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Classes of modules arising in algebraic geometry

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Abstract: This thesis summarises the author's results in representation theory of rings and schemes, obtained with several collaborators. First, we show that for a quasicompact semiseparated scheme X, the derived category of very flat quasicoherent sheaves is equivalent to the derived category of flat quasicoherent sheaves, and if X is affine, this is further equivalent to the homotopy category of projectives. Next, we prove that if R is a commutative Noetherian ring, then every countably generated flat module is *quite flat*, i.e., a direct summand of a transfinite extension of localizations of R in countable multiplicative subsets. Further, we investigate the relations between the geometric and categorical purity in categories of sheaves; we give a characterization of indecomposable geometric pure-injectives in both the quasicoherent and non-quasicoherent case. In particular, we describe the Ziegler spectrum and its geometric part for the category of quasicoherent sheaves on the projective line over a field. The final result is the equivalence of the following statements for a quasicompact quasiseparated scheme X: (1) the category $\mathrm{QCoh}(X)$ of all quasicoherent sheaves on X has a flat generator; (2) for every injective object $\mathcal{E}$ of $\mathrm{QCoh}(X)$, the internal Hom functor into $\mathcal{E}$ is exact; (3) for some injective cogenerator $\mathcal{E}$ of $\mathrm{QCoh}(X)$, the internal Hom functor into $\mathcal{E}$ is exact; (4) the scheme X is semiseparated.

Keywords: flat module, quasicoherent sheaf, transfinite extension, Quillen equivalence, Ziegler spectrum

Contents

1. Introduction to the thesis

It is a known fact that coherent sheaves on a scheme X carry a lot of information about X. The advantage of coherent sheaves over taking vector bundles (i.e. locally projective sheaves) only is that the category of coherent sheaves is abelian—an additive category with well-behaved kernels and cokernels of morphisms.

Similarly to the case of modules over a ring, when one drops finiteness conditions on modules, we can also relax our conditions on representations on X, obtaining *quasicoherent sheaves.* Let $\mathcal{O}_X$ denote the structure sheaf of the scheme X. Then a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is called *quasicoherent* if for every pair of open affine subsets $U, V \subseteq X$, $U \subseteq V$, the natural map

$$\mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U) \xrightarrow{\operatorname{res}^{\mathcal{M}}_{UV} \otimes \mathcal{O}_X(U)} \mathcal{M}(U) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U),$$

obtained by tensoring the restriction map $\mathcal{M}(V) \to \mathcal{M}(U)$ with $\mathcal{O}_X(U)$, is an isomorphism of $\mathcal{O}_X(U)$-modules. In this thesis we adopt the convention that instead of "quasicoherent sheaf of $\mathcal{O}_X$-modules" we simply write "quasicoherent sheaf on X".

The category $\mathrm{QCoh}(X)$ of all quasicoherent sheaves on X is also well behaved: It is a Grothendieck category [2], which is further locally finitely presented when X is "nice enough", i.e. quasicompact and quasiseparated. Let us recall here that a scheme is called *quasicompact* if the underlying, not necessarily Hausdorff topological space satisfies that each cover by open sets has a finite subcover. A *quasiseparated* scheme is one with the property that the intersection of two quasicompact open subsets is quasicompact.

If X is affine, i.e. isomorphic to the spectrum of a commutative ring R, then $\mathrm{QCoh}(X)$ is equivalent to the category R-Mod of all R-modules, which further supports the claim that $\mathrm{QCoh}(X)$ is the correct category assigned to X. However, in general, $\mathrm{QCoh}(X)$ lacks some properties of R-Mod: Most importantly, the category may not have enough projective objects, or even not any non-zero projective objects at all. This is closely related to the fact that (infinite) direct products are not necessarily exact in $\mathrm{QCoh}(X)$ [13].

The lack of projectives in $\mathrm{QCoh}(X)$ has led to search for replacements; for instance, the thesis [15] used *flat* sheaves to construct a triangulated category generalising the homotopy category of projective modules over a commutative ring. Here we need to clarify which notion of flatness we have in mind: A quasicoherent sheaf $\mathcal{M}$ on X is called *flat* if for every open affine set $U \subseteq X$, the module of sections $\mathcal{M}(U)$ is a flat $\mathcal{O}_X(U)$-module. Note that by standard facts from commutative algebra, this sort of flatness can be tested stalkwise, and as such it it also a Zariski-local property: A quasicoherent sheaf $\mathcal{M}$ on X is flat if and only if for some (equivalently, for any) open affine cover $(U_i)_{i \in I}$ of X, the modules $\mathcal{M}(U_i)$ are flat over their respective rings.

However, even in the affine case, the structure of flat modules can be complicated. Unless the ring is noetherian of finite Krull dimension, the projective dimension of flat modules can be arbitrarily large. Hence it is of interest whether one can work with a suitable smaller class instead.

A promising candidate for this refinement is the class of *very flat modules*, first introduced by Positselski in [17]. To introduce it properly, let us first recall several notions from homological algebra.

1.1 Cotorsion pairs and approximations

Let $\mathcal{G}$ be a category and $\mathcal{A} \subseteq \mathcal{G}$ a class (or subcategory) of objects. We say that a map $f\colon A \to M$ in $\mathcal{G}$ is an *$\mathcal{A}$-precover* of M provided that $A \in \mathcal{A}$ and for every $A' \in \mathcal{A}$, the map

$$\operatorname{Hom}_{\mathcal{G}}(A', A) \xrightarrow{\operatorname{Hom}_{\mathcal{G}}(A', f)} \operatorname{Hom}_{\mathcal{G}}(A', M)$$

is surjective. A class $\mathcal{A} \subseteq \mathcal{G}$ is called *precovering* if every $M \in \mathcal{G}$ has an $\mathcal{A}$-precover. Dualizing the definition, we obtain the notion of an *$\mathcal{A}$-preenvelope*: A map $f\colon M \to A$ such that $A \in \mathcal{A}$ and for every $A' \in \mathcal{A}$, the map

$$\operatorname{Hom}_{\mathcal{G}}(A, A') \xrightarrow{\operatorname{Hom}_{\mathcal{G}}(f, A')} \operatorname{Hom}_{\mathcal{G}}(M, A')$$

is surjective. A *preenveloping* class is defined in the obvious way.

An $\mathcal{A}$-precover $f\colon A \to M$ is called an *$\mathcal{A}$-cover* if any $g\colon A \to A$ satisfying $fg = f$ is an automorphism of A. This leads to a notion of a *covering* class and the dual notions of an *$\mathcal{A}$-envelope* and an *enveloping* class.

Let now $\mathcal{G}$ be an abelian category and $\mathcal{A} \subseteq \mathcal{G}$. Define

$$\begin{aligned} {}^{\perp}\mathcal{A} &\stackrel{\text{def}}{=} \{M \in \mathcal{G} \mid \operatorname{Ext}^1_{\mathcal{G}}(M, A) = 0 \text{ for all } A \in \mathcal{A}\}, \\ \mathcal{A}^{\perp} &\stackrel{\text{def}}{=} \{M \in \mathcal{G} \mid \operatorname{Ext}^1_{\mathcal{G}}(A, M) = 0 \text{ for all } A \in \mathcal{A}\}. \end{aligned}$$

It is not hard to see that an epimorphism $f\colon A \to M$ with $A \in \mathcal{A}$ and $\operatorname{Ker} f \in \mathcal{A}^{\perp}$ is an $\mathcal{A}$-precover; such a precover is called *special*. Dually, a monomorphism into an object of $\mathcal{A}$ with cokernel in ${}^{\perp}\mathcal{A}$ is a *special $\mathcal{A}$-preenvelope*.

Note that some authors call precovers and preenvelopes *right and left approximations*, respectively. The stronger versions without the prefix pre- are then called *minimal*.

The natural setting for constructing approximations is provided by cotorsion pairs. Let $\mathcal{G}$ be an abelian category. A pair $(\mathcal{A}, \mathcal{B})$ of subclasses $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$ is called a *cotorsion pair* (or a *cotorsion theory*) provided that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. Having a class of objects $\mathcal{S} \subseteq \mathcal{G}$, one can obtain a cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ which is said to be *generated by $\mathcal{S}$*.

Assume further that $\mathcal{G}$ is a Grothendieck category. The situation most suitable to work with is when $\mathcal{S}$ is only a set[1] of objects (as opposed to a proper class), for in such a case, the right class is always special preenveloping (see [1] for the module case). If, further, the left class contains a generator[2] of $\mathcal{G}$ (which

[1] A class containing only a set of isomorphism types of objects would work, too.

[2] Recall that a *generator* is an object G such that every object of the category is the epimorphic image of the direct sum of copies of G. This differs from the definition in the context of general category theory, but it is not hard to prove the equivalence of the two notions in the case of Grothendieck categories.

is always the case for the category of modules over a ring), then it is special precovering. Such cotorsion pairs are called *complete*.

A subclass need not be a left class of a cotorsion pair in order to be precovering, any deconstructible class will work as well. Let $\mathcal{E}$ be a class of objects; then an object $M \in \mathcal{G}$ is *$\mathcal{E}$-filtered* if there is an ordinal σ and a well-ordered chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M such that $M_0 = 0$, $M_\sigma = M$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each α limit ordinal $\leq \sigma$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{E}$ for each $\alpha < \sigma$. The chain $(M_\alpha \mid \alpha \leq \sigma)$ is called an *$\mathcal{E}$-filtration* of M. In addition, we say that M is a *transfinite extension of elements of* $\mathcal{E}$. The class of all $\mathcal{E}$-filtered objects is denoted by $\mathrm{Filt}(\mathcal{E})$. Classes of the form $\mathrm{Filt}(\mathcal{E})$ for $\mathcal{E}$ a set of objects are called *deconstructible* and by [27], such classes are always precovering.

If $\mathcal{G}$ is the category of modules over a ring and $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair (not necessarily generated by a set), then by Salce Lemma [9, Lemma 5.20], $\mathcal{A}$ is special precovering if and only if $\mathcal{B}$ is special preenveloping. Further, if $\mathcal{S} \subseteq \mathcal{G}$ is a set, then the class ${}^\perp(\mathcal{S}^\perp)$ is always deconstructible.

In the more general situation, the situation is more opaque. Recall that a scheme is *semiseparated* if the intersection of any two open affine subsets is again open affine. If X is a quasicompact quasiseparated scheme, which is not semiseparated (and, consequently, not affine), then the class of very flat quasicoherent sheaves is a left class of a cotorsion pair, but by the results of Chapter 5, it does *not* contain a generator of the category $\mathrm{QCoh}(X)$. However, this class *is* deconstructible by the results of [4] and is therefore precovering; note, however, that the precovers cannot be special in general, since they are not epimorphisms. On the other hand, if X *is* semiseparated, then $\mathrm{QCoh}(X)$ has very flat generators [17, Lemma 4.1.1], so the corresponding cotorsion pair is complete.

1.2 Abelian model category structures

As described in [11], cotorsion pairs are also an important ingredient for *abelian model category structures*. Such a model structure on a abelian category $\mathcal{G}$ is completely determined by a triple (called a *Hovey triple*) of classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, the objects of which are called *cofibrant*, *trivial* (also sometimes called *acyclic*), and *fibrant*, respectively.[3] To obtain a model structure in the original sense of Quillen [24], one defines (trivial) cofibrations as monomorphisms with (trivial) cofibrant cokernel, (trivial) fibrations as epimorphisms with (trivial) fibrant kernel, and weak equivalences as maps, which can be factored as pi, where p is a trivial fibration and i a trivial cofibration.

To go the other way, cofibrant objects are precisely those objects C such that the map $0 \to C$ is a cofibration, fibrant objects F are such that $F \to 0$ is a fibration, and trivial objects W satisfy that $0 \to W$ (equivalently, $W \to 0$) is a weak equivalence.

What conditions do the classes in a Hovey triple have to obey? By [11, Theorem 2.2] $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ corresponds to an abelian model category if and only if

[3] To be precise, [11] actually defines abelian model structures in a wider sense; one can also limit which exact sequences are "taken into account", i.e. pick a *proper class* of exact sequences and do all the computations with respect to this class. Although this approach allows more model structures to fit into the "abelian" framework, as [11, Examples 3.6 & 3.7] show, we will not need it here and so we stick to the simpler definition presented above.

(1) $\mathcal{W}$ is a thick subcategory and (2) the pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are hereditary complete cotorsion pairs in $\mathcal{G}$; recall that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* if $\mathrm{Ext}^i_{\mathcal{G}}(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $i \geq 1$.

The paper [11] lists many useful examples of model categories which are, in fact, abelian model categories. In particular, when R is a ring, its derived category (i.e. the derived category of R-Mod) arises as the homotopy category of several abelian model category structures on $\mathrm{Ch}(R)$, the category of unbounded complexes of R-modules with chain maps.

However, what we will need here is an extension of this notion due to Gillespie [6]: The *exact model structure*, which just carries Hovey's construction from abelian categories to exact categories[4] (in the sense of Quillen [23]); note that the definition of cotorsion pair makes sense in exact categories, too. This extension is fruitful in the sense that it allows to realise derived categories of exact categories as homotopy categories of exact model categories. In particular, Gillespie in [7] shows the way to construct recollements like the one of Murfet [15] using exact model structures.

Exact model structures serve us as a tool for proving the Quillen equivalence of Theorem A below.

1.3 Very flat modules and quasicoherent sheaves

Let R be a commutative ring and let $s \in R$. Then by $R[s^{-1}]$ we denote the localization of R in the multiplicative subset $\{1, s, s^2, \dots\}$. Let $(\mathcal{VF}, \mathcal{CA})$ be the cotorsion pair generated by the set of R-modules $\{R[s^{-1}] \mid s \in R\}$ (in the sense defined above); then, following [17], the modules in $\mathcal{VF}$ are called *very flat* and the ones in $\mathcal{CA}$ are *contraadjusted*. Because $R[s^{-1}]$ is a flat R-module of projective dimension ≤ 1 for each $s \in R$, all very flat modules are flat of projective dimension ≤ 1. In particular, the cotorsion pair is hereditary.

Note the geometric origin of the definition: $R[s^{-1}]$ is precisely the ring of sections on the principal open affine set $D(s)$ ("where s does not vanish"). Hence, speaking very informally, to construct $\mathcal{VF}$ one "starts with the basic ingredients of algebraic geometry and builds a homologically well-behaved class upon them". Then [17, Lemma 1.2.4] tells us that the resulting class encompasses the desired algebro-geometric examples: The ring of sections on *any* open affine subset of an affine scheme is a very flat module over the ring of global sections.

In fact, a host of modules naturally arising in algebraic geometry turns out to be very flat: In the author's joint paper with Positselski [19] (the results of which are not contained in this thesis), it is shown that any finitely presented commutative R-algebra, which is flat as an R-module, or more generally, any finitely presented module over such an algebra, is a very flat R-module.

While very flatness cannot be tested stalkwise, it is a Zariski-local property, hence one can define *very flat quasicoherent sheaves* by requiring them to be very flat on each open affine set, and the result is a well-behaved class. In fact, it

[4]In fact, the exact categories that Gillespie works with need to be *weakly idempotent complete*, meaning that every split monomorphism has a cokernel and every split epimorphism has a kernel.

is again a deconstructible, left class of a hereditary cotorsion pair generated by a set of sheaves. As outlined above, the completeness of this cotorsion pair is equivalent to the scheme being semiseparated (under the assumption that it is quasicompact and quasiseparated).

We refer the reader to the introduction of the paper [19] for a nice overview of properties that very flat modules and quasicoherent sheaves enjoy. The property we discuss in this thesis is the following:

Theorem A. *Let X be a quasicompact and semiseparated scheme. Then the derived category of the exact category of very flat quasicoherent sheaves is triangle equivalent to the derived category of the exact category of flat quasicoherent sheaves.*

This is a result of the joint paper with Estrada [5], which is the basis of the Chapter 2. More precisely, we realize these derived categories as homotopy categories of certain abelian model categories and obtain Quillen equivalences between them, using the methods of Gillespie [6, 7, 8]. Note that the derived category of flats is just another name for Murfet's mock homotopy category of projectives introduced in [15].

Actually, [5] lists conditions which are sufficient for a subclass of flat quasicoherent sheaves to have its derived category (Quillen) equivalent to the derived category of flats. Further, under the assumption that the corresponding cotorsion pair is complete, one can also use the class of (infinite-dimensional) vector bundles; a particularly interesting fact here is that the proof actually shows the Quillen equivalence with the derived category of very flats and we do not know a way to avoid this detour.

1.4 Quite flat modules

Another refinement of the class of flat modules is considered in Chapter 3, although this time only in the affine setting. Let R be a commutative ring. The class of quite flat modules is obtained in a way similar to the very flat modules; namely, one starts with the set

$$\{S^{-1}R \mid S \subseteq R \text{ is a countable multiplicative subset}\}$$

and considers the cotorsion pair which it generates. Then the modules in the left class are called *quite flat* and the right class consists of *almost cotorsion* modules. This definition comes from the joint paper with Positselski [18], which is not contained in this thesis, but some of its results are used in Chapter 3. This chapter is a part of yet another joint paper with Hrbek and Positselski [12].

One of the results of [18] is the following: If R is noetherian and its spectrum is (at most) countable, then every flat R-module is quite flat. Chapter 3 reproves this fact using another, new result:

Theorem B. *Let R be a commutative noetherian ring and F a countably generated flat module. Then F is quite flat.*

The previous result is then recovered by directly proving that for a commutative noetherian ring with countable spectrum, every flat module is a transfinite extension of countably generated flat modules. In fact, more is true:

Theorem C. *Let R be a noetherian commutative ring with spectrum of cardinality less than κ, where κ is an uncountable regular cardinal. Then every flat module is a transfinite extension of $< \kappa$-generated flat modules.*

1.5 Purity in categories of sheaves

Chapter 4 takes another approach to the category of quasicoherent sheaves. The starting point is the notion of purity, which has established as a prominent subject related to categories of modules. Recall that a short exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is called *pure-exact* if for every finitely presented left R-module M, the sequence

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is exact; equivalently, if for every[5] right R-module N, the sequence

$$0 \to N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$$

is exact. There are numerous other equivalent definitions—see e.g. [9, Lemma 2.19] or [20, 2.1.1], emphasising the "model-theoretic" origins of purity.

Using functor category methods, purity has expanded from categories of modules to locally finitely presented categories with products and more generally, definable categories [20, Part III]. Objects of such categories can be viewed as many-sorted (abelian) algebraic structures; informally, this approach allows one to talk about "elements" even when there is no underlying set present (which is the case e.g. for sheaves). Another point of view is that the objects are modules over a "ring with many objects", i.e. functors from a certain small preadditive category to the category of abelian groups; this has led to a smooth transfer of many notions originating in model theory of modules to the more general framework [21].

Pure-injective objects are defined as those objects N such that the contravariant Hom functor into N is exact on pure-exact sequences. Pure-injectives are model-theoretically important, too; in fact, they coincide with the *algebraically compact* modules [20, 4.3.2] (possibly over a ring with many objects). The isomorphism classes of indecomposable pure-injectives form a set, which can be equipped with a topology, forming a topological space called the *Ziegler spectrum*. Its closed subsets correspond to *definable subcategories*, i.e. subcategories closed under direct products, direct limits, and pure subobjects; see [20, 3.1] for further equivalent definitions of the topology. Ziegler spectrum has found applications even outside the model-theoretic branch of representation theory, e.g. [14].

Chapter 4, based on joint paper with Prest [22], deals with purity and the related notions in the category of all sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X$-Mod and the category of quasicoherent ones $\mathrm{QCoh}(X)$. If X is a quasicompact and quasiseparated scheme, then both these categories are locally finitely presented Grothendieck categories. Therefore one can introduce the notion of *categorical purity* in these categories: In accordance with the approach outlined above, a short exact sequence

[5] Finitely presented modules are sufficient here, too.

is *categorically pure-exact* (or *c-pure-exact*) if it stays exact after applying the Hom functor from a finitely presented object.

On the other hand, the category $\mathcal{O}_X$-Mod is equipped with the tensor product bifunctor, which $\mathrm{QCoh}(X)$ inherits. Being inspired by the equivalent definition for the module categories, we may call a short exact sequence *geometrically pure-exact* (or *g-pure-exact*), if it stays exact after applying the tensor product functor with any (quasicoherent) sheaf; under our assumptions on X, this is a weaker definition than the categorical one.

Both these notions of purity (although under different names) and some of their basic properties appeared in [3]; this paper (and the apparent lack of any examples in it) had been the primary motivation for our further research of the topic. The main goal is to provide a better understanding of pure-injective objects with respect to both the purities in question.

The investigation of purity in $\mathcal{O}_X$-Mod is mostly the auxiliary part of the chapter, serving for further investigation of the quasicoherent case. However, a nice structural result concerning g-pure-injectives is obtained:

Theorem D. *Let $\mathcal{N}$ be an indecomposable g-pure-injective sheaf in $\mathcal{O}_X$-Mod. Then there is $x \in X$ and an indecomposable pure-injective $\mathcal{O}_{X,x}$-module N such that $\mathcal{N} \cong \iota_{x,*}(N)$, i.e. $\mathcal{N}$ is the skyscraper sheaf with pure-injective module of sections. Conversely, for every $x \in X$ and a pure-injective $\mathcal{O}_{X,x}$-module N, the sheaf $\iota_{x,*}(N)$ is g-pure-injective.*

Moving on to quasicoherent sheaves, we start with observing that the geometric and categorical purity coincide if and only if the scheme is affine. Further, we describe the geometric part of the Ziegler spectrum similarly as in $\mathcal{O}_X$-Mod:

Theorem E. *Let X be a quasicompact quasiseparated scheme and $\mathcal{N}$ an indecomposable g-pure-injective quasicoherent sheaf on X. Then there is an open affine subset $U \subseteq X$ and an indecomposable pure-injective $\mathcal{O}_X(U)$-module N such that $\mathcal{N} \cong \iota_{U,*}(\tilde{N})$, i.e. $\mathcal{N}$ is the direct image of (the quasicoherent sheaf associated to) N with respect to the embedding $U \hookrightarrow X$. Conversely, for every $U \subseteq X$ open affine and a pure-injective $\mathcal{O}_X(U)$-module N, the sheaf $\iota_{U,*}(\tilde{N})$ is g-pure-injective.*

This shows that the geometric part is in fact "glued together" from the Ziegler spectra of the affine pieces in the same way as X is glued from open affine subsets. In particular, the geometric part, being the union of finitely many closed quasicompact sets (under our standing assumption X quasicompact quasiseparated), is itself quasicompact and closed in the whole (categorical) Ziegler spectrum. Consequently, there is a definable subcategory $\mathcal{D}_X \subseteq \mathrm{QCoh}(X)$ corresponding to the geometric part.

Lastly, we describe the categorical Ziegler spectrum of the category of quasicoherent sheaves over a projective line over a field—both the points and the topology. In addition to the indecomposable g-pure-injectives, which can be described by the methods described above and which are therefore analogous to pure-injectives over a Dedekind domain, one gets vector bundles $\mathcal{O}(n)$ as additional, isolated points. Because of them the Ziegler spectrum turns out not to be quasicompact, in contrast to the affine case. Finally, in this case we are also able to obtain an alternative description of the subcategory $\mathcal{D}_X$: A quasicoherent sheaf belongs to $\mathcal{D}_X$ if and only if for each $n \in \mathbb{Z}$,

$$\operatorname{Ext}^1_{\operatorname{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n), \mathcal{M}) = 0,$$

if and only if for each $n \in \mathbb{Z}$,

$$\operatorname{Hom}_{\operatorname{QCoh}(\mathbb{P}^1_k)}(\mathcal{M}, \mathcal{O}(n)) = 0.$$

1.6 Exactness of the internal Hom and the existence of flat generators

In addition to the tensor product, the category of quasicoherent sheaves on a scheme X is further equipped with the *internal Hom functor* $\mathcal{H}om^{\mathrm{qc}}$, which is right adjoint to the tensor product; these functors are part of the symmetric closed monoidal structure on $\operatorname{QCoh}(X)$.

Recall that when working with purity in the module categories, the following duality turns out to be of importance. For the sake of simplicity, let us work over a commutative ring R, let E be the injective cogenerator of R-Mod and put $(-)^* = \operatorname{Hom}_R(-, E)$; then a short exact sequence of R-modules $0 \to A \to B \to C \to 0$ is pure-exact if and only if the dual sequence $0 \to C^* \to B^* \to A^* \to 0$ splits [20, Proposition 4.3.30]. Furthermore, the canonical map $M \to M^{**}$ is always a pure embedding into a pure-injective module [20, Corollary 4.3.31].

It is therefore tempting to pick an injective cogenerator $\mathcal{E}$ of $\operatorname{QCoh}(X)$ and use the (contravariant) functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ in a similar fashion to produce similar assertions for geometric purity in $\operatorname{QCoh}(X)$. That is indeed possible and the authors of [3] take this approach, although with an intermediate step in $\mathcal{O}_X$-Mod. However, there is one property of the duality functor that has been overlooked so far: The exactness. Of course, when one considers the ordinary, categorical (contravariant) Hom, it is the very definition of injectivity that the corresponding functor is exact. However, there is no apparent reason for the internal Hom to have this property.

The final Chapter 5, which is the joint paper with Šťovíček [25], addresses this question in the usual setting of quasicoherent sheaves over a quasicompact quasiseparated scheme. It turns out that this is closely related to the question whether the category $\operatorname{QCoh}(X)$ has flat generators and to the properties of the scheme X, namely:

Theorem F. *Let X be a quasicompact and quasiseparated scheme. Then the following assertions are equivalent:*

(1) the category $\operatorname{QCoh}(X)$ of all quasicoherent sheaves on X has a flat generator;

(2) for every injective object $\mathcal{E}$ of $\operatorname{QCoh}(X)$, the contravariant internal Hom functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact;

(3) there exists an injective cogenerator $\mathcal{E}$ of $\operatorname{QCoh}(X)$ such that the contravariant internal hom functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact;

(4) the scheme X is semiseparated.

This result shows that the assumption on X to be semiseparated in Chapter 2 (and a number of other papers, e.g. [4], [15], [16]) indeed cannot be omitted, for otherwise the class of flat quasicoherent sheaves is not generating and, of course, the same holds for its subclasses—in particular the class of very flats, but also the class of vector bundles, a result of [28], which we recover using considerably less involved methods. Finally, the result can be viewed a reformulation of a purely scheme-theoretic property of semiseparatedness into the properties of the category $\mathrm{QCoh}(X)$.

Let us comment here informally on what goes wrong in the non-semiseparated case. The fundamental issue is the non-exactness of the direct image functor $\iota_{U,*}\colon \mathrm{QCoh}(U) \to \mathrm{QCoh}(X)$, where $\iota_U\colon U \hookrightarrow X$ is the embedding and U is an open affine subset of X whose intersection with some other open affine subset is not affine. However, this functor, being right adjoint to an exact functor (the restriction), preserves injectives, so one can pick the injective $\mathcal{E}$ to be of the form $\iota_{U,*}(\tilde{E})$ for E an injective $\mathcal{O}_X(U)$-module. We then show that

$$\mathcal{H}om^{\mathrm{qc}}(-, \iota_{U,*}(\tilde{E})) \cong \iota_{U,*}(\mathcal{H}om^{\mathrm{qc}}(-, \tilde{E})),$$

which "turns" the non-exactness of the direct image to the non-exactness of the internal Hom. Of course, in the actual proof one has to be more careful, since not only do we take the direct image, but also dualize.

1.7 Future research directions

Finally, let us comment on the possible extensions of the results mentioned above.

Perhaps the most attractive goal is to find alternative, hopefully homological description(s) of the definable subcategory $\mathcal{D}_X \subseteq \mathrm{QCoh}(X)$ defined in Section 1.5. Existence of such a description sounds plausible thanks to our known description of $\mathcal{D}_X$ in the case of projective line over a field . Speaking vaguely, $\mathcal{D}_X$ consists of "sheaves which think that X is affine", so it might well be the case that it coincides with the class Ext-orthogonal to vector bundles (as is the case for the projective line over a field, discussed in Section 1.5 above). However, it is not even clear from the original definition that $\mathcal{D}_X$ should be closed under extensions. Furthermore, all examples of elements of $\mathcal{D}_X$ currently known to us are direct images of quasicoherent sheaves on open affines, and direct sums or products of those.

Let us point out that it is a rather hard question when does X have enough vector bundles (see [10]). The study of $\mathcal{D}_X$ and more generally geometric purity offers a new way of attacking this problem with a brand new weaponry (set- and model-theoretical tools, homological algebra).

1.8 Relation of the thesis to author's papers

Most of the thesis is based on joint preprints of the author with several collaborators. These preprints have already been accepted for publication and published electronically, so they have a DOI, but they have not yet been assigned a particular issue of the journal in case. In particular, their final form may differ from the one presented here.

Chapter 2 is based on the paper [5] (also arXiv:1708.05913) electronically published in the Journal of the Australian Mathematical Society. Chapter 3 is roughly the first half of the paper [12] (arXiv:1907.00356), the other half of which does not feature many contributions by the author; this paper is to appear in the Journal of Commutative Algebra. Chapter 4 is based on [22] (arXiv:1809.08981), published in Mathematische Zeitschrift. Finally, Chapter 5 is based on the manuscript [25] (arXiv:1902.05740), whose revised version is now under review in the Bulletin of the London Mathematical Society.

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2. Quillen equivalent models for the derived category of flats and the resolution property

This chapter is based on the preprint doi:10.1017/S1446788720000075, to appear in the Journal of the Australian Mathematical Society. It is also available at arXiv:1708.05913.

2.1 Introduction

Throughout the paper R will denote a commutative ring. In [21] Neeman gives a new description of the homotopy category $\mathbf{K}(\mathrm{Proj}(R))$ as a Verdier quotient of $\mathbf{K}(\mathrm{Flat}(R))$. The main advantage of the new description is that it does not involve projective objects, so it can be generalized to non-affine schemes (see [20, Remark 3.4]). So, in his thesis [17], Murfet *mocks* the homotopy category of projectives on a non-affine scheme, by considering the category $\mathbf{D}(\mathrm{Flat}(X))$[1] defined as the Verdier quotient

$$\mathbf{D}(\mathrm{Flat}(X)) := \frac{\mathbf{K}(\mathrm{Flat}(X))}{\widetilde{\mathrm{Flat}}^{\mathbf{K}}(X)},$$

where $\widetilde{\mathrm{Flat}}^{\mathbf{K}}(X)$ denotes the class of acyclic complexes in $\mathbf{K}(\mathrm{Flat}(X))$ with flat cycles. In the language of model categories, Gillespie showed in [11] that $\mathbf{D}(\mathrm{Flat}(X))$ can be realized as the homotopy category of a Quillen model structure on the category $\mathrm{Ch}(\mathrm{QCoh}(X))$ of unbounded chain complexes of quasi-coherent sheaves on a quasi compact and semi-separated scheme, and that, in fact, in case $X = \mathrm{Spec}(R)$ is affine, both homotopy categories $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{K}(\mathrm{Proj}(R))$ are triangle equivalent, coming from a Quillen equivalence between the corresponding models.

However, from the homological point of view, flat modules are much more complicated than projective modules. For instance, for a general commutative ring, the exact category of flat modules has infinite homological dimension. In order to partially remedy these complications, recently Positselski in [22] has introduced a refinement of the class of flat quasi-coherent sheaves, the so-called *very flat* quasi-coherent sheaves (see Section 2.3 for the definition and main properties) and showed that this class shares many nice properties with the class of flat sheaves, but it has potentially several advantages with respect to it, for instance, it can be applied to matrix factorizations (see the introduction of the recent preprint [24] for a nice and detailed treatment of the goodness of the very flat sheaves).

