nonempty since it contains I. Let

$$\mathcal{F} := \Big\{(f,K) \in C(I,Q) \times \mathcal{K}(I)\,;\ f(\min K) = f(0)\,,\ f(\max K) = f(1)\Big\}.$$

Claim 2.5.3.1. The set $\mathcal{F}$ is closed in $C(I,Q) \times \mathcal{K}(I)$.

Proof. Define functions $M_1\colon \mathcal{K}(I) \to I$, $M_2\colon \mathcal{K}(I) \to I$ by $M_1(K) = \min K$ and $M_2(K) = \max K$. Define mappings $\Psi_1, \Psi_2, \Psi_3, \Psi_4\colon C(I,Q) \times \mathcal{K}(I) \to Q$ by

$$\begin{aligned}
\Psi_1(f,K) &= f(M_1(K)),\\
\Psi_2(f,K) &= f(M_2(K)),\\
\Psi_3(f,K) &= f(0),\\
\Psi_4(f,K) &= f(1).
\end{aligned}$$

Trivially, Ψ_3 and Ψ_4 are continuous. Moreover, since M_1 and M_2 are continuous, it is easy to see that so are Ψ_1 and Ψ_2. Therefore, the set

$$\Big\{(f,K) \in C(I,Q) \times \mathcal{K}(I)\,;\ \Psi_1(f,K) = \Psi_3(f,K)\,,\ \Psi_2(f,K) = \Psi_4(f,K)\Big\}$$

is closed. Clearly, this set is equal to $\mathcal{F}$. ■

Since $\mathcal{F}$ is closed and $\mathcal{K}(I)$ is compact, the vertical sections of $\mathcal{F}$ are compact. In other words, $\Lambda(f)$ is compact for every $f \in C(I,Q)$. Moreover, it follows from Lemma 2.2.8 that Λ is Borel measurable. □

For every $K \in \mathcal{K}(I)$, let

$$\begin{aligned}\alpha(K) : &= \Big\{(s,t) \in I \times I\,;\ s < t \text{ and } [s,t] \cap K = \{s,t\}\Big\} \\ &= \Big\{(s,t) \in K \times K\,;\ s < t \text{ and } \forall\, u \in I : s < u < t \implies u \notin K\Big\}.\end{aligned}$$

Lemma 2.5.4. *For every continuous mapping $f\colon I \to Q$, the set*

$$\Gamma(f) := \Big\{K \in \mathcal{K}(I)\,;\ \forall\,(s,t) \in \alpha(K) : f(s) = f(t)\Big\}$$

is nonempty and compact. Moreover, the mapping $\Gamma\colon C(I,Q) \to \mathcal{K}(\mathcal{K}(I))$ defined by $f \mapsto \Gamma(f)$ is Borel measurable.

Proof. Clearly, $\alpha(I) = \emptyset$. Hence, for every $f \in C(I,Q)$, the set $\Gamma(f)$ is nonempty as it contains I. Let

$$\mathcal{F} := \Big\{(f,K) \in C(I,Q) \times \mathcal{K}(I)\,;\ \forall\,(s,t) \in \alpha(K) : f(s) = f(t)\Big\}.$$

Claim 2.5.4.1. The set $\mathcal{F}$ is closed in $C(I,Q) \times \mathcal{K}(I)$.

Proof. Let us prove that the set $\mathcal{G} := (C(I,Q) \times \mathcal{K}(I)) \setminus \mathcal{F}$ is open. Given any $(f,K) \in \mathcal{G}$, there is a pair $(s,t) \in \alpha(K)$ such that $f(s) \neq f(t)$. Let U and V be disjoint open subsets of Q satisfying $f(s) \in U$ and $f(t) \in V$. By the continuity of f, there is $\delta > 0$ such that $f(u) \in U$ for every $u \in I$ with $|u - s| \leq \delta$ and $f(v) \in V$ for every $v \in I$ satisfying $|v - t| \leq \delta$. Since $s < t$, we can assume that $s + \delta < t - \delta$. Let $J_1 := I \cap [s-\delta, s+\delta]$ and $J_2 := I \cap [t-\delta, t+\delta]$. Then the set $\mathcal{U} := \{g \in C(I,Q)\,;\ g(J_1) \subseteq U,\ g(J_2) \subseteq V\}$ contains f and it is clear (by the compactness of J_1 and J_2) that $\mathcal{U}$ is open in $C(I,Q)$. Furthermore, the set

$$\begin{aligned}\mathcal{V} := \Big\{L \in \mathcal{K}(I)\,;\ & L \subseteq [0, s+\delta) \cup (t-\delta, 1], \\ & L \cap (s-\delta, s+\delta) \neq \emptyset,\ L \cap (t-\delta, t+\delta) \neq \emptyset\Big\}\end{aligned}$$

is open in $\mathcal{K}(I)$ and it contains K. Therefore, $\mathcal{U} \times \mathcal{V}$ is a neighbourhood of (f,K). It remains to verify that $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{G}$. Let $(g,L) \in \mathcal{U} \times \mathcal{V}$ be given and let

$$u := \max\Big(L \cap [0, s+\delta]\Big),\ v := \min\Big(L \cap [t-\delta, 1]\Big).$$

It is easy to see that $(u,v) \in \alpha(L)$. Also, as $u \in J_1$ and $v \in J_2$, we have $g(u) \in U$ and $g(v) \in V$. In particular, $g(u) \neq g(v)$, which proves that $(g,L) \in \mathcal{G}$. ■

Since $\mathcal{F}$ is closed and $\mathcal{K}(I)$ is compact, the vertical sections of $\mathcal{F}$ are compact. In other words, $\Gamma(f)$ is compact for every $f \in C(I,Q)$. Moreover, it follows from Lemma 2.2.8 that Γ is Borel measurable. □

Lemma 2.5.5. *Let $\mathcal{B} := \{(X,x,y) \in \mathsf{LC}(Q) \times Q \times Q\,; x,y \in X\}$. There exists a Borel measurable mapping $T\colon \mathcal{B} \to C(I,Q)$ such that for every $(X,x,y) \in \mathcal{B}$, if $T(X,x,y) = h$, then $h(0) = x$, $h(1) = y$ and $h(I) = X$.*

Proof. For any $f \in C(I,Q)$ and any two points $a,b \in I$, let us denote by $f_{a,b}$ the mapping from I to Q given by

$$f_{a,b}(t) = \begin{cases} f(a - 3at) & \text{if } 0 \leq t < \frac{1}{3} \\ f(3t-1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ f(3bt - 3t - 2b + 3) & \text{if } \frac{2}{3} < t \leq 1. \end{cases}$$

It is easy to see that $f_{a,b}$ is well-defined and continuous. Moreover, $f_{a,b}(0) = f(a)$, $f_{a,b}(1) = f(b)$ and $f_{a,b}(I) = f(I)$. Define a mapping $\phi\colon C(I,Q) \times I \times I \to C(I,Q)$ by $\phi(f,a,b) = f_{a,b}$. It is rather straightforward to prove that ϕ is continuous. Let $\mathcal{A} := \{(f,x) \in C(I,Q) \times Q\,;\ x \in f(I)\}$ and define a mapping $\gamma\colon \mathcal{A} \to \mathcal{K}(I)$ by $\gamma(f,x) = f^{-1}(\{x\})$. By Lemma 2.2.7, γ is Borel measurable. Let $\sigma\colon \mathcal{K}(I) \to I$ be a Borel measurable mapping such that $\sigma(K) \in K$ for every $K \in \mathcal{K}(I)$. Then $\tau := \sigma \circ \gamma$ is a Borel measurable mapping and it satisfies $f(\tau(f,x)) = x$ for every $(f,x) \in \mathcal{A}$. Let Φ be the mapping from Theorem 2.4.15. Finally, we can define the desired mapping $T\colon \mathcal{B} \to C(I,Q)$ by

$$T(X,x,y) = \phi\Big(\Phi(X), \tau(\Phi(X),x), \tau(\Phi(X),y)\Big).$$

It is easy to verify that all the requirements we put on T are met. □

Recall that a mapping $f\colon X \to Y$, where X and Y are topological spaces, is said to be *monotone* if f is continuous and $f^{-1}(\{y\})$ is connected for every $y \in f(X)$. The following lemma is an immediate consequence of [Nad92, 8.22].

Lemma 2.5.6. *If $g\colon I \to Q$ is a monotone mapping such that $g(0) \neq g(1)$, then $g(I)$ is an arc.*

Theorem 2.5.7. *Let $\mathcal{D} := \{(X,x,y) \in \mathsf{LC}(Q) \times Q \times Q\,;\ x,y \in X,\ x \neq y\}$. There exists a Borel measurable mapping $A\colon \mathcal{D} \to \mathcal{K}(Q)$ such that for every $(X,x,y) \in \mathcal{D}$, the set $A(X,x,y)$ is an arc in X with endpoints x and y.*

Proof. Define $\mathcal{B} := \{(X,x,y) \in \mathsf{LC}(Q) \times Q \times Q\,;\ x,y \in X\}$ and consider the mappings $\Lambda\colon C(I,Q) \to \mathcal{K}(\mathcal{K}(I))$, $\Gamma\colon C(I,Q) \to \mathcal{K}(\mathcal{K}(I))$ and $T\colon \mathcal{B} \to C(I,Q)$ from Lemmata 2.5.3, 2.5.4 and 2.5.5, respectively. Clearly, for every $f \in C(I,Q)$, we have $I \in \Lambda(f) \cap \Gamma(f)$ and, therefore, $\Lambda(f) \cap \Gamma(f) \neq \emptyset$. Hence, we can define a mapping $\Omega\colon C(I,Q) \to \mathcal{K}(\mathcal{K}(I))$ by $\Omega(f) = \Lambda(f) \cap \Gamma(f)$. By Lemma 2.2.4, Ω is Borel measurable. Let $\Upsilon\colon \mathcal{K}(\mathcal{K}(I)) \to \mathcal{K}(I)$ be the mapping provided by Lemma 2.5.2. Finally, by Lemma 2.2.5, the mapping $\Delta\colon C(I,Q) \times \mathcal{K}(I) \to \mathcal{K}(Q)$ defined by $\Delta(f,K) = f(K)$ is continuous. Define the desired mapping $A\colon \mathcal{D} \to \mathcal{K}(Q)$ by

$$A(X,x,y) = \Delta\Big(T(X,x,y),\ (\Upsilon \circ \Omega \circ T)(X,x,y)\Big).$$

Clearly, A is Borel measurable. It remains to show that $A(X,x,y)$ is an arc in X with endpoints x,y, whenever $(X,x,y) \in \mathcal{D}$. At this point, we could simply refer the reader to the corresponding part of the proof of [Nad92, 8.23], but, for the sake of completeness, let us present the proof here. Let $(X,x,y) \in \mathcal{D}$ be arbitrary and denote $f := T(X,x,y)$, $\mathcal{L} := \Omega(f)$ and $K := \Upsilon(\mathcal{L})$. Then $f(0) = x$, $f(1) = y$, $f(I) = X$, $A(X,x,y) = \Delta(f,K) = f(K)$, the family $\mathcal{L}$ is equal to

$$\Big\{L \in \mathcal{K}(I)\,;\ \forall\,(s,t) \in \alpha(L) : f(s) = f(t)\,,\ f(\min L) = f(0)\,,\ f(\max L) = f(1)\Big\}$$

and $K \in \mathcal{L}$ is a minimal element of $\mathcal{L}$ with respect to set inclusion.

Claim 2.5.7.1. For any two points $a,b \in K$ such that $a \leq b$ and $f(a) = f(b)$, we have $K \cap [a,b] = \{a,b\}$.

