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Differential equations with eigenvalue in boundary conditions

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Abstract: The goal of this thesis was to study Stokes problem with eigenvalue in boundary condition. We were in particular interested in determining the asymptotic behaviour of the sequence of eigenvalues. We approached this problem by modifying techniques used in several papers studying asymptotic behaviour of eigenvalues in boundary condition for Steklov problem and we wanted to conclude similar results. Firstly, we introduced some theoretical results yielding that the eigenvalue sequence of the problem is corresponding to an eigenvalue sequence of a certain compact and self-adjoint operator. Next, we explicitly calculated precise asymptotic behaviour of eigenvalues of auxiliary problems on simple domains, however, due to technical difficulties, we were only able to do in two and three dimensions. Finally, by using Min-max Theorem, we managed to get estimates of eigenvalues of the original problem on any bounded $\mathcal{C}^2$ domain by eigenvalues of considered auxiliary problems and thus by applying previous results, we managed to prove the desired asymptotic behaviour.

Keywords: partial differential equations, eigenvalues, asymptotic behaviour

Contents

Notation

n	Natural number.
$\mathbb{N}_0$	Set $\mathbb{N} \cup \{0\}$.
$\overline{z}$	Complex conjugate of a complex number z.
Ω	Bounded Lipschitz domain in $\mathbb{R}^n$ (by domain we mean open and connected set).
$\boldsymbol{g}$	Functions written in bold represent vector-valued functions $\boldsymbol{g}\colon \Omega \to \mathbb{R}^n$ while scalar functions are plain
$\boldsymbol{g}_i$	The i-th coordinate of vector-valued function $\boldsymbol{g}$.
$\boldsymbol{n}$	The outwards unit normal vector to Ω.
$\boldsymbol{A} : \boldsymbol{B}$	The inner product on space of complex matrices. For $\boldsymbol{A}_{ij}, \boldsymbol{B}_{ij} \in \mathbb{C}^{n\times n}$ it is defined as $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^{n} \boldsymbol{A}_{ij}\overline{\boldsymbol{B}_{ij}}$.
λ^n	n-dimensional Lebesgue measure.
$\mathcal{H}^{n-1}$	$(n-1)$-dimensional Hausdorff measure.
$\langle \cdot, \cdot \rangle_H$	Inner product on a Hilbert space H.
$\langle T, x \rangle_{X,Y}$	Linear operator $T\colon X \to Y$ applied to $x \in X$ where X, Y are Banach spaces.
$\sigma_p(T)$	Point spectrum of a linear bounded operator $T\colon X \to X$ where X is a Banach space.
$\left(\mathcal{C}^k(\Omega)\right)^n$	For $k \in \mathbb{N}$ denotes the space of real vector-valued functions $\boldsymbol{g}\colon \Omega \to \mathbb{R}^n$ for which are partial derivatives k-times continuously differentiable on Ω .
$\mathcal{C}^{n,\mu}(\Omega)$	Hölder space of real scalar functions which have continuous partial derivatives up to order n and such that the n-th partial derivatives are Hölder continuous with exponent $\mu \in \mathbb{R}, 0 < \mu \leq 1$.
$(L^p(\Omega))^n$	Lebesgue space of real vector valued measurable functions $\boldsymbol{g}\colon \Omega \to \mathbb{R}^n$ over the field of complex numbers where $p \in [1, \infty)$ (see Definition 1).
$(W^{1,p}(\Omega))^n$	Sobolev space of real vector valued measurable functions $\boldsymbol{g}\colon \Omega \to \mathbb{R}^n$ over the field of complex numbers where $p \in [1, \infty)$ (see Definition 2).
Tr	Trace operator $\mathrm{Tr}\colon\ (W^{1,p}(\Omega))^n \to (L^p(\partial\Omega))^n$.
$(W^{1,2}_{\boldsymbol{n}}(\Omega))^n$	Space $\{\boldsymbol{u} \in (W^{1,p}(\Omega))^n\,;\ \mathrm{Tr}(\boldsymbol{u}) \cdot \boldsymbol{n} = 0\}$.
$\operatorname{div} \boldsymbol{u}$	The divergence of a vector-valued function $\boldsymbol{u} \in (W^{1,p}(\Omega))^n$. It is defined as $\operatorname{div} \boldsymbol{u} = \sum_{i=1}^{n} \partial_i \boldsymbol{u}_i$.
$\left(W^{1,2}_{\boldsymbol{n},\mathrm{div}}(\Omega)\right)^n$	Space $\{\boldsymbol{u} \in (W^{1,2}_{\boldsymbol{n}}(\Omega))^n\,;\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}$.

∇p	Gradient of the function p. For a scalar function $p \in W^{1,p}(\Omega)$ it is defined as the matrix $(\partial_1,p, \ldots, \partial_n p)$.
$\nabla \boldsymbol{u}$	Gradient of the function $\boldsymbol{u}$. For $\boldsymbol{u} \in (W^{1,p}(\Omega))^n$ it is defined as the matrix $\nabla \boldsymbol{u} = (\partial_i \boldsymbol{u}_j)_{n \times n}$.
$\boldsymbol{D}\boldsymbol{u}$	The symmetric part of the gradient of $\boldsymbol{u} \in (W^{1,p}(\Omega))^n$. It is defined as $\boldsymbol{D}\boldsymbol{u} = 1/2 \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\right)$.
$\Delta \boldsymbol{u}$	The Laplace operator of a function $(\mathcal{C}^2(\Omega))^n$. It is vector valued function defined as $(\Delta \boldsymbol{u})_i = \sum_{j=1}^n \partial_j^2 \boldsymbol{u}_i$ where $i = 1, \ldots, n$.

Introduction

Motivation for this thesis comes from studying problems with dynamic boundary conditions. In our case we are interested in Stokes problem, i.e. for $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $I \subset \mathbb{R}$ non-empty interval, $\boldsymbol{u} \colon \overline{\Omega} \times I \to \mathbb{R}^n$, $p \colon \Omega \times I \to \mathbb{R}$, $\boldsymbol{f} \colon \Omega \times I \to \mathbb{R}^n$ $\boldsymbol{g} \colon \partial\Omega \times I \to \mathbb{R}^n$ we consider

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega \times I,\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega \times I,\\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega \times I,\\
\partial_t \boldsymbol{u} - [(\boldsymbol{D}\boldsymbol{u})\, \boldsymbol{n}]_\tau &= \boldsymbol{g} && \text{on } \partial\Omega \times I
\end{aligned}
$$

and some initial condition for $\boldsymbol{u}$. In the formulation $\boldsymbol{n}$ denotes the outwards unit normal vector and $\boldsymbol{v}_\tau$ denotes the projection of $\boldsymbol{v}$ to the tanget plane to Ω, i.e. $\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\, \boldsymbol{n}$. We now replace term $\partial_t \boldsymbol{u}$ with $\lambda \boldsymbol{u}$ for $\lambda \in \mathbb{R}$ and assume that right-hand sides of the system are zero. We will study this modified problem. Thus, in this thesis we will be dealing with stationary Stokes problem. More precisely, let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $\boldsymbol{u} \colon \overline{\Omega} \to \mathbb{R}^n$, $p \colon \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we consider the following system of partial differential equations

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= 0 && \text{in } \Omega, && (1)\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && (2)\\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, && (3)\\
[(\boldsymbol{D}\boldsymbol{u})\, \boldsymbol{n}]_\tau &= \lambda \boldsymbol{u} && \text{on } \partial\Omega. && (4)
\end{aligned}
$$

We will refer to the constant λ in the formulation of the problem (1)-(4) as an eigenvalue of the problem (1)-(4) if there exists a corresponding non-trivial weak solution to the problem. Specifically, we will be interested in studying the asymptotic behaviour of the sequence of eigenvalues $(\lambda_k)_{k=1}^{\infty}$ of the problem (1)-(4).

Inspiration for approaching this problem comes from a thesis by Sandgren [1] where they considered Steklov problem on a bounded $\mathcal{C}^2$ domain $\Omega \subset \mathbb{R}^n$, i.e.

$$
\begin{aligned}
-\Delta \boldsymbol{u} &= 0 && \text{in } \Omega, && (5)\\
(\nabla \boldsymbol{u})\, \boldsymbol{n} &= \mu \boldsymbol{u} && \text{on } \partial\Omega && (6)
\end{aligned}
$$

and studied asymptotic behaviour of the eigenvalue sequence $(\mu_k)_{k=1}^{\infty}$ of the problem (5)-(6). Firstly, they calculated μ_k explicitly on simple domains and then they improved the results to more general domains. Eventually, they managed to prove that

$$\mu_k = C_{\text{Stek}}(\Omega, n) k^{1/(n-1)} + o(k^{1/(n-1)}).$$

Later on, in a paper by von Below and François [2], they used this result to determine the growth order of $(\mu_k)_{k=1}^{\infty}$ and to obtain lower and upper bounds for the leading asymptotic coefficient in the following problem

$$
\begin{aligned}
-\Delta \boldsymbol{u} &= \mu \boldsymbol{u} && \text{in } \Omega, && (7)\\
(\nabla \boldsymbol{u})\, \boldsymbol{n} &= \mu \boldsymbol{u} && \text{on } \partial\Omega. && (8)
\end{aligned}
$$

More precisely, they were able to get

$$c_{\text{Stek}}(\Omega, n) \leq \liminf_{k\to\infty} \frac{\mu_k}{k^{1/(n-1)}} \leq \limsup_{k\to\infty} \frac{\mu_k}{k^{1/(n-1)}} \leq C_{\text{Stek}}(\Omega, n).$$

for some constants $c_{\text{Stek}}, C_{\text{Stek}}(\Omega, n) \in \mathbb{R}, c_{\text{Stek}}, C_{\text{Stek}}(\Omega, n) > 0$ depending on dimension n and domain Ω.

Our goal in this thesis will be to modify techniques used [1] and [2] for our problem (1)-(4) in order to obtain similar results about the asymptotic behaviour of $(\lambda_k)_{k=1}^{\infty}$. Our hypothesis is that the growth order will remain the same even for this more complex problem, i.e. we expect that

$$c_{\text{Stokes}}(\Omega, n) \leq \liminf_{k\to\infty} \frac{\lambda_k}{k^{1/(n-1)}} \leq \limsup_{k\to\infty} \frac{\lambda_k}{k^{1/(n-1)}} \leq C_{\text{Stokes}}(\Omega, n)$$

for some constants $c_{\text{Stokes}}, C_{\text{Stokes}}(\Omega, n) \in \mathbb{R}, c_{\text{Stokes}}, C_{\text{Stokes}}(\Omega, n) > 0$ depending on dimension n and domain Ω.

1. Preliminaries

In the chapter we introduce the most important definitions and Theorems that we will use extensively throughout the whole thesis.

Definition 1. (Lebesgue spaces). *Let $p \in \mathbb{R}, p \in [1,\infty]$, let $n, d \in \mathbb{N}$ and let Ω be a measurable set in $\mathbb{R}^n$. We define the Lebesgue space $(L^p(\Omega))^d$ as a set of all measurable functions $\boldsymbol{g}\colon \Omega \to \mathbb{R}^d$ such that $\|\boldsymbol{g}\|_{(L^p(\Omega))^d} < \infty$ where*

$$\|\boldsymbol{g}\|_{(L^p(\Omega))^d} = \begin{cases} \left(\int_\Omega \sum_{i=1}^{d} |\boldsymbol{g}_i(\boldsymbol{x})|^p \, d\lambda^n\right)^{1/p} & if\, p \in [1,\infty), \\ \text{ess-sup}_{x\in\Omega} \max\limits_{i\in\{1,\dots,d\}} |\boldsymbol{g}_i(\boldsymbol{x})| & if\, p = \infty. \end{cases}$$

Remark. Formally, Lebesgue space $(L^p(\Omega))^d$ should be a set of equivalence classes of functions which are equal λ^n almost everywhere in Ω in order for the pair $\left((L^p(\Omega))^d, \|\cdot\|_{(L^p(\Omega))^d}\right)$ to form a linear vector space. However, it is a common convention to leave out the notion of equivalence classes.

Definition 2. (Sobolev spaces). *Let $n, d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $k \in \mathbb{N}$ and $p \in \mathbb{R}, p \in [1,\infty]$. We define the space of Sobolev functions $\left(W^{k,p}(\Omega)\right)^d$ as the set*

$$\left(W^{k,p}(\Omega)\right)^d = \{\boldsymbol{g} \in (L^p(\Omega))^d ; \forall \alpha \in (\mathbb{N}_0)^d, |\alpha| \le k : \mathrm{D}^\alpha \boldsymbol{g} \in (L^p(\Omega))^d\},$$

where $\mathrm{D}^\alpha \boldsymbol{g}$ denotes the weak derivative of $\boldsymbol{g}$ with respect to α. We define the functional $\|\cdot\|_{\left(W^{k,p}(\Omega)\right)^d}$ as follows

$$\|\boldsymbol{g}\|_{\left(W^{k,p}(\Omega)\right)^d} = \begin{cases} \left(\sum\limits_{|\alpha|\le k} \int_\Omega \sum_{i=1}^{d} |\mathrm{D}^\alpha \boldsymbol{g}_i(\boldsymbol{x})|^p \, d\lambda^n\right)^{1/p} & if\, p \in [1,\infty), \\ \max\limits_{|\alpha|\le k} \text{ess-sup}_{x\in\Omega} \max\limits_{i\in\{1,\dots,d\}} |\mathrm{D}^\alpha \boldsymbol{g}_i(\boldsymbol{x})| & if\, p = \infty. \end{cases}$$

Remark. The set $\left(W^{k,p}(\Omega)\right)^d$ equipped with the functional $\|\cdot\|_{\left(W^{k,p}(\Omega)\right)^d}$ forms a normed linear space.

Definition 3. *($\mathcal{C}^{k,\mu}$* domains). *Let $k \in \mathbb{N}$ and $\mu \in \mathbb{R}, 0 < \mu \le 1$. Let $n \in \mathbb{N}$ and let Ω be a bounded domain in $\mathbb{R}^n$. We say that Ω is a $\mathcal{C}^{k,\mu}$ domain if there exist $\alpha, \beta \in \mathbb{R}, \alpha, \beta > 0$ and $M \in \mathbb{N}$ systems of Cartesian coordinates and $\mathcal{C}^{k,\mu}$ functions a_r for $r = 1, \dots, M$ such that*

- *for r-th system we denote $x = (x_{r_1}, \dots, x_{r_n})$ as (x'_r, x_{r_n}) and*

$$\Delta_r = \{x'_r \in \mathbb{R}^{n-1}; |x_{r_i}| < \alpha, i = 1, \dots, n-1\},$$

- *$a_r\colon \Delta_r \to \mathbb{R}$ and if we denote by T_r an orthogonal transformation from r-th system of Cartesian coordinates to global system of Cartesian coordinates then for each $x \in \partial\Omega$ there exists $r \in \{1, \dots, M\}$ and $x'_r \in \Delta_r$ such that $x = T_r(x'_r, a_r(x'_r))$,*

- *if we define*

$$\begin{aligned}
V_r^+ &= \{(x_r', x_{r_n}) \in \mathbb{R}^n;\ x_r' \in \Delta_r, a_r(x_r') < x_{r_n} < a_r(x_r') + \beta\}\,,\\
V_r^- &= \{(x_r', x_{r_n}) \in \mathbb{R}^n;\ x_r' \in \Delta_r, a_r(x_r') - \beta < x_{r_n} < a_r(x_r')\}\,,\\
\Lambda_r &= \{(x_r', x_{r_n}) \in \mathbb{R}^n;\ x_r' \in \Delta_r, a_r(x_r') = x_{r_n}\}\,,
\end{aligned}$$

then $T_r(V_r^+) \subset \Omega$, $T_r(V_r^-) \subset \mathbb{R}^n \setminus \overline{\Omega}$ *and* $\cup_{r=1}^{M} T_r(\Lambda_r) = \partial\Omega$.

Theorem 4. (Trace Theorem). *Let* $n \in \mathbb{N}$ *and let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, *i.e.* $\Omega \in \mathcal{C}^{0,1}$. *Let* $p \in \mathbb{R}, p \geq 1$. *Then there exists a bounded linear operator* Tr

$$\mathrm{Tr}\colon \left(W^{1,p}(\Omega)\right)^n \to \left(L^p(\partial\Omega)\right)^n$$

such that Tr *extends the classical trace operator, i.e.*

$$\mathrm{Tr}(\boldsymbol{u}) = \boldsymbol{u}|_{\partial\Omega}$$

for all $\boldsymbol{u} \in \left(W^{1,p}(\Omega)\right)^n \cap \left(\mathcal{C}(\overline{\Omega})\right)^n$.

Proof. See [3], Section 6.4. □

Theorem 5. (Riesz representation Theorem). *Let* H *be a Hilbert space with inner product* $\langle \cdot, \cdot \rangle_H$. *Let* $\varphi \in H^*$, *i.e.* φ *is a continuous linear functional on* H. *Then there exists a unique* $y \in H$ *such that*

$$\langle \varphi, u \rangle_{H^*,H} = \langle u, y \rangle_H$$

for all $u \in H$. *Moreover,*

$$\|y\|_H = \|\varphi\|_{H^*}.$$

Proof. See [4], Theorem 5.5, p. 135. □

Theorem 6. (Gauss's Theorem). *Let* $n \in \mathbb{N}$ *and let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$. *Let* $\boldsymbol{u} \in \left(W^{1,2}(\Omega)\right)^n$. *Then*

$$\int_\Omega \operatorname{div} \boldsymbol{u}\, d\lambda^n = \int_{\partial\Omega} \mathrm{Tr}(\boldsymbol{u}) \cdot \boldsymbol{n}\, d\mathcal{H}^{n-1}.$$

Proof. See [5], Theorem 1.1., p. 117. □

Remark. For bounded Lipschitz domains in $\mathbb{R}^n$ the outwards unit normal vector exists $\mathcal{H}^{n-1}$ almost everywhere on $\partial\Omega$ (see [5], Lemma 4.2., p. 83).

2. Basic properties and asymptotics on simple domains

2.1 Formulation of the problem

Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^n$, $p\colon \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we consider

$$-\Delta \boldsymbol{u} + \nabla p = 0 \qquad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{2.3}$$

$$[(\boldsymbol{D}\boldsymbol{u})\,\boldsymbol{n}]_\tau = \lambda \boldsymbol{u} \qquad \text{on } \partial\Omega, \tag{2.4}$$

where $\boldsymbol{n}$ denotes the outwards unit normal vector on $\partial\Omega$ and $\boldsymbol{v}_\tau$ denotes the projection to the tanget plane, i.e. $\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\,\boldsymbol{n}$.

First, we derive a weak formulation of the problem (2.1)-(2.4). Suppose for now that there exist $\boldsymbol{u} \in (\mathcal{C}^\infty(\overline{\Omega}))^n$, $p \in \mathcal{C}^\infty(\overline{\Omega})$ and $\lambda \in \mathbb{R}$ such that (2.1)-(2.4) hold. Assume that mapping $\boldsymbol{\varphi}\colon \overline{\Omega} \to \mathbb{R}^n$ satisfies $\boldsymbol{\varphi} \in (\mathcal{C}^\infty(\overline{\Omega}))^n$, $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Multiplying equation (2.1) by $\boldsymbol{\varphi}$ and integrating both sides over Ω we get

$$-\int_\Omega \Delta \boldsymbol{u} \cdot \overline{\boldsymbol{\varphi}}\, d\lambda^n + \int_\Omega \nabla p \cdot \overline{\boldsymbol{\varphi}}\, d\lambda^n = 0.$$

Sine $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\Delta \boldsymbol{u} = 2 \operatorname{div} \boldsymbol{D}\boldsymbol{u}.$$

Using Green's formula we get

$$2\int_\Omega \boldsymbol{D}\boldsymbol{u} : \overline{\boldsymbol{D}\boldsymbol{\varphi}}\, d\lambda^n - 2\int_{\partial\Omega} \overline{\boldsymbol{\varphi}} \cdot (\boldsymbol{D}\boldsymbol{u})\,\boldsymbol{n}\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} p\,(\overline{\boldsymbol{\varphi}} \cdot \boldsymbol{n})\, d\mathcal{H}^{n-1} - \int_\Omega p \operatorname{div} \overline{\boldsymbol{\varphi}}\, d\lambda^n = 0.$$

Using the assumptions on $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ we obtain

$$\int_\Omega \boldsymbol{D}\boldsymbol{u} : \overline{\boldsymbol{D}\boldsymbol{\varphi}}\, d\lambda^n = \lambda \int_{\partial\Omega} \boldsymbol{u} \cdot \overline{\boldsymbol{\varphi}}\, d\mathcal{H}^{n-1}.$$

This motivates the definition of weak solutions and appropriate function spaces where we will be looking for solutions.