Moreover, in the affine case $X = \mathrm{Spec}(R)$, the exact category of very flat modules has finite homological dimension (every very flat module has projective

[1]The original terminology in [17] for $\mathbf{D}(\mathrm{Flat}(X))$ was $\mathbf{K}_m(\mathrm{Proj}(X))$. This is referred in [19] as the *pure derived category of flat sheaves* on X and denoted by $\mathbf{D}(\mathrm{Flat}(X))$.

dimension ≤ 1). Therefore, in this case one easily obtains a triangulated equivalence between $\mathbf{D}(\mathcal{VF}(R))$ and $\mathbf{K}(\mathrm{Proj}(R))$ (here $\mathcal{VF}(R)$ denotes the class of very flat R-modules). In particular it is much less involved than the aforementioned triangulated equivalence between $\mathbf{D}(\mathrm{Flat}(R))$ and $\mathbf{K}(\mathrm{Proj}(R))$ [21, Theorem 1.2].

So, if we denote by $\mathcal{VF}(X)$ the class of very flat quasi-coherent sheaves, one can also consider "mocking" the homotopy category of projectives over a non-affine scheme by defining the Verdier quotient

$$\mathbf{D}(\mathcal{VF}(X)) := \frac{\mathbf{K}(\mathcal{VF}(X))}{\widetilde{\mathcal{VF}}^{\mathbf{K}}(X)}.$$

It is then natural to wonder whether or not the (indirect) triangulated equivalence between $\mathbf{D}(\mathrm{Flat}(R))$ and $\mathbf{D}(\mathcal{VF}(R))$ still holds over a non-affine scheme. This was already proved to be the case for a semi-separated Noetherian scheme of finite Krull dimension in [22, Corollary 5.4.3]. As a first consequence of the results in this paper, we extend in Corollary 2.6.2 this result for arbitrary (quasi-compact and semi-separated) schemes.

Corollary. *For any quasi-compact and semi-separated scheme X, the categories $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{D}(\mathcal{VF}(X))$ are triangle equivalent.*

Recall from Gross [13] that a scheme X satisfies the *resolution property* provided that X has enough infinite-dimensional vector bundles (in the sense of Drinfeld [4]; these are called locally free sheaves in [13]), that is, for every quasi-coherent sheaf $\mathcal{M}$ there exists an epimorphism $\bigoplus_i \mathcal{V}_i \to \mathcal{M}$ for some family $\{\mathcal{V}_i \mid i \in I\}$ of infinite-dimensional vector bundles.[2] In this case the class of infinite-dimensional vector bundles constitutes the natural extension of the class of projective modules for non-affine schemes and one can define the derived category of infinite-dimensional vector bundles again as the Verdier quotient

$$\mathbf{D}(\mathrm{Vect}(X)) := \frac{\mathbf{K}(\mathrm{Vect}(X))}{\widetilde{\mathrm{Vect}}^{\mathbf{K}}(X)}.$$

This definition trivially agrees with $\mathbf{K}(\mathrm{Proj}(R))$ in case $X = \mathrm{Spec}(R)$ is affine. By using the class of very flat sheaves we obtain in Corollary 2.6.4 the following meaningful consequence, which does not seem clearly to admit a direct proof (i.e. a proof without using very flat sheaves).

Corollary. *Let X be a quasi-compact and semi-separated scheme satisfying the resolution property (for instance if X is divisorial [18, Proposition 6(a)]). Murfet's and Neeman's derived category of flats, $\mathbf{D}(\mathrm{Flat}(X))$, is triangle equivalent to $\mathbf{D}(\mathrm{Vect}(X))$, the derived category of infinite-dimensional vector bundles.*

Indeed the methods developed in this paper go beyond the class of very flat quasi-coherent sheaves. More precisely, we investigate which are the conditions that a subclass $\mathcal{A}_{\mathrm{qc}}$ of flat quasi-coherent sheaves has to fulfil in order to get a triangle equivalent category to $\mathbf{D}(\mathrm{Flat}(X))$. In fact, we show that the triangulated

[2] The "original" statement of the resolution property, which can be found e.g. in Totaro [28], is that "every coherent sheaf is a quotient of a vector bundle". However, this property was considered in the setting of Noetherian schemes, where it agrees with the one we use.

equivalence comes from a Quillen equivalence between the corresponding models. We point out that there are well-known examples of non Quillen equivalent models with equivalent homotopy categories. The precise statement of our main result is in Theorem 2.6.1 (see the setup in Section 2.6 for unexplained terminology).

Theorem. *Let X be a quasi-compact and semi-separated scheme and let $\mathscr{P}$ be a property of modules and $\mathscr{A}$ its associated class of modules. Assume that $\mathscr{A} \subseteq$ Flat, and that the following conditions hold:*

(1) The class $\mathscr{A}$ is Zariski-local.

(2) For each $R = \mathcal{O}_X(U)$, $U \in \mathfrak{U}$, the pair $(\mathscr{A}_R, \mathscr{B}_R)$ is a hereditary cotorsion pair generated by a set.

(3) For each $R = \mathcal{O}_X(U)$, $U \in \mathfrak{U}$, every flat $\mathscr{A}_R$-periodic module is trivial.

(4) $j_*(\mathscr{A}_{\mathrm{qc}(U_\alpha)}) \subseteq \mathscr{A}_{\mathrm{qc}(X)}$, for each $\alpha \subseteq \{0, \dots, m\}$.

Then the class $\mathscr{A}_{\mathrm{qc}}$ defines an abelian model category structure in $\mathrm{Ch}(\mathrm{QCoh}(X))$ whose homotopy category $\mathbf{D}(\mathscr{A}_{\mathrm{qc}})$ is triangle equivalent to $\mathbf{D}(\mathrm{Flat}(X))$, induced by a Quillen equivalence between the corresponding model categories.

It is interesting to observe that conditions (1), (2) and (3) in the previous theorem only involve properties of modules. Thus we find useful and of independent interest to explicitly state in Theorem 2.5.1 the affine version of the previous theorem (and give an easy proof). Section 2.4 is meant to make abundantly clear the variety of examples of classes of modules that fit into those conditions. Of particular interest is the class $\mathscr{A}(\kappa)$ of *restricted flat Mittag-Leffler modules* considered in Theorem 2.4.5 which has been widely studied in the literature in the recent years (see, for instance, [6, 7, 12, 15, 26]). So regarding this class, we obtain the following meaningful consequences:

Corollary. *Let κ be an infinite cardinal and $\mathscr{A}(\kappa)$ be the class of κ-restricted flat Mittag-Leffler modules (notice that $\mathscr{A}(\kappa) = \mathrm{Proj}(R)$ in case $\kappa = \aleph_0$).*

(1) Every pure acyclic complex with components in $\mathscr{A}(\kappa)$ has cycles in $\mathscr{A}(\kappa)$.

(2) The categories $\mathbf{D}(\mathscr{A}(\kappa))$ and $\mathbf{K}(\mathrm{Proj}(R))$ are triangle equivalent.

The proof of (1) can be found in Theorem 2.4.5, whereas the proof of (2) is a particular instance of Theorem 2.5.1 with $\mathscr{A} = \mathscr{A}(\kappa)$. In the special case $\kappa = \aleph_0$, the statement (1) recovers a well-known result due to Benson and Goodearl [2, Theorem 1.1].

2.2 Preliminaries

2.2.1. Zariski-local classes of modules. Let $\mathscr{P}$ be a property of modules and let $\mathscr{A}$ be the corresponding class of modules satisfying $\mathscr{P}$, i.e. for any ring R, the class $\mathscr{A}_R$ consists of $M \in R$-Mod such that M satisfies $\mathscr{P}_R$. We define the class $\mathscr{A}_{\mathrm{qc}(X)}$ in $\mathrm{QCoh}(X)$ (or just $\mathscr{A}_{\mathrm{qc}}$ if the scheme is understood) as the class of all quasi-coherent sheaves $\mathcal{M}$ such that, for each open affine $U \subseteq X$, the module of sections $\mathcal{M}(U) \in \mathscr{A}_{\mathcal{O}_X(U)}$. We will be only interested in those properties of modules $\mathscr{P}$ such that the property of being in $\mathscr{A}_{\mathrm{qc}(X)}$ can be tested on an open affine covering

of X. In this case we will say that the class $\mathcal{A}$ of modules (associated to $\mathcal{P}$) is *Zariski-local*.
The following is a specialization of the *ascent-descent* conditions [7, Definition 3.4] that suffices to prove Zariski-locality (see Vakil [29, Lemma 5.3.2] and also [16, §27.4]):

Lemma 2.2.2. *The class of modules $\mathcal{A}$ associated to the property of modules $\mathcal{P}$ is Zariski-local if and only if satisfies the following:*

(1) If an R-module $M \in \mathcal{A}_R$, then $M_f \in \mathcal{A}_{R_f}$ for all $f \in R$.
(2) If $(f_1, \dots, f_n) = R$, and $M_{f_i} = R_{f_i} \otimes_R M \in \mathcal{A}_{R_{f_i}}$, for all $i \in \{1, \dots, n\}$, then $M \in \mathcal{A}_R$.

It is easy to see that the class Flat of flat modules is Zariski-local. A module M is *Mittag-Leffler* provided that the canonical map $M \otimes_R \prod_{i\in I} M_i \to \prod_{i\in I} M \otimes_R M_i$ is monic for each family of left R-modules $(M_i | \, i \in I)$. The classes FlatML (of flat Mittag-Leffler modules) and Proj (of projective modules) are also Zariski-local by 3.1.4(3) and 2.5.2 in [25, Seconde partie, 2.5.2]. The class rFlatML of *restricted* flat Mittag-Leffler modules (in the sense of [7, Example 2.1(3)]) is also Zariski-local by [7, Theorem 4.2].

2.2.3. Precovers, envelopes and complete cotorsion pairs. Throughout this section the symbol $\mathcal{G}$ will denote an abelian category. Let $\mathcal{C}$ be a class of objects in $\mathcal{G}$. A morphism $C \to M$ in $\mathcal{G}$ is called a $\mathcal{C}$*-precover* if C is in $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{G}}(C', C) \to \mathrm{Hom}_{\mathcal{G}}(C', M) \to 0$ is exact for every $C' \in \mathcal{C}$. If every object in $\mathcal{G}$ has a $\mathcal{C}$-precover, then the class $\mathcal{C}$ is called *precovering*. The dual notions are *preenvelope* and *preenveloping* class.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects in $\mathcal{G}$ is a *cotorsion pair* if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$, where, given a class $\mathcal{C}$ of objects in $\mathcal{A}$, the right orthogonal $\mathcal{C}^{\perp}$ is defined to be the class of all $Y \in \mathcal{G}$ such that $\mathrm{Ext}^1_{\mathcal{G}}(C, Y) = 0$ for all $C \in \mathcal{C}$. The left orthogonal ${}^{\perp}\mathcal{C}$ is defined similarly. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *hereditary* if $\mathrm{Ext}^i_{\mathcal{G}}(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $i \geq 1$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* if it has *enough projectives* and *enough injectives*, i.e. for each $D \in \mathcal{G}$ there exist short exact sequences $0 \to B \to A \to D \to 0$ (enough projectives) and $0 \to D \to B' \to A' \to 0$ (enough injectives) with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$. It is then easy to observe that $A \to D$ is an $\mathcal{A}$-precover of D (such precovers are called *special*). Analogously, $D \to B'$ is a *special* $\mathcal{B}$-preenvelope of D. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *generated by a set* provided that there exists a *set* $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S}^{\perp} = \mathcal{B}$. In case $\mathcal{G}$ is, in addition Grothendieck, it is known that a cotorsion pair generated by a set $\mathcal{S}$ which contains a generating set of $\mathcal{G}$ is automatically complete.

2.2.4. Exact model categories and Hovey triples. In [14] Hovey relates complete cotorsion pairs with abelian (or exact) model category structures.

An *abelian model structure* on $\mathcal{G}$, that is, a model structure on $\mathcal{G}$ which is compatible with the abelian structure in the sense of [14, Definition 2.1], corresponds by [14, Theorem 2.2] to a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of objects in $\mathcal{A}$ for which $\mathcal{W}$ is thick[3] and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are complete cotorsion pairs

[3]Recall that a class $\mathcal{W}$ in an abelian (or, more generally, in an exact) category $\mathcal{G}$ is *thick* if it is closed under direct summands and satisfies that whenever two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

in $\mathcal{G}$. In the model structure on $\mathcal{G}$ determined by such a triple, $\mathcal{C}$ is precisely the class of cofibrant objects, $\mathcal{F}$ is precisely the class of fibrant objects, and $\mathcal{W}$ is precisely the class of trivial objects (that is, objects weakly equivalent to zero). Such triple is often referred as a *Hovey triple*.

Gillespie extends in [9, Theorem 3.3] Hovey's correspondance, mentioned above, from the realm of abelian categories to the realm of weakly idempotent complete exact categories [9, Definition 2.2]. More precisely, if $\mathcal{G}$ is a weakly idempotent complete exact categories (not necessarily abelian), then an *exact model structure* on $\mathcal{G}$ (i.e. a model structure on $\mathcal{G}$ which is compatible with the exact structure in the sense of [9, Definition 3.1]) corresponds to a Hovey triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in $\mathcal{G}$.

2.2.5. Deconstructible classes. A well ordered direct system $(M_\alpha \mid \alpha \leq \lambda)$ of objects in $\mathcal{G}$ is called *continuous* if $M_0 = 0$ and, for each limit ordinal $\beta \leq \lambda$, we have $M_\beta = \varinjlim_{\alpha<\beta} M_\alpha$. If all the morphisms in the system are monomorphisms, then the system is called a *continuous directed union*.

Let $\mathcal{S}$ be a class of objects in $\mathcal{G}$. An object M in $\mathcal{G}$ is called $\mathcal{S}$*-filtered* if there is a continuous directed union $(M_\alpha \mid \alpha \leq \lambda)$ of subobjects of M such that $M = M_\lambda$ and for every $\alpha < \lambda$ the quotient $M_{\alpha+1}/M_\alpha$ is isomorphic to an object in $\mathcal{S}$. We denote by $\mathrm{Filt}(\mathcal{S})$ the class of all $\mathcal{S}$-filtered objects in $\mathcal{G}$. A class $\mathcal{C}$ is called *deconstructible* provided that there exists a set $\mathcal{S}$ such that $\mathcal{C} = \mathrm{Filt}(\mathcal{S})$ (see [27, Definition 1.4]). It is then known by [27, Theorem pg.195] that any deconstructible class is precovering.

2.2.6. Chain complexes of modules. We denote by $\mathrm{Ch}(\mathcal{G})$ the category of unbounded chain complexes of objects in $\mathcal{G}$, i.e. complexes $G_\bullet$ of the form

$$\cdots \to G_{n+1} \xrightarrow{d^G_{n+1}} G_n \xrightarrow{d^G_n} G_{n-1} \to \cdots .$$

We will denote by $Z_n G_\bullet$ the *n-cycle of* G, i.e. $Z_n G = \mathrm{Ker}(d^G_n)$. Given a chain complex G the *n-th suspension of* G, $\Sigma^n G$, is the complex defined as $(\Sigma^n G)_k = G_{k-n}$ and $d^{\Sigma^n G}_k = (-1)^n d_{k-n}$. And for a given object $A \in \mathcal{G}$, the *n-disc* complex $D^n(A)$ is the complex with the object A in the components n and $n-1$, d_n the identity map, and 0 elsewhere.

We denote by $\mathbf{K}(\mathcal{G})$ the homotopy category of $\mathcal{G}$, i.e. $\mathbf{K}(\mathcal{G})$ has the same objects as $\mathrm{Ch}(\mathcal{G})$ and the morphisms are the homotopy classes of morphisms of chain complexes.

In case $\mathcal{G} = R$-Mod, we will denote $\mathrm{Ch}(\mathcal{G})$ (resp. $\mathbf{K}(\mathcal{G})$) simply by $\mathrm{Ch}(R)$ (resp. $\mathbf{K}(R)$). Given a class $\mathcal{C}$ in $\mathcal{G}$, we shall consider the following classes of chain complexes:

- $\mathrm{Ch}(\mathcal{C})$ (resp. $\mathbf{K}(\mathcal{C})$) is the full subcategory of $\mathrm{Ch}(\mathcal{G})$ (resp. of $\mathbf{K}(\mathcal{C})$) of all complexes $C_\bullet \in \mathrm{Ch}(\mathcal{G})$ such that $C_n \in \mathcal{C}$.

- $\mathrm{Ch}_{\mathrm{ac}}(\mathcal{C})$ (resp. $\mathbf{K}_{\mathrm{ac}}(\mathcal{C})$) is the class of all acyclic complexes in $\mathrm{Ch}(\mathcal{C})$ (resp. in $\mathbf{K}(\mathcal{C})$).

- $\widetilde{\mathcal{C}}$ (resp. $\widetilde{\mathcal{C}}^{\mathbf{K}}$) is the class class of all complexes $C_\bullet \in \mathrm{Ch}_{\mathrm{ac}}(\mathcal{C})$ (resp. $C_\bullet \in \mathbf{K}_{\mathrm{ac}}(\mathcal{C})$) with the cycles $Z_n C_\bullet$ in $\mathcal{C}$ for all $n \in \mathbb{Z}$. A complex in $\widetilde{\mathcal{C}}$ is called a $\mathcal{C}$*-complex*.

- If $(\mathcal{A},\mathcal{B})$ is a cotorsion pair in $\mathcal{G}$, then $\mathrm{dg}(\mathcal{A})$ is the class of all complexes $A_\bullet \in \mathrm{Ch}(\mathcal{A})$ such that every morphism $f\colon A_\bullet \to B_\bullet$, with $B_\bullet$ a $\mathcal{B}$-complex, is null-homotopic. Since $\mathrm{Ext}^1_{\mathcal{G}}(A_n, B_n) = 0$ for every $n \in \mathbb{Z}$, a standard formula allows to infer that $\mathrm{dg}(\mathcal{A}) = {}^{\perp}\widetilde{\mathcal{B}}$. Analogously, $\mathrm{dg}(\mathcal{B})$ is the class of all complexes $B_\bullet \in \mathrm{Ch}(\mathcal{B})$ such that every morphism $f\colon A_\bullet \to B_\bullet$, with $A_\bullet$ an $\mathcal{A}$-complex, is null-homotopic. Hence $\mathrm{dg}(\mathcal{B}) = \widetilde{\mathcal{A}}^{\perp}$.

2.3 Very flat modules and sheaves

One of the main application of the results in this paper concerns the classes of very flat modules and very flat quasi-coherent sheaves, as defined by Positselski in [22]. In the present section we summarize all relevant definitions and properties regarding this class and that will be relevant in the sequel.

2.3.1. Very flat and contraadjusted modules. Let us consider the set

$$\mathcal{S} = \{R[r^{-1}] \mid r \in R\}$$

and let $(\mathcal{VF}(R), \mathcal{CA}(R))$ be the complete cotorsion pair generated by $\mathcal{S}$. The modules in the class $\mathcal{VF}(R)$ are called *very flat* and the modules in the class $\mathcal{CA}(R)$ are called *contraadjusted*. It is then clear that every projective module is very flat, and that every very flat module is, in particular, flat. In fact it is easy to observe that every very flat module has finite projective dimension ≤ 1. Thus, the complete cotorsion pair $(\mathcal{VF}, \mathcal{CA})$ is automatically hereditary and $\mathcal{CA}$ is closed under quotients. We finally notice that L is very flat in any short exact sequence $0 \to L \to V \to M \to 0$ in which V is very flat and $\mathrm{pd}_R(M) \leq 1$ (where $\mathrm{pd}_R(M)$ is the projective dimension of M).

Proposition 2.3.2 (Positselski). *The class of very flat modules is Zariski-local.*

Proof. Condition (1) of Lemma 2.2.2 holds by [22, Lemma 1.2.2(b)].
Condition (2) of Lemma 2.2.2 follows from [22, Lemma 1.2.6(a)]. □

2.3.3. Very flat and contraadjusted quasi-coherent sheaves. Let X be any scheme. A quasi-coherent sheaf $\mathcal{M}$ is *very flat* if there exists an open affine covering $\mathfrak{U}$ of X such that $\mathcal{M}(U)$ is a very flat $\mathcal{O}_X(U)$-module for each $U \in \mathfrak{U}$. By the previous proposition, the definition of very flat quasi-coherent sheaf is independent of the choice of the open affine covering. A quasi-coherent sheaf $\mathcal{N}$ is *contraadjusted* if $\mathrm{Ext}^n(\mathcal{M}, \mathcal{N}) = 0$ for each very flat quasi-coherent sheaf $\mathcal{M}$ and every integer $n \geq 1$.

Since the class of very flat modules is resolving (i.e. closed under kernels of epimorphisms) we infer that the class of very flat quasi-coherent sheaves is also resolving.

2.3.4. Very flat generators in $\mathrm{QCoh}(X)$. Let X be a quasi-compact and semi-separated scheme, with $\mathfrak{U} = \{U_0, \dots, U_d\}$ a semi-separated finite open affine covering of X. Let $U = U_{i_0} \cap \cdots \cap U_{i_p}$ be any intersection of open sets in the cover $\mathfrak{U}$ and let $j\colon U \hookrightarrow X$ be the inclusion of U in X. The inverse image functor j^* is just the restriction, so it is exact and preserves quasi-coherence. The direct image functor j_* is exact and preserves quasi-coherence because $j\colon U \hookrightarrow X$ is an affine morphism, due to the semi-separated assumption. Thus we have an adjunction (j^*, j_*) with $j_*\colon \operatorname{QCoh}(U) \to \operatorname{QCoh}(X)$ and $j^*\colon \operatorname{QCoh}(X) \to \operatorname{QCoh}(U)$.

The proof of the next proposition is implicit in [1, Proposition 1.1] (see also Murfet [17, Proposition 3.29] for a very detailed treatment) by noticing that the direct image functor j_* preserves not just flatness but in fact *very* flatness (by [22, Corollary 1.2.5(b)]). The reader can find a short and direct proof in [22, Lemma 4.1.1].

Proposition 2.3.5. *Let X be a quasi-compact and semi-separated scheme. Every quasi-coherent sheaf is a quotient of a very flat quasi-coherent sheaf. Therefore* $\operatorname{QCoh}(X)$ *possesses a family of very flat generators.*

2.3.6. The very flat cotorsion pair in $\operatorname{QCoh}(X)$. For any scheme X, the class $\mathcal{VF}(X)$ of very flat quasi-coherent sheaves is deconstructible (by [6, Corollary 3.14]). Therefore the class of very flat quasi-coherent sheaves is a precovering class (see 2.2.5). If, in addition, the scheme X is quasi-compact and semi-separated we infer from [6, Corollary 3.15] and [22, Corollary 4.1.2] that the pair $(\mathcal{VF}(X), \mathcal{CA}(X))$ is a complete hereditary cotorsion pair in $\operatorname{QCoh}(X)$ (where $\mathcal{CA}(X)$ denotes the class of all contraadjusted quasi-coherent sheaves on X).

By [22, Lemma 1.2.2(d)] the class of very flat modules (and hence the class of very flat quasi-coherent sheaves) is closed under tensor products. Thus, in case X is quasi-compact and semi-separated, [6, Theorem 4.5] yields a cofibrantly generated and monoidal model category structure in $\operatorname{Ch}(\operatorname{QCoh}(X))$ where the weak equivalences are the homology isomorphisms. The cofibrations (resp. trivial cofibrations) are monomorphisms whose cokernels are dg-very flat complexes (resp. very flat complexes). The fibrations (resp. trivial fibrations) are epimorphisms whose kernels are dg-contraadjusted complexes (resp. contraadjusted complexes). Therefore the corresponding triple is

$$(\operatorname{dg}(\mathcal{VF}(X)), \operatorname{Ch}_{\mathrm{ac}}(\operatorname{QCoh}(X)), \operatorname{dg}(\mathcal{CA}(X))).$$

2.4 The property of modules involved. Examples

As we will see in the next sections, we are mainly concerned in deconstructible classes of modules that are closed under certain periodic modules. We start by recalling the notion of $\mathcal{C}$-periodic module with respect to a class $\mathcal{C}$ of modules.

Definition 2.4.1. Let $\mathcal{C}$ be a class of modules. A module M is called $\mathcal{C}$-periodic if there exists a short exact sequence $0 \to M \to C \to M \to 0$, with $C \in \mathcal{C}$.

The following proposition relating flat $\mathcal{A}$-periodic modules and acyclic complexes with components in $\mathcal{A}$ is standard, but relevant for our purposes. The reader can find a proof in [5, Proposition 1 and Proposition 2].

Proposition 2.4.2. *Let $\mathcal{A}$ be a class of modules closed under direct sums and direct summands. The following are equivalent:*

(1) Every cycle of an acyclic complex with flat cycles and with components in $\mathcal{A}$ belongs to $\mathcal{A}$.

(2) Every flat $\mathcal{A}$-periodic module belongs to $\mathcal{A}$.

We are interested in deconstructible classes of modules $\mathcal{A}$ satisfying condition (2) in the previous proposition. Of course the first trivial example is the class $\mathrm{Flat}(R)$ of flat modules itself. Since the class of all flat Mittag-Leffler modules is closed under pure submodules, this class also trivially yields an example of a class $\mathcal{A}$ satisfying that every flat $\mathcal{A}$-periodic module is in $\mathcal{A}$. However this class has an important drawback: It is only deconstructible in the trivial case of a perfect ring (see Herbera and Trlifaj [15, Corollary 7.3]). This setback can be remedied by considering the *restricted* flat Mittag-Leffler modules, in the sense of [7, Example 2.1(3)], as we show in Theorem 2.4.5 below.

Now we will provide with other interesting non-trivial examples of such classes $\mathcal{A}$ satisfying condition (2) above, and that will be relevant in the applications of our main results in the next sections.

The first example is the class $\mathcal{A} = \mathrm{Proj}(R)$ of projective R-modules and goes back to Benson and Goodearl [2, Theorem 1.1].

Proposition 2.4.3. *Let* $\mathrm{Proj}(R)$ *be the class of all projective R-modules. Every flat* $\mathrm{Proj}(R)$*-periodic module is projective. As a consequence every pure acyclic complex of projectives is contractible (i.e. has projective cycles).*

The second application is the class $\mathcal{A} = \mathcal{VF}(R)$ of very flat modules (this is due to Šťovíček, personal communication).

Proposition 2.4.4. *Every flat $\mathcal{VF}(R)$-periodic module is very flat. As a consequence every pure acyclic complex of very flat modules has very flat cycles.*

Proof. Let $0 \to F \to G \to F \to 0$ be an exact sequence with F flat and G very flat. Let $0 \to F_1 \to P \to F \to 0$ be an exact sequence with P projective; then F_1 is flat. An application of the horseshoe lemma gives the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_1 & \to & Q & \to & F_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & P \oplus P & \to & P & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & G & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

where Q is projective, since $\mathrm{pd}_R(G) \leq 1$. Thus, by Proposition 2.4.3, F_1 is projective and therefore $\mathrm{pd}_R(F) \leq 1$. Let $C \in \mathcal{CA}(R)$. Then applying $\mathrm{Hom}_R(-, C)$ to the short exact sequence yields $0 = \mathrm{Ext}^1_R(G, C) \to \mathrm{Ext}^1_R(F, C) \to \mathrm{Ext}^2_R(F, C) = 0$, hence $F \in \mathcal{VF}(R)$. Finally, the consequence follows from Proposition 2.4.2(1) (with $\mathcal{A} = \mathcal{VF}(R)$). □

The last example is the announced deconstructible class of *restricted* flat Mittag-Leffler modules as defined in [7, Example 2.1(3)].

Theorem 2.4.5. *let κ be an infinite cardinal and $\mathcal{A}(\kappa)$ be the class of κ-restricted flat Mittag-Leffler modules. Every flat $\mathcal{A}(\kappa)$-periodic module is in $\mathcal{A}(\kappa)$. As a consequence every pure acyclic complex with components in $\mathcal{A}(\kappa)$ has cycles in $\mathcal{A}(\kappa)$.*

Proof. The proof mostly follows the pattern outlined in [2]; the main difference is that instead of direct sum decompositions, we work with filtrations and the Hill Lemma [12, Theorem 7.10]. Given a short exact sequence

$$0 \to F \to G \xrightarrow{f} F \to 0 \tag{2.1}$$

with F flat and $G \in \mathcal{A}(\kappa)$, we fix a Hill family $\mathcal{H}$ for G. The goal is to pick a filtration $(G_\alpha \mid \alpha \leq \sigma)$ from $\mathcal{H}$ such that for each $\alpha < \sigma$, $f(G_\alpha) = F \cap G_\alpha$, $f(G_\alpha) \subseteq_* F$, and $G_{\alpha+1}/G_\alpha$ is $\leq \kappa$-presented flat Mittag-Leffler; here $A \subseteq_* B$ means that A is a pure submodule of B.

Once this is achieved, we obtain a filtration of the whole short exact sequence (2.1) by short exact sequences of the form

$$0 \to F_{\alpha+1}/F_\alpha \to G_{\alpha+1}/G_\alpha \to F_{\alpha+1}/F_\alpha \to 0$$

(putting $F_\alpha = f(G_\alpha) = F \cap G_\alpha$). Since the property of being flat Mittag-Leffler passes to pure submodules [12, Corollary 3.20], this would make $F_{\alpha+1}/F_\alpha$ an $\leq \kappa$-presented flat Mittag-Leffler module and hence imply $F \in \mathcal{A}(\kappa)$. (Note that by [6, Lemma 2.7 (1)], each $\leq \kappa$-generated flat Mittag-Leffler module is (even strongly) $\leq \kappa$-presented.)

Put $G_0 = 0$. For limit ordinals α, it suffices to take unions of already constructed submodules G_β, $\beta < \alpha$; note that by property (H2) in the Hill Lemma, $G_\alpha \in \mathcal{H}$ then. Having constructed modules up to G_α (and assuming $G_\alpha \neq G$), we construct $G_{\alpha+1}$ as follows: We pass to the quotient short exact sequence

$$0 \to F/F_\alpha \to G/G_\alpha \xrightarrow{\overline{f}} F/F_\alpha \to 0,$$

which, by assumption, satisfies that F/F_α is flat and $G/G_\alpha \in \mathcal{A}(\kappa)$. Note that F/F_α, being (identified with) a pure submodule of G/G_α, is flat Mittag-Leffler. The Hill family $\mathcal{H}$ gives rise to family $\mathcal{H}'$ for G/G_α, which consists of factors of modules from $\mathcal{H}$ (containing G_α) by G_α.

Let us first show that any $\leq \kappa$-generated submodule Y of G/G_α can be enlarged to $\leq \kappa$-generated $\overline{G} \in \mathcal{H}'$ with the property that $\overline{f}(\overline{G}) \subseteq_* F/F_\alpha$ and $\overline{G} \cap F/F_\alpha$ is $\leq \kappa$-generated. To this end, we construct inductively a chain of submodules $\overline{G}_n \in \mathcal{H}'$ with union $\overline{G}$ (utilizing property (H2)). Let $\overline{G}_0$ be an arbitrary $\leq \kappa$-generated module $\overline{G}_0 \in \mathcal{H}'$ containing Y (obtained via (H4)). Assuming we have constructed $\overline{G}_n$, we get $\overline{G}_{n+1}$ by taking these steps:

(1) Enlarge $\overline{f}(\overline{G}_n)$ to a $\leq \kappa$-generated pure submodule X_n of F/F_α; this is possible by [6, Lemma 2.7 (2)] once we notice that F/F_α, being a pure submodule of G/G_α, is flat Mittag-Leffler.
(2) Take $\leq \kappa$-generated $\overline{G}_{n+1} \in \mathcal{H}'$ such that $X \subseteq \overline{f}(G'_n)$; this is again possible by property (H4) of the Hill Lemma.