Proof. Let $a, b \in K$ be arbitrary and assume that $a < b$ and $f(a) = f(b)$. Letting $K_0 := K \setminus (a, b)$, we have $K_0 \cap [a, b] = \{a, b\}$. Therefore, it suffices to show that $K_0 = K$. Clearly, K_0 is a nonempty compact subset of I. Also, $\min K_0 = \min K$, $\max K_0 = \max K$ and hence (since $K \in \mathcal{L}$) $f(\min K_0) = f(0)$, $f(\max K_0) = f(1)$. Given any $(s, t) \in \alpha(K_0)$, we are going to show that $f(s) = f(t)$. This equality is obvious if $(s, t) = (a, b)$. Thus, assume $(s, t) \neq (a, b)$. Then, as $[a, b] \cap K_0 = \{a, b\}$, it follows that either $s < t \leq a$, or $b \leq s < t$. In both cases it immediately follows from the definition of K_0 that $(s, t) \in \alpha(K)$ and, therefore, $f(s) = f(t)$. We have just shown that $K_0 \in \mathcal{L}$. Hence, since $K_0 \subseteq K$ and K is a minimal element of $\mathcal{L}$ with respect to set inclusion, it follows that $K_0 = K$. ■

Note that for every $u \in [\min K, \max K] \setminus K$, there is exactly one pair $(s_u, t_u) \in \alpha(K)$ such that $s_u < u < t_u$. Define a mapping $g \colon I \to Q$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \in K \\ f(0) & \text{if } 0 \leq u < \min K \\ f(1) & \text{if } \max K < u \leq 1 \\ f(s_u) & \text{if } u \in [\min K, \max K] \setminus K. \end{cases}$$

It is fairly easy to prove that g is continuous. Moreover, since we can clearly see that $g(I) = g(K) = f(K) = A(X, x, y)$ and $f(K) \subseteq f(I) = X$, it suffices to show that $g(I)$ is an arc with endpoints x, y.

Claim 2.5.7.2. The mapping g is monotone.

Proof. Since g is continuous, we just have to show that $g^{-1}(\{z\})$ is connected for every $z \in g(I)$. Let $z \in g(I)$ be arbitrary and denote $M := K \cap g^{-1}(\{z\})$. Since $g(I) = g(K)$ and g is continuous, M is nonempty and compact. Let $a := \min M$, $b := \max M$. Clearly, $f(a) = g(a) = z = g(b) = f(b)$. Hence, by Claim 2.5.7.1, we have $K \cap [a, b] = \{a, b\}$. Thus, by the definition of g,

$$[\min K, \max K] \cap g^{-1}(\{z\}) = [a, b].$$

It easily follows that $g^{-1}(\{z\})$ is equal to one of the intervals $[a, b]$, $[0, b]$, $[a, 1]$. ■

Since g is monotone and $g(0) = f(0) = x \neq y = f(1) = g(1)$, it follows from Lemma 2.5.6 that $g(I)$ is an arc. To show that x and y are the endpoints of the arc $g(I)$, let $a := \max(g^{-1}(\{x\}))$ and $b := \min(g^{-1}(\{y\}))$. Then $g^{-1}(\{x\}) = [0, a]$, $g^{-1}(\{y\}) = [b, 1]$ and $a < b$, which proves that $g((a, b)) = g(I) \setminus \{x, y\}$. Therefore, $g(I) \setminus \{x, y\}$ is a connected set. This clearly implies that x and y are the endpoints of the arc $g(I)$. □

3. The complexity of homeomorphism relations on some classes of compacta with bounded topological dimension

Abstract: We are dealing with the complexity of the homeomorphism equivalence relation on some classes of metrizable compacta from the viewpoint of invariant descriptive set theory. We prove that the homeomorphism equivalence relation for absolute retracts in the plane is Borel bireducible with the isomorphism equivalence relation for countable graphs. In order to stress the sharpness of this result, we prove that neither the homeomorphism relation for locally connected continua in the plane nor the homeomorphism relation for absolute retracts in $\mathbb{R}^3$ is Borel reducible to the isomorphism relation for countable graphs. We also improve recent results of Chang and Gao by constructing a Borel reduction from both the homeomorphism relation for compact subsets of $\mathbb{R}^n$ and the ambient homeomorphism relation for compact subsets of $[0,1]^n$ to the homeomorphism relation for n-dimensional continua in $[0,1]^{n+1}$.

3.1 Introduction

The task of measuring the complexity of an equivalence relation on a structure is very complex in itself. In this chapter we use the notion of Borel reducibility (see Definition 3.2.1) and the results of invariant descriptive set theory to compare the complexities of equivalence relations (abbreviated here to ERs) on Polish and standard Borel spaces. For getting familiar with the field of invariant descriptive set theory we recommend the book by Su Gao [Gao09].

Let us mention several ERs that have become milestones in the theory of Borel reductions (in ascending order with respect to their complexity):

- the equality on an uncountable Polish space (equivalently, on $\mathbb{R}$);
- the S_∞-universal orbit equivalence relation;
- the universal orbit equivalence relation;
- the universal analytic equivalence relation.

Let us present a few examples of "real-life" ERs, each with a complexity corresponding to one the four relations above. It was shown by Gromov that the isometry ER for compact metric spaces is Borel bireducible with the equality of real numbers (see e.g. [Gao09, Theorem 14.2.1]). The isomorphism ER for countable graphs is Borel bireducible with the S_∞-universal orbit ER (see [Gao09, Theorem 13.1.2]). Melleray [Mel07] proved that the isometry ER for separable Banach spaces is Borel bireducible with the universal orbit ER. Ferenczi, Louveau and Rosendal [FLR09] proved that the isomorphism ER for separable Banach spaces is Borel bireducible with the universal analytic ER.

In this chapter we study the homeomorphism ER on some subclasses of metrizable compacta. A crucial result by Zielinski (see [Zie16]) states that the homeomorphism ER for metrizable compacta is Borel bireducible with the universal orbit ER. Chang and Gao [CG17] have shown that the same applies to the homeomorphism ER for metrizable continua. This remains true even if we restrict ourselves to locally connected continua, as was proved by Cieśla [Cie19]. Finally, Krupski and Vejnar [KV20] proved that the homeomorphism ER for absolute retracts has the same complexity as the homeomorphism ER for metrizable compacta.

Now let us turn our attention to homeomorphism ERs which are less complex than the universal orbit ER. The homeomorphism ER for compacta in $\mathbb{R}$ is Borel bireducible with the S_∞-universal orbit ER (see [CG19, Theorem 4.2]). On the other hand, it is known (it follows e.g. from [CG19, Theorem 4.3]) that for every $n > 1$ the homeomorphism ER for compacta in $\mathbb{R}^n$ is strictly more complex, but it is not known whether it is as complex as the universal orbit ER. The homeomorphism ER for metrizable rim-finite continua was shown in [KV20] to be Borel bireducible with the S_∞-universal orbit ER. However, the homeomorphism ER for metrizable rim-finite compacta is strictly more complex, as was shown in the same paper. The homeomorphism ER for dendrites was shown in [CDM05] to be Borel bireducible with the S_∞-universal orbit ER.

Concerning the finite-dimensional homeomorphism classification problems, Chang and Gao proved in [CG19] that both the homeomorphism ER and the ambient homeomorphism ER for compacta in $[0,1]^n$ are Borel reducible to the homeomorphism ER for continua in $[0,1]^{n+2}$ for every $n \in \mathbb{N}$. They also showed that the ambient homeomorphism ER for compacta in $[0,1]^n$ is Borel reducible to the ambient homeomorphism ER for compacta in $[0,1]^{n+1}$ for every $n \in \mathbb{N}$ and it is strictly more complex than the S_∞-universal orbit ER when $n > 1$.

The main results of our investigation follow:

[A] The homeomorphism ER for absolute retracts in $\mathbb{R}^2$ is Borel bireducible with the isomorphism ER for countable graphs (Theorem 3.3.3).

[B] The homeomorphism ER for 2-dimensional absolute retracts in $\mathbb{R}^3$ is not classifiable by countable structures (Corollary 3.4.2).

[C] The homeomorphism ER for locally connected 1-dimensional continua in the plane is not classifiable by countable structures (Corollary 3.4.5).

[D] The homeomorphism ER for compacta in $\mathbb{R}^n$ is Borel reducible to the homeomorphism ER for n-dimensional continua in $\mathbb{R}^{n+1}$ (Theorem 3.3.10).

[E] The ambient homeomorphism ER for compacta in $[0,1]^n$ is Borel reducible to the homeomorphism ER for n-dimensional continua in $\mathbb{R}^{n+1}$ (Theorem 3.3.7).

Claim [A] is a generalization of [CDM05, Theorem 6.7]. Results [B] and [C] witness the sharpness of [A]. Claims [D] and [E] strengthen [CG19, Theorem 1].

It is worth noting that the Gelfand duality establishes a direct relationship between the complexity of compacta up to homeomorphism and the complexity of commutative unital C^*-algebras up to isomorphism. It is often the case that there are natural non-commutative versions of compacta (e.g. in the case of absolute retracts or dendrites [CD10]). Hence, complexity results in topology can serve as conjectures for complexity results on C^*-algebras and vice versa (compare e.g. [Sab16] and [Zie16]). However, the most intriguing question concerns the complexity of compacta with bounded dimension.

Question 3.1.1. What is the exact complexity of the homeomorphism ER for n-dimensional compacta for $n \geq 1$?

3.2 Preliminaries

In this section we introduce the notation, terminology, definitions and basic facts which will be used throughout this chapter. By a natural number we mean a strictly positive integer. We denote the set of all natural numbers by $\mathbb{N}$. If X is a topological space and A is a subset of X, we denote by $\operatorname{Int}(A)$, $\overline{A}$ and ∂A the interior, the closure and the boundary of A, respectively. Recall that a *standard Borel space* is a measurable space $(X, \mathcal{A})$ for which there exists a Polish topology τ on X such that the family of all Borel subsets of (X, τ) is equal to $\mathcal{A}$. By [Kec95, Theorem 13.1], for any Polish space X and any Borel set $B \subseteq X$, the measurable space $(B, \{A \subseteq B\,;\ A \text{ is Borel in } X\})$ is a standard Borel space. Hence, a Borel subset of a Polish space can be naturally viewed as a standard Borel space with the Borel structure inherited from the Polish space.

In order to compare the complexities of equivalence relations (ERs) on standard Borel spaces we use the notion of Borel reducibility.

Definition 3.2.1. Let E and F be ERs on sets X and Y, respectively.

(1) A mapping $f\colon X \to Y$ is called a *reduction* from E to F if for every two points $x, x' \in X$ we have $xEx' \iff f(x)Ff(x')$.
(2) If X, Y are Polish spaces and there is a continuous reduction from E to F, we say that E is *continuously reducible* to F and write $E{\leq_c}F$. We say that E is *continuously bireducible* with F if $E{\leq_c}F$ and $F{\leq_c}E$.
(3) If X, Y are standard Borel spaces and there is a Borel measurable reduction from E to F, then we say that E is *Borel reducible* to F and write $E{\leq_B}F$. We say that E is *Borel bireducible* with F if $E{\leq_B}F$ and $F{\leq_B}E$.

An equivalence relation E on a standard Borel space is said to be *classifiable by countable structures* if there is a countable relation language $\mathcal{L}$ such that E is Borel reducible to the isomorphism equivalence relation of $\mathcal{L}$-structures whose underlying set is $\mathbb{N}$.

For a class $\mathcal{C}$ of equivalence relations on standard Borel spaces and an element $E \in \mathcal{C}$, we say that E is *universal* for $\mathcal{C}$ if $F{\leq_B}E$ for every $F \in \mathcal{C}$.

