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Distance Magic Labelings

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Dedication. I would like to thank my supervisor Petr Gregor for his guidance and calmness while I have been casting magic over a distance. His timeliness and thorough editing remarks are what have made this thesis possible. My parents deserve a special thanks for all of their support throughout the years, and for inspiring me to pursue my interest in science and technology.

Title: Distance Magic Labelings

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Abstract: A distance magic labeling of a graph G is a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that the sum of labels on the neighbourhood of each vertex is constant. A framework based on linear algebra has been developed using the notion of neighbour balance to determine whether there exists a distance magic labeling for a hypercube with dimension n. In this thesis, we extend this framework to all Cayley graphs on $\mathbb{Z}_2^n$. We use this framework to reprove some known results from recent literature. We also use this framework to introduce the notion of *component-wise* distance magic labelings on Cayley graphs of $\mathbb{Z}_2^n$.

Keywords: distance magic labeling, Cayley graph, hypercube, neighbour balance

Contents

Introduction

Several practical problems in real life situations have motivated studying of labelings of graphs. One such labeling is the distance magic labeling, which in some sense is a generalization of magic squares. Formally, a distance magic labeling of a graph $G = (V, E)$ is a bijection $f : V \to \{1, 2 \ldots, |V|\}$ such that there is a constant m such that for each vertex $v \in V$, $\sum_{u \in N(v)} f(u) = m$. A survey of these problems conducted by Arumugam et al. [1] notes that the problem of existence of a distance magic labelings is wide and this field is ununified, with many papers using different terminology and sometimes yielding the same results. To clean up some results in this field, we establish a new framework for deducing the existence of distance magic labelings on the class of Cayley graphs of $\mathbb{Z}_2^n$.

One of the most common examples of distance magic labelings is in the design of *equal-sized incomplete fair tournaments* [1]. Consider the problem of scheduling a one-divisional sports tournament of n contestants where the skill level of each contestant is known in advance by a ranking from 1 to n, and there are constraints on the amount of time in which the tournament can take place. Furthermore, we wish for each contestant to face the same number of r opponents throughout the tournament, and for the total strength of opponents which a contestant v faces is a constant m. In this manner, the tournament is deemed *fair*. The problem of constructing the format of such a tournament can be resolved in the following manner. Let us define an r-regular graph G as the representation of the tournament, with each vertex v representing a contestant, and each neighbourhood $N(v)$ of v representing the set of r opponents of v. Finding a tournament format which satisfies these constraints is thus equivalent to finding a distance magic labeling f of graph G, where $f(v)$ is the strength of v.

The motivation for analyzing Cayley graphs of $\mathbb{Z}_2^n$ comes from the most well-studied subclass of this, the hypercube Q_n. The hypercube is a well-studied structure in multiple areas of software and hardware design. Its properties of symmetry and recursiveness give it a desirable structure to work with. Furthermore, the natural modelling of a hypercube by binary strings under bit flips gives an effective base to perform a variety of computations. One such example is that if parallel processing by an arrangement of 2^n single processors into a hypercube structure, called a hypercube interconnection scheme. By connecting each processor to its n neighbours, problems such as routing have efficient algorithms.

A distance magic labeling on a hypercube interconnection scheme can be compared to how to distribute tasks across individual processors, where each task has a known level of computational difficulty or priority. By ensuring that the sum of severity of each task on a given neighbourhood of each processor is constant, the tasks are distributed such that no collection of processors with a common neighbour receives too many "difficult" tasks, thus preventing parts of the network resulting in a bottleneck. Each collection is a set of processors (vertices) which share a common neighbour. We study a related way of distributing such tasks by redefining what consitutes a collection of processors. This is done via an application of Cayley graphs of $\mathbb{Z}_2^n$ with different generators to alter the edge structure of the hypercube network.

Some results on distance magic labelings of hypercube-like graphs have recently been discovered. Results by Kang et al. [2] have proven the existence of a distance magic labeling on the half cube, and results by Tian et al. [3] have proven the existence of a distance magic labeling of the folded half cube. These results use a different representation of the label set in the binary form, and study bijections from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$. This formulation of the label set gives us a nice property, the notion of *neighbour balance* first described by Gregor and Kovář [4]. A bijection from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^n$ is said to be neighbour balanced if for each vertex u of Q_n, and for each coordinate $i \in [n]$, the number of neighbours of u which contain the entry 0 at coordinate i is equal to the number of neighbours of u which contain the entry 1 at coordinate i. This was used to prove the conditions of n in which a distance magic labelings of Q_n exists. Neighbour balance is a key component of our developments, as we extend this definition to include all Cayley graphs of $\mathbb{Z}_2^n$.

We now describe the structure of this thesis. In the next chapter we introduce some notation related to graphs, groups, linear algebra, and hypercubes to give us the proper tools to develop the aformentioned new framework of this thesis. In Chapter 2, we discuss distance magic labelings on Q_n as a necessary step towards the generalized new framework. The new framework is then derived. In Chapter 3, we reprove some known results of hypercube-like graphs using our new framework, which we reformulate as Cayley graphs of $\mathbb{Z}_2^n$. In Chapter 4, we introduce a new notion of *component-wise* distance magic labelings, a generalization of distance magic labelings which have a natural formulation for Cayley graph of $\mathbb{Z}_2^n$.

1. Preliminaries

1.1 Graph Theory

We recall some relevant definitions and notations from graph theory. To gain familiarity with these concepts, we recommend the textbook Algebraic Graph Theory by Godsil and Royle [5].

For an undirected graph G, let $V(G)$ be the vertex set of G, and $E(G)$ be the edge set of G. The *distance* between two vertices u and v is the length of the shortest path between u and v. The *diameter* of a graph G, denoted by $diam(G)$, is the largest distance between any two vertices in G. A graph G is said to be r-regular if the degree of each vertex is exactly r. A vertex $u \in V(G)$ has a *self loop* if there is an edge from u to itself.

The *neighbourhood* of a vertex u, denoted by $N(u)$, is the set of all vertices v such that uv is an edge of G. Clearly, all vertices v in the neighbourhood of u are of distance exactly one. This concept can therefore be extended to a *D-distance* neighbourhood $N_D(v)$, defined as the set of vertices v of distance $d \in D$ from u, where $D \subseteq \{1, 2, \ldots, diam(G)\}$. Note that both of these definitions do not include the vertex u itself. If u is included, we speak about the *closed* neighbourhood and *closed* D-distance neighbourhood of u, denoted by $N[u]$ and $N_D[u]$, respectively. Equivalently, the closed neighbourhood of u is equal to the neighbourhood of u if u has a *self loop*. By altering the edge structure of a graph G such that every u becomes adjacent to its D-distance neighbourhood (and existing edges are removed), we obtain the D-distance graph of G.

A graph H is said to be an *induced subgraph* of G if and only if H is obtained exclusively by the deletion of vertices of $V(G)$, along with the deletion of incident edges.

A graph G is *bipartite* if there is a partition of $V(G)$ into two classes, such that for any two vertices u and v in a class, there is no edge between them. By taking the $\{2\}$-distance graph of a bipartite graph G, two components are yielded, with each component consisting of vertices of one partition classes. Each component is called the *halved graph* of G.

Graph theory is closely related to linear algebra through the study of graph spectra. Let A_G be the adjacency matrix of G. The spectrum of G is then defined to be the set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A_G.

An automorphism on a graph G is a permutation π which acts on vertices of G such that if uv is an edge of G, then $\pi(u)\pi(v)$ is an edge of G. That is, vertices of G are permuted in a way such that edges are preserved. We define the group of all automorphisms on a graph G as $AUT(G)$, with the group action being the usual composition of permutations. A graph G is said to be *vertex-transitive* if $AUT(G)$ acts transitively. That is, for any two vertices u, v of G, there exists an automorphism that maps u to v.

1.2 Group Theory and Cayley Graphs

It is assumed that the reader has elementary knowledge of group theory.

Let $(\Gamma, \cdot, ^{-1}, e)$ denote a group with identity element e, and binary operation $(\cdot)$, where an inverse of an element a is denoted by a^{-1}. Informally, we will simply refer to $(\Gamma, \cdot, ^{-1}, e)$ by Γ. A subset $S \subseteq \Gamma$ is said to be a *generating set* of Γ if and only if every element of Γ can be composed by elements of S. That is, $\Gamma = \langle S \rangle$.

Given a group Γ and a subset $S \subseteq \Gamma$, the *Cayley graph* $Cay(\Gamma, S)$ is obtained from G and S in the following way:

- the vertices of $Cay(\Gamma, S)$ are all elements of Γ,
- there is a directed edge (u, v) in $E(Cay(\Gamma, S))$ if and only if $uv^{-1} \in S$.

Note that if S is closed under inverses, then it is more straightforward to consider $Cay(\Gamma, S)$ as an undirected graph. Furthermore, $Cay(\Gamma, S)$ is connected if and only if S is a generating set. We also assume that the identity element e may be in S. If so, then each vertex of $Cay(\Gamma, S)$ has a *self loop*.

Furthermore, all Cayley graphs are vertex-transitive. Thus all connected components of $Cay(\Gamma, S)$ are isomorphic.

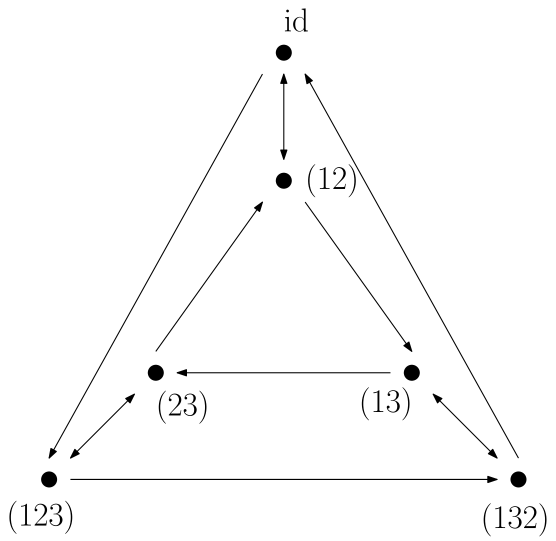

Figure 1.1: Example of a Cayley graph of the symmetric group S_3, with the form $Cay\left(S_3, \{(12), (123)\}\right)$.

We now recall the notion of direct products of two groups $(\Gamma, \cdot_\Gamma)$ and $(\Delta, \cdot_\Delta)$, denoted by $\Gamma \times \Delta$. The element set of $\Gamma \times \Delta$ is $\{(g, h) \mid g \in \Gamma, h \in \Delta\}$, and a new binary operation $(\cdot_{\Gamma\times\Delta})$ is characterized by $(g_1, h_1) \cdot_{\Gamma\times\Delta} (g_2, h_2) = (g_1 \cdot_\Gamma g_2, h_1 \cdot_\Delta h_2)$ for every $g_1, g_2 \in \Gamma$ and $h_1, h_2 \in \Delta$. It follows that the direct product of two groups follows the group axioms, and therefore is also a group.

While Cayley graphs can be defined on any group, we focus on the powers of the cyclic group $\mathbb{Z}_2$. $\mathbb{Z}_2$ is the simplest nontrivial group, and consists solely of elements $\{0, 1\}$, with the group operation being addition mod two (denoted by $\oplus$), and the identity element 0. The inverse of each element is then clearly the element itself.

Taking the n-fold direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, we obtain the group $\mathbb{Z}_2^n$ which is of particular importance. Namely, each element u of $\mathbb{Z}_2^n$ is a tuple from $\{0, 1\}^n$, with identity element $(0, 0, \ldots, 0)$. The group operation $\oplus$ acts on each index independently. In other words, this is the *xor* operation on binary strings of length n. Using this operation is analogous to bit flipping, where flipping a bit u_i for some element u and index $i \in [n]$ is the act of changing u_i to its opposite atomic value of 0 or 1.

We distinguish some interesting elements of $\mathbb{Z}_2^n$. For each $i \in [n]$, we define e_i as the element consisting of 0 in each index aside from the index i, consisting of 1. Any element $u = v \oplus e_i$ is obtained by flipping bit i of v. Likewise, we also define the element $\mathbf{1}$ as $(1, 1, \ldots, 1)$. Any element $u = v \oplus \mathbf{1}$ is obtained by flipping every bit of v.

We define the translation of a subset $A \subseteq \mathbb{Z}_2^n$ by a constant c by $A \oplus c := \{a \oplus c \mid a \in A\}$.

1.3 Linear Algebra

We use the notation of a matrix A with entries a_{ij} at row i and column j.

Matrices can be written in the *block form* with each block consisting of a submatrix A' of A. Simple block matrices for our purposes are the constant blocks $\mathbf{1}$ of 1's, and $\mathbf{0}$ of 0's, along with the identity matrix $\mathbf{I}$.

For a matrix A, the set of *eigenvalues* of A is a set of scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that each λ_i has a corresponding vector x_i^T (called an *eigenvector*) such that $Ax_i^T = \lambda_i x_i^T$.

We work with matrices over the field $\mathbb{Z}_2$. For a binary matrix A, we say that A is *balanced* if and only if the number of columns k of A is even, and each row a_i has exactly $\frac{k}{2}$ 1's.