We have $\overline{f}(\overline{G}) = \bigcup_{n\in\mathbb{N}} \overline{f}(\overline{G}) = \bigcup_{n\in\mathbb{N}} X_n \subseteq_* F/F_\alpha$. This also shows that $\overline{f}(\overline{G})$ is flat Mittag-Leffler, hence $\leq \kappa$-presented. The short exact sequence

$$0 \to \overline{G} \cap (F/F_\alpha) \to \overline{G} \to \overline{f}(\overline{G}) \to 0$$

now shows that $\overline{G} \cap (F/F_\alpha)$ is indeed $\leq \kappa$-generated.

Now iterate the claim as follows: Start with arbitrary $\leq \kappa$-generated non-zero $Y_0 \subseteq G/G_\alpha$ and obtain $\overline{G}_0$ from the claim. Enlarge it to $\overline{G}_1 \in \mathcal{H}'$ satisfying $\overline{G}_0 \cap (F/F_\alpha) \subseteq \overline{f}(\overline{G}_1)$ (which we may do using (H4), since $\overline{G}_0 \cap (F/F_\alpha)$ is $\leq \kappa$-generated). Taking $Y_1 = \overline{G}_1 + \overline{f}(\overline{G}_1)$ and applying the claim, we get $\overline{G}_2$ etc. This way we obtain a chain

$$\overline{G}_0 \cap (F/F_\alpha) \subseteq \overline{f}(\overline{G}_1) \subseteq \overline{G}_2 \cap (F/F_\alpha) \subseteq \overline{f}(\overline{G}_3) \subseteq \dots,$$

so for $\overline{G} = \bigcup_{n\in\mathbb{N}} \overline{G}_n \in \mathcal{H}'$ we have $\overline{G} \cap (F/F_\alpha) = \overline{f}(\overline{G})$. Also the purity of $\overline{f}(\overline{G})$ in F/F_α and being $\leq \kappa$-generated is ensured.

The desired module $G_{\alpha+1}$ is now the one satisfying $G_{\alpha+1}/G_\alpha = \overline{G}$. □

Note that in the case $\kappa = \aleph_0$, $\mathcal{A}(\kappa)$ is just the class of projective modules by [25, Seconde partie, Section 2.2], so this also covers the case of [2].

2.5 Quillen equivalent models for $\mathbf{K}(\mathrm{Proj}(R))$

It is known (see Bravo, Gillespie and Hovey [3, Corollary 6.4]) that the homotopy category of projectives $\mathbf{K}(\mathrm{Proj}(R))$ can be realized as the homotopy category of the model $\mathcal{M}_{\mathrm{proj}} = (\mathrm{Ch}(\mathrm{Proj}(R)), \mathrm{Ch}(\mathrm{Proj}(R))^\perp, \mathrm{Ch}(R))$ in $\mathrm{Ch}(R)$. Denote by $\mathrm{Cot}(R)$ the class of (Enochs) cotorsion modules, i.e. $\mathrm{Cot}(R) = \mathrm{Flat}(R)^\perp$. Now, by [11, Remark 4.2], the class $\mathrm{Ch}(\mathrm{Flat}(R))$ induces model category in $\mathrm{Ch}(R)$ given by the triple

$$(\mathrm{Ch}(\mathrm{Flat}(R)), \mathrm{Ch}(\mathrm{Proj}(R))^\perp, \mathrm{dg}(\mathrm{Cot}(R))).$$

This model is Quillen equivalent to $\mathcal{M}_{\mathrm{proj}}$. Therefore, its homotopy category, the derived category of flats $\mathbf{D}(\mathrm{Flat}(R))$, is triangulated equivalent to $\mathbf{K}(\mathrm{Proj}(R))$. The next theorem gives sufficient conditions on a class of modules $\mathcal{A}$ to get $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}(\mathrm{Flat}(R))$ to be triangulated equivalent. For concrete examples of such classes the reader should have in mind the classes of modules considered in Section 2.4.

Theorem 2.5.1. *Let $\mathcal{A} \subseteq \mathrm{Flat}(R)$ be a class of modules such that:*

(1) The pair $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair generated by a set.

(2) Every flat $\mathcal{A}$-periodic module is trivial.

Then there is an abelian model category structure

$$\mathcal{M} = (\mathrm{Ch}(\mathcal{A}), \mathrm{Ch}(\mathrm{Proj}(R))^\perp, \mathrm{dg}\,\widetilde{\mathcal{B}})$$

in $\mathrm{Ch}(R)$. If we denote by $\mathbf{D}(\mathcal{A})$ the homotopy category of $\mathcal{M}$, then $\mathbf{D}(\mathrm{Flat}(R))$, $\mathbf{D}(\mathcal{A})$ and $\mathbf{K}(\mathrm{Proj}(R))$ are triangulated equivalent, induced by a Quillen equivalence between the corresponding model categories.

Proof. Let $\mathcal{M} = (\mathrm{Ch}(\mathcal{A}), \mathcal{W}, \mathrm{dg}\,\widetilde{\mathcal{B}})$ be the model associated to the complete hereditary cotorsion pairs $(\mathrm{Ch}(\mathcal{A}), \mathrm{Ch}(\mathcal{A})^{\perp})$ and $(\widetilde{\mathcal{A}}, \mathrm{dg}\,\widetilde{B})$ in $\mathrm{Ch}(R)$. To get the claim it suffices to show that $\mathcal{W} = \mathrm{Ch}(\mathrm{Proj}(R))^{\perp}$. To this aim we will use [11, Lemma 4.3(1)], i.e. we need to prove:

(i) $\widetilde{\mathcal{A}} = \mathrm{Ch}(\mathcal{A}) \cap \mathrm{Ch}(\mathrm{Proj}(R))^{\perp}$.
(ii) $\mathrm{Ch}(\mathcal{A})^{\perp} \subseteq \mathrm{Ch}(\mathrm{Proj}(R))^{\perp}$.

Condition (ii) is clear because $\mathrm{Proj}(R) \subseteq \mathcal{A}$. Now, by Neeman [21, Theorem 8.6], $\mathrm{Ch}(\mathcal{A}) \cap \mathrm{Ch}(\mathrm{Proj}(R))^{\perp} = \widetilde{\mathrm{Flat}}(R) \cap \mathrm{Ch}(\mathcal{A})$. But, by the assumption (2), it follows that $\widetilde{\mathrm{Flat}}(R) \cap \mathrm{Ch}(\mathcal{A}) = \widetilde{\mathcal{A}}$. □

Remark 2.5.2. Starting with a class $\mathcal{A}$ in the assumptions of Theorem 2.5.1, we may construct, for each integer $n \geq 0$, the class $\mathcal{A}^{\leq n}$ of modules M possessing an exact sequence

$$0 \to A_n \to A_{n-1} \to \ldots \to A_0 \to M \to 0$$

with $A_i \in \mathcal{A}$, $i = 1, \ldots, n$. The derived categories $\mathbf{D}(\mathcal{A}^{\leq n})$ and $\mathbf{D}(\mathcal{A})$ are triangulated equivalent (see Positselski [22, Proposition A.5.6]). In particular we can infer from this a triangulated equivalence between $\mathbf{K}(\mathrm{Proj}(R))$ and $\mathbf{D}(\mathcal{VF}(R))$. By using a standard argument of totalization one can also check that $\mathbf{D}(\mathcal{A}^{\leq n})$ and $\mathbf{D}(\mathcal{A})$ can be realized as the homotopy categories of two models $\mathcal{M}_1$ and $\mathcal{M}_2$ and that these models are Quillen equivalent without using Neeman [21, Theorem 8.6]. From this point of view it seems that the triangulated equivalence between $\mathbf{K}(\mathrm{Proj}(R))$ and $\mathbf{D}(\mathcal{VF}(R))$ is much less involved than the one between $\mathbf{K}(\mathrm{Proj}(R))$ and $\mathbf{D}(\mathrm{Flat}(R))$.

2.6 Quillen equivalent models for $\mathbf{D}(\mathrm{Flat}(X))$

Setup: Throughout this section X will denote a quasi-compact and semi-separated scheme. If $\mathfrak{U} = \{U_0, \ldots, U_m\}$ is an affine open cover of X and $\alpha = \{i_0, \ldots, i_k\}$ is a finite sequence of indices in the set $\{0, \ldots, m\}$ (with $i_0 < \cdots < i_k$), we write $U_\alpha = U_{i_0} \cap \cdots \cap U_{i_k}$ for the corresponding affine intersection.

In [17] Murfet shows that the derived category of flat quasi-coherent sheaves on X, $\mathbf{D}(\mathrm{Flat}(X))$, constitutes a good replacement of the homotopy category of projectives for non-affine schemes, because in case $X = \mathrm{Spec}(R)$ is affine, the categories $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{K}(\mathrm{Proj}(R))$ are triangulated equivalent. There is a model for $\mathbf{D}(\mathrm{Flat}(X))$ in $\mathrm{Ch}(\mathrm{QCoh}(X))$ given by the triple

$$\mathcal{M}_{\mathrm{flat}} = (\mathrm{Ch}(\mathrm{Flat}(X)), \mathcal{W}, \mathrm{dg}(\mathrm{Cot}(X))).$$

(see [11, Corollary 4.1]). We devote this section to provide a general method to produce model categories $\mathcal{M}$ in $\mathrm{Ch}(\mathrm{QCoh}(X))$ which are Quillen equivalent to $\mathcal{M}_{\mathrm{flat}}$. In particular this implies that the homotopy category $\mathrm{Ho}(\mathcal{M})$ and $\mathbf{D}(\mathrm{Flat}(X))$ are triangulated equivalent.

Theorem 2.6.1. *Let X be a scheme and let $\mathcal{P}$ be a property of modules and $\mathcal{A}$ its associated class of modules. Assume that $\mathcal{A} \subseteq \mathrm{Flat}$, and that the following conditions hold:*

(1) *The class $\mathcal{A}$ is Zariski-local.*
(2) *For each $R = \mathcal{O}_X(U)$, $U \in \mathfrak{U}$, the pair $(\mathcal{A}_R, \mathcal{B}_R)$ is a hereditary cotorsion pair generated by a set.*
(3) *For each $R = \mathcal{O}_X(U)$, $U \in \mathfrak{U}$, every flat $\mathcal{A}_R$-periodic module is trivial.*
(4) $j_*(\mathcal{A}_{\mathrm{qc}(U_\alpha)}) \subseteq \mathcal{A}_{\mathrm{qc}(X)}$, *for each $\alpha \subseteq \{0, \ldots, m\}$.*

Then there is an abelian model category structure $\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}$ in $\mathrm{Ch}(\mathrm{QCoh}(X))$ given by the triple $(\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \mathcal{W}, \mathrm{dg}(\mathcal{B}))$. If we denote by $\mathbf{D}(\mathcal{A}_{\mathrm{qc}})$ the homotopy category of $\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}$, then the categories $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{D}(\mathcal{A}_{\mathrm{qc}})$ are triangulated equivalent, induced by a Quillen equivalence between the corresponding model categories. In case $X = \mathrm{Spec}(R)$ is affine, $\mathbf{D}(\mathcal{A}_R)$ is triangulated equivalent to $\mathbf{K}(\mathrm{Proj}(R))$.

Before proving the theorem, let us focus on one particular instance of it: If we take $\mathcal{A} = \mathcal{VF}$ (the class of very flat modules) the theorem gives us that $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{D}(\mathcal{VF}(X))$ are triangulated equivalent. This generalizes to arbitrary schemes [22, Corollary 5.4.3], where such a triangulated equivalence is obtained for a semi-separated Noetherian scheme of finite Krull dimension.

Corollary 2.6.2. *For any scheme X, the categories $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{D}(\mathcal{VF}(X))$ are triangulated equivalent.*

Let us prove Theorem 2.6.1. We firstly require the following useful lemma.

Lemma 2.6.3. *Suppose $\mathcal{A}$ is as in Theorem 2.6.1 (possibly without satisfying condition (3)). Then for any $\mathcal{M}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(X))$ there exists a short exact sequence*

$$0 \to \mathcal{K}_\bullet \to \mathcal{F}_\bullet \to \mathcal{M}_\bullet \to 0,$$

where $\mathcal{F}_\bullet \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(X)})$ and $\mathcal{K}_\bullet \in \widetilde{\mathrm{Flat}}(X)$.

Proof. We essentially follow the proof of [23, Lemma 4.1.1]; the main difference is that instead of sheaves, we are dealing with complexes of sheaves. Starting with the empty set, we gradually construct such a short exact sequence with the desired properties manifesting on larger and larger unions of sets from $\mathfrak{U}$, reaching X in a finite number of steps.

Assume that for an open subscheme T of X we have constructed a short exact sequence $0 \to \mathcal{L}_\bullet \to \mathcal{G}_\bullet \to \mathcal{M}_\bullet \to 0$ such that the restriction $h^*(\mathcal{G}_\bullet)$ belongs to $\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(T)})$ ($h\colon T \hookrightarrow X$ being the inclusion map) and $\mathcal{L}_\bullet \in \widetilde{\mathrm{Flat}}(X)$. Let $U \in \mathfrak{U}$ (with the inclusion map $j\colon U \hookrightarrow X$); our goal is to construct a short exact sequence $0 \to \mathcal{L}'_\bullet \to \mathcal{G}'_\bullet \to \mathcal{M}_\bullet \to 0$ with the same property with respect to the set $U \cup T$. Let us note that the adjoint pairs of functors on sheaves (j^*, j_*), (h^*, h_*) yield corresponding adjoint pairs of functors on complexes of sheaves.

Pick a short exact sequence

$$0 \to \mathcal{K}_\bullet \to \mathcal{Z}_\bullet \to j^*(\mathcal{G}_\bullet) \to 0 \tag{E}$$

of complexes of sheaves over the affine subscheme U, where $\mathcal{Z}_\bullet \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U)}) = \mathrm{Ch}(\mathcal{A}_{\mathcal{O}_U(U)})$ and $\mathcal{K}_\bullet \in \mathrm{Ch}(\mathcal{A}_{\mathcal{O}_U(U)})^\perp$, i.e. a special precover in the category of complexes of $\mathcal{O}_U(U)$-modules. In this (affine) setting we know from [21] that $\mathcal{K}_\bullet \in \widetilde{\mathrm{Flat}}(U)$, since $\mathcal{K}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(U)) \cap \mathrm{Ch}(\mathcal{A}_{\mathcal{O}_U(U)})^\perp \subseteq \mathrm{Ch}(\mathrm{Flat}(U)) \cap \mathrm{Ch}(\mathrm{Proj}(U))^\perp$. Using the direct image functor, we get $0 \to j_*(\mathcal{K}_\bullet) \to j_*(\mathcal{Z}_\bullet) \to j_*j^*(\mathcal{G}_\bullet) \to 0$

on X. Since $U \in \mathfrak{U}$ is affine, j_* is an exact functor taking flats to flats and also preserving $\mathcal{A}$ by condition (4), so $j_*(\mathcal{K}_\bullet) \in \widetilde{\mathrm{Flat}}(X)$, whence $j_*(\mathcal{Z}_\bullet)$ stays in $\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})$. Now considering the pull-back with respect to the adjunction morphism $\mathcal{G}_\bullet \to j_*j^*(\mathcal{G}_\bullet)$, one gets a new short exact sequence ending in $\mathcal{G}_\bullet$; let $\mathcal{G}'_\bullet$ be its middle term:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_*(\mathcal{K}_\bullet) & \longrightarrow & \mathcal{G}'_\bullet & \longrightarrow & \mathcal{G}_\bullet & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_*(\mathcal{K}_\bullet) & \longrightarrow & j_*(\mathcal{Z}_\bullet) & \longrightarrow & j_*j^*(\mathcal{G}_\bullet) & \longrightarrow & 0 \end{array}$$

Let us now check that $\mathcal{G}'_\bullet$ is in $\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U\cup T)})$; by Zariski-locality of $\mathcal{A}$, this is sufficient to check on U and T separately, i.e. $j^*(\mathcal{G}'_\bullet) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U)})$ and $h^*(\mathcal{G}'_\bullet) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(T)})$. The former is easy, since $j^*(\mathcal{G}'_\bullet) \cong \mathcal{Z}_\bullet$, which is in $\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U)})$ by the construction.

To verify the latter, let $k\colon U\cap T \hookrightarrow T$ and $l\colon U\cap T \hookrightarrow U$ be the embeddings; note that $U\cap T$ is open affine by semi-separatedness. Since $h^*(\mathcal{G}_\bullet) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(T)})$, we have $l^*j^*(\mathcal{G}_\bullet) = k^*h^*(\mathcal{G}_\bullet) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U\cap T)})$ ($\mathcal{A}$ being Zariski-local class). Furthermore, $l^*(\mathcal{G}_\bullet) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U\cap T)})$. By applying the exact functor l^* to the sequence (E), we now get that $l^*(\mathcal{K}_\bullet) \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(U\cap T)})$ by the resolving property of $\mathcal{A}$. Finally, $h^*j_*(\mathcal{K}_\bullet) = k_*l^*(\mathcal{K}_\bullet) \in \mathrm{Ch}(A_{\mathrm{qc}(T)})$ using the property (4). Therefore $h^*(\mathcal{G}'_\bullet)$ as an extension of $h^*j_*(\mathcal{K}_\bullet)$ by $h^*(\mathcal{G}_\bullet)$ belongs to $\mathrm{Ch}(A_{\mathrm{qc}(T)})$, too.

Finally, the kernel $\mathcal{L}'_\bullet$ of the composition of morphisms $\mathcal{G}'_\bullet \to \mathcal{G}_\bullet \to \mathcal{M}_\bullet$ is an extension of $\mathcal{L}_\bullet$ and $j_*(\mathcal{K}_\bullet)$, hence a complex from $\widetilde{\mathrm{Flat}}(X)$. This proves the existence of the short exact sequence from the statement. □

Proof of Theorem 2.6.1. First of all we notice that the class $\mathcal{A}_{\mathrm{qc}}$ contains a family of generators for $\mathrm{QCoh}(X)$; this is just a variation of the idea used in the proof of [22, Lemma 4.1.1], where we replace the class of very flat quasicoherent sheaves by $\mathcal{A}_{\mathrm{qc}}$ (and do not care about the kernel of the morphisms), which is possible thanks to property (4).

Then, by [6, Corollary 3.15] we get in $\mathrm{QCoh}(X)$ the complete hereditary cotorsion pair $(\mathcal{A}_{\mathrm{qc}}, \mathcal{B})$ generated by a set. Thus by [11, Theorem 4.10] we get the abelian model structure $\mathcal{M}^{\mathrm{qc}}_{\mathcal{A}_{\mathrm{qc}}} = (\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \mathcal{W}_1, \mathrm{dg}(\mathcal{B}))$ in $\mathrm{Ch}(\mathrm{QCoh}(X))$ given by the two complete hereditary cotorsion pairs:

$$(\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})^{\perp}) \quad \text{and} \quad (\widetilde{\mathcal{A}_{\mathrm{qc}}}, \mathrm{dg}(\mathcal{B})).$$

Since $\mathcal{A}_{\mathrm{qc}} \subseteq \mathrm{Flat}(X)$, we get the corresponding induced cotorsion pairs in the category $\mathrm{Ch}(\mathrm{Flat}(X))$ (with the induced exact structure from $\mathrm{Flat}(X)$):

$$(\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})^{\perp} \cap \mathrm{Ch}(\mathrm{Flat}(X))) \quad \text{and} \quad (\widetilde{\mathcal{A}_{\mathrm{qc}}}, \mathrm{dg}(\mathcal{B}) \cap \mathrm{Ch}(\mathrm{Flat}(X))).$$

To see that e.g. the former one is indeed a cotorsion pair, we have to check that $\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}) = {}^{\perp}(\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})^{\perp} \cap \mathrm{Ch}(\mathrm{Flat}(X))) \cap \mathrm{Ch}(\mathrm{Flat}(X))$. The inclusion "$\subseteq$" is clear. To see the other one, pick $\mathcal{X}_\bullet \in {}^{\perp}(\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})^{\perp} \cap \mathrm{Ch}(\mathrm{Flat}(X))) \cap \mathrm{Ch}(\mathrm{Flat}(X))$ and consider a short exact sequence $0 \to \mathcal{B}_\bullet \to \mathcal{A}_\bullet \to \mathcal{X}_\bullet \to 0$ with $\mathcal{A}_\bullet \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})$ and $\mathcal{B}_\bullet \in \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})^{\perp}$. As $\mathcal{A}_{\mathrm{qc}} \subseteq \mathrm{Flat}(X)$ and $\mathrm{Ch}(\mathrm{Flat}(X))$ is a resolving class, we infer that $\mathcal{B}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(X))$. Thus the sequence splits and $\mathcal{X}_\bullet$ is a direct summand of $\mathcal{A}_\bullet$, hence an element of $\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})$. The proof for the latter cotorsion pair goes in a similar way.

Now we will apply [11, Lemma 4.3] to these two complete cotorsion pairs in the category $\mathrm{Ch}(\mathrm{Flat}(X))$ and to the thick class $\mathscr{W} = \widetilde{\mathrm{Flat}}(X)$ in $\mathrm{Ch}(\mathrm{Flat}(X))$. So we need to check that the following conditions hold:

(i) $\widetilde{\mathscr{A}_{\mathrm{qc}}} = \mathrm{Ch}(\mathscr{A}_{\mathrm{qc}}) \cap \widetilde{\mathrm{Flat}}(X)$.

(ii) $\mathrm{Ch}(\mathscr{A}_{\mathrm{qc}})^{\perp} \cap \mathrm{Ch}(\mathrm{Flat}(X)) \subseteq \widetilde{\mathrm{Flat}}(X)$.

Since every flat $\mathscr{A}_R$-periodic module is trivial and the classes $\mathscr{A}$ and Flat are Zariski-local, we immediately infer that every flat $\mathscr{A}_{\mathrm{qc}}$-periodic quasi-coherent sheaf is trivial. Thus, from Proposition 2.4.2, we get condition (i). So let us see condition (ii). Let $\mathscr{L}_\bullet \in \mathrm{Ch}(\mathscr{A}_{\mathrm{qc}})^{\perp} \cap \mathrm{Ch}(\mathrm{Flat}(X))$. Since the pair

$$(\mathrm{Ch}(\mathrm{Flat}(X)), \mathrm{Ch}(\mathrm{Flat}(X))^{\perp})$$

in $\mathrm{Ch}(\mathrm{QCoh}(X))$ has enough injectives, there exists an exact sequence,

$$0 \to \mathscr{L}_\bullet \to \mathscr{P}_\bullet \to \mathscr{M}_\bullet \to 0,$$

with $\mathscr{P}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(X))^{\perp}$ and $\mathscr{M}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(X))$. Now, since $\mathscr{L}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(X))$, we get that

$$\mathscr{P}_\bullet \in \mathrm{Ch}(\mathrm{Flat}(X)) \cap \mathrm{Ch}(\mathrm{Flat}(X))^{\perp} = \widetilde{\mathrm{Flat\,Cot}}(X),$$

where $\widetilde{\mathrm{Flat\,Cot}}(X)$ is the class of all contractible complexes with components in $\mathrm{Flat}(X) \cap \mathrm{Flat}(X)^{\perp}$; this equality of classes is an easy consequence of [8, Proposition 3.2]. By Lemma 2.6.3, there exists an exact sequence

$$0 \to \mathscr{K}_\bullet \to \mathscr{F}_\bullet \to \mathscr{M}_\bullet \to 0,$$

where $\mathscr{F}_\bullet \in \mathrm{Ch}(\mathscr{A}_{\mathrm{qc}})$ and $\mathscr{K}_\bullet \in \widetilde{\mathrm{Flat}}(X)$. Now, we take the pull-back of $\mathscr{P}_\bullet \to \mathscr{M}_\bullet$ and $\mathscr{F}_\bullet \to \mathscr{M}_\bullet$, so we get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{K}_\bullet & = & \mathscr{K}_\bullet & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}_\bullet & \to & \mathscr{Q}_\bullet & \to & \mathscr{F}_\bullet & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}_\bullet & \to & \mathscr{P}_\bullet & \to & \mathscr{M}_\bullet & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In the middle column, the complexes $\mathscr{K}_\bullet$ and $\mathscr{P}_\bullet$ belong to $\widetilde{\mathrm{Flat}}(X)$. Therefore, the complex $\mathscr{Q}_\bullet$ also belongs to $\widetilde{\mathrm{Flat}}(X)$. Since $\mathscr{F}_\bullet \in \mathrm{Ch}(\mathscr{A}_{\mathrm{qc}})$ and $\mathscr{L}_\bullet \in \mathrm{Ch}(\mathscr{A}_{\mathrm{qc}})^{\perp}$, the exact sequence in the middle row splits. So, $\mathscr{L}_\bullet \in \widetilde{\mathrm{Flat}}(X)$ as desired.

Therefore by [11, Lemma 4.3] we have an exact model structure in the category $\mathrm{Ch}(\mathrm{Flat}(X))$ given by the triple

$$\mathscr{M}^{\mathrm{Ch_{flat}}}_{\mathscr{A}_{\mathrm{qc}}} = (\mathrm{Ch}(\mathscr{A}_{\mathrm{qc}}), \widetilde{\mathrm{Flat}}(X), \mathrm{dg}(\mathscr{B}) \cap \mathrm{Ch}(\mathrm{Flat}(X))).$$

Since it has the same class of trivial objects, this model is Quillen equivalent to the flat model in $\mathrm{Ch}(\mathrm{Flat}(X))$,

$$\mathcal{M}_{\mathrm{flat}}^{\mathrm{Ch}_{\mathrm{flat}}} = (\mathrm{Ch}(\mathrm{Flat}(X)), \widetilde{\mathrm{Flat}}(X), \mathrm{dg}(\mathrm{Cot}(X)) \cap \mathrm{Ch}(\mathrm{Flat}(X))).$$

This is, in turn, the restricted model of the model

$$\mathcal{M}_{\mathrm{flat}} = (\mathrm{Ch}(\mathrm{Flat}(X), \mathcal{W}, \mathrm{dg}(\mathrm{Cot}(X)))$$

in $\mathrm{Ch}(\mathrm{QCoh}(X))$ with respect to the exact category $\mathrm{Ch}(\mathrm{Flat}(X))$ of cofibrant objects. Thus, $\mathcal{M}_{\mathrm{flat}}$ and $\mathcal{M}_{\mathrm{flat}}^{\mathrm{Ch}_{\mathrm{flat}}}$ are canonically Quillen equivalent. To finish the proof, let us show that the model $\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}^{\mathrm{Ch}_{\mathrm{flat}}}$ is Quillen equivalent to

$$\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}} = (\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}(X)}), \mathcal{W}_1, \mathrm{dg}(\mathcal{B})).$$

But this model is canonically Quillen equivalent to its restriction to the cofibrant objects, i.e.

$$\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}^{\mathrm{Ch}_{\mathcal{A}_{\mathrm{qc}}}} = (\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \widetilde{\mathcal{A}}_{\mathrm{qc}}, \mathrm{dg}(\mathcal{B}) \cap \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}})).$$

Finally the Quillen equivalent cofibrant restricted model of

$$\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}^{\mathrm{Ch}_{\mathrm{flat}}} = (\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \widetilde{\mathrm{Flat}}(X), \mathrm{dg}(\mathcal{B}) \cap \mathrm{Ch}(\mathrm{Flat}(X)))$$

is given by the triple

$$(\mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \widetilde{\mathrm{Flat}}(X) \cap \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}), \mathrm{dg}(\mathcal{B}) \cap \mathrm{Ch}(\mathcal{A}_{\mathrm{qc}}))),$$

which by condition (i) above is precisely the previous model $\mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}^{\mathrm{Ch}_{\mathcal{A}_{\mathrm{qc}}}}$. In summary, we have the following chain of Quillen equivalences between the several models,

$$\mathcal{M}_{\mathrm{flat}} \simeq \mathcal{M}_{\mathrm{flat}}^{\mathrm{Ch}_{\mathrm{flat}}} \simeq \mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}^{\mathrm{Ch}_{\mathrm{flat}}} \simeq \mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}^{\mathrm{Ch}_{\mathcal{A}_{\mathrm{qc}}}} \simeq \mathcal{M}_{\mathcal{A}_{\mathrm{qc}}}.$$

The first and the last models give our desired Quillen equivalence. □

Recall from [4] that $\mathcal{M} \in \mathrm{QCoh}(X)$ is an *infinite-dimensional vector bundle* if, for each $U \in \mathcal{U}$, the $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ is projective. We will denote by $\mathrm{Vect}(X)$ the class of all infinite-dimensional vector bundles on X. In case $\mathrm{Vect}(X)$ contains a generating set of $\mathrm{QCoh}(X)$, we know from [6, Corollary 3.15 and 3.16] that the pair $(\mathrm{Vect}(X), \mathcal{B})$ (where $\mathcal{B} := \mathrm{Vect}(X)^{\perp}$) is a complete cotorsion pair generated by a set. It is hereditary, because the class $\mathrm{Vect}(X)$ is resolving. Thus by [11, Theorem 4.10] we get the abelian model structure $\mathcal{M}_{\mathrm{vect}} = (\mathrm{Ch}(\mathrm{Vect}(X)), \mathcal{W}_1, \mathrm{dg}(\mathcal{B}))$ in $\mathrm{Ch}(\mathrm{QCoh}(X))$ given by the two complete hereditary cotorsion pairs:

$$(\mathrm{Ch}(\mathrm{Vect}(X)), \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}) \quad \text{and} \quad (\widetilde{\mathrm{Vect}}(X), \mathrm{dg}(\mathcal{B})).$$

We will denote by $\mathbf{D}(\mathrm{Vect}(X))$ its homotopy category.

We are now in position to prove Corollary 2 in the Introduction. Let us remind the reader of our standing assumption that all schemes are quasi-compact and semi-separated.

Corollary 2.6.4. *Let X be a scheme with enough infinite-dimensional vector bundles. Then the categories $\mathbf{D}(\mathrm{Flat}(X))$ and $\mathbf{D}(\mathrm{Vect}(X))$ are triangle equivalent, the equivalence being induced by a Quillen equivalence between the corresponding model categories.*

Proof. The proof will follow by showing that $\mathbf{D}(\mathrm{Vect}(X))$ and $\mathbf{D}(\mathcal{VF}(X))$ are Quillen equivalent, and then by applying Corollary 2.6.2. To this end, we will prove that the model structures $\mathcal{M}_{\mathrm{vect}}$ and $\mathcal{M}_{\mathcal{VF}}$ have the same trivial objects. To achieve this, by [10, Theorem 1.2], it suffices to show that the trivial fibrant and cofibrant objects of one structure are trivial also in the other structure. This assertion is clearly satisfied by the trivial cofibrants of $\mathcal{M}_{\mathrm{vect}}$ and trivial fibrants of $\mathcal{M}_{\mathcal{VF}}$, as

$$\widetilde{\mathrm{Vect}}(X) \subseteq \widetilde{\mathcal{VF}}(X) \quad \text{and} \quad \mathrm{Ch}(\mathcal{VF}(X))^{\perp} \subseteq \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}.$$

Now let $\mathcal{V}_\bullet \in \widetilde{\mathcal{VF}}(X)$; since there are enough infinite-dimensional vector bundles, the cotorsion pair $(\widetilde{\mathrm{Vect}}(X), \mathrm{dg}(\mathscr{B}))$ has enough projectives, hence there is a short exact sequence

$$0 \to \mathcal{Q}_\bullet \to \mathcal{P}_\bullet \to \mathcal{V}_\bullet \to 0$$

with $\mathcal{P}_\bullet \in \widetilde{\mathrm{Vect}}(X)$. Restricting this to an open affine subset of X, we obtain a short exact sequence with a complex of projective modules in the middle and ending in a complex of very flat modules, and the objects of cycles also belonging to the respective classes. Since the projective dimension of very flat modules does not exceed 1, it follows that $\mathcal{Q}_\bullet$ has also projective cycles after this restriction, hence $\mathcal{Q}_\bullet \in \widetilde{\mathrm{Vect}}(X)$. We conclude that $\mathcal{V}_\bullet$, being a factor of two trivial objects, is itself trivial in $\mathcal{M}_{\mathrm{vect}}$.

