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Colorings of Infinite Graphs

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Abstract: This thesis focuses on the study of uncountable graphs in relation to Ramsey theory, the chromatic number, and the uncountable Hadwiger conjecture. A large part of this text deals with constructions of uncountable graphs in Cohen forcing extensions. We show that adding ω_2 Cohen reals forces the partition relation $\omega_2 \to (\omega_2, \omega : \omega)^2$ but it also forces that $\omega_2 \not\to (\omega_2, \omega : \omega_1)^2$. An unpublished result of Stevo Todorčević is proved—adding a single Cohen real forces that $\omega_1 \not\to (\omega_1, \omega : 2)^2$. From a single Cohen real, we also construct a triangle-free Hajnal–Máté graph, answering a question of Dániel Soukup. Using the same method, we construct a T-Hajnal–Máté graph with the same properties in ZFC, extending a result of Péter Komjáth and Saharon Shelah. Section 2.4.1 concentrates on a different generalization of HM graphs, the so-called δ-Hajnal–Máté graphs. We show that they do not exist under $\mathsf{MA}(\omega_1)$. In the same section, we also deduce a weak partition relation: $\omega_2 \to (\omega_1, \delta : 2)^2$, where δ is any countable ordinal, which holds in ZFC and is related to an old result of Fred Galvin. In Chapter 3, we focus on the uncountable Hadwiger conjecture. We introduce the cardinal invariant $\mathfrak{hc}$, the least size of a counterexample to the uncountable Hadwiger conjecture. We prove that it is equal to the special tree number.

The main results of this thesis are: Theorem 2.6, Proposition 2.7, Theorem 2.25, Theorem 2.26, Theorem 2.31, Proposition 2.32, Theorem 3.32, Theorem 3.34 and Theorem 3.35.

Keywords: uncountable graphs, partition relations, Hajnal–Máté graph, chromatic number, uncountable Hadwiger conjecture

Contents

Preliminaries

In this thesis, we will work within Zermelo–Fraenkel set theory with the axiom of choice, abbreviated as ZFC. Throughout the work, we use standard set-theoretic notation, and we will mostly rely on [Jec03] and [Kun80] as general references.

We recollect the most common notations and conventions used in the thesis for the reader's convenience.

Sets and Basic Combinatorial Notions

Let X be a set. By $|X|$ we mean the cardinal size of X. The least infinite cardinal is denoted as ω, and the first uncountable cardinal is ω_1. If α is an ordinal, ω_α denotes the α-th infinite cardinal. A cardinal is regular if it is equal to its cofinality. Otherwise, it is singular. Successor cardinals are denoted with a plus sign as a superscript, e.g., $\omega^+ = \omega_1$. The Greek lowercase letters $\kappa, \lambda, \mu, \ldots$ usually denote an arbitrary infinite cardinal. The lowercase letters $k, l, m, \ldots$ will denote elements of ω.

The set of all subsets of X (the power set of X) is $\wp(X)$. If μ is a cardinal, then $[X]^\mu$ denotes the set $\{Y \subseteq X \mid |Y| = \mu\}$; $[X]^{<\mu}$ is the set $\{Y \subseteq X \mid |Y| < \mu\}$ and $[X]^{\leq\mu}$ is the union of the previous two sets. For two sets X, Y we define $X \otimes Y := \{\{x, y\} \mid x \in X \wedge y \in Y\}$. If X, Y are subsets of some ordered set (e.g., subsets of ordinals), then $X < Y$ means that each element of X lies below each element of Y. By $\mathrm{ot}(X)$, we mean the order type of a well-ordered set X.

Given a regular cardinal κ a set $C \subseteq \kappa$ is called a *club set of* κ if C is unbounded in κ and closed, i.e., if $\bigcup(C \cap \alpha)$ is equal to α, then $\alpha \in C$. A subset $S \subseteq \kappa$ is called *stationary* if S intersects each club set of κ.

A collection of sets $\mathcal{X}$ is said to form a Δ-system if there is a set r such that $x \cap y = r$ for each $x, y \in \mathcal{X}$ distinct.

Graphs

A *graph* G is a pair (V, E), where V is an arbitrary set and $E \subseteq [V]^2$; the set V is called the vertex set and E the edge set of the graph. Unless otherwise specified, G will always denote a graph whose vertex set is V and the edge set is E. In case of ambiguity, we will use V_G and E_G instead. If κ, λ are cardinals, then K_κ denotes the *complete graph on* κ vertices, i.e., the graph $(\kappa, [\kappa]^2)$; by $K_{\kappa,\lambda}$ we denote the complete bipartite graph with partitions of size κ and λ. A *subgraph of* G is any graph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$; a subgraph is *induced* if $E_H = [V_H]^2 \cap E_G$. A set of vertices $X \subseteq V$ is called *independent* in G if $E \cap [X]^2 = \emptyset$.

The *degree of a vertex* v is the size of the set $\{u \in V \mid \{u, v\} \in E\}$. The degree of v is denoted as $\deg(v)$. By $\Delta(G)$, we denote the supremum of the degrees of vertices in G; we call this invariant the *maximum degree of* G. By $\mathrm{N}(v)$

we will denote the set $\{u \in V \mid \{u,v\} \in E\}$ and if V is ordered by $\prec$, then $\mathrm{N}^{<}(v) := \{u \prec v \mid \{u,v\} \in E\}$.

Two subsets $X, Y \subseteq V$ *are connected* if there are vertices $x \in X$ and $y \in Y$ such that $\{x,y\} \in E$. A *path* in G is a finite injective sequence $(v_i)_{i<k}$ of vertices of G such that for each $i < k-1$ the vertex v_i is connected to v_{i+1}. A *graph is connected* if a path connects any two vertices. A *component* of a graph is any maximal connected subgraph. A graph is *κ-connected* if it stays connected after removing $< \kappa$ many vertices. If X, Y, S are subsets of the vertex set, we say that S *separates* X *from* Y in G if after removing S from the graph, $X \setminus S$ and $Y \setminus S$ lie in different components. Given a natural number $n \geq 3$ and an injective sequence of vertices $(v_0, \ldots, v_{n-1})$ we say that $(v_0, \ldots, v_{n-1}, v_0)$ forms a *cycle* if $\{v_i, v_{i+1}\} \in E$ for each $i < n-2$ and $\{v_{n-1}, v_0\} \in E$.

The *chromatic number* of a graph G, denoted $\chi(G)$, is the least cardinal μ such that there is a proper coloring of G with μ colors. A proper coloring is a function $c : V \to \mu$ such that $c(u) \neq c(v)$ whenever $\{u,v\} \in E$.

The *coloring number* of a graph G, denoted $\mathrm{Col}(G)$, is the least cardinal μ such that there is a well-ordering of the vertex set where each vertex is connected to $< \mu$ vertices preceding it.

The *chromatic index* of a graph G, denoted as $\chi'(G)$, is the least cardinal μ such that there is a proper edge coloring $c : E \to \mu$. A proper edge coloring is a function $c : E \to \mu$ such that $c(e) \neq c(f)$ if $e \cap f \neq \emptyset$.

If G and H are graphs, we say that G *is a minor of* H if there are pairwise disjoint non-empty subsets $(X_u)_{u \in V_G}$ of V_H such that each induces a connected subgraph in H and for every $\{u,v\} \in E_G$, X_u and X_v are connected in H. Note also that being a minor is a transitive property. The graph H *is a subdivision of* G if H is constructed from G by replacing edges with paths. Note that if H is a subdivision of G, then G is a minor of H; we call G a *topological minor* of H.

A graph on ω_1 is called *Hajnal–Máté* if for each $\alpha < \beta$, the set of vertices $\gamma < \alpha$ connected to β is finite and the graph is uncountably chromatic.

For an introduction on (mostly finite) graphs, we refer the reader to [Die00]. For a decent overview of Erdős's results in infinite graph theory, we cite Komjáth's survey paper [Kom13].

Trees

A *tree* is a partially ordered set $(T, \leq)$ such that for each node $t \in T$, the set $\{s \in T \mid s < t\}$ is well-ordered, the order type of this set is called the *height of the node* t, denoted as $\mathrm{ht}(t)$. For a tree, the set T_α is the set of nodes of height α sometimes referred to as *level* α *of* T. Similarly, we denote $T_{<\alpha}$ the nodes whose height is strictly less than α, and $T_{\leq\alpha}$ is defined analogously. The *height of the tree*, $\mathrm{ht}(T)$, is the least α such that T_α is empty. By $\mathrm{pred}(t)$, we denote the set of nodes in the tree which are strictly below t in the tree order. By a *branch* in a tree, we mean a maximal chain. A *κ-branch* is a branch of size κ. A subset $A \subseteq T$ is called an *antichain* in T if for each distinct $s, t \in A$ neither $s \leq t$ nor $t \leq s$, in which case we say s and t are *incomparable*. A *κ-tree* is a tree of height

κ with levels of size less than κ.

A tree is called *κ-special* if there is a function $f : T \to \kappa$, which is injective on chains. An ω-special tree will be simply called a *special tree*.

A *κ-Suslin tree* is a tree of size κ with no κ-branches and no antichains of size κ. A *κ-Aronszajn tree* is a κ-tree with no κ-branch. A *κ-Kurepa tree* is a κ-tree with at least κ^+ different branches. When $\kappa = \omega_1$, we omit the cardinal specification.

The *comparability graph* of a tree is a graph whose vertex set is the domain of the tree, and the edge set is $\{\{s,t\} \in [T]^2 \mid s \leq t \vee t \leq s\}$.

If T is a tree, a graph is called a *T-graph* if it is isomorphic to a graph whose vertex set is T, is a subgraph of the comparability graph of T and for every $t \in T$ the vertices connected to t are cofinal in $\{s \in T \mid s < t\}$, if this set has a maximum we want t to be connected to it.

Todorčević's chapter in the Handbook of Set-Theoretic Topology [Tod84] contains an excellent overview of trees and presents their importance in set theory and topology. For applications of T-graphs, see the survey paper of Pitz [Pit22].

Partition Relations

Given a cardinal μ, an ordinal α and a sequence of ordinals $(\beta_i)_{i<\mu}$ the *partition relation* $\alpha \to ((\beta_i)_{i<\mu})^2_\mu$ says that given a function $c : [\alpha]^2 \to \mu$ there is a subset X of α and an ordinal $i < \mu$ such that the order type of X is β_i and $c''[X]^2 = \{i\}$. When the sequence $(\beta_i)_{i<\mu}$ is constant with value β we write $\alpha \to (\beta)^2_\mu$. For the most part, we will be assuming $\mu = 2$, in which case we omit the subscript and write $\alpha \to (\beta_0, \beta_1)^2$.

The weaker relation $\alpha \to (\beta, \gamma : \delta)^2$ says that for every function $c : [\alpha]^2 \to 2$ either there is an $X \subseteq \alpha$ such that the order type of X is β and $c''[X]^2 = \{0\}$ or there are sets $X, Y \subseteq \alpha$ such that $X < Y$, the order type of X is γ, the order type of Y is δ and $c''[X \otimes Y] = \{1\}$.

By a slight abuse of notation, we will write $c(\alpha, \beta)$ instead of $c(\{\alpha, \beta\})$ and when writing $c(\alpha, \beta)$ we also tacitly assume that $\alpha < \beta$ if a natural ordering is present.

Each function $c : [\alpha]^2 \to 2$ defines a graph on α, namely $(\alpha, c^{-1}[\{1\}])$. Thus it will sometimes be convenient to talk about arbitrary functions on α. Instead of looking for homogeneous sets for the coloring, we can consider independent sets and cliques, i.e., we can rephrase the notion of partition relations as follows: for ordinals α, β, γ the partition relation $\alpha \to (\beta, \gamma)^2$ says that given any graph whose vertex set is α and there is no independent set of order type β we can find a complete subgraph of order type γ. The relation $\alpha \to (\beta, \gamma : \delta)^2$ says that every graph on α either has an independent set of order type β or a subgraph whose vertex set is $A \cup B$, where A has order type γ and B has order type δ, $A < B$ and $A \otimes B \subseteq c^{-1}[\{1\}]$ (sometimes referred to as a subgraph of type $(\gamma : \delta)$). We will sometimes use the weak symmetric relation $\alpha \to (\gamma : \delta)^2$ which says that for each $c : [\alpha]^2 \to 2$, there are sets $X, Y \subseteq \alpha$ such that $X < Y$, the order type of X is γ, the order type of Y is δ and $|c''[X \otimes Y]| = 1$, in graph terms this relation says that any graph on α or its complement has a subgraph of type $(\gamma : \delta)$.

The failure of a partition relation will be denoted by crossing the arrow symbol, i.e., $2^\omega \not\to (\omega_1)^2$.

Hajnal and Larson's survey on partition relations in the Handbook of Set Theory [HL10] is a fitting reference for topics related to partition relations.

Cardinal Invariants

A *cardinal invariant of the continuum* is a cardinal $\mathfrak{x}$ such that provably in ZFC we have that $\mathfrak{x}$ is uncountable and at most 2^ω. Usually, the value of $\mathfrak{x}$ is not decided in ZFC. The most basic cardinal invariant is the value of the continuum itself, sometimes denoted as $\mathfrak{c}$. See [Hal17, Chapter 9] for an introduction on the most commonly occurring cardinal invariants.

We will only use a handful of cardinal invariants. The *bounding number*, denoted as $\mathfrak{b}$, is the least cardinal μ such that there is a family of μ functions from ω to ω with the property that given any $f : \omega \to \omega$ there is a function $g : \omega \to \omega$ in the family such that $g(n) \geq f(n)$ for infinitely many values of n.

The *special tree number*, $\mathfrak{st}$, is the least size of a non-special tree of height ω_1 without an uncountable branch. See [Swi23] for more details and the history of this invariant.

Beyond ZFC

The nature of problems we are interested in often requires us to step outside the scope of the usual axioms of ZFC and construct various models to prove independence results. The tool best fitted for our purposes is the method of *forcing* introduced by Paul Cohen [Coh63].

Our forcing notation is standard and mainly follows [Kun80]. If the reader is unfamiliar with forcing, they may also find Halbeisen's book [Hal17] helpful.

By a *forcing notion* (or just *forcing*), we mean a partially ordered set. Let $(\mathscr{P}, \leq)$ be a forcing notion. Elements of $\mathscr{P}$ are called *conditions*. Two conditions p, q are *compatible* if there is a condition r such that $r \leq p, q$ and we write $p \,||\, q$ otherwise, we say they are *incompatible* and write $p \perp q$. A subset $A \subseteq \mathscr{P}$ is called an *antichain* if each pair of distinct conditions from A are incompatible. A subset $D \subseteq \mathscr{P}$ is dense if, for each $p \in \mathscr{P}$, there is a $q \in D$ such that $q \leq p$.

In this thesis, given two forcing conditions p, q, we say that p *is stronger than* q if $p < q$.

A forcing is said to have the κ *chain condition* if each antichain has size strictly less than κ. We denote this property as κ-*cc*; when $\kappa = \omega_1$, we talk about the *countable chain condition* and abbreviate it as *ccc*.

By $\mathsf{MA}(\kappa)$, we denote the statement: if $(\mathscr{P}, \leq)$ is a ccc forcing notion and $\mathcal{D}$ is a family of $\leq \kappa$ dense subsets of $\mathscr{P}$, then there is a filter $G \subseteq \mathscr{P}$ such that $G \cap D \neq \emptyset$ for each $D \in \mathcal{D}$. By MA we denote the statement $\forall \kappa < 2^\omega(\mathsf{MA}(\kappa))$.

A forcing notion is said to be *κ-closed* if for each $\gamma < \kappa$ and each sequence of conditions $(p_\alpha \mid \alpha < \gamma)$ such that $\alpha < \beta$ implies $p_\alpha \geq p_\beta$, there exists a condition p with the property that $p \leq p_\alpha$ for each $\alpha < \gamma$.

Suppose $\kappa \leq \lambda$ are cardinals, the forcing for adding λ many Cohen subsets of κ will be denoted $\mathsf{Add}(\kappa, \lambda)$. Its underlying set is $\{p : \lambda \to 2 \mid |p| < \kappa\}$, and the ordering is reverse inclusion. If κ is regular and $2^{<\kappa} = \kappa$, then this forcing is κ^+-cc and κ-closed.

By $\mathsf{H}(\kappa)$, we denote the collection of all sets hereditarily of cardinality less than κ. These sets will be used for constructing suitable elementary submodels in the context of generic forcing extensions; for an introduction on this subject, we refer the reader to [JW97, Chapter 24.], and for applications of elementary submodels in infinite combinatorics we recommend [Sou11].

The combinatorial principle $\diamondsuit^+$ asserts the existence of a sequence $(\mathcal{A}_\alpha \mid \alpha < \omega_1)$, where $\mathcal{A}_\alpha \in [\wp(\alpha)]^{\leq\omega}$ such that for each $A \subseteq \omega_1$ there is a club $C \subseteq \omega_1$, such that for each $\alpha \in C$ both $A \cap \alpha$ and $C \cap \alpha$ belong to $\mathcal{A}_\alpha$.

1. Introduction

As the title of this thesis suggests, we will investigate various colorings of infinite graphs. Most of our results and inspiration come from attempting to generalize well-known theorems and conjectures in finite graph theory to infinite graphs.

1.1 Edge Colorings

For the most part, we will be interested in edge colorings in the context of partition relations. However, let us quickly observe that the chromatic index trivializes in the case of graphs with infinite maximum degree.

The chromatic index is closely related to the maximum degree of a graph. The following theorem nicely characterizes the case of finite graphs; see [Viz64].

Theorem 1.1 (Vizing's theorem)**.** *If G is a finite graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Using a standard compactness argument, this result can be extended to infinite graphs whose maximum degree is finite. The case when $\Delta(G)$ is infinite is an easy exercise, cf., [KT06, Problem 23.5.].

Proposition 1.2. *Suppose κ is an infinite cardinal. If G is a graph with $\Delta(G) = \kappa$, then $\chi'(G) = \kappa$.*

Proof. We can assume that the graph is connected as the colorings of distinct components are independent. The chromatic index must be at least κ. Either there is a vertex whose degree is κ, in which case we need κ colors for the edges incident to this vertex, or the supremum of the degrees of G is κ which means that for every $\lambda < \kappa$, there is a vertex whose degree is at least λ hence κ colors are needed.

To see that κ colors are enough, note that since $\Delta(G) = \kappa$, the size of G is at most κ. Suppose an arbitrary vertex is given. It is connected to at most κ vertices by our assumption. Each of these is again connected to at most κ other vertices. As paths are finite, we can reach any vertex from an arbitrary starting point in finitely many steps. Altogether, the number of vertices is $\sum_{i=0}^{\infty} \kappa^i$. This sum is equal to κ. As the graph has size at most κ it can have at most $|[\kappa]^2|$ many edges and $|[\kappa]^2| = \kappa$. Thus choose $c : E \to \kappa$ any injection. This will witness that $\chi'(G) \leq \kappa$. □

1.1.1 Partition Relations

"There are numerous theorems in mathematics which assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system." This famous quote by Harry Burkill and Leonid Mirsky reflects the part of set theory that concerns itself with

Ramsey-type theorems and partitions. Arguably the most prominent results in this branch are Ramsey's theorems [Ram30], one specific instance of the finite version states that in any group of 6 people, either at least 3 of them are (pairwise) mutual strangers or at least 3 of them are (pairwise) mutual friends. The infinite version asserts that every countably infinite graph contains an infinite clique or an infinite set of independent vertices.