When working with a matrix A, we are often interested in performing elementary row operations on A. These operations are (for rows R_i, R_j):

1. row switching, $R_i \leftrightarrow R_j$
2. row multiplication by a scalar k, $kR_i \rightarrow R_i$
3. row addition, $R_i + kR_j \rightarrow R_i$

These operations correspond to multiplication of A on the left by a corresponding row operation matrix. These matrices are well defined for each operation. In the following matrices, if not shown otherwise, each entry is 0.

The row switching operation matrix with rows i and j switched is obtained by swapping row i and j in the identity matrix.

$$\begin{pmatrix}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 &&&& 1 &&& \\
&&&& 1 &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& 1 &&&& \\
&&& 1 &&&& 0 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{pmatrix}$$

Likewise, row i multiplication by k is obtained by multiplying A by the identity matrix with the i-th diagonal entry replaced with k.

$$\begin{pmatrix}
1 &&&&&& \\
& \ddots &&&&& \\
&& 1 &&&& \\
&&& k &&& \\
&&&& 1 && \\
&&&&& \ddots & \\
&&&&&& 1
\end{pmatrix}$$

Lastly, the addition of k times row i to row j is obtained by multiplying A by the altered identity matrix with $a_{ij} = k$.

$$\begin{pmatrix}
1 &&&&&& \\
& \ddots &&&&& \\
&& 1 && k && \\
&&& \ddots &&& \\
&&&& 1 && \\
&&&&& \ddots & \\
&&&&&& 1
\end{pmatrix}$$

Note that when working over $\mathbb{Z}_2^n$, the only nontrivial row additions occur for $k = 1$, and row multiplication is not of concern.

The other aspect of linear algebra we will use is that of vector spaces. For a given vector space V, we define the basis of V to be a set of linearly independent vectors $B = \{b_1, b_2, \dots, b_d\}$ such that every element $v \in V$ can be written as a linear combination of vectors of B. The minimum such size d of B is called the *dimension* of V.

This formulation of vector spaces has a natural application when working over the vector space of elements of the field $\mathbb{Z}_2^n$. The standard basis of $\mathbb{Z}_2^n$ is naturally defined as the set $\{e_1, e_2, \ldots, e_n\}$. Furthermore, a subspace C of $\mathbb{Z}_2^n$ of dimension d can be represented by a $(d \times n)$ matrix S where the rows of S span C. That is, C is the *row space* of S. From the view of coding theory, C is a *linear code* over $\mathbb{Z}_2$ of dimension d [6]. The orthogonal complement of C is defined as the set of all elements $u \in \mathbb{Z}_2^n$ such that for every element $c \in C$, $cu^T = 0$. The dimension of the orthogonal complement is $n - d$.

1.4 Hypercubes

Definition 1. *Let n be a natural number. The hypercube of dimension n, denoted by Q_n is the graph with vertices defined by the set of all binary strings $\mathbb{Z}_2^n = \{0,1\}^n$. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}_2^n$. Then an edge of Q_n occurs between vertices u and v whenever $u = v \oplus e_i$ for some $i \in [n]$*

Note that the *Hamming distance* $d_H(u, v)$, the number of bits in which two vertices u and v differ, is the shortest distance between u and v in Q_n. Furthermore, if all n bits between u and v differ, i.e. u and v are of distance n, then u and v are said to be *antipodal*.

We define the size of a vertex u by $|u| := |\{i \in [n] \mid u_i = 1\}|$. This property of vertices of Q_n induces bipartition on Q_n based on the parity of size. This follows from the observation that each vertex $u \in V(Q_n)$ with an even size is only adjacent to vertices with odd size, and vice versa.

A useful observation about Q_n is that the subgraph induced by fixing any bit $i \in [n]$ is a hypercube of dimension $n - 1$. This is referred to as a *subcube* of dimension $n - 1$ of Q_n. In general, this process can be recursively repeated to show that Q_n contains all subcubes of dimension less than n. Specifically, Q_n has $\binom{n}{n-k} 2^{n-k}$ subcubes of dimension k, obtained by fixing $n - k$ bits as either 0 or 1, and thus the remaining k bits form Q_k.

For our purpose, it will be useful to have an equivalent definition of Q_n, defined as a Cayley graph on $\mathbb{Z}_2^n$. That is, $Q_n = Cay(\mathbb{Z}_2^n, \{e_1, e_2, \ldots, e_n\})$, under the group operation $(\oplus)$. Since each element of $\mathbb{Z}_2$ is its own inverse, each element of $\mathbb{Z}_2^n$ is also its own inverse. Therefore, each edge of $Cay(\mathbb{Z}_2^n, \{e_1, e_2, \ldots, e_n\})$ is undirected.

Using the Cayley graph definition, we can introduce the notion of hypercube-like graphs, with altered edge structure, namely folded cubes, half cubes, and folded half cubes.

Definition 2. *The folded n-cube FQ_{n-1} is the graph with the vertex set $\mathbb{Z}_2^{n-1}$, and vertices u and v are adjacent whenever $u = v \oplus e_i$ for some $i \in [n-1]$ or $u = v \oplus \mathbf{1}$.*

Equivalently, we can define FQ_{n-1} as a Cayley graph:

$$FQ_{n-1} = Cay(\mathbb{Z}_2^{n-1}, \{e_i \mid i \in [n-1]\} \cup \{\mathbf{1}\}).$$

On appearance, it may be strange why we work over $\mathbb{Z}_2^{n-1}$ instead of $\mathbb{Z}_2^n$, but there is a valid reasoning behind this. We define the *gluing* of two vertices u and v as the merging of u and v into a single vertex $w = \{u, v\}$. The neighbourhood of w is then defined as the union of $N(u)$ and $N(v)$. We claim that FQ_{n-1} can be alternatively defined in terms of gluing antipodal vertices of Q_n. Clearly, there are 2^{n-1} pairs of such vertices to glue, and the neighbourhood of each pair of glued vertices $\{u, v\}$ consists of all other pairs of glued vertices $\{s, t\}$ such that $d_H(u, s) = 1$ or $d_H(u, s) = n - 1$. Thus the generating set of FQ_{n-1} is the set of all vectors e_i of size 1, and the vector $\mathbf{1}$ of size $n - 1$.

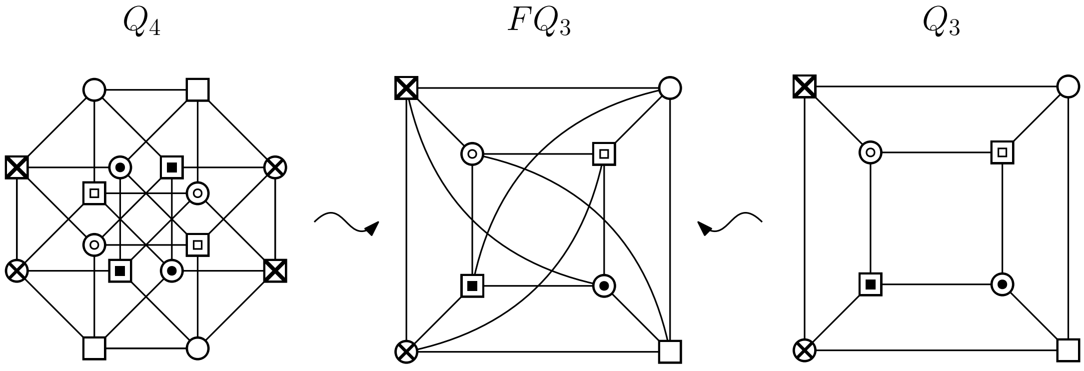

Figure 1.2: The folded 4-cube FQ_3 obtained by gluing of Q_4 or by additional edges to Q_3.

Definition 3. *The half cube $\frac{1}{2}Q_n$ is the component of the halved graph of Q_n, with the vertex set consisting of even size vertices of $\mathbb{Z}_2^n$, and vertices u and v are adjacent whenever $d_H(u, v) = 2$.*

Equivalently,

$$\frac{1}{2}Q_n = Cay(\{u \in \mathbb{Z}_2^n \mid |u| \equiv 0 \pmod 2\}, \{e_i \oplus e_j \mid i, j \in [n], i \neq j\}).$$

Note that the halved graph of Q_n consists of two isomorphic components, on the set of all even size vertices, and on the set of all odd size vertices. In some scenarios, it is not sufficient to exclude half of all binary strings of length n. Thus if we want to work over both components (denoted $\frac{1}{2}Q'_n$), we have the Cayley graph formulation

$$\frac{1}{2}Q'_n = Cay(\mathbb{Z}_2^n, \{e_i \oplus e_j \mid i, j \in [n], i \neq j\}).$$

Definition 4. *The folded half n-cube, denoted $\frac{1}{2}FQ_{n-1}$ (where n is even) is the graph with the vertex set $\{u \in \mathbb{Z}_2^{n-1} \mid |u| \equiv 0 \pmod 2\}$, and vertices u and v are adjacent whenever $d_H(u, v) = 2$ or $d_H(u, v) = n - 2$.*

Equivalently,

$$\frac{1}{2}FQ_{n-1} = Cay(\{u \in \mathbb{Z}_2^{n-1} \mid |u| \equiv 0 \pmod 2\}, \{e_i \oplus e_j \mid i, j \in [n-1], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-1]\}).$$

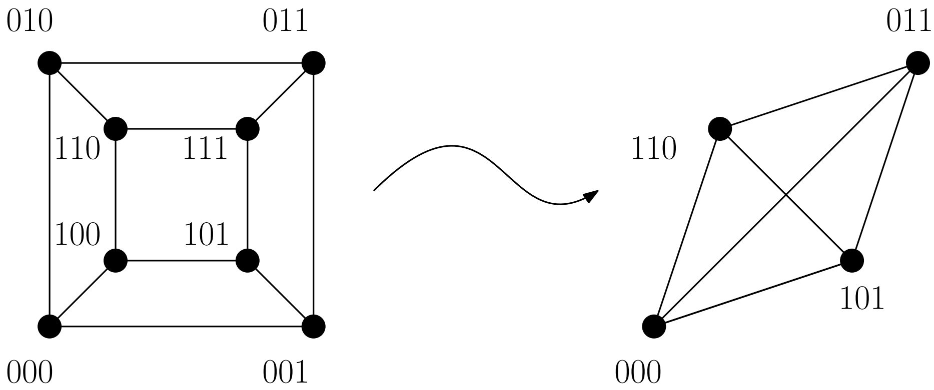

Figure 1.3: A single component of the half cube $\frac{1}{2}Q_3$.

It is clear to see that the folded half cube can be obtained from Q_n by first taking the folded cube FQ_{n-1} via the method of gluing antipodal vertices, and then taking the halved graph of the result (on the even size vertices). Similarly to the half cube, we can define a generalization of the folded half cube such that every vertex of $\mathbb{Z}_2^{n-1}$ is included, and thus we include both components. We denote this Cayley graph by $\frac{1}{2}FQ'_{n-1}$, which has the formulation

$$\frac{1}{2}FQ'_{n-1} = Cay(\mathbb{Z}_2^{n-1}, \{e_i \oplus e_j \mid i, j \in [n-1], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-1]\}).$$

2. Distance Magic Labelings on Cayley Graphs of $\mathbb{Z}_2^n$

2.1 Distance Magic Labelings

A distance magic labeling is in some sense a type of graph labeling inspired by magic squares.

Definition 5. *Let $G = (V, E)$ be a simple, undirected graph. A bijection $f : V \to \{1, 2, \ldots, |V|\}$ is a* distance magic labeling *if there exists a constant m such that for every vertex v,*

$$\sum_{u \in N(v)} f(u) = m.$$

This definition can be generalized to a *D-distance magic labeling*, in which for a subset $D \subseteq \{1, \ldots, diam(G)\}$, the constant m is obtained over the D-distance neighbourhood of every vertex v. That is, a bijective mapping $f : V \to \{1, 2, \ldots, |V|\}$ is a D-distance magic labeling if for every vertex v,

$$\sum_{u \in N_D(v)} f(u) = m.$$

A slight variation of distance magic labelings are those of *closed* distance magic labelings, where label $f(v)$ is included in the neighbourhood sum. That is, a bijective mapping $f : V \to \{1, 2, \ldots, |V|\}$ is a closed D-distance magic labeling if for every vertex v,

$$\sum_{u \in N_D[v]} f(u) = m.$$

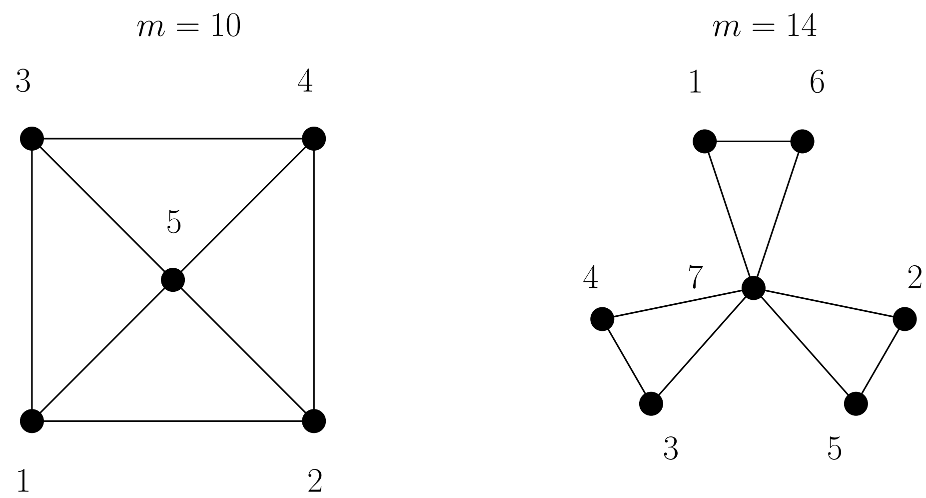

Figure 2.1: Example of a distance magic labeling (left) and a closed distance magic labeling (right).