Notation. Let V denote the space $\left(W^{1,2}_{\boldsymbol{n},\mathrm{div}}(\Omega)\right)^n$ and let $\tilde{V}$ denote the space $(W^{1,2}_{\boldsymbol{n}}(\Omega))^n$.

Remark. V and $\tilde{V}$ are Hilbert spaces with the following inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_V = \int_\Omega \boldsymbol{D}\boldsymbol{u} : \overline{\boldsymbol{D}\boldsymbol{v}}\, d\lambda^n + \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{v}})\, d\mathcal{H}^{n-1},$$

where $\boldsymbol{u}, \boldsymbol{v} \in \tilde{V}$. By $\|\cdot\|_V$ and $\|\cdot\|_{\tilde{V}}$ we denote their corresponding norms. Due to Korn's inequality (see [6], Proposition 3.13., p.271) and Trace Theorem (see Theorem 4), these norms are equivalent to the standard $\|\cdot\|_{(W^{1,2}(\Omega))^n}$ norm and we will use this equivalence throughout the work without mentioning.

Definition 7. *Let $\lambda \in \mathbb{R}$ be fixed. We say that $\boldsymbol{u} \in V$ is a weak solution to the problem (2.1)-(2.4) if*

$$\int_\Omega \boldsymbol{Du} : \overline{\boldsymbol{D\varphi}}\, d\lambda^n = \lambda \int_{\partial\Omega} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^{n-1} \tag{2.5}$$

holds for all $\boldsymbol{\varphi} \in V$.

Notation. Denote by Λ the set of all $\lambda \in \mathbb{R}$ such that there exist a non-trivial weak solution to the problem (2.1)-(2.4).

Remark. We will only be interested in non-trivial weak solutions hence by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ in (2.5) we immediately get that $\lambda \geq 0$. Thus $\Lambda \subset [0, \infty)$.

2.1.1 Existence of weak solutions

Now we are going to investigate the question of existence of non-trivial weak solutions to the problem (2.1)-(2.4). Since $\mathrm{Tr}(V) \subset \mathrm{Tr}\left((W^{1,2}(\Omega))^n\right) \subset (L^2(\partial\Omega))^n$ (see Theorem 4), we define a mapping $B\colon V \times V \to \mathbb{C}$ in the following way

$$B[\boldsymbol{u},\boldsymbol{v}] = \int_{\partial\Omega} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{v}})\, d\mathcal{H}^{n-1}.$$

Then B is clearly linear in the first coordinate, conjugate linear in the second coordinate and there exists $C \in \mathbb{R}, C > 0$ such that

$$\int_{\partial\Omega} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{v}})\, d\mathcal{H}^{n-1} \leq \|\mathrm{Tr}(\boldsymbol{u})\|_{(L^2(\partial\Omega))^n} \|\mathrm{Tr}(\overline{\boldsymbol{v}})\|_{(L^2(\partial\Omega))^n} \leq C\|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \tag{2.6}$$

holds for all $\boldsymbol{u}, \boldsymbol{v} \in V$ (see Theorem 4). Hence we get

$$\sup\{|B[\boldsymbol{u},\boldsymbol{v}]|;\ \|\boldsymbol{u}\|_V \leq 1, \|\boldsymbol{v}\|_V \leq 1\} < \infty.$$

Using a version of Riesz representation theorem it follows that there exists a unique $T \in L(V)$ such that

$$B[\boldsymbol{u},\boldsymbol{v}] = \langle T(\boldsymbol{u}), \boldsymbol{v}\rangle_V$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$. Operator T is thus almost a solution operator to the following slightly modified problem. The precise relation to the solution operator is addressed later in the proof of Lemma 10. For given $\boldsymbol{g}\colon \partial\Omega \to \mathbb{R}^n$ we seek $\boldsymbol{v}\colon \overline{\Omega} \to \mathbb{R}^n$ and $q\colon \Omega \to \mathbb{R}$ satisfying

$$-\Delta \boldsymbol{v} + \nabla q = 0 \qquad \text{in } \Omega, \tag{2.7}$$

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } \Omega, \tag{2.8}$$

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{2.9}$$

$$[(\boldsymbol{Dv})\,\boldsymbol{n}]_\tau + \boldsymbol{v} = \boldsymbol{g} \qquad \text{on } \partial\Omega, \tag{2.10}$$

Weak formulation of this problem would have been derived in the same way as for problem (2.1)-(2.4).

Definition 8. *Let $\boldsymbol{g} \in (L^2(\partial\Omega))^n$. We say that $\boldsymbol{v} \in V$ is a weak solution to the problem (2.7)-(2.10) if*

$$\int_\Omega \boldsymbol{Dv} : \overline{\boldsymbol{D\varphi}}\, d\lambda^n + \int_{\partial\Omega} \mathrm{Tr}(\boldsymbol{v}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} \boldsymbol{g} \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^{n-1} \tag{2.11}$$

holds for all $\boldsymbol{\varphi} \in V$.

Lemma 9. *Operator T is self-adjoint.*

Proof. Self-adjointness of T follows immediately from definition since

$$\langle T(\boldsymbol{u}), \boldsymbol{v}\rangle_V = B[\boldsymbol{u},\boldsymbol{v}] = \overline{B[\boldsymbol{v},\boldsymbol{u}]} = \overline{\langle T(\boldsymbol{v}), \boldsymbol{u}\rangle_V} = \langle \boldsymbol{u}, T(\boldsymbol{v})\rangle_V$$

holds for all $\boldsymbol{u}, \boldsymbol{v} \in V$.

□

Lemma 10. *Operator T is compact.*

Proof. Firstly, it holds that operator Tr

$$\mathrm{Tr}\colon \left(W^{1,2}(\Omega)\right)^n \to \left(L^2(\partial\Omega)\right)^n$$

is compact (see [5], Theorem 6.2., p. 103). Furthermore, like in (2.6), we get that there exists $c \in \mathbb{R}, c > 0$ such that

$$\overline{\int_{\partial\Omega} \boldsymbol{g} \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^{n-1}} \leq c\|\boldsymbol{g}\|_{(L^2(\partial\Omega))^n}\|\boldsymbol{\varphi}\|_V, \tag{2.12}$$

for all $\boldsymbol{\varphi} \in V$. Thus, for given $\boldsymbol{g}$ the left-hand side of (2.12) defines a continuous linear operator on V. Using Riesz representation theorem (see Theorem 5) we obtain a unique element $\tilde{\boldsymbol{v}} \in V$ satisfying

$$\langle \boldsymbol{\varphi}, \tilde{\boldsymbol{v}}\rangle_V = \overline{\int_{\partial\Omega} \boldsymbol{g} \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^{n-1}},$$

for all $\boldsymbol{\varphi} \in V$. Hence by using the properties of inner product, $\tilde{\boldsymbol{v}} \in V$ is a unique solution satisfying (2.11) for all $\boldsymbol{\varphi} \in V$. Define a mapping $\psi\colon \boldsymbol{g} \mapsto \tilde{\boldsymbol{v}}$, where $\tilde{\boldsymbol{v}} \in V$ is the unique solution from above. Then ψ is obviously linear. Furthermore, we know that (2.11) holds for $\boldsymbol{v} = \tilde{\boldsymbol{v}}$ and by taking $\boldsymbol{\varphi} = \tilde{\boldsymbol{v}}$ and using (2.12) we get

$$\|\tilde{\boldsymbol{v}}\|_V^2 = \int_{\partial\Omega} \boldsymbol{g} \cdot \mathrm{Tr}(\overline{\tilde{\boldsymbol{v}}})\, d\mathcal{H}^{n-1} \leq c\|\boldsymbol{g}\|_{(L^2(\partial\Omega))^n}\|\tilde{\boldsymbol{v}}\|_V.$$

Using Young's inequality we get

$$c\|\boldsymbol{g}\|_{(L^2(\partial\Omega))^n}\|\tilde{\boldsymbol{v}}\|_V \leq c\left(C(\varepsilon)\|\boldsymbol{g}\|^2_{(L^2(\partial\Omega))^n} + \varepsilon\|\tilde{\boldsymbol{v}}\|_V^2\right)$$

where $\varepsilon \in \mathbb{R}, \varepsilon > 0$ is chosen in a way that $\varepsilon c < 1$. Altogether this yields

$$\|\psi(\boldsymbol{g})\|_V \leq C\|\boldsymbol{g}\|_{(L^2(\partial\Omega))^n}$$

for some $C \in \mathbb{R}, C > 0$ which then implies $\psi \in L((L^2(\partial\Omega))^n, V)$. Finally, from construction, it holds that

$$T = \psi \circ \mathrm{Tr}. \tag{2.13}$$

Using compactness of Tr, continuity of ψ and (2.13) we indeed obtain that operator T is compact.

□

Lemma 11. *Let $\lambda \in \mathbb{R}, \lambda > 0$ be fixed. A non-trivial function $\boldsymbol{u}$ is a weak solution to the problem (2.7)-(2.10) with $\boldsymbol{g} = \lambda \mathrm{Tr}(\boldsymbol{u})$ if and only if $\boldsymbol{u}$ is an eigenfunction of T corresponding to eigenvalue $1/\lambda$.*

Proof. " $\Longrightarrow$ " For all $\boldsymbol{\varphi} \in V$ it holds that

$$\langle \boldsymbol{u}, \boldsymbol{\varphi} \rangle_V = \lambda B[\boldsymbol{u},\boldsymbol{\varphi}] = \lambda \langle T(\boldsymbol{u}), \boldsymbol{\varphi} \rangle_V ,$$

hence dividing both sides by λ and reorganizing we get

$$\left\langle \frac{1}{\lambda}\boldsymbol{u} - T(\boldsymbol{u}), \boldsymbol{\varphi} \right\rangle_V = 0.$$

Setting $\boldsymbol{\varphi} = (1/\lambda)\boldsymbol{u} - T(\boldsymbol{u})$ gives that

$$T(\boldsymbol{u}) = \frac{1}{\lambda}\boldsymbol{u}.$$

" $\Longleftarrow$ " For all $\boldsymbol{\varphi} \in V$ we have

$$\langle \boldsymbol{u}, \boldsymbol{\varphi} \rangle_V = \lambda \left\langle \frac{1}{\lambda}\boldsymbol{u}, \boldsymbol{\varphi} \right\rangle_V = \lambda \langle T(\boldsymbol{u}), \boldsymbol{\varphi} \rangle_V = \lambda B[\boldsymbol{u},\boldsymbol{\varphi}], \tag{2.14}$$

hence (2.11) holds for all $\boldsymbol{\varphi} \in V$.

□

Remark. If a function $\boldsymbol{u}$ is an eigenfunction of T corresponding to eigenvalue $1/\lambda$ then using (2.14) and setting $\boldsymbol{\varphi} = \boldsymbol{u}$ we get

$$\lambda = \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_V}{B[\boldsymbol{u},\boldsymbol{u}]} \geq 1.$$

Lemma 12. *Let $\lambda \in \mathbb{R}, \lambda \in (0,1]$. Then $\lambda \in \sigma_p(T)$ if and only if $1/\lambda - 1 \in \Lambda$.*

Proof. It follows immediately from Lemma 11 since $\lambda \in \sigma_p(T)$ if and only if for the corresponding eigenfunction $\boldsymbol{u}$ holds the following

$$\int_\Omega \boldsymbol{D}\boldsymbol{u} : \overline{\boldsymbol{D}\boldsymbol{\varphi}} \, d\lambda^n = \left(\frac{1}{\lambda} - 1\right) \int_{\partial\Omega} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^{n-1}$$

for all $\boldsymbol{\varphi} \in V$. The Lemma follows.

□

Theorem 13. *There exist at most countably many $\lambda \in \mathbb{R}$ such that there exist a non-trivial weak solution to the problem (2.1)-(2.4), i.e. set Λ is countable.*

Proof. Lemma 9 implies that $\sigma(T) \subset \mathbb{R}$ (see [4], Proposition 6.9., p. 165) and Lemma 10 implies that spectrum of T is at most countable (see [4], Theorem 6.8., p. 164). Lemma 12 completes the proof.

□

2.2 Reconstruction of pressure

Natural question that arises is whether we do not lose any information about the pressure. It is well known that for Stokes problem with Dirichlet boundary condition it is not the case and we would now like to modify this result to our problem. The core of the proof lies in the following Theorem and Lemmata.

Notation. For $n \in \mathbb{N}, q \in \mathbb{R}, q \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ measurable we denote $\tilde{L}^q(\Omega) = \{g \in L^q(\Omega);\, \int_\Omega g = 0\}$.

Theorem 14. *Let $n \in \mathbb{N}$ and let $q, q' \in \mathbb{R}$, $1 < q < \infty$ satisfying $1/q + 1/q' = 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $f \in \tilde{L}^q(\Omega)$. Then there exists a solution $\boldsymbol{v} \in (W_0^{1,q}(\Omega))^n$ to the following problem*

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= f \quad \text{in } \Omega, \\ \boldsymbol{v} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.15}$$

such that

$$\|\boldsymbol{v}\|_{(W_0^{1,q}(\Omega))^n} \le C \|f\|_{L^q(\Omega)}, \tag{2.16}$$

where constant $C \in \mathbb{R}$ is independent of f.

Proof. See [7], Theorem III.3.3, p. 179.

□

Remark. There exists a bounded linear operator $\operatorname{Bog}\colon \tilde{L}^q(\Omega) \to (W_0^{1,q}(\Omega))^n$ such that $\operatorname{div}(\operatorname{Bog}(f)) = f$ in Ω. We will refer to it as Bogovski operator.

Lemma 15. *Let $A\colon \tilde{V} \to \tilde{L}^2(\Omega)$ denote an operator defined by $A(\boldsymbol{v}) = \operatorname{div} \boldsymbol{v}$. Then*

$$R(A^*) = (\ker(A))^\perp = \{g \in \tilde{V}^*;\, \forall \boldsymbol{v} \in \ker(A) : \langle g, \boldsymbol{v} \rangle_{\tilde{V}^*, \tilde{V}} = 0\ \}.$$

Proof. Operator A is clearly linear and continuous. Firstly, using Gauss's theorem (see Theorem 6) we obtain for $\boldsymbol{v} \in \tilde{V}$ that

$$\int_\Omega A(\boldsymbol{v})\, d\lambda^n = \int_\Omega \operatorname{div} \boldsymbol{v}\, d\lambda^n = \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{v}) \cdot \boldsymbol{n}\, d\mathcal{H}^{n-1} = 0,$$

hence A is indeed a mapping to $\tilde{L}^2(\Omega)$. Due to Closed Range Theorem (see [8], Theorem II.18) it is enough to prove that the range of A is closed in $\tilde{L}^2(\Omega)$. Assume that $(f_k)_{k=1}^\infty$ is a sequence in $R(A)$ satisfying $f_k \to f$ in $\tilde{L}^2(\Omega)$. Owing to the fact that $R(A) \subset \tilde{L}^2(\Omega)$, we can use Theorem 14 to get $\boldsymbol{v}_k \in (W_0^{1,2}(\Omega))^n \subset \tilde{V}$ such that $A(\boldsymbol{v}_k) = f_k$ and

$$\|\boldsymbol{v}_k\|_{(W_0^{1,2}(\Omega))^n} \le C \|f_k\|_{L^2(\Omega)},$$

for all $k \in \mathbb{N}$ where constant $C \in \mathbb{R}$ is independent of f_k. Since $\left(\|f_k\|_{L^2(\Omega)}\right)_{k=1}^\infty$ is bounded, $(\|\boldsymbol{v}_k\|_{\tilde{V}})_{k=1}^\infty$ is also bounded and since $\tilde{V}$ is reflexive, we can extract a subsequence (denoted the same) such that $\boldsymbol{v}_k \rightharpoonup \boldsymbol{v}$ in $\tilde{V}$ and hence $\operatorname{div} \boldsymbol{v}_k \rightharpoonup \operatorname{div} \boldsymbol{v}$ in $L^2(\Omega)$. It follows that $A(\boldsymbol{v}) = \operatorname{div} \boldsymbol{v} = f$ in Ω which implies $f \in R(A)$ and thus $R(A)$ is indeed closed in $\tilde{L}^2(\Omega)$.

□

Lemma 16. *Suppose that* $\boldsymbol{G} \in \tilde{V}^*$ *is such that*

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} = 0$$

for all $\boldsymbol{\varphi} \in V$. *Then there exists exactly one* $p \in \tilde{L}^2(\Omega)$ *such that*

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} = \int_\Omega \overline{p} \operatorname{div} \boldsymbol{\varphi} \, d\lambda^n$$

holds for all $\boldsymbol{\varphi} \in \tilde{V}$.

Proof. The proof is almost identical to the case with zero Dirichlet boundary condition (see [7], Theorem III.5.3, p. 217) with slight modification proved in Lemma 15. We will show the proof for readers convenience. It clearly holds that $\ker(A) = V$. Thus $\boldsymbol{G} \in (\ker(A))^\perp = R(A^*)$, i.e. there exists $z \in \left(\tilde{L}^2(\Omega)\right)^*$ such that $A^*(z) = \boldsymbol{G}$. Using the definition of the adjoint operator we obtain

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} = \langle A^*(z), \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} = \langle z, A(\boldsymbol{\varphi}) \rangle_{\tilde{L}^2(\Omega)^*, \tilde{L}^2(\Omega)} .$$

Finally, by using Riesz representation theorem (see Theorem 5) we obtain a uniquely determined $p \in \tilde{L}^2(\Omega)$ satisfying

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} = \langle z, A(\boldsymbol{\varphi}) \rangle_{\tilde{L}^2(\Omega)^*, \tilde{L}^2(\Omega)} = \int_\Omega \overline{p} A(\boldsymbol{\varphi}) \, d\lambda^n = \int_\Omega \overline{p} \operatorname{div} \boldsymbol{\varphi} \, d\lambda^n$$

for all $\boldsymbol{\varphi} \in \tilde{V}$.

□

We now have all the tools ready to finally prove the reconstruction of pressure for our problem (2.1)-(2.4) as we show in the following Lemma.

Lemma 17. *Let* $\lambda \in \mathbb{R}$ *be fixed and let* $\boldsymbol{u} \in V$ *satisfy (2.5) for all* $\boldsymbol{\varphi} \in V$. *Then there exists exactly one* $p \in \tilde{L}^2(\Omega)$ *such that*

$$\int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}} \, d\lambda^n - \lambda \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^{n-1} = \int_\Omega p \operatorname{div} \overline{\boldsymbol{\varphi}} \, d\lambda^n \tag{2.17}$$

holds for all $\boldsymbol{\varphi} \in \tilde{V}$.

Proof. Define $\boldsymbol{G}$ for $\boldsymbol{\varphi} \in \tilde{V}$ as follows

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} = \overline{\int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}} \, d\lambda^n - \lambda \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^{n-1}}. \tag{2.18}$$

Like in (2.6), there exists $c \in \mathbb{R}, c > 0$ such that

$$\int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^{n-1} \leq c \|\boldsymbol{u}\|_{\tilde{V}} \|\boldsymbol{\varphi}\|_{\tilde{V}}$$

holds for all $\boldsymbol{u}, \boldsymbol{\varphi} \in \tilde{V}$. Using Hölder's inequality in the first term in (2.18) we obtain that there exists $C \in \mathbb{R}$ such that

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{\tilde{V}^*, \tilde{V}} \leq C \|\boldsymbol{u}\|_{\tilde{V}} \|\boldsymbol{\varphi}\|_{\tilde{V}}$$

holds for all $\boldsymbol{u}, \boldsymbol{\varphi} \in \tilde{V}$, i.e. $\boldsymbol{G} \in \tilde{V}^*$. Lemma 16 completes the proof.

□

2.3 Auxiliary problem

In order to investigate the asymptotic behaviour of eigenvalues of the original problem (2.1)-(2.4), we are firstly going to investigate the asymptotic behaviour of eigenvalues of a different problem on a simple domain on which we are going to be be able to calculate the asymptotics explicitly. This approach would work in any dimension but the calculations would get unbearably difficult and that is why we were not able to generalize it to arbitrary dimension. We will firstly investigate the two dimensional case.

Remark. Any symbolic calculation that we refer to in the thesis and are not explicitly shown were performed in the Wolfram Mathematica program and the corresponding files are added to the thesis externally.