Theorem 1.3 (Ramsey's theorem)**.** $\omega \to (\omega)^2$.

We are interested in generalizations of the infinite version of Ramsey's theorem. Unfortunately, any straightforward attempts are bound to fail by a classical result of Sierpiński we have that:

Theorem 1.4 (Sierpiński [Sie33])**.** *If κ is an infinite cardinal, then $2^\kappa \not\to (\kappa^+)^2$.*

Indeed, any cardinal number exhibiting the same behavior as ω, i.e., if κ is a cardinal and each graph on κ contains a κ-clique or a κ big set of independent vertices, already has large cardinal strength, namely κ is weakly compact (the following is sometimes taken as the definition).

Theorem 1.5 (Erdős and Tarski [ET61])**.** *If κ is an uncountable cardinal, then κ is weakly compact if and only if $\kappa \to (\kappa)^2$.*

It thus makes sense to consider unbalanced partition relations for more accessible cardinals, such as ω_1 or ω_2.

There have been numerous advances in this part of combinatorics in the last few decades. We aim to build upon various recent results in this area which grew out of a theorem by Dushnik and Miller.

Theorem 1.6 (Dushnik and Miller [DM41])**.** *If κ is an infinite cardinal, then $\kappa \to (\kappa, \omega)^2$.*

Considering only the size of the homogeneous set, this result is optimal. However, we can use the fact that the vertex set of a graph is an ordinal and analyze the order type of the homogeneous set. It was Erdős and Rado [ER56] who first took upon the quest of generalizing the result of Dushnik and Miller by improving it slightly to:

Theorem 1.7 (Erdős and Rado [ER56])**.** *If κ is a regular cardinal, then $\kappa \to (\kappa, \omega + 1)^2$.*

By a result of Hajnal, this is the best possible relation attainable in ZFC.

Theorem 1.8 (Hajnal [Haj60])**.** *If κ is a regular cardinal and $2^\kappa = \kappa^+$, then $\kappa^+ \not\to (\kappa^+, \kappa + 2)^2$.*

This result was later extended by Todorčević to cover even singular cardinals.

We will now focus on results for graphs defined on ω_1. Todorčević proved that the conclusion of Hajnal's theorem follows only from an assumption on the bounding number. Moreover, together with Raghavan, they proved that the assumption of the existence of a Suslin tree is also sufficient.

Theorem 1.9 (Todorčević [Tod89b], Raghavan and Todorčević [RT18])**.** *If* $\mathfrak{b} = \omega_1$ *or a Suslin tree exists, then* $\omega_1 \not\to (\omega_1, \omega + 2)^2$*.*

The best possible relation $\omega_1 \to (\omega_1, \alpha)^2$, where α is any countable ordinal, was shown to be consistent by Todorčević.

Theorem 1.10 (Todorčević [Tod83])**.** *If* PFA *holds, then* $\omega_1 \to (\omega_1, \alpha)^2$ *for each countable ordinal* α*.*

The case of ω_2 is far from being resolved. Under CH Theorem 1.7 generalizes.

Theorem 1.11 (Erdős and Rado [ER56])**.** *If* $2^\omega = \omega_1$*, then* $\omega_2 \to (\omega_2, \omega_1 + 1)^2$*.*

Erdős and Rado also showed that $(2^\omega)^+ \to (\omega_1)^2$, hence $\omega_2 \to (\omega_2, \omega_1)^2$ implies CH. Hajnal's result says that this result is sharp under the additional assumption that $2^{\omega_1} = \omega_2$. Similarly, the result of Raghavan and Todorčević covers ω_2.

Theorem 1.12 (Raghavan and Todorčević [RT18])**.** *Suppose an* ω_2*-Suslin tree exists, then the following hold:*

1. *If* CH *holds, then* $\omega_2 \not\to (\omega_2, \omega_1 + 2)^2$*,*
2. *if* ¬CH *holds, then* $\omega_2 \not\to (\omega_2, \omega + 2)^2$*.*

Rather surprisingly, Laver showed the following:

Theorem 1.13 (Laver [Lav75])**.** *Suppose* MA *holds and* $2^\omega = \omega_2$*, then* $\omega_2 \not\to (\omega_2, \omega + 2)^2$*.*

If we weaken the assumption on the coloring, there are other results. Baumgartner showed that after adding ω_2 Silver reals via countable support product to a model of CH, we obtain the following:

Theorem 1.14 (Baumgartner [Bau76])**.** *It is consistent with* ¬CH *that* $\omega_2 \to (\omega_2, \omega : \omega_2)^2$*.*

In Chapter 2, we focus on results in this spirit. We show the effect of adding Cohen reals and how it compares to Baumgartner's result. These results establish how much of the partition relation $\omega_2 \to (\omega_2, \omega_1)^2$, which holds under CH, remains after adding various kinds of reals.

1.2 Vertex Colorings and Coloring Number

In the context of vertex colorings, we will be interested in finding special subgraphs in graphs with large chromatic number or coloring number.

Whereas the case of the chromatic index was entirely determined by the maximum degree, for the chromatic number, it just implies an upper bound.

Proposition 1.15. *Suppose κ is an infinite cardinal. If G is a graph with $\Delta(G) = \kappa$, then $\chi(G) \leq \mathrm{Col}(G) \leq \kappa$.*

Proof. The fact that $\chi(G) \leq \mathrm{Col}(G)$ is well-known. The well-ordering witnessing the value of the coloring number can be used to color the vertex set using a greedy algorithm.

The inequality $\mathrm{Col}(G) \leq \kappa$ follows from the fact that components have size $\leq \kappa$ (see Proposition 1.2). Now any well-ordering of order type κ has the property that each vertex has less than κ predecessors. □

1.2.1 The Hadwiger Conjecture

In the second part of the thesis, we will consider the infinite version of Hadwiger's conjecture. The finite version of the conjecture states that if a finite simple graph has chromatic number at least $t \in \omega$, then it contains K_t as a minor. Recently there has been some interest in the infinite case; see [vdZ13] and [Kom17].

Dominic van der Zypen proved that there is a countable connected graph whose chromatic number is ω, but K_ω is not a minor of this graph, i.e., the straightforward generalization of the Hadwiger conjecture to infinite graphs fails.

Theorem 1.16 (van der Zypen [vdZ13])**.** *There is a countable connected graph whose chromatic number is ω, but K_ω is not a minor of this graph.*

Chapter 3 deals with the uncountable version of this conjecture. We also show that the previous theorem generalizes to limit cardinals.

Péter Komjáth showed that there are counterexamples for uncountably chromatic graphs as well.

Theorem 1.17 (Komjáth [Kom17])**.** *If κ is an infinite cardinal, then there is a graph of cardinality 2^κ, chromatic number κ^+, with no K_{κ^+} minor.*

Thus the Hadwiger conjecture fails also for uncountably chromatic graphs if there is no bound on the size of the witness. The conjecture may hold if we focus on graphs of size and chromatic number ω_1.

Theorem 1.18 (Komjáth [Kom17])**.** *If $\mathsf{MA}(\omega_1)$ holds, then every graph G with $|G| = \chi(G) = \omega_1$ contains a subdivision of K_{ω_1}.*

In chapter 3, we will characterize the cardinal specifying the least size of the graph that is the counterexample to the Hadwiger conjecture for the case when the chromatic number is ω_1. We will also show how the Hadwiger conjecture behaves on higher uncountable cardinals.

2. Partition Relations and Graph Constructions on Uncountable Cardinals

2.1 Partition Relations in Cohen Extensions

This section focuses on complete bipartite partition relations in Cohen forcing extensions. By the results of Erdős and Rado, CH is equivalent to $\omega_2 \to (\omega_2, \omega_1)^2$. Hence this relation must fail if $2^\omega = \omega_2$. In the first part of this section, we will investigate how much of $\omega_2 \to (\omega_2, \omega_1)^2$ is preserved if the continuum is raised by adding Cohen reals.

As the central tool of this section will be double Δ-systems, let us review a classical result about the existence of Δ-systems. The proof can be found in [Kun80].

Theorem 2.1 (Δ-system lemma)**.** *Suppose κ is an infinite cardinal, $\lambda > \kappa$ is regular, for each $\alpha < \lambda$ we have $|\alpha^{<\kappa}| < \lambda$ and $\mathcal{A}$ is a collection of sets such that $|\mathcal{A}| \geq \lambda$. If for all $x \in \mathcal{A}$ we have $|x| < \kappa$, then there is a $\mathcal{B} \subseteq \mathcal{A}$, such that $|\mathcal{B}| = \lambda$ and $\mathcal{B}$ forms a Δ-system.*

Double Δ-systems were introduced in [Tod86] and utilized in other papers of Stevo Todorčević. For an application in a similar context, see [Tod21, Theorem 3.3.]. An excellent exposition on double Δ-systems and their higher analogs can be found in [LH22].

Definition 2.2. Let Γ be a set of ordinals and $D := \{p_{\alpha\beta} \mid \{\alpha, \beta\} \in [\Gamma]^2\}$ a collection of sets. We say that D is a *double Δ-system* if the following hold:

1. for every $\alpha \in \Gamma$ $\{p_{\alpha\beta} \mid \beta \in \Gamma \setminus (\alpha + 1)\}$ is a Δ-system with root p^0_α
2. for every $\beta \in \Gamma$ $\{p_{\alpha\beta} \mid \alpha \in \Gamma \cap \beta\}$ is a Δ-system with root p^1_β
3. $\{p^0_\alpha \mid \alpha \in \Gamma\}$ is a Δ-system with root p^0
4. $\{p^1_\beta \mid \beta \in \Gamma\}$ is a Δ-system with root p^1

Remark. In our case, the sets $p_{\alpha\beta}$ will be conditions in the Cohen forcing. The notation $p_{\alpha\beta}$ implicitly assumes that $\alpha < \beta$. Note also that the conditions on the double Δ-system ensure that $\bigcap\{p_{\alpha\beta} \mid \{\alpha, \beta\} \in [\Gamma]^2\} = p^0 = p^1$, this condition will be called the root of the double Δ-system.

We will define the notion of isomorphism between forcing conditions.

Definition 2.3. Given $p, q \in \mathsf{Add}(\kappa, \lambda)$ we define the set type(p) as the sequence $(p_i)_{i<\mu}$, where $\mu = \mathrm{ot}(\mathrm{dom}(p))$ and $(p_i)_{i<\mu}$ is an enumeration of the values of p as a sequence respecting the ordering of its domain. Conditions p, q are isomorphic, $p \simeq q$, if type$(p) = $ type(q).

The type of a pair, $\mathrm{type}(p,q)$, is defined again as a sequence $(s_i)_{i<\eta}$, where $\eta = \mathrm{ot}(\mathrm{dom}(p) \cup \mathrm{dom}(q))$ and if $(r_i)_{i<\eta}$ is an enumeration of $\mathrm{dom}(p) \cup \mathrm{dom}(q)$ respecting the ordering, then $s_i = (v_p^i, v_q^i)$, where if $r_i \in \mathrm{dom}(p)$, then $v_p^i = p(r_i)$, else $v_p^i = 2$; analogously for v_q^i. Two pairs of conditions $(p,q), (r,s)$ are isomorphic, $(p,q) \simeq (r,s)$, if $\mathrm{type}(p,q) = \mathrm{type}(r,s)$.

Remark. The sets $\mathrm{type}(p)$ and $\mathrm{type}(p,q)$ just record all information about a condition (a pair of conditions). In other words, it codes them as structures. Also note that $\mathrm{type}(p,q) = \mathrm{type}(r,s)$ implies the equality of types coordinate-wise, i.e., $\mathrm{type}(p) = \mathrm{type}(r)$ and $\mathrm{type}(q) = \mathrm{type}(s)$.

We will need a more uniform version of Δ-systems of conditions in further applications.

Lemma 2.4. *Suppose $\kappa < \lambda$ are regular cardinals, $|2^\mu| < \lambda$ for all $\mu < \kappa$ and $\{p_\alpha \mid \alpha < \lambda\}$ is a set of conditions in $\mathsf{Add}(\kappa, \lambda^+)$ forming a Δ-system. There is an $X \in [\lambda]^\lambda$ and an $s \subseteq \mathrm{ot}(\mathrm{dom}(p_0))$ such that:*

1. *for all $\alpha, \beta \in X$ we have $\mathrm{type}(p_\alpha) = \mathrm{type}(p_\beta)$, and*
2. *for all $\alpha \in X$ if $(d_i \mid i < \mathrm{ot}(\mathrm{dom}(p_\alpha)))$ is an increasing enumeration of the domain of p_α, then $\{(d_i, p_\alpha(d_i)) \mid i \in s\}$ is exactly the root of the original Δ-system.*

Proof. The proof is a routine counting argument.

To ensure that the types of all the conditions are the same, note that the order type of the domain of any condition from $\mathsf{Add}(\kappa, \lambda^+)$ is an ordinal below κ. Let $f : \lambda \to \kappa$ be a function such that $f(\alpha) = \mathrm{ot}(\mathrm{dom}(p_\alpha))$. As $\kappa < \lambda$ and λ is regular we get a $\gamma_0 < \kappa$ and an $A \in [\lambda]^\lambda$ such that the order type of the domain of p_α is γ_0 for all $\alpha \in A$. Next, consider each function from γ_0 to 2. As $2^{\gamma_0} < \lambda$, there is less than λ many such functions. Given a condition p_α for $\alpha \in A$ let $\varphi_\alpha : \gamma_0 \to \mathrm{dom}(p_\alpha)$ be the unique increasing bijection and define a function $g : \lambda \to 2^{\gamma_0}$ such that $g(\alpha) = p_\alpha \circ \varphi_\alpha$. As before there is an $A' \in [A]^\lambda$ and a fixed function $q : \gamma_0 \to 2$ such that $\mathrm{type}(p_\alpha) = q$ for each $\alpha \in A'$.

To make sure that the relative position of the root of the Δ-system stays the same across all conditions, define another function $h : \lambda \to 2^{\gamma_0}$ such that $h(\alpha)(\beta) = 1$ if and only if $(\varphi_\alpha(\beta), p_\alpha(\varphi_\alpha(\beta))$ is in the root of the Δ-system. Analogously as before, we find a function $r : \gamma_0 \to 2$ such that for λ many α, we have $h(\alpha) = r$, which ensures the second condition. ◻

Lemma 2.5. *Suppose $\gamma_0 \leq \gamma_1$ are ordinals and $\{p_\alpha \mid \alpha < \gamma_0\}$, $\{q_\alpha \mid \alpha < \gamma_1\}$ are sets of conditions in $\mathsf{Add}(\kappa, \lambda)$. If $\{q_\alpha \mid \alpha < \gamma_1\}$ forms a Δ-system and for each $\alpha < \beta < \gamma_0$ we have $(p_\alpha, p_\beta) \simeq (q_\alpha, q_\beta)$, then $\{p_\alpha \mid \alpha < \gamma_0\}$ also forms a Δ-system.*

Proof. First enumerate in increasing order the domain of the condition q_0 as $(d_i \mid i < \mathrm{ot}(\mathrm{dom}(q_0)))$. As the conditions $\{q_\alpha \mid \alpha < \gamma_1\}$ are isomorphic and form a Δ-system let s be the set of indices $i < \mathrm{ot}(\mathrm{dom}(q_0))$ such that $\{(d_i, q_0(d_i)) \mid i \in s\}$ is exactly the root.

If we similarly enumerate the domain of p_0 as $(e_i \mid i < \text{ot}(\text{dom}(p_0)))$ (note that $\text{ot}(\text{dom}(q_0)) = \text{ot}(\text{dom}(p_0)))$, we claim that $\{(e_i, p_0(e_i)) \mid i \in s\}$ is the root of the Δ-system formed by the conditions $\{p_\alpha \mid \alpha < \gamma_0\}$.

Given any p_α and p_β as this pair is isomorphic to the pair (q_α, q_β) we have that $d_i \in \text{dom}(q_\alpha) \cap \text{dom}(q_\beta)$ if and only if $e_i \in \text{dom}(p_\alpha) \cap \text{dom}(p_\beta)$ and this happens exactly when $i \in s$, also when $d_i \in \text{dom}(q_\alpha) \cap \text{dom}(q_\beta)$ then d_i is also the i-th element of the domain of both q_α and q_β and the same holds for e_i and any p_α and p_β. Finally as the conditions $\{q_\alpha \mid \alpha < \gamma_1\}$ are isomorphic so are $\{p_\alpha \mid \alpha < \gamma_0\}$ so in particular $p_\alpha \simeq p_0 \simeq p_\beta$ and we are done. □

The main theorem follows.

Theorem 2.6. *Suppose $\kappa < \lambda$ are regular cardinals. If $\lambda^{<\lambda} = \lambda$ and for each $\alpha < \lambda$ we have $|\alpha^{<\kappa}| < \lambda$, then $\mathsf{Add}(\kappa, \lambda^+)$ forces the relation $\lambda^+ \to (\lambda^+, \mu : \mu)^2$ for any ordinal $\mu < \kappa^+$.*

Proof. Consider the extension by the Cohen forcing adding λ^+ subsets of κ. Fix a condition q and a name $\dot{c}$ such that q forces that in the extension c is a function from $[\lambda^+]^2$ to 2. Without loss of generality, we will assume that $q = \emptyset$. If it is the case that

$$\emptyset \Vdash \exists X \in [\lambda^+]^{\lambda^+} : \dot{c}''[X]^2 = \{0\}$$

we are done, so suppose this is not the case. Now an improved double Δ-system can be found.

Claim 2.6.1. *There is a set $X \in [\lambda^+]^\lambda$ of order type λ and a set of conditions $D := \{p_{\alpha\beta} \in \mathsf{Add}(\kappa, \lambda^+) \mid \{\alpha, \beta\} \in [X]^2\}$ such that the following hold:*

1. $p_{\alpha\beta} \Vdash \dot{c}(\alpha, \beta) = 1$,
2. *D forms a double Δ-system.*

Proof. In the ground model for every $\alpha < \beta$ in λ^+ either $\emptyset \Vdash \dot{c}(\alpha, \beta) = 0$ or there is a condition p such that $p \Vdash \dot{c}(\alpha, \beta) = 1$. For every pair fulfilling the second option, fix such a condition $p_{\alpha\beta}$, otherwise put $p_{\alpha\beta} := \emptyset$. Consider a regular cardinal θ large enough so that $\mathsf{H}(\theta)$ contains all the relevant objects we have considered so far. Choose an elementary submodel M of $\mathsf{H}(\theta)$ of size λ such that $M^{<\lambda} \subseteq M$ (we assume $\lambda^{<\lambda} = \lambda$ in the ground model) and $\delta := M \cap \lambda^+$ has cofinality λ. Fix also a λ-sequence converging to δ, say $(d_\alpha \mid \alpha < \lambda)$.