Finally, pick $\mathcal{M}_\bullet \in \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}$. Using the completeness of the cotorsion pair $(\mathrm{Ch}(\mathcal{VF}(X)), \mathrm{Ch}(\mathcal{VF}(X))^{\perp})$, we obtain a short exact sequence

$$0 \to \mathcal{K}_\bullet \to \mathcal{V}_\bullet \to \mathcal{M}_\bullet \to 0$$

with $\mathcal{V}_\bullet \in \mathrm{Ch}(\mathcal{VF}(X))$ and $\mathcal{K}_\bullet \in \mathrm{Ch}(\mathcal{VF}(X))^{\perp}$. As $\mathcal{K}_\bullet$ is trivial in $\mathcal{M}_{\mathrm{vect}}$, it suffices to show that $\mathcal{V}_\bullet$ is trivial, too. Furthermore, $\mathrm{Ch}(\mathcal{VF}(X))^{\perp} \subseteq \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}$ implies that in fact, $\mathcal{V}_\bullet \in \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}$. So as above, construct a short exact sequence

$$0 \to \mathcal{Q}_\bullet \to \mathcal{P}_\bullet \to \mathcal{V}_\bullet \to 0,$$

this time with $\mathcal{P}_\bullet \in \mathrm{Ch}(\mathrm{Vect}(X))$ and $\mathcal{Q}_\bullet \in \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}$. The same local argument as above shows that $\mathcal{Q}_\bullet \in \mathrm{Ch}(\mathrm{Vect}(X))$, and we also have $\mathcal{P}_\bullet \in \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}$ (being an extension of two objects from the class). Hence $\mathcal{V}_\bullet$ is a factor of two complexes from the class $\mathrm{Ch}(\mathrm{Vect}(X)) \cap \mathrm{Ch}(\mathrm{Vect}(X))^{\perp}$, which is a subclass of $\widetilde{\mathrm{Vect}}(X)$ and consequently $\widetilde{\mathcal{VF}}(X)$, therefore consisting of trivial objects of $\mathcal{M}_{\mathcal{VF}}$. ☐

Finally, the last consequence is also an application of Theorem 2.6.1 for the class of very flat quasi-coherent sheaves. It follows from Gillespie [11, Theorem 4.10]

Corollary 2.6.5. *There is a recollement*

$$\mathbf{D}_{\mathrm{ac}}(\mathcal{VF}(X)) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;j\;}}} \mathbf{D}(\mathcal{VF}(X)) \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;w\;}}} \mathbf{D}(X)$$

Remark 2.6.6. Murfet and Salarian deal in [19] with a suitable generalization of total acyclicity for schemes. Namely, they define the category $\mathbf{D}_{\text{F-tac}}(\text{Flat}(X))$[4] of *F-totally acyclic complexes* in $\mathbf{D}(\text{Flat}(X))$ and prove that, in case $X = \text{Spec}(R)$ is affine and R is Noetherian of finite Krull dimension, $\mathbf{D}_{\text{F-tac}}(\text{Flat}(X))$ is triangle equivalent to $\mathbf{K}_{\text{tac}}(\text{Proj}(R))$ (the homotopy category of totally acyclic complexes of projective modules) showing that $\mathbf{D}_{\text{F-tac}}(\text{Flat}(X))$ also constitutes a good replacement of $\mathbf{K}_{\text{tac}}(\text{Proj}(R))$ in a non-affine context. An analogous version of Theorem 2.6.1 allows to restrict the equivalence between $\mathbf{D}(\text{Flat}(X))$ and $\mathbf{D}(\mathscr{A}_{\text{qc}})$ to their corresponding categories of F-totally acyclic complexes $\mathbf{D}_{\text{F-tac}}(\mathscr{A}_{\text{qc}})$ and $\mathbf{D}_{\text{F-tac}}(\text{Flat}(X))$. In particular, the full subcategory $\mathbf{D}_{\text{F-tac}}(\mathcal{VF}(X))$ of F-totally acyclic complexes of very flat quasi-coherent sheaves in $\mathbf{D}(\mathcal{VF}(X))$ is triangle equivalent with Murfet's and Salarian's derived category of F-totally acyclic complexes of flats.

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3. Countably generated flat modules are quite flat

This chapter is based on the preprint available at Project Euclid, to appear in the Journal of Commutative Algebra. It is also available at arXiv:1907.00356.

3.1 Introduction

Over any ring, the Govorov–Lazard Theorem provides a description of flat modules as direct limits of finitely generated free modules. However, this description, while sometimes useful, does not give much insight into the properties of flat modules; for example, for the ring of integers, the theorem says that every torsion-free abelian group is the direct limit of finitely generated free abelian groups, which is clear from the fact that finitely generated subgroups of torsion-free groups are free. However, a more informative description of torsion-free groups is available, going back to Trlifaj [8] with a generalization due to Bazzoni–Salce [2] (see the beginning of the introduction to [7]). So one wishes, and sometimes can have, a more precise description of flat modules.

The descriptions of classes of modules (in particular, flat modules) that we have in mind are formulated in terms of transfinite extensions. Recall that if $\mathcal{C}$ is a class of R-modules, then an R-module M is a *transfinite extension* of modules from $\mathcal{C}$ if there is a well-ordered chain of submodules of M, $(M_\alpha \mid \alpha \leq \sigma)$, such that $M_0 = 0$, $M_\sigma = M$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for every limit ordinal $\alpha \leq \sigma$, and the quotient module $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for every for every $\alpha < \sigma$. We also say that M is *$\mathcal{C}$-filtered* in that case.

In particular, the class of quite flat modules over a commutative ring R was defined in the paper [7] as follows. We say that an R-module C is *almost cotorsion* if $\mathrm{Ext}^1_R(S^{-1}R, C) = 0$ for all (at most) countable multiplicative subsets $S \subseteq R$. An R-module F is said to be *quite flat* if $\mathrm{Ext}^1_R(F, C) = 0$ for all almost cotorsion R-modules C. By [6, Corollary 6.14], this means that quite flat modules are precisely the direct summands of transfinite extensions of modules of the form $S^{-1}R$, where S is a countable multiplicative subset of R.

It was shown in [7] that all flat modules over a commutative Noetherian ring with a countable spectrum are quite flat. In this paper we prove the following generalization of this result: For any commutative Noetherian ring, any countably generated flat module is quite flat. Then we offer an alternative proof of the mentioned theorem from [7], by explaining how to deduce the description of arbitrary flat modules over a commutative Noetherian ring with countable spectrum from the description of countably generated flat modules.

To be more specific, the theorem that all countably generated flat modules over a commutative Noetherian ring are quite flat is proved in Section 3.2. In Section 3.3 we work more generally with a commutative Noetherian ring R whose spectrum has cardinality smaller than κ, where κ is a regular uncountable cardinal. In this setting, we prove that every flat R-module is a transfinite extension of $< \kappa$-generated flat R-modules.[1]

We are grateful to Jan Trlifaj for the suggestion to include Remark 3.3.7. We also want to thank the anonymous referee for careful reading of the manuscript and several helpful suggestions on the improvement of the exposition.

3.2 Noetherian rings

In this section we prove the main result promised in the title of the paper: *All countably generated flat modules over a Noetherian commutative ring are quite flat.* There are two main ingredients: Firstly, there is the "Main Lemma" from [7], which makes it possible to check whether a module is quite flat by reducing the question to rings of smaller Krull dimension. We recall the statement for the convenience of the reader.

Lemma 3.2.1 ([7, Main Lemma 1.18]). *Let R be a Noetherian commutative ring and $S \subseteq R$ be a countable multiplicative subset. Then a flat R-module F is quite flat if and only if the R/sR-module F/sF is quite flat for all $s \in S$ and the $S^{-1}R$-module $S^{-1}F$ is quite flat.*

The second ingredient is a lemma ensuring that there is always a suitable countable multiplicative subset to be used in Lemma 3.2.1. Before formulating the lemma, we prove a proposition, which holds even for non-Noetherian commutative rings.

Proposition 3.2.2. *Let R be a commutative ring and F a countably presented flat R-module. Let $T \subseteq R$ be a multiplicative subset such that $T^{-1}F$ is a projective $T^{-1}R$-module. Then there is a countable multiplicative subset $S \subseteq T$ such that $S^{-1}F$ is a projective $S^{-1}R$-module.*

Proof. It is a standard fact that countably presented flat modules have projective dimension at most one. Furthermore, by [6, Corollary 2.23], F is the cokernel of a monomorphism between countable-rank free R-modules; let $f\colon R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ be this monomorphism. The monomorphism $T^{-1}f\colon T^{-1}R^{(\mathbb{N})} \to T^{-1}R^{(\mathbb{N})}$ splits by assumption; let $g\colon T^{-1}R^{(\mathbb{N})} \to T^{-1}R^{(\mathbb{N})}$ be a map of $T^{-1}R$-modules such that $(T^{-1}f)g = \mathrm{id}_{T^{-1}R^{(\mathbb{N})}}$.

The maps $T^{-1}f$ and g, being maps between free modules, can be represented by column-finite matrices of countable size of elements of $T^{-1}R$ (provided we view the elements of free modules as column vectors); denote by A and B the corresponding matrices, respectively, and let E be the identity matrix of countable size. Then $AB - E = 0$, a matrix equation which translates into countably many equations in $T^{-1}R$. Every such equation becomes a valid equation in R after multiplying by an appropriate element of T; pick such an element for each of the equations and let $V \subseteq T$ be the set of all these elements. Further, let $D \subseteq T$ be the set of all denominators appearing in the entries of the matrix B.

Both V and D are countable sets, therefore the multiplicative subset $S \subseteq R$ generated by $V \cup D$ is countable, too. As $D \subseteq S$, the entries of B are naturally elements of $S^{-1}R$ and since $V \subseteq S$, the matrix equation $AB - E = 0$ holds in

[1] The original introduction of the paper continues here with a decription of results over non-noetherian rings.

$S^{-1}R$, too. Hence B defines a splitting of the monomorphism $S^{-1}f\colon S^{-1}R^{(\mathbb{N})} \to S^{-1}R^{(\mathbb{N})}$, the cokernel of which is $S^{-1}F$, which is therefore a projective $S^{-1}R$-module. It remains to observe that $V \cup D \subseteq T$ implies $S \subseteq T$. □

Lemma 3.2.3. *Let R be a Noetherian commutative ring and F a countably generated flat module. Then there is a countable multiplicative subset $S \subseteq R$ such that $S \cap \mathfrak{q} = \emptyset$ for every minimal prime ideal $\mathfrak{q}$ of R and $S^{-1}F$ is a projective $S^{-1}R$-module.*

Proof. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ be the minimal prime ideals of R and put $T = R \setminus (\mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_k)$. Then T is a multiplicative subset intersecting all but the minimal primes of R, hence $T^{-1}R$ is an Artinian ring. It follows that $T^{-1}F$ is a projective $T^{-1}R$-module.

Since R is Noetherian, every countably generated module is countably presented, so, by Proposition 3.2.2, there is a countable multiplicative subset $S \subseteq T$ such that $S^{-1}F$ is a projective $S^{-1}R$-module. Finally, the inclusion $S \subseteq T$ implies $S \cap \mathfrak{q} = \emptyset$ for every minimal prime $\mathfrak{q}$ by the choice of T. □

We are now ready to prove the main result.

Theorem 3.2.4. *Let R be a Noetherian commutative ring and F a countably generated flat module. Then F is quite flat.*

Proof. The strategy, "Noetherian induction", is borrowed from the proof of [7, Theorem 1.17]. Assume that $F_0 = F$ is a countably generated flat module which is *not* quite flat. By Lemma 3.2.3, there is a countable multiplicative subset S_0 not intersecting the minimal primes of $R_0 = R$ and such that $S_0^{-1}F_0$ is a projective $S_0^{-1}R_0$-module. Therefore, by Lemma 3.2.1, since F_0 is not quite flat, there is $s_0 \in S_0$ such that F_0/s_0F_0, which is a countably generated flat R_0/s_0R_0-module, is not a quite flat R_0/s_0R_0-module.

The ring $R_1 = R_0/s_0R_0$ is a Noetherian commutative ring and by Lemma 3.2.3, we again obtain a multiplicative subset $S_1 \subseteq R_1$ with analogous properties with respect to the ring R_1 and the R_1-module $F_1 = F_0/s_0F_0$. Similarly, Lemma 3.2.1 produces an element $s_1 \in S_1$ such that F_1/s_1F_1 is not a quite flat R_1/s_1R_1-module. Repeating this procedure, we obtain an infinite sequence $s_0 \in R_0$, $s_1 \in R_1$ etc.

Denote by $\tilde{s}_n \in R$ any preimage of $s_n \in R_n$ for every $n \in \mathbb{N}_0$ and let I_n be the ideal generated by $\tilde{s}_0, \dots, \tilde{s}_n$. Since each s_n is picked from S_n, which avoids the minimal primes of R_n, the chain of ideals $I_0, I_1, \dots$ is strictly increasing, which contradicts Noetherianity of R. We conclude that F is a quite flat R-module. □

Corollary 3.2.5. *Let R be a Noetherian commutative ring. Then an R-module F is a countably generated flat module if and only if F is a direct summand of a transfinite extension, indexed by a countable ordinal, of R-modules of the form $S^{-1}R$, where S ranges over countable multiplicative subsets of R.*

Proof. The "if" part is clear. As for the "only if" part, by Theorem 3.2.4, F is quite flat, so as pointed out in [7, §1.6], it is a direct summand of a transfinite extension E of R-modules of the form $S^{-1}R$, where S are countable multiplicative subsets. Now by the Hill Lemma [6, Theorem 7.10] (taking $\kappa = \aleph_1$, $M = E$, $N = 0$, and X a countable generating set of F in (H4)), F is in fact contained in a countably generated module $E' \subseteq E$, again filtered by modules of the form $S^{-1}R$. An inspection of the last paragraph of the proof of [6, Theorem 7.10] then shows that the ordinal type of the filtration of E' is countable. □

3.3 Noetherian rings with bounded cardinality of spectrum

Let R be a Noetherian commutative ring with countable spectrum; then, by [7, Theorem 1.17], all flat R-modules are quite flat. In particular, all flat R-modules are transfinite extensions of countably generated flat modules. This result can be proved directly, which we are going to do now.

The following lemma is standard and holds also in the non-commutative case once the obvious alterations are made. We spell it out so we can refer to it easily.

Lemma 3.3.1. *Let R be a commutative ring and M, F R-modules such that $M \subseteq F$ and I an ideal of R. The following are equivalent:*

(1) the map $M \otimes_R (R/I) \to F \otimes_R (R/I)$ is injective,

(2) the map $M/IM \to F/IF$ is injective,

(3) $IF \cap M \subseteq IM$ (in which case necessarily $IF \cap M = IM$).

Proof. (1) ⇔ (2): By tensoring the short exact sequence $0 \to I \to R \to R/I \to 0$ by an R-module A and noting that the image of $A \otimes_R I \to A \otimes_R R \cong A$ is precisely IA, we get that $A \otimes_R (R/I)$ is naturally isomorphic to A/IA for any A and I.

(2) ⇔ (3): The kernel of the composition $M \hookrightarrow F \to F/IF$ is precisely $IF \cap M$, so $M/IM \to F/IF$ is injective if and only if $IF \cap M \subseteq IM$, and since $IF \cap M \supseteq IM$ holds always, this is also equivalent to $IF \cap M = IM$. □

The following is again a known result: The general (not necessarily commutative) case is e.g. [1, Lemma 19.18], and the Noetherian case was established in [3, Lemma 4.2 and the following paragraph], although the proof is quite different.

Lemma 3.3.2. *Let R be a commutative ring, F a flat R-module and M a submodule of F. Then M is a pure submodule of F if and only if for each finitely generated ideal I of R, the natural map*

$$M \otimes_R (R/I) \to F \otimes_R (R/I)$$

is injective. If R is a Noetherian commutative ring, then it suffices to take for I the prime ideals of R.

Proof. If the inclusion of M into F is pure, then it stays injective after tensoring with any R-module, in particular with R/I.

On the other hand, since F is flat, M is a pure submodule if and only if the factormodule $C = F/M$ is flat, i.e., $\mathrm{Tor}_1^R(C, A) = 0$ for every R-module A. However, the vanishing of Tor is preserved by transfinite extensions, and since every R-module is a transfinite extension of cyclic modules, it suffices to verify that $\mathrm{Tor}_1^R(C, R/I) = 0$ for every ideal of R. Moreover, since every ideal is the directed union of its finitely generated subideals, every cyclic module is the direct limit of modules of the form R/I for I finitely generated, and since Tor commutes with direct limits, we see that it is enough to test that $\mathrm{Tor}_1^R(C, R/I) = 0$ for every finitely generated ideal of R. Since F is flat, $\mathrm{Tor}_1^R(F, R/I) = 0$, so $\mathrm{Tor}_1^R(C, R/I)$ is precisely the kernel of the map $M \otimes_R (R/I) \to F \otimes_R (R/I)$, hence it is zero if and only if this map is injective.

If R is a Noetherian ring, then every module is a transfinite extension of modules of the form $R/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of R. Therefore it suffices to check only that $\mathrm{Tor}_1^R(C, R/\mathfrak{p}) = 0$ and the argument concludes in the same way. □

If F is not flat, Lemma 3.3.2 (even its weaker form) is no longer valid even in the Noetherian case, which we are most interested in:

Example 3.3.3. Let k be a field, $k[x, y]$ the ring of polynomials in two variables and $R = k[x, y]/(x^2, xy, y^2)$. We will denote the cosets of x and y in R again by x and y for simplicity. Let F be a k-vector space with five-element basis $\{a, b, s, t, e\}$, on which we define the actions of x and y as follows: $xs = ys = 0$, $xt = yt = 0$, $xa = s$, $ya = t$, $xb = t$, $yb = 0$, $xe = s$, $ye = 0$; it is easy to see that this makes F an R-module. Furthermore, the k-subspace generated by $\{a, b, s, t\}$ is an R-submodule of F, which we denote by M. We claim that $IM = IF \cap M$ for every ideal I of R, but M is not pure in F.

Firstly, observe that F/M is the simple R-module on which x and y act by zero. Since

$$x(\alpha a + \beta b + \varepsilon e) = (\alpha + \varepsilon)s + \beta t,$$
$$y(\alpha a + \beta b + \varepsilon e) = \alpha t$$

for $\alpha, \beta, \varepsilon \in k$, the only k-linear combination of a, b, e annihilated by both x and y is the trivial one. Therefore k-linear combinations of s and t are the only elements of F killed by both x and y. We conclude that there is no section of the R-module projection $F \to F/M$, hence M is not a direct summand and consequently, not a pure submodule of F.

Secondly, note that whenever I is an ideal of R such that $I \not\subseteq (y)$, then $s \in IM$: Either I contains an element i with a non-zero absolute term, in which case $is = s$, or $I \subseteq (x, y)$. In the latter case, there are $u, v \in k$, $u \neq 0$ such that $ux + vy \in I$; then one can find $\alpha, \beta \in k$ such that $(ux + vy)(\alpha a + \beta b) = s$ by solving a system of two linear equations with regular matrix.

A typical element q of IF is of the form

$$q = i_1(m_1 + \varepsilon_1 e) + \cdots + i_n(m_n + \varepsilon_n e),$$

where $i_1, \ldots, i_n \in I$, $m_1, \ldots, m_n \in M$ and $\varepsilon_1, \ldots, \varepsilon_n \in k$. The element $r = (i_1\varepsilon_1 + \cdots + i_n\varepsilon_n)e$ is a linear combination of s and e; for q to be in $IF \cap M$, r must be a multiple of s, therefore $r \in IM$ by the discussion above. Since $i_1 m_1 + \cdots + i_n m_n \in IM$, we conclude that $q \in IM$ as desired.

Finally, if an ideal I satisfies $I \subseteq (y)$, then $IM = IF$ and we are done.

Let κ be a cardinal. We say that a commutative ring R is $< \kappa$-*Noetherian* if every ideal of I is $< \kappa$-generated. Note that by [6, Lemma 6.31], submodules of $< \kappa$-generated modules over a $< \kappa$-Noetherian ring are $< \kappa$-generated; in particular, every $< \kappa$-generated module is $< \kappa$-presented.

Lemma 3.3.4. *Let κ be an uncountable regular cardinal, R a $< \kappa$-Noetherian commutative ring, I an ideal of R, F an R-module and X a subset of F of cardinality $< \kappa$. Then there is a $< \kappa$-generated submodule $M \subseteq F$ such that $X \subseteq M$ and $IM = IF \cap M$.*

Proof. Let $X_0 = X$. Denote by M_0 the submodule of F generated by X_0; this is a $< \kappa$-generated module. Since R is $< \kappa$-Noetherian, the submodule $IF \cap M_0$ of M_0 is $< \kappa$-generated, too; let Y_0 be a set of cardinality $< \kappa$ generating this module. Every $y \in Y_0$ can be written as

$$y = p_1 y_1 + \cdots + p_n y_n,$$

where $p_i \in I$ and $y_i \in F$ for $i = 1, \ldots, n$. Gathering these y_i's for all $y \in Y_0$, we obtain a subset $Z_0 \subseteq F$ of cardinality $< \kappa$. By the construction, the submodule $M_1 \subseteq F$ generated by $X_0 \cup Z_0$ has the property $IF \cap M_0 \subseteq IM_1$.

Now repeat this procedure, starting with the set $X_1 = X_0 \cup Z_0$ of cardinality $< \kappa$, obtaining a subset $Z_1 \subseteq F$ of cardinality $< \kappa$. Continuing in this fashion, i.e., repeating the procedure with $X_{i+1} = X_i \cup Z_i$, we obtain an $\mathbb{N}_0$-indexed chain $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ of subsets of F of cardinality $< \kappa$; let X be its union. Note that the cardinality of X is less than κ, since κ is uncountable and regular. We claim that the submodule $M \subseteq F$ generated by X has the desired property: This is because $M = \bigcup_{n \in \mathbb{N}_0} M_n$ and

$$IF \cap M = \bigcup_{n \in \mathbb{N}_0} (IF \cap M_n) \subseteq \bigcup_{n \in \mathbb{N}_0} IM_{n+1} = IM.$$

□

Lemma 3.3.5. *Let R be a Noetherian commutative ring with spectrum of cardinality less than κ, where κ is an uncountable regular cardinal. Let F be a flat R-module and X a subset of F of cardinality $< \kappa$. Then there is a pure submodule $M \subseteq F$ such that $X \subseteq M$ and M is $< \kappa$-generated.*

Proof. We prove the lemma by "iterating Lemma 3.3.4 sufficiently many times" for each prime ideal of R. More precisely, let λ_0 be the cardinality of the spectrum of R. Put $\lambda = \lambda_0$ if λ_0 is infinite, and let λ be the countable cardinality if λ_0 is finite. Let $\psi \colon \lambda \to \lambda$ be a surjective function such that for each ordinal $\alpha < \lambda$, the preimage $\psi^{-1}(\alpha)$ is unbounded in λ. Also let $\{\mathfrak{p}_\alpha \mid \alpha < \lambda\}$ be a numbering of the spectrum of R in which every prime ideal of R appears at least once. Finally, put $M_0 = 0$.

Now starting with $X_0 = X$, apply Lemma 3.3.4 with $I = \mathfrak{p}_{\psi(0)}$ to get $< \kappa$-generated submodule $M_1 \subseteq F$ such that $X_0 \subseteq M_1$ and $\mathfrak{p}_{\psi(0)} M_1 = \mathfrak{p}_{\psi(0)} F \cap M_1$. More generally, for every $\alpha < \lambda$, if M_α is constructed, let $M_{\alpha+1}$ be the result of applying Lemma 3.3.4 with the prime ideal $\mathfrak{p}_{\psi(\alpha)}$ and with a generating set of M_α of cardinality $< \kappa$. For every limit ordinal $\alpha < \lambda$, let $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$; since κ is regular, this keeps M_α $< \kappa$-generated for each $\alpha < \lambda$.

Put $M = \bigcup_{\beta < \lambda} M_\beta$. Since $\lambda < \kappa$, M is $< \kappa$-generated. Moreover, by the choice of ψ, for every $\gamma < \lambda$, M is the union of those $M_{\alpha+1}$ for which $\psi(\alpha) = \gamma$. Therefore, for every $\gamma < \lambda$,

$$\mathfrak{p}_\gamma F \cap M = \bigcup_{\substack{\alpha < \lambda \\ \psi(\alpha) = \gamma}} (\mathfrak{p}_\gamma F \cap M_{\alpha+1}) = \bigcup_{\substack{\alpha < \lambda \\ \psi(\alpha) = \gamma}} \mathfrak{p}_\gamma M_{\alpha+1} = \mathfrak{p}_\gamma M.$$

We conclude that $\mathfrak{p}M = \mathfrak{p}F \cap M$ holds for every prime $\mathfrak{p}$ as desired, which by Lemmas 3.3.1 and 3.3.2 means that M is a pure submodule of F. □

Note that in the case $\kappa = \aleph_1$, the lemma can be proved using already known results: Knowing that all flat modules are quite flat in this case [7, Theorem 1.17], it follows easily from the Hill Lemma [6, Theorem 7.10].

Remark 3.3.6. Let us comment here on the overall situation concerning "purifications": It is a standard fact that for a ring R of cardinality not exceeding an infinite cardinal λ, every R-module F and subset $X \subseteq F$ of cardinality at most λ, there is a pure submodule $M \subseteq F$ of cardinality at most λ containing X; see e.g. [6, Lemma 2.25(a)]. Lemma 3.3.5 shows that when R is commutative Noetherian and F is flat, then instead of the cardinality of the ring, one can take a potentially sharper bound, the cardinality of the spectrum (which, for Noetherian rings, cannot exceed the cardinality of the ring). This is thanks to Lemma 3.3.2.

Example 3.3.3 shows that when enlarging arbitrary submodules of non-flat modules to pure submodules, one has to add more than just "divisors", in particular, one cannot rely on Lemma 3.3.2. However, we do not know whether Lemma 3.3.5 holds for non-flat modules over commutative Noetherian rings or not.

Remark 3.3.7. In the special case when F is a flat and Mittag-Leffler module (see e.g. [4] or [6] for the definition), a stronger result than Lemma 3.3.5 is known [4, Lemma 2.7(2)]: For any ring R, a flat Mittag-Leffler module F, an uncountable cardinal κ, and a subset X in F of cardinality $< \kappa$, there exists a pure submodule $M \subseteq F$ such that $X \subseteq M$ and M is $< \kappa$-generated. Since free modules are flat Mittag-Leffler and a pure submodule of a flat Mittag-Leffler module is flat Mittag-Leffler [6, Corollary 3.20], this also covers the case of pure submodules of free modules settled by Osofsky [5, Theorem I.8.10].

Generally speaking, however, the bound of Lemma 3.3.5 is sharp. Indeed, let k be a field of infinite cardinality κ and $R = k[x]$ the ring of polynomials in one variable x with coefficients in k. Then the spectrum of R has cardinality κ, and the field of rational functions $Q = k(x)$ is a κ-generated flat R-module which has no nonzero proper pure submodules. Taking $X \subseteq Q$ to be the one-element set $X = \{1\}$, there does *not* exist a $< \kappa$-generated submodule M in Q containing X.

We are now ready to prove the improved deconstructibility of flat modules.

Theorem 3.3.8. *Let R be a Noetherian commutative ring with spectrum of cardinality less than κ, where κ is an uncountable regular cardinal. Then every flat module is a transfinite extension of $< \kappa$-generated flat modules.*

Proof. This is quite standard: Let F be a flat module; we are going to build a filtration of F by pure submodules such that the consecutive factors are $< \kappa$-generated. Let $F_0 = 0$. For every ordinal α, pick $x \in F \setminus F_\alpha$ (if it exists, otherwise the construction is finished) and let M be the $< \kappa$-generated pure submodule of the flat module F/F_α containing $x + F_\alpha$; this exists thanks to Lemma 3.3.5. Further let $F_{\alpha+1}$ be the preimage of M in the map $F \to F/F_\alpha$; then $F_{\alpha+1}$ is a pure submodule of F containing x and $F_{\alpha+1}/F_\alpha \cong M$ is $< \kappa$-generated. For every limit ordinal α, put $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$. This way we exhaust the module F as desired. □

Finally, as a special case, we obtain a new proof of [7, Theorem 1.17]:

Corollary 3.3.9. *Let R be a Noetherian commutative ring with countable spectrum. Then every flat module is quite flat.*

Proof. By Theorem 3.3.8 with $\kappa = \aleph_1$, every flat module is a transfinite extension of countably generated flat modules. By Theorem 3.2.4, countably generated flat modules are quite flat, hence all flat modules are quite flat. □

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4. Purity in categories of sheaves

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4.1 Introduction

Purity and related concepts (such as pure-injectivity, definable categories, the Ziegler spectrum) play a key role in the model theory of modules and have found many uses in the representation theory of finite-dimensional algebras and other parts of module theory.

This theory of purity, which in more particular form goes back to Prüfer [19] and Cohn [3], is based on that of a pure-exact sequence—that is, a direct limit of split-exact sequences. It extends to many additive categories, including categories of sheaves and categories of quasicoherent sheaves. Perhaps surprisingly, this *categorical* purity differs, outside the affine case, from the *geometric* purity which is implicit in the usual definitions of tensor product and flatness for sheaves. The general relationship between these has been explored in [5]; here we look more closely; we develop further results and use these to compute some examples which illustrate this relationship. In particular we give a completely explicit description of each of the geometric and categorical Ziegler spectrum for the category of quasicoherent sheaves over the projective line over a field.

Note that, for each (suitable) scheme X, there will be four Ziegler spectra: one for each of categorical and geometric purity, and for each category $\mathcal{O}_X$-Mod and $\mathrm{QCoh}(X)$. We reserve the term Ziegler spectrum for the categorical one, while the geometric one is called geometric part in the paper (as it is always a subset).

The paper is organised as follows: We start in Section 4.2 by briefly exploring the relations of the purities in $\mathcal{O}_X$-Mod and $\mathrm{QCoh}(X)$; note that while the pure-exact sequences (both categorical and geometric) turn out to be the same in these categories, the corresponding pure-injective objects are quite different.

Section 4.3 looks deeper into the purity-related notions in $\mathcal{O}_X$-Mod. We investigate which of them are preserved or reflected by the three functors associated to an open subset: the restriction, the extension by zero, and the direct image. The main result of the section is the description of the geometric pure-injectives in $\mathcal{O}_X$-Mod: They are precisely the skyscraper sheaves with an indecomposable pure-injective module of sections by Corollary 4.3.18. After this, Section 4.4 presents an example of the Ziegler spectrum of the category $\mathcal{O}_X$-Mod over a local affine 1-dimensional scheme X.