A property of distance magic labelings that is not immediately obvious is the uniqueness of the magic constant m. This is obtained from a relation to a fractional total domination function.

Definition 6. *Given a graph $G = (V, E)$, a function $g : V \to [0, 1]$ is a* fractional total domination function *in G if for every vertex v,*

$$\sum_{u \in N(v)} g(v) \geq 1.$$

The *size* of a total domination function g is defined by $|g| := \sum_{v \in V} g(v)$.

The concept of fractional total domination functions gives the fractional total domination number as follows.

Definition 7. *The fractional total domination number of a graph G, denoted by $\gamma_{g_t}(G)$, is the minimum size of g over all fractional total domination functions g of G.*

Theorem 1 ([7])**.** *If a graph G with n vertices admits a distance magic labeling with magic constant m, then $m = \frac{n(n+1)}{\gamma_{g_t}(G)}$.*

As a corollary, it is clear to see that since m has an explicit formula, then by the uniqueness of $\gamma_{g_t}(G)$, m is unique.

It is an interesting question to ask which graphs can have a distance magic labeling. An approach to this question turns to spectral graph theory. If a regular graph has a distance magic labeling, then the following property holds.

Proposition 2 ([8])**.** *If G is a regular graph with a distance magic labeling, then 0 is in the spectrum of G. Similarly, if G is a regular graph with a closed distance magic labeling, then -1 is in the spectrum of G.*

We present a rephrased proof originally given in [8] to illustrate the relation between distance magic labelings and algebraic methods.

Proof. Let G be an r-regular graph with the vertex set $\{v_1, v_2 \ldots, v_{|V|}\}$ and adjacency matrix A_G, and let f be a distance magic labeling represented as a vector $(f(v_1), f(v_2), \ldots, f(v_{|V|}))$. Since the neighbourhood size is exactly r for each vertex v of G, the average contribution of each neighbour u of v to $\sum_{u \in N(v)} f(u)$ is exactly $\frac{m}{r}$. Note that this contribution is constant for any neighbourhood in which u is included. Therefore the total sum of all labels is $\sum_{v \in V(G)} f(v) = \sum_{v \in V(G)} \frac{m}{r}$. Thus in the matrix multiplication form,

$$A_G f^T = m\mathbf{1} = A_G g^T$$

where $g = (\frac{m}{r}, \frac{m}{r}, \ldots, \frac{m}{r}) \in \mathbb{R}^n$. This implies that $A_G(f^T - g^T) = \mathbf{0}$, so 0 is an eigenvalue of A_G. Similarly, if $\hat{f}$ is a closed distance magic labeling of G with magic constant $\hat{m}$, then

$$(A_G + I)\hat{f}^T = m\mathbf{1} = (A_G + I)\hat{g}^T$$

where $\hat{g} = (\frac{m}{r+1}, \frac{m}{r+1}, \ldots, \frac{m}{r+1}) \in \mathbb{R}^n$. Thus $(A_G + I)(\hat{f}^T - \hat{g}^T) = 0$, so -1 is an eigenvalue of A_G. □

Note that if G is r-regular and f is a distance magic labeling, we can define a bijection $f' : V \to \{1+c, 2+c, \dots, |V|+c\}$ such that for every vertex v, $m' = \sum_{u \in N_D(v)} f(u) = m + rc$ by taking $f'(u) = f(u) + c$. Since m can easily be deduced from f', we say that f' is also a distance magic labeling of G with shifted labels by c. For example, if the labels of G are shifted by -1, then $m' = \frac{r(|V|-1)}{2}$.

We now focus on distance magic labelings of graphs defined on the same vertex set as the hypercube. Recall the uniqueness of the magic constant m, and that we can shift values of a distance magic labeling in a regular graph by -1. Since Q_n is n-regular, then clearly if Q_n has a distance magic labeling where the (shifted) magic constant is $m = \frac{n(2^n-1)}{2}$.

2.2 Neighbour Balance

Gregor and Kovář [4] first devised a method on how to obtain a magic labeling on hypercubes. This is the notion of a neighbour-balanced mapping as follows.

Definition 8. *For a given subset $A \subseteq \mathbb{Z}_2^n$, an index $i \in [n]$, and $b \in \mathbb{Z}_2$ let $A_i^b = \{(u_1, ..., u_n) \in A \mid u_i = b\}$. We say that A is* balanced *if $|A_i^0| = |A_i^1|$ for every index i. That is, splitting A across any dimension i results in a partition into sets of equal sizes.*

Note that translation of a balanced set A by a constant vector has no effect on balance.

Definition 9. *A bijection $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is said to be* neighbour-balanced *if the set $L(u) = \{f(v) \mid v \in N(u)\} = \{f(u \oplus e_i) \mid i \in [n]\}$ is balanced for every $u \in \mathbb{Z}_2^n$.*

We now discuss distance magic labelings as mapping vertices of Q_n to labels in their standard binary representation. Each neighbourhood sum is thus obtained through regular addition of binary strings. Together with the bijectivity of f, the neighbour balance property yields a distance magic labelling.

Proposition 3 ([4])**.** *Every neighbour-balanced $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is a distance magic labeling of Q_n.*

Proof. Recall that Q_n is n-regular. Thus a bijection which maps vertices to $\{0, 1, \dots, 2^n - 1\}$ is eligible to be a distance magic labeling. Now, consider f as a neighbour-balanced function. Then for every vertex u in $\mathbb{Z}_2^n$, the neighbourhood of u is exactly the set $N(u) = \{u \oplus e_i \mid i \in [n]\}$. Since f is neighbour-balanced, the neighbourhood of u has an equal number of labels from $\mathbb{Z}_2^n$ where bit i is equal to 1 as where bit i is equal to 0. Thus the contribution of all labels of the neighbourhood of u, $L(u)$ with a fixed bit i equal to 1 to the total sum of labels is $|L_i^1|2^{n-i}$. Since Q_n is n-regular, the size of a balanced partition of $L(u)$ along a fixed bit i is $\frac{n}{2}$. Therefore,

$$\sum_{v \in N(u)} f(v) = \sum_{i \in [n]} f(u \oplus e_i) = \sum_{i \in [n]} |L_i^1| 2^{n-i} = \sum_{i \in [n]} \frac{n}{2} 2^{n-i} = m.$$

□

In practice, since we are working exclusively in the group $\mathbb{Z}_2^n$ for both vertices and labels, it is convenient to construct a neighbour-balanced function f by a $n \times n$ invertible matrix M whose columns form a balanced set $C = \{c_1, c_2, \ldots, c_n\}$. Let f be defined by $f(u) := Mu^T$, where Mu^T is computed over $\mathbb{Z}_2$. This method results in the neighbour-balance property of f since for any vertex u of Q_n and index $i \in [n]$,

$$f(u \oplus e_i) = f(u) \oplus f(e_i) = f(u) \oplus c_i.$$

Therefore for every vertex u, the set $L(u) = C \oplus f(u)$ is a translation of a balanced set, and therefore is also balanced. The invertibility of M ensures that f is a bijective function. From this construction of f, we obtain the following.

Theorem 4 ([4]). *A neighbour-balanced mapping $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ exists for every $n \equiv 2 \pmod 4$.*

Proof. Assume that $n = 4p + 2$ for some integer p. Then we can construct a matrix M such that $m_{ij} = 1$ if and only if

$$i = j \text{ or } (i \leq 2p + 2 \text{ and } j > 2p + 2) \text{ or } (i > 2p + 2 \text{ and } j \leq 2p)$$

This gives M in the block representation

$$M = \begin{pmatrix} I & 0 & 1 \\ 0 & I & 1 \\ 1 & 0 & I \end{pmatrix}$$

with block sizes $(2p, 2, 2p) \times (2p, 2, 2p)$. The columns of M form a balanced set, since for the number of entries equal to 1 in the first block row is $1+0+2p = \frac{4p+2}{2}$. Likewise, the number of entries equal to 1 in the second row is $0+1+2p = \frac{4p+2}{2}$, and in the third row it is $2p + 0 + 1 = \frac{4p+2}{2}$. The invertibility of M is also obvious, since M can be transformed into an upper triangular matrix by Gaussian elimination on the bottom left block. □

In fact, it was proven by Fronček et al. that Q_n does not have any distance magic labeling if $n \not\equiv 2 \pmod 4$ [4]. We omit this proof as it does not fall under our framework.

2.3 New Framework

We now extend this framework of obtaining a neighbour-balanced mapping to all Cayley graphs of $\mathbb{Z}_2^n$. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a subset of $\mathbb{Z}_2^n$, inducing a Cayley graph $Cay(\mathbb{Z}_2^n, S)$. Note that S defines a neighbourhood of a vertex $u \in \mathbb{Z}_2^n$ by $N(u) = \{u \oplus s_i \mid i \in [k]\}$.

The definition of a neighbour-balanced function f is then altered in the following manner. A bijection $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is *neighbour-balanced (with respect to S)* if the set

$$L(u) = \{f(v) \mid v \in N(u)\} = \{f(u \oplus s_i) \mid i \in [k]\}$$

is balanced for every vertex u.

Assume that f is defined by the $f(u) = Mu^T$ over $\mathbb{Z}_2$, where M is an invertible matrix over $\mathbb{Z}_2$. Then we obtain that for each vertex u,

$$f(u \oplus s_i) = f(u) \oplus f(s_i) = f(u) \oplus Ms_i^T.$$

Therefore, putting the vectors of S into matrix form as columns, i.e. $S = (s_1^T, s_2^T, \ldots, s_k^T)$, we obtain that f is neighbour-balanced if and only if $MS = (Ms_1^T, Ms_2^T, \ldots, Ms_k^T)$ over $\mathbb{Z}_2^n$ is balanced, i.e. the number of 1s and 0s in each row is the same (in fact, this number is $\frac{k}{2}$).

Schematically, by taking the inner product over $\mathbb{Z}_2^n$, and the outer product over $\mathbb{Z}$, we obtain

$$(MS)1^T = \begin{pmatrix} \frac{k}{2} \\ \vdots \\ \frac{k}{2} \end{pmatrix}.$$

From this construction, a natural question to ask is whether such an invertible matrix M exists for a matrix S such that MS over $\mathbb{Z}_2^n$ is balanced. Such existence would be sufficient for neighbour-balanced labeling with respect to S, and therefore it is a sufficient condition for $Cay(\mathbb{Z}_2^n, S)$ to have a distance magic labeling.

2.4 Method of Row Balancing

We recall linear elementary matrix operations of swapping rows and adding a combination of rows to a row in a matrix A. Such operations correspond to multiplication of A by an invertible matrix A' on the left. Therefore, given the matrix S generating the Cayley graph $Cay(\mathbb{Z}_2^n, S)$, if an invertible matrix M exists such that MS is balanced, it can be obtained by the product of all matrices corresponding to elementary operations $M_1, M_2, \ldots, M_r$ on S that result in a balanced matrix MS. That is, if M_i corresponds to the ith operation, then $M = M_r M_{r-1} \cdots M_1$.

Consider for example the hypercube Q_n as a Cayley graph. Recall that the generating set of Q_n is the set of basis vectors $\{e_1, e_2, \ldots, e_n\}$, thus the matrix S is the identity matrix I. For $n = 6$, we obtain the following series of elementary operations.

$$S = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$\xrightarrow{(3)} \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&1&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \xrightarrow{(4)} \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \xrightarrow{(5)} \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1\\ 1&1&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$\xrightarrow{(6)} \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1\\ 1&1&0&0&1&0\\ 1&1&0&0&0&1 \end{pmatrix} = MS$$

This series of elementary operations is characterized by multiplication of the following matrices.

$$M_1 = \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&1&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying each M_i together in the correct order, we obtain

$$MS = (M_6 M_5 M_4 M_3 M_2 M_1)S,$$

which yields M with the $(2,2,2) \times (2,2,2)$ block form

$$M = \begin{pmatrix} I & 0 & 1 \\ 0 & I & 1 \\ 1 & 0 & I \end{pmatrix}.$$

Note that since M is obtained by a product of invertible matrices, M is also invertible.

It remains to be seen for which matrices $S \in \mathbb{Z}_2^{n \times k}$ a matrix M exists such that MS is balanced. Some necessary properties of S are obtained by the following conditions on its number of columns k.