Subclaim. *There is a set $B \subseteq \delta$, cofinal in δ of order type λ with the following properties for every $\alpha < \beta < \gamma$ in B:*

1. $p_{\alpha\beta} \simeq p_{\alpha\delta}$
2. $p_{\alpha\beta} \upharpoonright \beta = p_{\alpha\delta} \upharpoonright \delta$
3. $(p_{\alpha\gamma}, p_{\beta\gamma}) \simeq (p_{\alpha\delta}, p_{\beta\delta})$
4. $\text{dom}(p_{\alpha\beta}) \subseteq \gamma$

5. $p_{\alpha\beta} \Vdash \dot{c}(\alpha, \beta) = 1$

6. $p_{\alpha\delta} \Vdash \dot{c}(\alpha, \delta) = 1$

Proof. The set B cannot be an element of M, but any initial segment of such a set B belongs to M because M is closed under sequences of length $< \lambda$; this will be used in the inductive construction. Suppose we have constructed an initial segment of B, a sequence $b := (b_\xi \mid \xi < \beta)$ for some ordinal $\beta < \lambda$ satisfying all the conditions, and $b_\alpha \geq d_\alpha$ for all $\alpha < \beta$. We want to construct b_β above d_β. Note that $b \in M$ as the sequence has length $< \lambda$ and $d_\beta \in M$. Consider the following sequences/matrices for every $\eta \in \lambda^+ \setminus d_\beta$:

$$\begin{aligned} S_1^\eta &:= (\mathrm{type}(p_{\alpha\eta}) \mid \alpha \in b) \\ S_2^\eta &:= (p_{\alpha\eta} \upharpoonright \eta \mid \alpha \in b) \\ S_3^\eta &:= (\mathrm{type}(p_{\alpha\eta}, p_{\beta\eta}) \mid \alpha < \beta \in b) \end{aligned}$$

For any $\eta \in M$ all three sets S_1^η, S_2^η and S_3^η are definable in M and moreover the entire sequence $(S_i^\eta \mid \eta < \lambda^+)$ is in M for each i in $\{1, 2, 3\}$. Note also that S_i^δ is an element of M for $i \in \{1, 2, 3\}$ even though $\delta \notin M$, this follows from M being closed under sequences of length $< \lambda$. Now define the set:

$$\begin{aligned} S := \{\eta < \lambda^+ \mid \eta \geq d_\beta \wedge S_1^\eta = S_1^\delta \wedge S_2^\eta = S_2^\delta \wedge S_3^\eta = S_3^\delta \wedge \\ \wedge\, \eta > \sup\{\sup \mathrm{dom}(p_{\alpha\beta}) \mid \alpha < \beta \in b\} \wedge \forall \alpha \in b : p_{\alpha\eta} \Vdash \dot{c}(\alpha, \eta) = 1\} \end{aligned}$$

Clearly, S is definable in M and $\delta \in S$. Thus S is a stationary subset of λ^+.

Finally, consider the set:

$$T := \{\eta \in S \mid \forall \xi \in S \cap \eta : \emptyset \Vdash \dot{c}(\xi, \eta) = 0\}$$

If δ is not an element of T, then there is some $\xi \in S \cap \delta$ for which $\emptyset \nVdash \dot{c}(\xi, \delta) = 0$ so $p_{\xi\delta}$ was defined as some condition p such that $p \Vdash \dot{c}(\xi, \delta) = 1$, as ξ is also an element of S we can put $b_\beta := \xi$. Now each condition of the claim is satisfied as witnessed by ξ belonging to S and the fact that ξ witnesses that $\delta \notin T$.

On the other hand if $\delta \in T$, then T is unbounded in λ^+ and clearly $\emptyset \Vdash \dot{c}''[T]^2 = \{0\}$, a contradiction. □

Let B be the set constructed in the previous subclaim. We can also assume that $\{p_{\alpha\delta} \mid \alpha \in B\}$ is a Δ-system (we assume that for each $\alpha < \lambda$ we have $|\alpha^{<\kappa}| < \lambda$) so in particular $p_{\alpha\delta} \simeq p_{\beta\delta}$ for $\alpha, \beta \in B$. We now show that the set B can be refined so that the set of conditions $\{p_{\alpha\beta} \mid \{\alpha, \beta\} \in [B]^2\}$ will form a double Δ-system.

The previous paragraph, condition 3 and Lemma 2.5 imply that for each γ also the set $\{p_{\alpha\gamma} \mid \alpha \in B \cap \gamma\}$ is a Δ-system with root p_γ^1. We can now assume that $\left\{p_\gamma^1 \mid \gamma \in B\right\}$ also forms a Δ-system with root p^1.

Conditions 1, 2 and 4 imply that for each α in B the following set of conditions $\{p_{\alpha\beta} \mid \beta \in B \setminus (\alpha + 1)\}$ is a Δ-system with root $p_\alpha^0 := p_{\alpha\delta} \upharpoonright \delta$. Given any $p_{\alpha\beta}$

and $p_{\alpha\gamma}$ for $\alpha < \beta < \gamma$ in B consider the intersection $p_{\alpha\beta} \cap p_{\alpha\gamma}$, clearly $p_{\alpha\delta} \upharpoonright \delta \subseteq p_{\alpha\beta} \cap p_{\alpha\gamma}$ by condition 2. For the other direction if $(d, v) \in p_{\alpha\beta} \cap p_{\alpha\gamma}$, then $d < \gamma$ by condition 4 thus $(d, v) \in p_{\alpha\gamma} \upharpoonright \gamma$ and again by condition 2 $(d, v) \in p_{\alpha\delta} \upharpoonright \delta$. Finally, we can also assume that $\{p^0_\alpha \mid \alpha \in B\}$ forms a Δ-system with root p^0. Let X be the refined set B; this is our desired set. □

We will denote the root $p^0 = p^1$ of the double Δ-system as p.

Fix an ordinal μ such that $\kappa \leq \mu < \kappa^+$. Choose two sets: X_0 and X_1, subsets of X such that $X_0 < X_1$ and the order type of both sets is $\kappa \cdot \mu$. Fix also a bijection $g : \kappa \to \mu$.

Claim 2.6.2. *p forces a $(\mu : \mu)$ configuration in color* 1.

Proof. Let G be a generic set containing p, by induction we will construct sequences $(s_\alpha \mid \alpha < \kappa)$ in X_0 and $(t_\alpha \mid \alpha < \kappa)$ in X_1 such that s_α is in the $g(\alpha)$-th section of X_0, i.e., if $f : \kappa \cdot \mu \to X_0$ is the unique increasing bijection then

$$s_\alpha \in [f(\kappa \cdot g(\alpha)), f(\kappa \cdot (g(\alpha) + 1))),$$

denote this subset of X_0 as X^α_0, analogously for t_α and X_1. We will make sure that for all $\alpha, \beta \in \kappa$ we have $p_{s_\alpha t_\beta} \in G$; as $p_{s_\alpha t_\beta}$ forces the color of the pair $\{s_\alpha, t_\beta\}$ to be 1 this will ensure the conclusion of the claim.

To start the induction, note that by genericity for some $\alpha \in X^0_0$, we have $p^0_\alpha \in G$; this is because $\{p^0_\alpha \mid \alpha \in X^0_0\}$ is a Δ-system of size κ with root $p^0 \geq p$ and thus this set is predense below p. By the same argument there is some $\beta \in X^0_1$ such that $p_{\alpha\beta}$ is in G, so put $s_0 := \alpha$ and $t_0 := \beta$.

Suppose we have already constructed $(s_\alpha \mid \alpha < \gamma)$ and $(t_\alpha \mid \alpha < \gamma)$ for some $\gamma < \kappa$ such that $p_{s_\alpha t_\beta} \in G$ for all $\alpha, \beta < \gamma$. We will now find $\sigma \in X^\gamma_0$ such that $\{p_{\sigma t_\alpha} \mid \alpha < \gamma\} \subseteq G$ and this will be our s_γ.

Suppose that no σ satisfies our requirements, i.e., there is no σ in X^γ_0 such that $\{p_{\sigma t_\alpha} \mid \alpha < \gamma\} \subseteq G$. Then there must exist a condition $r \leq p$ forcing this (note that $\{p_{\sigma t_\alpha} \mid \alpha < \gamma\}$ is an element of the ground model because our forcing is κ-closed):

$$r \Vdash \forall \sigma \in X^\gamma_0 : \{p_{\sigma t_\alpha} \mid \alpha < \gamma\} \not\subseteq \dot{G}$$

This means that for all $\sigma \in X^\gamma_0$ there exists a $\beta < \gamma$ such that $r \perp p_{\sigma t_\beta}$. By going to a refinement, we can assume that for κ many σ, there is a fixed $\beta' < \gamma$ such that $r \perp p_{\sigma t_{\beta'}}$, call this set C. However note that the set $\{p_{\sigma t_{\beta'}} \mid \sigma \in C\}$ is a Δ-system with root $p^1_{t_{\beta'}}$ which is contained in the generic set G because $p_{s_0 t_{\beta'}} \leq p^1_{t_{\beta'}}$ and $p_{s_0 t_{\beta'}} \in G$. Since r has size $< \kappa$ and $r \parallel p^1_{t_{\beta'}}$, it cannot be incompatible with every condition from $\{p_{\sigma t_{\beta'}} \mid \sigma \in C\}$, a contradiction.

The construction of t_γ is almost verbatim. □

The claim concludes the proof. □

2.1.1 A Negative Partition Relation From Cohen Forcing

The result of the previous section cannot be strengthened so that the second partition of the bipartite graph has size κ^+.

Proposition 2.7. *If $\kappa < \lambda$ are regular cardinals, then $\mathsf{Add}(\kappa, \lambda) \Vdash \lambda \not\to (\kappa : \kappa^+)^2$.*

Proof. Consider an equivalent form of the forcing notion, specifically the poset $C_S := \{p : S \to 2 \mid |p| < \kappa\}$, where $S := [\lambda]^2$ and the ordering is reverse inclusion. We will prove that the generic graph, the union over the generic set G added this way, witnesses the failure of $\lambda \to (\kappa : \kappa^+)^2$.

Suppose, for contradiction, that in the extension there is a set X of size κ and a set Y of size κ^+ above it so that either X and Y form a complete bipartite graph or no edge is present between X and Y. Use the fact that when forcing with $\mathsf{Add}(\kappa, \lambda)$ any set of size κ can be decided already when forcing over a domain of size κ [Kun80, Lemma VIII.2.2.]. To be more precise denote by M the ground model; there is a set $I \subseteq [\lambda]^2$ of size κ so that $X \in M[G \cap C_I]$ (note that $C_{[\lambda]^2} \cong C_I \times C_{[\lambda]^2 \setminus I}$). Now working in the extension by C_I, there must exist a condition p in $C_{[\lambda]^2 \setminus I}$ so that $p \Vdash y \in \dot{Y}$ for some $y \notin \bigcup I$, otherwise $C_{[\lambda]^2 \setminus I} \Vdash \dot{Y} \subseteq \bigcup I$, which is not possible. Now p has size $< \kappa$ and $|X| = \kappa$ hence there must be an $x \in X \setminus \bigcup \mathrm{dom}(p)$. Now p can be extended by the pair $(\{x, y\}, i)$ for both $i \in \{0, 1\}$, which is a contradiction. □

2.1.2 Adding a Single Cohen Real Forces $\omega_1 \not\to (\omega_1, \omega : 2)^2$

In [Tod89a, §2] Stevo Todorčević used a coherent sequence of functions and the assumption that $\mathfrak{b} = \omega_1$ to construct a topological space closely related to the (S) conjecture. Using the same technique a witness to $\omega_1 \not\to (\omega_1, \omega : 2)^2$ can be obtained from $\mathfrak{b} = \omega_1$; for a proof see [Rin12]. Todorčević remarked at the end of [Tod89a, §2] that an analogous construction can be carried out if a Cohen real is used instead of assuming that the bounding number is small. Here we reproduce his result.

Lemma 2.8. *There is a function $d : [\omega_1]^2 \to \omega$ such that:*

1. *for any $X \in [\omega_1]^{\omega_1}$ we have $\sup d''[X]^2 = \omega$ and*
2. *for any distinct triple $\alpha, \beta, \gamma < \omega_1$ we have $d(\alpha, \beta)$[1] $\geq \min\{d(\alpha, \gamma), d(\beta, \gamma)\}$.*

Proof. An example of such a function can be constructed from a family of pairwise distinct functions $\{f_\alpha \colon \omega \to \omega \mid \alpha < \omega_1\}$. Let us define $d(\alpha, \beta) := \min\{n \in \omega \mid f_\alpha(n) \neq f_\beta(n)\}$. For the second requirement let α, β, γ be given, put $m := \min\{d(\alpha, \gamma), d(\beta, \gamma)\}$ and let $k < m$. Then $f_\alpha(k) = f_\gamma(k) = f_\beta(k)$, hence $d(\alpha, \beta) > k$.

For the first entry, let $n \in \omega$ and note that ${}^n\omega$ is countable. Hence there is a $Y \in [\omega_1]^{\omega_1}$ and some $t \in {}^n\omega$ such that for each $\alpha \in Y$, we have $t \subseteq f_\alpha$. Thus for

[1] Formally this should be $d(\{\alpha, \beta\})$, but for brevity we will omit the curly braces.

each $\alpha, \beta \in Y$, we have that the least n where they differ must be bigger or equal to n. Since n was arbitrary, we are done.

□

Theorem 2.9 (Todorčević)**.** *Adding a Cohen real forces that* $\omega_1 \not\to (\omega_1, \omega : 2)^2$.

Proof. In the ground model, fix injections $e_\beta : \beta \to \omega$ for each $\beta < \omega_1$ and let d be the function from Lemma 2.8.

Suppose we are in the generic extension obtained by adding a single Cohen real $r : \omega \to \omega$. Define a graph G such that for each $\alpha < \beta < \omega_1$ we have:

$$\alpha \in \mathrm{N}(\beta) \text{ if and only if } r(d(\alpha, \beta)) \geq e_\beta(\alpha).$$

Claim 2.9.1. *G has no uncountable independent set.*

Proof. Suppose p is a condition and $\dot{X}$ a name in the Cohen forcing such that $p \Vdash \dot{X}$ is an uncountable independent set in G. For $\alpha < \omega_1$ find conditions $p_\alpha \leq p$ and distinct ordinals x_α such that $p_\alpha \Vdash x_\alpha \in \dot{X}$. As our forcing is countable, there is a single condition $q \leq p$ and an uncountable set of ordinals Y in the ground model such that $q \Vdash Y \subseteq \dot{X}$. Thus we have that for each $\alpha, \beta \in Y$, it is true that $q \Vdash \alpha \notin \mathrm{N}(\beta)$. Unpacking the definition of G we obtain for each $\alpha, \beta \in Y$ that $q \Vdash \dot{r}(d(\alpha, \beta)) < e_\beta(\alpha)$. Using the properties of d obtain $\alpha^* < \beta^*$ in Y such that $d(\alpha^*, \beta^*) > |q|$. Consider any condition $q^* \leq q \cup \{(d(\alpha^*, \beta^*), e_{\beta^*}(\alpha^*))\}$. We have that $q^* \Vdash \alpha^* \in \mathrm{N}(\beta^*)$, a contradiction. □

The following claim finishes the proof.

Claim 2.9.2. *G has no subgraph of type $(\omega : 2)$.*

Proof. Let p be a condition, $\dot{X}$ a name and $\alpha < \beta$ countable ordinals such that $p \Vdash \dot{X} \in [\alpha]^\omega$ and $\dot{X} \cup \{\alpha, \beta\}$ forms a subgraph of type $(\omega : 2)$. Let $m := d(\alpha, \beta)$ and extend the condition p to a condition q such that $|q| > m$. Finally let $m^* \in \omega$ be greater than $\max\{q(i) \mid i < |q|\}$.

Find $a \in [\alpha]^{<\omega}$ such that for all $\gamma \in \alpha \setminus a$ we have $e_\alpha(\gamma) > m^*$ and $e_\beta(\gamma) > m^*$, this is possible as the functions e_α and e_β are injective.

We thus obtain the following:

$$q \Vdash \forall x \in \dot{X} \setminus a : \dot{r}(d(x, \alpha)) > m^* \wedge \dot{r}(d(x, \beta)) > m^*.$$

Next, we show that $q \Vdash \forall x \in \dot{X} \setminus a : d(x, \alpha) \geq m + 1 \wedge d(x, \beta) \geq m + 1$. Otherwise, there is some $q' \leq q$ and an $x \in \alpha \setminus a$ such that we have $q' \Vdash d(x, \alpha) \leq m$ or $d(x, \beta) \leq m$. Assume without loss of generality that $q' \Vdash d(x, \alpha) \leq m$. As $q' \Vdash x \in \dot{X}$ we have that $q' \Vdash \dot{r}(d(x, \alpha)) \geq e_\alpha(x)$, as $x \notin a$ we have that $q' \Vdash \dot{r}(d(x, \alpha)) \geq e_\alpha(x) > m^*$. Moreover since $m < |q|$ we obtain that $q' \Vdash \dot{r}(d(x, \alpha)) = q'(d(x, \alpha)) = q(d(x, \alpha)) > m^*$, a contradiction.

Now if we extend q to some q^* which decides an element of $\dot{X} \setminus a$ we get that there is an $x^* \in \alpha \setminus a$ such that $q^* \Vdash d(x^*, \alpha) \geq m + 1 \wedge d(x^*, \beta) \geq m + 1$, i.e., $d(\alpha, \beta) > m$, which contradicts the definition of m.

□

2.2 The Transversal Hypothesis and Countable Colorings

In this section, we concentrate on the consistency of the relation $\omega_2 \not\to (\omega_1 : 2)^2_\omega$. We show that an assumption closely related to the Kurepa hypothesis implies this negative partition relation and that the positive relation has large cardinal strength. Additionally, we reprove a result by András Hajnal on how the result can be forced.

We start by introducing the transversal hypothesis.

Definition 2.10. The *transversal hypothesis*, TH, is the following statement: there are functions $f_\alpha : \omega_1 \to \omega$ for $\alpha < \omega_2$ such that

$$|\{\gamma < \omega_1 : f_\beta(\gamma) = f_\alpha(\gamma)\}| \leq \omega,$$

whenever $\alpha < \beta < \omega_2$.

Sometimes the definition of TH considers regressive functions with range ω_1. However, these two notions coincide. For more information, see [Wal88]. It is also worth mentioning that the Kurepa hypothesis implies TH, see [BŠ86, III.3.68 Lemma].

The construction of a witness for $\omega_2 \not\to (\omega_1 : 2)^2_\omega$ uses higher order coherent sequences [BŠ86, III.3.46 Lemma].

Lemma 2.11 (Kunen)**.** *There are injective functions $e_\alpha : \alpha \to \omega_1$ for $\alpha < \omega_2$ such that*

$$|\{\gamma < \beta : e_\beta(\gamma) \neq e_\alpha(\gamma)\}| < \omega_1,$$

whenever $\beta < \alpha < \omega_2$.

Proposition 2.12. *If TH holds then $\omega_2 \not\to (\omega_1 : 2)^2_\omega$.*

Proof. Let $(f_\alpha : \alpha < \omega_2)$ witness TH and $(e_\alpha : \alpha < \omega_2)$ be a sequence satisfying the properties stated in Lemma 2.11. Define the coloring $c : [\omega_2]^2 \to \omega$ as follows:

$$(\alpha, \beta) \mapsto f_\beta(e_\beta(\alpha))$$

for $\alpha < \beta$. Assume there is a monochromatic $(\omega_1 : 2)$ configuration, i.e., there is some $X \in [\omega_2]^{\omega_1}$ and $\alpha, \beta > X$ such that $c(x, \alpha) = c(x, \beta)$ for every $x \in X$.

