School of Mathematics

University of Leeds
Leeds LS2 9JT UK

T +44 (0) 113 343 5166
F +44 (0) 113 343 5090
E h.d.macpherson@leeds.ac.uk

UNIVERSITY OF LEEDS

Monday, 28th August 2023

Dear Panel Members,

Report on the PhD thesis of Matěj Konečný

Introduction. The PhD thesis by Matěj Konečný is in the general area of combinatorics, with substantial connections to model theory, group theory, and to topological dynamics, and potential links to other fields. Konečný considers classes $\mathcal{C}$ of finite structures in a fixed first order language (very often Fraïssé classes, that is, amalgamation classes with a 'Fraïssé limit' which is homogeneous, that is, is a countably infinite structure such that any isomorphism between finite substuctures extends to an automorphism). Topics considered include: stationary independence relations, and simplicity of the automorphism group of the Fraïssé limit; EPPA – the extension property for partial automorphisms – the class $\mathcal{C}$ has EPPA if for every $A \in \mathcal{C}$ there is $B \in \mathcal{C}$ containing A as an induced substructure, such that every partial isomorphism between substructures of A extends to an automorphism of B; structural Ramsey theory for the class $\mathcal{C}$, as initiated in the 1970s by Nešetřil and Rödl; big Ramsey degrees (an infinitary version of structural Ramsey theory).

The original contributions of the thesis consist essentially of 10 papers (including three 'Extended Abstracts' in *Eurocomb*). Apart from two submitted papers, and one accepted Extended Abstract, all are published. Apart from the paper in Chapter 6, all the papers are multi-author. The publishing journals are of high quality (*Journal of Algebra*, *Transactions of the American Mathematical Society*, *Proceedings of the American Mathematical Society*, *Combinatorica*, *Discrete Mathematics*). In addition, Chapter 1 serves as an introduction to the subject, and is a survey of the area and the very strong interconnections between the themes. Chapter 2 is a stimulating discussion of a range of open questions and possible future directions. Konečný has around 10 further accepted or published papers on the same themes which could have been included, but were omitted presumably to avoid excessive length.

I am very familiar with the kind of material in the papers from Chapters 3-7, but am much less familiar with that of Chapters 8-12 on big Ramsey degrees (a rapidly-developing area, to which Konečný seems to have come quite recently). From my reading I have high confidence in the correctness of the results, though my confidence is lower for the big Ramsey degree material, for which I have no intuition. The writing style is excellent, with clear motivation given in introductory sections, precise definitions and statements, good overview of the general proof ideas, and detail given clearly, and plenty of conjectures and questions highlighted. Though all but one of the papers is multi-author (some with many authors), I have confidence that Konečný has contributed strongly in each case, and Chapters 1 and 2 show a very strong grasp of the wider field, and a distinctive and highly imaginative viewpoint.

I have not previously reported on a thesis which consists basically of (multi-author) papers stuck together. Though I have no major concerns on this, with so many authors involved it does make it hard, for example, to pick out a distinctive writing style of the candidate. There is a slight concern that a PhD thesis would usually include full proofs of the claimed results, but some proofs in the Extended Abstracts are omitted, or just sketched. Also, in a thesis one would usually expect non-standard concepts that are mentioned to be defined precisely, but for one or two potentially important terms, there is just a reference to a published definition elsewhere (e.g. 'universal completion flow' in Theorem 10.1.3).

Summary of results of thesis.

Chapter 1. Background. This is an introduction to the subject-matter from a very modern viewpoint. It describes the main results in the papers in the thesis, also conveying the key methods involved, with a good balance of technical detail and overview. There is strong emphasis on certain key examples, such as the universal homogeneous two-graph, and the H_4-free 3-hypertournament. There is also explanation of how some of Konečný's other papers (not included in the thesis) fit into this body of work, and of important related papers by his co-authors. The chapter includes background on connections to topological dynmics.

Chapter 2. Problems, questions and conjectures. This is a very useful description of open questions closely related to the thesis, and would furnish several stimulating problem sessions at a conference. It is likely to lead to further publications by others.

Chapter 3. Simplicity of the automorphism groups of generalised metric spaces (published paper with Evans, Hubička, Li, Ziegler). The paper applies the Tent-Ziegler method of stationary independence relations (SIRs) to prove simplicity of certain automorphism groups. Abstract conditions on such an SIR ('bounded 1-supported metric-like') are identified which address the 'moves almost maximally' condition of Tent-Zegler and so ensure simplicity. These are applied to prove simplicity of examples such as automorphism groups of certain metrically homogeneous graphs, and of certain 'generalised metric spaces' (structures with a metric into a partially ordered semigroup).