By the *orbit equivalence relation* induced by a group action $\varphi\colon G \times X \to X$ one understand the equivalence relation E on X given by

$$xEx' \iff \exists\, g \in G : \varphi(g, x) = x'.$$

It is known that for every Polish group G there exists an equivalence relation E_G which is universal for the class of all orbit ERs induced by Borel actions of G on standard Borel spaces (see e.g. [Gao09, Theorem 5.1.8]). Some of the most studied ERs in the field of invariant descriptive set theory include E_{S_∞} and E_{G_∞}, where S_∞ stands for the symmetric group on $\mathbb{N}$ and G_∞ stands for a universal Polish group. By [Hjo00, Theorem 2.39], an equivalence relation on a standard Borel space is Borel reducible to E_{S_∞} if and only if it is classifiable by countable structures. By [Gao09, Theorem 5.1.9], the equivalence relation E_{G_∞} is universal for the class of all orbit ERs induced by Borel actions of Polish groups on standard Borel spaces.

Throughout this chapter, we denote by I and $I_\circ$ the intervals $[0,1]$ and $(0,1)$, respectively. The space $I^{\mathbb{N}}$ with the product topology is called the *Hilbert cube*. Recall that every Polish space is homeomorphic to a G_δ subset of the Hilbert cube.

For a Polish space X, we denote by $\mathcal{K}(X)$ the space of all compact subsets of X equipped with the Vietoris topology. It is well-known that $\mathcal{K}(X)$ is a Polish space (see [Kec95, Theorems 4.22 and 4.25]). We consider various subspaces of $\mathcal{K}(X)$: We denote by $\mathsf{C}(X)$ the space of all continua in X and by $\mathsf{LC}(X)$ the space of all locally connected continua in X. For every $n \in \mathbb{N}$, we denote by $\mathsf{C}_n(X)$ the subspace of $\mathsf{C}(X)$ consisting of those members of $\mathsf{C}(X)$ which are n-dimensional. In a similar fashion, we define the space $\mathsf{LC}_n(X)$. It is well-known that $\mathsf{C}(X)$ is a closed set (this is very easy) and $\mathsf{LC}(X)$ is a Borel set in $\mathcal{K}(X)$ (see [GvM93]). Also, it is not difficult to show that $\{K \in \mathcal{K}(X)\,;\ \dim(K) \leq n\}$ is G_δ in $\mathcal{K}(X)$. It follows that $\{K \in \mathcal{K}(X)\,;\ \dim(K) = n\}$ is a Borel set in $\mathcal{K}(X)$ and (consequently) so are $\mathsf{C}_n(X)$ and $\mathsf{LC}_n(X)$.

An *absolute retract* is a topological space which is homeomorphic to a retract of the Hilbert cube. For every Polish space X, we denote by $\mathsf{AR}(X)$ the set of all absolute retracts contained in X. By [DR94, Theorem 2.2], the set $\mathsf{AR}(I^{\mathbb{N}})$ is $G_{\delta\sigma\delta}$ in $\mathcal{K}(I^{\mathbb{N}})$. Hence, the set $\mathsf{AR}(X)$ is Borel in $\mathcal{K}(X)$ for every Polish space X. We denote $\mathsf{AR}_n(X) := \{K \in \mathsf{AR}(X)\,;\ \dim(K) = n\}$, $n \in \mathbb{N}$.

For a Polish space X and a Borel set $B \subseteq \mathcal{K}(X)$ the equivalence relation

$$\Big\{(K,L) \in B \times B\,;\ K \text{ is homeomorphic to } L\Big\}$$

is called the *homeomorphism equivalence relation* on B. We also consider the equivalence relation

$$\Big\{(K,L) \in B \times B;\ \text{there is a self-homeomorphism of } X \text{ mapping } K \text{ onto } L\Big\}$$

and call it the *ambient homeomorphism equivalence relation* on B.

Notation 3.2.2. For every $n \in \mathbb{N}$, we denote by H_n and C_n the homeomorphism ERs on $\mathcal{K}(I^n)$ and $\mathsf{C}(I^n)$, respectively. In addition, R_n stands for the ambient homeomorphism ER on $\mathcal{K}(I^n)$.

Since the Hilbert cube contains a homeomorphic copy of every metrizable compact space, the homeomorphism ER on $\mathcal{K}(I^{\mathbb{N}})$ is often interpreted as the homeomorphism ER for all metrizable compacta. The homeomorphism ERs on $\mathsf{C}(I^{\mathbb{N}})$, $\mathsf{LC}(I^{\mathbb{N}})$ and $\mathsf{AR}(I^{\mathbb{N}})$ can be interpreted in a similar fashion.

Recall that a set $J \subseteq \mathbb{R}^2$ is called a *Jordan curve* if it is homeomorphic to a circle. By the Jordan curve theorem, for every Jordan curve J, the set $\mathbb{R}^2 \setminus J$ consists of exactly two connected components, one bounded and one unbounded. We denote the bounded one by $\mathrm{ins}(J)$ and the unbounded one by $\mathrm{out}(J)$. The

Jordan curve theorem also states that the boundary of both $\mathrm{ins}(J)$ and $\mathrm{out}(J)$ is equal to J. It immediately follows that for any two Jordan curves $J_1, J_2 \subseteq \mathbb{R}^2$ with $\mathrm{ins}(J_1) = \mathrm{ins}(J_2)$, we have $J_1 = J_2$.

Recall that the Schoenflies theorem asserts that any homeomorphism between any two Jordan curves can be extended to a self-homeomorphism of $\mathbb{R}^2$. We will need this result later on.

Given a metric space (M, d), a family $\mathcal{A}$ of subsets of M is said to be a *null family* if the set $\{A \in \mathcal{A}\,;\ \mathrm{diam}_d(A) \geq \varepsilon\}$ is finite for every $\varepsilon > 0$.

Lemma 3.2.3. *Let (X, ϱ) and (Y, σ) be metric spaces, $f\colon X \to Y$ a mapping, $F \subseteq X$ a closed set and $\mathcal{A}$ a family of subsets of X such that $X \setminus F \subseteq \bigcup \mathcal{A}$ and $F \cap \overline{A} \neq \emptyset$ for every $A \in \mathcal{A}$. Assume that $\mathcal{A}$ and $\{f(A)\,;\ A \in \mathcal{A}\}$ are null families in (X, ϱ) and (Y, σ), respectively. In addition, assume that $f(A) \neq f(B)$ for any two distinct sets $A, B \in \mathcal{A}$. Finally, assume that $f{\restriction_F}$ is continuous and so is $f{\restriction_{\overline{A}}}$ for every $A \in \mathcal{A}$. Then f is continuous.*

Proof. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence in X converging to some $x \in X$ such that $(f(x_n))_{n=1}^{\infty}$ does not converge to $f(x)$. We can assume that there is $\varepsilon > 0$ satisfying $\sigma(f(x_n), f(x)) > \varepsilon$ for every $n \in \mathbb{N}$ (otherwise we take a subsequence). Since f restricted to $F \cup \bigcup\{\overline{A};\ A \in \mathcal{F}\}$ is continuous for every finite family $\mathcal{F} \subseteq \mathcal{A}$, we can assume that there is an injective sequence $(A_n)_{n=1}^{\infty}$ of members of $\mathcal{A}$ such that $x_n \in A_n$ for every $n \in \mathbb{N}$ (we take a subsequence if this is not the case). Then $\mathrm{diam}_{\varrho}(A_n) \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$, fix a point $z_n \in F \cap \overline{A_n}$. By the triangle inequality, $z_n \to x$, hence $x \in F$ and so $f(z_n) \to f(x)$. Thus,

$$\sigma(f(x_n), f(x)) \leq \sigma(f(z_n), f(x)) + \mathrm{diam}_{\sigma}(f(A_n)) \to 0$$

by the assumptions on the sets $f(A)$, $A \in \mathcal{A}$. This is a contradiction. □

Recall that a compact metric space (K, d) is locally connected if and only if it has *property S*, that is, for every $\varepsilon > 0$, K can be expressed as the union of finitely many continua of diameter less than ε (see [Nad92, Theorem 8.4]).

Lemma 3.2.4. *Let (M, d) be a metric space, $F \subseteq M$ and $\mathcal{A}$ a null family of subsets of M with compact closures such that $F \cap \overline{A} \neq \emptyset$ and $\overline{A} \setminus A \subseteq F$ for every $A \in \mathcal{A}$.*

(i) *If F is compact, then so is $F \cup \bigcup \mathcal{A}$.*
(ii) *If F and every member of $\mathcal{A}$ are connected, then so is $F \cup \bigcup \mathcal{A}$.*
(iii) *If F and every member of $\mathcal{A}$ are locally connected continua, then so is $F \cup \bigcup \mathcal{A}$.*

Proof. Assertions (i) and (ii) are easy to prove. As for assertion (iii), assume that F and all members of $\mathcal{A}$ are locally connected continua and let $X := F \cup \bigcup \mathcal{A}$. By (i) and (ii), X is a continuum. To show that X has property S, let $\varepsilon > 0$ be given and let $\mathcal{A}_0 := \{A \in \mathcal{A}\,;\ \mathrm{diam}_d(A) \geq \varepsilon/3\}$ and $X_0 := F \cup \bigcup \mathcal{A}_0$. Since $\mathcal{A}_0$ is finite, there is a finite family $\mathcal{F}_0$ of continua of diameter less than $\varepsilon/3$ such that $\bigcup \mathcal{F}_0 = X_0$. For every $C \in \mathcal{F}_0$, define $K_C := C \cup \bigcup\{A \in \mathcal{A} \setminus \mathcal{A}_0\,;\ A \cap C \neq \emptyset\}$. Finally, let $\mathcal{F} := \{K_C\,;\ C \in \mathcal{F}_0\}$. Then $\mathcal{F}$ is finite and its union is equal to X. By (i), (ii) and the triangle inequality, each member of $\mathcal{F}$ is a continuum of diameter less than ε. □

Remark 3.2.5. Let X, Z be Polish spaces and d a compatible metric on X. Let $\psi_n\colon Z \to \mathcal{K}(X)$, $n \in \mathbb{N}$, be mappings such that $\sup_{z\in Z} \operatorname{diam}_d(\psi_n(z))$ converges to 0 as $n \to \infty$. Assume that $\psi_n(z) \cap \psi_1(z) \neq \emptyset$ for all $n \in \mathbb{N}$ and $z \in Z$. If ψ_n is continuous (resp. Borel measurable) for every $n \in \mathbb{N}$, then so is the mapping $z \mapsto \bigcup\{\psi_n(z)\,;\, n \in \mathbb{N}\}$. This is so because the mappings Ψ_n given by $z \mapsto \bigcup\{\psi_i(z)\,;\, i = 1, \dots, n\}$, $n \in \mathbb{N}$, are continuous (resp. Borel measurable) and, in the Hausdorff metric, the sequence $(\Psi_n)_{n=1}^{\infty}$ converges uniformly to the mapping in question.

Lemma 3.2.6. *Let X be a Polish space. Then the mapping from $\mathcal{K}(X)$ to $\mathcal{K}(X)$ given by $K \mapsto \partial K$ is Borel measurable.*

Proof. Denote by Φ the mapping in question and let ϱ be a metric on X compatible with the topology of X. For every $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}$, let

$$\partial_n K := \{x \in K\,;\ \exists\, y \in X \setminus K : \varrho(x, y) < 2^{-n}\}.$$

It is easy to see that

$$\forall K \in \mathcal{K}(X)\ \forall n \in \mathbb{N} : \partial_n K \supseteq \overline{\partial_{n+1} K}, \tag{3.1}$$

$$\forall K \in \mathcal{K}(X) : \bigcap_{n=1}^{\infty} \partial_n K = \bigcap_{n=1}^{\infty} \overline{\partial_n K} = \partial K. \tag{3.2}$$

For every $n \in \mathbb{N}$, define a mapping $\Phi_n\colon \mathcal{K}(X) \to \mathcal{K}(X)$ by $\Phi_n(K) = \overline{\partial_n K}$. Since any descending sequence of members of $\mathcal{K}(X)$ converges to its intersection, it follows from (3.1) and (3.2) that the sequence $(\Phi_n)_{n=1}^{\infty}$ converges pointwise to Φ. It remains to prove that Φ_n is Borel measurable for each $n \in \mathbb{N}$. The σ-algebra of Borel sets on $\mathcal{K}(X)$ is generated by sets of the form $\{K \in \mathcal{K}(X)\,;\, K \cap G \neq \emptyset\}$ for $G \subseteq X$ open (see e.g. [Kec95, Exercise 12.11 i)]). Thus, given $n \in \mathbb{N}$ and an open set $G \subseteq X$, it suffices to prove the following claim.