2.3.1 Formulation of the auxiliary problem in two dimensions and its basic properties

Notation. We denote $\Omega = (0,1)^2$ and $\Gamma = \partial\Omega$. Furthermore, let us denote

$$\begin{aligned}
\Gamma_1 &= \{(x_1,x_2) \in \mathbb{R}^2;\ x_2 = 0, x_1 \in (0,1)\},\\
\Gamma_2 &= \{(x_1,x_2) \in \mathbb{R}^2;\ x_1 = 0, x_2 \in (0,1)\},\\
\Gamma_3 &= \{(x_1,x_2) \in \mathbb{R}^2;\ x_1 = 1, x_2 \in (0,1)\},\\
\Gamma_4 &= \{(x_1,x_2) \in \mathbb{R}^2;\ x_2 = 1, x_1 \in (0,1)\}.
\end{aligned}$$

For $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^2$, $p\colon \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we consider a modified version of problem (2.1)-(2.4) in the following way

$$\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= 0 && \text{in } \Omega, && (2.19)\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && (2.20)\\
\boldsymbol{u} &= 0 && \text{on } \Gamma_1, && (2.21)\\
\frac{\partial u_1}{\partial x_2} = \lambda u_1 \text{ and } u_2 &= 0 && \text{on } \Gamma_4, && (2.22)\\
\boldsymbol{u}(0,x_2) - \boldsymbol{u}(1,x_2) &= 0 && \text{for } x_2 \in [0,1], && (2.23)\\
\frac{\partial \boldsymbol{u}}{\partial x_1}(0,x_2) - \frac{\partial \boldsymbol{u}}{\partial x_1}(1,x_2) &= 0 && \text{for } x_2 \in [0,1], && (2.24)\\
p(0,x_2) - p(1,x_2) &= 0 && \text{for } x_2 \in (0,1). && (2.25)
\end{aligned}$$

Thus, we consider periodic boundary conditions in direction x_1 on two parallel sides of the square for $\boldsymbol{u}$ and also for the pressure p. The weak formulation of this problem would have been derived in the same way as for the problem (2.1)-(2.4) only the function spaces of solutions and test functions will be different.

Remark. Condition (2.22) corresponds to condition $[(\nabla \boldsymbol{u})\, \boldsymbol{n}]_\tau = \lambda \boldsymbol{u}$ on Γ_4.

Definition 18. *We define space V_2 as follows*

$$\begin{aligned}
V_2 = \{\boldsymbol{u} \in (W^{1,2}_{\mathrm{div}}(\Omega))^2;\ &\mathrm{Tr}(\boldsymbol{u})|_{\Gamma_1} = 0,\\
&\mathrm{Tr}(\boldsymbol{u})(0,x_2) = \mathrm{Tr}(\boldsymbol{u})(1,x_2) \text{ for } x_2 \in [0,1], \mathrm{Tr}(\boldsymbol{u})_2|_{\Gamma_4} = 0\},
\end{aligned}$$

Remark. Space V_2 is again a Hilbert space. Due to the condition $\mathrm{Tr}(\boldsymbol{u})|_{\Gamma_1} = 0$ in Definition 18, the fact that $\mathcal{H}^1(\Gamma_1) > 0$ and the fact that Ω is connected, it follows that for $\boldsymbol{u}, \boldsymbol{v} \in V_2$ the expression

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V_2} = \int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}} \, d\lambda^2$$

defines a scalar product on V_2 and the corresponding norm $\|\cdot\|_{V_2}$ is equivalent to the standard $\|\cdot\|_{W^{1,2}}$ norm.

Definition 19. *Let $\lambda \in \mathbb{R}$ be fixed. We say that $\boldsymbol{u} \in V_2$ is a weak solution to the problem (2.19)-(2.25) if*

$$\int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}} \, d\lambda^2 = \lambda \int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^1 \tag{2.26}$$

holds for all $\boldsymbol{\varphi} \in V_2$.

Remark. We will only be interested in non-trivial weak solutions hence by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ we immediately get that $\lambda \geq 0$. Moreover, for $\lambda = 0$ we obtain by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ that $\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_2} = 0$ and hence the only weak solution is the trivial one.

In order to show that there exist only countably many values of $\lambda \in \mathbb{R}$ such that there exists a non-trivial weak solution to the problem (2.19)-(2.25), we would again define a mapping $B\colon V_2 \times V_2 \to \mathbb{C}$ by

$$B[\boldsymbol{u},\boldsymbol{v}] = \int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{v}}) \, d\mathcal{H}^1$$

and get the existence of a unique operator $T_2 \in L(V_2)$ satisfying

$$B[\boldsymbol{u},\boldsymbol{v}] = \langle T_2(\boldsymbol{u}), \boldsymbol{v} \rangle_{V_2}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V_2$. The procedure would now be the same, i.e we would show analogous Lemmata as in Section 2.1 with the only difference being the fact that due to inner product on V_2, we would not need to add additional term to the boundary condition in the formulation of the problem to show compactness of T_2. Hence the following Theorem holds.

Theorem 20. *Operator T_2 is compact and self-adjoint. For some fixed $\lambda \in \mathbb{R}, \lambda > 0$, a non-trivial function $\boldsymbol{u}$ is a weak solution to the problem (2.19)-(2.25) if and only if $\boldsymbol{u}$ is an eigenfunction of T_2 corresponding to eigenvalue $1/\lambda$. There exist at most countably many $\lambda \in \mathbb{R}$ such that there exist a non-trivial weak solution to the problem (2.19)-(2.25).*

2.3.2 Existence of solutions and asymptotic behaviour of eigenvalues in two dimensions

In order to determine the asymptotic behaviour of λ's for which there exist non-trivial weak solutions, we will now compute the solutions to the problem (2.19)-(2.25) explicitly and afterwards we will show that these solutions are in fact already all non-trivial weak solutions to this problem. The particular form of functions in which we will be looking for solutions was inspired by a paper by Rummler [9].

Proposition 21. *For any $k \in 2\pi\mathbb{N}_0$ there exist $\lambda(k) \in \mathbb{R}$ of the form*

$$\lambda(k) = \begin{cases} 1 & \text{if } k = 0, \\ \dfrac{2k\left(-1 + e^{4k} - 4e^{2k}k\right)}{1 - 2e^{2k} + e^{4k} - 4e^{2k}k^2} & \text{if } k \in 2\pi\mathbb{N} \end{cases} \tag{2.27}$$

and functions $a_{i,j}, p_i$ for $i, j \in \{1,2\}$ of variable x_2 such that functions

$$\boldsymbol{u_k}(x_1,x_2) = \begin{pmatrix} a_{1,1}(x_2)\cos(kx_1) + a_{1,2}(x_2)\sin(kx_1) \\ a_{2,1}(x_2)\cos(kx_1) + a_{2,2}(x_2)\sin(kx_1) \end{pmatrix}$$

and

$$p_k(x_1,x_2) = p_1(x_2)\cos(kx_1) + p_2(x_2)\sin(kx_1)$$

solve the problem (2.19)-(2.25) in the pointwise sense. Moreover, the multiplicity of $\lambda(0)$ is one and the multiplicity of $\lambda(k)$ for $k \in 2\pi\mathbb{N}$ is two.

Proof. Conditions (2.23)-(2.25) are satisfied trivially. First, we are going to deal with the case $k = 0$. Then

$$\boldsymbol{u_0}(x_1,x_2) = \begin{pmatrix} a_{1,1}(x_2) \\ a_{2,1}(x_2) \end{pmatrix}$$

and

$$p_0(x_1,x_2) = p_1(x_2).$$

Equation (2.20): Rewriting $\operatorname{div} \boldsymbol{u_0} = 0$ we want

$$a_{2,1}'(x_2) = 0$$

for all $x_2 \in (0,1)$. Thus $a_{2,1}(x_2) = c_{2,1}$ for some $c_{2,1} \in \mathbb{R}$.

Equation (2.21): We want

$$a_{1,1}(0) = a_{2,1}(0) = 0. \tag{2.28}$$

This already implies that $c_{2,1} = 0$ and thus $a_{2,1}(x_2) = 0$ for all $x_2 \in (0,1)$.

Equation (2.19): Rewriting $-\Delta \boldsymbol{u_0} + \nabla p_0 = 0$ we want

$$a_{1,1}''(x_2) = 0$$

for all $x_2 \in (0,1)$. Thus $a_{1,1}(x_2) = d_{1,1}x_2$ for some $d_{1,1} \in \mathbb{R}$ and all $x_2 \in (0,1)$ due to (2.28). Next, we want

$$p_1'(x_2) = 0$$

for all $x_2 \in (0,1)$. Thus $p_1(x_2) = p_1$ for some $p_1 \in \mathbb{R}$.

Equation (2.22): The condition that $u_2 = 0$ on Γ_4 is trivial. The second boundary condition for unknown λ gives

$$d_{1,1} = \lambda d_{1,1}.$$

Since we are only interested in non-trivial solutions, we require $d_{1,1} \neq 0$ and thus $\lambda = 1$. Consequently, the moreover part of the Proposition for $\lambda(0)$ follows.

Suppose now that $k \in 2\pi\mathbb{N}$. Formally applying divergence to (2.19) and assuming that (2.20) holds we get that

$$\operatorname{div}(-\Delta \boldsymbol{u} + \nabla p) = -\underbrace{\Delta(\operatorname{div} \boldsymbol{u})}_{=0} + \Delta p = \Delta p = 0 \tag{2.29}$$

holds in Ω. Using (2.29) for the special form of p_k we obtain that

$$p_1''(x_2)\cos(kx_1) + p_2''(x_2)\sin(kx_1) - k^2 p_1(x_2)\cos(kx_1) - k^2 p_2(x_2)\sin(kx_1) = 0$$

holds for $(x_1,x_2) \in \Omega$. Rearranging the equation yields

$$\cos(kx_1)(p_1''(x_2) - k^2 p_1(x_2)) + \sin(kx_1)(p_2''(x_2) - k^2 p_2(x_2)) = 0.$$

Since this equation should hold for any $(x_1,x_2) \in \Omega$ we deduce that for $i \in \{1,2\}$

$$p_i''(x_2) - k^2 p_i(x_2) = 0$$

holds for all $x_2 \in (0,1)$. Thus we can write p_i in the following way

$$p_i(x_2) = p_i^1 e^{kx_2} + p_i^2 e^{-kx_2},$$

where $x_2 \in (0,1)$ and $p_i^1, p_i^2 \in \mathbb{R}$ for $i \in \{1,2\}$. Plugging $\boldsymbol{u_k}$ and p_k into (2.19) we obtain (similarly as above) for $(x_1,x_2) \in \Omega$ the following relations

$$\cos(kx_1)(a_{1,1}''(x_2) - k^2 a_{1,1}(x_2) - kp_2(x_2)) + \\ \sin(kx_1)(a_{1,2}''(x_2) - k^2 a_{1,2}(x_2) + kp_1(x_2)) = 0$$

and

$$\cos(kx_1)(a_{2,1}''(x_2) - k^2 a_{2,1}(x_2) - p_1'(x_2)) + \\ \sin(kx_1)(a_{2,2}''(x_2) - k^2 a_{2,2}(x_2) - p_2'(x_2)) = 0.$$

Thus we again deduce that

$$a_{1,1}''(x_2) - k^2 a_{1,1}(x_2) - kp_2(x_2) = 0,$$
$$a_{1,2}''(x_2) - k^2 a_{1,2}(x_2) + kp_1(x_2) = 0$$

and for $i \in \{1,2\}$

$$a_{2,i}''(x_2) - k^2 a_{2,i}(x_2) - p_i'(x_2) = 0$$

holds for all $x_2 \in (0,1)$. Solving these equations we get

$$a_{1,1}(x_2) = a_{1,1}^1 e^{kx_2} + a_{1,1}^2 e^{-kx_2} + \frac{x_2}{2}\left(p_2^1 e^{kx_2} - p_2^2 e^{-kx_2}\right), \tag{2.30}$$

$$a_{1,2}(x_2) = a_{1,2}^1 e^{kx_2} + a_{1,2}^2 e^{-kx_2} + \frac{x_2}{2}\left(-p_1^1 e^{kx_2} + p_1^2 e^{-kx_2}\right) \tag{2.31}$$

and for $i \in \{1{,}2\}$

$$a_{2,i}(x_2) = a^1_{2,i} e^{kx_2} + a^2_{2,i} e^{-kx_2} + \frac{x_2}{2}\left(p^1_i e^{kx_2} + p^2_i e^{-kx_2}\right), \tag{2.32}$$

where $a^1_{i,j}, a^2_{i,j} \in \mathbb{R}$ for $i,j \in \{1{,}2\}$. Functions $\boldsymbol{u_k}$ and p_k of these forms thus satisfy (2.19).

Now we are going to determine for which values of λ there exist nonzero coefficients such that $\boldsymbol{u_k}$ also satisfy (2.20)-(2.22).

Equation (2.21): For all $x_1 \in (0,1)$ we want

$$\boldsymbol{u_k}(x_1{,}0) = \begin{pmatrix} a_{1,1}(0)\cos(kx_1) + a_{1,2}(0)\sin(kx_1) \\ a_{2,1}(0)\cos(kx_1) + a_{2,2}(0)\sin(kx_1) \end{pmatrix} = \mathbf{0}.$$

Thus we deduce that for $i,j \in \{1{,}2\}$ we need $a_{i,j}(0) = 0$. Plugging this into (2.30)-(2.32) we obtain for $i,j \in \{1{,}2\}$ that

$$a^2_{i,j} = -a^1_{i,j}. \tag{2.33}$$

Equation (2.20): Rewriting $\operatorname{div} \boldsymbol{u_k} = 0$ we want

$$-ka_{1,1}(x_2)\sin(kx_1) + ka_{1,2}(x_2)\cos(kx_1) + a'_{2,1}(x_2)\cos(kx_1) + a'_{2,2}(x_2)\sin(kx_1) = 0$$

for all $(x_1{,}x_2) \in \Omega$. Rearranging the equation yields

$$\cos(kx_1)(a'_{2,1}(x_2) + ka_{1,2}(x_2)) + \sin(kx_1)(a'_{2,2}(x_2) - ka_{1,1}(x_2)) = 0.$$

Thus we deduce that

$$a'_{2,1}(x_2) + ka_{1,2}(x_2) = 0, \tag{2.34}$$

$$a'_{2,2}(x_2) - ka_{1,1}(x_2) = 0 \tag{2.35}$$

for all $x_2 \in (0,1)$. Using (2.32) and (2.33) we deduce for $i \in \{1{,}2\}$ that

$$a'_{2,i}(x_2) = ka^1_{2,i}\left(e^{kx_2} + e^{-kx_2}\right) + \frac{1}{2}\left(p^1_i e^{kx_2} + p^2_i e^{-kx_2}\right) + \frac{kx_2}{2}\left(p^1_i e^{kx_2} - p^2_i e^{-kx_2}\right).$$

Plugging this into (2.34)-(2.35) we obtain

$$ka^1_{2,1}\left(e^{kx_2} + e^{-kx_2}\right) + \frac{1}{2}\left(p^1_1 e^{kx_2} + p^2_1 e^{-kx_2}\right) + ka^1_{1,2}\left(e^{kx_2} - e^{-kx_2}\right) = 0, \tag{2.36}$$

$$ka^1_{2,2}\left(e^{kx_2} + e^{-kx_2}\right) + \frac{1}{2}\left(p^1_2 e^{kx_2} + p^2_2 e^{-kx_2}\right) - ka^1_{1,1}\left(e^{kx_2} - e^{-kx_2}\right) = 0 \tag{2.37}$$

for all $x_2 \in (0,1)$. By comparing coefficients for each term e^{kx_2}, e^{-kx_2} in (2.36)-(2.37) we deduce the following relations

$$ka^1_{2,1} + \frac{p^1_1}{2} + ka^1_{1,2} = 0, \tag{2.38}$$

$$ka^1_{2,1} + \frac{p^2_1}{2} - ka^1_{1,2} = 0, \tag{2.39}$$

$$ka^1_{2,2} + \frac{p^1_2}{2} - ka^1_{1,1} = 0, \tag{2.40}$$

$$ka^1_{2,2} + \frac{p^2_2}{2} + ka^1_{1,1} = 0. \tag{2.41}$$

Equation (2.22): The condition that $u_2 = 0$ on Γ_4 is treated in the same way as for equation (2.21) and hence we obtain that $a_{2,i}(1) = 0$ for $i \in \{1,2\}$. Using (2.32) and (2.33) we get

$$a^1_{2,i}\left(e^k - e^{-k}\right) + \frac{1}{2}\left(p^1_i e^k + p^2_i e^{-k}\right) = 0 \tag{2.42}$$

for $i \in \{1,2\}$. The second boundary condition for unknown λ gives

$$a'_{1,1}(1)\cos(kx_1) + a'_{1,2}(1)\sin(kx_1) = \lambda\left(a_{1,1}(1)\cos(kx_1) + a_{1,2}(1)\sin(kx_1)\right)$$

for all $x_1 \in (0,1)$. Equivalently

$$\cos(kx_1)(a'_{1,1}(1) - \lambda a_{1,1}(1)) + \sin(kx_1)(a'_{1,2}(1) - \lambda a_{1,2}(1)) = 0.$$

Hence we finally deduce that

$$a'_{1,i}(1) - \lambda a_{1,i}(1) = 0$$

for $i \in \{1,2\}$. Using (2.30)-(2.31) we get

$$\begin{aligned} ka^1_{1,1}\left(e^k + e^{-k}\right) + \frac{1}{2}\left(p^1_2 e^k - p^2_2 e^{-k}\right) + \frac{k}{2}\left(p^1_2 e^k + p^2_2 e^{-k}\right) \\ - \lambda a^1_{1,1}\left(e^k - e^{-k}\right) - \frac{\lambda}{2}\left(p^1_2 e^k - p^2_2 e^{-k}\right) = 0, \end{aligned} \tag{2.43}$$

$$\begin{aligned} ka^1_{1,2}\left(e^k + e^{-k}\right) + \frac{1}{2}\left(-p^1_1 e^k + p^2_1 e^{-k}\right) - \frac{k}{2}\left(p^1_1 e^k + p^2_1 e^{-k}\right) \\ - \lambda a^1_{1,2}\left(e^k - e^{-k}\right) - \frac{\lambda}{2}\left(-p^1_1 e^k + p^2_1 e^{-k}\right) = 0. \end{aligned} \tag{2.44}$$

Thus we end up with 8 equations for 8 unknown coefficients, i.e. equations (2.38)-(2.44). Let us denote the matrix of the considered problem by A_k. The corresponding system then can be rewritten like

$$\underbrace{\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & k & k & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & k & -k & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -k & 0 & 0 & k \\ 0 & 0 & 0 & \frac{1}{2} & k & 0 & 0 & k \\ \frac{e^{2k}}{2} & \frac{1}{2} & 0 & 0 & 0 & e^{2k}-1 & 0 & 0 \\ 0 & 0 & \frac{e^{2k}}{2} & \frac{1}{2} & 0 & 0 & 0 & e^{2k}-1 \\ 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ -b_1 & -b_2 & 0 & 0 & 0 & 0 & b_3 & 0 \end{pmatrix}}_{A_k} \begin{pmatrix} p^1_1 \\ p^2_1 \\ p^1_2 \\ p^2_2 \\ a^1_{1,1} \\ a^1_{2,1} \\ a^1_{1,2} \\ a^1_{2,2} \end{pmatrix} = \mathbf{0}, \tag{2.45}$$

where

$$b_1 = \frac{e^{2k}}{2}\left(1 + k - \lambda\right), \quad b_2 = \frac{1}{2}\left(-1 + k + \lambda\right), \quad b_3 = k\left(e^{2k} + 1\right) - \lambda\left(e^{2k} - 1\right).$$

In order for non-trivial solutions to exist, the matrix of this system must be singular. We thus want

$$0 = \det(A_k) = -\frac{1}{16}\left(2k + \lambda + e^{4k}(-2k+\lambda) - 2e^{2k}\left(\lambda + 2k^2(\lambda - 2)\right)\right)^2$$

from which we obtain

$$\lambda(k) = \frac{2k\left(-1 + e^{4k} - 4e^{2k}k\right)}{1 - 2e^{2k} + e^{4k} - 4e^{2k}k^2},$$

which completes the proof of (2.27) and the first part of the Proposition. Concerning the moreover part of the Proposition for $\lambda(k)$ where $k \in 2\pi\mathbb{N}$, computations yield that the kernel of the matrix A_k for $\lambda = \lambda(k)$ has dimension two which completes the whole proof.