Theorem 5. *Let S be a $n \times k$ generating matrix of a Cayley graph $Cay(\mathbb{Z}_2^n, S)$. Assume that $Cay(\mathbb{Z}_2^n, S)$ admits a neighbour-balanced labeling by matrix multiplication with a matrix M. Clearly, k is even since MS is balanced. Then*

- *if $k \equiv 0 \pmod 4$, then S has all rows of even parity,*
- *if $k \equiv 2 \pmod 4$, then S has a row of odd parity.*

Proof. **Case 1:** If $k \equiv 0 \pmod 4$, then $\frac{k}{2} \equiv 0 \pmod 2$. By taking the product of MS and 1^T over $\mathbb{Z}_2$, we have

$$(MS)1^T = \begin{pmatrix} \frac{k}{2} \pmod 2 \\ \vdots \\ \frac{k}{2} \pmod 2 \end{pmatrix} = 0^T.$$

Thus $S1^T = M^{-1}0^T = 0^T$ over $\mathbb{Z}_2$, so all rows of S have even parity.

Case 2: Likewise, if $k \equiv 2 \pmod 4$, then $\frac{k}{2} \equiv 1 \pmod 2$, so $S1^T = M^{-1}1^T$. This implies that all rows of S have the same parity as the rows of M^{-1}. Suppose that all rows of S have even parity. Then the rows of M^{-1} span only a subspace of vectors of even parity, which is a contradiction with the rank n of M^{-1}. Thus S has some row of odd parity.

□

These results provide necessary conditions on S for the existence of M.

When determining the existence of M, instead of greedily trying all elementary row operations on S, it is useful to find a way to group columns of S together such that each group can be analyzed independently to determine some properties of M, such as the size of each row m_i.

Definition 10. *The* contribution *of a set of columns $C_j \subseteq S$ given a row $m_i \in M$ is the number of columns $c^T \in C_j$ such that $m_i c^T = 1$.*

Ideally, we construct a special partition of columns of S into sets called *pairings*, such that each pairing is balanced for any given row m_i. That is, the contribution of any pairing is exactly half the size of the pairing.

Lemma 6. *Let S be a $n \times k$ generating matrix of a Cayley graph $Cay(\mathbb{Z}_2^n, S)$, and let M be an $n \times n$ binary matrix. Then MS is balanced if and only if for each row m_i of M, there exists a partition of columns of S into sets $C_1, C_2, \ldots, C_p$ each of even size such that for each $C_j = \{c_1^T, c_2^T, \ldots, c_q^T\}, j \in [p]$, $\sum_{c^T \in C_j} m_i c^T = \frac{q}{2}$. That is, for each column s^T of S we can choose a* pairing *of columns C_j including s^T such that the number of 1's and 0's contributed by C_j to $m_i S$ is equal.*

Proof. Suppose that for an arbitrary row m_i of M, S has a partition of columns $C = \{C_1, C_2, \ldots, C_p\}$ of sizes $q_1, q_2, \ldots, q_p$ such that each $C_j \in C$ is balanced with $\sum_{c^T \in C_j} m_i c^T = \frac{q_j}{2}$. Then

$$|m_i S| = \sum_{s^T \in S} m_i s^T = \sum_{C_j \in C} \sum_{c^T \in C_j} m_i c^T = \sum_{j=1}^{p} \frac{q_j}{2} = \frac{q_1 + q_2 + \cdots + q_p}{2} = \frac{k}{2}.$$

On the other hand, suppose that MS is balanced. Then for each row m_i of M, let s^T be a column of S such that $m_i s^T = 1$. Due to the balance of MS, there exists a column $\hat{s}^T$ such that $m_i \hat{s}^T = 0$, then we pair s^T with $\hat{s}^T$. Then construct C_1 as the pair $\{s^T, \hat{s}^T\}$. C_1 is then balanced as $m_i(s^T \hat{s}^T) = (10)$ is balanced. Let $S' = S \backslash \{s^T, \hat{s}^T\}$, and let $k' = |S'|$. Since $m_i S$ is balanced, and $\{s^T, \hat{s}^T\}$ contribute by only one 1 to $m_i S$, then $m_i S'$ contributes $\frac{k}{2} - 1 = \frac{k'+2}{2} = \frac{k'}{2}$, and thus $m_i S'$ is also balanced. We proceed inductively on S' to obtain a partition of columns $\{C_1, C_2, \ldots, C_{\frac{k}{2}}\}$ such that $|C_j| = 2$ for all $j \in [\frac{k}{2}]$. □

While any pairing gives a nice structure to determining balance of a row $m_i S$, we are often not so lucky in having an obvious way to construct such a strong pairing from an observation of S. However, there may be other such ways to partition columns of S depending on its structure that turn out to be easy to analyze for a given M. Some particular cases of this are analyzed in the next chapter.

3. Application to Previous Results

We now show the utility of the row balancing framework by reproving some known results.

3.1 D-distance Magic Labelings of Q_n

It might be asked whether there are D-distance magic labelings of Q_n, and for which values of n such a labeling is possible. By recasting the problem of finding D-distance labelings of Q_n as finding 1-distance labelings of Cayley graphs of $\mathbb{Z}_2^n$ by constructing edges between vertices of Hamming distance d for every $d \in D$, the problem is sufficiently simplified. We then obtain some notable results for which sets of D that a hypercube Q_n can have a D-distance magic labeling.

Let us first focus on the case when $D = \{d\}$. Let S be the set of all vectors of $\mathbb{Z}_2^n$ of size exactly d. Recall that S can be seen as a matrix with its elements as columns. Then we obtain the following.

Proposition 7 ([9])**.** *A d-distance magic labeling of Q_n exists if $n \equiv 2 \pmod 4$ and $1 \leq d \leq n$ is odd.*

Proof. Recall that the d-distance neighbourhood of a vertex u is $N_{\{d\}}(u) = \{u \oplus s \mid s^T \in S\}$. Thus finding a $\{d\}$-distance magic labeling of Q_n is equivalent to finding a $\{1\}$-distance magic labeling of $Cay(\mathbb{Z}_2^n, S)$. The number of columns $k = \binom{n}{d}$ of S is even if d is odd. Furthermore, for each column vector u in S, if a coordinate $u_i = 1$ is fixed, then there are $\binom{n-1}{d-1}$ ways to arrange the remaining 1's in the column. Therefore there are $\binom{n-1}{d-1}$ 1's in each row of S.

Claim 8. *If $\binom{n}{d} \equiv 0 \pmod 4$, then $\binom{n-1}{d-1}$ is even. Furthermore, if $\binom{n}{d} \equiv 2 \pmod 4$, then $\binom{n-1}{d-1}$ is odd.*

Proof. We use the well known formula from combinatorics

$$d\binom{n}{d} = n\binom{n-1}{d-1}.$$

- If $\binom{n}{d} \equiv 0 \pmod 4$-then $d\binom{n}{d} \equiv 0 \pmod 4$. Thus we obtain that $n\binom{n-1}{d-1} \equiv 0 \pmod 4$. However, since $n \equiv 2 \pmod 4$, we have that $\binom{n-1}{d-1}$ is even.

- If $\binom{n}{d} \equiv 2 \pmod 4$, then since d is odd, $d\binom{n}{d} \equiv 2 \pmod 4$. Therefore since $n \equiv 2 \pmod 4$ and $n\binom{n-1}{d-1} \equiv 2 \pmod 4$, we have that $\binom{n-1}{d-1}$ is odd.

□

Thus the parity conditions in Theorem 5 of a distance magic labeling are satisfied.

Suppose that $n = 4p + 2$. Consider taking M as before in the form of a $(2p, 2, 2p) \times (2p, 2, 2p)$ block matrix

$$M = \begin{pmatrix} I & 0 & 1 \\ 0 & I & 1 \\ 1 & 0 & I \end{pmatrix}$$

where each row m_i of M has $2p + 1$ 1's. Then the linear combination of $m_i S$ over $\mathbb{Z}_2^n$ will also be balanced by a construction of pairings for each m_i as follows.

Assume that we have a permutation π such that the permutation of n indices of 1's and 0's of the row m_i results in a row vector $m_{i,\pi} = (111 \dots 100 \cdots 0)$. Consider a permuted column $s_\pi^T = \begin{pmatrix} s_1^T \\ s_2^T \end{pmatrix}$ of S such that each $s_{\pi,j}^T$ has length $2p+1$ (exactly half of s_π^T). Since d is odd, then one s_j^T will have an odd parity, and the other will have an even parity. Without loss of generality, assume that s_1^T has an odd parity. We then pair s^T with the column c^T such that $c_\pi^T = \begin{pmatrix} s_2^T \\ s_1^T \end{pmatrix}$. Since s_1^T has an odd size and s_2^T has an even size, then by the assumption that $m_{i,\pi} = (111 \dots 100 \cdots 0)$, $m_i s^T = m_{i,\pi} s_\pi^T = m_{i,\pi} s_1^T = 1$, and $m_i c^T = m_{i,\pi} c_\pi^T = m_{i,\pi} s_2^T = 0$, thus the pair $\{s^T, c^T\}$ is a pairing. By Lemma 6, since each of the $\frac{k}{2}$ pairs is balanced, then $m_i S$ is balanced. This also holds if s_1^T has even parity and s_2^T has odd parity.

Lastly, the invertibility of M is easy to see, since it is also obtained by a product of invertible matrices, as seen in Chapter 2.4.

□

Corollary 9. *Q_n has a D-distance magic labeling if $n \equiv 2 \pmod 4$ and $D \subseteq \{1, 3, \dots, n-1\}$.*

Proof. This follows from constructing S as the union of d-sized vectors for every $d \in D$. We then partition S into matrices $C_1, C_2, \dots, C_{|D|}$ for each fixed sized d of columns. By Proposition 7, each of $m_i C_j$ is balanced. By Lemma 6, MS is balanced and therefore there exists a D-distance magic labeling of Q_n. □

Proposition 10 ([9])**.** *Q_n has a D-distance magic labeling if $n \equiv 2 \pmod 4$ and $D = E \uplus \bigcup_{i \in I} \{i, n-i\}$, where $E \subseteq \{1, 3, \dots, n-1\}$ is nonempty, $I \subseteq \{0, 1 \dots, \frac{n}{2}\}$ and $\uplus$ denotes* disjoint union.

Proof. Suppose we partition S into submatrices $C_1, C_2, \dots, C_{|E|}, L$ where each submatrix C_j denotes the sets of columns of fixed size from E, and the submatrix L is formed by the set of columns $L = \bigcup_{i \in I} \{i, n-i\}$. From Corollary 9, we have that MC_j is balanced for each set C_j of columns of fixed odd size given

$$M = \begin{pmatrix} I & 0 & 1 \\ 0 & I & 1 \\ 1 & 0 & I \end{pmatrix}$$

and each row m_i of M has $2p+1$ many 1's. It suffices to show that the remaining columns of the form from L are also balanced with a row m_i. The intuition is as follows. Let L_i denote the set of columns of length i for each $i \in I$. Note that the size of L_i is exactly $\binom{n}{i}$. Likewise, the size of L_{n-i} is $\binom{n}{n-i} = \binom{n}{i} = |L_i|$. Therefore for each column $s^T \in L_i$, we pair s^T with $c^T \in L_{n-i}$ such that $c^T = s^T \oplus \mathbf{1}^T$. That is, each pairing is of complimentary columns. Since $|m_i| = 2p+1$ is odd, then each pair contribute only one 1 to the row $m_i S$. Therefore, MS is balanced.

Recall from Chapter 2.4 that M is invertible, and therefore Q_n has a D-distance magic labeling for $n \equiv 2 \pmod 4$ and $D = E \uplus \bigcup_{i \in I}\{i, n-i\}$. □

3.2 The Folded Cube

We now move on to the first example of a cube-like graph, the folded cube FQ_{n-1}. Recall the Cayley graph definition of the folded cube FQ_{n-1}. That is,

$$FQ_{n-1} = Cay(\mathbb{Z}_2^{n-1}, \{e_i \mid i \in [n-1]\} \cup \{\mathbf{1}\}).$$

Proposition 11 ([10])**.** *The folded cube FQ_{n-1} has a neighbour balanced distance magic labeling if and only if $n \equiv 0 \pmod 4$.*

Proof. The Cayley graph of FQ_{n-1} gives us the generating matrix S with the $(n-1) \times (n-1, 1)$ block representation

$$S = \begin{pmatrix} \mathbf{I} & \mathbf{1} \end{pmatrix}$$

Note that the number of 1's in each row of S is exactly 2 for any value of n. By Theorem 5, we must then have that $n \equiv 0 \pmod 4$. We construct M by row balancing as follows. Suppose we want to fix the number of 1's in each row m_i of M. By taking a set $\hat{C}$ of p rows of S, the addition of all rows s_i in $\hat{C}$ together result in a row $\hat{s_i}$ with the number of 1's determined by the contribution of columns from $\mathbf{I}$ plus the contribution of the column $\mathbf{1}^T$. The summation of p rows of $\mathbf{I}$ give $\hat{s_i}$ exactly p many 1's in the first $n-1$ coordinates. Likewise, the summation of p rows of $\mathbf{1}^T$ over $\mathbb{Z}_2$ is either 0 if p is even, or 1 if p is odd. Suppose that p is even. Thus to count the total contribution of p rows of S by row balancing, we set $p = \frac{n}{2}$. Therefore by taking M with the $(\frac{n-2}{2}, 1, \frac{n-2}{2}) \times (\frac{n-2}{2}, 1, \frac{n-2}{2})$ block representation as

$$M = \begin{pmatrix} I & 0 & 1 \\ 0 & I & 1 \\ 1 & 0 & I \end{pmatrix}$$

is sufficient. Therefore, the folded cube has a distance magic labeling for $n \equiv 0 \pmod 4$.