Unpacking the definition we get that $f_\alpha(e_\alpha(x)) = f_\beta(e_\beta(x))$ for all $x \in X$. Note that by the properties of the sequence $(e_\alpha : \alpha < \omega_2)$, for all but countably many $x \in X$, we have $e_\alpha(x) = e_\beta(x)$, call this common value $e(x)$ (note also that this mapping is injective) and assume it holds for all $x \in X$. Now we have $f_\alpha(e(x)) = f_\beta(e(x))$ for all $x \in X$, however, the set $\{\gamma \in \omega_1 : f_\alpha(\gamma) = f_\beta(\gamma)\}$ is countable by the transversal hypothesis, a contradiction. □

Remark. Note that we only used that $c(x, \alpha) = c(x, \beta)$ for all $x \in X$ and did not need that $c(x, \alpha) = c(x', \alpha)$ for $x, x' \in X$. This is, however, not a strengthening because if we have an $X \in [\omega_2]^{\omega_1}$ and $\alpha, \beta > X$ such that $c(x, \alpha) = c(x, \beta)$ for every $x \in X$, then for some $Y \in [X]^{\omega_1}$ there is a fixed natural number k such that $c(y, \alpha) = c(y, \beta) = k$ for every $y \in Y$.

Since the negation of the Kurepa hypothesis implies that ω_2 is inaccessible in the constructible universe, see, e.g., [Jec71, §3.], we get the following.

Corollary 2.13. *If $\omega_2 \to (\omega_1 : 2)^2_\omega$ holds, then a strongly inaccessible cardinal consistently exists.*

The conclusion of Proposition 2.12 can also be easily forced. This result is stated in [EH74, p. 272] and is originally due to Hajnal.

Proposition 2.14 (Hajnal). *If* CH *holds then there is a ω_1-closed, ω_2-cc poset forcing $\omega_2 \not\to (\omega_1 : 2)^2_\omega$.*

Proof. Consider the poset consisting of pairs (s, g), where $s \in [\omega_2]^\omega$, $g : [s]^2 \to \omega$ and the ordering is defined as:

$$(t, h) \leq (s, g) \equiv s \subseteq t \wedge g \subseteq h \wedge \forall x, y \in s \forall z \in t \setminus s : h(\{z, x\}) \neq h(\{z, y\}).$$

This poset is obviously ω_1-closed as the union of countably many conditions is easily seen to be a condition extending all of them.

For the chain condition, assume we are given a sequence of ω_2 conditions and assume the first coordinates form a Δ-system and that the second coordinates all agree on the root (CH is needed here). Given any two conditions, $(s, g), (t, h)$ from this thinned-out sequence, consider the condition $(s \cup t, g \cup h \cup f)$, where $f : [s \setminus t \cup t \setminus s]^2 \to \omega$ is any injection. This is a condition, and it extends both (s, g) and (t, h) as the only conflict could arise if two points from $s \setminus t$ would be connected to one point in $t \setminus s$ with the same color, but as f is injective this cannot happen.

The sets $D_\alpha := \{(s, g) : \alpha \in s\}$ are dense because if (t, h) is such that $\alpha \notin t$, then $(t \cup \{\alpha\}, h \cup f)$, where $f : t \times \{\alpha\} \to \omega$ is again an arbitrary one-to-one function, is a condition in D_α.

The union over the second coordinates in the generic set defines a function $c : [\omega_2]^2 \to \omega$ with the desired properties. Suppose

$$(s, g) \Vdash A \cup \{x, y\} \text{ forms a homogeneous } (\omega_1 : 2) \text{ in } \dot{c}$$

and assume $x, y \in s$. Then the condition (s, g) forces that there is no $z \in \omega_2 \setminus s$ connected to x and y with the same color in $\dot{c}$, i.e., $A \subseteq s$, a contradiction. □

2.3 A Short Proof of Galvin's Theorem About Graphs on ω_2

We will prove a result by Fred Galvin about graphs on ω_2 without uncountable independent sets, see [EH74, p. 271].

The result will follow from the following observation.

Lemma 2.15. *If κ is a regular cardinal, then $\kappa^+ \to (\kappa^+, \kappa : 1)^2$.*

Proof. Suppose a graph on κ^+ is given with no configuration $(\kappa : 1)$. Consider the set $S := \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$ and note that it is stationary. Now since there is no $(\kappa : 1)$ configuration we can define a function f so that for each $\alpha \in S$ we have $f(\alpha) < \alpha$ and for each $\gamma \in (f(\alpha), \alpha)$ there is no edge between γ and α. Hence by Fodor's lemma, there is a stationary set $T \subseteq S$ and a fixed $\gamma < \kappa^+$ such that for all $\alpha \in T$, we have $f(\alpha) = \gamma$. Now the set $T \setminus \gamma$ is stationary and independent. Take $\alpha < \beta$ in $T \setminus \gamma$ now as $f(\beta) = \gamma < \alpha$ there can be no edge between α and β. □

Note that the independent set we found is a stationary subset of κ^+. A possible strengthening of the previous lemma would require the $\kappa : 1$ configuration so that the first κ vertices form a closed set cofinal in the last vertex; this is, however, not possible. The following example is due to Assaf Rinot[2].

Proposition 2.16 (Rinot)**.** *If κ is a regular cardinal, then there is a graph on κ^+ with no independent set of size κ^+, and for each $\beta < \kappa^+$ and each club $C \subseteq \beta$ of size κ there is some $\gamma \in C$ such that $\{\gamma, \beta\}$ is not an edge in the graph.*

Proof. For each α of cofinality larger than ω choose a club $C_\alpha \subseteq \alpha$ of order type $\mathrm{cf}(\alpha)$. Now we define a graph $G := (\kappa^+, E)$ as follows, for $\alpha < \beta < \kappa^+$:

$$\alpha E \beta \equiv \alpha \notin C_\beta.$$

Assume X is an independent set of size κ^+. Let $\alpha \in X$ be such that $\mathrm{ot}(X \cap \alpha) > \kappa$. Consider the club C_α and note that the order type of C_α is at most κ. Hence there is some $\gamma \in X \setminus C_\alpha$. Now $\{\gamma, \alpha\}$ forms an edge in G.

Suppose there is a $\beta < \kappa^+$ of uncountable cofinality and a club subset D of β such that $D \otimes \{\beta\}$ is a subgraph of G. Notice that $D \cap C_\beta$ is not empty and consider any $\gamma \in D \cap C_\beta$. Now $\{\gamma, \beta\}$ is not an edge in G. □

The following proposition is a more general form of Galvin's theorem.

Proposition 2.17. *Suppose κ is an uncountable regular cardinal and $\gamma, \delta \leq \kappa$. Then the following hold:*

1. *If $\kappa \to (\kappa, \gamma)$, then $\kappa^+ \to (\kappa, \gamma + 1)$.*
2. *If $\kappa \to (\kappa, \gamma : \delta)$, then $\kappa^+ \to (\kappa, \gamma : (\delta + 1))$.*

Proof. Given a graph on κ^+ with no independent set of size κ, from the previous lemma, find a $(\kappa : 1)$ configuration and consider the subgraph on the first κ vertices, as this cannot be independent using the assumption that $\kappa \to (\kappa, \gamma)$ we obtain a complete subgraph of type γ. With the last vertex of the configuration, this yields a $\gamma + 1$ subgraph.

The proof of the second statement is the same. □

Corollary 2.18 (Galvin)**.** $\omega_2 \to (\omega_1, \omega + 2)$.

Proof. Apply the first part of the previous proposition with $\kappa = \omega_1$ using the Dushnik–Miller theorem. □

[2]Personal communication.

2.4 Hajnal–Máté Graphs and Their Generalizations

András Hajnal and Attila Máté considered graphs on ω_1, which are sparse in a certain sense. In particular, they demanded that the set $\mathrm{N}^<(\alpha)$ is finite or cofinal in α with order type ω. We will investigate this class of graphs.

This section is a result of joint work with Chris Lambie-Hanson.

We start with a generalized definition of Hajnal–Máté graphs.

Definition 2.19. Let T be a tree of height ω_1. A graph $G = (T, E)$ is called a T-*Hajnal–Máté* graph if it is a subgraph of the comparability graph of T, the chromatic number of G is uncountable, and for every $\alpha < \beta < \omega_1$ and $t \in T_\beta$ the set $\{s \in T_{<\alpha} \mid \{s, t\} \in E\}$ is finite.

Remark. If T is the tree $(\omega_1, \in)$ we omit the parameter T. Thus an $(\omega_1, \in)$-Hajnal–Máté graph will be called a Hajnal–Máté graph, or HM graph for short. Note that the vertex set of these graphs is always a tree, hence ordered.

The first existence result about HM graphs is due to Hajnal and Máté [HM75]. They showed that under $\diamondsuit^+$ an HM graph exists. In the same paper, they showed that under $\mathsf{MA}(\omega_1)$ there are no such graphs.

Since then, many new constructions have been discovered. Let us mention a few. Komjáth, in his series of papers about HM graphs [Kom80, Kom84, Kom89], showed that one can construct a triangle-free HM graph just from the $\diamondsuit$ principle. From $\diamondsuit^+$, he constructed an HM graph with no cycles formed from two monotone paths. Komjáth, together with Shelah [KS88], constructed further examples of HM graphs. Given a natural number s, they constructed an HM graph having no odd cycles of length less than or equal to $2s + 1$ for which the complete bipartite graph $K_{\omega,\omega}$ is not a subgraph. They provide one forcing construction and one from just the $\diamondsuit$ principle. Using $\diamondsuit$ Lambie-Hanson and Soukup [LHS21] constructed a coloring $c : [\omega_1]^2 \to \omega$ such that $c^{-1}(n)$ is a triangle-free HM graph for each n. Soukup studied an interesting and natural generalization of HM graphs to trees [Sou15a].

Adding a single Cohen real adds a Suslin tree. It was known that a similar construction can be used to construct an HM graph in a model obtained by adding one Cohen real. Dániel Soukup asked [Sou15b, Problem 5.2] whether this HM graph can be triangle-free. We provide a positive answer. First, we need a few preliminary definitions and lemmas.

The following graphs are a generalization of the Specker graph.

Definition 2.20. Let $s, n \in \omega$ be such that $1 \leq s < n$. Then $\mathsf{S}^n_s = ([\omega_1]^n, E)$ is a graph defined as follows: if $x := \{x_i \mid i < n\}$ and $y := \{y_i \mid i < n\}$ are disjoint sets, both enumerated in increasing order, then $\{x, y\} \in E$ if and only if

$$x_s < y_0 < x_{s+1} < y_1 < x_{s+2} < \cdots < y_{n-s-2} < x_{n-1} < y_{n-s-1}$$

or vice versa.

Remark. S_1^3 is the Specker graph.

For a proof of the following proposition, see [LH20a, Theorem 5.].

Proposition 2.21 (Erdős and Hajnal)**.** *For each $s \geq 1$, there is an $n > s$ such that S_s^n has no cycles of length $2s + 1$ or less.*

In our construction of the HM graph, we will ensure that there is a graph homomorphism of the constructed graph onto a suitable S_s^n, preventing the graph from having short odd cycles. The proof of the following lemma is in [LH20b, Proposition 2.8.].

Lemma 2.22. *Suppose G and H are graphs and there is a map $f : V_G \to V_H$ such that if $\{x, y\} \in E_G$, then $\{f(x), f(y)\} \in E_H$. If $s \in \omega$ and H has no odd cycles of length $2s + 1$ or shorter, then G has no odd cycles of length $2s + 1$ or shorter.*

We will also forbid cycles formed by two monotone paths.

Definition 2.23. Suppose G is a graph with an ordered vertex set. A cycle $(x_0, \ldots, x_{n-1}, x_0)$ is called *special* if there is an $r < n$ such that $x_i > x_{i+1}$ for $i < r$ and $x_i < x_{i+1}$ for $r \leq i$. A graph is said to be *special cycle–free* if it contains no special cycles.

Recall that $H_{\omega,\omega+2}$ is the graph with the vertex set made up of disjoint sets $\{x_i \mid i < \omega\}$ and $\{y_i \mid i < \omega + 2\}$, where for each $i < \omega$ and $j < \omega + 2$ such that $i \leq j$ the vertex x_i is connected to y_j. The graph $H_{\omega,\omega+1}$ is the same graph with the vertex $y_{\omega+1}$ omitted. Hajnal and Komjáth [HK84, Theorem 1.] showed that $H_{\omega,\omega+1}$ is a subgraph of each uncountably chromatic graph.

Note that if G is special cycle–free, it is triangle-free. Additionally, the following lemma says that being special cycle–free also forbids $H_{\omega,\omega+2}$ in specific T-HM graphs [HK84, Theorem 3.].

Lemma 2.24 (Hajnal and Komjáth)**.** *Suppose T is a tree of height ω_1 that does not split on limit levels, and G is a special cycle–free T-HM graph. Then G is triangle-free and $H_{\omega,\omega+2}$ is not a subgraph of G.*

Proof. Clearly, G has no triangles. Suppose $\{x_i \mid i < \omega\}$ and $\{y_i \mid i < \omega + 2\}$ form an $H_{\omega,\omega+2}$ subgraph in G. First, note that $\{x_i \mid i < \omega\}$ lies on a branch. Otherwise, there are incomparable x_i and x_j with infinitely many common neighbors. However, as T does not split on limit levels, there must be $\alpha(i) \neq \alpha(j)$ such that $x_i \in T_{\alpha(i)}$ and $x_j \in T_{\alpha(j)}$. This contradicts the definition of a T-HM graph. Assume that $x_i \in T_{\alpha(i)}$.

Next, note that $\{x_i \mid i < \omega\} \subseteq \mathrm{N}(y_\omega) \cap \mathrm{N}(y_{\omega+1})$. The nodes y_ω and $y_{\omega+1}$ cannot lie in the same level of T as the tree does not split on limit levels, and they share infinitely many common neighbors. Suppose that of $y_\omega \in T_\gamma$, $y_{\omega+1} \in T_\delta$ and $\gamma < \delta$, then as G is T-HM we get that only finitely many elements of $\{x_i \mid i < \omega\}$ lie below level γ. Hence it must be the case that there is $z \in \{y_\omega, y_{\omega+1}\}$ such that $z \in T_\beta$ for some $\beta < \sup\{\alpha_i \mid i < \omega\}$. Choose $k \in \omega$ such that $\alpha_k > \beta$. Note that x_k and x_{k+1} have infinitely many common neighbors whose levels are above α_{k+1}. Choose one such y, now (y, x_k, z, x_{k+1}, y) is a special cycle. □

Remark. In particular, $K_{\omega,\omega}$ is also forbidden in special cycle–free T-HM graphs.

Lemma 2.24 will be applicable to the graphs constructed in Theorems 2.25 and 2.26. In particular, those graphs will not have $H_{\omega,\omega+2}$ as a subgraph.

We are ready to answer Soukup's question. See [LH20b, Theorem B.] for a similar construction used in a different context.

Theorem 2.25. *Adding a Cohen real forces that for each $s \in \omega$ there is a special cycle–free Hajnal–Máté graph without odd cycles of length at most $2s+1$.*

Proof. In the ground model, fix bijections $e_\beta : \omega \to \beta$ for each infinite countable ordinal β and let $n \in \omega$ be such that S^n_s (the generalization of the Specker graph) has no odd cycles of length $2s+1$ or shorter. Fix further a partition of countable limit ordinals into stationary sets $\mathscr{S} := \{S_{x,\delta} \mid x \in [\omega_1]^n \wedge \delta \in \omega_1 \setminus (\max x + 1)\}$ such that $S_{x,\delta} \subseteq \omega_1 \setminus (\delta+1)$. Fix also an increasing ω-sequences $(C_\beta(n) \mid n < \omega)$ for each limit ordinal β such that $(C_\beta(n) \mid n < \omega)$ converges to β and $C_\beta(0) = \delta$ for the unique x and δ such that $\beta \in S_{x,\delta}$.

Let $r : \omega \to \omega$ be the Cohen real in the extension. We will define a graph G on ω_1 by specifying the set of smaller neighbors $\mathrm{N}^<(\beta)$ for each $\beta < \omega_1$. We proceed by induction on β. Given $\delta < \beta$ let us say that β is *δ-covered* if there is a monotone decreasing path from β to an ordinal α such that $\alpha \leq \delta$. Our construction will ensure that for each $\beta \in S_{x,\delta}$, we have that β is not δ-covered.

Suppose we have constructed G up to some β, i.e., $\mathrm{N}^<(\alpha)$ is defined for each $\alpha < \beta$. If β is a successor ordinal, it will have no neighbors below β. Assume β is limit and $\beta \in S_{x,\delta}$. We also inductively assume that for each $\alpha < \beta$ we have that if $\alpha \in S_{y,\varepsilon}$, then α is not ε-covered. By induction on $k \in \omega$ we construct a set $K_\beta \in [\omega]^{\leq\omega}$ and ordinals $\{\beta_k \mid k \in K_\beta\}$ such that:

1. for all $k \in K_\beta$ we have $\beta_k = e_\beta(r(k))$.
2. for all $k \in K_\beta$ if $\beta_k \in S_{y,\varepsilon}$, then $\varepsilon > \max(\{\beta_i \mid i \in K_\beta \cap k\} \cup \{\delta\})$,
3. for all $k \in K_\beta$ we have $\beta_k > \max(\{\beta_i \mid i \in K_\beta \cap k\} \cup \{C_\beta(k)\})$
4. for all $k \in K_\beta$ if $\beta_k \in S_{y,\varepsilon}$, then $\{x, y\}$ is an edge in S^n_s.

At stage k of the construction, we consider the ordinal $e_\beta(r(k))$ and $k \notin K_\beta$ unless it satisfies all of the conditions above. In which case we put $\beta_k := e_\beta(r(k))$ and k will be an element of K_β. Finally we let $\mathrm{N}^<(\beta)$ be $\{\beta_k \mid k \in K_\beta\}$. Note that β is not δ-covered. Each $\alpha \in \mathrm{N}^<(\beta)$ is an element of some $S_{y,\varepsilon}$ with $\varepsilon > \delta$ by the second condition. The induction hypothesis now gives that there is no monotone decreasing path from any such α to an ordinal below ε. In particular, there is no monotone path from β to an ordinal less than or equal to δ.

By the third condition we obtain that for each $\alpha < \beta$ we have $|\mathrm{N}(\beta) \cap \alpha| < \omega$. The second condition ensures that if $\alpha < \alpha' < \beta$ and α, α' are both elements of $\mathrm{N}^<(\beta)$, then α' is not α-covered, in particular α, α', β cannot be the three topmost elements of a special cycle, hence G is special cycle–free. Last but not least, the fourth condition ensures that G can have no odd cycles of length $2s+1$ or less.

Consider the map which takes $\beta \in S_{x,\delta}$ to x (as a vertex in S^n_s), then by Lemma 2.22 the graph G can have no odd cycle of length $2s+1$ or shorter.

We need to prove that G is uncountably chromatic.