□

Remark. The denominator of $\lambda(k)$ is nonzero for all $k \in 2\pi\mathbb{N}$. Solving

$$1 - 2e^{2k} + e^{4k} - 4e^{2k}k^2 = e^{4k} - e^{2k}\left(2 + 4k^2\right) + 1 = 0$$

gives

$$e^{2k} = \frac{2 + 4k^2 \pm \sqrt{16k^4 + 16k^2}}{2} = 1 + 2k^2 \pm 2k\sqrt{k^2+1}.$$

However, it clearly holds that

$$1 + 2k^2 \pm 2k\sqrt{k^2+1} \leq 1 + 6k^2$$

for all $k \in 2\pi\mathbb{N}$. It can be proven by induction that

$$1 + 6k^2 < e^{2k}$$

for all $k \in 2\pi\mathbb{N}$, hence the claim follows.

Corollary 22. *Let $k \in 2\pi\mathbb{N}$ and denote $\boldsymbol{u}_{\boldsymbol{k}}^1, \boldsymbol{u}_{\boldsymbol{k}}^2$ the linearly independent solutions from Proposition 21 corresponding to $\lambda(k)$. Then the first coordinates of the solutions on Γ_4 can be chosen in such a way that*

$$\boldsymbol{u}_{\boldsymbol{k}}^1(x_1{,}1)_1 = c_1(k)\cos(kx_1),$$
$$\boldsymbol{u}_{\boldsymbol{k}}^2(x_1{,}1)_1 = c_2(k)\sin(kx_1),$$

for $x_1 \in (0,1)$ where $c_1(k)$ and $c_2(k)$ are some non-zero constants. For $k = 0$ denote $\boldsymbol{u}_{\mathbf{0}}$ as the solution corresponding to $\lambda(0)$ and for $i \in \{1{,}2\}$ denote $\boldsymbol{u}_{\mathbf{0}}^i = \boldsymbol{u}_{\mathbf{0}}$. Then for $x_1 \in (0,1)$ it holds that $\boldsymbol{u}_{\mathbf{0}}(x_1{,}1)_1 = c$ for some non-zero constant c.

Proof. Proof eventually follows from computing the kernel of the matrix A_k for $\lambda = \lambda(k)$.

□

Proposition 23. *The values*

$$\lambda(k) = \begin{cases} 1 & \text{if } k = 0, \\ \dfrac{2k\left(-1 + e^{4k} - 4e^{2k}k\right)}{1 - 2e^{2k} + e^{4k} - 4e^{2k}k^2} & \text{if } k \in 2\pi\mathbb{N} \end{cases}$$

are exactly all the values of λ for which there exist a non-trivial weak solution to the problem (2.19)-(2.25).

Remark. Throughout the proof we will use results and notation from Corollary 22.

Proof. One implication is trivial since for these values of $\lambda(k)$ we calculated corresponding smooth solutions $\boldsymbol{u_k}$ in Proposition 21 and hence they are also weak solutions.

On the other hand, suppose that there exists $\lambda^* \in \mathbb{R}, \lambda^* > 0$ such that

$$\lambda^* \notin \{\lambda(k);\, k \in 2\pi\mathbb{N}_0\}$$

and a corresponding non-trivial weak solution $\boldsymbol{u}^* \in V_2$. Taking $\boldsymbol{u}_{\boldsymbol{k}}^i$ for $i \in \{1,2\}$ and $k \in 2\pi\mathbb{N}_0$ as a test functions in (2.26) we obtain

$$\left\langle \boldsymbol{u}^*, \boldsymbol{u}_{\boldsymbol{k}}^i \right\rangle_{V_2} = \lambda^* \int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}^*) \cdot \mathrm{Tr}(\overline{\boldsymbol{u}_{\boldsymbol{k}}^i})\, d\mathcal{H}^1. \tag{2.46}$$

Furthermore, taking $\boldsymbol{u}^*$ as a test function in (2.26) we obtain for $i \in \{1,2\}$ that

$$\overline{\langle \boldsymbol{u}_{\boldsymbol{k}}^i, \boldsymbol{u}^* \rangle_{V_2}} = \overline{\lambda(k) \int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}_{\boldsymbol{k}}^i) \cdot \mathrm{Tr}(\overline{\boldsymbol{u}^*})\, d\mathcal{H}^1} = \lambda(k) \int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}^*) \cdot \mathrm{Tr}(\overline{\boldsymbol{u}_{\boldsymbol{k}}^i})\, d\mathcal{H}^1. \tag{2.47}$$

Relations (2.46)-(2.47) imply that for $i \in \{1,2\}$ it holds

$$0 = \left\langle \frac{1}{\lambda^*}\boldsymbol{u}^*, \boldsymbol{u}_{\boldsymbol{k}}^i \right\rangle_{V_2} - \left\langle \frac{1}{\lambda(k)}\boldsymbol{u}^*, \boldsymbol{u}_{\boldsymbol{k}}^i \right\rangle_{V_2} = \left\langle \boldsymbol{u}^*, \boldsymbol{u}_{\boldsymbol{k}}^i \right\rangle_{V_2} \left(\frac{1}{\lambda^*} - \frac{1}{\lambda(k)} \right),$$

thus using the assumption on λ^* we get

$$\left\langle \boldsymbol{u}^*, \boldsymbol{u}_{\boldsymbol{k}}^i \right\rangle_{V_2} = 0 \tag{2.48}$$

for $i \in \{1,2\}$. Using (2.46) and (2.48) we obtain

$$\int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}^*)_1 \mathrm{Tr}(\overline{\boldsymbol{u}_{\boldsymbol{k}}^i})_1\, d\mathcal{H}^1 = 0$$

for all $k \in 2\pi\mathbb{N}_0$. Thus

$$\int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}^*)_1 \cos(kx_1) d\mathcal{H}^1 = 0, \tag{2.49}$$

$$\int_{\Gamma_4} \mathrm{Tr}(\boldsymbol{u}^*)_1 \sin(kx_1) d\mathcal{H}^1 = 0 \tag{2.50}$$

for all $k \in 2\pi\mathbb{N}_0$. Since the set $\{\cos(kx_1), \sin(kx_1);\, k \in 2\pi\mathbb{N}_0\}$ forms an orthogonal basis in $L^2((0,1))$, relations (2.49)-(2.50) imply $\mathrm{Tr}(\boldsymbol{u}^*)_1 = 0$ on Γ_4 and hence

$$\mathrm{Tr}(\boldsymbol{u}^*)|_{\Gamma_4} = \boldsymbol{0}. \tag{2.51}$$

Plugging (2.51) into (2.26) we get that

$$\int_{\Omega} \nabla \boldsymbol{u}^* : \overline{\nabla \boldsymbol{\varphi}}\, d\lambda^2 = \langle \boldsymbol{u}^*, \boldsymbol{\varphi} \rangle_{V_2} = 0 \tag{2.52}$$

holds for all $\boldsymbol{\varphi} \in V_2$. By setting $\boldsymbol{\varphi} = \boldsymbol{u}^*$ in (2.52) we obtain that $\boldsymbol{u}^* = 0$ in Ω which is a contradiction.

□

Remark. Assume for now that $\tilde{\lambda}\colon [0,\infty) \to \mathbb{R}$ is defined by $\tilde{\lambda}(0) = 1$ and

$$\tilde{\lambda}(x) = \frac{2x\left(-1 + e^{4x} - 4e^{2x}x\right)}{1 - 2e^{2x} + e^{4x} - 4e^{2x}x^2}$$

for $x \in (0,\infty)$. It can be shown that $\tilde{\lambda}'(x) > 0$ for $x \in (0,\infty)$. Using this and showing e.g. numerically that $\lambda(2\pi) > \lambda(0)$, it follows that $\lambda(k+1) > \lambda(k)$ for all $k \in 2\pi\mathbb{N}_0$.

In order to formally determine the asymptotic growth of $\lambda(k)$, let us now arrange $\lambda(k)$ into a non-decreasing sequence of positive numbers, i.e. we consider the sequence $(\lambda_k)_{k=1}^{\infty}$ where each eigenvalue λ appears according its multiplicity and $\{\lambda_k;\, k \in \mathbb{N}\} = \{\lambda(k);\, k \in 2\pi\mathbb{N}_0\}$.

Definition 24. *Let $(\mu_k)_{k=1}^{\infty}$ and $(\nu_k)_{k=1}^{\infty}$ be a sequences of real numbers. We say that*

$$\mu_k \sim \nu_k$$

as $k \to \infty$ if

$$\lim_{k\to\infty} \frac{\mu_k}{\nu_k} = 1.$$

Proposition 25. *The asymptotic growth of the sequence $(\lambda_k)_{k=1}^{\infty}$ is linear and it holds that*

$$\lambda_k \sim 2\pi k$$

as $k \to \infty$.

Notation. For any $x \in \mathbb{R}$ we denote $\lfloor x \rfloor$ as the floor function, i.e. $\lfloor x \rfloor$ gives as output the greatest integer less than or equal to x.

Proof. Using the notation and results from Propositions 21 we know that eigenvalue $\lambda(0) = 1$ has multiplicity one and $\lambda(k)$ for $k \in 2\pi\mathbb{N}$ has multiplicity two. Hence the sequence $(\lambda_k)_{k=1}^{\infty}$ has the following form

$$\lambda_k = \begin{cases} 1 & \text{if } k = 1, \\ \lambda\left(2\pi\lfloor k/2 \rfloor\right) & \text{if } k \geq 2. \end{cases} \tag{2.53}$$

Now the claim follows easily from Proposition 23 and Remark afterwards. Using (2.27) and (2.53) we obtain for $k \in \mathbb{N}, k \geq 2$ that

$$\lambda_k = \frac{4\pi\lfloor k/2 \rfloor \left(-1 + e^{8\pi\lfloor k/2 \rfloor} - 8\pi e^{4\pi\lfloor k/2 \rfloor}\lfloor k/2 \rfloor\right)}{1 - 2e^{4\pi\lfloor k/2 \rfloor} + e^{8\pi\lfloor k/2 \rfloor} - 16\pi^2 e^{4\pi\lfloor k/2 \rfloor}\lfloor k/2 \rfloor^2}.$$

For any $k \in \mathbb{N}$ it holds that $k/2 - 1 \leq \lfloor k/2 \rfloor \leq k/2$ hence

$$\lim_{k\to\infty} \frac{\lfloor k/2 \rfloor}{k} = \frac{1}{2}.$$

Using

$$\lim_{k\to\infty} \left(\frac{-1 + e^{8\pi\lfloor k/2 \rfloor} - 8\pi e^{4\pi\lfloor k/2 \rfloor}\lfloor k/2 \rfloor}{1 - 2e^{4\pi\lfloor k/2 \rfloor} + e^{8\pi\lfloor k/2 \rfloor} - 16\pi^2 e^{4\pi\lfloor k/2 \rfloor}\lfloor k/2 \rfloor^2} \right) = 1$$

we indeed get

$$\lambda_k \sim 2\pi k$$

as $k \to \infty$.

□

Proposition 26. *The sequence* $(\lambda_k)_{k=1}^{\infty}$ *does not depend on the position of the rectangle, i.e. if we consider* $\Omega' = (a_1, b_1) \times (a_2, b_2)$ *for some* $a_i, b_i \in \mathbb{R}, a_i < b_i$ *where* $i \in \{1,2\}$*, then the sequence* $(\lambda_k)_{k=1}^{\infty}$ *for this domain is the same as for domain* $\Omega'' = (0, b_1 - a_1) \times (0, b_2 - a_2)$*. Moreover, let* $\varepsilon_i \in \mathbb{R}, \varepsilon_i > 0$ *for* $i \in \{1,2\}$ *and denote* $\Omega = (0, \varepsilon_1) \times (0, \varepsilon_2)$ *and* $S_d = (0, \varepsilon_1)$*. Then there exists* $c \in \mathbb{R}, c > 0$ *independent of the domain such that the sequence* $(\lambda_k)_{k=1}^{\infty}$ *for* Ω *satisfies*

$$\lambda_k \sim \frac{ck}{\mathcal{H}^1(S_d)} \tag{2.54}$$

as $k \to \infty$*.*

Proof. The fact that the sequence does not depend on the position of the square is trivial since one can get solutions on different squares by appropriate affine translations of variables.

Next, assuming that $\Omega = (0, \varepsilon_1) \times (0, \varepsilon_2)$, we need to suppose that $k \in 2\pi\mathbb{N}/\varepsilon_1$ in order for (2.23)-(2.25) to hold. We could now go through the computations from Proposition 21 again and we would eventually end up with the condition for determinant to equal zero being

$$\lambda(k) = \frac{2k\left(-1 + e^{4\varepsilon_2 k} - 4e^{2\varepsilon_2 k}\varepsilon_2 k\right)}{1 - 2e^{2\varepsilon_2 k} + e^{4\varepsilon_2 k} - 4e^{2\varepsilon_2 k}\varepsilon_2^2 k^2}.$$

Arrange $\lambda(k)$ again into a non-decreasing sequence $(\lambda_k)_{k=1}^{\infty}$ where each eigenvalue λ is counted with its multiplicity and $\{\lambda_k;\ k \in \mathbb{N}\} = \{\lambda(k);\ k \in 2\pi\mathbb{N}_0/\varepsilon_1\}$. Using the same calculations as in Proposition 25 yields that indeed

$$\lambda_k \sim \frac{2\pi k}{\varepsilon_1} = \frac{2\pi k}{\mathcal{H}^1(S_d)}$$

as $k \to \infty$.

□

2.3.3 Formulation of the auxiliary problem in three dimensions and its basic properties

Now we are going to deal with the three dimensional case, however, the approach is going to be the same apart from some changes in determining the asymptotics so we will omit details and focus on parts that are different.

Notation. We denote $\Omega = (0,1)^3$ and $\Gamma = \partial\Omega$. Furthermore, let us denote

$$\begin{aligned}
\Gamma_1 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_3 = 0, x_1, x_2 \in (0,1)\},\\
\Gamma_2 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_2 = 0, x_1, x_3 \in (0,1)\},\\
\Gamma_3 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 = 1, x_2, x_3 \in (0,1)\},\\
\Gamma_4 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_2 = 1, x_1, x_3 \in (0,1)\},\\
\Gamma_5 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 = 0, x_2, x_3 \in (0,1)\},\\
\Gamma_6 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_3 = 1, x_1, x_2 \in (0,1)\}.
\end{aligned}$$

For $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^3$, $p\colon \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we now consider the following problem

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= 0 && \text{in } \Omega, && (2.55)\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, && (2.56)\\
\boldsymbol{u} &= 0 && \text{on } \Gamma_1, && (2.57)\\
\frac{\partial u_i}{\partial x_3} = \lambda u_i \text{ for i} \in \{1{,}2\} \text{ and } u_3 &= 0 && \text{on } \Gamma_6, && (2.58)\\
\boldsymbol{u}(x_1{,}0{,}x_3) - \boldsymbol{u}(x_1{,}1{,}x_3) &= 0 && \text{for } x_1, x_3 \in [0,1], && (2.59)\\
\boldsymbol{u}(0{,}x_2{,}x_3) - \boldsymbol{u}(1{,}x_2{,}x_3) &= 0 && \text{for } x_2, x_3 \in [0,1], && (2.60)\\
\frac{\partial \boldsymbol{u}}{\partial x_1}(0{,}x_2{,}x_3) - \frac{\partial \boldsymbol{u}}{\partial x_1}(1{,}x_2{,}x_3) &= 0 && \text{for } x_2, x_3 \in [0,1], && (2.61)\\
\frac{\partial \boldsymbol{u}}{\partial x_2}(x_1{,}0{,}x_3) - \frac{\partial \boldsymbol{u}}{\partial x_2}(x_1{,}1{,}x_3) &= 0 && \text{for } x_1, x_3 \in [0,1], && (2.62)\\
p(x_1{,}0{,}x_3) - p(x_1{,}1{,}x_3) &= 0 && \text{for } x_1, x_3 \in (0,1), && (2.63)\\
p(0{,}x_2{,}x_3) - p(1{,}x_2{,}x_3) &= 0 && \text{for } x_2, x_3 \in (0,1). && (2.64)
\end{aligned}
$$

Thus, we again consider periodic boundary conditions on four parallel sides of the cube for $\boldsymbol{u}$ and also for the pressure p.

Definition 27. *We define space V_3 as follows*

$$
\begin{aligned}
V_3 = \{\boldsymbol{u} \in (W^{1,2}_{\mathrm{div}}(\Omega))^3;\ &\mathrm{Tr}(\boldsymbol{u})|_{\Gamma_1} = 0,\\
&\mathrm{Tr}(\boldsymbol{u})(x_1{,}0{,}x_3) = \mathrm{Tr}(\boldsymbol{u})(x_1{,}1{,}x_3) \text{ for } x_1, x_3 \in [0,1],\\
&\mathrm{Tr}(\boldsymbol{u})(0{,}x_2{,}x_3) = \mathrm{Tr}(\boldsymbol{u})(1{,}x_2{,}x_3) \text{ for } x_2, x_3 \in [0,1], \mathrm{Tr}(u_3)|_{\Gamma_6} = 0\}.
\end{aligned}
$$

Remark. As for space V_2, space V_3 is again a Hilbert space, the expression

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V_3} = \int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}}\, d\lambda^3$$

defines a scalar product on V_3 and the corresponding norm $\|\cdot\|_{V_3}$ is equivalent to the standard $\|\cdot\|_{W^{1,2}}$ norm.

Definition 28. *Let $\lambda \in \mathbb{R}$ be fixed. We say that $\boldsymbol{u} \in V_3$ is a weak solution to the problem (2.55)-(2.64) if*

$$\int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}}\, d\lambda^3 = \lambda \int_{\Gamma_6} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^2 \tag{2.65}$$

holds for all $\boldsymbol{\varphi} \in V_3$.

Remark. We will only be interested in non-trivial weak solutions hence by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ we immediately get that $\lambda \geq 0$. Moreover, for $\lambda = 0$ we obtain by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ that $\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_3} = 0$ and hence the only weak solution is the trivial one.

We again define a mapping $B\colon V_3 \times V_3 \to \mathbb{C}$ by

$$B[\boldsymbol{u}{,}\boldsymbol{v}] = \int_{\Gamma_6} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{v}})\, d\mathcal{H}^2$$

and get the existence of a unique operator $T_3 \in L(V_3)$ satisfying

$$B[\boldsymbol{u}{,}\boldsymbol{v}] = \langle T_3(\boldsymbol{u}), \boldsymbol{v} \rangle_{V_3}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V_3$. Completely analogous version of Theorem 20 holds also in this case, however, we will formulate it nevertheless so we can refer to it later on in this subsection.

Theorem 29. *Operator T_3 is compact and self-adjoint. For some fixed $\lambda \in \mathbb{R}, \lambda > 0$, a non-trivial function $\boldsymbol{u}$ is a weak solution to the problem (2.55)-(2.64) if and only if $\boldsymbol{u}$ is an eigenfunction of T_3 corresponding to eigenvalue $1/\lambda$. There exist at most countably many $\lambda \in \mathbb{R}$ such that there exist a non-trivial weak solution to the problem (2.55)-(2.64).*

2.3.4 Existence of solutions and asymptotic behaviour of eigenvalues in three dimensions

In order to determine the asymptotic behaviour of λ's for which there exist non-trivial weak solutions, we will now (similarly to two dimensional case) compute the solutions to the problem (2.55)-(2.64) explicitly and afterwards we will show that these solutions are in fact already all non-trivial solutions to this problem.

Notation. Denote by $\hat{\mathbb{N}}^2$ the set $(2\pi\mathbb{N}_0 \times 2\pi\mathbb{N}_0) \setminus \{(0,0)\}$.