In Section 4.5, we turn our attention to quasicoherent sheaves, restricting to the case of quasicompact quasiseparated schemes. We start by showing that such schemes are affine if and only if the two purities coincide in the category $\mathrm{QCoh}(X)$. We continue by describing the geometric part of the Ziegler spectrum of $\mathrm{QCoh}(X)$, showing that this is always "glued from affine pieces" and forms a quasicompact closed subset of the spectrum. To this closed set we assign a definable subcategory $\mathcal{D}_X \subseteq \mathrm{QCoh}(X)$, whose objects enjoy the property—among others—that every geometrically pure-exact sequence starting in them is categorically pure.

Finally, Section 4.6 is devoted to the computation of the Ziegler spectrum of the category of quasicoherent sheaves over a projective line. We describe both the points and the topology, noting that unlike the affine case, the Ziegler spectrum turns out not to be quasicompact. We also show that in this case, the subcategory $\mathcal{D}_X$ allows a more explicit description.

Setup. Throughout, X denotes a scheme, $\mathcal{O}_X$-Mod the category of all $\mathcal{O}_X$-modules and $\mathrm{QCoh}(X)$ the category of all quasicoherent modules (sheaves) on X. Whenever $U \subseteq X$ is an open set, $\mathcal{O}_U$ is a shorthand for the restriction $\mathcal{O}_X|_U$.

For the majority of our results, we will assume that the scheme in question has some topological property:

Definition 4.1.1. A scheme is called *quasiseparated* if the intersection of any two quasicompact open sets is quasicompact. A scheme is *concentrated* if it is quasicompact and quasiseparated.

We continue by defining all the purity-related notions used in the paper.

Definition 4.1.2. A short exact sequence in $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ in $\mathcal{O}_X$-Mod is *geometrically pure* (or *g-pure* for short) if it stays exact after applying the sheaf tensor product functor $- \otimes \mathcal{Y}$ for all $\mathcal{Y} \in \mathcal{O}_X$-Mod; equivalently, [5, Proposition 3.2], if the sequence of $\mathcal{O}_{X,x}$-modules $0 \to \mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x \to 0$ is pure-exact for each $x \in X$. We define *g-pure monomorphisms* and *g-pure epimorphisms* in the obvious way. An $\mathcal{O}_X$-module $\mathcal{N}$ is *g-pure-injective* if the functor $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(-, \mathcal{N})$ is exact on g-pure exact sequences of $\mathcal{O}_X$-modules.

Definition 4.1.3. Recall that $\mathrm{QCoh}(X)$ is a full subcategory of $\mathcal{O}_X$-Mod. Following [5], we say that a short exact sequence of quasicoherent sheaves is *g-pure* if it is g-pure in the larger category $\mathcal{O}_X$-Mod. By [5, Propositions 3.3 & 3.4], this notion of g-purity for quasicoherent sheaves is equivalent to purity after restricting either to all open affine subsets, or to a chosen open affine cover of X. We say that a quasicoherent sheaf $\mathcal{N}$ is *g-pure-injective in* $\mathrm{QCoh}(X)$ if the functor $\mathrm{Hom}_{\mathrm{QCoh}(X)}(-, \mathcal{N})$ is exact on g-pure exact sequences of quasicoherent sheaves.

Remark 4.1.4. The notion of g-purity in $\mathrm{QCoh}(X)$ could also be established via the property that tensoring with any *quasicoherent* sheaf preserves exactness. By [5, Remark 3.5], this would give the same for quasiseparated schemes.

Definition 4.1.5. If X is a quasiseparated scheme, it(s underlying topological space) has a basis of quasicompact open sets closed under intersections. Therefore, by [18, 3.5], the category $\mathcal{O}_X$-Mod is locally finitely presented and as such has a notion of purity: This is defined by exactness of the functors $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{F}, -)$, where $\mathcal{F}$ runs over all finitely presented objects (i.e. $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{F}, -)$ commutes with direct limits). We will call this notion *categorical purity* or *c-purity* for short, defining c-pure-injectivity etc. in a similar fashion as in Definition 4.1.2.

Definition 4.1.6. If X is a concentrated scheme, then the category $\mathrm{QCoh}(X)$ is locally finitely presented by [7, Proposition 7]. Again, all the c-pure notions are defined for $\mathrm{QCoh}(X)$ in a natural way.

We will use [14] as a convenient reference for many of the results that we use concerning purity and definability.

Remark 4.1.7. If X is a concentrated scheme, then by [5, Proposition 3.9], c-pure-exact sequences in $\mathrm{QCoh}(X)$ are g-pure-exact, and it is easy to see that the proof carries mutatis mutandis to the category $\mathcal{O}_X$-Mod for X quasiseparated. Therefore, in these cases, g-pure-injectivity is a stronger notion than c-pure-injectivity.

Let us point out that for having a well-behaved (categorical) purity, one does not need a locally finitely presented category; a *definable* category (in the sense of [14, Part III]) would be enough. However, we are not aware of any scheme X for which $\mathrm{QCoh}(X)$ or $\mathcal{O}_X$-Mod would be definable, but not locally finitely presented. Furthermore, the question of when exactly are these categories locally finitely presented does not seem to be fully answered. It is also an open question whether every definable Grothendieck category is locally finitely presented (there is a gap in the argument for this at [16, 3.6]).

In any case, the situation in this paper is as follows: Section 4.3, dealing with the category $\mathcal{O}_X$-Mod, does not need any special assumptions on the scheme X for most of its propositions, therefore it starts with the (slightly obscure) assumption on mere definability of $\mathcal{O}_X$-Mod. On the other hand, Section 4.5 really needs X to be concentrated almost all the time, a fact that is stressed in all the assertions.

If X is an affine scheme, then the category $\mathrm{QCoh}(X)$ is equivalent to the category of modules over the ring of global sections of X, and both g-purity and c-purity translate to the usual purity in module categories. A converse to this for X concentrated is Proposition 4.5.1. However, as the Section 4.4 shows, even for very simple affine schemes, the purities do not coincide in the category $\mathcal{O}_X$-Mod.

4.2 Relation between purity in $\mathcal{O}_X$-Mod and $\mathrm{QCoh}(X)$

Recall that the (fully faithful) forgetful functor $\mathrm{QCoh}(X) \to \mathcal{O}_X$-Mod has a right adjoint $\mathcal{C}\colon \mathcal{O}_X\text{-Mod} \to \mathrm{QCoh}(X)$, usually called the *coherator*. If we need to specify the scheme X, we use notation like $\mathcal{C}_X$.

Since g-purity in $\mathrm{QCoh}(X)$ is just "restricted" g-purity from $\mathcal{O}_X$-Mod, each quasicoherent sheaf which is g-pure-injective as an $\mathcal{O}_X$-module is also g-pure-injective as a quasicoherent sheaf. The example at the end of Section 4.4 shows that even in a quite simple situation, the converse is not true: A g-pure-injective quasicoherent sheaf need not be g-pure-injective in $\mathcal{O}_X$-Mod.

The following was observed in [5]:

Lemma 4.2.1 ([5, Lemma 4.7]). *Let $\mathcal{N}$ be a g-pure-injective $\mathcal{O}_X$-module. Then its coherator $\mathcal{C}(\mathcal{N})$ is a g-pure-injective quasicoherent sheaf.*

It is not clear whether the coherator preserves g-pure-exact sequences or at least g-pure monomorphisms. A partial result in this direction is Lemma 4.5.4.

The relation between c-purity in $\mathcal{O}_X$-Mod and $\mathrm{QCoh}(X)$ is in general not so clear as for g-purity. Note that while $\mathrm{QCoh}(X)$ is closed under direct limits in $\mathcal{O}_X$-Mod (indeed, arbitrary colimits), it is usually not closed under direct products and hence it is not a definable subcategory. However, in the case of X concentrated, much more can be said. We start with an important observation.

Definition 4.2.2 ([14, Part III]). A functor between definable categories is called *definable* if it commutes with products and direct limits.

Lemma 4.2.3. *Let X be a concentrated scheme. Then the coherator functor is definable.*

Proof. As a right adjoint, $\mathcal{C}$ always commutes with limits and in particular products. By [21, Lemma B.15], it also commutes with direct limits for X concentrated. □

Recall that by [14, Corollary 18.2.5], definable functors preserve pure-exactness and pure-injectivity. Hence we obtain the following properties for concentrated schemes:

Lemma 4.2.4. *Let X be a concentrated scheme.*

(1) The coherator functor preserves c-pure-exact sequences and c-pure-injectivity.
(2) A short exact sequence of quasicoherent sheaves is c-pure-exact in $\mathrm{QCoh}(X)$ *if and only if it is also c-pure-exact in the larger category* $\mathcal{O}_X$-Mod.
(3) A quasicoherent sheaf c-pure-injective in $\mathcal{O}_X$-Mod *is c-pure-injective in* $\mathrm{QCoh}(X)$.
(4) A quasicoherent sheaf finitely presented in $\mathrm{QCoh}(X)$ *is finitely presented in* $\mathcal{O}_X$-Mod.

Proof. (1) follows from definability of coherator.

(2) Since $\mathrm{QCoh}(X)$ is locally finitely presented, every c-pure-exact sequence is the direct limit of split short exact sequences. However, direct limits in $\mathrm{QCoh}(X)$ are the same as in $\mathcal{O}_X$-Mod, so we get the "only if" part. To see the "if" part, note that the coherator acts as the identity when restricted to $\mathrm{QCoh}(X)$, so the statement follows from (1).

(3) This is a consequence of (2), or we can again argue that the coherator is the identity on $\mathrm{QCoh}(X)$ and use (1).

(4) Let $\mathcal{F}$ be a finitely presented object of $\mathrm{QCoh}(X)$, I be a directed set and $(\mathcal{M}_i)_{i\in I}$ a directed system in $\mathcal{O}_X$-Mod. Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}\big(\mathcal{F}, \varinjlim_{i\in I} \mathcal{M}_i\big) &\cong \mathrm{Hom}_{\mathrm{QCoh}(X)}\big(\mathcal{F}, \mathcal{C}(\varinjlim_{i\in I} \mathcal{M}_i)\big)\\ &\cong \mathrm{Hom}_{\mathrm{QCoh}(X)}\big(\mathcal{F}, \varinjlim_{i\in I} \mathcal{C}(\mathcal{M}_i)\big)\\ &\cong \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{QCoh}(X)}\big(\mathcal{F}, \mathcal{C}(\mathcal{M}_i)\big)\\ &\cong \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{F}, \mathcal{M}_i),\end{aligned}$$

where the natural isomorphisms are due to (in this order) $\mathcal{C}$ being a right adjoint, $\mathcal{C}$ commuting with direct limits for concentrated schemes, $\mathcal{F}$ being finitely presented in $\mathrm{QCoh}(X)$, and finally the adjointness again. □

Note, however, that the examples at the end of Section 4.5 show that the pure-injectives in $\mathrm{QCoh}(X)$ have little in common with the pure-injectives of $\mathcal{O}_X$-Mod.

4.3 Purity in $\mathcal{O}_X$-Mod

Setup. If, in this section, any assertion involves c-purity or definability, then it is assumed that the scheme X is such that $\mathcal{O}_X$-Mod is a definable category (e.g. X is quasiseparated); similarly for $\mathcal{O}_U$-Mod if U is involved, too.

We start with a more general lemma, which is of its own interest. Recall that a full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is called *reflective* provided that the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ has a left adjoint (usually called the *reflector*).

Lemma 4.3.1. *Let $\mathcal{B}$ be an additive category with arbitrary direct sums and products and $\mathcal{A}$ a reflective subcategory. Then an object of $\mathcal{A}$ is pure-injective if and only if it is pure-injective as an object of $\mathcal{B}$.*

Proof. Recall the Jensen-Lenzing criterion for pure-injectivity (see [14, 4.3.6]): An object M is pure-injective if and only if for any set I, the summation map $M^{(I)} \to M$ factors through the natural map $M^{(I)} \to M^I$. Also recall that while limits of diagrams in $\mathcal{A}$ coincide with those in $\mathcal{B}$, colimits in $\mathcal{A}$ are computed via applying the reflector to the colimit in $\mathcal{B}$.

Let M be an object of $\mathcal{A}$, I any set and denote by $M^{(I)}$ the coproduct in $\mathcal{A}$ and $M^{(I)_\mathcal{B}}$ the coproduct in $\mathcal{B}$. By the adjunction, maps from $M^{(I)}$ to objects of $\mathcal{A}$ naturally correspond to maps from $M^{(I)_\mathcal{B}}$ to objects of $\mathcal{A}$, hence the summation map $M^{(I)} \to M$ factors through $M^{(I)} \to M^I$ if and only if $M^{(I)_\mathcal{B}} \to M$ factors through $M^{(I)_\mathcal{B}} \to M^I$: the naturality of adjunction ensures that the factorising map fits into the commutative diagram regardless of which direct sum we pick. □

Corollary 4.3.2. *A sheaf of $\mathcal{O}_X$-modules is c-pure-injective regardless if it is viewed as a sheaf or a presheaf.*

Proof. The category of sheaves is always a reflective subcategory of presheaves, sheafification being the reflector. □

Corollary 4.3.3. *Let $\mathcal{N}$ be a c-pure-injective $\mathcal{O}_X$-module and $U \subseteq X$ open. Then $\mathcal{N}(U)$ is a pure-injective $\mathcal{O}_X(U)$-module.*

Proof. Let I be any set. By the previous corollary, $\mathcal{N}$ is also a (c-)pure-injective object in the category of presheaves, so the summation map from the presheaf coproduct $\mathcal{N}^{(I)\,\mathrm{pre}} \to \mathcal{N}$ factors through $\mathcal{N}^{(I)\,\mathrm{pre}} \hookrightarrow \mathcal{N}^I$. However, for presheaves we have $\mathcal{N}^I(U) \cong \mathcal{N}(U)^I$ and (unlike for sheaves) $\mathcal{N}^{(I)\,\mathrm{pre}}(U) \cong \mathcal{N}(U)^{(I)}$, so we have the desired factorisation for $\mathcal{N}(U)$ as well. □

In the case of concentrated open sets we can say even more:

Lemma 4.3.4. *Let $U \subseteq X$ be a concentrated open set. Then the functor of sections over U, $\mathcal{M} \mapsto \mathcal{M}(U)$, is definable.*

Proof. The functor of sections commutes with products for any open set. Furthermore, if the open set is concentrated, then this functor also commutes with direct limits by [22, 009E]. □

Since g-purity is checked stalk-wise, it is useful to overview the related properties of skyscrapers; recall that if $x \in X$ and M is an $\mathcal{O}_{X,x}$-module, then the *skyscraper (sheaf)* $\iota_{x,*}(M)$ is an $\mathcal{O}_X$-module given by

$$\iota_{x,*}(M)(U) = \begin{cases} M & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}$$

for $U \subseteq X$ open. This gives a functor $\mathcal{O}_{X,x}\text{-Mod} \to \mathcal{O}_X\text{-Mod}$, which is a fully faithful right adjoint to taking stalks at x.

Lemma 4.3.5. *For every $x \in X$ the functor $\iota_{x,*}$ is definable.*

Proof. Commuting with products is clear. If $(M_i)_{i \in I}$ is a directed system of $\mathcal{O}_{X,x}$-modules, then $\iota_{x,*}(\varinjlim_{i \in I} M_i)$ coincides with the "section-wise direct limit", i.e. direct limit in the presheaf category. Since this direct limit is again a skyscraper, hence a sheaf, it is the direct limit also in the sheaf category. □

From Lemma 4.3.5 we know that skyscrapers built from pure-injective modules are c-pure-injective. It is easy to see that they are even g-pure-injective:

Lemma 4.3.6. *Let $x \in X$ and N be a pure-injective $\mathcal{O}_{X,x}$-module. Then $\iota_{x,*}(N)$ is a g-pure-injective $\mathcal{O}_X$-module.*

Proof. Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a g-pure monomorphism in $\mathcal{O}_X$-Mod. Using the adjunction, checking that

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{B}, \iota_{x,*}(N)) \to \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{A}, \iota_{x,*}(N))$$

is surjective is equivalent to checking that

$$\mathrm{Hom}_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{B}_x, N) \to \mathrm{Hom}_{\mathcal{O}_{X,x}\text{-Mod}}(\mathcal{A}_x, N)$$

is surjective. But this is true since $\mathcal{A}_x \hookrightarrow \mathcal{B}_x$ is a pure mono in $\mathcal{O}_{X,x}$-Mod. □

The preceding observation allows us to establish a property of g-purity similar to that of c-purity:

Corollary 4.3.7. *A short exact sequence in* $\mathcal{O}_X$-Mod *is g-pure-exact if and only if it remains exact after applying the functor* $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(-, \mathcal{N})$ *for every g-pure-injective $\mathcal{O}_X$-module $\mathcal{N}$.*

Proof. The "only if" part is clear. The "if" follows from the fact that purity in $\mathcal{O}_{X,x}$-Mod (where $x \in X$) can be checked using the functors $\mathrm{Hom}_{\mathcal{O}_{X,x}\text{-Mod}}(-, N)$, where N is pure-injective, the adjunction between stalk and skyscraper, and Lemma 4.3.6.

Alternatively, one can also modify the standard proof of this fact to the present setting, since by [5, Proposition 4.4 & Corollary 4.6] or Lemma 4.3.20, every sheaf can be g-purely embedded into a g-pure-injective one. □

We recall the very general argument for the fact referred to in the above proof, since we use it elsewhere. If $f : A \to B$ is an embedding and the notion of purity is such that there is a pure embedding $h : A \to N$ for some pure-injective N, and if this factors through $g\colon B \to N$, then, using gf pure implies f pure, we deduce purity of f.

We proceed with investigating what purity-related notions are preserved under various functors between sheaf categories. Let $U \subseteq X$ be open; then there are the following three functors:

(1) $\iota_{U,!}\colon \mathcal{O}_U\text{-Mod} \to \mathcal{O}_X\text{-Mod}$, the *extension by zero*,
(2) $(-)|_U\colon \mathcal{O}_X\text{-Mod} \to \mathcal{O}_U\text{-Mod}$, the *restriction*,
(3) $\iota_{U,*}\colon \mathcal{O}_U\text{-Mod} \to \mathcal{O}_X\text{-Mod}$, the *direct image* (also called the *pushforward*).

These three form an adjoint triple

$$\iota_{U,!} \dashv (-)|_U \dashv \iota_{U,*}$$

with the outer two functors fully faithful—composing any of them with the restriction gives the identity. Consequently, all the three functors preserve stalks at any point of U. Finally, $\iota_{U,!}$ is always exact; see e.g. [9, §§II.4, II.6] for details.

Lemma 4.3.8. *Let $U \subseteq X$ be an open set. Then the restriction functor $(-)|_U$ is definable and preserves g-pure-exactness and g-pure-injectivity.*

Proof. Definability follows from the fact that this functor has both a right and a left adjoint. Since restriction preserves stalks, it preserves g-pure-exactness. Finally, let $\mathcal{N} \in \mathcal{O}_X$-Mod be g-pure-injective and $\mathcal{A} \hookrightarrow \mathcal{B}$ a g-pure monomorphism of $\mathcal{O}_U$-modules. Then the monomorphism $\iota_{U,!}(\mathcal{A}) \hookrightarrow \iota_{U,!}(\mathcal{B})$ is g-pure, because on U the stalks remain the same, whereas outside U they are zero (see, e.g., [9, p. 106]). Straightforward use of the adjunction then implies that $\mathcal{N}|_U$ is g-pure-injective in $\mathcal{O}_U$-Mod. □

Note that extension by zero does not commute with products in general, as the following example shows, therefore there is no hope for definability. It also preserves neither c- nor g-pure-injectivity:

Example 4.3.9. Let $p \in \mathbb{Z}$ be a prime number, $X = \operatorname{Spec}(\mathbb{Z})$ and

$$U = \operatorname{Spec}(\mathbb{Z}[p^{-1}]) \subseteq X.$$

For a prime number $q \neq p$, let $\mathcal{N}^q = \iota_{(q),*}(\mathbb{Z}[p^{-1}]/(q))$, i.e. the skyscraper coming from the q-element group, regarded as an $\mathcal{O}_U$-module. By Lemma 4.3.6, this is a g-pure-injective $\mathcal{O}_U$-module, and so is $\mathcal{N} = \prod_{q \neq p} \mathcal{N}^q$. Let $\mathcal{M} = \iota_{U,!}(\mathcal{N})$; we are going to show that the global sections of $\mathcal{M}$ are not a pure-injective $\mathbb{Z}$-module, thus (Corollary 4.3.3) $\mathcal{M}$ is not even c-pure-injective.

Since extension by zero preserves all sections within U, $\mathcal{M}(U)$ is the product of all q-element groups for q a prime distinct from p. By [9, Definition 6.1], $\mathcal{M}(X)$ consists of those elements of $\mathcal{M}(U)$, whose support in U is closed in X. Now every element of $\mathcal{M}(U)$ is either torsion, in which case its support is a finite union of closed points, therefore closed in X, or torsion-free, which has a non-zero stalk at the generic point and hence on each point of U, but U is not closed in X. We infer that $\mathcal{M}(X)$ is the torsion part of $\mathcal{M}(U)$, in other words the direct sum inside the direct product. However, this is not pure-injective, as it is torsion and reduced, but not bounded.

Finally, the global sections of the sheaf $\iota_{U,!}(\mathcal{N}^q)$ form the q-element group, hence we see that the global sections of the product $\mathcal{P} = \prod_{q\neq p} \iota_{U,!}(\mathcal{N}^q)$ are different from $\mathcal{M}(X)$ (namely, $\mathcal{P}(X)$ is the product of q-element groups over all primes $q \neq p$), hence $\iota_{U,!}$ does not commute with products.

Here is what we can actually prove:

Lemma 4.3.10. *Let $U \subseteq X$ be an open set. Then the functor $\iota_{U,!}$ preserves and reflects c- and g-pure-exact sequences, and reflects c- and g-pure-injectivity.*

Proof. A monomorphism $\mathcal{A} \hookrightarrow \mathcal{B}$ in $\mathcal{O}_X$-Mod is c-pure (g-pure) if and only if the contravariant functor $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(-, \mathcal{N})$ turns it into a surjection for every c-pure-injective (g-pure-injective) $\mathcal{O}_X$-module $\mathcal{N}$ (for g-purity, this is Corollary 4.3.7); similarly for monomorphisms in $\mathcal{O}_U$-Mod.

Due to the adjunction, the surjectivity of

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\iota_{U,!}(\mathcal{B}), \mathcal{N}) \to \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\iota_{U,!}(\mathcal{A}), \mathcal{N})$$

is equivalent to the surjectivity of

$$\mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{B}, \mathcal{N}|_U) \to \mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{A}, \mathcal{N}|_U)$$

and by Lemma 4.3.8, $\mathcal{N}|_U$ is c-pure-injective (g-pure-injective) if $\mathcal{N}$ is, so the preservation of pure-exactness follows.

If a short exact sequence in $\mathcal{O}_U$-Mod becomes c-pure-exact (g-pure-exact) after applying $\iota_{U,!}$, then the original sequence has to be c-pure-exact (g-pure-exact), too, since it can be recovered by applying the restriction functor, which preserves both types of pure-exactness by Lemma 4.3.8. The same argument shows that $\iota_{U,!}$ reflects pure-injectivities as well. ☐

Finally, we investigate the properties of direct image.

Lemma 4.3.11. *Let $U \subseteq X$ be an open set. Then the functor $\iota_{U,*}$ preserves and reflects c- and g-pure-injectivity.*

Proof. That $\iota_{U,*}$ reflects c-pure-injectivity (g-pure-injectivity) is clear because its composition with restriction to U produces the identity functor. If $\mathcal{A} \hookrightarrow \mathcal{B}$ is c-pure (g-pure) in $\mathcal{O}_X$-Mod, then so is $\mathcal{A}|_U \hookrightarrow \mathcal{B}|_U$, hence

$$\mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{B}|_U, \mathcal{N}) \to \mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{A}|_U, \mathcal{N})$$

is surjective for $\mathcal{N} \in \mathcal{O}_U$-Mod c-pure-injective (g-pure-injective). The adjunction implies the surjectivity of

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{B}, \iota_{U,*}(\mathcal{N})) \to \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{A}, \iota_{U,*}(\mathcal{N}))$$

and we conclude that $\iota_{U,*}(\mathcal{N})$ is c-pure-injective (g-pure-injective).

Alternatively, for the c-pure-injectivity, we may argue as follows: Note that the full faithfulness of $\iota_{U,*}$ allows us to view $\mathcal{O}_U$-Mod as a reflective subcategory of $\mathcal{O}_X$-Mod, restriction functor being the reflector. The statement about c-pure-injectivity thus follows from Lemma 4.3.1. ☐

For special open sets U we obtain a definable functor:

Lemma 4.3.12. *Let $U \subseteq X$ be an open set such that $\iota_U : U \hookrightarrow X$ is a concentrated morphism (i.e. for every $V \subseteq X$ affine open, the intersection $U \cap V$ is concentrated). Then the functor $\iota_{U,*}$ is definable. If X is concentrated, then the essential image of $\iota_{U,*}$ is a definable subcategory of $\mathcal{O}_X$-Mod.*

Proof. Commuting with products follows from the fact that $\iota_{U,*}$ is right adjoint. Commuting with direct limits is a special case of [21, Lemma B.6].

If X is concentrated, then U is concentrated as well by [12, Lemma 16], and the intersection of any concentrated open $V \subseteq X$ with U is concentrated, too. Therefore, by Lemma 4.3.4, for each concentrated $V \subseteq X$, we have the definable functors of sections over V and $U \cap V$. Since direct limits and direct products are exact in categories of modules, we have another two definable functors, K_V and C_V, assigning to $\mathcal{M}$ the kernel and the cokernel of the restriction $\mathcal{M}(V) \to \mathcal{M}(U \cap V)$, respectively.

As concentrated open sets form a basis of X, we see that $\mathcal{M} \in \mathcal{O}_X$-Mod is in the essential image of $\iota_{U,*}$ if and only if $K_V(\mathcal{M}) = C_V(\mathcal{M}) = 0$ for every $V \subseteq X$ open concentrated. Therefore the essential image in question, being the intersection of kernels of definable functors, is a definable subcategory. □

Remark 4.3.13. Let us point out that [21, Lemma B.6] has a much wider scope: Indeed, for any concentrated map of schemes f, the direct image functor f_* is definable.

Let us now focus on the weaker notion of geometric purity. We start with observing that there is a plenty of naturally arising g-pure monomorphisms around.

Lemma 4.3.14. *Let $U \subseteq V$ be open subsets of X and denote by ι_U, ι_V their inclusions into X. Then for any $\mathcal{M} \in \mathcal{O}_X$-Mod the natural map*

$$\iota_{U,!}(\mathcal{M}|_U) \to \iota_{V,!}(\mathcal{M}|_V)$$

is a g-pure monomorphism.

Proof. Passing to stalks at $x \in X$, we see that the map is either the identity (if $x \in U$) or a map from the zero module (otherwise), hence a pure monomorphism of $\mathcal{O}_{X,x}$-modules. Hence the map is even stalkwise split. □

Recall that a sheaf is called *flasque* if all its restriction maps are surjective. Recall also that for $\mathcal{O}_X$-modules $\mathcal{A}$, $\mathcal{B}$, the *sheaf hom*, which we denote by $\mathcal{H}om_X(\mathcal{A}, \mathcal{B})$, is the $\mathcal{O}_X$-module defined via

$$\mathcal{H}om_X(\mathcal{A}, \mathcal{B})(U) = \mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{A}|_U, \mathcal{B}|_U)$$

for every open $U \subseteq X$, with the obvious restriction maps.

Corollary 4.3.15. *If $\mathcal{N} \in \mathcal{O}_X$-Mod is g-pure-injective and $\mathcal{A} \in \mathcal{O}_X$-Mod arbitrary, then the sheaf hom $\mathcal{O}_X$-module $\mathcal{H}om_X(\mathcal{A}, \mathcal{N})$ is flasque. In particular, $\mathcal{N} \cong \mathcal{H}om_X(\mathcal{O}_X, \mathcal{N})$ is flasque.*

Proof. Apply the functor $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(-, \mathcal{N})$ to the g-pure monomorphism

$$\iota_{U,!}(\mathcal{A}|_U) \to \iota_{V,!}(\mathcal{A}|_V)$$

(Lemma 4.3.14) and use the adjunction

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\iota_{U,!}(\mathcal{A}|_U), \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{A}|_U, \mathcal{N}|_U).$$

□

Lemma 4.3.16. *Let $U \subseteq X$ be open. Then the natural map*

$$\iota_{U,!}(\mathcal{M}|_U) \to \iota_{U,*}(\mathcal{M}|_U)$$

*is a g-pure monomorphism for every $\mathcal{M} \in \mathcal{O}_X$-*Mod.

Proof. Exactly the same as for Lemma 4.3.14—passing to stalks at $x \in X$, we see that the map is either the identity (if $x \in U$) or a map from the zero module (otherwise), hence a pure monomorphism of $\mathcal{O}_{X,x}$-modules. □

Corollary 4.3.17. *Let $\mathcal{N}$ be g-pure-injective $\mathcal{O}_X$-module. Every restriction map in $\mathcal{N}$ is a split epimorphism. For every $U \subseteq X$ open, $\iota_{U,*}(\mathcal{N}|_U)$ is a direct summand of $\mathcal{N}$.*

Proof. Let $\mathcal{N} \in \mathcal{O}_X$-Mod be g-pure-injective and U, V be open subsets of X. Using Lemma 4.3.16, we obtain a surjection

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\iota_{U,*}(\mathcal{N}|_U), \mathcal{N}) \to \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\iota_{U,!}(\mathcal{N}|_U), \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_U\text{-Mod}}(\mathcal{N}|_U, \mathcal{N}|_U).$$

Hence there is a map $f \colon \iota_{U,*}(\mathcal{N}|_U) \to \mathcal{N}$ which, after restricting to the subsheaf $\iota_{U,!}(\mathcal{N}|_U)$, corresponds to the identity map in the adjunction, thus being the identity when restricted to subsets of U. Note that

$$\iota_{U,*}(\mathcal{N}|_U)(V) = \mathcal{N}(U \cap V);$$

put $W = U \cap V$. Since the action of f on sections on V and W commutes with restriction maps (i.e. f is a sheaf homomorphism), we obtain a commutative square

$$\begin{array}{ccc} \mathcal{N}(W) = \iota_{U,*}(\mathcal{N}|_U)(V) & \xrightarrow{\quad f^V \quad} & \mathcal{N}(V) \\ {\scriptstyle \mathrm{res}_{WV}^{\iota_{U,*}(\mathcal{N}|_U)} = \mathrm{id}_{\mathcal{N}(W)}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{WV}^{\mathcal{N}}} \\ \mathcal{N}(W) = \iota_{U,*}(\mathcal{N}|_U)(W) & \xrightarrow{f^W = \mathrm{id}_{\mathcal{N}(W)}} & \mathcal{N}(W) \end{array}$$

We see that $\mathrm{res}_{WV}^{\mathcal{N}} \circ f^V$ is an epi-mono factorization of the identity map on $\mathcal{N}(W)$, from which both assertions in the statement follow—$\mathrm{res}_{WV}^{\mathcal{N}}$ is a split epimorphism and f is a split embedding of $\iota_{U,*}(\mathcal{N}|_U)$ into $\mathcal{N}$. □

Corollary 4.3.18. *Let $\mathcal{N}$ be an indecomposable g-pure-injective $\mathcal{O}_X$-module. Then there is $x \in X$ and an indecomposable pure-injective $\mathcal{O}_{X,x}$-module N such that $\mathcal{N} \cong \iota_{x,*}(N)$.*

Proof. Let us first show that restriction maps in $\mathcal{N}$ are either isomorphisms or maps to the zero module (but not maps *from* a zero module to a non-zero one). Assume this is not the case, so let U be an open subset of X such that $\ker \mathrm{res}_{XU}^{\mathcal{N}}$ is a proper non-zero direct summand of $\mathcal{N}(X)$. In that case, by Corollary 4.3.17, $\iota_{U,*}(\mathcal{N}|_U)$ is a non-trivial proper direct summand of $\mathcal{N}$, a contradiction.