Let p be a condition in the Cohen forcing and $\dot{c}$ a name such that $p \Vdash \dot{c} : \omega_1 \to \omega$ is a coloring of G. Consider a finite chain of countable elementary submodels $(N_i \mid i < n+1)$ of some large enough $H(\theta)$ such that $\dot{c}, \mathscr{S} \in N_0$ and if $i < j < n+1$, then $N_i \in N_j$. Put $x_i := N_i \cap \omega_1$, $x := \{x_i \mid i < n\}$ and $\delta := N_n \cap \omega_1$. For each $\alpha \in S_{x,\delta}$ find a condition $q_\alpha \leq p$ and a natural number k_α such that $q_\alpha \Vdash \dot{c}(\alpha) = k_\alpha$. Next, let $T_{x,\delta} \subseteq S_{x,\delta}$ be stationary such that there is a fixed condition $q \leq p$ and a natural number k such that for each $\alpha \in T_{x,\delta}$ we have that $q \Vdash \dot{c}(\alpha) = k$, this is possible as the Cohen forcing is countable. We can now find a countable elementary submodel M of $H(\theta)$ such that $N_n \in M$ and $\beta := M \cap \omega_1 \in T_{x,\delta}$. We thus have that $q \Vdash \dot{c}(\beta) = k$. Put $m := |q|$ and note that q decides the first m candidates for neighbors of β as $q \Vdash e_\beta(\dot{r}(i)) = e_\beta(q(i))$ for all $i < m$.

We have the following.

$$H(\theta) \models \beta \in S_{x,\delta} \wedge q \Vdash \dot{c}(\beta) = k.$$

Thus $\alpha = \beta$ witnesses that the following formula is true.

$$H(\theta) \models \forall \xi < \beta \exists \alpha < \omega_1 : \xi < \alpha \wedge \alpha \in S_{x,\delta} \wedge q \Vdash \dot{c}(\alpha) = k.$$

By elementarity and the fact that $M \cap \omega_1 = \beta$ we obtain that:

$$M \models \forall \xi < \omega_1 \exists \alpha < \omega_1 : \xi < \alpha \wedge \alpha \in S_{x,\delta} \wedge q \Vdash \dot{c}(\alpha) = k.$$

Let $\varphi(z, \sigma)$ denote the previous sentence where x and δ are parameters z and σ. Still in M we have:

$$M \models \forall \zeta < \delta \exists \varepsilon < \omega_1 : \zeta < \varepsilon \wedge \varphi(x, \varepsilon)$$

this is witnessed by $\varepsilon = \delta$ for each ζ. Again applying elementarity, we have:

$$N_n \models \forall \zeta < \omega_1 \exists \varepsilon < \omega_1 : \zeta < \varepsilon \wedge \varphi(x, \varepsilon).$$

Let us abbreviate by $\exists^\infty \varepsilon : \psi(\varepsilon)$ the formula $\forall \zeta < \omega_1 \exists \varepsilon < \omega_1 : \zeta < \varepsilon \wedge \psi(\varepsilon)$. Thus

$$N_n \models \exists^\infty \varepsilon : \varphi(x, \varepsilon).$$

Continuing analogously, using the elementarity of the models N_i, we can prove the following:

$$N_0 \models \exists^\infty \gamma_0 \cdots \exists^\infty \gamma_{n-1} \exists^\infty \varepsilon : \varphi(\{\gamma_0, \ldots, \gamma_{n-1}\}, \varepsilon).$$

Put $\lambda := \max\{e_\beta(q(i)) \mid i < m\}$, note that $\lambda < \beta$. Now we choose $\{\gamma^*_0, \ldots, \gamma^*_{n-1}\}$ and the ordinal ε^* such that $\{x, \{\gamma^*_0, \ldots, \gamma^*_{n-1}\}\}$ is an edge in S^n_s, $\varepsilon^* > \max\{\delta, \lambda\}$ and $\varphi(\{\gamma^*_0, \ldots, \gamma^*_{n-1}\}, \varepsilon^*)$ holds. We use elementarity to choose the elements while we work ourselves up from N_0 to M. Choose the set $\{\gamma^*_i \mid i \leq s\}$ in N_0 increasing, then choose γ^*_{s+j} in N_j above x_{j-1} for $1 \leq j \leq n-s-1$.

In M we thus have that $\exists^\infty \varepsilon : \varphi(\{\gamma_0^*, \dots, \gamma_{n-1}^*\}, \varepsilon)$ so choose ε^* in M above both δ and λ. Still in M we have the following $M \models \exists^\infty \alpha : \alpha \in S_{\{\gamma_0^*, \dots, \gamma_{n-1}^*\}, \varepsilon^*} \wedge q \Vdash \dot{c}(\alpha) = k$. It is enough to choose α^* in M such that α^* is above $C_\beta(m)$ and $\alpha^* \in S_{\{\gamma_0^*, \dots, \gamma_{n-1}^*\}, \varepsilon^*} \wedge q \Vdash \dot{c}(\alpha^*) = k$. Put $q^* := q \cup \{(m, e_\beta^{-1}(\alpha^*))\}$.

By construction we have that $q^* \Vdash \dot{c}(\alpha^*) = \dot{c}(\beta)$. However, observe that $q^* \Vdash \alpha^* \in \mathrm{N}^<(\beta)$. Suppose we are in a generic extension by a Cohen real r and $q^* \subseteq r$. At stage β in the construction consider the m-th step, the ordinal $e_\beta(r(m))$ at that point is exactly α^* as $q^* \Vdash e_\beta(\dot{r}(m)) = \alpha^*$. Note that by carefully choosing the set $\{\gamma_0^*, \dots, \gamma_{n-1}^*\}$ and ε^* we made sure that α^* satisfies all the conditions so $\beta_m := \alpha^*$, hence $\alpha^* \in \mathrm{N}^<(\beta)$. We showed that the set of conditions forcing that $\dot{c}$ is not a name for a proper coloring is dense. Thus G cannot be countably chromatic. □

Using the same technique, we can construct a ZFC example of a T-HM graph with no special cycles and no short odd cycles. The first such graph was constructed by Hajnal and Komjáth [HK84, Theorem 2.], but the construction used CH. An interesting ZFC example is due to Soukup and uses a tree of the form $\{t \subseteq S \mid t \text{ is closed}\}$ where $S \subseteq \omega_1$ is stationary, co-stationary, and the tree order is end extension [Sou15a, Theorem 5.5.]. The tree we will consider is ${}^{<\omega_1}\omega$. Our theorem extends Komjáth and Shelah's result [KS88, Theorem 10.] by excluding all special cycles. Note that ${}^{<\omega_1}\omega$ does not split on limit levels.

Theorem 2.26. *For each $s \in \omega$, there is a special cycle–free ${}^{<\omega_1}\omega$-HM graph without odd cycles of length at most $2s+1$.*

Proof. The idea of the proof is the same as in Theorem 2.25. We point out the differences. In the beginning, we fix the same objects: the collection of stationary sets $\mathscr{S}$ and for each $\beta < \omega_1$ an ω-sequence $(C_\beta(n) \mid n \in \omega)$. For $\delta < \beta$ and $f \in {}^\beta\omega$ we say that f *is δ-covered* if there is a monotone decreasing path from f to $f \restriction \gamma$ for some $\gamma \leq \delta$. By $\mathrm{N}^<(f)$ we will denote the set $\{f \restriction \gamma \mid \gamma < \beta \wedge \{f \restriction \gamma, f\} \in E\}$.

The proof proceeds by induction on the levels of the tree, and the induction hypothesis is that if $\beta \in S_{x,\delta}$ and $f \in {}^\beta\omega$ then f is not δ-covered. If β is a successor ordinal $\mathrm{N}^<(f)$ will be empty. If β is limit and $\beta \in S_{x,\delta}$, the construction for each $f \in {}^\beta\omega$ is identical. Let us fix an $f \in {}^\beta\omega$. We use induction on $k \in \omega$ to construct a set $K_f \in [\omega]^{\leq\omega}$ and ordinals $\{\beta_k^f \mid k \in K_f\}$ such that:

1. for all $k \in K_f$ we have $f(\beta_k^f) = k$.
2. for all $k \in K_f$ if $\beta_k^f \in S_{y,\varepsilon}$, then $\varepsilon > \max(\{\beta_i^f \mid i \in K_f \cap k\} \cup \{\delta\})$,
3. for all $k \in K_f$ we have $\beta_k^f > \max(\{\beta_i^f \mid i \in K_\beta \cap k\} \cup \{C_\beta(k)\})$
4. for all $k \in K_f$ if $\beta_k^f \in S_{y,\varepsilon}$, then $\{x, y\}$ is an edge in S_s^n.

At stage k if there is some α satisfying the required properties, we put β_k^f the minimal such α and $k \in K_f$, otherwise β_k^f is undefined and $k \notin K_f$. Lastly we put $\mathrm{N}^<(f) := \{f \restriction \beta_k^f \mid k \in K_f\}$.

The fact that this graph has no special cycles and no short odd cycles is proved analogously as in Theorem 2.25. We only comment on the chromaticity. Fix a coloring $c : {}^{<\omega_1}\omega \to \omega$ and define $g : \omega_1 \to \omega$ by recursion as $g(\beta) = c(g \restriction \beta)$.

Find the models $(N_i \mid i < n+1)$ and define x and δ as in Theorem 2.25 and make sure that N_0 contains g. Find M such that $M \cap \omega_1 \in S_{x,\delta}$. Define $\beta := M \cap \omega_1$ and $k := g(\beta)$. Similarly as before we arrive at:

$$N_0 \models \exists^\infty \gamma_0 \cdots \exists^\infty \gamma_{n-1} \exists^\infty \varepsilon \exists^\infty \alpha : \alpha \in S_{\{\gamma_0,\ldots,\gamma_{n-1}\},\varepsilon} \wedge g(\alpha) = k.$$

We then choose the γ_i^* and ε^* to satisfy the fourth and second conditions. Then choose α^* large enough so that we have a witness that at stage k in the construction of the set $\mathrm{N}^<(g \restriction \beta)$ the element $\beta_k^{g\restriction\beta}$ was defined. Hence $c(g \restriction \beta) = g(\beta) = k = g \restriction \beta(\beta_k^{g\restriction\beta}) = g(\beta_k^{g\restriction\beta}) = c(g \restriction \beta_k^{g\restriction\beta})$ and c is not proper. □

2.4.1 δ-HM Graphs

Here we focus on so-called δ-HM graphs for δ arbitrary countable ordinal.

Definition 2.27. Let δ be a countable ordinal. A graph $G = (\omega_1, E)$ is called a *δ-Hajnal–Máté* graph if the chromatic number of G is uncountable and for every $\alpha < \beta < \omega_1$ the set $\{\gamma < \alpha \mid \{\gamma, \beta\} \in E\}$ has order type less than δ.

Recall Hajnal and Máté's result on the effect of Martin's axiom on HM graphs, [HM75, §8.2.].

Theorem 2.28 (Hajnal and Máté)**.** *If $\mathsf{MA}(\omega_1)$ holds, there are no Hajnal–Máté graphs.*

We will show that this result also extends to δ-HM graphs. We also show that they always contain uncountable independent sets.

Definition 2.29. An ordinal δ is called *indecomposable* if for each subset $X \subseteq \delta$ the order type of X is δ or the order type of $\delta \setminus X$ is δ.

Lemma 2.30. *Let δ be an ordinal. The following are equivalent:*

1. *δ is indecomposable.*
2. *For any $1 \leq n < \omega$ and any $f : \delta \to n$ there is some $i < n$ such that $f^{-1}(i)$ has order type δ.*

Proof. Taking $n = 2$ in the second formula, we obtain the definition of an indecomposable ordinal.

Suppose then that δ is indecomposable. We prove by induction on n that for any $f : \delta \to n$, there is some $i < n$ such that $f^{-1}(i)$ has order type δ. When $n \leq 2$, this is clear. Suppose then that $n \geq 3$, $f : \delta \to n+1$ and define a function $g : \delta \to n$ such that $g(\gamma) = f(\gamma) \pmod n$. Now there is some $i < n$ such that $g^{-1}(i)$ has order type δ. If $i > 0$, then as $f^{-1}(i) = g^{-1}(i)$ we get the desired result. If $i = 0$ let $X := g^{-1}(0)$ and consider $f|X$. Note that the range of this restriction is a subset of $\{0, n\}$. Since we showed that the formula holds for functions whose range has size 2, we are done if we apply it to the function $f : X \to \{0, n\}$. □

Theorem 2.31. *Suppose δ is a countable ordinal. If $\mathsf{MA}(\omega_1)$ holds, there are no δ-Hajnal–Máté graphs.*

Proof. Suppose $G = (\omega_1, E)$ is a graph with the property that for every $\alpha < \beta < \omega_1$ the set $\mathrm{N}(\beta) \cap \alpha$ has order type less than δ. We will show that the chromatic number of G is countable.

We construct a ccc poset $(\mathscr{P}, \leq)$ such that a $\mathscr{P}$-generic will define a proper countable coloring of G. A condition p in $\mathscr{P}$ is a finite partial coloring of G, i.e., $\mathsf{dom}(p) \in [\omega_1]^{<\omega}$, $p : \mathsf{dom}(p) \to \omega$ and for each $\alpha, \beta \in \mathsf{dom}(p)$ such that $\{\alpha, \beta\} \in E$ we have that $p(\alpha) \neq p(\beta)$. A condition p is stronger than q if $q \subseteq p$.

Claim 2.31.1. *$\mathscr{P}$ is ccc.*

Proof. Suppose $(p_\alpha \mid \alpha < \omega_1)$ forms an antichain in $\mathscr{P}$. Use the Δ-system lemma to ensure that the domains of the conditions are of the same size and form a Δ-system with root r. Additionally, make sure that on the root, the conditions agree. The fact that these conditions form an antichain means that for each $\alpha < \beta$, the union $p_\alpha \cup p_\beta$ is not a partial proper coloring of G. In particular there is some $\eta \in \mathsf{dom}(p_\alpha) \setminus r$ and $\xi \in \mathsf{dom}(p_\beta) \setminus r$ such that $\{\eta, \xi\} \in E$. This implies that if we omit the root of the Δ-system in these conditions, we are still left with an antichain.

We can also make sure that the domains are increasing, i.e., we want to make sure that $\sup\{\max \mathsf{dom}(p_\beta) \mid \beta < \alpha\} < \min \mathsf{dom}(p_\alpha)$ for each $\alpha < \omega_1$.

Let $\delta^* > \delta$ be indecomposable. Consider the condition p_{δ^*} and suppose we have $\mathsf{dom}(p_{\delta^*}) = \{\xi_0, \ldots, \xi_{n-1}\}$. Define a function $f : \delta^* \to n$ as follows: $f(\gamma) = i$ if and only if i is the least natural number with the property that $\{\eta, \xi_i\} \in E$ for some $\eta \in \mathsf{dom}(p_\gamma)$, such an η always exists as the conditions form an antichain. Using Lemma 2.30 we obtain an $i < n$ such that $f^{-1}(i)$ has order type δ^*. We thus have a set of ordinals of order type δ^*, all of which are connected to the same ξ_i; this contradicts the property of the graph we started with. □

To finish the proof let $\alpha < \omega_1$ and consider $D_\alpha := \{p \in \mathscr{P} \mid \alpha \in \mathsf{dom}(p)\}$.

Claim 2.31.2. *D_α is dense in $\mathscr{P}$ for each $\alpha < \omega_1$.*

Proof. Let p be a condition in $\mathscr{P}$. If $\alpha \in \mathsf{dom}(p)$ we are done, else let $n < \omega$ be such that $n > \max\{i < \omega \mid \exists \alpha \in \mathsf{dom}(p) : p(\alpha) = i\}$, now $p \cup \{(\alpha, n)\}$ is an extension of p in D_α. □

If H is a generic filter intersecting each D_α, then $\bigcup H$ is a proper coloring of G with countably many colors. □

Let us also remark that even though δ-Hajnal–Máté graphs are not so sparse anymore, they always contain uncountable independent sets.

Proposition 2.32. *Suppose δ is a countable ordinal and G is a graph on ω_1 with the property that for each $\beta < \omega_1$, the set $\mathrm{N}^<(\beta)$ has order type at most δ. Then G has an uncountable independent set.*

Proof. Let $G = (\omega_1, E)$ be given. For contradiction, assume G has no uncountable independent sets.

Claim 2.32.1. *For club many α the set $\mathrm{N}^<(\alpha)$ is cofinal in α.*

Proof. Otherwise, there is a stationary set T such that for all $\alpha \in T$, the set $\mathrm{N}^<(\alpha)$ is bounded in α. Use Fodor's lemma to the regressive function $f : T \to \omega_1$ defined as $f(\alpha) = \gamma$ if and only if γ is the least ordinal such that $\mathrm{N}^<(\alpha) \cap [\gamma, \alpha) = \emptyset$. There is a stationary set $T' \subseteq T$ and a fixed γ^* such that $f(\alpha) = \gamma^*$ for all $\alpha \in T'$. Now the set $T' \setminus \gamma^*$ is independent. □

Call the club set from the previous claim C. For $i \leq \delta$ put $X_i := \{\alpha < \omega_1 \mid \exists \beta \in C : \alpha \in \mathrm{N}^<(\beta) \wedge \mathrm{ot}(\mathrm{N}^<(\beta) \cap (\alpha + 1)) = i\}$.

Claim 2.32.2. *There is an $i \leq \delta$ such that X_i is uncountable.*

Proof. If all sets X_i were countable, their supremum would also be countable. Consider α above this supremum. Now $\mathrm{N}^<(\alpha)$ would have to be bounded in α. This is a contradiction, given the previous claim. □

Let i^* be the least such i and put $\delta := \sup\{\sup X_i \mid i < i^*\}$. By induction, we will find an uncountable independent set $\{x_\alpha \mid \alpha < \omega_1\}$. Let x_0 be any vertex in C above δ. Suppose we have constructed $\{x_\alpha \mid \alpha < \beta\}$ for some $\beta < \omega_1$. Choose x_β such that there is some $\xi_\beta \in \mathrm{N}^<(x_\beta)$ with the property that $\mathrm{N}^<(x_\beta) \cap (\xi_\beta + 1)$ has order type i^* and $\xi_\beta > \sup\{x_\alpha \mid \alpha < \beta\}$. This is always possible as X_{i^*} is unbounded.

To finish the proof, we show that the set $\{x_\alpha \mid \alpha < \omega_1\}$ forms an independent set in G. Suppose that for some $\alpha < \beta < \omega_1$ we have $x_\alpha E x_\beta$. As x_α is above δ it must be the case that $\mathrm{N}^<(x_\beta) \cap (x_\alpha + 1)$ has order type at least i^*. However, x_β was chosen so that there was some $\xi_\beta < x_\beta$ connected to it above x_α. Now $\mathrm{N}^<(x_\beta) \cap (\xi_\beta + 1)$ has order type i^* and $x_\alpha < \xi_\beta$, a contradiction. □

Corollary 2.33. *If $G = (\omega_1, E)$ is a graph with no uncountable independent set, then for every $\alpha < \omega_1$, there is a $\beta \geq \alpha$ such that $\mathrm{N}^<(\beta)$ has order type at least α.*

We also obtain a weak partition relation from the proposition.

Corollary 2.34. *Suppose δ is a countable ordinal. Then $\omega_1 \to (\omega_1, \delta : 1)^2$.*

Similarly to our proof of Galvin's Theorem, we obtain a partition result.

