COLORINGS OF INFINITE GRAPHS

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Abstract: This thesis focuses on the study of uncountable graphs in relation to Ramsey theory, the chromatic number, and the uncountable Hadwiger conjecture. A large part of this text deals with constructions of uncountable graphs in Cohen forcing extensions. We show that adding ω_2 Cohen reals forces the partition relation $\omega_2 \to (\omega_2, \omega : \omega)^2$ but it also forces that $\omega_2 \not\to (\omega_2, \omega : \omega_1)^2$. An unpublished result of Stevo Todorčević is proved—adding a single Cohen real forces that $\omega_1 \not\to (\omega_1, \omega : 2)^2$. From a single Cohen real, we also construct a triangle-free Hajnal–Máté graph, answering a question of Dániel Soukup. Using the same method, we construct a T-Hajnal–Máté graph with the same properties in ZFC, extending a result of Péter Komjáth and Saharon Shelah. Section 2.4.1 concentrates on a different generalization of HM graphs, the so-called δ-Hajnal–Máté graphs. We show that they do not exist under $\mathsf{MA}(\omega_1)$. In the same section, we also deduce a weak partition relation: $\omega_2 \to (\omega_1, \delta : 2)^2$, where δ is any countable ordinal, which holds in ZFC and is related to an old result of Fred Galvin. In Chapter 3, we focus on the uncountable Hadwiger conjecture. We introduce the cardinal invariant $\mathfrak{hc}$, the least size of a counterexample to the uncountable Hadwiger conjecture. We prove that it is equal to the special tree number.

The main results of this thesis are: Theorem 2.6, Proposition 2.7, Theorem 2.25, Theorem 2.26, Theorem 2.31, Proposition 2.32, Theorem 3.32, Theorem 3.34 and Theorem 3.35.