Secondly, let S be the support of $\mathcal{N}$, i.e. the set of those points $x \in X$ such that $\mathcal{N}_x$ is non-zero; because of the above-described nature of restriction maps in $\mathcal{N}$, S coincides with the set of those $x \in X$ whose *every* open neighbourhood has non-zero sections. This is a closed subset of X, because if $x \in X \setminus S$, then there is an open set $U \subseteq X$ containing x with $\mathcal{N}(U) = 0$, hence $\mathcal{N}_y = 0$ for all $y \in U$ and $U \subseteq X \setminus S$.

Next we show that S is irreducible. Let $U, V \subseteq X$ be open sets satisfying $U \cap V \cap S = \emptyset$ ($\Leftrightarrow \mathcal{N}(U \cap V) = 0$), we want to show that $U \cap S = \emptyset$ ($\Leftrightarrow \mathcal{N}(U) = 0$) or $V \cap S = \emptyset$ ($\Leftrightarrow \mathcal{N}(V) = 0$). Since $\mathcal{N}$ is a sheaf the assumption implies that the map $\mathcal{N}(U \cup V) \to \mathcal{N}(U) \times \mathcal{N}(V)$ is an isomorphism but then, by the first paragraph, only one of $\mathcal{N}(U)$, $\mathcal{N}(V)$ can be non-zero, as desired.

We conclude that for any open set $U \subseteq X$, $\mathcal{N}(U)$ is non-zero if and only if U intersects the irreducible closed set S, i.e. contains its generic point x. Given the description of restriction maps above, we infer that $\mathcal{N}$ is indeed a skyscraper based on x. The associated $\mathcal{O}_{X,x}$-module has to be pure-injective by Corollary 4.3.3 and clearly has to be indecomposable, too. □

Lemma 4.3.19. *For any $\mathcal{M} \in \mathcal{O}_X$-*Mod *the natural map*

$$\mathcal{M} \to \prod_{x \in X} \iota_{x,*}(\mathcal{M}_x)$$

is g-pure-monomorphism.

Proof. Pick $x \in X$. Then $\mathcal{M}_x \cong \left(\iota_{U_x}(\mathcal{M}|_{U_j})\right)_x$, hence the projection on the x-th coordinate of the product is a splitting and the map is stalkwise split. □

The following has been already observed in [5], where character modules were used to give a proof. We give a different, slightly more constructive proof.

Lemma 4.3.20. *Every $\mathcal{M} \in \mathcal{O}_X$-*Mod *embeds g-purely into a g-pure-injective $\mathcal{O}_X$-module. This module can be chosen to be a product of indecomposable g-pure-injectives.*

Proof. For each $x \in X$, pick a pure embedding $\mathcal{M}_x \hookrightarrow N_x$, where N_x is a pure-injective $\mathcal{O}_{X,x}$-module; N_x can be chosen to be a product of indecomposables by [14, Corollary 5.3.53]. This gives rise to a map $\mathcal{M} \to \iota_{x,*}(N_x)$, the skyscraper being g-pure-injective by Lemma 4.3.6. Taking the diagonal of these maps we obtain a map

$$\mathcal{M} \to \prod_{x \in X} \iota_{x,*}(N_x).$$

To show that this is a g-pure monomorphism, pick $y \in X$ and passing to stalks at y we have

$$\mathcal{M}_y \to N_y \oplus \Big(\prod_{\substack{x \in X \\ x \neq y}} \iota_{x,*}(N_x)\Big)_y$$

which is a pure monomorphism after projecting on the left-hand direct summand, hence a pure monomorphism. □

Lemma 4.3.21. *Let $(U_i)_{i \in I}$ be an open cover of X and $\mathcal{M} \in \mathcal{O}_X$-*Mod*. Then the natural map*

$$\mathcal{M} \to \prod_{i \in I} \iota_{U_i,*}(\mathcal{M}|_{U_i})$$

is a g-pure-monomorphism.

Proof. Pick $x \in X$ and assume that $x \in U_j$, where $j \in I$. Then

$$\mathcal{M}_x \cong \left(\iota_{U_j,*}(\mathcal{M}|_{U_j})\right)_x,$$

therefore as in the previous lemma the projection on the j-th coordinate of the product is a splitting and the map is stalkwise split. □

4.4 Example: Spectrum of $\mathbb{Z}_{(p)}$

In this section we investigate the properties of sheaves over the affine scheme $\mathrm{Spec}(\mathbb{Z}_{(p)})$, where $p \in \mathbb{Z}$ is any prime number and $\mathbb{Z}_{(p)}$ denotes the localisation of the ring of integers $\mathbb{Z}$ at the prime ideal (p). Below, X denotes $\mathrm{Spec}(\mathbb{Z}_{(p)})$.

Since X is affine, the category $\mathrm{QCoh}(X)$ is equivalent to the category of modules over the discrete valuation ring $\mathbb{Z}_{(p)}$. Purity in such a category is well understood, hence we will not focus on it here at all.

As a topological space, X has two points, (p) and 0. Its non-empty open sets are $Y = \{0\}$ and X, with $\mathcal{O}_X(Y) = \mathbb{Q}$ and $\mathcal{O}_X(X) = \mathbb{Z}_{(p)}$. It is straightforward that any presheaf of $\mathcal{O}_X$-modules is automatically a sheaf[1]; therefore, the objects $\mathcal{M}$ of $\mathcal{O}_X$-Mod are described by the following data: a $\mathbb{Z}_{(p)}$-module $\mathcal{M}(X)$, a $\mathbb{Q}$-module (vector space) $\mathcal{M}(Y)$, and a $\mathbb{Z}_{(p)}$-module homomorphism $\mathcal{M}(X) \to \mathcal{M}(Y)$. This category is also easily seen to be equivalent to the category of right modules over the ring

$$R = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

This equivalence translates all c-pure notions in $\mathcal{O}_X$-Mod to ordinary purity in Mod-R. As for g-purity, note that in this simple setting, passing to stalks at a point corresponds to passing to the smallest open subset containing the point. Therefore, a short exact sequence of $\mathcal{O}_X$-modules is g-pure-exact if and only if it is pure exact after passing to global sections (g-purity on Y holds always).

The (right) Ziegler spectrum of R was described in [20, §4.1]. Let us give here an overview of the points (where CB denotes Cantor-Bendixson rank in the Ziegler spectrum):

$\mathcal{N}(X)$	$\mathcal{N}(Y)$	CB rank	injective	g-pure-inj.	quasicoh.
$\mathbb{Z}_{p^\infty}$	0	1	✓	✓	✓
$\mathbb{Q}$	0	2	✓	✓	
$\mathbb{Q}$	$\mathbb{Q}$	1	✓	✓	✓
$\mathbb{Z}_{(p)}/(p^k)$	0	0		✓	✓
$\overline{\mathbb{Z}_{(p)}}$	0	1		✓	
$\overline{\mathbb{Z}_{(p)}}$	$\overline{\mathbb{Q}_{(p)}}$	0			✓
0	$\mathbb{Q}$	1			

[1]Let us point out here that even though "there is no non-trivial covering of any open set", the sheaf axiom in general has the extra consequence that sections over the empty set are the final object of the category. Therefore, e.g. sheaves of abelian groups over this two-point space form a proper subcategory of presheaves, which need not assign the zero group to the empty set (!). However, since we always assume $\mathcal{O}_X$ to be a *sheaf* of rings, its ring of sections over the empty set is the zero ring, over which any module is trivial.

The map $\mathcal{N}(X) \to \mathcal{N}(Y)$ is always the obvious one (identity in the first non-trivial case, inclusion in the second). In all the positive cases, g-pure-injectivity follows directly from Lemma 4.3.6. On the other hand, the remaining two $\mathcal{O}_X$-modules are not flasque and therefore cannot be g-pure-injective (Corollary 4.3.15).

Note, however, that the penultimate module *is* g-pure-injective in the category $\mathrm{QCoh}(X)$, since it corresponds to the pure-injective $\mathbb{Z}_{(p)}$-module $\overline{\mathbb{Z}_{(p)}}$.

4.5 Purity in $\mathrm{QCoh}(X)$

While the previous section showed that even for very simple schemes X, the two purities differ in the category $\mathcal{O}_X$-Mod, it is not so difficult to answer when they coincide in $\mathrm{QCoh}(X)$, at least for concentrated schemes (cf. the introduction to [6], where the proof of this assertion is outlined for projective schemes):

Proposition 4.5.1. *Let X be a concentrated scheme. Then X is affine if and only if c-purity and g-purity in $\mathrm{QCoh}(X)$ coincide.*

Proof. The "only if" part is clear. On the other hand, observe that the structure sheaf $\mathcal{O}_X$ is locally flat (i.e. flat on every open affine subset of X), hence every short exact sequence of quasicoherent sheaves ending in it is g-pure. If, moreover, the sequence is c-pure, then it splits, since $\mathcal{O}_X$ is finitely presented for X concentrated. This means that if the two purities coincide, the first cohomology functor $\mathrm{H}^1(X, -) = \mathrm{Ext}^1_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -)$ vanishes on all quasicoherent sheaves. By Serre's criterion [22, 01XF], this implies that X is affine. □

The following example shows that some sort of finiteness condition on the scheme is indeed necessary to obtain the result.

Example 4.5.2. Let k be a field and $X = (\operatorname{Spec} k)^{(\mathbb{N})}$ the scheme coproduct of countably many copies of $\operatorname{Spec} k$. As a topological space, X is a countable discrete space. Every sheaf of $\mathcal{O}_X$-modules is quasicoherent and the category $\mathrm{QCoh}(X)$ is actually equivalent to the category of countable collections of k-vector spaces with no relations at all. Such a category is semisimple, hence all short exact sequences are both c-pure- and g-pure-exact.

We proceed with deeper investigation of geometric purity in $\mathrm{QCoh}(X)$. The following analogue of Lemma 4.3.7 and the similar criterion for c-purity will be useful; note that there is no restriction imposed on the scheme.

Lemma 4.5.3. *A short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ in $\mathrm{QCoh}(X)$ is g-pure-exact if and only if for every g-pure-injective $\mathcal{N} \in \mathrm{QCoh}(X)$, the sequence*

$$0 \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{C}, \mathcal{N}) \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{B}, \mathcal{N}) \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{A}, \mathcal{N}) \to 0$$

is exact.

Proof. The "only if" part is clear. To verify the "if" part, recall that by Corollary 4.3.7, the sequence is g-pure exact (in $\mathcal{O}_X$-Mod, which is equivalent), if it stays exact after applying $\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(-, \mathcal{N})$ for every $\mathcal{N}$ g-pure-injective in $\mathcal{O}_X$-Mod. The result now follows by using the coherator adjunction and the fact that coherator preserves g-pure-injectives. □

Lemma 4.5.4. *Let $\mathcal{M} \in \mathrm{QCoh}(X)$ be g-purely embedded into $\mathcal{N} \in \mathcal{O}_X$-Mod. Then this embedding factors through a g-pure monomorphism $\mathcal{M} \hookrightarrow \mathcal{C}(\mathcal{N})$.*

Proof. Since $\mathcal{C}$ is a right adjoint, we have a map $f\colon \mathcal{M} \to \mathcal{C}(\mathcal{N})$, through which the original embedding factorises. Passing to stalks at $x \in X$, we have a factorisation of pure embeddings of $\mathcal{O}_{X,x}$-modules and an appeal to [14, Lemma 2.1.12] shows that f_x is a pure monomorphism, hence f is a g-pure monomorphism. □

Lemma 4.5.5. *Let X be a quasicompact scheme, $\mathcal{N} \in \mathrm{QCoh}(X)$ indecomposable g-pure-injective and $U_1, \dots, U_n$ a finite open affine cover of X. Then there is $i \in \{1, \dots, n\}$ such that $\mathcal{N}$ is a direct summand of $\mathcal{C}(\iota_{U_i,*}(\mathcal{N}|_{U_i}))$.*

Proof. By Lemma 4.3.21, there is a g-pure monomorphism

$$\mathcal{N} \to \bigoplus_{1 \le i \le n} \iota_{U_i,*}(\mathcal{N}|_{U_i}),$$

which, by Lemma 4.5.4, yields another g-pure monomorphism of quasicoherent sheaves

$$\mathcal{N} \to \bigoplus_{1 \le i \le n} \mathcal{C}\big(\iota_{U_i,*}(\mathcal{N}|_{U_i})\big).$$

Since $\mathcal{N}$ is g-pure-injective, this map splits; moreover, $\mathcal{N}$ has local endomorphism ring, therefore $\mathcal{N}$ is a direct summand of one of the summands. □

However, we are not able to say much more for the general quasicompact case. Let us therefore restrict our attention further to concentrated schemes. At this point, it is convenient to clarify the role of the direct image functor.

Remark 4.5.6. Recall that for X concentrated, the functor $\iota_{U,*}$ preserves quasicoherence for every $U \subseteq X$ open affine (even open concentrated is enough, [22, 01LC]). Lemma 4.3.12 teaches us that under the same assumptions on X and U, $\iota_{U,*}$ is a definable functor from $\mathcal{O}_U$-Mod to $\mathcal{O}_X$-Mod. Since direct limits in the category of quasicoherent sheaves are the same as those in the larger category of all sheaves of modules, there is no need to care about direct limits. However, direct products do not agree, so some caution has to be exercised here.

Fortunately, if we view $\iota_{U,*}$ solely as a functor from $\mathrm{QCoh}(U)$ to $\mathrm{QCoh}(X)$, then this functor *does* commute with direct products, simply for the reason that it is the right adjoint to the restriction functor from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(U)$ (cf. the discussion in [21, B.13]). Therefore, the restricted direct image functor, which we further denote by $\iota^{\mathrm{qc}}_{U,*}\colon \mathrm{QCoh}(U) \to \mathrm{QCoh}(X)$, is definable.

For each $U \subseteq X$ open affine, the fully faithful functor $\iota^{\mathrm{qc}}_{U,*}$ identifies $\mathrm{QCoh}(U)$ with a definable subcategory of $\mathrm{QCoh}(X)$; the closure under c-pure subsheaves follows from Lemma 4.3.12 and the fact that by Lemma 4.2.4 (2), c-purity in $\mathrm{QCoh}(X)$ is the same as in $\mathcal{O}_X$-Mod.

Corollary 4.5.7. *Let X be a concentrated scheme, $\mathcal{N} \in \mathrm{QCoh}(X)$ indecomposable g-pure-injective and $U_1, \dots, U_n$ a finite open affine cover of X. Then there is $i \in \{1, \dots, n\}$ such that $\mathcal{N} \cong \iota^{\mathrm{qc}}_{U_i,*}(\mathcal{N}|_{U_i})$.*

Proof. Building on Lemma 4.5.5, we have an i such that the adjunction unit $n\colon \mathcal{N} \to \iota_{U_i,*}(\mathcal{N}|_{Ui})$ is a split monomorphism. By the discussion above, the essential image of $\iota^{\mathrm{qc}}_{U_i,*}$ is a definable subcategory, hence it contains $\mathcal{N}$. However, n restricted to U is the identity map, therefore n is actually an isomorphism. □

Theorem 4.5.8. *Let X be concentrated scheme. Then the indecomposable g-pure-injective quasicoherent sheaves form a closed quasicompact subset of the Ziegler spectrum* $\mathrm{Zg}(\mathrm{QCoh}(X))$.

Proof. Let $U_1, \ldots, U_n$ be a finite open affine cover of X. By Corollary 4.5.7, every indecomposable g-pure-injective quasicoherent sheaf is of the form $\iota^{\mathrm{qc}}_{U_i,*}(\mathcal{M})$ for some $i \in \{1, \ldots, n\}$ and $\mathcal{M} \in \mathrm{QCoh}(U_i)$. We infer that the geometric part of $\mathrm{Zg}(\mathrm{QCoh}(X))$ is the union of finitely many closed sets corresponding to the ($\iota^{\mathrm{qc}}_{U_i,*}$-images of) definable categories $\mathrm{QCoh}(U_i)$, hence a closed set. Furthermore, since the U_i are affine, we have equivalences $\mathrm{QCoh}(U_i) \cong \mathcal{O}_X(U_i)$-Mod leading to homeomorphisms $\mathrm{Zg}(\mathrm{QCoh}(U_i)) \cong \mathrm{Zg}(\mathcal{O}_X(U_i))$. Therefore, by [14, Corollary 5.1.23], the sets in the union are all quasicompact and their (finite) union as well. □

Definition 4.5.9. Let X be a concentrated scheme. We denote by $\mathcal{D}_X$ the definable subcategory of $\mathrm{QCoh}(X)$ corresponding to the closed subset of $\mathrm{Zg}(\mathrm{QCoh}(X))$ from Theorem 4.5.8.

Remark 4.5.10. Note that by the proof of Theorem 4.5.8, for every open affine $U \subseteq X$ and every $\mathcal{M} \in \mathrm{QCoh}(U)$, the direct image $\iota^{\mathrm{qc}}_{U,*}(\mathcal{M})$ belongs to $\mathcal{D}_X$.

Proposition 4.5.11. *Let X be a concentrated scheme.*

(1) A c-pure-injective quasicoherent sheaf is g-pure-injective if and only if it belongs to $\mathcal{D}_X$.

(2) Any g-pure monomorphism (g-pure-exact sequence) starting in an object of $\mathcal{D}_X$ is c-pure.

Proof. It is a standard fact (see [14, Corollary 5.3.52]) about definable subcategories that their objects are precisely pure subobjects of products of indecomposable pure-injectives. Therefore, if $\mathcal{N} \in \mathcal{D}_X$ is c-pure-injective, then it is a direct summand of a product of indecomposable c-pure-injective objects in $\mathcal{D}_X$, all of which are g-pure-injective, a property passing both to products and direct summands, hence $\mathcal{N}$ is g-pure-injective.

On the other hand, if $\mathcal{N}$ is g-pure-injective and $U_1, \ldots, U_n$ a finite open affine cover of X, then the g-pure monomorphism

$$\mathcal{N} \to \bigoplus_{1 \le i \le n} \iota^{\mathrm{qc}}_{U_i,*}(\mathcal{N}|_{U_i})$$

splits. Since $\iota^{\mathrm{qc}}_{U_i,*}(\mathcal{N}|_{U_i}) \in \mathcal{D}_X$ for each $1 \le i \le n$, we infer that $\mathcal{N} \in \mathcal{D}_X$.

For the second claim, let $f \colon \mathcal{M} \hookrightarrow \mathcal{A}$ be a g-pure monomorphism with $\mathcal{M} \in \mathcal{D}_X$. Assume first $\mathcal{A}$ is g-pure-injective; then $\mathcal{A} \in \mathcal{D}_X$ by the first part. Recall that in any definable category, purity of a monomorphism can be tested by applying the contravariant Hom functor with every pure-injective object. Since the pure-injectives of $\mathcal{D}_X$ are precisely g-pure-injectives, f "passes" this test and is therefore c-pure. If $\mathcal{A} \in \mathrm{QCoh}(X)$ is arbitrary, pick a g-pure embedding $g \colon \mathcal{A} \hookrightarrow \mathcal{N}$ with $\mathcal{N}$ g-pure-injective; such an embedding exists by [5, Corollary 4.8] (or combining Lemmas 4.3.20 and 4.5.4). Then gf is a g-pure monomorphism and by the argument above, it is c-pure. We infer that f is c-pure as well. □

Corollary 4.5.12. *A concentrated scheme X is affine if and only if the subcategory $\mathcal{D}_X$ is the whole category* $\mathrm{QCoh}(X)$*.*

Proof. If X is affine, then the two purities coincide, therefore c-pure-injectives are g-pure-injectives. On the other hand, if every c-pure-injective is g-pure-injective then, since c-pure-injectives determine which short exact sequences are c-pure-exact, we see that all g-pure-exact sequences are c-pure-exact hence, by Proposition 4.5.1, X is affine. □

Corollary 4.5.13. *Let X be a concentrated scheme. Then every indecomposable g-pure-injective $\mathcal{N} \in \mathrm{QCoh}(X)$ is the coherator of some indecomposable g-pure-injective $\mathcal{M} \in \mathcal{O}_X$-*Mod*.*

Proof. By Corollary 4.5.7, there is an open affine $U \subseteq X$ such that $\mathcal{N}$ is the direct image of an indecomposable g-pure-injective object of $\mathrm{QCoh}(U)$, which further corresponds to an indecomposable pure-injective $\mathcal{O}_X(U)$-module N. By [13, Theorem 2.Z8], N is in fact a module over the localisation in some maximal ideal of $\mathcal{O}_X(U)$; let this maximal ideal correspond to a point $x \in U$. Then clearly $\mathcal{N}|_U = \mathcal{C}_U(\iota_{x,*}(N)|_U)$ and by Lemmas 4.3.6 and 4.3.8, $\mathcal{M} = \iota_{x,*}(N)$ and $\mathcal{M}|_U$ are g-pure-injective (and clearly indecomposable).

By [21, B.13], coherator commutes with direct images of concentrated maps, hence we have

$$\mathcal{N} = \iota_{U,*}^{\mathrm{qc}}\big(\mathcal{C}_U(\mathcal{M}|_U)\big) = \mathcal{C}_X\big(\iota_{U,*}(\mathcal{M}|_U)\big) = \mathcal{C}_X(\mathcal{M}),$$

where the last equality follows from the fact that $\mathcal{M}$ is a skyscraper. □

Note, however, that the preimage $\mathcal{M}$ from the preceding Corollary is far from being unique; Section 4.4 gives a couple of examples of indecomposable c-pure-injective $\mathcal{O}_X$-modules with the same module of global sections (and therefore the same coherator).

Proposition 4.5.14. *Let X be a concentrated scheme. The following are equivalent for $\mathcal{M} \in \mathrm{QCoh}(X)$:*

(1) $\mathcal{M} \in \mathcal{D}_X$.
(2) $\mathcal{M}$ is a c-pure subsheaf of a g-pure-injective quasicoherent sheaf.
(3) For every finite open affine cover $U_1, \dots, U_n$ of X, the monomorphism

$$\mathcal{M} \to \bigoplus_{1 \le i \le n} \iota_{U_i,*}^{\mathrm{qc}}(\mathcal{M}|_{U_i})$$

is c-pure.
(4) There exists a finite open affine cover $U_1, \dots, U_n$ of X such that the monomorphism

$$\mathcal{M} \to \bigoplus_{1 \le i \le n} \iota_{U_i,*}^{\mathrm{qc}}(\mathcal{M}|_{U_i})$$

is c-pure.

Proof. (1) ⇒ (2): The definable subcategory $\mathcal{D}_X$ is closed under taking (c-)pure-injective envelopes of its objects; since the pure-injectives in $\mathcal{D}_X$ are g-pure-injective by Proposition 4.5.11 (1), this produces a c-pure embedding into a g-pure-injective quasicoherent sheaf.

(2) ⇒ (1): $\mathcal{D}_X$ contains all g-pure-injectives and is closed under taking c-pure subsheaves.

(1) ⇒ (3): The monomorphism in question is always g-pure; by assumption, its domain belongs to $\mathcal{D}_X$, therefore it is c-pure by Proposition 4.5.11 (2).

(3) ⇒ (4) is clear.

(4) ⇒ (1): The codomain of the monomorphism belongs to $\mathcal{D}_X$ (Remark 4.5.10), which is closed under taking c-pure subsheaves. □

The following lemma, ensuring that the sheaf cohomology vanishes on $\mathcal{D}_X$, will prove useful in Section 4.6.

Lemma 4.5.15. *Let X be a concentrated scheme. Then $\mathrm{H}^1(X, \mathcal{M}) = 0$ for every $\mathcal{M} \in \mathcal{D}_X$.*

Proof. Observe that the subcategory of $\mathrm{QCoh}(X)$, where the first cohomology vanishes, is definable: The closure under direct limits follows from [21, Lemma B.6]. The closure under direct products follows from [4, Corollary A.2] and the fact that $\mathrm{H}^1(X, -) = \mathrm{Ext}^1_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -)$. Finally, let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a c-pure-exact sequence in $\mathrm{QCoh}(X)$, where $\mathrm{H}^1(X, \mathcal{B}) = 0$. Using c-pure-exactness and Lemma 4.5.16, this sequence stays exact after applying the global sections functor, therefore the map $\mathrm{H}^1(X, \mathcal{A}) \to \mathrm{H}^1(X, \mathcal{B})$ is injective. Since its codomain vanishes, the same holds for its domain.

Thanks to this observation, it suffices to prove the assertion only for the pure-injectives of $\mathcal{D}_X$, i.e. g-pure-injectives. To show that

$$\mathrm{H}^1(X, \mathcal{N}) = \mathrm{Ext}^1_{\mathrm{QCoh}}(\mathcal{O}_X, \mathcal{N}) = 0$$

for every g-pure-injective, consider a short exact sequence starting in $\mathcal{N}$ and ending in $\mathcal{O}_X$; as $\mathcal{O}_X$ is flat on each open affine set, the sequence is g-pure-exact, and because $\mathcal{N}$ is g-pure-injective, the sequence splits as desired. □

Let us end this section by observing that the pure-injective objects of the category $\mathrm{QCoh}(X)$ are a bit "mysterious" from the sheaf point of view. The best we can say is that by the following lemma, their module of global sections is pure-injective over the ring $\mathcal{O}_X(X)$, but that can be far from true on the rest of the open sets.

Lemma 4.5.16. *Let X be a concentrated scheme. Then the functor of global sections, viewed as a functor from $\mathrm{QCoh}(X)$ to $\mathcal{O}_X(X)$-modules, is definable.*

Proof. The functor in question is naturally isomorphic to the representable functor $\mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -)$, which clearly commutes with products. Commuting with direct limits is [22, 009E]. □

Note that this lemma is very similar to Lemma 4.3.4; however, in the quasicoherent case, the functor of taking sections on an open set usually does not commute with direct products.

Example 4.5.17. Proposition 4.6.3 below shows that the structure sheaf of the projective line is c-pure-injective in $\mathrm{QCoh}(\mathbb{P}^1_k)$. However, on any open affine set (or stalk at any closed point), this sheaf is not pure injective. Since this violates Corollary 4.3.3, it is not c-pure-injective in $\mathcal{O}_X$-Mod.

A slightly more sophisticated example exhibits a similar behaviour even for g-pure-injectives in $\mathrm{QCoh}(X)$:

Example 4.5.18. Let k be a field, $R = k[\![x, y]\!]$ the ring of power series in two commuting variables and $X = \operatorname{Spec} R$. Since R is a commutative noetherian complete local domain, it is a pure-injective module over itself by [10, Theorem 11.3]. Therefore the structure sheaf $\mathcal{O}_X$ is a g-pure-injective quasicoherent sheaf. However, for every proper distinguished open affine subset $U \subset X$, $\mathcal{O}_X(U)$ is the localisation of R in a single element, which is a commutative noetherian domain, but not even local, hence not pure-injective by [10, Theorem 11.3] again.

4.6 Example: Projective line

This section is devoted to investigating purity in one of the simplest non-affine schemes, the projective line $\mathbb{P}^1_k$, where k is any field. Our primary aim is to describe $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$, but this scheme also provides several important examples. Since $\mathbb{P}^1_k$ is a noetherian, hence a concentrated, scheme all the results of the previous section apply.

First of all, recall that the finitely presented objects of $\mathrm{QCoh}(\mathbb{P}^1_k)$ are precisely the coherent sheaves [11, Remark 10 & Proposition 75], and each coherent sheaf decomposes uniquely (in the sense of the Krull-Schmidt Theorem) into a direct sum of indecomposable ones. These indecomposables are of two kinds [2, Section 5]:

- line bundles, i.e., the structure sheaf and its twists, which we denote $\mathcal{O}(n)$ $(n \in \mathbb{Z})$,
- torsion sheaves, i.e., skyscrapers $\iota_{x,*}(F)$, where $x \in \mathbb{P}^1_k$ is a closed point and F is a cyclic torsion module over the DVR $\mathcal{O}_{\mathbb{P}^1_k,x}$.

In the following, we simply use $\mathcal{O}$ for $\mathcal{O}(0)$, which is also the same thing as the structure sheaf $\mathcal{O}_{\mathbb{P}^1_k}$.

Example 4.6.1. For every $a, b, c, d \in \mathbb{Z}$ such that $a < b < d$, $a < c < d$ and $a + d = b + c$, there is a short exact sequence

$$0 \to \mathcal{O}(a) \to \mathcal{O}(b) \oplus \mathcal{O}(c) \to \mathcal{O}(d) \to 0$$

in $\mathrm{QCoh}(\mathbb{P}^1_k)$, which is non-split, since $\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(d), \mathcal{O}(b) \oplus \mathcal{O}(c)) = 0$. This sequence is not c-pure-exact, since it ends in a finitely presented object but does not split. On the other hand, passing to any stalk or any open affine set, we obtain a split short exact sequence of free modules, therefore the sequence of sheaves is g-pure-exact.

For the projective line, we are able to give a better characterization of g-pure-exactness:

Proposition 4.6.2. *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence in* $\mathrm{QCoh}(\mathbb{P}^1_k)$. *Then the following statements are equivalent:*

(1) The sequence is g-pure-exact.

(2) For each (indecomposable) torsion coherent sheaf $\mathcal{T}$, the sequence

$$0 \to \mathcal{T} \otimes \mathcal{A} \to \mathcal{T} \otimes \mathcal{B} \to \mathcal{T} \otimes \mathcal{C} \to 0$$

is exact.

(3) For each (indecomposable) torsion coherent sheaf $\mathcal{T}$, the sequence

$$0 \to \mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{T}, \mathcal{A}) \to \mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{T}, \mathcal{B}) \to \mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{T}, \mathcal{C}) \to 0$$

is exact.

Proof. It is clear from the description of coherent sheaves over $\mathbb{P}^1_k$ that we may restrict to the indecomposable coherent sheaves in (2) and (3).

(1) $\Rightarrow$ (2) is clear from the definition of g-pure-exactness. For (2) $\Rightarrow$ (1) it suffices to recall that the tensor product preserves direct limits, every quasicoherent sheaf is the direct limit of coherent ones, and tensoring by a line bundle is always exact.