Proposition 2.35. *Suppose δ is a countable ordinal. Then $\omega_2 \to (\omega_1, \delta : 2)^2$.*

Proof. Apply Proposition 2.17 with the previous corollary. □

3. The Uncountable Hadwiger Conjecture and Characterizations of Trees Using Graphs

The Hadwiger conjecture is a deep unsolved problem in finite graph theory with far-reaching consequences. It states that if G is a simple finite graph and the chromatic number of G is t, then the complete graph on t vertices is a minor of G. Paul Erdős even called it "one of the deepest unsolved problems in graph theory." We are interested in generalizations of this conjecture to uncountable graphs. This chapter will extend recent results by Dominic van der Zypen [vdZ13] and Péter Komjáth [Kom17] on the infinite version of the Hadwiger conjecture.

The tools used to prove our results on the uncountable Hadwiger conjecture are of separate interest. They provide a way of translating tree properties onto graphs and vice versa. Our main tools will be the comparability graph of a tree and a lesser-known construction of a tree from a graph by Brochet and Diestel [BD94]. We will then apply the general results to solve the uncountable Hadwiger conjecture.

3.1 Generalization of the Hadwiger conjecture

It was van der Zypen [vdZ13] who first observed that the straightforward generalization of the Hadwiger conjecture does not hold for infinite graphs.

Theorem 3.1 (van der Zypen [vdZ13])**.** *There is a countable connected graph whose chromatic number is ω, but K_ω is not a minor of this graph.*

The proof generalizes to limit cardinals.

Proposition 3.2. *Suppose κ is a limit cardinal. There exists a connected graph whose size and chromatic number is κ, but K_κ is not a minor of this graph.*

Proof. Suppose $(\kappa_\alpha \mid \alpha < \mu)$ is an increasing cofinal sequence of cardinals in κ, where μ may be equal to κ. Let the vertex set of the graph be the union $\{0\} \cup \{(\alpha, \beta) \mid \alpha \in \mu \wedge \beta \in \kappa_\alpha\}$. The edge set is defined as follows: connect the vertex 0 to every other vertex and for $\alpha < \mu$ connect (α, β) to (α, β') for every $\beta < \beta' < \kappa_\alpha$. The graph is the disjoint union of complete graphs of increasing size, cofinal in κ, plus a vertex connected to every other vertex.

The chromatic number of this graph is bounded by κ but also larger than κ_α for every $\alpha < \mu$ as the complete graph K_{κ_α} embeds into the graph. Thus the chromatic number is exactly κ.

Now assume $\{C_\gamma \mid \gamma < \kappa\}$ is a collection of pairwise disjoint connected subgraphs forming a K_κ minor.

Suppose first that 0 does not belong to any of the sets C_γ, then necessarily $\bigcup_{\gamma<\kappa} C_\gamma$ is a subset of some complete graph K_{κ_α} this is, of course, a contradiction as $\kappa_\alpha < \kappa$ and each C_γ contains at least one element.

So assume $0 \in C_{\gamma_0}$ for some $\gamma_0 < \kappa$. Now given any $\gamma \neq \gamma_0$, we must have that $C_\gamma \subseteq K_{\kappa_\alpha}$ for some κ_α otherwise, they cannot be connected by an edge as 0 is in C_{γ_0} and it is the only vertex connecting the disjoint cliques. Hence we get a contradiction as in the previous case. □

Remark. Note that for κ singular, we can deduce that the counterexample has no independent set of size κ; however, this is not true for κ regular.

Thus the Hadwiger conjecture fails unconditionally for ω and, in general, for every limit cardinal.

Komjáth [Kom17] showed it can consistently hold for graphs of size and chromatic number ω_1.

Theorem 3.3 (Komjáth [Kom17])**.** *If* $\mathsf{MA}(\omega_1)$ *holds, then every graph* G *with* $|G| = \chi(G) = \omega_1$ *contains a subdivision of* K_{ω_1}*.*

Remark. The proof shows how to kill counterexamples to the Hadwiger conjecture. Given an uncountable graph G, one can consider the poset of finite ω-colorings of the vertex set. Either this poset is ccc, in which case we force a countable coloring for G, or else a K_{ω_1} topological minor can be constructed from an uncountable antichain of this poset.

On the other hand, counterexamples exist in ZFC.

Theorem 3.4 (Komjáth [Kom17])**.** *For every cardinal* κ*, there is a graph of size* 2^κ *whose chromatic number is* κ^+ *but* K_{κ^+} *is not a minor of the graph.*

3.1.1 Positive Examples

Before proving our main result, we analyze several well-known examples of uncountably chromatic graphs and prove that they all satisfy the desired conclusion, i.e., they all contain K_{ω_1} as a minor.

The following lemma will be useful.

Lemma 3.5. *Suppose* κ *is an infinite cardinal and* G *is a graph. If there is a set* $X \in [V]^\kappa$ *such that for every* $x, y \in X$*, there is a set* $\mathbf{p}_{x,y}$ *of size* κ *of pairwise disjoint paths connecting* x *to* y*, then* K_κ *is a topological minor of* G*.*

Proof. We will inductively choose elements $\{x_\alpha \mid \alpha < \kappa\}$ from the set X and pairwise disjoint $\{c_{\alpha,\beta} \mid \alpha < \beta < \kappa\}$ such that $c_{\alpha,\beta} \in \mathbf{p}_{x_\alpha,x_\beta}$ and $x_\alpha{}^\frown c_{\alpha,\beta}{}^\frown x_\beta$ is a path in G.

Suppose we have constructed $\{x_\alpha \mid \alpha < \gamma\}$ for some $\gamma < \kappa$. Choose any $x_\gamma \in X \setminus (\{x_\alpha \mid \alpha \in \gamma\} \cup \bigcup\{c_{\alpha,\beta} \mid \alpha < \beta < \gamma\})$, this is possible as κ is a cardinal and paths are finite.

Again by induction, choose the elements $\{c_{\alpha,\gamma} \mid \alpha < \gamma\}$ each from $\mathbf{p}_{x_\alpha,x_\gamma}$ disjoint from all the vertices and paths chosen so far. The choice of these paths is possible. Note that $\mathbf{p}_{x_\alpha,x_\gamma}$ has size κ, and at each step of the construction we only used $< \kappa$ vertices, these can intersect only $< \kappa$ paths, so we have κ available choices. □

Remark. A special case of the previous lemma occurs when the sets $\mathbf{p}_{x,y}$ are singletons, i.e., the set $\mathrm{N}(x) \cap \mathrm{N}(y)$ has size κ.

The following construction is due to Erdős and Hajnal.

Corollary 3.6. *There is a graph whose vertex set has order type $\omega_1{}^2$, which is uncountably chromatic, but every set of order type ω_1 induces a countably chromatic graph. Moreover, K_{ω_1} is a topological minor of this graph.*

Proof. The vertex set is $V := \bigcup_{\alpha\in\omega_1} V_\alpha$, where $V_\alpha := [\omega_1 \cdot \alpha + \alpha, \omega_1 \cdot (\alpha + 1))$. Vertices $\omega_1 \cdot \alpha + \beta$ and $\omega_1 \cdot \alpha' + \beta'$ are connected if and only if $\alpha < \alpha'$ and $\beta > \beta'$. For the proof of chromaticity, see [KT06, Problem 23.11.].

The corollary easily follows with $X := \{\omega_1 \cdot \alpha + \alpha \mid 0 < \alpha < \omega_1\}$. Given $\alpha < \alpha'$ notice that the set $\mathrm{N}(\omega_1 \cdot \alpha + \alpha) \cap \mathrm{N}(\omega_1 \cdot \alpha' + \alpha')$ is a superset of the uncountable set $\{\omega_1 + \beta \mid \alpha' < \beta < \omega_1\}$. □

Another example tackled by this lemma is the shift graph, denoted as $\mathrm{Sh}_2(\mathfrak{c}^+)$. The vertex set is $[\mathfrak{c}^+]^2$ and two vertices $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ such that $\alpha < \beta$ and $\gamma < \delta$ are connected if $\beta = \gamma$ or $\alpha = \delta$. It is a well-known fact [KT06, Problem 23.26.] that this graph is uncountably chromatic and has no triangles.

Corollary 3.7. *The graph $\mathrm{Sh}_2(\mathfrak{c}^+)$ has uncountable chromatic number, contains no triangles, and K_{ω_1} is a topological minor of this graph.*

Proof. To prove the last part, consider $\mathfrak{c}^+$ as an ordinal, obviously $\omega_1 < \mathfrak{c}^+$. Put $X := \{(\alpha, \omega_1) \mid \alpha < \omega_1\}$ and note that any vertex from $\{(\omega_1, \gamma) \mid \omega_1 < \gamma < \mathfrak{c}^+\}$ is connected to any (β, ω_1) for $\beta < \omega_1$. □

Next, we consider the original construction of a Hajnal–Máté graph, see [HM75].

Proposition 3.8. *Assuming $\diamondsuit^+$, there is an uncountably chromatic Hajnal–Máté graph having K_{ω_1} as a minor.*

Proof. Let $(\mathcal{A}_\xi \mid \xi < \omega_1)$ be our $\diamondsuit^+$ sequence. For any limit $\alpha < \omega_1$, let $(A^n_\xi \mid n < \omega)$ be an enumeration of all elements $A \in \mathcal{A}_\xi$ such that $\bigcup A = \alpha$. Choose elements $\gamma^n_\alpha \in A^n_\xi$ such that $\lim_{n\to\infty} \gamma^n_\alpha = \alpha$ and define the graph as follows: the pair $\beta < \alpha$ is connected by an edge if and only if α is a limit ordinal and $\beta = \gamma^n_\alpha$ for some $n \in \omega$.

This graph is uncountably chromatic, as shown in [HM75]. We will prove K_{ω_1} is a minor of this graph.

Claim 3.8.1. *If $S \subseteq \omega_1$ is stationary, then a stationary subset T of S exists such that T induces a connected subgraph.*

Proof. Let S be given. Using $\Diamond^+$, we get that the set $C := \{\xi < \omega_1 \mid S \cap \xi \in A_\xi\}$ is a club. Let D be the set of limit points of $S \cap C$; this set is also a club. Hence $S' := S \cap C \cap D$ is again stationary. For any $\xi \in S'$ we have $S \cap \xi \in A_\xi$, so $S \cap \xi = A_\xi^{n(\xi)}$ for some $n(\xi) \in \omega$, in particular $\gamma_\xi^{n(\xi)} \in S$ and the pair $\{\gamma_\xi^{n(\xi)}, \xi\}$ is connected by an edge. We can thus construct a regressive function $f : S' \to \omega_1$ defined as $f(\xi) = \gamma_\xi^{n(\xi)} \in S$. Using Fodor's lemma, there exists a stationary $S'' \subseteq S'$ such that f is constant on S'', i.e., there is some fixed $\gamma \in \omega_1$ such that $f(\xi) = \gamma$ for all $\xi \in S''$. To finish put $T := S'' \cup \{\gamma\}$ and note that every $\xi \in S''$ is connected to γ. □

Claim 3.8.2. *If $S, T \subseteq \omega_1$ are stationary, then $\alpha \in S$ and $\beta \in T$ are connected by an edge.*

Proof. We proceed analogously. Define C and D as previously and consider any $\beta \in C \cap D \cap T$. Note that $\beta \in T$. As $\beta \in C$ we get that $S \cap \beta = A_\beta^{n(\beta)}$ for some $n(\beta) < \omega$, thus $\gamma_\beta^{n(\beta)} \in S$ and $\{\gamma_\beta^{n(\beta)}, \beta\}$ forms an edge. □

To finish the proof, take uncountably many pairwise disjoint stationary sets. By the first claim, we can assume they induce connected graphs, and by the second claim, each pair is connected by at least one edge. □

Remark. If the first claim in the previous proposition is iterated, we get that every stationary set contains K_ω as a subgraph.

Another well-known example of an uncountably chromatic graph is the Specker graph [KT06, Problem 23.25.]. The vertex set of the Specker graph is $[\omega_1]^3$ and two triples (α, β, γ), $(\alpha', \beta', \gamma')$ (in both cases assume the enumeration is in increasing order) are connected if and only if $\alpha < \beta < \alpha' < \gamma < \beta' < \gamma'$.

Proposition 3.9. *K_{ω_1} is a minor of the Specker graph.*

Proof. We prove two claims from which the result will follow.

Claim 3.9.1. *If $X \subseteq \omega_1$ is uncountable, then there is an uncountable $Y \subseteq X$ and $Z \subseteq [X]^3$ such that $[Y]^3 \cup Z$ induces a connected graph.*

Proof. Let Y be any uncountable subset of X with the property that infinitely many elements in X separate any two elements of Y. Next, we will show that any two elements in $[Y]^3$ are connected by a finite path using only vertices from $[X]^3$.

Let $(\alpha, \beta, \gamma), (\alpha', \beta', \gamma') \in [Y]^3$ be given and assume $\gamma \leq \gamma'$. Choose an increasing sequence of 9 elements from X, $(x_i)_{i<9}$, so that the following holds:

1. $\beta < x_0 < \gamma$,
2. $\beta' < x_1 < \gamma'$,
3. $\gamma' < x_i$ for all $2 \leq i < 9$.

Consider the three vertices $(x_0, x_3, x_5), (x_1, x_2, x_6), (x_4, x_7, x_8)$. The chain of inequalities proves that $(\alpha, \beta, \gamma)^\frown(x_0, x_3, x_5)^\frown(x_4, x_7, x_8)^\frown(x_1, x_2, x_6)^\frown(\alpha', \beta', \gamma')$ is a path.

$$\begin{gathered}
\alpha < \beta < x_0 < \gamma < x_3 < x_5 \\
x_0 < x_3 < x_4 < x_5 < x_7 < x_8 \\
x_1 < x_2 < x_4 < x_6 < x_7 < x_8 \\
\alpha' < \beta' < x_1 < \gamma' < x_2 < x_6
\end{gathered}$$

Thus let Z consist of these triples of vertices for pairs in $[Y]^3$, separated in $[Y]^3$. □

Claim 3.9.2. *If $X, Y \subseteq \omega_1$ are uncountable, then there is a vertex in $[X]^3$ connected to a vertex in $[Y]^3$.*

Proof. As both X and Y are unbounded, two vertices $(\alpha, \beta, \gamma) \in [X]^3$ and $(\alpha', \beta', \gamma') \in [Y]^3$ with the property that: $\alpha < \beta < \alpha' < \gamma < \beta' < \gamma'$ can be chosen easily, proving the claim. □

Given any partition of ω_1 into uncountably many pieces of uncountable size, use the first claim to find Y and Z for each piece, these will define a connected subgraph, and by the second claim, an edge connects any $[Y]^3$ to any other $[Y']^3$. □

3.1.2 Negative Examples

To find graphs that are uncountably chromatic and do not contain a K_{ω_1} minor, we turn to trees.

Proposition 3.10. *The comparability graph of a Suslin tree is uncountably chromatic and has no K_{ω_1} minor.*

Proof. Let S be a Suslin tree and G_S its comparability graph. Suppose first that G_S is countably chromatic, then there is a subset of G_S which is uncountable and independent, but this exact same subset forms an uncountable antichain in S, a contradiction.

The proof that G_S has no K_{ω_1} minor is the same as in the proof of [Kom17, Theorem 2]. □

Next, we make an observation about the connectedness of the graphs, which are counterexamples to the uncountable Hadwiger conjecture. The following proposition states that a graph with no K_κ minor cannot be κ-connected.

Proposition 3.11. *Suppose κ is an infinite cardinal. If G is κ-connected, then G contains a subdivision of K_κ.*

Proof. This proposition can be seen as a corollary of Lemma 3.5. The fact that G is κ-connected allows us to construct the paths in the assumption of the lemma.

We will provide a direct proof not appealing to Lemma 3.5. We proceed by induction, choose elements $\{v_\alpha \mid \alpha < \kappa\} \subseteq V$ and finite sequences of vertices $\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \kappa\}$ so that for each pair of vertices v_α, v_β the sequence $v_\alpha{}^\frown\mathbf{p}_{\alpha\beta}{}^\frown v_\beta$ is a path and the collection of all these $\mathbf{p}_{\alpha\beta}$'s is pairwise disjoint; this forms a subdivision of K_κ.

Suppose we have constructed $\{v_\alpha \mid \alpha < \gamma\}$ for some $\gamma < \kappa$ and we also have the collection of pairwise disjoint paths $\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \gamma\}$. Consider the subgraph of G induced by the vertices $V_\gamma := V \setminus \bigcup\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \gamma\}$. By our assumption, this still induces a connected graph, choose any vertex v from this set different from any vertex included in $\{v_\alpha \mid \alpha < \gamma\}$, this will be the vertex v_γ.

By induction again, we choose the finite sequences $\mathbf{p}_{\alpha\gamma}$. Suppose we have constructed $\{\mathbf{p}_{\beta\gamma} \mid \beta < \alpha\}$ for some $\alpha < \gamma$. As paths are finite, we have that the set $V_\gamma \setminus \bigcup\{\mathbf{p}_{\beta\gamma} \mid \beta < \alpha\}$ still induces a connected graph as we assume G is κ-connected, so choose any path between v_α and v_γ using only these vertices excluding the so far chosen $\{v_\alpha \mid \alpha < \gamma\}$ and this will be our $\mathbf{p}_{\alpha\gamma}$. □

3.1.3 The Hadwiger Conjecture Number

Because of Theorem 3.4, the least size of a graph having chromatic number ω_1 and no K_{ω_1} minor is at most continuum. Obviously, such a graph must be uncountable. Thus we are justified to define a new cardinal invariant.

$$\mathfrak{hc} := \min\{|G| \mid \chi(G) = \omega_1 \wedge K_{\omega_1} \text{ is not a minor of } G\}$$

The Hadwiger conjecture number is related to the special tree number, $\mathfrak{st}$; this is the least size of a non-special (possibly wide) Aronszajn tree. The following fact is implicit in [Kom17, Theorem 2].

Proposition 3.12 (Komjáth)**.** $\mathfrak{hc} \leq \mathfrak{st}$.

Ultimately we will show that $\mathfrak{hc} = \mathfrak{st}$.

The following proposition summarizes some basic facts about $\mathfrak{hc}$.

Proposition 3.13. *The number $\mathfrak{hc}$ has the following properties:*

1. $\omega_1 \leq \mathfrak{hc} \leq \mathfrak{st} \leq \mathfrak{c}$,
2. MA *implies* $\mathfrak{hc} = \mathfrak{c}$,
3. *for any $\kappa > \omega_1$ of uncountable cofinality it is consistent that $\omega_1 = \mathfrak{hc} < \mathfrak{c} = \kappa$.*

Proof. The number $\mathfrak{hc}$ cannot be countable as the graph must be uncountably chromatic. The upper bound comes from Theorem 3.12.

Assuming full MA in the proof of Theorem 3.3, we obtain that in any uncountably chromatic graph of size $< \mathfrak{c}$ a subdivision of K_{ω_1} can be found.

For 3, consider the comparability graph of a Suslin tree. By Proposition 3.10, this graph is uncountably chromatic and has no K_{ω_1} minor. As is well known [Kun11, Corollary V.4.14], the continuum can be arbitrarily large while a Suslin tree exists. □

We will proceed by showing some facts about this newly introduced cardinal invariant.