Claim 3.2.6.1. The set $\{K \in \mathcal{K}(X)\,;\ \overline{\partial_n K} \cap G \neq \emptyset\}$ is open in $\mathcal{K}(X)$.

Proof. Given a set $K \in \mathcal{K}(X)$ with $\overline{\partial_n K} \cap G \neq \emptyset$, we need to find a neighborhood $\mathcal{V}$ of K in $\mathcal{K}(X)$ such that $\overline{\partial_n L} \cap G \neq \emptyset$ for every $L \in \mathcal{V}$. Since G is open, there is a point $x \in \partial_n K \cap G$. By the definition of $\partial_n K$, there exists $y \in X \setminus K$ such that $\varrho(x, y) < 2^{-n}$. Let $\varepsilon_1 := 2^{-n} - \varrho(x, y)$. Since G is open and $x \in G$, there is $\varepsilon_2 > 0$ such that every $z \in X$ with $\varrho(x, z) < \varepsilon_2$ belongs to G. Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and $B := \{z \in X;\ \varrho(x, z) < \varepsilon\}$. Define

$$\mathcal{V} := \{L \in \mathcal{K}(X)\,;\ L \subseteq X \setminus \{y\}\,,\ L \cap B \neq \emptyset\}.$$

Then $\mathcal{V}$ is open in $\mathcal{K}(X)$ and it contains K. Given any $L \in \mathcal{V}$, let us show that $\partial_n L \cap G \neq \emptyset$. By the definition of $\mathcal{V}$, there exists a point $z \in L \cap B$. We have $\varrho(x, z) < \varepsilon \leq \varepsilon_1 = 2^{-n} - \varrho(x, y)$, and hence $\varrho(z, y) < 2^{-n}$. Therefore, $z \in \partial_n L$. Since $z \in B \subseteq G$, it follows that $\partial_n L \cap G \neq \emptyset$. ■

This completes the proof of Lemma 3.2.6. □

3.3 Classification results

In this section we prove that the homeomorphism ER for absolute retracts in the plane is Borel bireducible with the isomorphism ER for countable graphs. We also prove that both the homeomorphism and ambient homeomorphism ERs for compacta in I^n are Borel reducible to the homeomorphism ER for n-dimensional continua in I^{n+1}.

Let us start by presenting a few results on absolute retracts in the plane. Recall that a topological space is said to be *rim-finite* if it has an open base consisting of sets with finite boundaries. By [Why42, p. 33, 2.2], [Bor67, p. 132] and [Kur68, p. 512, Theorem 4], we have the following.

Lemma 3.3.1. *If $X \subseteq \mathbb{R}^2$ is an absolute retract, then*

(1) *X is a locally connected continuum and $\mathbb{R}^2 \setminus X$ is connected;*
(2) *∂X is a rim-finite locally connected continuum.*

Lemma 3.3.2. *Assume that $X \subseteq \mathbb{R}^2$ is an absolute retract. Then the family $\{\text{ins}(J) \,;\, J \subseteq \partial X \text{ is a Jordan curve}\}$ is a null family and it coincides with the family of connected components of $\text{Int}(X)$.*

Proof. It is easy to see that every connected component of $\text{Int}(X)$ is a connected component of $\mathbb{R}^2 \setminus \partial X$. Hence, since ∂X is locally connected (by Lemma 3.3.1), it follows from [Kur68, p. 515, Theorem 10] that the family of connected components of $\text{Int}(X)$ is a null family. It remains to prove that this family is equal to $\{\text{ins}(J) \,;\, J \subseteq \partial X \text{ is a Jordan curve}\}$.

Let us start by showing that $\text{ins}(J)$ is a subset of $\text{Int}(X)$ for every Jordan curve $J \subseteq \partial X$. Assume $\text{ins}(J) \nsubseteq \text{Int}(X)$ for a Jordan curve $J \subseteq \partial X$. Then, as $\text{ins}(J)$ is open, we have $\text{ins}(J) \nsubseteq X$. Hence, $(\mathbb{R}^2 \setminus X) \cap \text{ins}(J) \neq \emptyset$. On the other hand, since X is compact (and hence bounded in $\mathbb{R}^2$), we have $(\mathbb{R}^2 \setminus X) \cap \text{out}(J) \neq \emptyset$. Moreover, the sets $\text{ins}(J)$, $\text{out}(J)$ are connected and, by Lemma 3.3.1, so is $\mathbb{R}^2 \setminus X$. Hence, the set $\text{ins}(J) \cup (\mathbb{R}^2 \setminus X) \cup \text{out}(J)$ is connected as well. However, this set is equal to $\mathbb{R}^2 \setminus J$. Therefore, $\mathbb{R}^2 \setminus J$ is connected, which contradicts the Jordan curve theorem.

Now, given a Jordan curve $J \subseteq \partial X$, let us prove that $\text{ins}(J)$ is a connected component of $\text{Int}(X)$. Since $\text{ins}(J)$ is a connected subset of $\text{Int}(X)$, there is a connected component U of $\text{Int}(X)$ containing $\text{ins}(J)$. Then $\text{ins}(J)$ and $U \cap \text{out}(J)$ are disjoint open sets and their union is equal to U. Thus, by the connectedness of U, we have $U \cap \text{out}(J) = \emptyset$ and $U = \text{ins}(J)$.

Finally, given a connected component G of $\text{Int}(X)$, let us prove that there exists a Jordan curve $J \subseteq \partial X$ with $\text{ins}(J) = G$. By Lemma 3.3.1, ∂X is a locally connected continuum. Also, it is easy to see that G is a connected component of $\mathbb{R}^2 \setminus \partial X$. Fix $a \in G$ and $b \in \mathbb{R}^2 \setminus X$. By [Kur68, p. 513, Theorem 5], there is a Jordan curve $J \subseteq \partial X$ such that one of the points a, b is in $\text{ins}(J)$ and the other in $\text{out}(J)$. By the previous paragraph, $\text{ins}(J)$ is a connected component of $\text{Int}(X)$. In particular, $b \notin \text{ins}(J)$. Hence, $a \in \text{ins}(J)$, which shows that $G \cap \text{ins}(J) \neq \emptyset$. Thus, as both of the sets G and $\text{ins}(J)$ are connected components of $\text{Int}(X)$, we conclude that $G = \text{ins}(J)$. □

Theorem 3.3.3. *The homeomorphism equivalence relation on $\mathsf{AR}(\mathbb{R}^2)$ is Borel bireducible with the isomorphism equivalence relation for countable graphs.*

Proof. By [CDM05, Theorem 6.7], the homeomorphism ER for dendrites is Borel bireducible with the isomorphism ER for countable graphs. Moreover, by a careful reading of [CDM05, Lemma 6.6] and its proof we conclude that the homeomorphism ER for dendrites in I^2 is Borel bireducible with the isomorphism ER for countable graphs (in fact, by [Nad92, paragraph 10.37], every dendrite can be embedded into the plane). By [Kur68, p. 344, Theorem 16], every dendrite is an absolute retract. Therefore, the isomorphism ER for countable graphs is Borel reducible to the homeomorphism ER on $\mathsf{AR}(\mathbb{R}^2)$.

Conversely, let us prove that the homeomorphism ER on $\mathsf{AR}(\mathbb{R}^2)$ is Borel reducible to the isomorphism ER for countable graphs. By [KV20], the isomorphism ER for countable graphs is Borel bireducible with the homeomorphism ER for rim-finite continua. Hence, it suffices to find a Borel reduction Φ from the homeomorphism ER on $\mathsf{AR}(\mathbb{R}^2)$ to the homeomorphism ER on $\mathcal{K}(\mathbb{R}^2)$ such that $\Phi(X)$ is a rim-finite continuum for each $X \in \mathsf{AR}(\mathbb{R}^2)$. Define the desired mapping $\Phi\colon \mathsf{AR}(\mathbb{R}^2) \to \mathcal{K}(\mathbb{R}^2)$ by $\Phi(X) = \partial X$. By Lemmata 3.2.6 and 3.3.1, Φ is Borel measurable and $\Phi(X)$ is a rim-finite continuum for every $X \in \mathsf{AR}(\mathbb{R}^2)$. It remains to show that Φ is a reduction.

Suppose $X_1, X_2 \in \mathsf{AR}(\mathbb{R}^2)$ are homeomorphic and let $f\colon X_1 \to X_2$ be a homeomorphism. By the domain invariance theorem, $f(\mathrm{Int}(X_1)) = \mathrm{Int}(X_2)$. Therefore, $f(\partial X_1) = \partial X_2$, which shows that $\Phi(X_1)$ is homeomorphic to $\Phi(X_2)$.

Conversely, assume we are given $X_1, X_2 \in \mathsf{AR}(\mathbb{R}^2)$ such that $\Phi(X_1)$ is homeomorphic to $\Phi(X_2)$. Let $h\colon \partial X_1 \to \partial X_2$ be a homeomorphism. For each $i \in \{1, 2\}$, denote by Γ_i the set of all Jordan curves contained in ∂X_i. Since h is a homeomorphism, the mapping $J \mapsto h(J)$, $J \in \Gamma_1$, is a bijection between Γ_1 and Γ_2. Moreover, trivially, for every $J \in \Gamma_1$ the mapping $h\restriction_J$ is a homeomorphism between the Jordan curves J and $h(J)$. Therefore, by the Schoenflies theorem, for every $J \in \Gamma_1$ there is a homeomorphism $g_J\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g_J\restriction_J = h\restriction_J$. Clearly, $g_J(\mathrm{ins}(J)) = \mathrm{ins}(h(J))$ for each $J \in \Gamma_1$. By Lemma 3.3.2, for all $J \in \Gamma_i$ and $i \in \{1, 2\}$, the set $\mathrm{ins}(J)$ is a subset of $\mathrm{Int}(X_i)$. Moreover, by Lemma 3.3.2 together with the Jordan curve theorem, for every $i \in \{1, 2\}$ and every $x \in \mathrm{Int}(X_i)$, there is a unique Jordan curve $J \in \Gamma_i$ with $x \in \mathrm{ins}(J)$. The mapping $f\colon X_1 \to X_2$ given by

$$f(x) = \begin{cases} h(x) & \text{if } x \in \partial X_1 \\ g_J(x) & \text{if } x \in \mathrm{ins}(J),\ J \in \Gamma_1 \end{cases}$$

is thus a well-defined bijection. Moreover, f is continuous by Lemmata 3.2.3 and 3.3.2. Hence, since X_1 is compact, f is a homeomorphism. □

Let us start the preparation for the proof of Theorem 3.3.7.

Lemma 3.3.4. *Let $n \in \mathbb{N}$, let G_1, G_2 be disjoint open subsets of $\mathbb{R}^n$ and let C, D be countable dense subsets of $G := G_1 \cup G_2$. Then there exists a homeomorphism $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $g(C \cap G_i) = D \cap G_i$ for $i \in \{1, 2\}$ and $g(x) = x$ for every $x \in \mathbb{R}^n \setminus G$.*

Proof. If $G_1 = G_2 = \emptyset$, the assertion is clear. Next assume that exactly one of the sets G_1, G_2 is empty. For each open Euclidean ball $B \subseteq G$ and any two points $x, y \in B$, there is a self-homeomorphism u of $\mathbb{R}^n$ such that $u(x) = y$ and u is the identity outside B, and hence outside G. This allows us to replicate the proof of [Ben72, Theorem 3] with $X = G$ in such a way that all the homeomorphisms constructed will actually be restrictions (to G) of self-homeomorphisms of $\mathbb{R}^n$ which are the identity on $\mathbb{R}^n \setminus G$.

