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Garching, February 14, 2024

Report on the PhD thesis of Michael Skotnica

Dear members of the thesis committee,

I am writing to report on my assessment of the thesis titled *Combinatorics, group theory, computational complexity & topology*.

The topic of the thesis lies in the area of combinatorial and PL topology, covering problems related to shellability, collapsibility, Lusternik–Schnirelmann category, and computing homotopy groups. The problems are well motivated and of clear relevance to the field, and parts of the thesis have been published in a highly regarded journals (SIAM Journal on Discrete Mathematics).

Contents

The thesis concerns several combinatorial aspects of the topology of simplicial complexes.

Chapter 1 provides a brief and concise discussion of the preliminaries and the required background, clarifying the terminology and conventions without discussing details that are familiar to experts. The content is well-chosen and suitable for the natural audience of this work, with only a few topics that would have deserved a more detailed introduction, as mentioned below.

The topic of **Chapter 2**, establishing a principled connection between shellability and collapsibility, has drawn significant interest in recent years due to the celebrated NP-hardness proof of shellability in [GPP+19]. A critical step in that proof is a result by Hachimori providing an equivalent condition for shellability of the second barycentric subdivision of a complex in terms of collapsibility, which eventually places the problem in the realm of simplicial collapses and discrete Morse theory. This argument by Hachimori is rather ad hoc and specific to 2-complexes, and so it is very important to have a full and systematic understanding of this connection in arbitrary dimension. The challenge of obtaining a shelling of the second barycentric subdivision from a discrete gradient of the original

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complex (with critical cell only in the top dimension, in addition to the mandatory critical 0-cell) lies in the inherent complexity of the large number of simplices in the second barycentric subdivision, making it difficult to generate a suitable shelling order. The key idea is the use of certain more structured types of decomposition that imply shellability but are described using different primitive operations. In particular, the central notion introduced in this work is *star decomposability*, a notion that can be linked to the well-established property of vertex decomposability. This connection is not immediate from the definitions, and the implication is an interesting result by itself. Of particular relevance for the purpose of the thesis is the special case of *star decomposability in vertices*, which applies to barycentric subdivisions only and concerns decompositions by closed stars of the original vertices. Finally, the pairs in an elementary collapse in the original complex can be brought into correspondence with pairs of vertex stars in the second barycentric subdivision, in such a way that the collapsing order induces a star decomposition. While I would still hope for a simpler approach to the main result (there are many variants to the proposed method one could try to investigate), the approach followed in the thesis is carried out in a way that makes the numerous technical details well navigable.

Although the topic of **Chapter 3** appears quite different from the previous chapter, the mentioned hardness result for shellability provides a common motivation for both. In this chapter, Skotnica discusses a variant of the Lusternik–Schnirelmann category, asking for the minimum number of collapsible PL subspaces covering a given polyhedron. The key insight relating this problem to shellability is the fact that shellable 2-dimensional polyhedra can always be covered by two collapsible subpolyhedra. The converse implication however does not hold, and in fact, the complexes constructed in the reduction of shellability from [GPP+19] always such a covering by two collapsible subspaces, regardless of their shellability. The surprising strategy used to address this issue is to glue a torus along a meridian to the boundary of each triangle. This increases the number of required collapsible subspaces to 3 precisely for the non-shellable spaces arising form the reduction. This way, the original argument for hardness of shellability is revamped to show that the decision problem whether a polyhedron can be covered by two collapsible subpolyhedra is NP-hard.

Finally, **Chapter 4** concerns a problem that is not of topological nature itself, but is motivated by a topological question: the *VEST* problem asks for counting all possible choice of matrices from a given list (with repetitions allowed) whose product maps a given vector to the kernel of a given matrix. This problem appears in the proof of a seminal result in computational topology by Anick, establishing the #P-hardness of computing rational homotopy groups. In this thesis, Skotnica considers several variants of the problem, establishing their parametrized complexity, where the parameter is the number of matrices chosen. In particular, Skotnica improves a previous result by Matousek about W[1]-hardness of VEST, showing that the counting problem is #W[2]-hard by a reduction from the #W[2]-complete counting problem for dominating sets of a given cardinality in a graph. Since the complexity classes considered here are not too well known, I would have liked to see a discussion in the preliminaries. Moreover, [Mat13] does not mention #W[1]-hardness, so the claim that this paper actually establishes #W[1]-hardness should be substantiated. Besides, the thesis does not explicitly discuss the notion of a parsimonious reduction, which is required for establishing #W[2]-hardness; I am assuming that the nonstandard term *FPT counting reduction* is used to describe this property. (The reduction used is indeed shown to be parsimonious.) The results are complemented by a fixed-parameter tractability result for matrices over finite fields (instead of rationals) and a constraint on the nonzero rows of the matrices. Finally, the problem without parameter, asking for whether the *VEST* problem has a solution regardless of the number of matrices chosen, is shown to be undecidable by a slick reduction from the classical Post Correspondence Problem.

Assessment

In the submitted thesis, Michael Skotnica demonstrates the ability to carry out research in topics that span a wide range of mathematical skills, combining expertise from combinatorial and PL topology with complexity theory. He demonstrated deep and solid knowledge in the requisite background and managed to achieve substantial findings that lead to notable progress in the area of combinatorial topology. The methods developed in this thesis show an original and creative approach to well-motivated problems. I especially appreciate his systematic approach in the results of Chapter 2, generalizing previous specific results obtained through somewhat ad hoc methods to stronger and more general results by much more principled and transparent methods, significantly improving our understanding of the relationship between the various notions of decomposition of simplicial complexes and polyhedra, a central topic of the field.

In summary, I consider the submitted work to clearly and fully meet the expectations and criteria for awarding the doctoral degree, and I do not hesitate to recommend that it should be accepted by the Department of Mathematics and defended in front of the examination committee.

Sincerely,

Ulrich Bauer

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