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Review of the thesis "Geometric function theory and its application in nonlinear elasticity" by Mgr. Ondřej Bouchala

Continuum mechanics has posed intricate mathematical challenges for centuries, with many foundational problems persisting despite the relentless efforts of the scientific community. The demand for injectivity and the preservation of orientation in admissible deformations within elasticity naturally stems from our daily experiences. The imperative for the continuity of deformations and their inverses is another crucial requirement firmly grounded in the realm of contemporary approaches to problems in continuum multiphysics, exemplified in areas such as magnetoelasticity where the problems are formulated in both configurations, i.e., in the reference domain as well as in its deformed image.

The notions of homeomorphism and bijections are the central themes of the thesis that consists of two main parts. In the first part, the author addresses the question whether a sequence of homeomorphisms converges (and in which sense) to an injective map. The second part of the thesis studies relations of quasiconformal maps and Hardy spaces. The first part of Ondřej's work is based on a paper written jointly with S. Hencl and A. Molchanova published in the Jornal of Functional Analysis, the second part of the thesis follows from the joint paper with P. Koskela. It is published in Proceedings of the AMS. It follows that *the scientific contributions of Mgr. Bouchala withstand scientific scrutini in respected international journals.* Additionally, the first paper has already collected seven citations. All this means that the work of O. Bouchala is recognized by the scientific community.

Injectivity of Sobolev deformations in $W^{1,p}$ is relatively well-understood if p is greater than the spatial dimension and the Jacobian determinant is almost everywhere positive. Then the so-called Ciarlet-Nečas condition ensures only injectivity almost everywhere of the deformation and various examples of pathological behavior of such deformations are known in the literature. This thesis discusses deformations that are integrable only with a smaller power. Such cases have been treated by several authors, in particular, in connection with cavitation in rubber-like materials. The question of almost everywhere injectivity of the limit of homeomorphisms is completely resolved in the thesis. Namely, counterexamples are given showing that if $p \leq n-1$ (n is the spatial dimension) then even *strong* limits of homeomorphism does not have to be almost everywhere injective. The results are striking because strong limits are generically out of reach in the direct-method approach to minimization problems. Moreover, the boundary conditions of the involved maps are are just identity maps, and then it is shown that for a set of positive Lebesgue measure the inverse "deformation" is set-valued with the image of the cardinality of continuum. The construction of counterexamples is a very sophisticated combination of analysis and geometry. *It would be interesting to investigate if there is an additional assumption ensuring that the limit (strong/weak) is almost everywhere injective and perhaps whether such assumption is realistic for elasticity.*

For $p > n-1$, the thesis of Mr. Bouchala provides positive answers in two theorems. They also carefully distinguish injectivity almost everywhere in the domain and in the image and the limits of homeomorphisms are considered in the weak sense, which is more suitable for variational problems. Theorem 1.4. generalizes a well-known result by Müller and Spector by removing the assumption on the sign of the Jacobian which is a very nice improvement.

The second part of the thesis is devoted to quasiconformal maps in Hardy spaces. The results summarized in Theorems 2.5 and 2.6 are, however very interesting and strong. The proofs rely on deep results from complex analysis.

The thesis clearly shows applicant's ability to work in various fields of mathematical analysis and to obtain new and deep results. He is a strong mathematical analyst that is able to combine analysis with geometry to prove his statements. All relevant references are included and properly quoted. In summary, I clearly see that the PhD dissertation of O. Bouchala satisfies high scientific standards, it is clearly written and based on already published papers (with co-authors) in prestigious journals. All obtained results are new and their proofs require advanced analytical tools and require complex knowledge of the subject. As we already see, the first paper is well-quoted by the experts in the field. In my judgment, O. Bouchala's work unquestionably satisfies all the prerequisites for a doctoral thesis, warranting the conferment of a well-deserved PhD degree.

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