In the general case, we use the above to get self-homeomorphisms g_1 and g_2 of $\mathbb{R}^n$ such that $g_i(C \cap G_i) = D \cap G_i$ and $g_i(x) = x$ for all $x \in \mathbb{R}^n \setminus G_i$ and $i \in \{1, 2\}$. Clearly, $g := g_1 \circ g_2$ is as desired. □

Proposition 3.3.5 below is a simple enhancement of [KV20, Proposition 10], which is a modification of the well-known Arsenin-Kunugui measurable selection theorem [Kec95, Theorem 35.46].

Proposition 3.3.5. *Let X be a standard Borel space, Y a Polish space and let $B \subseteq X \times Y$ be a Borel set whose nonempty vertical sections are infinite and σ-compact. Then $\pi_X(B)$ is a Borel set and there exist Borel measurable mappings $f_k \colon \pi_X(B) \to Y$, $k \in \mathbb{N}$, such that, for every $x \in \pi_X(B)$,*

(i) *the set $\{f_k(x)\,;\, k \in \mathbb{N}\}$ is a dense subset of the vertical section B_x;*
(ii) *the sequence $(f_k(x))_{k=1}^{\infty}$ is injective.*

Recall that we use $I_\circ$ to denote the open interval $(0, 1)$. The following lemma can be proven using Lemma 3.2.6 together with [Kec95, Theorem 28.8]. We leave the details for the reader.

Lemma 3.3.6. *Let $n \in \mathbb{N}$. Then both of the sets*

$$\{(K, y) \in \mathcal{K}(I^n) \times I^n\,;\; y \in \operatorname{Int}(K)\},$$
$$\{(K, y) \in \mathcal{K}(I^n) \times I^n\,;\; y \in I_\circ^n \setminus \partial K\}$$

are Borel in $\mathcal{K}(I^n) \times I^n$, where ∂K and $\operatorname{Int}(K)$ are taken with respect to the whole space $\mathbb{R}^n$.

Theorem 3.3.7. *For every $n \in \mathbb{N}$, the ambient homeomorphism equivalence relation for compacta in I^n is Borel reducible to the homeomorphism equivalence relation for n-dimensional continua in I^{n+1}. In particular, $R_n \leq_B C_{n+1}$ for every $n \in \mathbb{N}$ (see Notation 3.2.2).*

Proof. By [Hjo00, Exercise 4.13] (see [CG19] for more details), the equivalence relation R_1 is Borel bireducible with the isomorphism ER for countable graphs, which (as explained at the beginning of the proof of Theorem 3.3.3) is Borel bireducible with the homeomorphism ER for dendrites in I^2. Since dendrites are 1-dimensional continua, the case $n = 1$ follows.

Let $n > 1$, denote $X := \mathcal{K}(I^n)$ and consider the set

$$B := \{(K, y) \in X \times I^n\,;\; y \in I_\circ^n \setminus \partial K\}.$$

Then B is Borel by Lemma 3.3.6 and, clearly, $\pi_X(B) = X$. As the vertical sections of B are open in $\mathbb{R}^n$, they are σ-compact and infinite. Hence, by Proposition 3.3.5, there exist Borel measurable mappings $f_k \colon X \to I^n$, $k \in \mathbb{N}$, such that, for each $K \in X$,

- $\{f_k(K)\,;\, k \in \mathbb{N}\}$ is a dense subset of $I_\circ^n \setminus \partial K$;
- the sequence $(f_k(K))_{k=1}^{\infty}$ is injective.

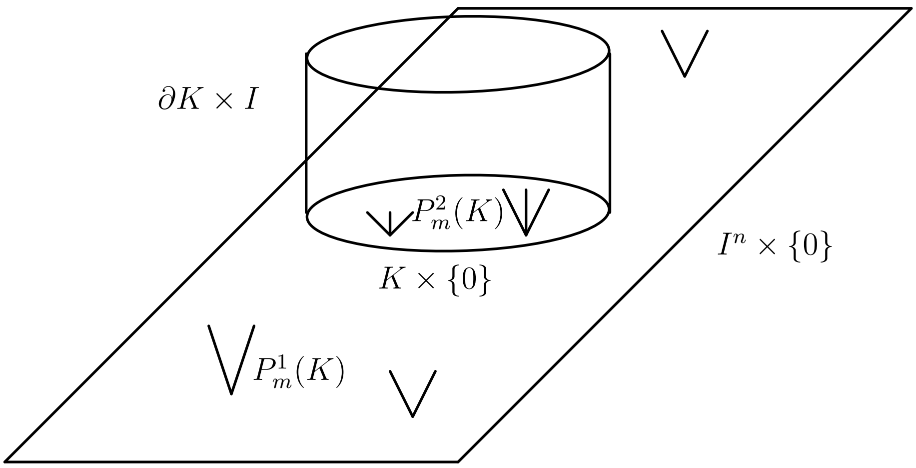


Figure 3.1: Compact set $\varphi(K)$ topologically.

In order to simplify the notation let us identify $\mathbb{R}^{n+1}$ with $\mathbb{R}^n \times \mathbb{R}$. Denote $F := (\mathbb{R}^n \setminus I^n_\circ) \times \mathbb{R}$ and let $p_1, p_2, p_3 \in \mathbb{R}^n \times \{1/2\}$ be distinct points of norm (in $\mathbb{R}^{n+1}$) less than 1. Let

$$T_1 := \Big\{tp\,;\ p \in \{p_1, p_2\},\, t \in I\Big\},\ T_2 := \Big\{tp\,;\ p \in \{p_1, p_2, p_3\},\, t \in I\Big\}$$

and for $z \in \mathbb{R}^n$, $s > 0$ and $i \in \{1, 2\}$, let $T_i(z, s) := \{(z, 0) + sx\,;\ x \in T_i\}$. Observe that $T_i(z, s)$ is a continuum contained in the open s-ball centered at $(z, 0)$ and it intersects $\mathbb{R}^n \times \{0\}$ at $(z, 0)$ only. Consider the following inductive definition. For every $K \in X$ and $m \in \mathbb{N}$, let $t_m(K)$ be the minimum of $1/m$ and the distance from the point $(f_m(K), 0)$ to the set

$$F \cup (\partial K \times \mathbb{R}) \cup \bigcup_{j=1}^{m-1} T_2(f_j(K), t_j(K)).$$

Also, for all $K \in X$ and $m \in \mathbb{N}$, define $P_m(K) := T_{i(m,K)}(f_m(K), t_m(K))$, where $i(m, K) = 1$ if $f_m(K) \in I^n_\circ \setminus K$ and $i(m, K) = 2$ if $f_m(K) \in \mathrm{Int}(K)$. For every $K \in X$, these definitions and the above observation show that $t^i_m(K) > 0$ for each $m \in \mathbb{N}$ and $\{P_m(K)\,;\, m \in \mathbb{N}\}$ is a null family of pairwise disjoint subcontinua of I^{n+1} which are disjoint from $\partial K \times \mathbb{R}$. Moreover, for every $K \in X$ and $m \in \mathbb{N}$, since $(f_m(K), 0) \in P_m(K)$ and $f_m(K) \in I^n_\circ \setminus \partial K$, an easy connectedness argument shows that $P_m(K) \subseteq \mathrm{Int}(K) \times I$ when $i(m, K) = 2$ and $P_m(K) \subseteq (I^n \setminus K) \times I$ if $i(m, K) = 1$. Finally, for every $K \in X$, let $P(K) := \bigcup\{P_m(K)\,;\ m \in \mathbb{N}\}$ and $\varphi(K) := (I^n \times \{0\}) \cup (\partial K \times I) \cup P(K)$, see Figure 3.1.

Claim 3.3.7.1. Given any $K \in X$, the set $\varphi(K)$ is an n-dimensional subcontinuum of I^{n+1} .

Proof. Clearly, $\varphi(K)$ is a connected subset of I^{n+1} and it is compact by Lemma 3.2.4. As $I^n \times \{0\} \subseteq \varphi(K)$, the dimension of $\varphi(K)$ is at least n. Since $\varphi(K)$ is meagre in $\mathbb{R}^{n+1}$, the interior of $\varphi(K)$ is empty. Thus, by [Eng95, Theorem 1.8.11], the dimension of $\varphi(K)$ is equal to n. ■

We claim that the mapping $\varphi\colon X \to \mathsf{C}_n(I^{n+1})$ given by $K \mapsto \varphi(K)$ is a Borel reduction from R_n to the homeomorphism ER on $\mathsf{C}_n(I^{n+1})$.

Claim 3.3.7.2. Let $(K, L) \in R_n$. Then $\varphi(K)$ is homeomorphic to $\varphi(L)$.

Proof. Let g_0 be a self-homeomorphism of I^n such that $g_0(K) = L$. For each $Z \in \{K, L\}$, denote $G_1^Z := I_\circ^n \setminus Z$, $G_2^Z := \mathrm{Int}(Z)$, $G^Z := G_1^Z \cup G_2^Z = I_\circ^n \setminus \partial Z$ and $D^Z := \{f_m(Z)\,;\, m \in \mathbb{N}\}$. By the domain invariance theorem, $g_0(G_2^K) = G_2^L$ and $g_0(I_\circ^n) = I_\circ^n$. Therefore, $g_0(G_1^K) = G_1^L$ and $g_0(G^K) = G^L$. Note that both of the sets $g_0(D^K)$ and D^L are countable and dense in G^L. Thus, by Lemma 3.3.4, there is a homeomorphism $g_1 \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $g_1(g_0(D^K) \cap G_i^L) = D^L \cap G_i^L$ for $i \in \{1, 2\}$ and $g_1(x) = x$ for each $x \in \mathbb{R}^n \setminus G^L = (\mathbb{R}^n \setminus I_\circ^n) \cup \partial L$. In particular, g_1 maps I^n onto I^n. This allows us to define a homeomorphism $g \colon I^n \to I^n$ by $g(x) = g_1(g_0(x))$. Clearly, $g(\partial K) = \partial L$ and $g(D^K \cap G_i^K) = D^L \cap G_i^L$ for $i = 1, 2$. Hence, there exists a bijection $\mu \colon \mathbb{N} \to \mathbb{N}$ such that $g(f_m(K)) = f_{\mu(m)}(L)$ and $i(m, K) = i(\mu(m), L)$ for each $m \in \mathbb{N}$. For every $m \in \mathbb{N}$, fix a homeomorphism $h_m \colon P_m(K) \to P_{\mu(m)}(L)$ sending $(f_m(K), 0)$ to $(f_{\mu(m)}(L), 0)$. Define a mapping $h \colon \varphi(K) \to \varphi(L)$ by

$$h(x,t) = \begin{cases} (g(x), t) & \text{if } (x,t) \in (\partial K \times I) \cup (I^n \times \{0\}) \\ h_m(x,t) & \text{if } (x,t) \in P_m(K),\ m \in \mathbb{N}. \end{cases}$$

It is clear that h is a bijection and, by Lemma 3.2.3, it is continuous. Hence, since $\varphi(K)$ is compact, h is a homeomorphism. ■

Claim 3.3.7.3. Let $K, L \in X$ and assume that $\varphi(K)$ is homeomorphic to $\varphi(L)$. Then $(K, L) \in R_n$.