Proposition 30. *For any $k_1, k_2 \in 2\pi\mathbb{N}_0$ there exist $\lambda(k_1,k_2) \in \mathbb{R}$ of the form*

$$\lambda_1(k_1,k_2) = \frac{\left(1 + e^{2\sqrt{k_1^2+k_2^2}}\right)\sqrt{k_1^2+k_2^2}}{e^{2\sqrt{k_1^2+k_2^2}} - 1} \tag{2.66}$$

and

$$\lambda_2(k_1,k_2) = \frac{2\left(1 - e^{4\sqrt{k_1^2+k_2^2}} + 4e^{2\sqrt{k_1^2+k_2^2}}\sqrt{k_1^2+k_2^2}\right)\sqrt{k_1^2+k_2^2}}{e^{2\sqrt{k_1^2+k_2^2}}\left(2 + 4k_1^2 + 4k_2^2\right) - e^{4\sqrt{k_1^2+k_2^2}} - 1} \tag{2.67}$$

for $(k_1,k_2) \in \hat{\mathbb{N}}^2$ and $\lambda = 1$ for $k_1 = k_2 = 0$ and functions $a_{i,j}, p_i$ for $i \in \{1,2,3\}, j \in \{1,2,3,4\}$ of variable x_3 such that functions

$$\boldsymbol{u_{k_1,k_2}}(x_1,x_2,x_3) = \begin{pmatrix} u^1_{k_1,k_2,1} + u^2_{k_1,k_2,1} + u^3_{k_1,k_2,1} + u^4_{k_1,k_2,1} \\ u^1_{k_1,k_2,2} + u^2_{k_1,k_2,2} + u^3_{k_1,k_2,2} + u^4_{k_1,k_2,2} \\ u^1_{k_1,k_2,3} + u^2_{k_1,k_2,3} + u^3_{k_1,k_2,3} + u^4_{k_1,k_2,3} \end{pmatrix}$$

and

$$p_{k_1,k_2}(x_1,x_2,x_3) = p^1_{k_1,k_2} + p^2_{k_1,k_2} + p^3_{k_1,k_2} + p^4_{k_1,k_2}$$

solve the problem (2.55)-(2.64) in the pointwise sense. For $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$, functions $u^j_{k_1,k_2,i}$ and $p^j_{k_1,k_2}$ of variables x_1,x_2,x_3 are defined as follows

$$\begin{aligned} u^1_{k_1,k_2,i}(x_1,x_2,x_3) &= a_{i,1}(x_3)\cos(k_1x_1)\cos(k_2x_2), \\ u^2_{k_1,k_2,i}(x_1,x_2,x_3) &= a_{i,2}(x_3)\cos(k_1x_1)\sin(k_2x_2), \\ u^3_{k_1,k_2,i}(x_1,x_2,x_3) &= a_{i,3}(x_3)\sin(k_1x_1)\cos(k_2x_2), \\ u^4_{k_1,k_2,i}(x_1,x_2,x_3) &= a_{i,4}(x_3)\sin(k_1x_1)\sin(k_2x_2) \end{aligned}$$

and

$$\begin{aligned} p^1_{k_1,k_2}(x_1,x_2,x_3) &= p_1(x_3)\cos(k_1x_1)\cos(k_2x_2),\\ p^2_{k_1,k_2}(x_1,x_2,x_3) &= p_2(x_3)\cos(k_1x_1)\sin(k_2x_2),\\ p^3_{k_1,k_2}(x_1,x_2,x_3) &= p_3(x_3)\sin(k_1x_1)\cos(k_2x_2),\\ p^4_{k_1,k_2}(x_1,x_2,x_3) &= p_4(x_3)\sin(k_1x_1)\sin(k_2x_2).\end{aligned}$$

Moreover, eigenvalues of form $\lambda_i(k_1,k_2)$ *where* $i \in \{1,2\}$ *and* $(k_1,k_2) \in (2\pi\mathbb{N}_0 \times \{0\}) \cup (\{0\} \times 2\pi\mathbb{N}_0)$ *have multiplicity two and eigenvalues of form* $\lambda_i(k_1,k_2)$ *where* $i \in \{1,2\}$ *and* $(k_1,k_2) \in 2\pi\mathbb{N} \times 2\pi\mathbb{N}$ *have multiplicity four.*

Proof. Conditions (2.59)-(2.64) are satisfied trivially. First, we are going to deal with the case $k_1 = k_2 = 0$. Then

$$\boldsymbol{u_{0,0}}(x_1,x_2,x_3) = \begin{pmatrix} a_{1,1}(x_3)\\ a_{2,1}(x_3)\\ a_{3,1}(x_3)\end{pmatrix}$$

and

$$p_0(x_1,x_2,x_3) = p_1(x_3).$$

Equation (2.56): Rewriting $\operatorname{div}\boldsymbol{u_0} = 0$ we want

$$a'_{3,1}(x_3) = 0$$

for all $x_3 \in (0,1)$. Thus $a_{3,1}(x_3) = c_{3,1}$ for some $c_{3,1} \in \mathbb{R}$.

Equation (2.57): We want

$$a_{1,1}(0) = a_{2,1}(0) = a_{3,1}(0) = 0. \tag{2.68}$$

This already implies that $c_{3,1} = 0$ and thus $a_{3,1}(x_3) = 0$ for all $x_3 \in (0,1)$.

Equation (2.55): Rewriting $-\Delta\boldsymbol{u_{0,0}} + \nabla p_0 = 0$ we want

$$a''_{1,1}(x_3) = a''_{2,1}(x_3) = 0$$

for all $x_3 \in (0,1)$. Thus $a_{1,1}(x_3) = d_{1,1}x_3$ and $a_{2,1}(x_3) = d_{2,1}x_3$ for some $d_{1,1}, d_{2,1} \in \mathbb{R}$ due to (2.68). Next, we want

$$p'_1(x_3) = 0$$

for all $x_3 \in (0,1)$. Thus $p_1(x_3) = p_3$ for some $p_3 \in \mathbb{R}$.

Equation (2.58): The condition $u_3 = 0$ on Γ_6 is trivial. The second boundary condition for unknown λ gives

$$d_{i,1} = \lambda d_{i,1}$$

for $i \in \{1{,}2\}$. Since we are only interested in non-trivial solutions, we require $d_{i,1} \neq 0$ for at least one $i \in \{1{,}2\}$ and thus $\lambda = 1$. Consequently, the moreover part of Proposition 30 for $k_1 = k_2 = 0$ follows.

Suppose now that $k_1 \in 2\pi\mathbb{N}$ and $k_2 \in 2\pi\mathbb{N}_0$ or $k_1 \in 2\pi\mathbb{N}_0$ and $k_2 \in 2\pi\mathbb{N}$. Identically as in (2.29) we formally get that

$$\Delta p = 0 \tag{2.69}$$

holds in Ω. Using (2.69) for the special form of p_{k_1,k_2} we obtain that

$$\frac{\partial^2 p_{k_1,k_2}}{\partial x_3^2} - (k_1^2 + k_2^2) p_{k_1,k_2} = 0$$

holds in Ω. Similarly to proof of Proposition 21 we deduce from this that for $i \in \{1{,}2{,}3{,}4\}$

$$p_i''(x_3) - (k_1^2 + k_2^2) p_i(x_3) = 0$$

holds for all $x_3 \in (0, 1)$. Thus we can write p_i in the following way

$$p_i(x_3) = p_i^1 e^{\sqrt{k_1^2+k_2^2}x_3} + p_i^2 e^{-\sqrt{k_1^2+k_2^2}x_3},$$

where $p_i^1, p_i^2 \in \mathbb{R}$ for $i \in \{1{,}2{,}3{,}4\}$. Plugging p_{k_1,k_2} into (2.55) we obtain the following relations

$$\begin{aligned}
&\cos(k_1x_1)\cos(k_2x_2)(a_{1,1}''(x_3) - (k_1^2 + k_2^2)a_{1,1} - k_1 p_3(x_3))+\\
&\cos(k_1x_1)\sin(k_2x_2)(a_{1,2}''(x_3) - (k_1^2 + k_2^2)a_{1,2} - k_1 p_4(x_3))+\\
&\sin(k_1x_1)\cos(k_2x_2)(a_{1,3}''(x_3) - (k_1^2 + k_2^2)a_{1,3} + k_1 p_1(x_3))+\\
&\sin(k_1x_1)\sin(k_2x_2)(a_{1,4}''(x_3) - (k_1^2 + k_2^2)a_{1,4} + k_1 p_2(x_3)) = 0,
\end{aligned} \tag{2.70}$$

next

$$\begin{aligned}
&\cos(k_1x_1)\cos(k_2x_2)(a_{2,1}''(x_3) - (k_1^2 + k_2^2)a_{2,1} - k_2 p_2(x_3))+\\
&\cos(k_1x_1)\sin(k_2x_2)(a_{2,2}''(x_3) - (k_1^2 + k_2^2)a_{2,2} + k_2 p_1(x_3))+\\
&\sin(k_1x_1)\cos(k_2x_2)(a_{2,3}''(x_3) - (k_1^2 + k_2^2)a_{2,3} - k_2 p_4(x_3))+\\
&\sin(k_1x_1)\sin(k_2x_2)(a_{2,4}''(x_3) - (k_1^2 + k_2^2)a_{2,4} + k_2 p_3(x_3)) = 0
\end{aligned} \tag{2.71}$$

and finally

$$\begin{aligned}
&\cos(k_1x_1)\cos(k_2x_2)(a_{3,1}''(x_3) - (k_1^2 + k_2^2)a_{3,1} - p_1'(x_3))+\\
&\cos(k_1x_1)\sin(k_2x_2)(a_{3,2}''(x_3) - (k_1^2 + k_2^2)a_{3,2} - p_2'(x_3))+\\
&\sin(k_1x_1)\cos(k_2x_2)(a_{3,3}''(x_3) - (k_1^2 + k_2^2)a_{3,3} - p_3'(x_3))+\\
&\sin(k_1x_1)\sin(k_2x_2)(a_{3,4}''(x_3) - (k_1^2 + k_2^2)a_{3,4} - p_4'(x_3)) = 0.
\end{aligned} \tag{2.72}$$

From (2.70) we deduce that for $i \in \{1{,}2\}$ and all $x_3 \in (0, 1)$

$$a_{1,i}''(x_3) - (k_1^2 + k_2^2)a_{1,i} - k_1 p_{i+2}(x_3) = 0 \tag{2.73}$$

and for $i \in \{3{,}4\}$ and all $x_3 \in (0, 1)$

$$a_{1,i}''(x_3) - (k_1^2 + k_2^2)a_{1,i} + k_1 p_{i-2}(x_3) = 0. \tag{2.74}$$

From (2.71) we deduce that

$$a''_{2,1}(x_3) - (k_1^2 + k_2^2)a_{2,1} - k_2 p_2(x_3) = 0, \tag{2.75}$$
$$a''_{2,2}(x_3) - (k_1^2 + k_2^2)a_{2,2} + k_2 p_1(x_3) = 0, \tag{2.76}$$
$$a''_{2,3}(x_3) - (k_1^2 + k_2^2)a_{2,3} - k_2 p_4(x_3) = 0, \tag{2.77}$$
$$a''_{2,4}(x_3) - (k_1^2 + k_2^2)a_{2,4} + k_2 p_3(x_3) = 0 \tag{2.78}$$

holds for all $x_3 \in (0,1)$. Finally from (2.72) we deduce that for $i \in \{1,2,3,4\}$

$$a''_{3,i}(x_3) - (k_1^2 + k_2^2)a_{3,i} - p'_i(x_3) = 0 \tag{2.79}$$

holds for all $x_3 \in (0,1)$. Solving equations (2.73)-(2.74) we obtain for $i \in \{1,2\}$

$$a_{1,i}(x_3) = a_{1,i}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + a_{1,i}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{k_1 x_3}{2\sqrt{k_1^2+k_2^2}} \left(p_{i+2}^1 e^{\sqrt{k_1^2+k_2^2}x_3} - p_{i+2}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} \right), \tag{2.80}$$

for $i \in \{3,4\}$

$$a_{1,i}(x_3) = a_{1,i}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + a_{1,i}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{k_1 x_3}{2\sqrt{k_1^2+k_2^2}} \left(-p_{i-2}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + p_{i-2}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} \right). \tag{2.81}$$

Next, solving equations (2.75)-(2.78) we obtain

$$a_{2,1}(x_3) = a_{2,1}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + a_{2,1}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{k_2 x_3}{2\sqrt{k_1^2+k_2^2}} \left(p_2^1 e^{\sqrt{k_1^2+k_2^2}x_3} - p_2^2 e^{-\sqrt{k_1^2+k_2^2}x_3} \right), \tag{2.82}$$

$$a_{2,2}(x_3) = a_{2,2}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + a_{2,2}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{k_2 x_3}{2\sqrt{k_1^2+k_2^2}} \left(-p_1^1 e^{\sqrt{k_1^2+k_2^2}x_3} + p_1^2 e^{-\sqrt{k_1^2+k_2^2}x_3} \right), \tag{2.83}$$

$$a_{2,3}(x_3) = a_{2,3}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + a_{2,3}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{k_2 x_3}{2\sqrt{k_1^2+k_2^2}} \left(p_4^1 e^{\sqrt{k_1^2+k_2^2}x_3} - p_4^2 e^{-\sqrt{k_1^2+k_2^2}x_3} \right), \tag{2.84}$$

$$a_{2,4}(x_3) = a_{2,4}^1 e^{\sqrt{k_1^2+k_2^2}x_3} + a_{2,4}^2 e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{k_2 x_3}{2\sqrt{k_1^2+k_2^2}} \left(-p_3^1 e^{\sqrt{k_1^2+k_2^2}x_3} + p_3^2 e^{-\sqrt{k_1^2+k_2^2}x_3} \right). \tag{2.85}$$

Finally, solving equations (2.79) we obtain for $i \in \{1,2,3,4\}$

$$a_{3,i}(x_3) = a^1_{3,i} e^{\sqrt{k_1^2+k_2^2}x_3} + a^2_{3,i} e^{-\sqrt{k_1^2+k_2^2}x_3} + \frac{x_3}{2}\left(p^1_i e^{\sqrt{k_1^2+k_2^2}x_3} + p^2_i e^{-\sqrt{k_1^2+k_2^2}x_3}\right), \quad (2.86)$$

where $a^1_{i,j}, a^2_{i,j} \in \mathbb{R}$ for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$. Functions $\boldsymbol{u_{k_1,k_2}}$ and p_{k_1,k_2} of these forms thus satisfy (2.55). Now we are going to determine for which values of λ there exist nonzero coefficients such that $\boldsymbol{u_{k_1,k_2}}$ also satisfy (2.56)-(2.58).

Equation (2.57): For all $x_1, x_2 \in (0,1)$ we require

$$\boldsymbol{u_{k_1,k_2}}(x_1,x_2,0) = \begin{pmatrix} (u^1_{k_1,k_2,1} + u^2_{k_1,k_2,1} + u^3_{k_1,k_2,1} + u^4_{k_1,k_2,1})(x_1,x_2,0) \\ (u^1_{k_1,k_2,2} + u^2_{k_1,k_2,2} + u^3_{k_1,k_2,2} + u^4_{k_1,k_2,2})(x_1,x_2,0) \\ (u^1_{k_1,k_2,3} + u^2_{k_1,k_2,3} + u^3_{k_1,k_2,3} + u^4_{k_1,k_2,3})(x_1,x_2,0) \end{pmatrix} = \mathbf{0}.$$

After rewriting these terms using definitions we again deduce that for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$ we need $a_{i,j}(0) = 0$. Plugging this into (2.80)-(2.86) we obtain for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$ that

$$a^2_{i,j} = -a^1_{i,j}.$$

Equation (2.56): Rewriting $\operatorname{div} \boldsymbol{u_{k_1,k_2}} = 0$ we want

$$\begin{aligned} &\cos(k_1x_1)\cos(k_2x_2)(k_1a_{1,3}(x_3) + k_2a_{2,2}(x_3) + a'_{3,1}(x_3)) + \\ &\cos(k_1x_1)\sin(k_2x_2)(k_1a_{1,4}(x_3) - k_2a_{2,1}(x_3) + a'_{3,2}(x_3)) + \\ &\sin(k_1x_1)\cos(k_2x_2)(-k_1a_{1,1}(x_3) + k_2a_{2,4}(x_3) + a'_{3,3}(x_3)) + \\ &\sin(k_1x_1)\sin(k_2x_2)(-k_1a_{1,2}(x_3) - k_2a_{2,3}(x_3) + a'_{3,4}(x_3)) = 0 \end{aligned} \quad (2.87)$$

for all $(x_1,x_2,x_3) \in \Omega$. Thus we deduce

$$k_1a_{1,3}(x_3) + k_2a_{2,2}(x_3) + a'_{3,1}(x_3) = 0, \quad (2.88)$$
$$k_1a_{1,4}(x_3) - k_2a_{2,1}(x_3) + a'_{3,2}(x_3) = 0, \quad (2.89)$$
$$-k_1a_{1,1}(x_3) + k_2a_{2,4}(x_3) + a'_{3,3}(x_3) = 0, \quad (2.90)$$
$$-k_1a_{1,2}(x_3) - k_2a_{2,3}(x_3) + a'_{3,4}(x_3) = 0 \quad (2.91)$$

for all $x_3 \in (0,1)$. For $i \in \{1,2,3,4\}$ it holds that

$$a'_{3,i}(x_3) = \sqrt{k_1^2+k_2^2}\, a_{3,i}(x_3)\left(e^{\sqrt{k_1^2+k_2^2}x_3} + e^{-\sqrt{k_1^2+k_2^2}x_3}\right) + \frac{1}{2}\left(p^1_i e^{\sqrt{k_1^2+k_2^2}x_3} + p^2_i e^{-\sqrt{k_1^2+k_2^2}x_3}\right) + \frac{\sqrt{k_1^2+k_2^2}\,x_3}{2}\left(p^1_i e^{\sqrt{k_1^2+k_2^2}x_3} - p^2_i e^{-\sqrt{k_1^2+k_2^2}x_3}\right).$$

Plugging this into (2.88)-(2.91) and by comparing coefficients for each term

$e^{\sqrt{k_1^2+k_2^2}x_3}, e^{-\sqrt{k_1^2+k_2^2}x_3}$ we deduce the following relations

$$k_1 a_{1,3}^1 + k_2 a_{2,2}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,1}^1 + \frac{p_1^1}{2} = 0, \tag{2.92}$$

$$-k_1 a_{1,3}^1 - k_2 a_{2,2}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,1}^1 + \frac{p_1^2}{2} = 0, \tag{2.93}$$

$$k_1 a_{1,4}^1 - k_2 a_{2,1}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,2}^1 + \frac{p_2^1}{2} = 0, \tag{2.94}$$

$$-k_1 a_{1,4}^1 + k_2 a_{2,1}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,2}^1 + \frac{p_2^2}{2} = 0, \tag{2.95}$$

$$-k_1 a_{1,1}^1 + k_2 a_{2,4}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,3}^1 + \frac{p_3^1}{2} = 0, \tag{2.96}$$

$$k_1 a_{1,1}^1 - k_2 a_{2,4}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,3}^1 + \frac{p_3^2}{2} = 0, \tag{2.97}$$

$$-k_1 a_{1,2}^1 - k_2 a_{2,3}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,4}^1 + \frac{p_4^1}{2} = 0, \tag{2.98}$$

$$k_1 a_{1,2}^1 + k_2 a_{2,3}^1 + \sqrt{k_1^2+k_2^2}\, a_{3,4}^1 + \frac{p_4^2}{2} = 0. \tag{2.99}$$

Equation (2.58): The condition that $u_3 = 0$ on Γ_6 is treated in the same way as before and hence we obtain that $a_{3,i}(1) = 0$ for $i \in \{1,2,3,4\}$. Using (2.86) we get

$$a_{3,i}^1 \left(e^{\sqrt{k_1^2+k_2^2}} - e^{-\sqrt{k_1^2+k_2^2}} \right) + \frac{1}{2} \left(p_i^1 e^{\sqrt{k_1^2+k_2^2}} + p_i^2 e^{-\sqrt{k_1^2+k_2^2}} \right) = 0$$

for $i \in \{1,2,3,4\}$. The second boundary condition gives for $i \in \{1,2\}$

$$\begin{aligned} \cos(k_1x_1)\cos(k_2x_2)(a_{i,1}'(1) - \lambda a_{i,1}(1)) &= 0,\\ \cos(k_1x_1)\sin(k_2x_2)(a_{i,2}'(1) - \lambda a_{i,2}(1)) &= 0,\\ \sin(k_1x_1)\cos(k_2x_2)(a_{i,3}'(1) - \lambda a_{i,3}(1)) &= 0,\\ \sin(k_1x_1)\sin(k_2x_2)(a_{i,4}'(1) - \lambda a_{i,4}(1)) &= 0. \end{aligned} \tag{2.100}$$

Hence we finally deduce that

$$a_{i,j}'(1) - \lambda a_{i,j}(1) = 0$$

for $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$.