To prove (1) $\Leftrightarrow$ (3), first recall that for quasicoherent sheaves, g-pure-exactness is equivalent to pure exactness on each open affine. Observe that the support of an indecomposable torsion coherent sheaf $\mathcal{T}$ is a single closed point of $\mathbb{P}^1_k$; let U be any open affine set containing this point. In such a case, not only is $\mathcal{T}$ the direct image of its restriction to U, but it is also the extension by zero of this restriction. The adjunction now implies that the exactness of the sequence in (3) is equivalent to the exactness of

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathcal{T}|_U, \mathcal{A}|_U) \to \mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathcal{T}|_U, \mathcal{B}|_U) \to \\ \to \mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathcal{T}|_U, \mathcal{C}|_U) \to 0. \end{aligned}$$

Since U is affine and the ring of sections over U is a PID, this new sequence is exact for every indecomposable torsion coherent $\mathcal{T}$ if and only if the sequence of $\mathcal{O}(U)$-modules

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U) \to 0$$

is pure-exact—recall that over a PID, purity can be checked just by using finitely generated (= presented) indecomposable torsion modules, which follows from the description of finitely generated modules. Since $\mathcal{T}$ runs over all torsion indecomposable coherent sheaves, U runs over all open affine subsets of $\mathbb{P}^1_k$, and every torsion $\mathcal{O}(U)$-module extends to a torsion coherent sheaf, we are done. □

Let us proceed with describing the indecomposable g-pure-injectives. This is easy thanks to Corollary 4.5.7, taking into account that $\mathbb{P}^1_k$ is covered by affine lines. The Ziegler spectrum of a PID (more generally, a Dedekind domain) is described e.g. in [14, 5.2.1]. Therefore, for each closed point $x \in \mathbb{P}^1_k$, there are the following sheaves:

- all the indecomposable torsion coherent sheaves based at x (an $\mathbb{N}$-indexed family),

- the "Prüfer" sheaf: $\iota_{x,*}(P)$, where P is the injective envelope of the unique simple $\mathcal{O}_{X,x}$-module,
- the "adic" sheaf: for every $U \subseteq X$ open, the module of sections is either the completion $\overline{\mathcal{O}_{X,x}}$ (if $x \in U$), or the fraction field of this completion (if $x \notin U$); in other words, this is the coherator of $\iota_{x,*}(\overline{\mathcal{O}_{X,x}})$.

Finally, there is the (g-pure-)injective constant sheaf, assigning to each open set the residue field of the generic point of $\mathbb{P}^1_k$. We will refer to this sheaf as the generic point of $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$.

This observation can be informally summarised by saying that the geometric part of $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$ is the "projectivization" of the Ziegler spectrum of an affine line.

However, by Proposition 4.5.1, there have to be c-pure-injectives which are not g-pure-injective. An example of this phenomenon is the structure sheaf $\mathcal{O}$:

Proposition 4.6.3. *The structure sheaf is Σ-c-pure-injective, i.e. any direct sum of its copies is c-pure-injective. The same holds for all line bundles.*

We give two proofs, one rather direct, illustrating the technique of computing the coherator over concentrated schemes, the other requiring more framework from model theory. Since twisting is an autoequivalence of $\mathrm{QCoh}(\mathbb{P}^1_k)$, we may restrict to $\mathcal{O}$ in both proofs.

Elementary proof. We prove that for any set I, the inclusion $\mathcal{O}^{(I)} \hookrightarrow \mathcal{O}^I$ splits. Let U, V be open affine subsets of $\mathbb{P}^1_k$ such that $\mathcal{O}(U) = k[x]$, $\mathcal{O}(V) = k[x^{-1}]$, and $\mathcal{O}(U \cap V) = k[x, x^{-1}]$. For the direct sum, the computation is easy: $\mathcal{O}^{(I)}(U) = k[x]^{(I)}$, $\mathcal{O}^{(I)}(V) = k[x^{-1}]^{(I)}$, and $\mathcal{O}^{(I)}(U \cap V) = k[x, x^{-1}]^{(I)}$.

To compute the direct product in $\mathrm{QCoh}(\mathbb{P}^1_k)$, we have to compute the coherator of the product in the category of all sheaves. The way to do that is described in [21, B.14]:

$$\begin{aligned}
\mathcal{O}^I(U) &= \ker\big(\mathcal{O}(U)^I \oplus (\mathcal{O}(V)^I \otimes_k k[x]) \to \mathcal{O}(U \cap V)^I\big),\\
\mathcal{O}^I(V) &= \ker\big((\mathcal{O}(U)^I \otimes_k k[x^{-1}]) \oplus \mathcal{O}(V)^I \to \mathcal{O}(U \cap V)^I\big),\\
\mathcal{O}^I(U \cap V) &= \ker\big((\mathcal{O}(U)^I \otimes_k k[x^{-1}]) \oplus (\mathcal{O}(V)^I \otimes_k k[x]) \to \mathcal{O}(U \cap V)^I\big).
\end{aligned}$$

Therefore $\mathcal{O}^I(U)$ is the submodule of $\mathcal{O}(U)^I = k[x]^I$ consisting of sequences of polynomials with bounded degree, similarly for $\mathcal{O}^I(V)$; $\mathcal{O}^I(U \cap V)$ consists of sequences of polynomials in x and x^{-1} with degree bounded both from above and below.

Observe that $\mathcal{O}^I(U)$ can be identified with $k^I[x]$, polynomials over the ring k^I of arbitrary sequences, whereas $\mathcal{O}^{(I)}(U)$ corresponds to $k^{(I)}[x]$; analogous assertions hold for V and $U \cap V$. The inclusion of k-vector spaces $k^{(I)} \hookrightarrow k^I$ splits, and this splitting naturally lifts to each of $\mathcal{O}(U)$, $\mathcal{O}(V)$, $\mathcal{O}(U \cap V)$, commuting with the restriction maps, hence defining a splitting of the inclusion $\mathcal{O}^{(I)} \hookrightarrow \mathcal{O}^I$ as desired. □

Model-theoretic proof. The category $\mathrm{QCoh}(\mathbb{P}^1_k)$ being locally finitely presented, is equivalent to the category of flat contravariant functors on its subcategory, $\mathcal{C} = \mathrm{coh}(\mathbb{P}^1_k)$, of finitely presented objects, via the embedding taking a quasicoherent sheaf $\mathcal{M}$ to the representable functor $(-,\mathcal{M})$ restricted to $\mathcal{C}$. That is a definable subcategory of the category Mod-$\mathcal{C}$ of all contravariant functors from $\mathrm{QCoh}(\mathbb{P}^1_k)$ to k (see, e.g., [15, §18]). We regard such functors as multisorted modules (as in [17]), with the set of elements of $\mathcal{M} \in \mathrm{QCoh}(\mathbb{P}^1_k)$ in sort $(\mathcal{F},-)$, for $\mathcal{F} \in \mathcal{C}$, being $(\mathcal{F},\mathcal{M})$. The model theory of modules then applies. In particular $\mathcal{M}$ is Σ-pure-injective exactly if each sort $(\mathcal{F},\mathcal{M})$ has the descending chain condition on pp-definable subgroups (cf. [14, Theorem 4.4.5]). Since, for each $\mathcal{F} \in \mathcal{C}$, $\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{F},\mathcal{O})$ is finite-dimensional over k, this descending chain condition is satisfied (every pp-definable subgroup is a k-subspace). □

In fact, the list is now complete. To prove this, we use Ziegler spectra of derived categories, but first we need the following observations for the Ext functors and cohomology:

Lemma 4.6.4. *For every $n \in \mathbb{Z}$, the class of objects $\mathcal{M} \in \mathrm{QCoh}(\mathbb{P}^1_k)$ such that*

$$\mathrm{Ext}^1_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n),\mathcal{M}) = 0$$

is a definable subcategory of $\mathrm{QCoh}(\mathbb{P}^1_k)$, containing all g-pure-injectives.

Proof. Taking into account that twisting is an autoequivalence of $\mathrm{QCoh}(\mathbb{P}^1_k)$, we have natural equivalence

$$\mathrm{Ext}^1_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n),\mathcal{M}) \cong \mathrm{Ext}^1_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O},\mathcal{M}\otimes\mathcal{O}(-n)) \cong \mathrm{H}^1(\mathbb{P}^1_k,\mathcal{M}\otimes\mathcal{O}(-n))$$

and the proof of definability of the vanishing class continues as in the proof of Lemma 4.5.15.

The statement about g-pure-injectives suffices to be checked only for the indecomposable ones. Since these are invariant under twists, this follows from Lemma 4.5.15. □

We are now prepared to prove that we have successfully identified all indecomposable c-pure-injectives.

Theorem 4.6.5. *Every indecomposable c-pure-injective object of $\mathrm{QCoh}(\mathbb{P}^1_k)$ is either g-pure-injective or a line bundle.*

Proof. Let $R = k\tilde{A}_1$ be the path algebra of the Kronecker quiver. Let $\mathbf{D}(R)$ denote the (unbounded) derived category of the category of right R-modules and $\mathbf{D}(\mathbb{P}^1_k)$ the (also unbounded) derived category of $\mathrm{QCoh}(\mathbb{P}^1_k)$. By the results of [1], the functor

$$F = \mathbf{R}\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}\oplus\mathcal{O}(1),-)\colon \mathbf{D}(\mathbb{P}^1_k) \to \mathbf{D}(R)$$

is a triangulated equivalence. Since R is hereditary, by [14, Theorem 17.3.22], its Ziegler spectrum is just the union of all shifts of the Ziegler spectrum of R, embedded into $\mathbf{D}(R)$ as complexes with cohomology concentrated in degree 0.

Let Z be the (representative) set containing all twists of the structure sheaf and all indecomposable g-pure-injectives of $\mathrm{QCoh}(\mathbb{P}^1_k)$. The Ziegler spectrum of $\mathbf{D}(\mathbb{P}^1_k)$ contains all shifts of Z; therefore, since F is an equivalence, to show that Z is indeed the Ziegler spectrum of $\mathrm{QCoh}(\mathbb{P}^1_k)$ it suffices to show that the shifts of $F(Z)$ cover $\mathrm{Zg}(\mathbf{D}(R))$.

The Ziegler spectrum of R is described in [14, 8.1], so the checking is only a matter of computation. From Lemma 4.6.4 we get that

$$\mathrm{Ext}^1_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{N}) = 0$$

for every indecomposable g-pure-injective $\mathcal{N}$, therefore F simply sends the torsion, "Prüfer", "adic" and generic sheaves to the corresponding points of $\mathrm{Zg}(R)$ put into the same cohomological degree in $\mathbf{D}(R)$.

Using Serre duality [8, Theorem 7.1], one obtains the same Ext-vanishing for $\mathcal{O}(n)$ for all $n \geq 0$, and it is easy to observe that these line bundles are sent to preprojective R-modules via F. On the other hand, we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O}(n)) = 0$$

for $n < 0$, and using Serre duality again we get that these line bundles are mapped to preinjective R-modules, just shifted to the neighbouring cohomological degree in $\mathbf{D}(R)$.

We conclude that each point of $\mathrm{Zg}(\mathbf{D}(R))$ is a shift of an object from $F(Z)$ as desired. □

Having now the complete description of the points of $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$, let us investigate the topology. The geometric part (which forms a closed subset by Theorem 4.5.8) is easy to handle and basically follows the description of the Ziegler spectrum of a Dedekind domain (cf. [14, Theorem 5.2.3]). The following observation describes how the line bundles sit in the Ziegler topology:

Proposition 4.6.6. *Every line bundle is a closed and isolated point of the Ziegler spectrum* $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$. *Any set of line bundles* $\mathcal{O}(n)$, *where* n *is bounded from above, is closed. Any set of line bundles* $\mathcal{O}(n)$, *where* n *is not bounded from above, additionally contains all the "adic" sheaves and the generic point in its closure.*

Proof. This could be deduced from the description of the topology of the Ziegler spectrum over the Kronecker algebra R but we can argue directly as follows. For each $n \in \mathbb{Z}$, $\mathcal{O}(n)$ is the only indecomposable c-pure-injective for which both functors

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n+1), -) \quad \text{and} \quad \mathrm{Ext}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n-1), -)$$

vanish; the vanishing class of the former is clearly definable, whereas for the latter we use Lemma 4.6.4. Therefore the single-point set containing $\mathcal{O}(n)$ is closed.

On the other hand, $\mathcal{O}(n)$ is the only indecomposable c-pure-injective for which neither of the functors

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n), -) \quad \text{and} \quad \mathrm{Ext}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n+2), -)$$

vanishes; the former vanishes on all $\mathcal{O}(m)$ for $m < n$, whereas the latter vanishes on g-pure-injectives and all $\mathcal{O}(m)$ with $m > n$.

We see that line bundles form a discrete subspace of $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$, therefore taking the closure of any set of line bundles can possibly add only points from the geometric part.

If a set S of line bundles $\mathcal{O}(n)$ has n strictly bounded above by some m, then this set is contained in the definable vanishing class of $\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(m), -)$, which contains no g-pure-injectives. Hence S is closed.

If, on the other hand, a set S of line bundles contains $\mathcal{O}(n)$ with arbitrarily large $n \in \mathbb{Z}$, additional points appear in the closure. Denote by $\mathcal{X}$ the smallest definable subcategory of $\mathrm{QCoh}(X)$ containing S. Firstly, let U be the complement of a single closed point $x \in \mathbb{P}^1_k$. For every pair $m < n$ such that $\mathcal{O}(m), \mathcal{O}(n) \in S$, pick a monomorphism $\mathcal{O}(m) \to \mathcal{O}(n)$ which is an isomorphism on U. This way we obtain a chain of monomorphisms, the direct limit of which is the sheaf $\mathcal{M} = \iota^{\mathrm{qc}}_{U,*}(\mathcal{O}|_U)$. Therefore $\mathcal{M} \in \mathcal{X}$.

Since $\mathcal{O}(U)$ is a PID, the pure-injective hull N of $\mathcal{O}(U)$ in the category of $\mathcal{O}(U)$-modules is the direct product of completions of the local rings $\mathcal{O}_y$ for each closed $y \in U$; applying the (definable by 4.5.6) direct image functor $\iota^{\mathrm{qc}}_{U,*}$ to the corresponding map in $\mathrm{QCoh}(U)$ produces a c-pure-injective hull $\mathcal{M} \hookrightarrow \mathcal{N}$ in $\mathrm{QCoh}(\mathbb{P}^1_k)$, therefore $\mathcal{N} \in \mathcal{X}$. However, $\mathcal{N}$ is just the product of "adic" sheaves, which therefore belong to $\mathcal{X}$, too. Since the choice of the point x was arbitrary, we obtain all "adic" sheaves in the closure of S. Furthermore, any such point has the generic point in its closure.

Finally, observe that $\mathcal{X}$ consists only of torsion-free sheaves, i.e. the sheaves for which the definable functors $\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{T}, -)$, $\mathcal{T}$ torsion coherent, vanish. All the remaining indecomposable pure-injectives are not torsion-free and therefore cannot be in the closure of S. □

Corollary 4.6.7. *The Ziegler spectrum of* $\mathrm{QCoh}(\mathbb{P}^1_k)$ *is not quasicompact.*

Proof. By Proposition 4.6.6, the sets of line bundles $S_n = \{\mathcal{O}(m) \mid m < n\}$ are closed in $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$. The intersection of any finite collection of such sets is non-empty, but the intersection over all $n \in \mathbb{Z}$ is empty, which shows that $\mathrm{Zg}(\mathrm{QCoh}(\mathbb{P}^1_k))$ is not quasicompact. □

Remark 4.6.8. Corollary 4.6.7 can be easily generalised: For example, let R be an $\mathbb{N}_0$-graded ring finitely generated as an R_0-algebra and $X = \operatorname{Proj} R$. Consider for each $n \in \mathbb{Z}$ the set

$$S_n = \{\mathcal{M} \in \mathrm{Zg}(\mathrm{QCoh}(X)) \mid \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X(n), \mathcal{M}) = 0\}.$$

Since $\mathcal{O}_X(n)$ is finitely presented, each S_n is Ziegler-closed. If they are also non-empty, then they necessarily form a collection with the finite intersection property, but empty intersection, since the line bundles form a generating set of the category. For example, if $R = A[x_0, \dots, x_t]$ for A a commutative ring and $t \geq 1$, then $\mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X(n), \mathcal{O}_X(n-1)) = 0$, so $S_n \neq \emptyset$ because the non-trivial definable subcategory, where $\mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X(n), -)$ vanishes, has to intersect the Ziegler spectrum non-trivially. This shows that for $X = \mathbb{P}^t_A$, $\mathrm{Zg}(\mathrm{QCoh}(X))$ is not quasicompact.

Turning back to the projective line, we are also able to give an alternative description of the subcategory $\mathcal{D}_{\mathbb{P}^1_k}$, which we denote here just $\mathcal{D}$ for short; this description was suggested to us by Jan Šťovíček:

Proposition 4.6.9. *An object* $\mathcal{M}$ *of* $\mathrm{QCoh}(\mathbb{P}^1_k)$ *belongs to* $\mathcal{D}$ *if and only if for each* $n \in \mathbb{Z}$,

$$\mathrm{Ext}^1_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n), \mathcal{M}) = 0,$$

if and only if for each $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{M}, \mathcal{O}(n)) = 0.$$

Proof. Let $\mathcal{E}$ be the subcategory of $\mathrm{QCoh}(\mathbb{P}^1_k)$ consisting of those $\mathcal{M}$ such that $\mathrm{Ext}^1_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{O}(n), \mathcal{M}) = 0$ for all $n \in \mathbb{Z}$. By Lemma 4.6.4, $\mathcal{E}$ is the intersection of definable subcategories, therefore it is itself definable.

The proof that $\mathcal{D} = \mathcal{E}$ is now just a matter of checking that $\mathcal{D}$ and $\mathcal{E}$ contain the same indecomposable c-pure-injectives. By Lemma 4.6.4, all indecomposable g-pure-injectives belong to $\mathcal{E}$, while Example 4.6.1 shows that no line bundle belongs to $\mathcal{E}$, which is precisely the case for $\mathcal{D}$ as well.

To check the second claim, pick $\mathcal{M} \notin \mathcal{D}$ and let $i \colon \mathcal{M} \hookrightarrow \mathcal{N}$ be a c-pure embedding into the direct product of indecomposable c-pure-injectives. Taking into account the description of these indecomposables (Theorem 4.6.5), some of the terms in the product have to be line bundles, for otherwise we would have $\mathcal{M} \in \mathcal{D}$, and for the same reason the composition of i with the projection on at least one of the line bundles has to be non-zero, showing that $\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{M}, \mathcal{O}(n)) \neq 0$ for some $n \in \mathbb{Z}$.

On the other hand, let $\mathcal{M} \in \mathcal{D}$. The description of $\mathcal{D}$ as the intersection of Ext-vanishing classes shows that $\mathcal{D}$ is closed under arbitrary factors as Ext^2 vanishes on $\mathrm{QCoh}(\mathbb{P}^1_k)$. The image of any non-zero map $\mathcal{M} \to \mathcal{O}(n)$ would be a line bundle, too, but no line bundle belongs to $\mathcal{D}$, therefore we conclude that $\mathrm{Hom}_{\mathrm{QCoh}(\mathbb{P}^1_k)}(\mathcal{M}, \mathcal{O}(n)) = 0$ for each $n \in \mathbb{Z}$ in this case. □

Therefore, for the projective line, we get some extra properties of $\mathcal{D}$:

Corollary 4.6.10. *The subcategory $\mathcal{D}$ is a torsion class and a right class of a cotorsion pair. In particular, $\mathcal{D}$ is closed under arbitrary colimits, factors, and extensions.*

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5. On flat generators and Matlis duality for quasicoherent sheaves

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5.1 Introduction

Let X be a scheme. It is well-known that unless the scheme is affine, the category $\mathrm{QCoh}(X)$ of all quasicoherent sheaves on X does not usually have enough projective objects. In fact, a quasiprojective scheme over a field is affine if and only if $\mathrm{QCoh}(X)$ has enough projective objects, [12, Theorem 1.1], and a direct proof that $\mathrm{QCoh}(X)$ has no non-zero projective objects if X is the projective line over a field can be found in [5, Corollary 2.3].

Such an issue is often fixed using some flat objects; recall that a quasicoherent sheaf $\mathcal{M}$ is called *flat* if for any open affine set $U \subseteq X$, the $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ is flat. Murfet in his thesis showed that for X quasicompact and semiseparated, every quasicoherent sheaf is a quotient of a flat one [14, Corollary 3.21] (recall that a scheme is called *semiseparated* if the intersection of any two open affine sets is affine; this differs from the original definition [20, B.7], but turns out to be equivalent, cf. [1, Remark after 2.5]). A short proof of the same fact, attributed to Neeman, can be found in [8, Appendix A], which was (under the same assumptions) later improved by Positselski [16, Lemma 4.1.1] by showing that so-called very flat quasicoherent sheaves are sufficient for this job.

Note that by [22, 077K] (cf. also the introduction to [4]), $\mathrm{QCoh}(X)$ is a Grothendieck category for any scheme X and as such, it has a generator. Therefore the assertion that "$\mathrm{QCoh}(X)$ has enough flat sheaves" can be equivalently rephrased that "$\mathrm{QCoh}(X)$ has a flat generator"; we will use these two statements interchangeably.

It was hoped for a long time that the existence of a flat generator can be extended at least to the case of quasicompact *quasiseparated* schemes (i.e. those for which the intersection of any two open affine sets is quasicompact), which encompass a considerably wider class of "natural" examples arising in algebraic geometry, while being an assumption rather pleasant to work with. However, our results show that for quasicompact quasiseparated schemes, semiseparatedness is in fact *necessary* for the existence of enough flat quasicoherent sheaves.

In this context, we note that it has been already known that semiseparatedness is necessary for the existence of a generating set consisting of vector bundles. This is a consequence of much more involved structure theorems for stacks, see [21, Proposition 1.3] and [9, Theorem 1.1(iii)]. Here we present a stronger version of that consequence with a much simpler proof.

A question closely related to the existence of a flat generator turns out to be the exactness of the Matlis duality functor. If R is a commutative ring and E an injective cogenerator of the category R-Mod, the Matlis duality functor

$\mathrm{Hom}_R(-, E)\colon R\text{-Mod}^{\mathrm{op}} \to R\text{-Mod}$ has been considered on numerous occasions in the literature and one of its fundamental properties is that it is exact.

If X is a possibly non-affine scheme, we can consider an analogous duality of the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves. Namely, $\mathrm{QCoh}(X)$ has an internal hom functor $\mathcal{H}om^{\mathrm{qc}}$ which is right adjoint to the usual tensoring of sheaves of $\mathcal{O}_X$-modules, and we can consider the functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})\colon \mathrm{QCoh}(X)^{\mathrm{op}} \to \mathrm{QCoh}(X)$ for an injective cogenerator $\mathcal{E} \in \mathrm{QCoh}(X)$; note that the existence of such a cogenerator follows by [13, Theorem 9.6.3] from the previously mentioned fact that $\mathrm{QCoh}(X)$ is a Grothendieck category. For a simple formal reason which we discuss below, $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact provided that $\mathrm{QCoh}(X)$ has a flat generator, so in particular if X is quasicompact and semiseparated. Perhaps somewhat surprisingly, we prove that for quasicompact quasiseparated schemes, semiseparatedness is again a *necessary* condition for the exactness.

To summarize, our main result reads as follows:

Main Theorem (see Theorems 5.2.2 and 5.3.10 and Corollary 5.3.13)**.** *Let X be a quasicompact and quasiseparated scheme. Then the following assertions are equivalent:*

(1) the category $\mathrm{QCoh}(X)$ of all quasicoherent sheaves on X has a flat generator;

(2) for every injective object $\mathcal{E}$ of $\mathrm{QCoh}(X)$, the contravariant internal hom functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact;

(3) there exists an injective cogenerator $\mathcal{E}$ of $\mathrm{QCoh}(X)$ such that the contravariant internal hom functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact;

(4) the scheme X is semiseparated.

The paper is organized as follows. In Section 5.2, we give a direct proof that for a non-semiseparated scheme X, the category $\mathrm{QCoh}(X)$ does not have a flat generator. The proof is rather constructive, producing a quasicoherent sheaf which is not a quotient of a flat one. Section 5.3 then provides the characterization of semiseparated schemes using the exactness of the internal hom $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ for every injective quasicoherent sheaf $\mathcal{E}$. This in fact gives another, less explicit proof of the results of Section 5.2.

Notation. If R is a commutative ring and M an R-module, then by $\tilde{M}$ we denote the quasicoherent sheaf on $\operatorname{Spec} R$ with M as the module of global sections. If the formula describing the module is too long and the tilde would not be wide enough, we use the notation like $M^{\sim}$.

If U is an open subset of a scheme X, which is usually clear from the context, then $\iota_U\colon U \to X$ denotes the inclusion and $\iota_{U,*}$ the direct image functor. Since we are dealing only with quasicompact open sets, ι_U is a quasicompact and quasiseparated map, hence $\iota_{U,*}$ sends quasicoherent sheaves to quasicoherent sheaves by [22, 01LC].

5.2 Non-existence of flat generators

In this section we show that if a quasicompact quasiseparated scheme X is not semiseparated, then $\mathrm{QCoh}(X)$ cannot have a flat generator by exhibiting a quasicoherent sheaf on X, which is not a quotient of a flat quasicoherent sheaf. We begin with an easy observation.

Let $\mathcal{M}$ be a quasicoherent sheaf on an affine scheme X. If U is an open affine subset of X then it is a part of the very definition of a quasicoherent sheaf on X that the map $\mathcal{M}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) \to \mathcal{M}(U)$ is an isomorphism of $\mathcal{O}_X(U)$-modules. If U is not affine, this may not be the case. However, more can be said for flat sheaves:

Lemma 5.2.1. *Let U be a quasicompact open subset of an affine scheme X and $\mathcal{F}$ a flat quasicoherent sheaf on X. Then the map*

$$\begin{aligned} \mathrm{res}^{\mathcal{F}}_{UX} \otimes \mathcal{O}_X(U) \colon \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U) &\longrightarrow \mathcal{F}(U), \\ f \otimes s &\longmapsto \mathrm{res}^{\mathcal{F}}_{UX}(f) \cdot s \end{aligned}$$

is an isomorphism.

Proof. We may assume that $X = \operatorname{Spec} R$. The assertion is clearly true for the structure sheaf and all its finite direct sums (i.e. all finite rank free R-modules). Since U is a quasicompact open subset of an affine scheme, it is also quasiseparated and the functor of sections over U commutes with direct limits [22, 009F]. By the Govorov-Lazard Theorem [22, 058G], any flat R-module is the direct limit of finite rank free modules, and since tensor product commutes with colimits, the desired property holds for all flat modules. □

Theorem 5.2.2. *Let X be a quasicompact quasiseparated scheme. Then X is semiseparated if and only if each quasicoherent sheaf on X is a quotient of a flat quasicoherent sheaf (equivalently:* $\mathrm{QCoh}(X)$ *has a flat generator).*

Proof. If X is semiseparated, then the assertion holds by the results mentioned in the introduction.

If X is not semiseparated, let U, V be two open affine subsets of X such that the intersection $W = U \cap V$ is *not* affine. Since X is quasiseparated, W is quasicompact; therefore, there are sections $f_1, \dots, f_n \in \mathcal{O}_X(U)$ such that $W = U_{f_1} \cup \dots \cup U_{f_n}$, where U_f denotes the distinguished open subset of the affine subscheme U where f does not vanish. Denote by I the ideal of $\mathcal{O}_X(U)$ generated by $f_1, \dots, f_n$ and $\mathcal{I} = \iota_{U,*}(\tilde{I})$ the direct image of $\tilde{I}$ with respect to the inclusion ι_U.

Since $\mathcal{I}(U_{f_i}) = \mathcal{O}_X(U_{f_i})$ for each $i = 1, \dots, n$, the sheaf axiom implies that $\mathcal{I}(W) = \mathcal{O}_X(W)$. On the other hand, by [10, Chapter II, Exercise 2.17(b)], the restrictions of $f_1, \dots, f_n$ to W do not generate the unit ideal of the ring $\mathcal{O}_X(W)$.

Assume that there is a flat quasicoherent sheaf $\mathcal{F}$ and an epimorphism $f \colon \mathcal{F} \to \mathcal{I}$. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(V) & \xrightarrow{\mathrm{res}^{\mathcal{F}}_{WV}} & \mathcal{F}(W) & \xleftarrow{\mathrm{res}^{\mathcal{F}}_{WU}} & \mathcal{F}(U) \\ \downarrow{\scriptstyle f(V)} & & \downarrow{\scriptstyle f(W)} & & \downarrow{\scriptstyle f(U)} \\ \mathcal{I}(V) & \overset{\mathrm{res}^{\mathcal{I}}_{WV}}{=\!=\!=} & \mathcal{I}(W) & \xleftarrow{\mathrm{res}^{\mathcal{I}}_{WU}} & \mathcal{I}(U) \end{array}$$

with the outer vertical arrows being epimorphisms due to U, V being affine and the middle arrow due to commutativity of the left-hand square.

The right-hand square induces the following commutative diagram, where the horizontal maps are given by the same formula as in Lemma 5.2.1:

$$\begin{array}{ccc} \mathcal{F}(W) & \xleftarrow{\operatorname{res}^{\mathcal{F}}_{WU}\otimes \mathcal{O}_X(W)} & \mathcal{F}(U)\otimes \mathcal{O}_X(W) \\ \downarrow{\scriptstyle f(W)} & & \downarrow{\scriptstyle f(U)\otimes \mathcal{O}_X(W)} \\ \mathcal{I}(W) & \xleftarrow{\operatorname{res}^{\mathcal{I}}_{WU}\otimes \mathcal{O}_X(W)} & \mathcal{I}(U)\otimes \mathcal{O}_X(W) \end{array}$$

Since $\mathcal{F}$ is flat, by Lemma 5.2.1, the top arrow is an isomorphism. Consequently, the bottom arrow is an epimorphism. However, this cannot be so: As $\mathcal{I}(U) = I$, the map in question factors as

$$I \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(W) \longrightarrow I\mathcal{O}_X(W) \longrightarrow \mathcal{I}(W),$$

and as observed above, the image is a proper submodule of $\mathcal{I}(W) = \mathcal{O}_X(W)$. This is a contradiction and, hence, the quasicoherent sheaf $\mathcal{I}$ cannot be a quotient of a flat quasicoherent sheaf. $\square$

Example 5.2.3. Let k be any field. A handy (and possibly the easiest) example of a non-semiseparated scheme X is the *plane with double origin*, obtained by gluing two copies of $\operatorname{Spec} k[x, y]$ along the *punctured plane*, i.e. the non-affine open subset containing everything except the maximal ideal (x, y). It may be illuminating to trace the proof of Theorem 5.2.2 in this particular case.

Let $R = k[x, y]$ for brevity. Then U, V are the two copies of $\operatorname{Spec} R$ and W is the punctured plane. Then $I = (x, y)$, as W can be covered by the two distinguished open subsets U_x, U_y. The resulting sheaf $\mathcal{I}$ then satisfies $\mathcal{I}(U) = I$, $\mathcal{I}(V) = \mathcal{I}(W) = R$. Since, by Lemma 5.2.1, for any flat sheaf on X, both restrictions from U and V to W are the identity morphisms, it is easy to see that the image of any map from a flat sheaf is contained in I on all three open sets.

5.3 Exactness of the internal Hom

For any scheme X, the category $\operatorname{QCoh}(X)$ has a closed symmetric monoidal structure given by the usual sheaf tensor product $\otimes$ together with its right adjoint, which we denote by $\mathcal{H}om^{\mathrm{qc}}$. This bifunctor is just the usual sheaf hom composed with the coherator functor [14, Proposition 6.15]. In this section we investigate the exactness of the contravariant functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$, where $\mathcal{E}$ is an injective object of $\operatorname{QCoh}(X)$.

