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Subject: Report on Thesis of Dalimil Peša

To whom it concerns

The doctoral thesis by Dalimil Peša focuses on fine properties of specific function spaces, supervised by prof. RNDr. Luboš Pick. It includes six papers covering abstract functions spaces classes, a particular function spaces, and applications in function space theory, such as quasi-Banach spaces and Wiener–Luxemburg amalgam spaces, Lorentz–Karamata spaces, and application on nonlinear PDEs.

The main topics discussed in thesis are:

(a) Function spaces in Abstract in papers (I) and (II)

(b) Lorentz–Karamata spaces in papers (III) and (IV),

(c) detailing slowly varying functions and applications of function spaces in Papers (V) and (VI).

Here are quick notes about papers which are part of thesis:

Paper I.
Title: On the properties of quasi-Banach function spaces1
Authors: Aleš Nekvinda and Dalimil Peša

This paper was a part of Peša's Master thesis.

The paper explores some basic properties of quasi-Banach function spaces, such as completeness, embeddings, separability, and boundedness of the dilation operator. The paper also extends the classical Riesz-Fischer theorem to the context of quasinormed spaces and shows that quasi-Banach function spaces satisfy a generalised Riesz-Fischer property (Theorem 3.3 and Theorem 3.6).

The paper shows that an embedding between quasi-Banach function spaces is always continuous and that the dilation operator is bounded on rearrangement invariant quasi-Banach function spaces (Theorem 3.8 and Theorem 3.22).

Also this paper provides a characterisation of separability for quasi-Banach function spaces over the Euclidean space in terms of the absolute continuity of the quasinorm (Theorem 3.18).

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Paper II.
Title: Wiener–Luxemburg amalgam spaces12
Author: Dalimil Peša
Journal: Journal of Functional Analysis
Year: 2022

This paper is an extension of results from author's Master thesis.

The paper introduces a new concept of Wiener–Luxemburg amalgam spaces, which are a modification of the classical Wiener amalgam spaces that allow for separate prescription of the local and global behaviour of functions in the context of rearrangement-invariant Banach and quasi-Banach function spaces. The properties of Wiener–Luxemburg amalgam spaces, such as their normability, embeddings, associate spaces, and relation to the Hardy–Littlewood–Pólya principle is studied.

The developed theory is used to resolve some open questions in the field, such as the validity of the Hardy–Littlewood–Pólya principle for all rearrangement-invariant quasi-Banach function spaces.

The paper also illustrates the shortcomings of Wiener amalgam spaces by providing counterexamples to certain properties of Banach function spaces and rearrangement invariance.

Paper III.
Title: Lorentz-Karamata Spaces1
Author: Dalimil Pesa

The paper studies the properties of Lorentz-Karamata spaces, which are function spaces defined using slowly varying functions and generalise several classical spaces such as Lebesgue, Lorentz, Zygmund, and Orlicz spaces. The paper covers topics such as non-triviality, embeddings, associate spaces, fundamental functions, endpoints, Boyd indices, and equivalence to Banach function spaces for Lorentz-Karamata spaces.

The Main Results are: Several characterisations and classifications of Lorentz-Karamata spaces, such as:
-A complete description of the cases when a Lorentz-Karamata space is non-trivial, i.e. contains non- zero functions.
-A complete characterisation of the embeddings between Lorentz-Karamata spaces in terms of the parameters and the slowly varying functions involved.
-A complete description of the associate spaces of Lorentz-Karamata spaces, which are dual spaces with respect to the Hardy-Littlewood-Polya inequality.
-A computation of the fundamental functions of Lorentz-Karamata spaces, which measure the growth of the norms of characteristic functions of sets of given measure.
-A description of the endpoint spaces of Lorentz-Karamata spaces, which are the limits of the spaces as the parameters approach zero or infinity.
-A computation of the Boyd indices of Lorentz-Karamata spaces, which are important quantities in the theory of interpolation of operators.
-A complete characterisation of the cases when a Lorentz-Karamata space is equivalent to a Banach function space, which is a space with a norm that satisfies the triangle inequality

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Paper IV.
Title: On the smoothness of slowly varying functions
Author: Dalimil Pesa

The paper studies the question of smoothness of slowly varying functions, which are functions that satisfy a certain asymptotic condition involving monotone functions.