Proof. First note that for every $z \in \varphi(K)$, the set $\varphi(K) \setminus \{z\}$ has $i > 2$ connected components if and only if $z = (f_m(K), 0)$ for some $m \in \mathbb{N}$. Moreover, $i = 4$ if and only if $z = (f_m(K), 0)$ for some $m \in \mathbb{N}$ satisfying $f_m(K) \in \mathrm{Int}(K)$. Also, the set $A_K := \{f_m(K)\,;\ m \in \mathbb{N}\}$ is dense in I^n and $A_K \cap \mathrm{Int}(K)$ is dense in $\mathrm{Int}(K)$. In addition, $(I^n \times \{0\}) \cup (\partial K \times I)$ is the complement in $\varphi(K)$ of the union of those components of $\varphi(K) \setminus (A_K \times \{0\})$ which are homeomorphic to an interval. Since the same is true with K everywhere replaced by L, it follows that if $h \colon \varphi(K) \to \varphi(L)$ is a homeomorphism, then

$$h(I^n \times \{0\}) = I^n \times \{0\}\,,\ h\Big(\overline{\mathrm{Int}(K)} \times \{0\}\Big) = \overline{\mathrm{Int}(L)} \times \{0\} \qquad (*)$$

and h maps $(I^n \times \{0\}) \cup (\partial K \times I)$ onto $(I^n \times \{0\}) \cup (\partial L \times I)$. As a consequence, $h(\partial K \times (0, 1]) = \partial L \times (0, 1]$, hence $h(\partial K \times \{0\}) = \partial L \times \{0\}$. This equality and $(*)$ imply that h restricted to $I^n \times \{0\}$ is a self-homeomorphism of $I^n \times \{0\}$ sending $K \times \{0\}$ onto $L \times \{0\}$. ■

With the help of Remark 3.2.5 and Lemma 3.2.6, it is easy to show that φ is Borel measurable. □

We will need the following two lemmata for the proof of Theorem 3.3.10.

Lemma 3.3.8. *Let $n \in \mathbb{N}$. The homeomorphism equivalence relation for compacta in I^n is continuously reducible to the homeomorphism equivalence relation for compacta in $I_\circ^n$ with nonempty interior.*

Proof. Clearly, we can replace above $I_\circ^n$ by $\mathbb{R}^n$. Denote by X the space of all the compact subsets of $\mathbb{R}^n$ which have nonempty interior. It is easy to see that

X is a Borel (in fact, it is F_σ) set in $\mathcal{K}(\mathbb{R}^n)$. Thus, X forms a standard Borel space. Let $Z := [2,3]^n$ and define a mapping $\Phi\colon \mathcal{K}(I^n) \to X$ by $\Phi(K) = K \cup Z$. Obviously, Φ is continuous. Now, let $K, L \in \mathcal{K}(I^n)$ be given. Clearly, if K is homeomorphic to L, then $\Phi(K)$ is homeomorphic to $\Phi(L)$. Conversely, assume that there is a homeomorphism $h\colon \Phi(K) \to \Phi(L)$. Note that both Z and $h(Z)$ are connected components of $\Phi(L)$. Therefore, if $Z \cap h(Z) \neq \emptyset$, then $h(Z) = Z$ and, consequently, $h(K) = L$, which shows that K is homeomorphic to L. If $Z \cap h(Z) = \emptyset$, then $h(Z) \subseteq L$, $h^{-1}(Z) \subseteq K$ and one can easily verify that the mapping $g\colon K \to L$ given by

$$g(x) = \begin{cases} h(x) & \text{if } x \in K \setminus h^{-1}(Z) \\ h(h(x)) & \text{if } x \in h^{-1}(Z) \end{cases}$$

is a homeomorphism. Thus, K is homeomorphic to L. □

Lemma 3.3.9. *Let $n \in \mathbb{N}$ and denote by X the space of all the compact subsets of $I^n_\circ$ which have nonempty interior. Then there exist Borel measurable mappings $f_k\colon X \to I^n_\circ$, $k \in \mathbb{N}$, such that, for each $K \in X$,*

(i) the sequence $(f_k(K))_{k=1}^\infty$ is injective;
(ii) $S(K) := \{f_k(K)\,;\, k \in \mathbb{N}\}$ is a relatively discrete subset of $I^n_\circ \setminus K$;
(iii) $\partial K = \overline{S(K)} \setminus S(K)$, where $S(K)$ is as above.

Proof. For every $k \in \mathbb{N}$, fix a finite set $M_k \subseteq I^n_\circ$ such that M_k is a 2^{-k}-net for $I^n_\circ$. For every $K \in X$ and every $k \in \mathbb{N}$, let

$$S_k(K) := \{z \in M_k \setminus K\,;\ \operatorname{dist}(z, K) < 3 \cdot 2^{-k}\}.$$

Moreover, for every $K \in X$ define $S(K) := \bigcup_{k=1}^{\infty} S_k(K)$.

Claim 3.3.9.1. For every $K \in X$ we have $\overline{S(K)} = S(K) \cup \partial K$.

Proof. Let $K \in X$ be given and let $R_j := \bigcup\{S_k(K);\, k \geq j\}$ for every $j \in \mathbb{N}$. Then $S(K) \setminus R_j$ is a finite set for each $j \in \mathbb{N}$. Thus, every accumulation point of $S(K)$ belongs to $\bigcap\{\overline{R_j};\, j \in \mathbb{N}\}$. Consequently, as $\operatorname{dist}(z, K) < 3 \cdot 2^{-j}$ for every $z \in R_j$ and every $j \in \mathbb{N}$, we have $\overline{S(K)} \setminus S(K) \subseteq K$. Therefore, since $\overline{S(K)} \cap \operatorname{Int}(K) = \emptyset$, it follows that $\overline{S(K)} \subseteq S(K) \cup \partial K$.

It remains to prove that $\partial K \subseteq \overline{S(K)}$. Given any $x \in \partial K$ and $\varepsilon \in (0,1)$, we are going to show that there is $z \in S(K)$ with $\|z - x\| < 2\varepsilon$. Since $x \in \partial K$ and $K \subseteq I^n_\circ$, there is $y \in I^n_\circ \setminus K$ satisfying $\|y - x\| < \varepsilon$. Let $k \in \mathbb{N}$ be such that $2^{-k} \leq \operatorname{dist}(y, K) < 2^{1-k}$. Since M_k is a 2^{-k}-net for $I^n_\circ$, there is $z \in M_k$ with $\|z - y\| < 2^{-k}$. Then $\|z - y\| < \operatorname{dist}(y, K) \leq \|y - x\|$, which shows that $z \notin K$ and $\|z - y\| < \varepsilon$. Therefore, $z \in M_k \setminus K$ and, by the triangle inequality, $\operatorname{dist}(z, K) \leq \|z - y\| + \operatorname{dist}(y, K) < 3 \cdot 2^{-k}$ and $\|z - x\| < 2\varepsilon$. ■

Let $M := \bigcup\{M_k\,;\ k \in \mathbb{N}\}$. Since it is clear that M is countably infinite, we can write $M = \{z_j;\, j \in \mathbb{N}\}$, where $z_i \neq z_j$ for $i \neq j$. For every $k \in \mathbb{N}$ and $K \in X$, let $f_k(K) = z_{m(k,K)}$, where $m(k, K)$ is the least $m \in \mathbb{N}$ such that the set $\{z_1, \ldots, z_m\}$ contains exactly k members of $S(K)$. For every $K \in X$, it is clear that $\{f_k(K);\, k \in \mathbb{N}\} = S(K)$. Hence, by Claim 3.3.9.1, condition (iii) holds and it is easy to see that so do conditions (i) and (ii). Given $k \in \mathbb{N}$, it remains to prove that f_k is Borel measurable. Since M is countable and $f_k(K) \in M$ for every $K \in X$, it suffices to show that the set $\{K \in X\,;\ f_k(K) = z_j\}$ is Borel in X for each $j \in \mathbb{N}$. Let $j \in \mathbb{N}$ be given and define

$$\Gamma := \Big\{P \subseteq \{1, \dots, j\}\,;\ |P| = k\,,\ j \in P\Big\}.$$

Then Γ is finite (possibly empty) and it is straightforward to verify that

$$\{K \in X\,;\ f_k(K) = z_j\} = \bigcup_{P\in\Gamma} \bigcap_{i=1}^{j} \{K \in X\,;\ z_i \in S(K) \iff i \in P\}.$$

Therefore, we are done once we prove the following claim.

Claim 3.3.9.2. For every $x \in I^n_\circ$, the set $\{K \in X;\, x \in S(K)\}$ is Borel in X.

Proof. Let $x \in I^n_\circ$ be given. Then

$$\{K \in X\,;\ x \in S(K)\} = \bigcup_{l=1}^{\infty} \{K \in X\,;\ x \in S_l(K)\}$$
$$= \bigcup_{l=1}^{\infty} \Big(\{K \in X\,;\ x \in M_l \setminus K\} \cap \{K \in X\,;\ \mathrm{dist}(x, K) < 3 \cdot 2^{-l}\}\Big),$$

which shows that the set in question is, in fact, open in X. ■

This completes the proof of Lemma 3.3.9. □

Theorem 3.3.10. *For every $n \in \mathbb{N}$, the homeomorphism equivalence relation for compacta in I^n is Borel reducible to the homeomorphism equivalence relation for n-dimensional continua in I^{n+1}. In particular, we have $H_n \leq_B C_{n+1}$ for every $n \in \mathbb{N}$ (see Notation 3.2.2).*

Proof. Let $n \in \mathbb{N}$ be given. By [CG19, Theorem 4.2], the equivalence relation H_1 is Borel bireducible with the isomorphism ER for countable graphs. Thus, repeating the arguments presented at the beginning of the proof of Theorem 3.3.7, we conclude that H_1 is Borel reducible to the homeomorphism ER for 1-dimensional continua in I^2. Hence, we will assume that $n > 1$. Let X denote the space of all the compact subsets of $I^n_\circ$ which have nonempty interior and let $Y := \mathsf{C}_n(I^n \times [-1, 1])$. Obviously, the homeomorphism ER on $\mathsf{C}_n(I^{n+1})$ is continuously bireducible with the homeomorphism ER on Y. Hence, by Lemma 3.3.8, it suffices to find a Borel reduction from the homeomorphism ER on X to the homeomorphism ER on Y.

By Lemma 3.3.9, there exist Borel measurable mappings $f^1_k \colon X \to I^n_\circ$, $k \in \mathbb{N}$, such that, for every $K \in X$,

(1) the sequence $(f^1_k(K))_{k=1}^{\infty}$ is injective;
(2) $S(K) := \{f^1_k(K)\,;\ k \in \mathbb{N}\}$ is a relatively discrete subset of $I^n_\circ \setminus K$;
(3) $\partial K = \overline{S(K)} \setminus S(K)$.

Let $B := \{(K, y) \in X \times I^n_\circ\,;\ y \in \mathrm{Int}(K)\}$. By Lemma 3.3.6, B is a Borel subset of $X \times I^n_\circ$. Moreover, it is clear that $\pi_X(B) = X$ and that vertical sections of B are infinite and σ-compact. Hence, by Proposition 3.3.5, there are Borel measurable mappings $f^2_k \colon X \to I^n_\circ$, $k \in \mathbb{N}$, such that, for each $K \in X$,

(i) the set $\{f_k^2(K)\,;\, k \in \mathbb{N}\}$ is a dense subset of $\mathrm{Int}(K)$;
(ii) the sequence $(f_k^2(K))_{k=1}^{\infty}$ is injective.

For every $k \in \mathbb{N}$, define $f_{2k-1} := f_k^1$ and $f_{2k} := f_k^2$. Then, for every $K \in X$, the sequence $(f_k(K))_{k=1}^{\infty}$ is injective and $f_k(K)$ belongs to $I_\circ^n \setminus \partial K$ for each $k \in \mathbb{N}$. For every $m \in \mathbb{N}$ and $K \in X$, construct the set $P_m(K)$ in exactly the same way as in the proof of Theorem 3.3.7, before Claim 3.3.7.1. Take an arbitrary point $p_0 \in I^n \times \{-1\}$ and, for every $K \in X$, let

$$\begin{aligned} P(K) &:= \bigcup\{P_m(K);\, m \in \mathbb{N}\}, \\ R(K) &:= \Big\{tx + (1-t)p_0\,;\, x \in \overline{S(K)} \times \{0\},\, t \in I\Big\}, \\ \Phi(K) &:= (K \times \{0\}) \cup P(K) \cup R(K). \end{aligned}$$

Claim 3.3.10.1. The set $\Phi(K)$ is in Y for each $K \in X$.