□

As a remark, in fact Miklavič and Šparl [10] proved that the this proposition also holds for distance magic labelings which are not neighbour balanced. That is, if $n \equiv 0 \pmod 4$, then FQ_{n-1} has a distance magic labeling. To create this proof, the authors used Proposition 2. We omit the rest of this proof.

3.3 The Half Cube

While proving the existence of a distance magic labeling on a Cayley graph G with vertex set $V \subseteq \mathbb{Z}_2^n, |V| = 2^l$ for $0 \leq l \leq n$ directly is usually possible, it can sometimes be easier to prove a stronger condition. That is, proving that there exists a distance magic labeling for the Cayley graph G' on $\mathbb{Z}_2^n$ consisting of components $G_1, G_2, \ldots, G_2^{n-l}$ such that each component $G_i \cong G$.

Such an example of a Cayley graph with multiple components is that of the half cube $\frac{1}{2}Q'_n$, where each neighbourhood of a vertex u is defined by all vertices v such that $d_H(u,v) = 2$. Recall the formulation as

$$\frac{1}{2}Q'_n = Cay(\mathbb{Z}_2^n, \{e_i \oplus e_j \mid i,j \in [n], i \neq j\}).$$

If S is the matrix with columns of exactly all vectors of size 2, then $k = \binom{n}{2} = \frac{n(n-1)}{2}$, and each row of S will have $n-1$ 1's since for each column s^T, if $s_i^T = 1$, there are $n-1$ ways to choose the coordinate j where $s_j^T = 1$. By Theorem 5, if $n-1$ is odd, then it must be that $k \equiv 2 \pmod 4$ for MS to be balanced. Since $k = \frac{n(n-1)}{2}$, this occurs when $\frac{n}{2} \equiv 2 \pmod 4$, and thus $n \equiv 0 \pmod 4$. Likewise, if $n-1$ is even, then we require $k \equiv 0 \pmod 4$ for MS to be balanced. If $\frac{n(n-1)}{2} \equiv 0 \pmod 4$, we have $n(n-1) \equiv 0 \pmod 8$. But since n is odd, then $n - 1 \equiv 0 \pmod 8$, and thus $n \equiv 1 \pmod 8$. These two conditions are satisfied when $n = q^2$ for some natural number q such that $q \geq 2$.

Proposition 12 ([2])**.** *The half cube $\frac{1}{2}Q_n$ has a distance magic labeling if $n = q^2$ for some natural number $q \geq 2, q \not\equiv 0 \pmod 4$.*

Proof. We first show that both component $\frac{1}{2}Q'_n$ has a distance magic labeling. Recall that given S as the matrix of vectors from $\mathbb{Z}_2^n$ all of size 2, then $k = |S| = \binom{n}{2} = \binom{q^2}{2} = \frac{n(n-1)}{2}$.

Let $f(u) = Mu^T$, and suppose we take M as the $q^2 \times q^2$ band matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & & & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 \\ & & & \ddots & & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

with each row m_i of M having exactly $\frac{q^2-q}{2}$ many 1's. Suppose that we have a permutation π of coordinates such that for a fixed row m_i, $m_{i,\pi} = (1, \ldots, 1, 0, \ldots, 0)$. For a given column s_π^T, we then have that $m_i s^T = m_{i,\pi} s_\pi = 1$ if and only if one of the two 1's in s_π is in the first $\frac{q^2-q}{2}$ coordinates, and the other is in the latter $\frac{q^2-q}{2} + q$ coordinates. So for any given column s^T of S, we partition s_π^T into

(s_1^T, s_2^T, s_3^T) where $|s_1^T| = \frac{q^2-q}{2}$, $|s_2^T| = q$, and $|s_3^T| = \frac{q^2-q}{2}$. If s_π has exactly one 1 in s_2, then the other 1 in s^T is either in s_1^T or s_3^T. Thus if we pair s^T with a column c^T of S where c_π^T has the partition (s_3^T, s_2^T, s_1^T), then we guarantee that only one of s^T, c^T contribute by 1 to m_iS.

Likewise, if s_2 has both or no 1's, consider this entire set of columns as a pairing. There are only $\binom{q}{2} + \binom{q^2-q}{2} = \frac{q(q-1)}{2} + \frac{(q^2-q)(q^2-q-1)}{2} = \frac{(q^2-q)^2}{2} = \frac{q^2(q-1)^2}{2}$ such columns of this type. So pair the columns s^T from this set where s_1 has one 1 and s_2 has zero 1's with the other columns of this set where $m_{i,\pi}s_\pi^T = 0$. Since there are $\frac{q^2-q}{2}$ choices for the first 1 to be in s_1^T, and $\frac{q^2-q}{2}$ choices from the second 1 to be in s_3^T, the total number of columns that contribute a 1 to this set is $\left(\frac{q^2-q}{2}\right)^2 = \frac{q^2(q-1)^2}{4} = \frac{1}{2}\left(\frac{q^2(q-1)^2}{2}\right)$, and thus is exactly half of the size of the set.

Thus m_iS is balanced for each row m_i, and by Lemma 6, MS is balanced. Lastly, we check that M is invertible. Since we have that $q \not\equiv 0 \pmod 4$, $\frac{q^2-q}{2}$ is odd. Therefore the size of each row and the size of each column is odd. Thus for a subset of columns to be equal to $\mathbf{0}^T$ under addition, the subset must be of even size. But since the column sizes are also odd and any two columns intersect on at most $\frac{q^2-q}{2} - 2$ coordinates, no such subset exists. Thus all columns of M are linearly independent, and thus M is invertible.

To show that the one component $\frac{1}{2}Q_n$ has a distance magic labeling, we must show that there is a bijection $f' : \{u \in \mathbb{Z}_2^n \mid |u| \equiv 0 \pmod 2\} \to \mathbb{Z}_2^{n-1}$ which maps the 2^{n-1} vertices of the even component of $\frac{1}{2}Q'_n$ to the labels from $\mathbb{Z}_2^{n-1}$. To do this, we show that Mu^T preserves the parity of components. That is, if a vertex u has $|u| \equiv x \pmod 2$, then $|Mu^T| \equiv x \pmod 2$. Therefore all even vertices receive even labels. As $f(u) = Mu^T$ is a bijection, there are no other unassigned labels of even parity. By then removing the first bit of each label, we obtain the entire element set of labels $\mathbb{Z}_2^{n-1}$ for the component of even vertices, and therefore f' is a bijection.

The preservation of parity of f follows from the structure of M. Suppose for each even vertex u we split u into its component basis vectors $u = e_i \oplus e_j \oplus e_l \oplus \ldots$. Then

$$f(u) = Mu^T = Me_i^T + Me_j^T + Me_l^T \oplus \ldots.$$

Note that the output of each product Me_i^T yields a resultant vector with a band form $(0\ldots01\ldots10\ldots0)$ with $\frac{q^2-q}{2}$ many 1's. Since each Me_i^T has an odd size, and the number of such e_i's is even, then the addition of all such Me_i^T's is even, and therefore the vertex parity is preserved by f. Thus M is a distance magic labeling of $\frac{1}{2}Q_n$.

□

3.4 The Folded Half-Cube

Recall that the Cayley graph definition of the folded half-cube $\frac{1}{2}FQ_{n-1}$ where n is even.

$$\frac{1}{2}FQ_{n-1} = Cay(\{u \in \mathbb{Z}_2^{n-1} \mid |u| \equiv 0 \pmod 2\}, \{e_i \oplus e_j \mid i,j \in [n-1], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-1]\}).$$

For ease of computation, we construct an equivalent formulation of $\frac{1}{2}FQ_{n-1}$ by ignoring the first bit of each vertex. The original formulation of $\frac{1}{2}FQ_{n-1}$ is by taking all vertices from $\mathbb{Z}_2^{n-1}$ of even size, and applying edges uv if $d_H(u,v) = 2$ or $d_H(u,v) = n-2$. Unfortunately, this formulation is not a Cayley graph of $\mathbb{Z}_2^{n-1}$, but a Cayley graph of all vertices of even size, a subgroup of $\mathbb{Z}_2^{n-1}$. This is precisely the reason we choose the formulation of $\frac{1}{2}FQ_{n-1}$ over the vertex set $\mathbb{Z}_2^{n-2}$ for the row balancing framework in the following manner.

$$\frac{1}{2}FQ_{n-1} = Cay(\mathbb{Z}_2^{n-2}, \{e_i \mid i \in [n-2]\} \cup \{e_i \oplus e_j \mid i,j \in [n-2], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-2]\} \cup \{\mathbf{1}\})$$

The connection between these two formulations is in removal of (without loss of generality) the first bit of each even sized vertex u of $\mathbb{Z}_2^{n-1}$ (otherwise known as the parity bit), and defining a Cayley graph on the suffixes of length $n-2$, where each neighbourhood is preserved. To distinguish the generators of these two formulations of the folded half-cube, let S be the matrix of columns of length $n-2$

$$S = \{e_i \mid i \in [n-2]\} \cup \{e_i \oplus e_j \mid i,j \in [n-2], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-2]\} \cup \{\mathbf{1}\},$$

and let S' be the matrix of columns of length $n-1$

$$S' = \{e_i \oplus e_j \mid i,j \in [n-1], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-1]\}$$

representing the original formulation of $\frac{1}{2}FQ_{n-1}$. This matrix S' will have utility later, when we work over the Cayley graph $\frac{1}{2}FQ'_{n-1}$.

Since we are ignoring the first bit of each Cayley graph generator, S has the same number of columns k from the original formulation S'. The computation of the degree k is as follows. We have $n-2$ possible elements of size 1, $\binom{n-2}{2}$ elements of size 2, and $n-2$ elements of size $n-2$, along with the constant element $\mathbf{1}$. Therefore

$$k = 2(n-2) + \binom{n-2}{2} + 1 = \frac{n(n-1)}{2}.$$

Theorem 13 ([3])**.** *$\frac{1}{2}FQ_{n-1}$ has a distance magic labeling if $n = 16q^2$ for some natural number q.*

The original proof of this theorem relies on determining the rank of the $(n-2) \times (n-1)$ matrix M where $f(u) = Mu^T$. Tian et al. [3] showed that given a specific construction of M, and the $(n-1) \times k$ matrix S' such that the columns of S' are generators of $\frac{1}{2}FQ_{n-1}$ (one the even size vertices of $\mathbb{Z}_2^{n-1}$), then since MS is balanced and the rank of M is $n-2$, then a distance magic labeling for $\frac{1}{2}FQ_{n-1}$ exists. Note that the invertibility of M is not shown.

We construct an alternative proof by redefining the folded half cube as a Cayley graph on all vertices of $\mathbb{Z}_2^{n-2}$, and defining M as a $(n-2)\times(n-2)$ matrix such that $f(u) = Mu^T$ along with a $(n-2)\times k$ matrix S of generators of this altered Cayley graph. We then proceed by analyzing the contribution of specified sets of columns of S to the total number of 1's in each row of MS.

Proof. Recall that $k = \frac{n(n-1)}{2}$. If $n = 16q^2$, then $k = \frac{(16q^2)(16q^2-1)}{2} = 128q^4 - 8q^2$. Thus we need to obtain $\frac{k}{2} = 64q^4 - 4q^2$ 1's for each row m_iS.

Suppose we construct M such that the number of 1's in each row m_i is $8q^2 - 2q - 1$. Then as M is a $(16q^2-2)\times(16q^2-2)$ matrix, the number of 0's in each row m_i is $8q^2 + 2q - 1$. We show that this suffices for MS to be balanced.

For convenience, let us construct a partition of S from a partition of its columns into the following sets.

$$
\begin{aligned}
C_1 &= \{e_i \mid i \in [n-2]\}\\
C_2 &= \{e_i \oplus e_j \mid i, j \in [n-2], i \neq j\}\\
C_3 &= \{\mathbf{1} \oplus e_i \mid i \in [n-2]\}\\
C_4 &= \{\mathbf{1}\}
\end{aligned}
$$

We proceed by analyzing the contribution of each set C_j given M as above to the total sum of $64q^4 - 4q^2$ for each row m_i of M.

For each $e_j^T \in C_1$, we have that $m_is^T = 1$ if and only if $m_{ij} = 1$. Thus there are $8q^2 - 2q - 1$ such choices of j such that this holds. Therefore the total number of 1's contributed by C_1 is $8q^2 - 2q - 1$.