The main results is Theorem 1.2: Every slowly varying function is equivalent to a smooth function that is also slowly varying.

The paper also develops some new lemmas for decomposing and combining slowly varying functions.

Paper V.
Title: Reduction principle for a certain class of kernel-type operators1
Author: Dalimil Peša

The paper studies the boundedness of integral operators of the form

$$\int_t^\infty f(s)I(s)\left(\int_t^s \frac{1}{I(r)}\,dr\right)^{m-1} ds$$

where *I* is a non-decreasing function and *m* is a positive integer. The paper shows that the boundedness of such operators on rearrangement invariant Banach function spaces is equivalent to their boundedness on the cone of non-increasing functions, under some mild conditions on *I*. This generalizes a previous result by Carro et al. (2001) for finite intervals.

The main result of the paper is Theorem 3.11, which states the equivalence of the two boundedness statements mentioned above. The proof relies on some auxiliary lemmas, such as Lemma 3.9, which characterizes the set of points where the inequality between the operator and its non-increasing rearrangement is strict. The paper also provides some examples and applications of the main result, such as Sobolev embeddings and potential operators.

Paper VI.
Title: Nonlinear Gagliardo–Nirenberg inequality and a priori estimates for nonlinear elliptic eigenvalue problems
Authors: Agnieszka Kalamajska, Dalimil Peša, and Tomáš Roskovec.

The paper presents some identities and inequalities involving elliptic operators, Sobolev norms, and nonlinear transformations of functions. The main applications are to obtain a priori estimates for solutions of nonlinear eigenvalue problems with various types of nonlinearities, such as power, power-logarithmic, or exponential. The paper also discusses some connections with probability and potential theory, harmonic analysis, and analytic semigroups.

The paper contains several theorems that establish identities and inequalities of the form

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$$\int_\Omega |\nabla u(x)|^2 A\, h(u(x))dx = -\int_\Omega P\, u \cdot H(u)dx - \int_\Omega div\, A \cdot \nabla u \cdot H(u)dx + \Theta,$$

and

$$\int_\Omega |\nabla u(x)|^2 A\, h(u(x))dx \leq \int_\Omega |Pu||H(u)|dx + \Theta,$$

where **Ω** is a bounded Lipschitz domain, *u* is a nonnegative function in a Sobolev space, *P* is a uniformly elliptic operator in nondivergent form, *A* is the ellipticity matrix, *h* is a positive continuous function, *H* is an antiderivative of *h*, and *Θ* is a boundary term. The paper also shows how to apply these inequalities to obtain a priori estimates for solutions of nonlinear eigenvalue problems of the form

$$Pu = f(x)\tau(u),$$

where *f* is a given function in $L^1(\Omega)$ and τ is a nonlinear function. The paper provides several examples of such problems, including Emden–Fowler and Liouville-type equations.

Summary: The thesis consists of several papers that present original and novel results in various areas of mathematics related to the theory of quizi-Banach spaces, special Banach function spaces and their applications (on PDE, etc.). The results demonstrate exceptional quality and depth, exceeding the standards of originality and rigor expected from a graduate-level thesis. This is also evidenced by the high-quality journals where some of the papers have been published, as well as by the fact that Peša is the sole author of most of the submitted papers.

Conclusion: The thesis surpasses the level of quality and originality required for a PhD thesis in mathematics, and I strongly recommend it to be accepted and awarded the grade of “exceptional”. If you have any questions, please feel free to contact me.

Sincerely,

Jan Lang

Prof. Jan Lang