Proposition 3.14. *The cofinality of* $\mathfrak{hc}$ *is uncountable.*

Proof. Suppose κ is a cardinal and $(\kappa_n \mid n \in \omega)$ is an increasing sequence of cardinals converging to κ. Assume that $\kappa_n < \mathfrak{hc}$ for all $n \in \omega$. We will show that κ is also strictly smaller than $\mathfrak{hc}$.

Take a graph G of size κ whose chromatic number is ω_1 and which has no K_{ω_1} minor. Partition the vertex set into countably many parts of increasing size corresponding to the cofinal sequence $(\kappa_n \mid n \in \omega)$ and consider the induced subgraphs $\{G_n \mid n \in \omega\}$. Now each graph G_n has size less than $\mathfrak{hc}$. Thus it is either countably chromatic or contains K_{ω_1} as a minor. As the entire graph has no K_{ω_1} minor, the latter is impossible, so it must be the case that G_n is countably chromatic for every $n \in \omega$. Fix the countable colorings $c_n : G_n \to \omega$. We will show there is a countable coloring of the entire G. Define $c : G \to \omega \times \omega$ as $c(v) = (n, k)$ if $v \in G_n$ and $c_n(v) = k$. Now if $\{u, v\}$ is an edge in G and $c(u) = c(v)$, then by the equality in the first coordinate, both vertices are part of the same G_n and by the equality in the second coordinate $c_n(u) = c_n(v)$. However, since these were induced subgraphs, the edge $\{u, v\}$ is present in G_n, so c_n is not a good coloring, a contradiction. □

3.2 Trees and Graphs

In this section, we will be investigating the relationship between trees and graphs. We will be looking at constructions of graphs from trees and trees from graphs; these constructions will provide tools to translate various graph properties into properties of the constructed tree and vice versa.

First, we introduce a new generalized notion of connectedness and a so-called *Kurepa minor family* useful for a more succinct characterization of Kurepa trees.

Definition 3.15. Suppose κ, λ are infinite cardinals. A graph is (κ, λ)-connected if, after arbitrary removal of less than κ vertices, the number of components is non-zero and less than λ.

Remark. Note that the proof of Proposition 3.2 shows that if κ is singular, then the graph constructed in the proposition is (κ, κ)-connected; this is not the case for regular κ.

Proposition 3.16. *Suppose* κ *is an infinite cardinal. If* G *is a graph of size at least* κ *and* G *has no independent set of size* κ*, then* G *is* (κ, κ)*-connected.*

Proof. Suppose G is not (κ, κ)-connected, then there exists a set $X \subseteq V$ of size less than κ such that if X is removed from G, the graph splits into at least κ many components, $\{C_\gamma \mid \gamma < \kappa\}$. From each component choose a vertex, $x_\gamma \in C_\gamma$. Now $\{x_\gamma \mid \gamma < \kappa\}$ forms an independent set in G. □

Definition 3.17. Suppose κ and λ are infinite cardinals. Let G be a graph and $\{W_\alpha \mid \alpha < \lambda\}$ a collection of K_κ minors of G. We say that $\{W_\alpha \mid \alpha < \lambda\}$ forms a *κ-Kurepa minor family* of size λ if for each α and β a set of size less than κ separates W_α from W_β in G.

Remark. Note that W_α and W_β need not be disjoint.

3.2.1 Comparability Graph

We prove how the properties of a given tree translate to the properties of its comparability graph.

Proposition 3.18. *Suppose κ, λ, μ are infinite cardinals. If T is a tree of height κ and G_T is its comparability graph, then the following hold:*

1. *T has a cofinal branch if and only if G_T has a K_κ minor,*
2. *T is λ-special if and only if the chromatic number of G_T is at most λ,*
3. *T has an antichain of size λ if and only if G_T has an independent set of size λ,*
4. *if κ is regular, then T is a κ-tree if and only if G_T is (κ, κ)-connected,*
5. *T has μ many different branches if and only if G_T has a κ-Kurepa graph minor family of size μ.*

Proof. A branch of size κ in T defines a complete graph K_κ in G_T. The proof of the other direction is in [Kom17, Theorem 2].

Next, suppose T is λ-special, i.e., there is a specializing function $f : T \to \lambda$. By the properties of the specializing function, the map f is also a proper coloring of G_T. On the other hand, a proper coloring $c : T \to \lambda$ of G_T is also a specializing function of T as edges in G_T correspond to comparable elements in T.

If T has an antichain of size λ, then this antichain forms an independent set in G_T. The same is true for the other direction.

Assume the levels of T have size less than κ. Let X be a subset of the vertex set of G_T of size less than κ, then $X \subseteq T_{<\alpha}$ for some $\alpha < \kappa$ as κ is regular. Removing the vertices from G_T corresponding to nodes in $T_{<\alpha}$, we obtain less than κ many components, each defined by some node $t \in T_\alpha$. Suppose G_T is (κ, κ)-connected. We want to see that T has levels of size less than κ. The proof is by induction on the levels of T. As G_T is (κ, κ)-connected, the set T_0 has size less than κ. Suppose an $\alpha < \kappa$ is given, then by the induction hypothesis and the fact that α is less than κ, we get that $|T_{<\alpha}| < \kappa$. Removing $T_{<\alpha}$ from G_T we obtain less than κ many components in G_T, thus T_α must have size less than κ.

Assuming T has μ many branches, clearly, each defines a complete graph K_κ in G_T. As T has height κ, each pair of branches splits on some level $\alpha < \kappa$. Thus they are separated by a set of size less than κ. Using item 1, we obtain that each K_κ minor defines a branch in T, and as these minors are separated by a set of size less than κ, these branches must be pairwise different. □

Using the previous proposition, many known tree constructions can be characterized by their comparability graph.

Corollary 3.19. *Suppose κ is a regular cardinal. If T is a tree of height κ and G_T is its comparability graph, then the following holds:*

1. *T is a κ-Aronszajn tree if and only if G_T has no K_κ minor and is (κ, κ)-connected,*
2. *T is a non-κ-special tree of height κ^+ with no cofinal branch if and only if G_T has no K_{κ^+} minor and its chromatic number is κ^+,*
3. *T is a κ-Kurepa tree if and only if G_T is (κ, κ)-connected and has a κ-Kurepa graph minor family of size κ,*
4. *T is a κ-Suslin tree if and only if G_T has no K_κ minor and no independent set of size κ.*

3.2.2 Partition Tree

In the previous section, we showed how the comparability graph inherits certain properties of the tree it was constructed from. Using the notion of a partition tree, we show that starting with a graph, a tree can be constructed, reflecting many useful properties of the graph. The construction we will be using is originally due to Brochet and Diestel [BD94, Theorem 4.2.].

Before proving the main theorem, we need a result about T-graphs. The following is folklore; see [Pit22, §2].

Proposition 3.20. *Suppose κ is an uncountable regular cardinal. If G is a T-graph and T has a κ-branch, then G contains a subdivision of K_κ.*

Proof. For simplicity, assume that the vertex set of G is T and let b be a branch of size κ in T.

Notice that for each $t \in b$, the set $\{s \in b \mid s \geq t\}$ forms a connected subgraph of G. Given any u, v in this set, assume that $u \leq v$. The path connecting u to v can be constructed as follows: the first vertex of the path is v. If the height of v in T is a successor ordinal, the next vertex on the path will be the predecessor of v in T. As G is a T-graph, v is connected to its predecessor in T. If the height of v is a limit ordinal, choose as the next vertex the minimum of the set $\{s \in \mathrm{pred}(v) \mid s \geq u \wedge s \text{ is connected to } v \text{ in } G\}$, this is always possible as G is a T-graph; then proceed by induction. Note that this construction must be finite as there is no infinite decreasing sequence of ordinals and must terminate at the vertex u.

The fact that $\{s \in b \mid s \geq t\}$ is connected in G and has size κ implies that there must be a vertex in this set whose degree is κ. Assume each vertex has degree less than κ. Start with an arbitrary vertex v_0. It is connected to at most $\lambda_0 < \kappa$ vertices and each of these is then again connected to at most $\lambda_1 < \kappa$ (λ_1 is the supremum over the number of neighbors for each of the λ_0 neighbors of v_0) other vertices and as paths are finite we can reach any vertex from an arbitrary starting point in finitely many steps. Altogether we get that the number of vertices is $\sum_{i=0}^{\infty} \lambda_i$, where $\lambda_i < \kappa$ for each $i \in \omega$. As κ is regular, this sum is strictly less than κ, a contradiction. In particular, b has κ many vertices whose degree is κ in the subgraph b alone. Let $\bar{b}$ be a κ sized subset of b such that each vertex in $\bar{b}$ has degree κ in b.

Inductively we will construct a subdivision of K_κ in G. We will find a set of vertices $\{v_\alpha \mid \alpha < \kappa\} \subseteq \bar{b}$ and sequences $\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \kappa\}$ of vertices, all of them contained in b, so that for each pair of vertices v_α, v_β the sequence $v_\alpha{}^\frown \mathbf{p}_{\alpha\beta}{}^\frown v_\beta$ is a finite path in G and the collection of all these $\mathbf{p}_{\alpha\beta}$'s is pairwise disjoint. This forms a subdivision of K_κ.

Suppose we have constructed $\{v_\alpha \mid \alpha < \gamma\}$ for some $\gamma < \kappa$, and we also have the collection of pairwise disjoint paths $\{\mathbf{p}_{\alpha\beta} \mid \alpha < \beta < \gamma\}$. Choose the vertex v_γ in $\bar{b}$ above all vertices considered so far; this is possible since κ is regular.

To construct the paths $\{\mathbf{p}_{\alpha\gamma} \mid \alpha < \gamma\}$, we again use induction and proceed analogously. Suppose we have constructed the paths $\{\mathbf{p}_{\alpha\gamma} \mid \alpha < \beta\}$ for some $\beta < \gamma$. To construct $\mathbf{p}_{\beta\gamma}$ choose a vertex u in $b \cap N(v_\beta)$ above all of the vertices considered so far and a vertex v in $b \cap N(v_\gamma)$ above u. The vertices u and v are connected by a path, $\mathbf{p}$, using only vertices w such that $u \leq w \leq v$. The path $\mathbf{p}_{\beta\gamma}$ is then $\mathbf{p}$. □

We are ready to reproduce Brochet and Diestel's result and analyze the construction.

Theorem 3.21 (Brochet and Diestel [BD94]). *Suppose κ is an uncountable cardinal. If $G = (V, E)$ is a graph of size κ, then there exists a tree T_G of height at most κ and a partition $(V_t)_{t \in T_G}$ of V such that the following holds:*

1. *$|V_t| \leq \mathrm{cf}(\mathrm{ht}(t))$ and V_t induces a connected subgraph in G for each $t \in T_G$.*
2. *Define a graph (T_G, F) such that for $s, t \in T_G$ distinct we have $\{s, t\} \in F$ if and only if V_s is connected to V_t. Then (T_G, F) is a T_G-graph and a minor of G.*

Proof. We reproduce the construction by Brochet and Diestel [BD94, Theorem 4.2.] and show that it has the required properties.

Let G be a graph of size κ and assume that the vertex set of G is κ. We will build a tree T level-by-level using induction. To each node $t \in T$ we will associate two subsets $V_t \subseteq C_t$ of V, both inducing connected subgraphs. The graph obtained by contracting the sets V_t to a point will be a T-graph. For every α, the following conditions will be satisfied:

1. if $\alpha = \beta + 1$, $t \in T_\alpha$ and s is the predecessor of t in T_β, then C_t is a component of $C_s \setminus V_s$ and $V_t = \{x_t\}$ such that $x_t \in C_t$ and x_t is connected

to some element in V_s; also if C is a component of $C_s \setminus V_s$, then there is a unique node $r \in T_\alpha$ above s, such that $C = C_r$,

2. if α is limit and $t \in T_\alpha$, then C_t is a component of $\bigcap_{s\in \text{pred}(t)} C_s$ and V_t induces a non-empty connected subgraph of C_t such that the following set $\{s \in \text{pred}(t) \mid V_s \text{ is connected to } V_t \text{ in } G\}$ is cofinal in pred(t); also if C is a component of $\bigcap_{s\in \text{pred}(t)} C_s$, then there is a unique node $r \in T_\alpha$ above pred(t), such that $C = C_r$,

3. if $t \in T_\alpha$, then $s \in \text{pred}(t)$ if and only if $C_t \subsetneq C_s$.

The first step is to define the root $T_0 := \{t_i \mid i < \mu(G)\}$, where G has $\mu(G)$ many components, $\{C_i \mid i < \mu(G)\}$. Define $V_{t_i} := \{x_i\}$, where x_i is the least element of C_i and $C_{t_i} := C_i$. We now continue the construction by levels.

If we are at a successor stage, i.e., we have defined the tree up to level $\alpha+1$, we do the following: for every node t in T_α consider the graph induced by $C_t \setminus V_t$ and let $\{C_i \mid i < \mu(t)\}$ be all of its components. In $T_{\alpha+1}$ the node t will have $\mu(t)$ many successors $\{t^i \mid i < \mu(t)\}$ and we put $C_{t^i} := C_i$. Let y be the least element of C_{t^i} (we assumed $V = \kappa$) and put $V_{t^i} := \{x^i_t\}$, where $x^i_t \in C_{t^i}$ is a vertex connected to V_t whose distance to y is minimal. If $C_t \setminus V_t$ is empty, t will have no successors in T.

At limit stages, we proceed similarly. Suppose we have defined the tree up to level α and α is a limit ordinal. We consider every branch b of $T_{<\alpha}$ and determine the set $\bigcap_{s\in b} C_s$. If it is empty, b will be a branch in T as well; otherwise, let $\{C_i \mid i < \mu(b)\}$ be all of its components. We put nodes $(b^i)_{i<\mu(b)}$ into T_α, all of them extending b and pairwise incomparable. Fix a $j < \mu(b)$. The set C_{b^j} is simply C_j again. It is enough now to define the set V_{b^j}.

Claim 3.21.1. *There is a subset V' of C_j inducing a connected subgraph in G of size at most* cf$|\alpha|$ *such that the set of nodes $\{t \in b | V_t$ is connected to V' in $G\}$ is cofinal in b.*

Proof. First, we show that $\{t \in b | V_t \text{ is connected to } C_j \text{ in } G\}$ is cofinal in b. Suppose s is a node in b. Note that $C_j \subseteq C_s$, so there must be vertices $u \in C_j$ and $v \in C_s \setminus C_j$ which are connected. As $v \notin C_j$ let $s' \in b$ be the least node such that $v \notin C_{s'}$, note that we have $u \in C_{s'}$.

First, suppose ht(s') is a successor ordinal, and the predecessor of s' is $\overline{s}$. We obtain that $u, v \in C_{\overline{s}}$ but $v \notin C_s$ and since u is connected to v in G we must have that $v \in V_{\overline{s}}$. Hence $V_{\overline{s}}$ is connected to C_j.

The case when the height of s' is limit, we obtain that $u, v \in \bigcap_{r\in \text{pred}(s')} C_r$. However, since $\{u, v\}$ forms an edge, both vertices must be contained in the same component of $\bigcap_{r\in \text{pred}(s')} C_r$, i.e., v would have to be an element of $C_{s'}$.

Let $b' \subseteq b$ be a cofinal subset of the branch of size cf$|\alpha|$ such that each $s \in b'$ has the property that V_s is connected to C_j. For each $s \in b'$, choose a witness $x^i_s \in C_j$ connected to V_s. Let V' be a minimal subset of C_i inducing a connected graph of size at most cf$|\alpha|$ containing all the vertices x^i_s. ☐

The claim defines the sets V_{b^i} and finishes the construction.

It is clear that in each step of the induction, we use up at least one vertex of G that we put in some V_t, so the length of the induction is some ordinal δ such that $|\delta| = \kappa$. However, note that at successor steps of the induction, we always chose a vertex with minimal distance from the least vertex, which was not part of some V_t defined before. This implies that the vertex $\alpha < \kappa$ was the least vertex at the latest in step $\omega \cdot \alpha$ of the induction and was put into some V_t after finitely many steps, i.e., α belongs to some V_t for a t such that $\mathrm{ht}(t) < \omega \cdot \alpha + \omega$. From this, we also obtain that the height of the constructed tree is at most κ.

We make a few observations. The sets $(V_t)_{t \in T}$ form a partition of V, and each V_t induces a connected subgraph of G. Also the size of the set V_t is bounded by $\mathrm{cf}(\mathrm{ht}(t))$, in particular $|V_t| < \kappa$ for each $t \in T$. The graph (T, F), where

$$\{s,t\} \in F \ \equiv \ V_s \text{ is connected to } V_t \text{ in } G$$

is a T-graph and also a minor of G.

To see that it is a T-graph, note that due to the way, we defined the sets V_t, it is clear that each t is connected to its predecessor by an F-edge. If the height of t is limit, the previous claim implies that it is cofinally often connected to its predecessors.

Thus it is enough to see that (T, F) is a subgraph of the comparability graph of T. Consider any $u, v \in G$ connected and the corresponding V_s and V_t so that $u \in V_s, v \in V_t$. Suppose s is incomparable with t.

In the first case, assume that there is a limit α and a branch b in $T_{<\alpha}$ such that there are nodes $\bar{s}$ and $\bar{t}$ directly above b such that $\bar{s} \leq s$ and $\bar{t} \leq t$. We have that $u, v \in \bigcap_{r \in b} C_r$. Since $\{u, v\}$ is an edge, both u and v belong to the same component of $\bigcap_{r \in b} C_r$ and by the second induction hypothesis we obtain that $\bar{s} = \bar{t}$.

The case when the split happens on a successor is proven analogously. Assume there is a node $t \in T$ with two successors $\bar{s}$ and $\bar{t}$ such that $\bar{s} \leq s$ and $\bar{t} \leq t$. Both u and v belong to C_t and since $\{u, v\} \cap V_t = \emptyset$ we obtain that u and v belong to the same component of $C_t \setminus V_t$, thus $\bar{s} = \bar{t}$ by the first induction hypothesis. □

Remark. We will refer to T_G as the *partition tree of G*.

In what follows, we will analyze this construction and prove its usefulness. We will reference objects from the construction without explicitly defining them.

Proposition 3.22. *Suppose κ is an uncountable regular cardinal. If G is a graph of size κ, then G has a K_κ minor if and only if T_G has a κ-branch.*

Proof. Assume first that G has a K_κ minor. Suppose $\{U_\alpha \mid \alpha < \kappa\}$ are the disjoint sets of vertices forming a K_κ minor. For each α we define a node in the tree T_G, put $t_\alpha := \min \{t \in T_G \mid U_\alpha \cap V_t \neq \emptyset\}$.