Next, for each $(e_j^T \oplus e_l^T) \in C_2$, $m_i(e_j \oplus e_l)^T = 1$ if and only if (without loss of generality) $m_{ij} = 1$ and $m_{il} = 0$. Since m_i has $8q^2 - 2q - 1$ entries of 1, and $8q^2 + 2q - 1$ entries of 0, there are $(8q^2 - 2q - 1) \times (8q^2 + 2q - 1) = 64q^4 - 20q^2 + 1$ ways that a column $e_j \oplus e_l \in C_2$ can contribute 1 to m_iS.

For each $s^T \in C_3$, note that the number of 1's in m_i is odd. Since each s^T contains only one zero, if $m_is^T = 1$, then the coordinate in which s^T has the entry of 0 must be one of the coordinates of m_i that have an entry of 0. Therefore the number of ways to choose this is the number of ways to choose the one 0 in s^T to have coordinate j such that $m_{ij} = 0$. There are $8q^2 + 2q - 1$ ways to choose such a coordinate. Therefore the total contribution of 1's to m_iS by C_3 is $8q^2 + 2q - 1$.

For C_4, since every coordinate l such that $m_{il} = 1$ also has $\mathbf{1}_l^T = 1$, and since $8q^2 - 2q - 1$ is odd, we obtain that $m_i\mathbf{1}^T = 1$. Thus the contribution of C_4 is 1.

For the folded half-cube, we obtain that the total sum of contributions of each C_j is

$$(8q^2 - 2q - 1) + (64q^4 - 20q^2 + 1) + (8q^2 + 2q - 1) + (1) = 64q^4 - 4q^2 = \frac{k}{2}.$$

Therefore MS is balanced.

The structure of M remains to be shown. Tian et. al [3] showed that if we restrict the task of finding a distance magic labeling of the half folded cube to one component of the vertex set $\mathbb{Z}_2^{n-1}$, then the $(8q^2-2q-2, 8q^2-2q-2, 2q, 2q, 2) \times (1, 8q^2 - 2q - 2, 8q^2 - 2q - 2, 2q, 2q, 2)$ matrix

$$M' = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$

is a suitable matrix such that the mapping f defined by $f(u) = Mu^T$ for every even sized $u \in \mathbb{Z}_2^{n-1}$ is a distance magic labeling of $\frac{1}{2}FQ_{n-1}$. Note that their formulation of M' is that of a non-square $(n-2) \times (n-1)$ matrix but with rank $n-2$. For our purpose, this is insufficient as M' is non-invertible. This encodes a distance magic labeling on the component with vertex set corresponding to the elements of even size in $\mathbb{Z}_2^{n-1}$. Thus only the labels $\{0, 1, \ldots, 2^{n-2}-1\}$ are utilized.

We claim that defining M as the $(n-2) \times (n-2)$ matrix such that M is equivalent to M' with the first column of 0's removed is sufficient. That is, M has the $(8q^2 - 2q - 2, 8q^2 - 2q - 2, 4q + 2) \times (8q^2 - 2q - 2, 8q^2 - 2q - 2, 4q + 2)$ block representation

$$M = \begin{pmatrix} \mathbf{I} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{I} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

As before, each column m_i has exactly $8q^2 - 2q - 1$ many 1's and thus m_iS is balanced. Through elementary row operations, we can reduce M to the lower triangular form

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{I} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$

in which each column is linearly independent. Therefore M is invertible.

It remains to be shown that M can be obtained via row balancing. As before, we require that the contribution of 1's to each row of each set $\{C_1, C_2, C_3, C_4\}$ after row balancing is correct. Again, this is $8q^2 - 2q - 1$ for C_1, $64q^4 - 20q^2 + 1$ for C_2, $8q^2 + 2q - 1$ for C_3, and 1 for C_4. We proceed by showing that the binary addition of a row s_i with any subset $\hat{C}$ of $8q^2-2q-2$ rows from each partition C_j results in the correct contribution of C_j. The illustration below summarizes this proof sketch. In the first step, we show the structure of S. In the second step, we illustrate taking a fixed set of rows $\hat{C}$ and a given row s_i partitioned in the same manner as each type of column. In the third step, we show the total number of 1's in each partition of $s_i \oplus \sum_{c\in\hat{C}} c$ is the same amount as above.

$$S = \left[\begin{array}{c|c|c|c} C_1 & C_2 & C_3 & C_4 \end{array}\right]$$

$$\rightarrow \quad \begin{array}{r} \text{row } s_i \left\{ \right. \\ \\ \text{set } \hat{C} \text{ of } 8q^2-2q-2 \text{ rows} \left\{ \right. \end{array} \left[\begin{array}{c|c|c|c} \hline s_{i,1} & s_{i,2} & s_{i,3} & s_{i,4} \\ \hline & & & \\ \hline \hat{C}_1 & \hat{C}_2 & \hat{C}_3 & \hat{C}_4 \\ \hline & & & \end{array}\right]$$

$$\rightarrow \quad s_i \oplus \sum_{c\in\hat{C}} c \left\{ \right. \left[\begin{array}{c|c|c|c} & & & \\ \hline 8q^2-2q-1 & 64q^4-20q^2+1 & 8q^2+2q-1 & 1 \\ \hline & & & \end{array}\right]$$

For C_1, the row addition of a subset of $8q^2-2q-2$ rows $\hat{C}$ to s_i will contribute exactly $8q^2-2q-1$, since each row contains only one entry of 1, each in different coordinates.

For C_2, note that each row has exactly $16q^2-3$ 1's, and intersects any other row in exactly one coordinate. Furthermore, these coordinates of intersection are all unique, otherwise there would be a contradiction in the size of each column being 2. Therefore each row intersects the other $8q^2-2q-2$ rows in $8q^2-2q-2$ coordinates. Thus the binary addition of all $8q^2-2q-1$ rows of $\hat{C} \cup s_i$ will result in the number of 1's being

$$\begin{aligned} &(16q^2-3)(8q^2-2q-1)-(8q^2-2q-2)(8q^2-2q-1) \\ &= 128q^4-32q^3-40q^2+6q+3-64q^4+32q^3+20q^2-6q-2 \\ &= 64q^4-20q^2-1 \end{aligned}$$

For C_3, each row has exactly one 0. Thus adding $8q^2-2q-1$ rows together will result in a 1 at coordinate i if none of the chosen rows contain a 0 at coordinate i

since $8q^2 - 2q - 1$ is odd, and will result in 0 otherwise (exactly one row will have 0 at coordinate i and thus the number of 1's added together is even). Since there are $8q^2 - 2q - 1$ coordinates which will become 0 after all of the row addition of $\hat{C} \cup s_i$, the number of 1's remaining is $(16q^2 - 2) - (8q^2 - 2q - 1) = 8q^2 + 2q - 1$, which is the required number of 1's needed for row balance.

For C_4, adding any odd number of rows from C_4 to a row will not change the value 1. Therefore adding $8q^2 - 2q - 2$ rows $\hat{C}$ to s_i is sufficient.

Now that it has been determined that we can add any $8q^2 - 2q - 2$ rows $\hat{C}$ to any row s_i (and thus $8q^2 - 2q - 1$ rows together total) and each partition will contribute the correct number of 1's such that s_i is balanced, the choice of $8q^2 - 2q - 2$ rows to add to row s_i of S does not affect the end balance. Therefore, we can choose for each of $s_1, s_2, \ldots, s_{8q^2-2q-2}$, we add all rows from $s_{8q^2-2q-1}, \ldots, s_{16q^2-4q-4}$. Likewise, for every other row, we add all of $s_1, \ldots, s_{8q^2-2q-2}$. These elementary row additions are precisely described by M.

The computation of M is as follows. For simplicity, we represent row m_i by the row index i, and the column m_j^T by the column index j to the left and above the matrix M, respectively.

$$M_1 = \begin{pmatrix} 1 & 0 & \dots & \overset{8q^2-2q-1}{1} & \dots & 0 \\ 0 & 1 & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \dots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & \dots & \overset{8q^2-2q}{1} & \dots & 0 \\ 0 & 1 & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \dots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$\vdots$$

$$M_{8q^2-2q-2} = \begin{pmatrix} 1 & 0 & \dots & \overset{16q^2-4q-4}{1} & \dots & 0 \\ 0 & 1 & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \dots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$M_{8q^2-2q-1} = \begin{pmatrix} 1 & 0 & \dots & \overset{8q^2-2q-1}{0} & \dots & 0 \\ 0 & 1 & \dots & 1 & \dots & 0 \\ 0 & 0 & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \dots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

$$\vdots$$

$$M_{(8q^2-2q-2)^2} = \begin{matrix} \\ \\ \\ m_{8q^2-2q-2} \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & 0 & \dots & \overset{16q^2-4q-4}{0} & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \dots & 1 & \ddots & \vdots \\ 0 & 0 & \dots & \vdots & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

The first $(8q^2-2q-2)^2$ M_i's therefore give

$$M_{(8q^2-2q-1)^2} M_{(8q^2-2q-1)^2-1} \cdots M_2 M_1 = \begin{pmatrix} \mathbf{I} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$

The other intermediate M_i's are found in a similar manner to form the other $\mathbf{1}$-blocks of M. In total, we perform $8q^2 - 2q - 2$ operations per row, over all $16q^2 - 2$ rows of M. Thus we can compute M by row balancing using $(8q^2 - 2q - 1) \times (16q^2 - 2)$ matrices corresponding to elementary row addition operations.

□

While the trick of working on the Cayley graph of the folded half-cube by reducing the vertex set $\mathbb{Z}_2^{n-1}$ to $\mathbb{Z}_2^{n-2}$ is convenient (as no vertices become irrelevant), an interesting question to pose is whether there is a distance magic labeling on the Cayley graph

$$\frac{1}{2}FQ'_{n-1} = Cay(\mathbb{Z}_2^{n-1}, \{e_i \oplus e_j \mid i, j \in [n-1], i \neq j\} \cup \{\mathbf{1} \oplus e_i \mid i \in [n-1]\})$$

of all vertices $u \in \mathbb{Z}_2^{n-1}$.

This Cayley graph has two components isomorphic to $\frac{1}{2}FQ_{n-1}$, one on the even vertices, and one on the odd vertices. Note that if we have a distance magic labeling for a single component of a graph where all components are isomorphic, the fact that each component can have the same labeling is trivial. Nonetheless, we illustrate the linear algebra technique to translate labels across components of a Cayley graph of $\mathbb{Z}_2^n$ using the example of the folded half cube.

Recall from the formulation of $\frac{1}{2}FQ'_{n-1}$ we have the generating matrix S'. The method of constructing a matrix M for the vertices of size $n-2$ as above can be used to construct a matrix M' such that $M'S'$ is balanced. We show that this construction of M' results in the same labeling on each component.

Proposition 14. *There exists a $(n-1) \times (n-1)$ matrix M' such that each component of $\frac{1}{2}FQ'_{n-1}$ receives the same labeling, where $n = 16q^2$ for some nonzero integer q.*

Proof. We construct the $n-1 \times n-1$ matrix M' from M as

$$M' = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ 0 & & & & \\ \vdots & & M & & \\ 0 & & & & \end{pmatrix}.$$

where

$$M = \begin{pmatrix} \mathbf{I} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{I} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$

Due to the first column of 0's, the first bit of each vertex u (the parity bit) is ignored in the calculation of $f(u)$. Furthermore, the first row of 0's ensures that the first bit of $f(u)$ is set to 0. Therefore each odd size vertex v recieves the label $f(u)$ when u and v differ in only the first bit. Note that this is not a distance magic labeling since M' is not invertible, and clearly f is not bijective.

□

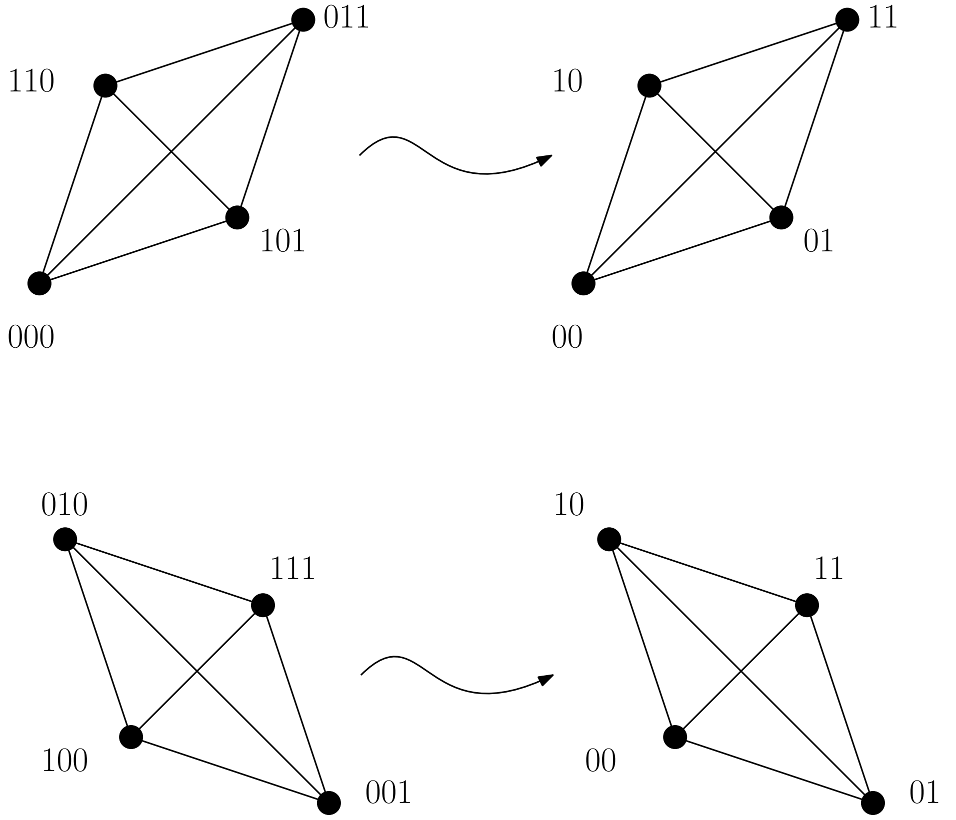

Figure 3.1: Ignoring the parity bit in $\mathbb{Z}_2^3$ of $\frac{1}{2}FQ_2$ results in both components having the same vertex set $\mathbb{Z}_2^2$.