Chapter 4. All those EPPA classes (strengthenings of the Herwig-Lascar theorem) (published paper with Hubička, Nešetřil). This major paper on EPPA provides a common generalisation of results of Herwig-Lascar, Hodkinson-Otto, Solecki-Siniora, Hubička-Nešetřil. It includes many results on bounds on sizes of EPPA witnesses, and for the first time makes explicit the connections between EPPA and the Ramsey property, identifying strong common themes. Key ideas developed strongly include: the action of a group permuting symbols of the same type and arity in a language (already seen in Herwig's work and in other papers, but developed and used strongly here); allowing unary function symbols in the language as in work of Hubička and Nešetřil and other authors, but allowing these to take values in the power set of the universe of the structure; careful analysis of when the EPPA witness is 'coherent' (a condition which ensures that the automorphism group of the Fraïssé limit has a dense locally finite subgroup); a beautiful, very natural and enlightening tree-amalgamation construction; some results involving function symbols of higher arity.

Chapter 5. EPPA for two-graphs and antipodal metric spaces (published paper with Evans, Hubička, Nešrtřil). The main result is a beautiful proof of EPPA for the class of all finite two-graphs (3-hypergraphs such that every 4-set contains an even number of hyperedges), and coherent EPPA for integer-valued antipodal metric spaces of diameter 3. Two-graphs provide an interesting example involving somewhat new techniques. There are positive results on Ramsey expansions of the universal homogeneous two-graph, and a negative result on the 'amalgamation property with automorphisms'.

Chapter 6. Extending partial isometries of antipodal graphs (published paper). This single-author paper combines results and methods from the previous two papers to complete the proof of EPPA for all the metrically homogeneous graphs (from the catalogue of Cherlin) which are *antipodal* (an important subclass), filling in two families of examples not handled in a previous paper co-authored by Konečný.

Chapter 7. Ramsey expansions of 3-hypertournaments (published Extended Abstract, with Cherlin, Hubička, Nešetřil). There are many classification theorems for classes of binary homogeneous structures, but very little in higher arity. Cherlin proposed a challenge to classify homogeneous 3-hypertournaments (structures equipped with a single irreflexive ternary relation, holding on every k-set in some ordering, with the realisations on any k-set invariant under $\mathrm{Alt}(k)$). In particular Cherlin classified the homogeneous

.

3-hypertournaments which are 4-constrained, i.e. whose age is determined by 4-vertex forbidden configurations. The authors identify a Ramsey expansion for all but one such structure, and for this structure (the H_4-free 3-hypertournament), show that the problem of identifying an optimal Ramsey expansion is unclear, and a very interesting test case. The main interest here is in the process of identifying easily described examples of Fraïssé classes and homogeneous structures which present new features for problems such as finding a Ramsey expansion in a finite language, or proving EPPA.

Chapter 8. Big Ramsey degrees of 3-uniform hypergraphs are finite (published paper with Balko, Chodounský, Hubička, Vena). Papers 8-12 concern the topic of big Ramsey degrees, a topic in structural Ramsey theory of more infinitary nature and high current activity. Given a countable structure M and finite induced substructure A, the *big Ramsey degree* of A in M is the least $l \in \omega \cup \{\omega\}$ such that for any positive integer k and colouring χ of the embeddings of A in M with at most k colours, there exists an embedding $f : M \to M$ such that χ takes at most l values on the set of embeddings of A into $f(M)$. We say M has *finite big Ramsey degrees* if such l is finite for each finite A. The goal is to identify whether a given structure M has finite big Ramsey degrees, and when it does, to find bounds or exact values for such degrees. Such results go back to Laver in 1969 (for $(\mathbb{Q}, <)$) and have received impetus through recent connections to topological dynamics (and similar activity in finite structural Ramsey theory). Methods of proof typically involve infinite coding results with countable trees, and depend on specific and canonical previous infinitary structural Ramsey theorems such as Milliken's tree theorem.

In this chapter, the authors prove that the universal homogeneous 3-hypergraph (or any countable universal 3-hypergraph) has finite big Ramsey degrees. They indicate that the same method works for d-uniform hypergraphs for any $d \geq 2$.

Chapter 9. Big Ramsey degrees and infinite languages (submitted paper, with Braunfeld, Chodounský, de Rancourt, Hubička, Kawach). The main result here is that if L is a relational language with finitely many relation symbols of each arity at least two, with interpretations of such symbols assumed to be irreflexive, then any countable 'unrestricted' L-structure has finite big Ramsey degrees. This is the first such result in an infinite language.