Claim 3.22.1. *For each α, the node t_α is well-defined.*

Proof. Suppose s, t are incomparable such that $U_\alpha \cap V_s \neq \emptyset$ and also $U_\alpha \cap V_t \neq \emptyset$. Let x_s be an element of $U_\alpha \cap V_s$ and x_t an element of $U_\alpha \cap V_t$. As U_α is connected, there is a finite path $(x_i)_{i<n}$ in U_α such that $x_0 = x_s$ and $x_{n-1} = x_t$. To each x_i, we can associate a node r_i such that $x_i \in V_{r_i}$ (this mapping need not be injective). Since x_i is connected to x_{i+1} we obtain that r_i is comparable with r_{i+1} for each $i < n-1$. There must exist some r_j such that $r_j \leq r_i$ for each $i \neq j$. We can prove this by induction. If $n = 3$, this is clear. Suppose $k < n$ and $(r_i)_{i<k}$ has a least element, say r_l. Now $r_l \leq r_{k-1}$ and r_k is comparable with r_{k-1}, hence r_k is comparable with r_l and the least element of $(r_i)_{i \leq k}$ is $\min \{r_k, r_l\}$.

Let r_j be the least element of $(r_i)_{i<n}$, then $r_j \leq s, t$ and $x_j \in V_{r_j} \cap U_\alpha$. □

We claim that for every $\alpha, \beta < \kappa$, the nodes t_α and t_β are comparable. If α and β are given consider the nodes $s_\alpha, s_\beta \in T_G$ such that $x \in U_\alpha$ and $y \in U_\beta$ are connected in G and $x \in V_{s_\alpha}$ and $y \in V_{s_\beta}$, so s_α is comparable with s_β. We have that $t_\alpha, t_\beta \leq \max \{s_\alpha, s_\beta\}$, hence t_α and t_β are comparable. We are almost done but notice that the mapping $\alpha \mapsto t_\alpha$ need not be injective. However, as the sets $\{U_\alpha \mid \alpha < \kappa\}$ are pairwise disjoint and the sets V_t have size less than κ, this mapping has the property that the preimage of each node has size $< \kappa$ and so there must be κ many unique nodes t_α all comparable to each other forming a branch in T_G.

If T_G has a κ-branch then the graph (T, F) has a K_κ minor, see Proposition 3.20. By transitivity, G also has a K_κ minor. □

Remark. The uncountability of κ is only used in the implication from left to right.

Proposition 3.23. *Suppose $\kappa \geq \lambda$ are infinite cardinals, and κ is uncountable. If G is a graph of size κ and T_G is λ-special, then G has chromatic number at most $\lambda \cdot \sup \{\mathrm{cf}(\mathrm{ht}(t)) \mid t \in T_G\}$.*

Proof. Suppose T_G is λ-special, i.e. there is a specializing function $f : T_G \to \lambda$. Let $\eta := \sup \{|V_t| \mid t \in T_G\}$ and $\{A_\alpha \mid \alpha < \lambda\}$ pairwise disjoint η-sized subsets of $\lambda \cdot \eta$. For every $t \in T_G$ let $g_t : V_t \to A_{f(t)}$ be any injection and define a coloring $c : G \to \lambda \cdot \eta$ as follows: for every $u \in G$ there is a unique $t \in T_G$ such that $u \in V_t$, let $c(u)$ be $g_t(u)$. To see that c is proper, consider any two vertices $u, v \in G$ which are connected. If both are in the same V_t, then as g_t is injective, they get different colors. Otherwise, there are different $s, t \in T$ so that $u \in V_s$ and $v \in V_t$. Since (T, F) is a T-graph, we get that $s < t$ or vice versa. Then however $f(s) \neq f(t)$, and subsequently $c(u) \in A_{f(s)}$ and $c(v) \in A_{f(t)}$. The color of u is again different from the color of v. Thus the chromatic number of G is at most $\lambda \cdot \eta$. □

The implication of the last proposition cannot be reversed.

Proposition 3.24. *There is a countably chromatic graph such that T_G has an uncountable branch.*

Proof. Consider the complete graph on ω_1 vertices and subdivide each edge once. This is a bipartite graph; hence 2-colorable, and from Proposition 3.22, we obtain that T_G has an ω_1-branch. □

Proposition 3.25. *Suppose $\kappa \geq \lambda$ are infinite cardinals, and κ is uncountable. If G is a graph of size κ and T_G has an antichain of size λ, then G has an independent set of size λ.*

Proof. Let $\{t_i \mid i \in \lambda\}$ be an antichain in T_G and consider the sets $\{V_{t_i} \mid i \in \lambda\}$. We showed that if s is incomparable with t, then there is no edge connecting the sets V_s and V_t, thus choose any $x_i \in V_{t_i}$ as all of these are non-empty, now it is clear that $\{x_i \mid i \in \lambda\}$ forms an independent set in G. □

As with chromaticity, the other implication does not hold in the case of independent sets either.

Proposition 3.26. *There is a graph with an independent set of size ω_1 such that T_G is isomorphic to $(\omega_1, \in)$.*

Proof. The witness is simply the complete bipartite graph with both partitions of size ω_1. Clearly, either partition forms an independent set of size ω_1. Note that this graph is ω_1-connected. Following the construction of T_G, it is easy to see that ω_1-connectedness implies that at no step of the construction does the graph split, i.e., the tree T_G is simply a single ω_1-branch. □

Proposition 3.27. *Suppose $\kappa \geq \lambda$ are regular cardinals. If G is a (κ, λ)-connected graph of size κ, then T_G has levels of size $< \lambda$.*

Proof. We will proceed by induction, clearly by our assumption that the set of roots of T_G has size $< \lambda$. Let $\alpha < \kappa$ and consider the level $(T_G)_\alpha$. Take the union $\bigcup_{t \in (T_G)_{<\alpha}} V_t$ and note that by the construction of T_G and the induction hypothesis, this set has size $< \kappa$. Removing these vertices from G leaves us with less than λ many components. Hence at stage α in the construction of T_G, there are fewer than λ many components to consider and hence less than λ many nodes to extend $(T_G)_{<\alpha}$. □

Corollary 3.28. *Suppose κ is an uncountable regular cardinal. If G is a graph of size κ and there exists a cardinal $\lambda < \kappa$ such that G is (κ, λ)-connected, then K_κ is a minor of G.*

Proof. Given G with these properties, consider the tree T_G. The size of the levels of this tree is less than λ, but the size of the entire tree is κ. By a result of Kurepa [Tod84, Theorem 2.7.] each tree of height κ whose levels have size less than λ has a cofinal branch, hence T_G has a branch of size at least κ so by Proposition 3.20 G has a K_κ minor. □

Proposition 3.29. *Suppose κ is an uncountable regular cardinal. If G is a graph of size κ, then G is (κ, κ)-connected if and only if T_G is a κ-tree.*

Proof. If G is (κ, κ)-connected of size κ with no K_κ minor, then by the previous proposition T_G has levels of size less than κ. We also have that T_G has height κ as G has size κ, hence T_G is a κ-tree.

To see the converse holds as well, T_G being a κ-tree implies that G is (κ, κ)-connected. If less than κ many vertices are removed from G, then there is a level $\alpha < \kappa$ such that all of these vertices are contained in the sets V_t for $t \in (T_G)_{<\alpha}$. However, as T_G has levels of size less than κ, removing all of these vertices leaves us with less than κ many cones in T_G. These define connected subgraphs of G. □

Proposition 3.30. *Suppose κ is a regular cardinal. If G is a graph of size κ, then G has a κ-Kurepa graph minor family of size μ if and only if T_G has at least μ many κ-branches.*

Proof. Suppose a graph G with the required properties is given. Each minor defines a κ-branch in T_G by Proposition 3.22. Each pair of minors is separated by a set X of size less than κ. Thus we obtain that there is some $\alpha < \kappa$ such that the set of nodes t with the property that V_t intersects X lie in $(T_G)_{<\alpha}$. This implies that the branches defined from the minors are indeed different.

For the other direction, we again use Proposition 3.22 to define the minors. Let b, c be any two distinct branches and note that $b \cap c \subseteq (T_G)_{<\alpha}$ for some $\alpha < \kappa$. It is enough to observe that $\bigcup\{V_t \mid t \in b \cap c\}$ has size less than κ, and this set separates the minors defined from b and c. □

Analogously as in Corollary 3.19, we obtain the following using the previous propositions.

Corollary 3.31. *Suppose κ is a regular cardinal. If G is a graph of size κ and T_G is its partition tree, then the following hold:*

1. *G has no K_κ minor and is (κ, κ)-connected if and only if T_G is a κ-Aronszajn tree,*
2. *if G has no K_{κ^+} minor and its chromatic number is κ^+, then T_G is a non-κ-special tree of height κ^+ with no cofinal branch,*
3. *G is (κ, κ)-connected and has a κ-Kurepa graph minor family of size κ if and only if T_G is a κ-Kurepa tree,*
4. *if G has no K_κ minor and no independent set of size κ, then T_G is a κ-Suslin tree.*

3.2.3 $\mathfrak{hc} = \mathfrak{st}$

The previous section presents a complete picture of the Hadwiger conjecture on ω_1.

Theorem 3.32. $\mathfrak{hc} = \mathfrak{st}$.

Proof. By Proposition 3.12, we have $\mathfrak{hc} \leq \mathfrak{st}$. Hence we only need to show the other inequality. Given an uncountably chromatic graph of size $\mathfrak{hc}$ which does not have a K_{ω_1} minor, use Theorem 3.21 to construct the tree T_G. The size of T_G is clearly at most $\mathfrak{hc}$. Now by Corollary 3.31(2), the tree T_G is a non-special tree with no uncountable branch, hence $\mathfrak{st} \leq \mathfrak{hc}$. □

3.2.4 The κ-Hadwiger Conjecture

We used Corollary 3.31 to prove that $\mathfrak{hc} = \mathfrak{st}$. Additionally, this corollary clarifies what models of the Hadwiger conjecture on higher cardinals look like.

Definition 3.33. The κ-Hadwiger conjecture states that every graph of size κ whose chromatic number is κ has a K_κ minor.

The case when $\kappa = \omega_1$ consistently holds, see Theorem 3.3, and by our result is equivalent to $\mathfrak{st} > \omega_1$. The conjecture always fails for κ limit, as shown in Proposition 3.2. The generalized continuum hypothesis implies that the κ-Hadwiger conjecture fails for every infinite κ; this follows from Theorem 3.4.

Using the technique of Laver and Shelah (see the closing remarks in [LS81]), we get a model where each tree with no κ^+-branch of size κ^+ is κ-special, except possibly at successors of singular cardinals. By Corollary 3.31, this also models the κ^+-Hadwiger conjecture.

Theorem 3.34. *Suppose κ is a regular cardinal. It is consistent that the κ^+-Hadwiger conjecture holds.*

3.2.5 Graph Characterizations of Trees

Using Corollaries 3.19 and 3.31, we quickly obtain the following graph characterizations of well-known classes of trees.

Theorem 3.35. *Suppose κ is a regular cardinal.*

1. *The existence of a κ-Suslin tree is equivalent to the existence of a graph of size κ, which has no independent set of size κ and has no K_κ minor,*

2. *the existence of a κ-Aronszajn tree is equivalent to the existence of a (κ, κ)-connected graph of size κ, which has no K_κ minor,*

3. *the existence of a κ-Kurepa tree is equivalent to the existence of a (κ, κ)-connected graph of size κ, which has a κ-Kurepa minor family of size at least κ^+.*

Remark. We would like to point out that 1 has been independently discovered by Komjáth and Shelah in their recent paper [KS21]. It is also worth mentioning that it answers a question of Erdős and Hajnal [EH64]: if κ is uncountable, is it true that every graph on κ either contains an independent set of size κ or a K_κ minor? The fact that the answer is always negative for κ singular can be proven in the same way as Proposition 3.2.

Bibliography

[Bau76] James E. Baumgartner. Almost-disjoint sets, the dense set problem and the partition calculus. *Ann. Math. Logic*, 9:401–439, 1976.

[BD94] Jean-Michel Brochet and Reinhard Diestel. Normal tree orders for infinite graphs. *Trans. Amer. Math. Soc.*, 345(2):871–895, 1994.

[BŠ86] Bohuslav Balcar and Petr Štěpánek. *Teorie množin.* Academia [Publishing House of the Czech Academy of Sciences], Prague, 1986.

[Coh63] Paul Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[Die00] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2000.

[DM41] Ben Dushnik and Edwin W. Miller. Partially ordered sets. *Amer. J. Math.*, 63(3):600–610, 1941.

[EH64] Paul Erdős and András Hajnal. On complete topological subgraphs of certain graphs. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 7:143–149, 1964.

[EH74] Paul Erdős and András Hajnal. Unsolved and solved problems in set theory. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 269–287. Amer. Math. Soc., Providence, R.I., 1974.

[ER56] Paul Erdős and Richard Rado. A partition calculus in set theory. *Bull. Amer. Math. Soc.*, 62(5):427–489, 1956.

[ET61] Paul Erdős and Alfred Tarski. On some problems involving inaccessible cardinals. In *Essays on the foundations of mathematics*, pages 50–82. Magnes Press, Hebrew Univ., Jerusalem, 1961.

[Haj60] András Hajnal. Some results and problems on set theory. *Acta Math. Acad. Sci. Hungar.*, 11:277–298, 1960.

[Hal17] Lorenz J. Halbeisen. *Combinatorial set theory.* Springer Monographs in Mathematics. Springer, Cham, 2017. With a gentle introduction to forcing, Second edition.

[HK84] András Hajnal and Péter Komjáth. What must and what need not be contained in a graph of uncountable chromatic number? *Combinatorica*, 4(1):47 – 52, 1984.

[HL10] András Hajnal and Jean A. Larson. Partition relations. In *Handbook of set theory. Vols. 1, 2, 3*, pages 129–213. Springer, Dordrecht, 2010.

[HM75] András Hajnal and Attila Máté. Set mappings, partitions, and chromatic numbers. In *Logic Colloquium '73 (Bristol, 1973)*, Studies in Logic and the Foundations of Mathematics, Vol. 80, pages 347–379. North-Holland, Amsterdam, 1975.

[Jec71] Thomas Jech. Trees. *J. Symbolic Logic*, 36:1–14, 1971.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[JW97] Winfried Just and Martin Weese. *Discovering modern set theory. II*, volume 18 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Set-theoretic tools for every mathematician.

[Kom80] Péter Komjáth. A note on Hajnal-Máté graphs. *Studia Sci. Math. Hungar.*, 15(1-3):275–276, 1980.

[Kom84] Péter Komjáth. A second note on Hajnal-Máté graphs. *Studia Sci. Math. Hungar.*, 19(2-4):245–246, 1984.

[Kom89] Péter Komjáth. Third note on Hajnal-Máté graphs. *Studia Sci. Math. Hungar.*, 24(4):403–406, 1989.

[Kom13] Péter Komjáth. Erdős's work on infinite graphs. In *Erdős centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 325–345. János Bolyai Math. Soc., Budapest, 2013.

[Kom17] Péter Komjáth. Hadwiger's conjecture for uncountable graphs. *Abh. Math. Semin. Univ. Hambg.*, 87(2):337–341, 2017.

[KS88] Péter Komjáth and Saharon Shelah. Forcing constructions for uncountably chromatic graphs. *J. Symbolic Logic*, 53(3):696–707, 1988.

[KS21] Péter Komjáth and Saharon Shelah. Monocolored topological complete graphs in colorings of uncountable complete graphs. *Acta Math. Hungar.*, 163(1):71–84, 2021.

[KT06] Péter Komjáth and Vilmos Totik. *Problems and theorems in classical set theory*. Problem Books in Mathematics. Springer, New York, 2006.

[Kun80] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.

[Kun11] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[Lav75] Richard Laver. Partition relations for uncountable cardinals $\leq 2^{\aleph_0}$. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday)*, volume 2, page 1029–1042, Amsterdam, 1975. North-Holland.

[LH20a] Chris Lambie-Hanson. Disjoint type graphs with no short odd cycles. `https://arxiv.org/pdf/2004.04189.pdf`, 2020. Unpublished note. Accessed: 2023-04-27.

[LH20b] Chris Lambie-Hanson. On the growth rate of chromatic numbers of finite subgraphs. *Adv. Math.*, 369:107176, 13, 2020.

[LH22] Chris Lambie-Hanson. Higher-dimensional delta-systems. *Order*, 2022.

[LHS21] Chris Lambie-Hanson and Dániel T. Soukup. Extremal triangle-free and odd-cycle-free colourings of uncountable graphs. *Acta Math. Hungar.*, 163(1):174–193, 2021.

[LS81] Richard Laver and Saharon Shelah. The $\aleph_2$-Souslin hypothesis. *Trans. Amer. Math. Soc.*, 264(2):411–417, 1981.

[Pit22] Max Pitz. Applications of order trees in infinite graphs. *Order*, 2022.

[Ram30] Frank P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 2(30):264–286, 1930.

[Rin12] Assaf Rinot. Dushnik-Miller for regular cardinals (part 2). `https://blog.assafrinot.com/?p=652`, 2012. Blog post. Accessed: 2023-04-27.

[RT18] Dilip Raghavan and Stevo Todorčević. Suslin trees, the bounding number, and partition relations. *Israel J. Math.*, 225:771–796, 2018.

[Sie33] Wacław Sierpiński. Sur une probleme de la theorie des relations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2(2):285–287, 1933.

[Sou11] Lajos Soukup. Elementary submodels in infinite combinatorics. *Discrete Math.*, 311(15):1585–1598, 2011.

[Sou15a] Dániel T. Soukup. Trees, ladders and graphs. *J. Combin. Theory Ser. B*, 115:96–116, 2015.

[Sou15b] Dániel T. Soukup. Open problems around uncountable graphs. `https://danieltsoukup.github.io/academic/norwich_handout.pdf`, 2015. Unpublished note. Accessed: 2023-04-27.

[Swi23] Corey B. Switzer. The special tree number. `https://arxiv.org/pdf/2203.04186.pdf`, 2023. Preprint. Accessed: 2023-04-27.

[Tod83] Stevo Todorčević. Forcing positive partition relations. *Trans. Amer. Math. Soc.*, 280(2):703–720, 1983.

[Tod84] Stevo Todorčević. Trees and linearly ordered sets. In *Handbook of set-theoretic topology*, pages 235–293. North-Holland, Amsterdam, 1984.

[Tod86] Stevo Todorčević. Reals and positive partition relations. In *Logic, methodology and philosophy of science, VII (Salzburg, 1983)*, volume 114 of *Stud. Logic Found. Math.*, pages 159–169. North-Holland, Amsterdam, 1986.

[Tod89a] Stevo Todorčević. *Partition problems in topology*, volume 84 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[Tod89b] Stevo Todorčević. *A partition calculus in set theory*, volume 84 of *Contemporary Mathematics*. Amer. Math. Soc., Providence, RI, 1989.

[Tod21] Stevo Todorčević. Erdős-Kakutani phenomena for paths. *Acta Math. Hungar.*, 163(1):168–173, 2021.

[vdZ13] Dominic van der Zypen. Hadwiger's conjecture for graphs with infinite chromatic number. *Adv. Develop. Math. Sci.*, 4:1–4, 2013.

[Viz64] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3:25–30, 1964.

[Wal88] David J. Walker. On the transversal hypothesis and the weak Kurepa hypothesis. *J. Symbolic Logic*, 53(3):854–877, 1988.

List of publications

1. On a Lindenbaum composition theorem. (with Jaroslav Šupina) *Tatra Mountains Mathematical Publications*, 74:145–158, 2019.

2. The Uncountable Hadwiger Conjecture and Characterizations of Trees Using Graphs. 2022. — Submitted

3. Complete Bipartite Partition Relations in Cohen Extensions. 2023. — Submitted