Constructing an isomorphic labeling on $\frac{1}{2}FQ'_n$ leads to a hint of a distance magic labeling on $\frac{1}{2}FQ_n$. Since we know that MS is balanced, every row of $M'S'$ after the first is balanced. Thus if we find a $(n-1)\times(n-1)$ matrix M' such that the first row of m'_1S' is balanced, then we obtain a distance magic labeling of $\frac{1}{2}FQ'_{n-1}$ over both components.

Proposition 15. *There exists a distance magic labeling of $\frac{1}{2}FQ'_{n-1}$ for every $n = 16q^2$ for some nonzero integer q.*

Proof. We construct the $(n-1)\times(n-1)$ matrix M' from M by

$$M' = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 0 & & & & & & \\ \vdots & & & M & & & \\ 0 & & & & & & \end{pmatrix}.$$

Where the first row m'_1 has $8q^2 - 2q - 1$ 1's. The size of each row of M' is therefore the same as the row size of M. We create a partition of S' similarly to a partition of S to show that $M'S'$ is also balanced.

$$C'_1 = \{e_i \oplus e_j \mid i, j \in [n-1], i \neq j\},$$
$$C'_2 = \{\mathbf{1} \oplus e_i \mid i \in [n-1]\}.$$

For each $(e_j^T \oplus e_l^T) \in C_1'$, $m_i(e_j \oplus e_l)^T = 1$ if and only if (without loss of generality) $m'_{ij} = 1$, and $m'_{il} = 0$. Since m'_i has $8q^2 - 2q - 1$ entries of 1, and $8q^2 + 2q$ entries of 0, there are $(8q^2 - 2q - 1) \times (8q^2 + 2q) = 64q^4 - 12q^2 - 2$ such ways to choose j and l such that $e_j \oplus e_l \in C_1'$ can contribute 1 to $m_i S$. Therefore the total number of columns of C_1' which add to the contribution is $64q^4 - 12q^2 - 2$.

For each $s'^T \in C_2'$, note that the number of 1's in m_i is odd. Since each s'^T contains only one zero, the coordinate in which s'^T has the 0 entry must be one of the coordinates of m'_i that have a 0 entry. Therefore number of ways to choose this is the number of ways to choose the one 0 in s'^T to have coordinate j such that $m'_{ij} = 0$. There are $8q^2 + 2q$ ways to choose such a coordinate. Therefore the total contribution of 1's to $m'_i S'$ by C_2' is $8q^2 + 2q$.

The total sum of contributions of the partition is therefore $(64q^4 - 12q^2 - 2q) + (8q^2 + 2q) = 64q^4 - 4q^2$, and thus $M'S'$ is balanced.

Lastly, we check the invertibility of M'. Recall that there is a reduction of M to the $(8q^2 - 2q - 2, 8q^2 - 2q - 2, 4q + 2) \times (8q^2 - 2q - 2, 8q^2 - 2q - 2, 4q + 2)$ lower triangular form

$$M = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{I} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

By adding the first $8q^2 - 2q - 2$ rows of M (the rows from the index range of 2 to $8q^2 - 2q - 1$ of M') to m'_1, we then reduce M' to have the lower triangular form

$$M' = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{1} & \mathbf{I} & \mathbf{0} \\ 0 & \mathbf{1} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$

Therefore M' is invertible, and $f(u) = M'u^T$ is thus a distance magic labeling of $\frac{1}{2}FQ'_{n-1}$. □

4. Component-Wise Distance Magic Labelings

One natural question that arises from working over Cayley graphs which contain multiple disconnected components is whether a distance magic labeling is still possible for the entire graph. While each component G_i may have a unique magic constant m_i given a bijection f on all vertices of G, there is no guarantee that $m_i = m_j$ for all components G_i, G_j. If such a labeling is still possible, then we call this generalized distance magic labeling as a *component-wise* distance magic labeling.

Definition 11. *Let G be a graph with components $G_1, G_2, \ldots, G_k$. A bijection $f : V \to \{1, 2, \ldots, |V|\}$ is a* component-wise distance magic labeling *if there exists constants $m_1, m_2, \ldots, m_k$ such that for every component G_i, f is a distance magic labeling of G_i with magic constant m_i.*

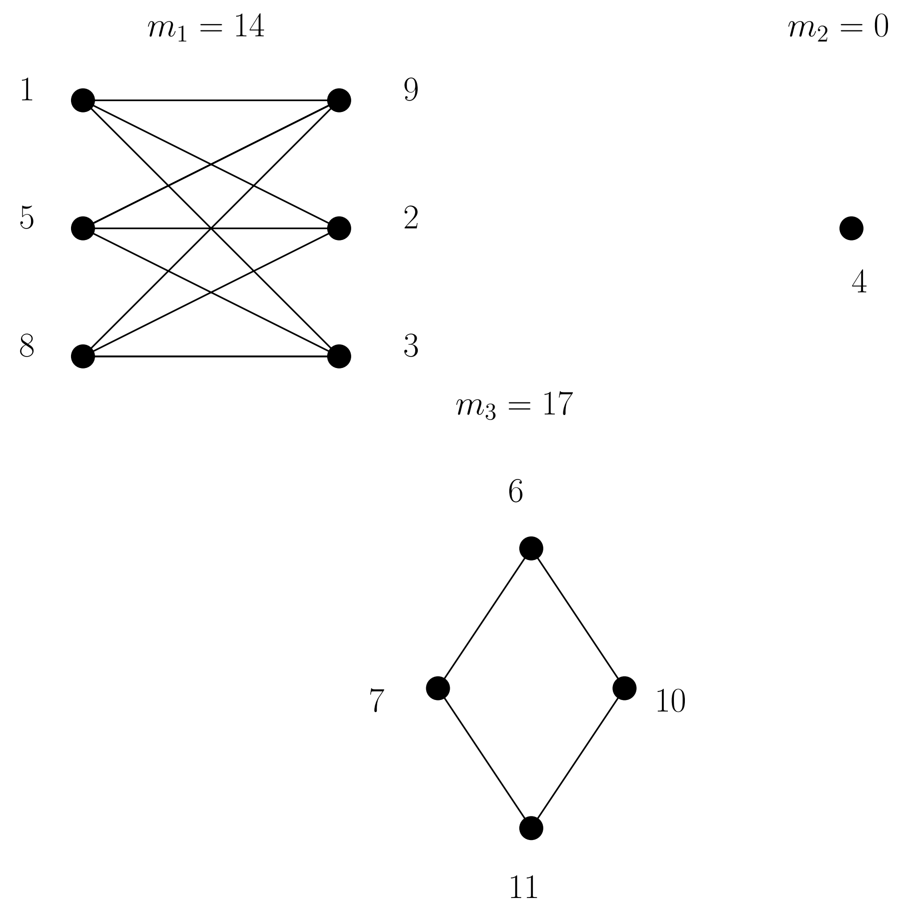

Figure 4.1: A component-wise distance magic labeling over three components with different magic constants m_1, m_2, and m_3.

4.1 Prefix Identification

Recall that each component of a Cayley graph is isomorphic to every other component. This implies that for every component G_i of $Cay(\mathbb{Z}_2^n, S)$, $|V(G_i)|$ divides 2^n, and therefore each component contains 2^d vertices for some $0 \leq d \leq n$. If there exists a $(n-d) \times n$ matrix P such that for every two vertices $u, v \in \mathbb{Z}_2^n$, $Pu^T = Pv^T$ when u, and v are in the same component G_i and $Pu^T \neq Pv^T$ otherwise, then each component is said to have a unique $(n-d)$-bit *prefix*. The matrix P is called the *prefix matrix*.

Proposition 16. *Consider a Cayley graph $Cay(\mathbb{Z}_2^n, S)$ with components $G_1, G_2, \ldots, G_{2^{n-d}}$ each of size 2^d. Suppose that there exists a $(n-d) \times n$ prefix matrix P, and there exists a $d \times n$ matrix M such that (without loss of generality) G_1 has a distance magic labeling by from $\mathbb{Z}_2^n \to \mathbb{Z}_2^d$ by $f(u) = Mu^T$. Then $Cay(\mathbb{Z}_2^n, S)$ has a component-wise distance magic labeling.*

Proof. If we already have a $d \times n$ matrix M that defines a distance magic labeling for G_1 to $\mathbb{Z}_2^d$ with magic constant m_1, then taking $M' = \begin{pmatrix} \mathbf{P} \\ \mathbf{M} \end{pmatrix}$ appends a $(n-d)$-bit prefix that is unique to each component. This yields 2^{n-d} different prefixes, one to each component, and thus f is a bijection. All labels in a given component G_i are then shifted by the $(n-d)$-bit prefix c_i with a d-length suffix of $\mathbf{0}$ appended. Thus each component G_i receives the magic constant $m_i = m_1 + d_i 2^d$ where d_i is the decimal representation of c_i. Thus f is a component-wise distance magic labeling. □

Take for example, a component-wise distance magic labeling for the folded half-cube $\frac{1}{2}FQ'_{n-1}$ where $n = 16q^2$ for some natural number q. Recall from Chapter 3.4 that for the Calyey graph on one component $\frac{1}{2}FQ_{n-1}$, we have that a $(n-2) \times (n-2)$ labeling matrix M for $Cay(\mathbb{Z}_2^{n-2}, S)$ has the $(8q^2-2q-2, 8q^2-2q-2, 4q+2) \times (8q^2-2q-2, 8q^2-2q-2, 4q+2)$ block form

$$M = \begin{pmatrix} \mathbf{I} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{I} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Note that if we add a column of 0's to the left of M, then we obtain a surjective labeling over both components of $\frac{1}{2}FQ'_{n-1}$ from $\mathbb{Z}_2^{n-1}$ to $\mathbb{Z}_2^{n-2}$. We can then construct a component-wise distance magic labeling by adding the row $\mathbf{1} = (1, 1, \ldots, 1)$ to the top of M. That is,

$$M' = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & & \\ \vdots & M & \\ 0 & & \end{pmatrix}.$$

This row separates the labelings in the following manner. If a vertex $u \in \mathbb{Z}_2^{n-1}$ has an even size, then $\mathbf{1}u^T = 0$, and if a vertex v has odd size, then $\mathbf{1}v^T = 1$. This results in each odd sized vertex v where $v = u \oplus (1, 0, \ldots, 0)$ receiving the label $f(u) \oplus (1, 0, \ldots, 0)$. That is, each even sized vertex of $\mathbb{Z}_2^{n-1}$ receives the labels

$\{0, \ldots, 2^{n-2} - 1\}$, and odd sized vertices receive the labels $\{2^{n-2}, \ldots, 2^{n-1} - 1\}$. Since each vertex recieves a unique label, then $f'(u) = M'u^T$ is a bijection.

While Cayley graphs whose components are defined on the parity of each vertex yields a simple row addition to M, the general question as to which Cayley graphs which yield a prefix matrix P resulting $n-d$-bit prefixes for some $d \in [n]$ is non-trival. To determine such Cayley graphs, we turn to orthogonal spaces.

Proposition 17. *A $(n-d) \times n$ prefix matrix P exists for every Cayley graph $Cay(\mathbb{Z}_2^n, S)$ with components $G_1, G_2, \ldots, G_2^{n-d}$ such that the prefix of G_1 is the $(n-d)$-length vector $\boldsymbol{0}$.*

Proof. Recall that any matrix has two spaces, the row space and column space. Furthermore, recall that we originally viewed S as the generating set of columns of the Cayley graph $Cay(\mathbb{Z}_2^n, S)$, and that G_1 is the component of $Cay(\mathbb{Z}_2^n, S)$ containing the element $\mathbf{0}$. Let $\langle S \rangle$ be the column space of S. Let us take $\langle P \rangle$ as the matrix whose rows form a base of the orthongonal space to $\langle S \rangle$. That is, for the row space $\langle P \rangle$ of P, we have $\langle P \rangle = \langle S \rangle^\perp$. The dimension of P is thus $n-d$ [6]. In another view, $\langle S \rangle = ker(P)$.