Notation. We denote

$$C_{k_1,k_2,\lambda} = \sqrt{k_1^2+k_2^2}\left(e^{2\sqrt{k_1^2+k_2^2}} + 1\right) - \lambda \left(e^{2\sqrt{k_1^2+k_2^2}} - 1\right).$$

Using (2.80)-(2.85) we get for $j \in \{1,2\}$

$$a_{1,j}^1 C_{k_1,k_2,\lambda} + \frac{p_{j+2}^1 k_1 e^{2\sqrt{k_1^2+k_2^2}}}{2\sqrt{k_1^2+k_2^2}} \left(1 + \sqrt{k_1^2+k_2^2} - \lambda\right) + \frac{p_{j+2}^2 k_1}{2\sqrt{k_1^2+k_2^2}} \left(-1 + \sqrt{k_1^2+k_2^2} + \lambda\right) = 0 \tag{2.101}$$

and for $j \in \{3,4\}$

$$a^1_{1,j} C_{k_1,k_2,\lambda} + \frac{p^1_{j-2} k_1 e^{2\sqrt{k_1^2+k_2^2}}}{2\sqrt{k_1^2+k_2^2}} \left(-1 - \sqrt{k_1^2+k_2^2} + \lambda\right) + \frac{p^2_{j-2} k_1}{2\sqrt{k_1^2+k_2^2}} \left(1 - \sqrt{k_1^2+k_2^2} - \lambda\right) = 0. \quad (2.102)$$

Finally, the last four equations are

$$a^1_{2,1} C_{k_1,k_2,\lambda} + \frac{p^1_2 k_2 e^{2\sqrt{k_1^2+k_2^2}}}{2\sqrt{k_1^2+k_2^2}} \left(1 + \sqrt{k_1^2+k_2^2} - \lambda\right) + \frac{p^2_2 k_2}{2\sqrt{k_1^2+k_2^2}} \left(-1 + \sqrt{k_1^2+k_2^2} + \lambda\right) = 0, \quad (2.103)$$

$$a^1_{2,2} C_{k_1,k_2,\lambda} + \frac{p^1_1 k_2 e^{2\sqrt{k_1^2+k_2^2}}}{2\sqrt{k_1^2+k_2^2}} \left(-1 - \sqrt{k_1^2+k_2^2} + \lambda\right) + \frac{p^2_1 k_2}{2\sqrt{k_1^2+k_2^2}} \left(1 - \sqrt{k_1^2+k_2^2} - \lambda\right) = 0, \quad (2.104)$$

$$a^1_{2,3} C_{k_1,k_2,\lambda} + \frac{p^1_4 k_2 e^{2\sqrt{k_1^2+k_2^2}}}{2\sqrt{k_1^2+k_2^2}} \left(1 + \sqrt{k_1^2+k_2^2} - \lambda\right) + \frac{p^2_4 k_2}{2\sqrt{k_1^2+k_2^2}} \left(-1 + \sqrt{k_1^2+k_2^2} + \lambda\right) = 0, \quad (2.105)$$

$$a^1_{2,4} C_{k_1,k_2,\lambda} + \frac{p^1_3 k_2 e^{2\sqrt{k_1^2+k_2^2}}}{2\sqrt{k_1^2+k_2^2}} \left(-1 - \sqrt{k_1^2+k_2^2} + \lambda\right) + \frac{p^2_3 k_2}{2\sqrt{k_1^2+k_2^2}} \left(1 - \sqrt{k_1^2+k_2^2} - \lambda\right) = 0. \quad (2.106)$$

Thus we end up with 20 equations for 20 unknown coefficients, i.e. equations (2.92)-(2.106). Let us denote the matrix of the corresponding problem by A_{k_1,k_2}. In order for non-trivial solutions to exist, the matrix of this system must be singular. We thus want $\det(A_{k_1,k_2}) = 0$. Solving this equations yields two solutions

$$\lambda_1(k_1,k_2) = \frac{\left(1 + e^{2\sqrt{k_1^2+k_2^2}}\right)\sqrt{k_1^2+k_2^2}}{e^{2\sqrt{k_1^2+k_2^2}} - 1}$$

and

$$\lambda_2(k_1,k_2) = \frac{2\left(1 - e^{4\sqrt{k_1^2+k_2^2}} + 4e^{2\sqrt{k_1^2+k_2^2}}\sqrt{k_1^2+k_2^2}\right)\sqrt{k_1^2+k_2^2}}{e^{2\sqrt{k_1^2+k_2^2}}\left(2 + 4k_1^2 + 4k_2^2\right) - e^{4\sqrt{k_1^2+k_2^2}} - 1}$$

which completes the proof of relations (2.66)-(2.67) and the first part of the Proposition. Concerning the moreover part of the Proposition, computations yield that for $(k_1,k_2) \in (2\pi\mathbb{N} \times \{0\}) \cup (\{0\} \times 2\pi\mathbb{N})$ and $\lambda = \lambda_i(k_1,k_2)$ for $i \in \{1,2\}$, the matrix A_{k_1,k_2} has dimension two. Further, for $(k_1,k_2) \in 2\pi\mathbb{N}\times 2\pi\mathbb{N}$ and $\lambda = \lambda_i(k_1,k_2)$ for $i \in \{1,2\}$, the matrix A_{k_1,k_2} has dimension four which completes the whole proof.

□

Proposition 31. *The values*

$$\lambda_1(k_1,k_2) = \frac{\left(1 + e^{2\sqrt{k_1^2+k_2^2}}\right)\sqrt{k_1^2+k_2^2}}{e^{2\sqrt{k_1^2+k_2^2}} - 1}$$

and

$$\lambda_2(k_1,k_2) = \frac{2\left(1 - e^{4\sqrt{k_1^2+k_2^2}} + 4e^{2\sqrt{k_1^2+k_2^2}}\sqrt{k_1^2+k_2^2}\right)\sqrt{k_1^2+k_2^2}}{e^{2\sqrt{k_1^2+k_2^2}}\left(2+4k_1^2+4k_2^2\right) - e^{4\sqrt{k_1^2+k_2^2}} - 1}$$

for $(k_1,k_2) \in \hat{\mathbb{N}}^2$ and $\lambda = 1$ for $k_1 = k_2 = 0$ are exactly all the values of λ for which there exist non-trivial weak solution to the problem (2.55)-(2.64).

Remark. The idea of the proof is the same as in Proposition 23, however, there are slight technical changes.

Proof. One implication is trivial since for these values of $\lambda_1(k_1,k_2)$ and $\lambda_2(k_1,k_2)$ we calculated corresponding smooth non-trivial $\boldsymbol{u_{k_1,k_2}}$ in Proposition 30 and hence they are also weak solutions.

On the other hand, suppose that there exists $\lambda^* \in \mathbb{R}, \lambda^* > 0$ such that

$$\lambda^* \notin \left(\{\lambda_i(k_1,k_2);\ i \in \{1,2\},\ (k_1,k_2) \in \hat{\mathbb{N}}^2\} \cup \{1\}\right)$$

and a corresponding non-trivial weak solutions $\boldsymbol{u}^* \in V_3$. We are going to prove that $\mathrm{Tr}(\boldsymbol{u}^*)$ is orthogonal to a certain orthogonal basis W in $L^2(\Gamma_6)\times L^2(\Gamma_6)$ where we consider natural scalar product, i.e. for $(v_1,v_2), (w_1,w_2) \in L^2(\Gamma_6) \times L^2(\Gamma_6)$ we consider

$$\langle (v_1,v_2), (w_1,w_2)\rangle_{L^2(\Gamma_6)\times L^2(\Gamma_6)} = \int_{\Gamma_6} v_1\overline{w_1}\, d\mathcal{H}^2 + \int_{\Gamma_6} v_2\overline{w_2}\, d\mathcal{H}^2.$$

Denote

$$\begin{aligned}
z_{k_1,k_2,1}(x_1,x_2) &= \cos(k_1x_1)\cos(k_2x_2),\\
z_{k_1,k_2,2}(x_1,x_2) &= \cos(k_1x_1)\sin(k_2x_2),\\
z_{k_1,k_2,3}(x_1,x_2) &= \sin(k_1x_1)\cos(k_2x_2),\\
z_{k_1,k_2,4}(x_1,x_2) &= \sin(k_1x_1)\sin(k_2x_2).
\end{aligned}$$

Then the set $\{z_{k_1,k_2,1}, z_{k_1,k_2,2}, z_{k_1,k_2,3}, z_{k_1,k_2,4};\ k_1, k_2 \in 2\pi\mathbb{N}_0\}$ forms an orthogonal basis of $L^2((0,1)^2)$. We will consider

$$W = \{(z_{k_1,k_2,i},0), (0,z_{k_1,k_2,i});\ i \in \{1,2,3,4\}, k_1, k_2 \in 2\pi\mathbb{N}_0\}$$

which obviously forms an orthogonal basis in $L^2(\Gamma_6) \times L^2(\Gamma_6)$. We will prove orthogonality of $\mathrm{Tr}(\boldsymbol{u}^*)$ to the elements of W only for the case $k_1, k_2 \in 2\pi\mathbb{N}$ since the other cases would be proved in a similar way. Denote

$$W' = \{(z_{k_1,k_2,i},0), (0,z_{k_1,k_2,i});\ i \in \{1,2,3,4\}, k_1, k_2 \in 2\pi\mathbb{N}\}$$

Using Proposition 30 we know that $\lambda_i(k_1,k_2)$ for $i \in \{1,2\}$ has multiplicity 4 and calculations yield that the first two coordinates of the corresponding solutions $\boldsymbol{u}^1_{\boldsymbol{k_1,k_2}}, \boldsymbol{u}^2_{\boldsymbol{k_1,k_2}}, \boldsymbol{u}^3_{\boldsymbol{k_1,k_2}}, \boldsymbol{u}^4_{\boldsymbol{k_1,k_2}}$ on Γ_6 can be chosen so that

$$\boldsymbol{u}^1_{\boldsymbol{k_1,k_2}}(x_1,x_2,1) = \begin{pmatrix} a_{1,1}(1)\cos(k_1x_1)\cos(k_2x_2) \\ a_{2,4}(1)\sin(k_1x_1)\sin(k_2x_2) \\ 0 \end{pmatrix},$$

$$\boldsymbol{u}^2_{\boldsymbol{k_1,k_2}}(x_1,x_2,1) = \begin{pmatrix} a_{1,2}(1)\cos(k_1x_1)\sin(k_2x_2) \\ a_{2,3}(1)\sin(k_1x_1)\cos(k_2x_2) \\ 0 \end{pmatrix},$$

$$\boldsymbol{u}^3_{\boldsymbol{k_1,k_2}}(x_1,x_2,1) = \begin{pmatrix} a_{1,3}(1)\sin(k_1x_1)\cos(k_2x_2) \\ a_{2,2}(1)\cos(k_1x_1)\sin(k_2x_2) \\ 0 \end{pmatrix},$$

$$\boldsymbol{u}^4_{\boldsymbol{k_1,k_2}}(x_1,x_2,1) = \begin{pmatrix} a_{1,4}(1)\sin(k_1x_1)\sin(k_2x_2) \\ a_{2,1}(1)\cos(k_1x_1)\cos(k_2x_2) \\ 0 \end{pmatrix}$$

for $x_1, x_2 \in (0, 1)$ where $a_{i,j}$ are some non-zero constants and $a_{i,j}$ differs depending on $\lambda_i(k_1,k_2), i \in \{1,2\}$. From now on we will distinguish $\boldsymbol{u}^j_{\boldsymbol{k_1,k_2}}$ for $j \in \{1,2,3,4\}$ and $a_{p,q}(1)$ for $p, q \in \{1,2,3,4\}$ depending on $\lambda_i(k_1,k_2), i \in \{1,2\}$ by adding λ_i to the notation, i.e. $\boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}}$ and $a_{p,q,\lambda_i}(1)$. For convenience, let us denote $\boldsymbol{u_{0,0}} = \boldsymbol{u}^j_{\boldsymbol{0,0,\lambda_i}}$ for $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$. Completely analogously as in Proposition 23 for two dimensional case, i.e. relations (2.46)-(2.48), we would prove that

$$\left\langle \boldsymbol{u}^*, \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}} \right\rangle_{V_3} = 0 \tag{2.107}$$

for any $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$. Next, taking $\boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}}$ as a test function in (2.65) and using (2.107) we obtain

$$0 = \left\langle \boldsymbol{u}^*, \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}} \right\rangle_{V_3} = \int_\Omega \nabla \boldsymbol{u}^* : \overline{\nabla \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}}}\, d\lambda^3$$
$$= \lambda_i(k_1,k_2) \int_{\Gamma_6} \mathrm{Tr}(\boldsymbol{u}^*) \cdot \mathrm{Tr}(\overline{\boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}}})\, d\mathcal{H}^2$$

for all $k_1, k_2 \in 2\pi\mathbb{N}$ and hence also

$$\alpha_i \left(\int_{\Gamma_6} \mathrm{Tr}(\boldsymbol{u}^*)_1 \mathrm{Tr}(\overline{\boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}}})_1\, d\mathcal{H}^2 + \int_{\Gamma_6} \mathrm{Tr}(\boldsymbol{u}^*)_2 \mathrm{Tr}(\overline{\boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_i}}})_2\, d\mathcal{H}^2 \right) = 0$$

for any $\alpha_i \in \mathbb{R}$, $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$. From this we deduce that

$$\begin{aligned}\int_{(0,1)^2} \mathrm{Tr}(\boldsymbol{u}^*)_1 \left(\alpha_1 \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_1}}(x_1,x_2,1)_1 + \alpha_2 \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_2}}(x_1,x_2,1)_1\right) d(x_1,x_2) \\ + \int_{(0,1)^2} \mathrm{Tr}(\boldsymbol{u}^*)_2 \left(\alpha_1 \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_1}}(x_1,x_2,1)_2 + \alpha_2 \boldsymbol{u}^j_{\boldsymbol{k_1,k_2,\lambda_2}}(x_1,x_2,1)_2\right) d(x_1,x_2) = 0.\end{aligned} \tag{2.108}$$

It can be computationally verified that for any $j \in \{1,2,3,4\}$ the matrix

$$\begin{pmatrix} a_{1,j,\lambda_1}(1) & a_{1,j,\lambda_2}(1) \\ a_{2,5-j,\lambda_1}(1) & a_{2,5-j,\lambda_2}(1) \end{pmatrix}$$

is regular and hence we can choose $\alpha_1, \alpha_2 \in \mathbb{R}$ in such a way that (2.108) yields

$$\langle (\mathrm{Tr}(\boldsymbol{u}^*)_1, \mathrm{Tr}(\boldsymbol{u}^*)_2), (w_1,w_2) \rangle_{L^2(\Gamma_6)\times L^2(\Gamma_6)} = 0 \tag{2.109}$$

for any $(w_1,w_2) \in W'$. Suppose further, that we proved (2.109) for all $(w_1,w_2) \in W$. This implies that $\mathrm{Tr}(\boldsymbol{u}^*)_1 = \mathrm{Tr}(\boldsymbol{u}^*)_2 = 0$ on Γ_6 and hence

$$\mathrm{Tr}(\boldsymbol{u}^*)|_{\Gamma_6} = \boldsymbol{0}. \tag{2.110}$$

Plugging (2.110) into (2.65) we get that

$$\int_\Omega \nabla \boldsymbol{u}^* : \overline{\nabla \boldsymbol{\varphi}}\, d\lambda^3 = \langle \boldsymbol{u}^*, \boldsymbol{\varphi} \rangle_{V_3} = 0 \tag{2.111}$$

holds for all $\boldsymbol{\varphi} \in V_3$. By setting $\boldsymbol{\varphi} = \boldsymbol{u}^*$ in (2.111) we obtain that $\boldsymbol{u}^* = 0$ in Ω which is a contradiction.

□

In order to determine the asymptotic growth of the eigenvalue sequence to the problem (2.55)-(2.64), we consider $(\lambda_k)_{k=1}^\infty$ as a non-decreasing sequence of eigenvalues appearing according to their multiplicity such that

$$\{\lambda_k;\, k \in \mathbb{N}\} = \{\lambda_i(k_1,k_2);\, i \in \{1,2\},\, (k_1,k_2) \in \hat{\mathbb{N}}^2\} \cup \{1\}.$$

Notation. Denote μ^1 as a counting measure on $\mathcal{P}(\mathbb{N})$ and μ^2 as a counting measure on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$.

Proposition 32. *Sequence $(\lambda_k)_{k=1}^\infty$ satisfies*

$$\lambda_k \sim \frac{4\sqrt{\pi}}{\sqrt{5}} k^{1/2} \tag{2.112}$$

as $k \to \infty$.

Proof. For convenience we will assume that mappings λ_i for $i \in \{1,2\}$ are defined on $\mathbb{N}_0 \times \mathbb{N}_0$ via the composition $\psi\colon \mathbb{N}_0 \times \mathbb{N}_0 \to 2\pi\mathbb{N}_0 \times 2\pi\mathbb{N}_0$ where $\psi(k_1,k_2) = (2\pi k_1, 2\pi k_2)$ for (k_1,k_2) in $\mathbb{N}_0 \times \mathbb{N}_0$. More precisely, for $i \in \{1,2\}$ and (k_1,k_2) in $\mathbb{N}_0\times\mathbb{N}_0$ we define mappings $\lambda_i^*(k_1,k_2) = \lambda(\psi(k_1,k_2))$ and we will further refer to λ_i^* as λ_i. This requires that $k_1, k_2 \in \mathbb{N}_0$ instead of $k_1, k_2 \in 2\pi\mathbb{N}_0$. For $(k_1,k_2) \in (\mathbb{N}_0 \times \mathbb{N}_0) \setminus \{(0,0)\}$ denote

$$D_1(k_1,k_2) = \frac{2\pi\left(1 + e^{4\pi\sqrt{k_1^2+k_2^2}}\right)}{e^{4\pi\sqrt{k_1^2+k_2^2}} - 1}$$

and

$$D_2(k_1,k_2) = \frac{4\pi\left(1 - e^{8\pi\sqrt{k_1^2+k_2^2}} + 8\pi e^{4\pi\sqrt{k_1^2+k_2^2}}\sqrt{k_1^2+k_2^2}\right)}{e^{4\pi\sqrt{k_1^2+k_2^2}}\left(2 + 16\pi^2 k_1^2 + 16\pi^2 k_2^2\right) - e^{8\pi\sqrt{k_1^2+k_2^2}} - 1}.$$

First of all, for $i \in \{1,2\}$ it holds

$$\lambda_i(k_1,k_2) \sim d_i\sqrt{k_1^2 + k_2^2} \tag{2.113}$$

as $\|(k_1,k_2)\|_2 \to \infty$ where $d_1 = 2\pi$ and $d_2 = 4\pi$ since

$$\lim_{\|(k_1,k_2)\|_2 \to \infty} D_i(k_1,k_2) = d_i.$$

The first step to prove this Proposition will be to prove that

$$\mu^1\left(\{k \in \mathbb{N};\ \lambda_k \leq \lambda\}\right) \sim \frac{5}{16\pi}\lambda^2 \tag{2.114}$$

as $\lambda \to \infty$. Knowing (2.114), we will be able to prove (2.112).