Proof. Let $K \in X$. Clearly, $\Phi(K) \subseteq I^n \times [-1,1]$. Since K is a compact subset of $\mathbb{R}^n$, every connected component of K intersects ∂K. Moreover, we have

$$\partial K \times \{0\} \subseteq \overline{S(K)} \times \{0\} \subseteq R(K)$$

and it is clear that $R(K)$ is connected. Hence, $(K \times \{0\}) \cup R(K)$ is connected. For each $m \in \mathbb{N}$, $P_m(K)$ is a connected set intersecting $(K \times \{0\}) \cup R(K)$. Thus, $\Phi(K)$ is connected. Since $R(K)$ is the image of the compact set $(\overline{S(K)} \times \{0\}) \times I$ under the continuous mapping $(x,t) \mapsto tx + (1-t)p_0$, it is compact. The compactness of $\Phi(K)$ now follows from Lemma 3.2.4. Since $\mathrm{Int}(K) \neq \emptyset$, the dimension of $\Phi(K)$ is at least n. Thus, by the same reasoning as in the proof of Claim 3.3.7.1, $\Phi(K)$ is n-dimensional. ■

Claim 3.3.10.2. The mapping $\Phi\colon X \to Y$ given by $K \mapsto \Phi(K)$ is a reduction from the homeomorphism ER on X to the homeomorphism ER on Y.

Proof. For every $K \in X$, denote $D_K := \{f_k^2(K)\,;\, k \in \mathbb{N}\}$. Let $K, L \in X$ and let $h\colon K \to L$ be a homeomorphism. By the domain invariance theorem, h maps $\mathrm{Int}(K)$ onto $\mathrm{Int}(L)$ and ∂K onto ∂L. By (2), (3) and [KV20, Proposition 2], there is a homeomorphism $h_0\colon \overline{S(K)} \to \overline{S(L)}$ with $h_0(S(K)) = S(L)$ and $h_0(x) = h(x)$, $x \in \partial K$. Define a mapping $f\colon K \cup S(K) \to L \cup S(L)$ by

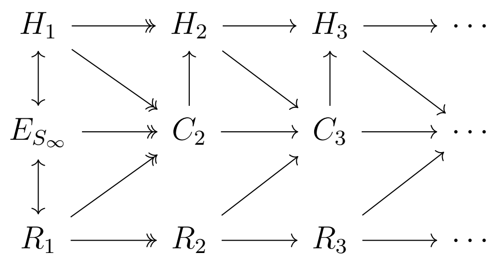
$$f(x) = \begin{cases} h(x) & \text{if } x \in K; \\ h_0(x) & \text{if } x \in S(K). \end{cases}$$

Clearly, f is a homeomorphism. Also, as both of the sets $f(D_K)$, D_L are countable and dense in $\mathrm{Int}(L)$, there is, by Lemma 3.3.4, a homeomorphism $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $g(f(D_K)) = D_L$ and $g(x) = x$ for every $x \in \mathbb{R}^n \setminus \mathrm{Int}(L)$. It follows that $g \circ f$ is a homeomorphism between $K \cup S(K)$ and $L \cup S(L)$ sending $S(K)$ onto $S(L)$, D_K onto D_L, K onto L and ∂K onto ∂L. At this point, it is not difficult to show that $(K \times \{0\}) \cup R(K)$ is homeomorphic to $(L \times \{0\}) \cup R(L)$. Proceeding similarly as in the corresponding part of the proof of Theorem 3.3.7 (see the proof of Claim 3.3.7.2), we conclude that $\Phi(K)$ is homeomorphic to $\Phi(L)$.

Conversely, assume we are given $K, L \in X$ with $\Phi(K)$ homeomorphic to $\Phi(L)$. Let $h\colon \Phi(K) \to \Phi(L)$ be a homeomorphism. Similarly as in the corresponding part of the proof of Theorem 3.3.7 (see the proof of Claim 3.3.7.3) we eventually find that $h(S(K) \times \{0\}) = S(L) \times \{0\}$ and $h(D_K \times \{0\}) = D_L \times \{0\}$. Therefore, $h(\overline{\mathrm{Int}(K)} \times \{0\}) = \overline{\mathrm{Int}(L)} \times \{0\}$ and $h(\partial K \times \{0\}) = \partial L \times \{0\}$. Consequently, we obtain $h(K \times \{0\}) = L \times \{0\}$, which proves that K is homeomorphic to L. ■

Using Remark 3.2.5, it is easy to verify that Φ is Borel measurable, completing the proof of Theorem 3.3.10. □

Remark 3.3.11. From Theorems 3.3.7 and 3.3.10, combined with [CG19, Theorem 3.9] and [CG19, Theorem 4.2], we obtain the following diagram, where simple arrows denote (non-strict) Borel reducibility, two-headed arrows denote strict Borel reducibility and bidirectional arrows denote Borel bireducibility. It is not known which of the simple arrows (if any) could actually be replaced by two-headed arrows.



3.4 Non-classification results

Throughout this section we write J for $[-1, 2]$ and E for the ER on $I^{\mathbb{N}}$ given by $xEy \iff \lim_{n\to\infty} |x_n - y_n| = 0$. It is known that E is not classifiable by countable structures; see [KV20, Lemma 17] for the proof. In this section we prove that E is continuously reducible to both the homeomorphism ER for 2-dimensional absolute retracts in $\mathbb{R}^3$ and the homeomorphism ER for 1-dimensional locally connected continua in $\mathbb{R}^2$. Thus, we get the main results of this section (see Corollaries 3.4.2 and 3.4.5) which say that the above two ERs are not classifiable by countable structures.

Theorem 3.4.1. *E is continuously reducible to the homeomorphism equivalence relation for 2-dimensional absolute retracts in $\mathbb{R}^3$.*

Proof. We need to construct a Borel measurable mapping $\Phi\colon I^{\mathbb{N}} \to \mathsf{AR}_2(\mathbb{R}^3)$ such that, for all $x, y \in I^{\mathbb{N}}$, xEy if and only if $\Phi(x)$ is homeomorphic to $\Phi(y)$. Fixing a suitable 2-cell in $\mathbb{R}^3$, each of the sets $\Phi(x)$, $x \in I^{\mathbb{N}}$, will be obtained by attaching a null family of x-dependent cones (over finite sets) to the 2-cell in such a way that each of the cones intersects the 2-cell only at its vertex. Note that this is similar to what was done in the proof of Theorem 3.3.7 with $n = 2$. This time, however, the cones will be topologically more diverse and their vertices will not form a dense subset of the 2-cell anymore.

Denote $P := [0, 1/2] \times J$ and let $\{q_n\,;\ n \in \mathbb{N}\}$ be a dense subset of J, where $q_i \neq q_j$ for $i \neq j$. For every $n \in \mathbb{N}$, let $A_n \subseteq \{-2^{-n}\} \times \{0\} \times [0, 2^{-n}]$ and $B_n \subseteq \{0\} \times [0, 2^{-n}] \times \{2^{-n}\}$ be arbitrary sets with $|A_n| = 2n + 1$, $|B_n| = 2n$. For all $n \in \mathbb{N}$ and $x \in I^{\mathbb{N}}$, define

$$R_n := \Big\{tp + (0, q_n, 0)\,;\ t \in I,\ p \in A_n\Big\},$$
$$T_n(x) := \Big\{tp + (2^{-n}, x_n, 0)\,;\ t \in I,\ p \in B_n\Big\}.$$

Also, for each $x \in I^{\mathbb{N}}$, let $\mathcal{S}(x) := \{R_n\,;\, n \in \mathbb{N}\} \cup \{T_n(x)\,;\, n \in \mathbb{N}\}$. It is easy to check that $\mathcal{S}(x)$ is a null family of pairwise disjoint cones and that, for every $S \in \mathcal{S}(x)$, the only point where S intersects $P \times \{0\}$ is the vertex of the cone S. This vertex is equal to $(0, q_n, 0)$ if $S = R_n$ and to $(2^{-n}, x_n, 0)$ if $S = T_n(x)$. For every $x \in I^{\mathbb{N}}$, let $\Phi(x) := (P \times \{0\}) \cup \bigcup \mathcal{S}(x)$.

Claim 3.4.1.1. Given any $x \in I^{\mathbb{N}}$, the set $\Phi(x)$ is in $\mathsf{AR}_2(\mathbb{R}^3)$.

Proof. Since $\Phi(x)$ contains $P \times \{0\}$, the dimension of $\Phi(x)$ is no less than 2. The same reasoning as in the proof of Claim 3.3.7.1 shows that $\Phi(x)$ is 2-dimensional and compact. By [Bor67, 10.5], it remains to show that $\Phi(x)$ is contractible and locally contractible. Since each $S \in \mathcal{S}(x)$ intersects $P \times \{0\}$ only at one point, with respect to which S is starlike, $\Phi(x)$ is contractible and every $p \in P \times \{0\}$ has a local base consisting of contractible sets (we use the fact that $P \times \{0\}$ is a cell and that $\mathcal{S}(x)$ is a null family). Of course, $\Phi(x)$ is locally contractible also at every point of the set $\Phi(x) \setminus (P \times \{0\})$. ■

Let us show that the mapping $\Phi\colon I^{\mathbb{N}} \to \mathsf{AR}_2(\mathbb{R}^3)$ given by $x \mapsto \Phi(x)$ is a reduction from E to the homeomorphism ER on $\mathsf{AR}_2(\mathbb{R}^3)$.

Claim 3.4.1.2. For all $x, y \in I^{\mathbb{N}}$, if xEy, then $\Phi(x)$ is homeomorphic to $\Phi(y)$.

Proof. For every $x \in I^{\mathbb{N}}$ and $k \in \mathbb{N}$, let $h_k^x\colon J \to J$ be the unique function which is affine on $[-1, 0]$ and on $[0, 2]$ and which sends -1 to -1, 0 to x_k and 2 to 2. For every $s \in (0, 1/2]$, there are unique $c(s) \in (0, 1]$ and $n(s) \in \mathbb{N}$ such that

$$s = c(s) \cdot 2^{-n(s)} + (1 - c(s)) \cdot 2^{-n(s)-1}.$$

For all $x \in I^{\mathbb{N}}$ and $s \in (0, 1/2]$, denote

$$\varphi_s^x := c(s) \cdot h_{n(s)}^x + (1 - c(s)) \cdot h_{n(s)+1}^x.$$

Clearly, φ_s^x is a self-homeomorphism of J. Observe that, for all $x, y \in I^{\mathbb{N}}$ and $k \in \mathbb{N}$, since $t \mapsto h_k^x(t) - h_k^y(t)$ is affine on both $[-1, 0]$ and $[0, 2]$ and since it maps the set $\{-1, 0, 2\}$ onto $\{0, x_k - y_k\}$, we have $|h_k^x(t) - h_k^y(t)| \leq |x_k - y_k|$ for every $t \in J$. Consequently, for all $x, y \in I^{\mathbb{N}}$, $s \in (0, 1/2]$ and $t \in J$,

$$|\varphi_s^x(t) - \varphi_s^y(t)| \leq \max\Big\{|x_{n(s)} - y_{n(s)}|, |x_{n(s)+1} - y_{n(s)+1}|\Big\}. \qquad (*)$$

Furthermore, note that the function $s \mapsto \varphi_s^x(t)$ is continuous on $(0, 1/2]$ for every $x \in I^{\mathbb{N}}$ and every $t \in J$. In fact, observing that $s \mapsto c(s)$ is 2^{k+1}-Lipschitz on $(2^{-k-1}, 2^{-k}]$ for every $k \in \mathbb{N}$, it follows that $s \mapsto \varphi_s^x(t)$ is $(3 \cdot 2^{k+1})$-Lipschitz on $[2^{-k-1}, 2^{-k}]$ for all $k \in \mathbb{N}$ and $t \in J$. Therefore, since $t \mapsto \varphi_s^x(t)$ is 2-Lipschitz on J for every $s \in (0, 1/2]$ (because so is each h_k^x), the function $(s, t) \mapsto \varphi_s^x(t)$ is $(2 + 3 \cdot 2^{k+1})$-Lipschitz on $[2^{-k-1}, 2^{-k}] \times J$ for every $k \in \mathbb{N}$.