If $(\mathcal{G}, \otimes)$ is a general abelian category with a symmetric monoidal structure, we call an object $F \in \mathcal{G}$ *flat* if the functor $F \otimes - \colon \mathcal{G} \to \mathcal{G}$ is exact. This is well-known to be consistent with the previous definition of flatness for $\mathcal{G} = \operatorname{QCoh}(X)$.

Since we do not assume that $\mathcal{G}$ has infinite coproducts, we follow [13, Definition 5.2.1] and call an object $G \in \mathcal{G}$ a *generator* if $\operatorname{Hom}_{\mathcal{G}}(G, -) \colon \mathcal{G} \to \mathcal{A}b$ is a conservative functor, i.e. reflects isomorphisms. By [13, Proposition 2.2.3], we can equivalently require $\operatorname{Hom}_{\mathcal{G}}(G, -)$ to be faithful, and if $\mathcal{G}$ happens to have all coproducts, [13, Proposition 5.2.4] matches this definition with the more usual one. If G is a generator, $\operatorname{Hom}_{\mathcal{G}}(G, -)$ reflects both epimorphisms and monomorphisms by [13, Proposition 1.2.12]; in fact, by [15, Section 3.1, Exercise 4], $\operatorname{Hom}_{\mathcal{G}}(G, -)$ also reflects exactness.

Now we can make a general observation:

Lemma 5.3.1. *Let $\mathcal{G}$ be an abelian category with a closed symmetric monoidal structure, with the internal hom denoted $[-,-]$. Assume that $\mathcal{G}$ has a flat generator G and let E be an injective object of $\mathcal{G}$. Then the functor*

$$[-,E]\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{G}$$

is exact.

Proof. The internal hom $[-,E]\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{G}$ is a right adjoint for any $E \in \mathcal{G}$. Indeed, the natural isomorphisms

$$\mathrm{Hom}_{\mathcal{G}^{\mathrm{op}}}([B,E],A) = \mathrm{Hom}_{\mathcal{G}}(A,[B,E]) \cong \mathrm{Hom}_{\mathcal{G}}(A \otimes B, E) \cong \mathrm{Hom}_{\mathcal{G}}(B,[A,E])$$

show that $[-,E]\colon \mathcal{G} \to \mathcal{G}^{\mathrm{op}}$ is the corresponding left adjoint. In particular, $[-,E]\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{G}$ is always left-exact.

Having a monomorphism $A \hookrightarrow B$ in $\mathcal{G}$, to test that $[B,E] \to [A,E]$ is an epimorphism, we can check it against the epimorphism-reflecting functor $\mathrm{Hom}_{\mathcal{G}}(G,-)$,

$$\mathrm{Hom}_{\mathcal{G}}(G,[B,E]) \to \mathrm{Hom}_{\mathcal{G}}(G,[A,E]),$$

which, using the adjunction, is surjective if and only if

$$\mathrm{Hom}_{\mathcal{G}}(G \otimes B, E) \to \mathrm{Hom}_{\mathcal{G}}(G \otimes A, E)$$

is, where $\otimes$ denotes the tensor product in $\mathcal{G}$. Since G is flat, $G \otimes A \to G \otimes B$ is a monomorphism, and injectivity of E implies that the map in question is indeed surjective. □

Example 5.3.2. Let $\mathcal{G}$ be the category of chain complexes of vector spaces over a field. This is a Grothendieck category where the injective objects are precisely the contractible complexes. However, the internal hom is exact for any arguments, hence there is in general no converse to Lemma 5.3.1 in the sense that if $[-,E]$ is exact, then E is injective.

Corollary 5.3.3. *Let X be a quasicompact and semiseparated scheme. Then for every injective $\mathcal{E} \in \mathrm{QCoh}(X)$, the functor*

$$\mathcal{H}om^{\mathrm{qc}}(-,\mathcal{E})\colon\ \mathrm{QCoh}(X)^{\mathrm{op}} \to \mathrm{QCoh}(X)$$

is exact.

Proof. As pointed out in the introduction, the category $\mathrm{QCoh}(X)$ has a flat generator whenever X is quasicompact semiseparated, hence Lemma 5.3.1 applies. □

For the sake of completeness, we also record the following result.

Proposition 5.3.4. *Let X be a scheme and $\mathcal{E} \in \mathrm{QCoh}(X)$ an injective quasicoherent sheaf such that $\mathcal{E}$ is also an injective object of the category $\mathcal{O}_X$-*Mod *of all sheaves of $\mathcal{O}_X$-modules. Then the functor $\mathcal{H}om^{\mathrm{qc}}(-,\mathcal{E})$ is exact on short exact sequences of quasicoherent sheaves of finite presentation.*

Proof. By [22, 05NI], the category $\mathcal{O}_X$-Mod has a flat generator; one possible choice is the direct sum of the extensions by zero of the restrictions $\mathcal{O}_X|_U$, where $U \subseteq X$ runs over all open subsets. Therefore, by the assumption on $\mathcal{E}$ and Lemma 5.3.1, the usual sheaf hom $\mathcal{H}om(-, \mathcal{E})$ is exact on $\mathcal{O}_X$-Mod. Finally, by [22, 01LA], $\mathcal{H}om(\mathcal{A}, \mathcal{E})$ is quasicoherent for every $\mathcal{A}$ of finite presentation, hence it coincides with $\mathcal{H}om^{\mathrm{qc}}(\mathcal{A}, \mathcal{E})$ and we are done. □

Note that the assumption of $\mathcal{E}$ being injective in $\mathcal{O}_X$-Mod is satisfied e.g. whenever X is locally Noetherian, [11, §II.7]. This shows that to produce a counterexample to Corollary 5.3.3 with X locally Noetherian non-semiseparated, one has to work with sheaves not of finite presentation.

Furthermore, in the locally Noetherian case, the proof shows that the sheaf hom into an injective sheaf is exact, so it is the coherator that is "responsible" for the failure of exactness in general.

Next, following [3, Appendix B], let us briefly recall some relevant facts about the map of schemes $\iota_p\colon \operatorname{Spec} \mathcal{O}_{X,p} \to X$, where p is a point of X. This map arises as the composition of the natural maps

$$\operatorname{Spec} \mathcal{O}_{X,p} \hookrightarrow \operatorname{Spec} \mathcal{O}_X(U) \cong U \hookrightarrow X,$$

where $U \subseteq X$ is some open affine neighbourhood of p. By [22, 0816], ι_p is a quasicompact quasiseparated map, hence the direct image functor $\iota_{p,*}$ preserves quasicoherence by [22, 01LC]. (Note that the assumption on X being locally Noetherian in [3] is superfluous.) To ease the notation, let us further compose this functor with the standard equivalence of categories

$$\mathcal{O}_{X,p}\text{-Mod} \cong \operatorname{QCoh}(\operatorname{Spec} \mathcal{O}_{X,p}),$$

obtaining the functor

$$\tilde{\iota}_{p,*}\colon \mathcal{O}_{X,p}\text{-Mod} \to \operatorname{QCoh}(X).$$

This functor can also be viewed as the right adjoint to the functor of taking stalks at p, $(-)_p\colon \operatorname{QCoh}(X) \to \mathcal{O}_{X,p}$-Mod.

In the sequel, we will need the following "enriched version" of this adjunction:

Lemma 5.3.5. *Let X be a quasicompact and quasiseparated scheme and $p \in X$ a point. Then, for every $\mathcal{M} \in \operatorname{QCoh}(X)$ and $N \in \mathcal{O}_{X,p}$-Mod we have the natural isomorphism*

$$\mathcal{H}om^{\mathrm{qc}}(\mathcal{M}, \tilde{\iota}_{p,*}(N)) \cong \tilde{\iota}_{p,*}\big(\operatorname{Hom}_{\mathcal{O}_{X,p}\text{-Mod}}(\mathcal{M}_p, N)\big).$$

Proof. Since taking the stalks at p commutes with the tensor product, the following two functors $\operatorname{QCoh}(X) \to \mathcal{O}_{X,p}$-Mod are naturally isomorphic:

$$\mathcal{M}_p \otimes_{\mathcal{O}_{X,p}} (-)_p \cong (\mathcal{M} \otimes -)_p.$$

Both functors are compositions of left adjoints—taking the stalks and the tensor product. Hence composing the corresponding right adjoints produces the isomorphism from the statement. □

Let us recall further relevant definitions, which we are going to use: A subcategory of a Grothendieck category is called a *Giraud subcategory* if the inclusion functor has a kernel-preserving left adjoint (if the inclusion functor has a left adjoint, the subcategory has kernels). A Giraud subcategory is always an abelian category *per se* and the left adjoint to the inclusion is exact, but the inclusion functor itself is only left exact in general (see [19, §X.1] for details).

Further, a subcategory of a locally finitely presented category with products is *definable* provided it is closed under direct products, direct limits, and pure subobjects; this follows the definition in [17, Section 16.1]. Let us make clear that the purity we have in mind here is the "categorical purity" or "c-purity" in the language of [18].

Lemma 5.3.6. *Let R be a commutative ring, $X = \operatorname{Spec} R$ and U a quasicompact open subset of X. Then the R-modules of the form $\tilde{M}(U)$, where $M \in R\text{-Mod}$, form a Giraud, definable subcategory of $R\text{-Mod}$, which we denote $\mathcal{G}_U$. The inclusion functor $i\colon \mathcal{G}_U \hookrightarrow R\text{-Mod}$ is exact if and only if U is affine.*

Proof. We have the following solid diagram of categories and functors:

$$\begin{array}{ccc}
\operatorname{QCoh}(U) & \underset{\iota_{U,*}}{\overset{(-)|_U}{\leftrightarrows}}\ (\perp) & \operatorname{QCoh}(X) \\
\simeq \updownarrow & & \updownarrow \simeq \\
\mathcal{G}_U & \underset{i}{\overset{\eta}{\leftrightarrows}}\ (\perp) & R\text{-Mod}
\end{array}$$

The left-hand vertical equivalence follows from [22, 0EHM], utilizing the assumptions on U, and the right-hand one is the standard one; in both cases the passage from sheaves to modules is just taking the global sections, so the diagram is clearly commutative. The (fully faithful) direct image functor $\iota_{U,*}$ identifies $\operatorname{QCoh}(U)$ with a full subcategory of $\operatorname{QCoh}(X)$ with the restriction to U being the exact left adjoint. We define η to be the composition of the sheaf restriction with the two vertical equivalences, hence η is the exact left adjoint to the inclusion i. This shows that $\mathcal{G}_U$ is a Giraud subcategory of R-Mod.

Similarly, to show that $\mathcal{G}_U$ is definable, we need to show that the essential image of the functor $\iota_{U,*}$ is a definable subcategory of $\operatorname{QCoh}(X)$. Basically, one has to observe that [18, Remark 4.6] generalizes to (possibly non-affine) quasicompact open subsets of X: As a right adjoint, $\iota_{U,*}$ commutes with direct products, and by [20, Lemma B.6], it commutes with direct limits. The closedness of the essential image under pure subobjects follows from [18, Lemma 2.12] and the fact that by [18, Lemma 1.4(2)], categorical pure-exactness in $\operatorname{QCoh}(X)$ is inherited from the larger category of all sheaves of $\mathcal{O}_X$-modules.

For the final claim, note that if U is affine, then since X is semiseparated, the functor $\iota_{U,*}$ is exact, which via the vertical equivalences implies the exactness of i. On the other hand, if U is not affine, then by Serre's criterion [22, 01XF], the sections over U, i.e. the composition of the left-hand equivalence with i, is not an exact functor $\operatorname{QCoh}(U) \to R\text{-Mod}$, therefore i is not exact. □

The following two lemmas shows that if a quasicompact quasiseparated scheme X is not semiseparated, this can be detected even at the level of stalks of closed points.

Lemma 5.3.7. *Let $X = \operatorname{Spec} R$ be an affine scheme and $U \subseteq X$ a quasicompact open subset. Then the following are equivalent:*

(1) U is affine,
(2) $\iota_p^{-1}(U) \subseteq \operatorname{Spec} R_p$ is affine for every point $p \in X$,
(3) $\iota_p^{-1}(U) \subseteq \operatorname{Spec} R_p$ is affine for every closed point $p \in X$,
(4) $\iota_p^{-1}(U) \subseteq \operatorname{Spec} R_p$ is affine for every closed point $p \in X \setminus U$.

Proof. (1) $\Rightarrow$ (2) follows from [22, 01JQ]. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1): Assume that U is not affine. As in the proof of Theorem 5.2.2, let us express U as the union $X_{f_1} \cup \cdots \cup X_{f_n}$ of distinguished open affine subsets of X and put $I = (f_1, \ldots, f_n)$ the corresponding ideal of R. Then [10, Chapter II, Exercise 2.17(b)] implies that $I\mathcal{O}_X(U) \subsetneq \mathcal{O}_X(U)$. In particular, there is a maximal ideal p of R such that $I_p\mathcal{O}_X(U)_p \subsetneq \mathcal{O}_X(U)_p$. This p cannot belong to U, for otherwise $p \in X_{f_i}$ or equivalently $f_i \notin p$ for some $i \in \{1, \ldots, n\}$, so I_p would be the unit ideal of R_p.

Let $Y = \operatorname{Spec} R_p$, $W = \iota_p^{-1}(U) \subseteq Y$, and denote by $f_1', \ldots, f_n'$ the images of $f_1, \ldots, f_n$ under the canonical map $R \to R_p$. Then, on one hand, $Y_{f_i'} = \iota_p^{-1}(X_{f_i})$ for every $i \in \{1, \ldots, n\}$, so $W = Y_{f_1'} \cup \cdots \cup Y_{f_n'}$. On the other hand, $\mathcal{O}_Y(Y_{f_i'}) = \mathcal{O}_X(X_{f_i})_p$ for every $i \in \{1, \ldots, n\}$, hence $\mathcal{O}_Y(W) \cong \mathcal{O}_X(U)_p$ by the sheaf axiom. Since $I_p = (f_1', \ldots, f_n')$, we infer that W is not affine, once more by [10, Chapter II, Exercise 2.17(b)]. □

Lemma 5.3.8. *Let X be a quasicompact quasiseparated scheme, which is not semiseparated. Then there is an affine open subset $V \subseteq X$ and a closed point $p \in X$ such that $\iota_p^{-1}(V)$ is not affine.*

Proof. Let U, V be open affine subsets of X such that $U \cap V$ is not affine. By Lemma 5.3.7, there is a point $p \in U \setminus V$ such that $\iota_p^{-1}(V)$ is not affine. Now pick a closed point $p' \in \overline{\{p\}}$. Such a point exists by [22, 005E] and clearly it is also a closed point in X. Moreover, any open subset of X containing p' also contains p, and hence $\iota_p = \iota_{p'} \circ j_p$, where $j_p \colon \operatorname{Spec} \mathcal{O}_{X,p} \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,p'}$ denotes the obvious inclusion. Since $j_p^{-1}(\iota_{p'}^{-1}(V)) = \iota_p^{-1}(V)$ is not affine, $\iota_{p'}^{-1}(V) \subseteq \operatorname{Spec} \mathcal{O}_{X,p'}$ is not affine by Lemma 5.3.7 (with $X = \operatorname{Spec} \mathcal{O}_{X,p'}$). □

As a last step of preparation before proving the main theorem, we recall a standard construction of simple quasicoherent sheaves on X.

Lemma 5.3.9. *Let X be a quasicompact quasiseparated scheme and $p \in X$ a closed point. Then the sheaf $\tilde{\iota}_{p,*}(\mathcal{O}_{X,p}/p)$ is a simple object of* $\operatorname{QCoh}(X)$ *(where we identify p with the sole maximal ideal of the ring $\mathcal{O}_{X,p}$).*

Proof. Note that the simple $\mathcal{O}_{X,p}$-module $\mathcal{O}_{X,p}/p$ is supported only in the maximal ideal p. As p is closed in X, by the definition of the direct image functor, $\tilde{\iota}_{p,*}(\mathcal{O}_{X,p}/p)$ is actually a skyscraper sheaf with $\mathcal{O}_{X,p}/p$ as the module of sections on every neighbourhood of p. In other words, for every point $q \in X$, the module of stalks at q of $\tilde{\iota}_{p,*}(\mathcal{O}_{X,p}/p)$ is either zero (if $q \neq p$), or the simple module $\mathcal{O}_X/p$ (if $q = p$). This readily implies that the only non-zero subsheaf of $\tilde{\iota}_{p,*}(\mathcal{O}_{X,p}/p)$ is the sheaf itself. □

Theorem 5.3.10. *Let X be a quasicompact quasiseparated scheme. Then the following are equivalent:*

(1) for each $\mathcal{E} \in \mathrm{QCoh}(X)$ injective, the contravariant functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact;

(2) for each simple quasicoherent sheaf $\mathcal{S}$ and its injective envelope $\mathcal{E}$ in the category $\mathrm{QCoh}(X)$, the contravariant functor $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is exact;

(3) X is semiseparated.

Proof. (1) $\Rightarrow$ (2) is obvious and (3) $\Rightarrow$ (1) follows from Corollary 5.3.3.

To prove (2) $\Rightarrow$ (3), assume that X is not semiseparated; then, by Lemma 5.3.8, there are a closed point $p \in X$ and an open affine set $V \subseteq X$ such that $W = \iota_p^{-1}(V) \subseteq \operatorname{Spec} \mathcal{O}_{X,p}$ is not affine. By Serre's criterion [22, 01XF], there is $\mathcal{A}' \in \mathrm{QCoh}(W)$ satisfying $\mathrm{H}^1(W, \mathcal{A}') \neq 0$. Since $\mathrm{QCoh}(W)$ is a Grothendieck category, there is an embedding $\mathcal{A}' \hookrightarrow \mathcal{B}'$ with $\mathcal{B}' \in \mathrm{QCoh}(W)$ injective; in particular, $\mathrm{H}^1(W, \mathcal{B}') = 0$. Let $\mathcal{C}' \in \mathrm{QCoh}(W)$ be the cokernel of this embedding; hence we have a short exact sequence $0 \to \mathcal{A}' \to \mathcal{B}' \to \mathcal{C}' \to 0$ in $\mathrm{QCoh}(W)$ with non-exact sequence of sections over W.

Let ι_W be the composition of the inclusions $W \hookrightarrow \operatorname{Spec} \mathcal{O}_{X,p}$ and $\operatorname{Spec} \mathcal{O}_{X,p} \hookrightarrow X$. By [22, 0816], this is a quasicompact quasiseparated morphism, hence the corresponding direct image functor $\iota_{W,*} \colon \mathrm{QCoh}(W) \to \mathrm{QCoh}(X)$ preserves quasicoherence [22, 01LC]. This is the right adjoint to the composition of taking the stalks at p and restricting to W; let us denote this composition, with slight abuse of notation, by $(-)|_W \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(W)$, and call it the *restriction to W*. Being a composition of exact functors, it is an exact functor.

Abusing the notation and the terminology even further, for each quasicoherent sheaf $\mathcal{M}$ on X, let us denote $\mathcal{M}|_W(W)$ by $\mathcal{M}(W)$ and call this $\widetilde{\mathcal{O}_{X,p}}(W)$-module *sections on W*.

Put $\mathcal{A} = \iota_{W,*}(\mathcal{A}')$, $\mathcal{B} = \iota_{W,*}(\mathcal{B}')$. The direct image functor is left exact; let $\mathcal{C}$ be the cokernel of $\mathcal{A} \hookrightarrow \mathcal{B}$. Since the restriction to W is an exact functor and $\mathcal{A}|_W = \mathcal{A}'$, $\mathcal{B}|_W = \mathcal{B}'$, it follows that $\mathcal{C}|_W = \mathcal{C}'$. However, $\mathcal{C} \neq \iota_{W,*}(\mathcal{C}')$, for the sequence $0 \to \mathcal{A} \to \mathcal{B} \to \iota_{W,*}(\mathcal{C}') \to 0$ is not exact, as passing to stalks at p shows. Anyway, we have obtained a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ in $\mathrm{QCoh}(X)$ such that the sequence of sections on W, $0 \to \mathcal{A}(W) \to \mathcal{B}(W) \to \mathcal{C}(W) \to 0$, is not exact.

Let E be the injective envelope of the sole simple $\mathcal{O}_{X,p}$-module $\mathcal{O}_{X,p}/p$ (where we identify p with the unique maximal ideal of the ring $\mathcal{O}_{X,p}$). By [2, Corollary 18.19], this is an injective cogenerator of $\mathcal{O}_{X,p}$-Mod, easily seen to be indecomposable. Further, put $\mathcal{E} = \tilde{\iota}_{p,*}(E)$. Since $\tilde{\iota}_{p,*}$ is a right adjoint to an exact functor, it preserves injectives, hence $\mathcal{E}$ is an injective object of $\mathrm{QCoh}(X)$.

Note that by Lemma 5.3.9, $\mathcal{S} = \tilde{\iota}_{p,*}(\mathcal{O}_{X,p}/p)$ is a simple quasicoherent sheaf. (This is actually the only place where we use that p is closed in X.) Moreover, the functor $\tilde{\iota}_{p,*}$ is fully faithful, therefore $\mathcal{E}$ is indecomposable, hence the injective envelope of $\mathcal{S}$ must be the whole of $\mathcal{E}$.

For $\mathcal{M} \in \mathrm{QCoh}(X)$, denote by $\mathcal{M}^+$ the sheaf $\mathcal{H}om^{\mathrm{qc}}(\mathcal{M}, \mathcal{E})$. We are going to show that the sequence

$$0 \to \mathcal{A}^{++} \to \mathcal{B}^{++} \to \mathcal{C}^{++} \to 0 \qquad (++)$$

cannot be exact by showing that the sections over the open affine set V are not exact.

Denote further $M^* = \mathrm{Hom}_{\mathcal{O}_{X,p}}(M, E)$ for $M \in \mathcal{O}_{X,p}$-Mod. By Lemma 5.3.5, we have

$$\mathcal{M}^+ = \mathcal{H}om^{\mathrm{qc}}(\mathcal{M}, \mathcal{E}) = \tilde{\iota}_{p,*}\big(\mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{M}_p, E)\big) = \tilde{\iota}_{p,*}(\mathcal{M}_p^*),$$

therefore

$$(\mathcal{M}^+)_p = \mathcal{M}_p^*$$

and

$$\mathcal{M}^{++} = \mathcal{H}om^{\mathrm{qc}}(\mathcal{M}^+, \mathcal{E}) = \tilde{\iota}_{p,*}\big(\mathrm{Hom}_{\mathcal{O}_{X,p}}((\mathcal{M}^+)_p, E)\big) = \tilde{\iota}_{p,*}(\mathcal{M}_p^{**})$$

for every $\mathcal{M} \in \mathrm{QCoh}(X)$. We further get

$$(\mathcal{M}^{++})_p = \mathcal{M}_p^{**},$$

and so

$$\mathcal{M}^{++}(V) = \mathcal{M}^{++}(W).$$

Let $\mathcal{G}_W$ denote the subcategory of $\mathcal{O}_{X,p}$-Mod from Lemma 5.3.6 (with $R = \mathcal{O}_{X,p}$ and $U = W$). Further put $A = \mathcal{A}_p$, $B = \mathcal{B}_p$, and $C = \mathcal{C}_p$. By the construction, $A = \mathcal{A}(W)$, $B = \mathcal{B}(W)$, so $A, B \in \mathcal{G}_W$. On the other hand $C \subsetneq \mathcal{C}(W)$, hence $C \notin \mathcal{G}_W$. Indeed, C is the cokernel of $A \hookrightarrow B$ in $\mathcal{O}_{X,p}$-Mod and $\mathcal{C}(W)$ is the cokernel of $A \hookrightarrow B$ in $\mathcal{G}_W$, so $C \in \mathcal{G}_W$ would imply $C = \mathcal{C}(W)$.

Since by Lemma 5.3.6 $\mathcal{G}_W$ is a definable subcategory of $\mathcal{O}_{X,p}$-Mod, [17, Corollary 3.4.21] implies that $A^{**}, B^{**} \in \mathcal{G}_W$, but $C^{**} \notin \mathcal{G}_W$ by [17, Corollary 1.3.16] and the fact that $C \notin \mathcal{G}_W$. In other words,

$$\mathcal{A}^{++}(V) = \mathcal{A}^{++}(W) = (\mathcal{A}^{++})_p = A^{**},$$
$$\mathcal{B}^{++}(V) = \mathcal{B}^{++}(W) = (\mathcal{B}^{++})_p = B^{**},$$

but, as $\mathcal{C}^{++}(W) \in \mathcal{G}_W$,

$$\mathcal{C}^{++}(V) = \mathcal{C}^{++}(W) \ncong (\mathcal{C}^{++})_p = C^{**}.$$

However, as $(-)^*$ is an exact contravariant functor on $\mathcal{O}_{X,p}$-Mod, the sequence $0 \to A^{**} \to B^{**} \to C^{**} \to 0$ is exact. This shows that the sequence $(++)$ is not exact after passing to sections over V as desired. □

Remark 5.3.11. Note that we have re-proved Theorem 5.2.2: If X is quasicompact and quasiseparated, but not semiseparated, then by Theorem 5.3.10, there is an injective $\mathcal{E} \in \mathrm{QCoh}(X)$ such that $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ is not exact; by Lemma 5.3.1, this means that the category $\mathrm{QCoh}(X)$ cannot have a flat generator.

Example 5.3.12. As in Example 5.2.3, it may be instructive to consider the plane with double origin. We again put $R = k[x, y]$, the open subsets U, V will be the two copies of $\operatorname{Spec} R$, W will be the punctured plane and p the origin belonging to U. In this situation, p and V fit into the roles assigned in the statement of Lemma 5.3.8, and $W = \iota_p^{-1}(V)$ is an actual open subset of X.

The following example was suggested to us by Leonid Positselski. The sheafification $0 \to \mathcal{O}_U \to \mathcal{O}_U \to \mathcal{M} \to 0$ of the short exact sequence $0 \to k[x, y] \xrightarrow{y} k[x, y] \to k[x] \to 0$ of R-modules has non-exact sections on W. Indeed, a direct computation using the sheaf axiom reveals that the sections on W form only the left exact sequence

$$0 \to k[x,y] \xrightarrow{y} k[x,y] \to k[x^{\pm 1}]. \tag{$*$}$$

If we take the sheafification of the same exact sequence of R-modules in $\mathrm{QCoh}(V)$ and glue it together with the sequence in $\mathrm{QCoh}(U)$, we obtain a short exact sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ of quasicoherent sheaves on $X = U \cup V$ whose sections on W look like $(*)$.

Let $E \in R\text{-Mod}$ be the injective envelope of the simple R-module $R/(x,y)$. The arguments in the proof of Theorem 5.3.10 show that the stalks at p of the double dual $(++)$, where $(-)^+ = \mathcal{H}om^{\mathrm{qc}}(-,\mathscr{E})$ for $\mathscr{E} = \iota_{U,*}(\tilde{E})$, are of the form

$$0 \to k[\![x,y]\!] \xrightarrow{y} k[\![x,y]\!] \to k[\![x]\!] \to 0.$$

Here we also use [7, Theorem 3.4.1(8)] to compute double duals of modules of stalks at p with respect to $\mathrm{Hom}_{\mathcal{O}_{X,p}}(-,E)$. Consequently, following the arguments in the proof of Theorem 5.3.10 further, the sections of $(++)$ on the open sets W and V coincide and the sheaf axiom implies that they look like

$$0 \to k[\![x,y]\!] \xrightarrow{y} k[\![x,y]\!] \to k(\!(x)\!).$$

This is again a left, but not right exact sequence and so is $(++)$ since V is affine.

Although, for the sake of simplicity, this example does not fully follow the proof of Theorem 5.3.10 (the sheaf $\mathscr{B}|_W$ is not injective in $\mathrm{QCoh}(W)$), it is still sufficient to explicitly illustrate the non-exactness of $\mathcal{H}om^{\mathrm{qc}}(-,\mathscr{E})$ on $\mathrm{QCoh}(X)$.

Corollary 5.3.13. *Let X be a quasicompact quasiseparated scheme and $\mathscr{E} \in \mathrm{QCoh}(X)$ be an injective cogenerator. Then $\mathcal{H}om^{\mathrm{qc}}(-,\mathscr{E})$ is exact if and only if X is semiseparated.*

Proof. Suppose that X is non-semiseparated. Theorem 5.3.10(2) asserts that there is a simple quasicoherent sheaf $\mathcal{S}$ and its injective envelope $\mathscr{E}'$ such that $\mathcal{H}om^{\mathrm{qc}}(-,\mathscr{E}')$ is not exact. Clearly $\mathcal{S}$ embeds into $\mathscr{E}$ and this embedding extends to a (necessarily split) embedding $\mathscr{E}' \hookrightarrow \mathscr{E}$. It follows that $\mathcal{H}om^{\mathrm{qc}}(-,\mathscr{E})$ is not exact. □

Remark 5.3.14. Let R be a commutative ring and E an injective cogenerator of the category R-Mod. The contravariant functor $\mathrm{Hom}_R(-,E)$ has found many applications in the model theory of modules and its generalizations (cf. [17, 1.3.3]; this also works in greater generality over non-commutative rings). This has led to a natural generalization to symmetric closed monoidal Grothendieck categories; in particular, in [6], the functor $\mathcal{H}om^{\mathrm{qc}}(-,\mathscr{E})$, where $\mathscr{E}$ is an injective cogenerator of $\mathrm{QCoh}(X)$, has been used to investigate the properties of "geometric" purity.[1]

Corollary 5.3.13 shows that this "duality" is, perhaps surprisingly, not exact for non-semiseparated schemes. However, it turns out that this is not really an

[1]To be precise, [6] first considers the ordinary sheaf hom functor $\mathcal{H}om(-,\mathscr{E}')$, where $\mathscr{E}'$ is an injective cogenerator of the category $\mathcal{O}_X$-Mod of all sheaves of $\mathcal{O}_X$-modules, and then composes this with the coherator to obtain a functor to $\mathrm{QCoh}(X)$. Using the fact that the tensor product of quasicoherent sheaves is just the ordinary sheaf tensor product, the "composition-of-adjoints argument" as in the proof of Lemma 5.3.5 shows that we can replace the second argument with the coherator of $\mathscr{E}'$, which turns out to be an injective cogenerator of $\mathrm{QCoh}(X)$.

obstacle to using this functor for investigating purity in the same way as in the classical situation, cf. [6, Proposition 4.5]. Furthermore, this functor at least reflects exactness, as the next proposition shows.

Proposition 5.3.15. *Let* $\mathcal{E}$ *be an injective cogenerator of* $\mathrm{QCoh}(X)$*. Then the functor* $\mathcal{H}om^{\mathrm{qc}}(-, \mathcal{E})$ *reflects exactness.*

Proof. Assume that

$$0 \to \mathcal{H}om^{\mathrm{qc}}(\mathcal{C}, \mathcal{E}) \to \mathcal{H}om^{\mathrm{qc}}(\mathcal{B}, \mathcal{E}) \to \mathcal{H}om^{\mathrm{qc}}(\mathcal{A}, \mathcal{E}) \to 0$$

is exact. By [6, Proposition 4.4 & Lemma 4.7], all the terms are geometrically pure-injective (or g-pure-injective, or just pure-injective in the language of [6]). In particular, by [18, Proposition 4.14 & Lemma 4.15], $\mathcal{H}om^{\mathrm{qc}}(\mathcal{C}, \mathcal{E})$ has vanishing sheaf cohomology, therefore taking global sections produces a short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{C}, \mathcal{E}) \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{B}, \mathcal{E}) \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{A}, \mathcal{E}) \to 0.$$

As $\mathcal{E}$ is a cogenerator (cf. the discussion before Lemma 5.3.1), this implies that $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact. □

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