Choose $\varepsilon \in \mathbb{R}, \varepsilon > 0$ small enough so that $d_i - \varepsilon > 0$ for $i \in \{1,2\}$ and find appropriate $r \in \mathbb{R}, r > 0$ such that for all $k_1, k_2 \in \mathbb{N}$ satisfying $\|(k_1,k_2)\|_2 > r$ it holds that $D_i(k_1,k_2) \in B(d_i, \varepsilon)$ for $i \in \{1,2\}$. Choose $\lambda \in \mathbb{R}$ so that

$$\lambda \geq \max\{\max\{\lambda_i(k_1,k_2);\ i \in \{1,2\}, k_1, k_2 \in \mathbb{N}_0, \|(k_1,k_2)\|_2 \leq r\},\\ \max\{(d_i - \varepsilon)\, r;\ i \in \{1,2\}\}\}.$$

Hence for $i \in \{1,2\}$ we have

$$\{(k_1,k_2) \in \mathbb{N}_0\times\mathbb{N}_0;\ \|(k_1,k_2)\|_2 \leq r\} \subset \{(k_1,k_2) \in \mathbb{N}_0\times\mathbb{N}_0;\ \lambda_i(k_1,k_2) \leq \lambda\} \tag{2.115}$$

and also

$$\{(k_1,k_2) \in \mathbb{N}_0\times\mathbb{N}_0;\ \|(k_1,k_2)\|_2 \leq r\} \subset \{(k_1,k_2) \in \mathbb{N}_0\times\mathbb{N}_0;\ (d_i - \varepsilon)\sqrt{k_1^2 + k_2^2} \leq \lambda\}. \tag{2.116}$$

Knowing the multiplicity for each eigenvalue of the problem (2.55)-(2.64) (see Proposition 30) and using the definition of the sequence $(\lambda_k)_{k=1}^{\infty}$ we have

$$\mu^1\left(\{k \in \mathbb{N};\ \lambda_k \leq \lambda\}\right) = 4\sum_{i=1}^{2}\mu^2\left(\{(k_1,k_2) \in \mathbb{N}\times\mathbb{N};\ \lambda_i(k_1,k_2) \leq \lambda\}\right)\\ + 2\sum_{i=1}^{2}\mu^2\left(\{(k_1,k_2) \in (\mathbb{N}_0\times\{0\}) \cup (\{0\}\times\mathbb{N}_0);\ \lambda_i(k_1,k_2) \leq \lambda\}\right) \tag{2.117}$$

Using (2.115) we get for $i \in \{1,2\}$

$$\mu^2\left(\{(k_1,k_2) \in \mathbb{N}\times\mathbb{N};\ (d_i + \varepsilon)\sqrt{k_1^2 + k_2^2} \leq \lambda\}\right) \leq\\ \mu^2\left(\{(k_1,k_2) \in \mathbb{N}\times\mathbb{N};\ \lambda_i(k_1,k_2) \leq \lambda\}\right). \tag{2.118}$$

Using (2.115) and (2.116) we get for $i \in \{1,2\}$

$$\mu^2 \left(\{(k_1,k_2) \in \mathbb{N}_0 \times \mathbb{N}_0;\ \lambda_i(k_1,k_2) \le \lambda\}\right) \le \mu^2 \left(\{(k_1,k_2) \in \mathbb{N}_0 \times \mathbb{N}_0;\ (d_i - \varepsilon)\sqrt{k_1^2 + k_2^2} \le \lambda\}\right). \quad (2.119)$$

Let $M \subset \mathbb{N}_0 \times \mathbb{N}_0$. We can view the number of pairs satisfying $(k_1,k_2) \in M$ as the area of the corresponding unit squares $[k_1 - 1, k_1] \times [k_2 - 1, k_2]$, i.e. $\sum_{(k_1,k_2)\in M} 1$. Using this interpretation we obtain for $i \in \{1,2\}$

$$\mu^2 \left(\{(k_1,k_2) \in \mathbb{N} \times \mathbb{N};\ (d_i - \varepsilon)\sqrt{k_1^2 + k_2^2} \le \lambda\}\right) \le \frac{\pi\lambda^2}{4\,(d_i - \varepsilon)^2} \quad (2.120)$$

since we can restrict only to a quarter circle. Furthermore, we obtain for $i \in \{1,2\}$

$$\mu^2 \left(\{(k_1,k_2) \in (\mathbb{N}_0 \times \{0\}) \cup (\{0\} \times \mathbb{N}_0)\,;\ (d_i - \varepsilon)\sqrt{k_1^2 + k_2^2} \le \lambda\}\right) \le \frac{2\lambda}{d_i - \varepsilon} + 1. \quad (2.121)$$

For the lower estimate of $\mu^1(\{k \in \mathbb{N};\ \lambda_k \le \lambda\})$, it is enough to observe that for $\lambda \in \mathbb{R}, \lambda \ge 32\pi$ and $i \in \{1,2\}$ any point from the set $\{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+;\ (d_i + \varepsilon)\sqrt{x^2 + y^2} \le (\lambda - 16\pi)\}$ is contained in some unit square corresponding to some pair in the set $\{(k_1,k_2) \in \mathbb{N} \times \mathbb{N};\ (d_i + \varepsilon)\sqrt{k_1^2 + k_2^2} \le \lambda\}$. This holds since for the difference of radii of these two corresponding circles we have

$$\frac{\lambda}{d_i + \varepsilon} - \left(\frac{\lambda}{d_i + \varepsilon} - \frac{16\pi}{d_i + \varepsilon}\right) = \frac{16\pi}{d_i + \varepsilon} \ge \frac{16\pi}{\max\{d_i;\ i \in \{1,2\}\} + \varepsilon}.$$

Using that $\max\{d_i;\ i \in \{1,2\}\} = d_2 = 4\pi$ and the assumption that $d_i > \varepsilon$ for $i \in \{1,2\}$ and thus $2\pi > \varepsilon$ we obtain

$$\frac{16\pi}{\max\{d_i;\ i \in \{1,2\}\} + \varepsilon} > \frac{16\pi}{6\pi} > 2.$$

Using this results we get the following estimate

$$\mu^2 \left(\{(k_1,k_2) \in \mathbb{N} \times \mathbb{N};\ (d_i + \varepsilon)\sqrt{k_1^2 + k_2^2} \le \lambda\}\right) \ge \frac{\pi\,(\lambda - 16\pi)^2}{4\,(d_i + \varepsilon)^2}. \quad (2.122)$$

Combining (2.118)-(2.122) and plugging it back to (2.117) we get

$$(\lambda - 16\pi)^2 \sum_{i=1}^{2} \frac{\pi}{(d_i + \varepsilon)^2} \le \mu^1(\{k \in \mathbb{N};\ \lambda_k \le \lambda\}) \le \lambda^2 \sum_{i=1}^{2} \frac{\pi}{(d_i - \varepsilon)^2} + \sum_{i=1}^{2} \frac{4\lambda}{d_i - \varepsilon} + 2 \quad (2.123)$$

where for the lower estimate we forgot about the second sum on the right-hand side of (2.117). Dividing (2.123) by λ^2 we get

$$\frac{(\lambda - 16\pi)^2}{\lambda^2} \sum_{i=1}^{2} \frac{\pi}{(d_i + \varepsilon)^2} \le \frac{\mu^1(\{k \in \mathbb{N};\ \lambda_k \le \lambda\})}{\lambda^2} \le \sum_{i=1}^{2} \frac{\pi}{(d_i - \varepsilon)^2} + \sum_{i=1}^{2} \frac{4}{\lambda\,(d_i - \varepsilon)} + \frac{2}{\lambda^2}.$$

Since

$$\lim_{\lambda\to\infty}\frac{(\lambda-16\pi)^2}{\lambda^2}=1,$$

$$\lim_{\lambda\to\infty}\frac{2}{\lambda^2}=0$$

and for $i\in\{1{,}2\}$

$$\lim_{\lambda\to\infty}\frac{4}{\lambda\left(d_i-\varepsilon\right)}=0,$$

we get that for any $\varepsilon\in\mathbb{R},\varepsilon>0$ it holds that

$$\sum_{i=1}^{2}\frac{\pi}{\left(d_i+\varepsilon\right)^2}\leq\limsup_{\lambda\to\infty}\frac{\mu^1\left(\{k\in\mathbb{N};\ \lambda_k\leq\lambda\}\right)}{\lambda^2}\leq\sum_{i=1}^{2}\frac{\pi}{\left(d_i-\varepsilon\right)^2}.$$

Since the same estimates could have been done also for limes inferior and

$$\lim_{\varepsilon\to0^+}\left(\sum_{i=1}^{2}\frac{\pi}{\left(d_i+\varepsilon\right)^2}-\sum_{i=1}^{2}\frac{\pi}{\left(d_i-\varepsilon\right)^2}\right)=0$$

we obtain

$$\lim_{\lambda\to\infty}\frac{\mu^1\left(\{k\in\mathbb{N};\ \lambda_k\leq\lambda\}\right)}{\lambda^2}=\sum_{i=1}^{2}\frac{\pi}{d_i^2}=\frac{5}{16\pi},$$

i.e.

$$\mu^1\left(\{k\in\mathbb{N};\ \lambda_k\leq\lambda\}\right)\sim\frac{5}{16\pi}\lambda^2$$

as $\lambda\to\infty$ hence we indeed proved (2.114).

Finally, we are going to prove (2.112). Let $d=5/(16\pi)$. Choose $\varepsilon\in\mathbb{R},\varepsilon>0$ such that $d-\varepsilon>0$. For any $\lambda\in\mathbb{R},\lambda>0$ denote

$$P_\lambda=\mu^1\left(\{k\in\mathbb{N};\ \lambda_k\leq\lambda\}\right).$$

Using (2.114) there exists $\tilde{\lambda}_0\in\mathbb{R},\tilde{\lambda}_0>1$ such that for all $\lambda\in\mathbb{R},\lambda>\tilde{\lambda}_0$ it holds

$$d-\varepsilon<\frac{P_\lambda}{\lambda^2}<d+\varepsilon. \tag{2.124}$$

We find $k_0\in\mathbb{N}$ such that $\left(d+\varepsilon\right)\tilde{\lambda}_0^2<k_0$. Fix any $k\in\mathbb{N},k\geq k_0$. For $\tilde{\lambda}_1\in\mathbb{R}$ defined by

$$\tilde{\lambda}_1=\sqrt{\frac{k}{d-\varepsilon}}+\varepsilon$$

we have that $k<\left(d-\varepsilon\right)\tilde{\lambda}_1^2$ and $\tilde{\lambda}_1>\tilde{\lambda}_0$. Using (2.124) we get that $k<P_{\tilde{\lambda}_1}$ and by the definition of $P_{\tilde{\lambda}_1}$ this implies that $\lambda_k<\tilde{\lambda}_1$. Next, let $\varepsilon_1\in\mathbb{R},\varepsilon_1>0$ satisfy $\varepsilon_1<\sqrt{k_0/\left(d+\varepsilon\right)}$. For $\tilde{\lambda}_2\in\mathbb{R}$ defined by

$$\tilde{\lambda}_2=\sqrt{\frac{k}{d+\varepsilon}}-\varepsilon_1$$

we have that $k>\left(d+\varepsilon\right)\tilde{\lambda}_2^2$. Since by definition of k_0 it holds that

$$\tilde{\lambda}_0<\sqrt{\frac{k_0}{d+\varepsilon}}\leq\sqrt{\frac{k}{d+\varepsilon}}$$

we can choose ε_1 sufficiently small so that we also have $\tilde{\lambda}_2 > \tilde{\lambda}_0$. Again, using (2.124) we get that $P_{\tilde{\lambda}_2} < k$ and by the definition of $P_{\tilde{\lambda}_2}$ this implies that $\lambda_k > \tilde{\lambda}_2$. Thus we proved that for any $\varepsilon \in \mathbb{R}, \varepsilon > 0$ there exists sufficiently small $\varepsilon_1 \in \mathbb{R}, \varepsilon_1 > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}, k \geq k_0$ we have that

$$\frac{\lambda_k}{\sqrt{k}} \in \left(\frac{1}{\sqrt{d+\varepsilon}} - \frac{\varepsilon_1}{\sqrt{k}}, \frac{1}{\sqrt{d-\varepsilon}} + \frac{\varepsilon}{\sqrt{k}} \right) \subset \left(\frac{1}{\sqrt{d+\varepsilon}} - \varepsilon, \frac{1}{\sqrt{d-\varepsilon}} + \varepsilon \right) \quad (2.125)$$

since $k \in \mathbb{N}, k \geq k_0 \geq 1$ and thus $1/\sqrt{k} \leq 1$ and ε_1 can be chosen so that $\varepsilon_1 < \varepsilon$. Sending $\varepsilon \to 0^+$, relation (2.125) implies that

$$\lim_{k\to\infty} \frac{\lambda_k}{\sqrt{k}} = \frac{1}{\sqrt{d}} = \frac{4\sqrt{\pi}}{\sqrt{5}}$$

thus we indeed proved (2.112).

□

Definition 33. *Let $n \in \{2,3\}$ and Ω be a domain in $\mathbb{R}^n$. We say that Ω is a cuboid in $\mathbb{R}^2$ if the set Ω can be written as $\Omega = (a_1, b_1) \times (a_2, b_2)$ for some $a_i, b_i \in \mathbb{R}, a_i < b_i$ where $i \in \{1,2\}$. We say that Ω is a cuboid in $\mathbb{R}^3$ if the set Ω can be written as $\Omega = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ for some $a_i, b_i \in \mathbb{R}, a_i < b_i$ where $i \in \{1,2,3\}$.*

Proposition 34. *The sequence $(\lambda_k)_{k=1}^\infty$ does not depend on the position of the cuboid, i.e. if we consider $\Omega' = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ for some $a_i, b_i \in \mathbb{R}, a_i < b_i$ where $i \in \{1,2,3\}$, then the sequence $(\lambda_k)_{k=1}^\infty$ for this domain is the same as for domain $\Omega'' = (0, b_1 - a_1) \times (0, b_2 - a_2) \times (0, b_3 - a_3)$. Moreover, let $\varepsilon_i \in \mathbb{R}, \varepsilon_i > 0$ for $i \in \{1,2,3\}$ and denote $\Omega = (0, \varepsilon_1) \times (0, \varepsilon_2) \times (0, \varepsilon_3)$ and $S_d = (0, \varepsilon_1) \times (0, \varepsilon_2)$. Then there exists $c \in \mathbb{R}, c > 0$ independent of the domain such that the sequence $(\lambda_k)_{k=1}^\infty$ for Ω satisfies*

$$\lambda_k \sim \frac{ck^{1/2}}{\mathcal{H}^2(S_d)} \quad (2.126)$$

as $k \to \infty$.

Proof. The invariance of position is trivial as in Proposition 26. Next, assuming that $\Omega = (0, \varepsilon_1) \times (0, \varepsilon_2) \times (0, \varepsilon_3)$, we need to assume that $k_i \in 2\pi\mathbb{N}/\varepsilon_i$ for $i \in \{1,2\}$ in order for (2.59)-(2.64) to hold. We could now go through the computations from Proposition 30 again and we would once again end up with the fact that asymptotic behaviour does not depend on ε_3. Moreover, we could do similar estimates as in Proposition 32 with the only difference being that we would work on an ellipse, not circle. Nevertheless, we would obtain that there exists $d \in \mathbb{R}, d > 0$ independent of the domain such that

$$\mu^1\left(\{k \in \mathbb{N};\, \lambda_k \leq \lambda\}\right) \sim d \left(\mathcal{H}^2(S_d)\right)^2 \lambda^2,$$

hence it indeed follows that there exists $c \in \mathbb{R}, c > 0$ independent of the domain such that

$$\lambda_k \sim \frac{ck^{1/2}}{\mathcal{H}^2(S_d)}$$

as $k \to \infty$.

□

3. Upper estimate of eigenvalues on general domains

The goal of this Chapter will be to obtain upper estimates of eigenvalues of the problem (2.1)-(2.4) by using the results we managed to get in Chapter 2. We will state the following theorem for future references since we will be using it extensively throughout this Chapter.

Theorem 35. *Let A be a linear, self-adjoint and compact operator on a Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$. Denote $\sigma_p(A) = (\lambda_k)_{k=1}^N$ where N is either finite natural number or infinity. Moreover, suppose that eigenvalues of A are non-negative, they are sorted in a non-increasing order and each eigenvalue appear according to its multiplicity. For $k \in \mathbb{N}, k \leq N$ denote by H_k the set of all k-dimensional subspaces of H. Then*

$$\lambda_k = \max_{E \in H_k} \min_{\boldsymbol{u} \in E \setminus \{\boldsymbol{0}\}} \frac{\langle A(\boldsymbol{u}), \boldsymbol{u} \rangle_H}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_H} \tag{3.1}$$

for all admissible $k \in \mathbb{N}$.

Proof. See [10], Chapter 28, Theorem 4., p. 318 □

Suppose that the assumptions of Theorem 35 hold and let $k \in \mathbb{N}, k \leq N$. Suppose that $\lambda_k \in (0, +\infty)$. Then it is easy to check that (3.1) implies

$$\frac{1}{\lambda_k} = \min_{E \in \mathcal{H}_k} \max_{\boldsymbol{u} \in E \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_H}{\langle A(\boldsymbol{u}), \boldsymbol{u} \rangle_H}. \tag{3.2}$$

after the following slight adjustments. For some non-zero $\tilde{\boldsymbol{u}} \in H$ it may happen that $\langle A(\tilde{\boldsymbol{u}}), \tilde{\boldsymbol{u}} \rangle_H = 0$ and the denominator in (3.1) would not be defined. However, since we suppose that $\lambda_k \in (0, \infty)$ there exists a specific subspace $E_k \in H_k$ such that

$$\min_{\boldsymbol{u} \in E_k \setminus \{\boldsymbol{0}\}} \frac{\langle A(\boldsymbol{u}), \boldsymbol{u} \rangle_H}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_H} > 0.$$

This motivates the following definition. If for some $k \in \mathbb{N}, k \leq N$, $E \in H_k$ and some non-zero $\tilde{\boldsymbol{u}} \in E$ holds that $\langle A(\tilde{\boldsymbol{u}}), \tilde{\boldsymbol{u}} \rangle_H = 0$ we define

$$\max_{\boldsymbol{u} \in E \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_H}{\langle A(\boldsymbol{u}), \boldsymbol{u} \rangle_H} = +\infty.$$

With this definition formula (3.2) indeed works. We will use this convention throughout the rest of the thesis.

3.1 Generalization to cylindrical domain

Firstly, we are going to generalize our results to cylindrical domain in the following way.

Definition 36. *Let $n \in \{2,3\}$ and Ω be a domain in $\mathbb{R}^n$. We say that Ω is a cylinder in $\mathbb{R}^n$ if the set Ω can be written as $\Omega = B(x,r) \times I$ where $B(x,r) \subset \mathbb{R}^{n-1}$ is an open ball for some $x \in \mathbb{R}^{n-1}$, $r \in \mathbb{R}, r > 0$ and I is some non-empty open interval $I \subset \mathbb{R}$.*

Throughout this Section, we will assume that $n \in \{2,3\}$ and that Ω denotes a cylinder in $\mathbb{R}^n$.

Notation. Denote by $S_{u,\Omega}$ the upper disk part of the boundary $\partial\Omega$, by $S_{l,\Omega}$ the lower disk part of the boundary $\partial\Omega$ and denote $S_{r,\Omega} = \partial\Omega \setminus (S_{l,\Omega} \cup S_{u,\Omega})$. For cuboid $\Omega' \subset \mathbb{R}^n$ we introduce the same notation but instead of S we use S'.

Notation. Let $\Omega' \subset \mathbb{R}^n$ be a cuboid. In order to emphasize on which set we consider the space V_n we will write $V_n(\Omega')$ instead of V_n.

For $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^n$, $p\colon \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we consider the following problem

$$-\Delta \boldsymbol{u} + \nabla p = 0 \qquad \text{in } \Omega, \tag{3.3}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{3.4}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega, \tag{3.5}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega \setminus S_{u,\Omega}, \tag{3.6}$$

$$[(\nabla \boldsymbol{u})\, \boldsymbol{n}]_\tau = \lambda \boldsymbol{u} \qquad \text{on } S_{u,\Omega}. \tag{3.7}$$

The weak formulation of this problem would have been again derived in the same way as for the problem (2.1)-(2.4) only the function spaces of solutions and test functions will be different.

Definition 37. *We define space $V_c(\Omega)$ as follows*

$$V_c(\Omega) = \left\{ \boldsymbol{u} \in \left(W^{1,2}_{\boldsymbol{n},\mathrm{div}}(\Omega) \right)^n ;\ \mathrm{Tr}(\boldsymbol{u})|_{\partial\Omega \setminus S_{u,\Omega}} = \boldsymbol{0} \right\}.$$

Remark. As for spaces V_2 and V_3, space $V_c(\Omega)$ is again a Hilbert space, the expression

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V_c} = \int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}}\, d\lambda^n$$

defines a scalar product on $V_c(\Omega)$ and the corresponding norm $\|\cdot\|_{V_c}$ is equivalent to the standard $\|\cdot\|_{W^{1,2}}$ norm.

Definition 38. *Let $\lambda \in \mathbb{R}$ be fixed. We say that $\boldsymbol{u} \in V_c(\Omega)$ is a weak solution to the problem (3.3)-(3.7) if*

$$\int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}}\, d\lambda^n = \lambda \int_{S_{u,\Omega}} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}})\, d\mathcal{H}^{n-1} \tag{3.8}$$

holds for all $\boldsymbol{\varphi} \in V_c(\Omega)$.

Remark. We will only be interested in non-trivial weak solutions hence by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ we immediately get that $\lambda \geq 0$. Moreover, for $\lambda = 0$ we obtain by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ that $\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_c} = 0$ and hence the only weak solution is the trivial one.

Remark. We will refer to the constant λ in the formulation of the problem (3.3)-(3.7) as an eigenvalue of the problem (3.3)-(3.7) if there exists a corresponding non-trivial weak solution to the problem. This convention will be used also for other problems.