For all $x \in I^{\mathbb{N}}$, define a mapping $\Psi^x\colon P \to P$ by $\Psi^x(s, t) = (s, \varphi_s^x(t))$ for $(s, t) \in (0, 1/2] \times J$ and by $\Psi^x(0, t) = (0, t)$ for $t \in J$. Then Ψ^x is bijective and it is $(3 + 3 \cdot 2^{k+1})$-Lipschitz on $[2^{-k-1}, 2^{-k}] \times J$ for each $k \in \mathbb{N}$. It easily follows that the restriction of Ψ^x to $[a, 1/2] \times J$ is a self-homeomorphism of $[a, 1/2] \times J$ for each $a \in (0, 1/2]$. In particular, Ψ^x and $(\Psi^x)^{-1}$ are continuous at each point of $(0, 1/2] \times J$. For all $x, y \in I^{\mathbb{N}}$, let $\Gamma^{x,y} := \Psi^y \circ (\Psi^x)^{-1}$. Then $\Gamma^{x,y}$ is a bijection on

P and it is continuous at every point of $(0, 1/2] \times J$. Moreover, for all $x, y \in I^{\mathbb{N}}$, $t \in J$ and $s \in (0, 1/2]$, it follows from $(*)$ that

$$\left|\varphi_s^y\Big((\varphi_s^x)^{-1}(t)\Big) - t\right| \leq \max\Big\{|x_{n(s)} - y_{n(s)}|, |x_{n(s)+1} - y_{n(s)+1}|\Big\}.$$

Hence, assuming xEy for the rest of this paragraph (and using the obvious fact that $n(s) \to \infty$ as $s \to 0^+$), we conclude that $\Gamma^{x,y}$ is continuous also at every point of the set $\{0\} \times J$. Therefore, by the compactness of P, $\Gamma^{x,y}$ is a homeomorphism. Since $\Gamma^{x,y}(2^{-k}, x_k) = (2^{-k}, y_k)$ and $\Gamma^{x,y}(0, q_k) = (0, q_k)$ for every $k \in \mathbb{N}$, the same argument as the one used at the end of the proof of Claim 3.3.7.2 (and later in the proof of Claim 3.3.10.1) shows that $\Gamma^{x,y}$ can be extended to a homeomorphism between $\Phi(x)$ and $\Phi(y)$. ■

Claim 3.4.1.3. For all $x, y \in I^{\mathbb{N}}$, if $\Phi(x)$ is homeomorphic to $\Phi(y)$, then xEy.

Proof. Let us proceed by contradiction. Assume that $\Phi(x)$ is homeomorphic to $\Phi(y)$, and yet $(x, y) \notin E$, i.e. the sequence $x - y$ does not converge to 0. Let $\Psi\colon \Phi(x) \to \Phi(y)$ be a homeomorphism. It is easy to see that for every $w \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$, the point $(0, q_n, 0)$ is the unique point $p \in \Phi(w)$ such that the set $\Phi(w) \setminus \{p\}$ has exactly $2n + 2$ connected components. Similarly, the point $(2^{-n}, w_n, 0)$ is the unique point $p \in \Phi(w)$ such that the set $\Phi(w) \setminus \{p\}$ has exactly $2n+1$ connected components. Using this observation together with the fact that Ψ is a homeomorphism and that the set $\{q_n;\ n \in \mathbb{N}\}$ is dense in J, we conclude that $\Psi(p) = p$ for every $p \in \{0\} \times J \times \{0\}$ and that $\Psi(2^{-n}, x_n, 0) = (2^{-n}, y_n, 0)$ for every $n \in \mathbb{N}$. Since the sequence $x - y$ does not converge to 0, standard compactness arguments show that there are $\alpha, \beta \in I$ with $\alpha \neq \beta$ and an increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that $x_{n_k} \to \alpha$ and $y_{n_k} \to \beta$ as $k \to \infty$. Then, however,

$$(0, \alpha, 0) = \Psi(0, \alpha, 0) = \lim_{k\to\infty} \Psi(2^{-n_k}, x_{n_k}, 0) = \lim_{k\to\infty} (2^{-n_k}, y_{n_k}, 0) = (0, \beta, 0),$$

which is a contradiction. ■

It easily follows from Remark 3.2.5 that Φ is continuous, completing the proof of Theorem 3.4.1. □

Corollary 3.4.2. *The homeomorphism equivalence relation for 2-dimensional absolute retracts in $\mathbb{R}^3$ is not classifiable by countable structures.*

Before proving the final theorem of this chapter, let us state the following lemma, which is a reformulation of a well-known result originally proved by Whyburn [Why58]. Other references include [Bor66, p. 82], [Can73] and [DV09, Theorem 7.5.7.].

Lemma 3.4.3. *Let $P \subseteq \mathbb{R}^2$ be homeomorphic to I^2 and let $\mathcal{V}$ be an infinite family of open subsets of $\mathbb{R}^2$ such that:*

1. *$\overline{V} \subseteq \operatorname{Int}(P)$ for every $V \in \mathcal{V}$;*
2. *$\overline{U} \cap \overline{V} = \emptyset$ for all $U, V \in \mathcal{V}$ with $U \neq V$;*
3. *∂V is a Jordan curve for every $V \in \mathcal{V}$;*
4. *the set $\bigcup \mathcal{V}$ is dense in P;*
5. *$\mathcal{V}$ is a null family in $\mathbb{R}^2$.*

Then $P \setminus \bigcup \mathcal{V}$ is homeomorphic to the Sierpiński carpet.

Theorem 3.4.4. *E is continuously reducible to the homeomorphism equivalence relation for 1-dimensional locally connected continua in $\mathbb{R}^2$.*

Proof. For every $n \in \mathbb{N}$, let $O_n := (2^{-n-1}, 2^{-n-1} + 4^{-n-2}) \times (-4^{-n-2}, 4^{-n-2})$ and $P_n := [2^{-n-1}, 2^{-n}] \times J$. Recalling the classical iterative construction of the Sierpiński Carpet, it is clear that one can construct a family $\mathcal{V}$ satisfying the assumptions of Lemma 3.4.3 with $P := [0, 1/2] \times J$ in such a way that the following additional conditions hold:

(i) $O_n \in \mathcal{V}$ for each $n \in \mathbb{N}$;
(ii) for every $V \in \mathcal{V}$, there is $n \in \mathbb{N}$ such that $V \subseteq P_n$;
(iii) $\operatorname{diam}(V) < 4^{-n}$ for all $n \in \mathbb{N}$ and $V \in \mathcal{V}$ with $V \subseteq P_n$.

For every $x \in I^{\mathbb{N}}$, using the notation from the proof of Claim 3.4.1.2, let $\mathcal{W}(x) := \{\Psi^x(V)\,;\ V \in \mathcal{V}\}$. Since Ψ^x is $(3 + 3 \cdot 2^{n+1})$-Lipschitz on P_n for each $n \in \mathbb{N}$, it easily follows from (5), (ii) and (iii) that $\mathcal{W}(x)$ is a null family. Recalling that the restriction of Ψ^x to $[a, 1/2] \times J$ is a self-homeomorphism of $[a, 1/2] \times J$ for every $a \in (0, 1/2]$, it is now clear that $\mathcal{W}(x)$ satisfies all the assumptions of Lemma 3.4.3. Hence, $P \setminus \bigcup \mathcal{W}(x)$ is homeomorphic to the Sierpiński carpet (and thus belongs to $\mathsf{LC}_1(\mathbb{R}^2)$) for every $x \in I^{\mathbb{N}}$.

Let $\{q_n\,;\ n \in \mathbb{N}\}$ be a dense subset of J such that $q_i \neq q_j$ when $i \neq j$. For every $n \in \mathbb{N}$, let $A_n \subseteq \{-1\} \times I$ and $B_n \subseteq \{4^{-n-3}\} \times [0, 4^{-n-3}]$ be arbitrary sets satisfying $|A_n| = 2n + 1$ and $|B_n| = 2n$. Clearly, it is possible to construct a sequence $(r_n)_{n=1}^{\infty}$ of positive real numbers converging to 0 such that the sets $R_n := \{tp + (0, q_n)\,;\ 0 \leq t \leq r_n,\ p \in A_n\}$, $n \in \mathbb{N}$, are pairwise disjoint. For every $n \in \mathbb{N}$, let $T_n := \{tp + (2^{-n}, 0)\,;\ t \in I,\ p \in B_n\}$. Moreover, for every $x \in I^{\mathbb{N}}$, define $\mathcal{S}(x) := \{R_n\,; n \in \mathbb{N}\} \cup \{\Psi^x(T_n)\,; n \in \mathbb{N}\}$. Clearly, members of $\mathcal{S}(x)$ are pairwise disjoint sets and each of them is (homeomorphic to) a cone. Observing that $T_n \setminus \{(2^{-n}, 0)\} \subseteq O_{n-1}$ for each $n \in \mathbb{N} \setminus \{1\}$, it is clear that every $S \in \mathcal{S}(x)$ intersects $P \setminus \bigcup \mathcal{W}(x)$ only at one point – the vertex of S. This vertex is equal to $(0, q_n)$ if $S = R_n$ and to $\Psi^x(2^{-n}, 0) = (2^{-n}, x_n)$ if $S = \Psi^x(T_n)$. Since $\{\Psi^x(O_n)\,;\ n \in \mathbb{N}\} \subseteq \mathcal{W}(x)$ is a null-family (and since $r_n \to 0$ as $n \to \infty$), so is $\mathcal{S}(x)$. Hence, as every member of $\mathcal{S}(x)$ belongs to $\mathsf{LC}_1(\mathbb{R}^2)$, it follows from Lemma 3.2.4 and from the countable sum theorem for topological dimension that

$$\Phi(x) := \Big(P \setminus \bigcup \mathcal{W}(x)\Big) \cup \bigcup \mathcal{S}(x)$$

belongs to $\mathsf{LC}_1(\mathbb{R}^2)$.

We claim that the mapping $\Phi\colon I^{\mathbb{N}} \to \mathsf{LC}_1(\mathbb{R}^2)$ given by $x \mapsto \Phi(x)$ is a reduction from E to the homeomorphism ER on $\mathsf{LC}_1(\mathbb{R}^2)$. Indeed, since (for all $x, y \in I^{\mathbb{N}}$) the mapping $\Gamma^{x,y}$ from the proof of Claim 3.4.1.2 maps $\bigcup \mathcal{W}(x)$ onto $\bigcup \mathcal{W}(y)$ and since the Sierpiński carpet has no cut points, it is clear that the proofs of Claims 3.4.1.2 and 3.4.1.3 apply to our situation. Moreover, since it is easy to show (with the help of $(*)$ from the proof of Claim 3.4.1.2) that the mapping $x \mapsto P \setminus \bigcup \mathcal{W}(x)$ is continuous, it follows from Remark 3.2.5 that Φ is continuous. □

Corollary 3.4.5. *The homeomorphism equivalence relation for 1-dimensional locally connected continua in $\mathbb{R}^2$ is not classifiable by countable structures.*

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