Chapter 10. Characterisation of the big Ramsey degree of the generic partial order (submitted paper with Balko, Chodounský, Dobrinen, Hubička, Vena, Zucker). The authors calculate the big Ramsey degrees of the universal homogeneous partial order.

Chapter 11. Big Ramsey degrees and forbidden cycles, (published Extended Abstract with Balko, Chodounský, Hubička, Nešetřil, Vena). Tha authors find a new condition which ensures that a structure in a finite binary relational language has finite big Ramsey degrees.

Chapter 12. Type-respecting amalgamation and big Ramsey degrees (accepted Extended Abstract with Aranda, Braunfeld, Chodounský, Hubička, Nešetřil, Zucker). The authors identify an analogue for big Ramsey degrees of a classical theorem of Nešetřil and Rödl concerning (finitary) structural Ramsey theory.

Overall evaluation of novelty and impact. The thesis operates with combinatorial concepts (such as stationary independence relations and simplicity of the automorphism group, EPPA, Ramsey property, big Ramsey degrees) already identified and considered to be significant. The achievement of the thesis is to successfully explore such concepts for interesting classes of homogeneous structures, or Fraïssé classes, and to develop methods for such exploration. (Generalised) metric spaces and metrically homogeneous graphs recur in the thesis, as natural examples to explore, especially in Chapter 1-7.

There are ingenious new ideas and constructions in every chapter, and the chapters complement each other beautifully. For me, the highlight is the work on EPPA, especially Chapter 4, which seems to take current methods related to EPPA to their natural endpoint. This chapter has many results of high interest: bounds on sizes of EPPA witnesses, proofs of EPPA in new cases, the tree-amalgamation construction, and significant exploitation of the idea of a group permuting the symbols in a language.

.

As mentioned earlier, I am much less familiar with the material surrounding big Ramsey degrees, but believe that these papers make a major advance here too, pushing positive results beyond binary languages, and identifying new combinatorial techniques.

Overall, I find the thesis creative and original, attractive in the focus on examples, and of great importance in the general area of homogeneous structures and structural Ramsey theory. The wider impact is hard to predict, but for example, EPPA, beyond its intrinsic combinatorial interest, seems to be significant from several viewpoints: there are connections to group theory (with many applications to automorphism groups, and also close connections to the Ribes-Zalesskii Theorem, and to sofic groups), to topological dynamics, and to theoretical computer science.

Suggested questions for thesis defence.

1. I was slightly confused by the use of the term 'amenable group'. The definition given in Remark 1.7.1 appears to be the traditional one, with no reference to a G-flow. I understand that any group containing a rank 2 free group is not amenable, so no closed oligomorphic group is amenable, contradicting the assertion, supposedly in [KR07] that EPPA implies amenability. In fact, a word search in the published version of [KR07] does not find 'amenable' though Theorem 2.26 of [EHN19] refers to a different version of Proposition 6.4 of [KR07] which does mention amenability. But the definition of 'amenable' in [EHN19] is different – a topological group G is amenable if, whenever Y is a G-flow, then there is a Borel probability measure μ on Y which is invariant under the action of G. Was this the intended meaning of 'amenable' in Remark 1.7.1?

2. The univeral homogeneous two-graph considered in Chapter 5 is a reduct of Rado's graph. There are other nice examples of reducts of structures – higher arity versions of two-graphs, D-relations as reducts of C-relations, circular orders as reducts of linear orders, and of course many others. To what extent is a systematic treatment possible, e.g. of the Ramsey theory of such reducts (e.g. presumably a Ramsey expansion of the reduct has to involve the structure it is a reduct of)?

3. You mention in Section 1.6.1 that Hrushovski showed that any EPPA witness for a half-graph with two parts each of size n has to have at least 2^n vertices. Is it possible that finite graphs with no large induced half-graphs have very small EPPA witnesses – rather as Malliaris-Shelah give a very strong version of the Regularity Lemma for such graphs?

4. You briefly mention Otto's [Ott20]. Do you see connections between your work on EPPA and his?

5. On p.95 you mention a conjecture that any primitive homogeneous structure in a finite binary relational language with trivial algebraic closure can be viewed as a semigroup-valued metric space (see also Conjecture 3.5.3). What is the evidence for this?

Recommendation. Overall, I consider this to be an exceptionally strong PhD thesis, with important results likely to be influential in the future, technically difficult arguments carefully written, some beautiful ideas, and a broad vision bringing the themes together. On the basis of the thesis I very strongly recommend Matěj Konečný for the award of PhD.

Your Sincerely,

Dugald Macpherson (Professor of Pure
Mathematics, School of Mathematics,
University of Leeds)