Since each component G_i has size 2^d, and G_i is generated by S, then it follows that the dimension of the linear space $\langle S \rangle$ generated by S is d. Furthermore, the vectors spanned by S compose the component G_1 of $Cay(\mathbb{Z}_2^n, S)$ containing the vertex $\mathbf{0}$. The dimension of $\langle P \rangle$ is $n-d$, and thus P is an $(n-d) \times n$ matrix.

Since $\langle P \rangle = \langle S \rangle^\perp$, and since $\mathbf{0} \in G_1 = \langle S \rangle$, then for every element u in G_1, and every row p_i of P, $p_i u^T = 0$. Thus G_1 has the prefix $\mathbf{0}$.

Now, suppose we take an affine space $G_1 \oplus x = G_i$ for some $x \notin G_1$. Since $x \notin \langle S \rangle$, then $Px^T \neq 0$.

Let u, v be elements of G_1 such that two elements $\hat{u} = u \oplus x$ and $\hat{v} = v \oplus x$ are elements of G_i for some $i \neq 1$. Then we have that

$$P(\hat{u}^T \oplus \hat{v}^T) = P((u^T \oplus x^T) \oplus (v^T \oplus x^T)) = P(u^T \oplus v^T) = Pu^T \oplus Pv^T = \mathbf{0}.$$

Thus any two elements $\hat{u}$ and $\hat{v}$ of the same component G_i receive the same prefix. □

Here are two examples of component structures of $\mathbb{Z}_2^n$ that have a prefix matrix P such that $g(u) = Pu^T$ maps each component to a unique element $v \in \mathbb{Z}_2^{n-d}$:

1. Two components, one on even vertices, one on odd vertices.

2. Components with a fixed $(n-d)$-bit prefix (subcubes) for some $d \in [n]$.

In the first case, we have already seen P as the $1 \times n$ matrix $(1, 1, \ldots, 1)$. This is a sufficient matrix as every vertex of even size will intersect this row an even number of times, mapping all even vertices to 0. It follows that $\langle P \rangle$ forms the orthogonal space to $\langle S \rangle$. Likewise, every vertex of odd size will intersect this row an odd number of times, mapping all odd vertices to 1.

In the second case, each component already has a d-bit prefix, thus taking P as the $d \times n$ matrix with block representation $d \times (d, n-d)$,

$$P = \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix}$$

is sufficient as each prefix is maintained. Taking G_1 as the component containing the $\mathbf{0}$ vector, for any $u \in G_1$, $Pu^T = 0$, and thus $\langle P \rangle$ is the orthogonal space to $\langle S \rangle$.

Putting Propositions 16 and 17 together, the following lemma is obtained.

Lemma 18. *A component-wise distance magic labeling exists for a Cayley graph $Cay(\mathbb{Z}_2^n, S)$ with components $G_1, G_2, \ldots, G_{2^{n-d}}$ if there exists a distance magic labeling of G_1.*

4.2 The Fixed Half Cube

Consider the following Cayley graph of $\mathbb{Z}_2^n$ where the first bit of each component is fixed, and the vertex set of each component is either of even or odd size. One such method to achieve this component structure is via the Cayley graph

$$Cay(\mathbb{Z}_2^n, \{e_i \oplus e_j \mid i, j \in \{2, \ldots, n\}, i \neq j\}).$$

This Cayley graph consists of four half cubes of dimension $n-2$, called the *fixed half cube* (due to the first bit being fixed).

Proposition 19. *There exists a component-wise distance magic labeling for the n-dimensional fixed half cube for $n \equiv 2 \pmod 8$ or $n \equiv 1 \pmod 4$.*

Proof. Let S' be the $n \times k'$ generating matrix of the fixed half cube. Note that the number of columns k' is $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$. Working over one component is then equivalent to working over $\mathbb{Z}_2^{n-2}$ by ignoring the first two bits of each vertex, with the column set $S = \{e_i \mid i \in [n-2]\} \cup \{e_i \oplus e_j \mid i, j \in [n-2]\}$. Note again that the number of columns k of S is equal to the number of columns k' of S' since there are $\binom{n-2}{1}$ elements of size one and $\binom{n-2}{2}$ elements of size two. In total this gives us

$$\begin{aligned} k &= \binom{n-2}{1} + \binom{n-2}{2} \\ &= n - 2 + \frac{(n-2)(n-3)}{2} \\ &= \frac{2(n-2) + (n-3)(n-2)}{2} \\ &= \frac{(n-1)(n-2)}{2} \\ &= k'. \end{aligned}$$

Clearly, if one component has a distance magic labeling, then by Theorem 5 either $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ for $\frac{(n-1)(n-2)}{2}$ to be even. Since each row has $n-2$ many 1's, if $\frac{(n-1)(n-2)}{2} \equiv 0 \pmod 4$, then $n-2$ is even and therefore $n \equiv 2 \pmod 8$. Likewise, if $\frac{(n-1)(n-2)}{2} \equiv 2 \pmod 4$, then $n-2$ is odd, and therefore $n \equiv 1 \pmod 4$.

Suppose we want to obtain M via row balancing. Recall that if we add a set $\hat{C}$ of $p-1$ rows of S to an arbitrary row s_i for some natural number p such that s_i becomes balanced, then the contribution of $s_i \oplus \sum_{c \in \hat{C}} c$ is equal to the contribution of each subset of a partition C of S. This is equivalent to each row m_i of M having p many 1's. Thus the construction of M such that each row m_i has exactly p many 1's yields the following.

Let us partition S into

$$C_1 = \{e_i \mid i \in [n-2]\},$$
$$C_2 = \{e_i \oplus e_j \mid i, j \in [n-2]\}.$$

We proceed by analyzing the contribution of each set C_j given that each row m_i of M has p many 1's.

For C_1, the addition of p rows together yields a contribution of p.

For C_2, the addition of p rows together yields a contribution of $p\left((n-2)-p\right)$ since $m_i(e_j \oplus e_l)^T = 1$ if and only if (without loss of generality) $m_{ij} = 1$ and $m_{il} = 0$.

Therefore the total contribution of p rows of S added together is $p+p(n-2-p)$. Since we would like for MS to be balanced, then we set $p+p(n-2-p) = \frac{(n-1)(n-2)}{4}$ which is half of the total number of columns of S. Therefore MS is balanced when $p+p(n-2-p) = \frac{(n-1)(n-2)}{4}$ for natural numbers n and p.

Now, we can construct the $n \times n$ matrix M' for S' using the $2 \times n$ matrix

$$P = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \mathbf{1} \end{pmatrix}.$$

Which yields the matrix

$$M' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & 0 \\ \vdots & \rlap{\Large M} & & \\ 0 & & & \end{pmatrix}.$$

Therefore, $Cay(\mathbb{Z}_2^n, \{e_i \oplus e_j \mid i, j \in \{2, \ldots, n\}\})$ has a component-wise distance magic labeling for every n such that $p+p(n-2-p) = \frac{(n-1)(n-2)}{4}$ has an integer solution. Some such solutions are $n = 5, p = 1$ and $n = 10, p = 6$. □

For illustration, consider this Cayley graph on the vertex set $\mathbb{Z}_2^5$. This yields the generating matrix S' with the values

$$S' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

By using the 2×5 matrix P as before, the 2-bit prefixes are assigned to components of this Cayley graph in the following manner.

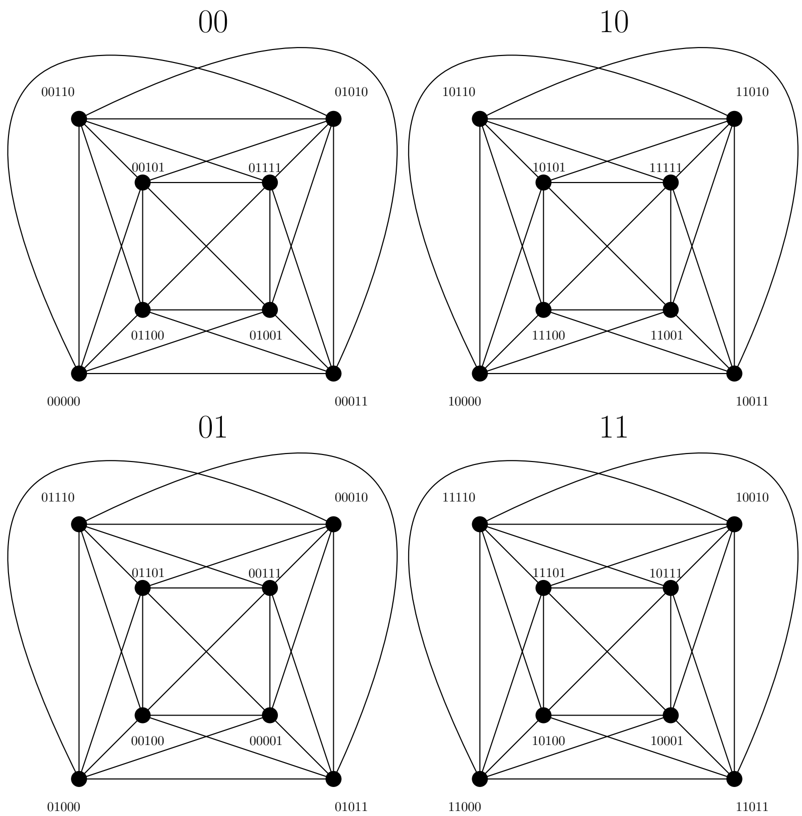

Figure 4.2: Prefix assignment of the fixed half cube of dimension 5 with the first bit fixed and the next bit as a parity bit.

By ignoring the first two rows of S', we obtain the generators of one component over the vertex set $\mathbb{Z}_2^3$. That is,

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Clearly each row of S is already balanced, thus taking M as the 3×3 identity matrix is sufficient. Therefore $Cay(\mathbb{Z}_2^5, \{e_i \oplus e_j \mid i, j \in \{2, \ldots, 5\}\})$ has a component-wise distance magic labeling with $f'(u) = M'u^T$ where

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Conclusion

In this thesis we have seen the current techniques used to obtain distance magic labeling of hypercube-like graphs, the most important being the notion of neighbour balance originally developed by Gregor and Kovář. The framework of row balancing was then introduced as an application of neighbour balance. This framework was then generalized to be suitable for determining the existence of distance magic labelings for all Cayley graphs of $\mathbb{Z}_2^n$. To achieve this, we used the generating matrix S of the Cayley graph to obtain a matrix M such that a function $f(u) = Mu^T$ is a distance magic labeling if MS is balanced. We introduced the pairing lemma (Lemma 6) and the method of counting contributions of sets of columns of S with a common structure as tools to show whether a given matrix M gives a neighbour balanced distance magic labeling.

Next, we used this framework of row balancing to reprove some notable recent results for which values of n a distance magic labeling of a specific Cayley graph of $\mathbb{Z}_2^n$ exists. In the examples of the half cube and folded half cube, new distance magic labelings f' were obtained in the Cayley graphs $\frac{1}{2}Q'_n$ (half cube) and $\frac{1}{2}FQ'_{n-1}$ (folded half cube) over multiple components.

Finally, the quest for determining the existence of a distance magic labeling over graphs with multiple components was relaxed to finding the existence of a component-wise distance magic labeling, where each component has a different magic constant. For Cayley graphs of $\mathbb{Z}_2^n$ with multiple components, the existence of a component-wise magic labeling was shown to be dependent on the structure of components. It was stated that a component-wise distance magic labeling for a Cayley graph $Cay(\mathbb{Z}_2^n, S)$ with components $G_1, G_2, \ldots, G_k$ exists if and only if G_1 has a distance magic labeling. The linear space orthogonal to the space generated by S was used to obtain prefixes which are unique to each component G_i, which are then added as a constant to all labels $f(u)$ where $u \in V(G_i)$.

We now discuss some possible continuations of the row balancing framework.

Consider the augmented cube AQ_n where there exists an edge between vertices u and v if u and v differ by one bit or if for some coordinate l, $u_i = v_i$ for $i < l$ and $u_i = v_i \oplus 1$ for $i \geq l$ [11]. The augmented cube has desirable properties, such as being pancyclic. Currently, it is unknown for which values of n AQ_n has a distance magic labeling. Unlike the other examples we have seen, the Cayley graph which generates AQ_n has the generating set of columns S with a different number of 1's in each column. More work is required to determine the order in which to balance the rows of S.

In this thesis, we have restricted ourselves only to Cayley graphs of $\mathbb{Z}_2^n$ on the n-length binary strings. Consider Cayley graphs of $\mathbb{Z}_p^n$ for some $p > 2$. We could possibly use this framework if we work over the group $\mathbb{Z}_p^n$, although the definition of neighbour balance would need to be generalized in the following manner. A bijection $f : \mathbb{Z}_p^n \to \mathbb{Z}_p^n$ is neighbour balanced if for every vertex $v \in \mathbb{Z}_p^n$ and for every coordinate $i \in [n]$, $|\{u \in N(v) \mid u_i = 0\}| = |\{u \in N(v) \mid u_i = 1\}| = \cdots = |\{u \in N(v) \mid u_i = p-1\}|$. Additional work is needed to check if this formulation of neighbour balance is appropriate for Cayley graphs of $\mathbb{Z}_p^n$.

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