We again define a mapping $B\colon V_c(\Omega)\times V_c(\Omega)\to\mathbb{C}$ by

$$B[\boldsymbol{u},\boldsymbol{v}]=\int_{S_{u,\Omega}}\operatorname{Tr}(\boldsymbol{u})\cdot\operatorname{Tr}(\overline{\boldsymbol{v}})\,d\mathcal{H}^{n-1}$$

and get the existence of a unique operator $T_c\in L(V_c(\Omega))$ satisfying

$$B[\boldsymbol{u},\boldsymbol{v}]=\langle T_c(\boldsymbol{u}),\boldsymbol{v}\rangle_{V_c} \tag{3.9}$$

for all $\boldsymbol{u},\boldsymbol{v}\in V_c(\Omega)$. Again, completely analogous version of Theorem 20 holds also in this case, however, we will formulate it nevertheless for future references.

Theorem 39. *Operator T_c is compact and self-adjoint. For some fixed $\lambda\in\mathbb{R},\lambda>0$, a non-trivial function $\boldsymbol{u}$ is a weak solution to the problem (3.3)-(3.7) if and only if $\boldsymbol{u}$ is an eigenfunction of T_c corresponding to eigenvalue $1/\lambda$. There exist at most countably many $\lambda\in\mathbb{R}$ such that there exist a non-trivial weak solution to the problem (3.3)-(3.7).*

Proposition 40. *The set of eigenvalues of the problem (3.3)-(3.7) is countably infinite.*

Proof. From Theorem 39 we know that there exist at most countably many eigenvalues of the problem (3.3)-(3.7). Suppose for a contradiction that there are only finitely many eigenvalues of the problem (3.3)-(3.7). Since T_c is compact and self-adjoint operator, we can use Hilbert-Schmidt Theorem (see [4], Theorem 6.11., p. 167) which gives that there exists an orthonormal basis W_c of $V_c(\Omega)$ consisting of eigenvectors of T_c. Since $\dim\left(V_c(\Omega)\right)=\infty$, we get that $\dim\left(\ker(T_c)\right)=\infty$ by our assumption and compactness of T_c. Let $\boldsymbol{u}\in\ker(T_c)$. Using (3.9) for $\boldsymbol{v}=\boldsymbol{u}$ we get that $B[\boldsymbol{u},\boldsymbol{u}]=0$ and hence $\operatorname{Tr}(\boldsymbol{u})=\boldsymbol{0}$ on $\partial\Omega$. Thus

$$\ker(T_c)\subset\left(W^{1,2}_{\boldsymbol{0},\mathrm{div}}(\Omega)\right)^n.$$

However, using Theorem 14 it is not hard to construct an infinite sequence $(\boldsymbol{u}_k)_{k=1}^{\infty}$ of linearly independent functions in $V_c(\Omega)$ such that

$$\boldsymbol{u}_k\in\left(\ker(T_c)\right)^{\perp}$$

for $k\in\mathbb{N}$. This is a contradiction since by our assumptions we have only finitely many linearly independent functions $\boldsymbol{v}$ satisfying $\boldsymbol{v}\in\left(\ker(T_c)\right)^{\perp}$.

□

Notation. Let Ω be a cylinder $\mathbb{R}^n$ and let Ω_1 be a cuboid in $\mathbb{R}^n$. When we refer to the operators T_n and T_c we always assume that they are defined on corresponding functions spaces, i.e. $V_n(\Omega_1)$ for T_n and $V_c(\Omega)$ for T_c. Moreover, for $m\in\{n,c\}$ denote $\sigma_p(T_m)=\left(\lambda'_{k,T_m}\right)_{k=1}^{\infty}$ and $1/\sigma_p(T_m)=\left(\lambda_{k,T_m}\right)_{k=1}^{\infty}$ where $\left(\lambda'_{k,T_m}\right)_{k=1}^{\infty}$ is sorted in a non-increasing order, $\left(\lambda_{k,T_m}\right)_{k=1}^{\infty}$ is sorted in a non-decreasing order and each eigenvalue in both sequences appear according to its multiplicity.

The following lemma will be crucial in proving the upper estimate for cylindrical domain.

Lemma 41. *Let $\Omega_1 \subset \Omega$ be an n-dimensional cuboid satisfying $S_{l,\Omega'} \subset S_{l,\Omega}$, $S_{u,\Omega'} \subset S_{u,\Omega}$ and $\overline{\Omega_1} \cap S_r = \emptyset$. Then there exists a linear operator $E\colon V_n(\Omega_1) \to V_c(\Omega)$ such that $E(\boldsymbol{u})|_{\Omega_1} = \boldsymbol{u}$. Moreover, there exists $C(\Omega, \Omega_1) \in \mathbb{R}, C(\Omega, \Omega_1) > 0$ depending on Ω and Ω_1 such that*

$$\int_{\Omega\setminus\overline{\Omega_1}} |\nabla E(\boldsymbol{u})|^2 \, d\lambda^n \leq C(\Omega, \Omega_1) \int_{\Omega_1} |\nabla \boldsymbol{u}|^2 \, d\lambda^n \tag{3.10}$$

for all $\boldsymbol{u} \in V_n(\Omega_1)$.

Remark. Estimate (3.10) implies that E is also bounded.

Proof. Choose arbitrary cylinders $\Omega_i \subset \mathbb{R}^n$ for $i \in \{2,3,4\}$ with the same axis as Ω that satisfy $\Omega_1 \subset \Omega_i \subset \Omega$, $\Omega_m \neq \Omega_{m+1}$ for $m \in \{1,2,3\}$ and $\Omega_4 \neq \Omega$. Let $\psi\colon \Omega \to \mathbb{R}$ be a cutoff function satisfying $\psi \in \mathcal{C}^\infty(\Omega)$, $\psi \in [0,1]$ on Ω, $\psi = 1$ on Ω_2 and $\psi = 0$ on $\Omega \setminus \Omega_3$, i.e. it also holds that $\nabla\psi = 0$ on $\Omega_2 \cup \left(\Omega \setminus \overline{\Omega_3}\right)$. Moreover, there exists $c \in \mathbb{R}, c > 0$ depending on Ω and Ω_1 such that $|\nabla\psi| \leq c$ on Ω. Let $\boldsymbol{u} \in V_n(\Omega_1)$ and denote $h \in \mathbb{R}, h > 0$ as the height of the cuboid. Since $\boldsymbol{u}$ is periodic on the sides of the cuboid, we can extend $\boldsymbol{u}$ periodically to $\mathbb{R}^{n-1} \times I$ where I is an appropriate interval of length h and then restrict $\boldsymbol{u}$ to Ω. Denote this periodic extension by $E^*(\boldsymbol{u})$. Then $E^*(\boldsymbol{u}) \in (W^{1,2}(\Omega))^n$. Due to periodicity, it still holds that $\operatorname{div} E^*(\boldsymbol{u}) = 0$ in Ω. It follows that

$$\operatorname{div}\left(E^*(\boldsymbol{u})\psi\right) = \underbrace{\operatorname{div}\left(E^*(\boldsymbol{u})\right)}_{=0} \psi + E^*(\boldsymbol{u}) \cdot \nabla\psi,$$

in Ω, i.e. $\operatorname{div}\left(E^*(\boldsymbol{u})\psi\right) = 0$ except from the set $\Omega_3 \setminus \Omega_2$. Furthermore, function $E^*(\boldsymbol{u})\psi$ has zero trace on $S_{r,\Omega} \cup S_{l,\Omega}$ by construction and also $(E^*(\boldsymbol{u})\psi) \cdot \boldsymbol{n} = 0$ in the trace sense on $S_{u,\Omega}$ since $E^*(\boldsymbol{u}) \cdot \boldsymbol{n} = 0$ in the trace sense on $S_{u,\Omega}$ by construction. Using this, the definition of ψ and Gauss's theorem (see Theorem 6) we obtain that

$$\int_{\Omega\setminus\overline{\Omega_1}} E^*(\boldsymbol{u}) \cdot \nabla\psi \, d\lambda^n = \int_\Omega \operatorname{div}\left(E^*(\boldsymbol{u})\psi\right) \, d\lambda^n = \int_{\partial\Omega} \left(E^*(\boldsymbol{u})\psi\right) \cdot \boldsymbol{n} \, d\mathcal{H}^{n-1} = 0,$$

hence $E^*(\boldsymbol{u}) \cdot \nabla\psi \in \tilde{L}^2(\Omega \setminus \overline{\Omega_1})$. Using remark after Theorem 14 we obtain a solution

$$\boldsymbol{h} = \operatorname{Bog}(E^*(\boldsymbol{u}) \cdot \nabla\psi),$$

$\boldsymbol{h} \in (W^{1,2}_{\boldsymbol{0}}(\Omega \setminus \overline{\Omega_1}))^n$ to the following problem

$$\begin{aligned} \operatorname{div} \boldsymbol{h} &= E^*(\boldsymbol{u}) \cdot \nabla\psi && \text{in } \Omega \setminus \overline{\Omega_1}, \\ \boldsymbol{h} &= 0 && \text{on } \partial\left(\Omega \setminus \overline{\Omega_1}\right) \end{aligned}$$

such that

$$\|\boldsymbol{h}\|_{(W^{1,2}_0(\Omega\setminus\overline{\Omega_1}))^n} \leq C\|E^*(\boldsymbol{u}) \cdot \nabla\psi\|_{L^2(\Omega)}, \tag{3.11}$$

for some constant $C \in \mathbb{R}$. Moreover, we extend $\boldsymbol{h}$ by zero on Ω_1 and we will further refer to $\boldsymbol{h}$ as this extended function defined on whole Ω. Define $E(\boldsymbol{u}) = E^*(\boldsymbol{u})\psi - \boldsymbol{h}$. Since operator Bog is linear, E is also linear. By construction it follows that $E(\boldsymbol{u}) \in \left(W^{1,2}(\Omega)\right)^n$, $\operatorname{div} E(\boldsymbol{u}) = 0$ in Ω, $\operatorname{Tr}\left(E(\boldsymbol{u})\right) = \boldsymbol{0}$ on $S_{r,\Omega} \cup S_{l,\Omega}$ and $\operatorname{Tr}\left(E(\boldsymbol{u})\right) \cdot \boldsymbol{n} = \boldsymbol{0}$ on $S_{u,\Omega}$ hence indeed $E(\boldsymbol{u}) \in V_c(\Omega)$.

For the moreover part of the Lemma, firstly, for any $\boldsymbol{u} \in V_n(\Omega_1)$ it holds that

$$|\nabla E(\boldsymbol{u})|^2 = \nabla \left(E^*(\boldsymbol{u})\psi - \boldsymbol{h}\right) : \overline{\nabla \left(E^*(\boldsymbol{u})\psi - \boldsymbol{h}\right)} = \\ \left(\psi \left(\nabla E^*(\boldsymbol{u})\right) + \left(\nabla \psi\right) E^*(\boldsymbol{u}) - \nabla \boldsymbol{h}\right) : \left(\overline{\psi \left(\nabla E^*(\boldsymbol{u})\right)} + \overline{\left(\nabla \psi\right) E^*(\boldsymbol{u})} - \overline{\nabla \boldsymbol{h}}\right).$$

Using linearity, Cauchy-Schwarz inequality and Young's inequality we eventually obtain that there exists $K \in \mathbb{R}, K > 0$ such that

$$|\nabla E(\boldsymbol{u})|^2 \leq K \left(|\psi \left(\nabla E^*(\boldsymbol{u})\right)|^2 + |\left(\nabla \psi\right) E^*(\boldsymbol{u})|^2 + |\nabla \boldsymbol{h}|^2\right)$$

holds in Ω. Using the assumptions on ψ we get

$$|\nabla E(\boldsymbol{u})|^2 \leq K \left(|\nabla E^*(\boldsymbol{u})|^2 + c^2|E^*(\boldsymbol{u})|^2 + |\nabla \boldsymbol{h}|^2\right).$$

Next, using (3.11) we obtain

$$\int_{\Omega \setminus \overline{\Omega_1}} |\nabla E(\boldsymbol{u})|^2 \, d\lambda^n \leq K \left(\int_{\Omega \setminus \overline{\Omega_1}} |\nabla E^*(\boldsymbol{u})|^2 \, d\lambda^n + \int_{\Omega \setminus \overline{\Omega_1}} |E^*(\boldsymbol{u})|^2 \, d\lambda^n\right) \tag{3.12}$$

for a different constant $K \in \mathbb{R}, K > 0$ now depending on Ω and Ω_1. Since Ω is bounded and $E^*(\boldsymbol{u})$ is a periodic extension of $\boldsymbol{u}$ to Ω, it clearly holds that there exists constant $\tilde{K} \in \mathbb{R}, \tilde{K} > 0$ depending on Ω and Ω_1 such that

$$\int_{\Omega \setminus \overline{\Omega_1}} |\nabla E^*(\boldsymbol{u})|^2 \, d\lambda^n \leq \tilde{K} \int_{\Omega_1} |\nabla \boldsymbol{u}|^2 \, d\lambda^n. \tag{3.13}$$

Next, by construction we know that $\operatorname{Tr}\left(E^*(\boldsymbol{u})\right)|_{S_{l,\Omega} \setminus \left(S_{l,\Omega} \cap \partial \Omega_1\right)} = 0$, hence we can use Poincaré's inequality (see [11], Section 5.8.1., Theorem 1 for $\mathcal{C}^1$ domains, however, can be easily extended to Lipschitz domains by using Theorem 6.1., p. 102 from [5]) to get

$$\int_{\Omega \setminus \overline{\Omega_1}} |E^*(\boldsymbol{u})|^2 \, d\lambda^n \leq D \int_{\Omega \setminus \overline{\Omega_1}} |\nabla E^*(\boldsymbol{u})|^2 \, d\lambda^n \tag{3.14}$$

for some constant $D \in \mathbb{R}, D > 0$ depending on Ω and Ω_1 and for the right-hand side in (3.14) we can now use (3.13). Using (3.13) and (3.14) for the right-hand side in (3.12) we indeed obtain that there exists constant $C(\Omega, \Omega_1) \in \mathbb{R}, C(\Omega, \Omega_1) > 0$ such that (3.10) holds for all $\boldsymbol{u} \in V_n(\Omega_1)$.

□

Corollary 42. *Let $k \in \mathbb{N}, k \geq 1$. A finitely generated subspace $W_k = \operatorname{span}\{\boldsymbol{u}_1,...,\boldsymbol{u}_k\}$ of space $V_n(\Omega_1)$ has dimension k if and only if subspace $W_k' = \operatorname{span}\{E(\boldsymbol{u}_1),...,E(\boldsymbol{u}_k)\}$ of space $V_c(\Omega)$ has dimension k.*

Proof. Firstly, case $k = 1$ is trivial. Assume now that $k \geq 2$.
" $\Longrightarrow$ " By construction we know that for any $\boldsymbol{u} \in V_n(\Omega_1)$ it holds that $E(\boldsymbol{u})|_{\Omega_1} = \boldsymbol{u}$ hence the linear independence of $(\boldsymbol{u}_1,...,\boldsymbol{u}_k)$ in $V_n(\Omega_1)$ will be preserved for $(E(\boldsymbol{u}_1),...,E(\boldsymbol{u}_k))$ in $V_c(\Omega)$.

" $\Longleftarrow$ " Suppose for a contradiction that $(E(\boldsymbol{u}_1),...,E(\boldsymbol{u}_k))$ is linearly independent in $V_c(\Omega)$ and $(\boldsymbol{u}_1,...,\boldsymbol{u}_k)$ is linearly dependent in $V_n(\Omega_1)$. Without loss of generality, assume that $\boldsymbol{u}_k$ can be written as a linear combination of $(\boldsymbol{u}_1,...,\boldsymbol{u}_{k-1})$. By construction we know that if for $\boldsymbol{u}, \boldsymbol{v} \in V_n(\Omega_1)$ holds that $\boldsymbol{u} = \boldsymbol{v}$ in $V_n(\Omega_1)$ then $E(\boldsymbol{u}) = E(\boldsymbol{v})$ in $V_c(\Omega)$. Since E is linear, $E(\boldsymbol{u}_k)$ can be written as a linear combination of $(E(\boldsymbol{u}_1),...,E(\boldsymbol{u}_{k-1}))$ which is a contradiction.

□

Proposition 43. *Let Ω be a cylinder in $\mathbb{R}^n$ where $n \in \{2,3\}$. Let Ω_1 be an n-dimensional cuboid satisfying the same assumptions as in Lemma 41 and suppose moreover that $\lambda^n(\Omega_1) = \lambda^n(\Omega)/4$. Then there exists a constant $C(\Omega) \in \mathbb{R}, C(\Omega) > 0$ depending on Ω such that*

$$\lambda_{k,T_c} \le C(\Omega)\lambda_{k,T_n} \tag{3.15}$$

for all $k \in \mathbb{N}$.

Remark. The condition $\lambda^n(\Omega_1) = \lambda^n(\Omega)/4$ is introduced in order to relate sets Ω_1 and Ω so that constants from Lemma 41 and λ_{k,T_n} depend only on the set Ω.

Proof. Combining Theorem 35, Theorem 39 and (3.1) we get for $k \in \mathbb{N}$ and $m \in \{n,c\}$ that

$$\lambda'_{k,T_m} = \max_{D \in (V_m)_k} \min_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle T_m(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_m}}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_m}}.$$

Using (3.2) it also holds

$$\lambda_{k,T_m} = \min_{D \in (V_m)_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_m}}{\langle T_m(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_m}}. \tag{3.16}$$

Using the embedding of spaces from Lemma 41 and definitions we get

$$\begin{aligned}
\lambda_{k,T_c} &= \min_{D \in (V_c(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_c}}{\langle T_c(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_c}} \overset{\text{(L41)}}{\le} \min_{D \in (E(V_n(\Omega_1)))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_c}}{\langle T_c(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_c}} \\
&\overset{\text{(C42)}}{=} \min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle E(\boldsymbol{u}), E(\boldsymbol{u}) \rangle_{V_c}}{\langle T_c(E(\boldsymbol{u})), E(\boldsymbol{u}) \rangle_{V_c}} \\
&= \min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega |\nabla E(\boldsymbol{u})|^2 \, d\lambda^n}{\int_{S_{u,\Omega}} |\mathrm{Tr}(E(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1}} \\
&\overset{\left(S'_{u,\Omega_1} \subset S_{u,\Omega}\right)}{\le} \min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega |\nabla E(\boldsymbol{u})|^2 \, d\lambda^n}{\int_{S'_{u,\Omega_1}} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&= \min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\Omega_1} |\nabla \boldsymbol{u}|^2 \, d\lambda^n + \int_{\Omega \setminus \overline{\Omega_1}} |\nabla E(\boldsymbol{u})|^2 \, d\lambda^n}{\int_{S'_{u,\Omega_1}} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}}.
\end{aligned} \tag{3.17}$$

Combining (3.10) and (3.17) we obtain that

$$\lambda_{k,T_c} \le C(\Omega) \min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\Omega_1} |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{S'_{u,\Omega_1}} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}}$$

thus using (3.16) we get

$$\min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\Omega_1} |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{S'_{u,\Omega_1}} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} = \min_{D \in (V_n(\Omega_1))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_n}}{\langle T_n(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_n}} = \lambda_{k,T_n}$$

hence we indeed obtain

$$\lambda_{k,T_c} \le C(\Omega)\lambda_{k,T_n}$$

for all $k \in \mathbb{N}$.

□

Remark. If minimum (maximum) somewhere throughout the calculations in Proposition 43 does not exist we replace it with infimum (supremum), however, we leave the notation like this since in the formulas for λ_{k,T_c} and λ_{k,T_n} we know that minimum and maximum exist by Theorem 35. We will use this convention throughout the rest of the thesis.

3.2 Generalization to $\mathcal{C}^2$ domain

In this Section we are going to prove upper estimate for eigenvalues of the problem (2.1)-(2.4) for $n \in \{2,3\}$ on any bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$ using the results obtained in Section 3.1. Throughout this Section, we will assume that $n \in \{2,3\}$ and that Ω denotes a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$.

Proposition 44. *The set Λ of eigenvalues of the problem (2.1)-(2.4) is countably infinite.*

Proof. By Theorem 13 we know that there exist at most countably many eigenvalues of the problem (2.1)-(2.4). The rest of the proof would be almost identical to the proof of Proposition 40.

□

Notation. If we want to emphasize on which set we consider the space V, we will write $V(\Omega)$ instead of V. When we refer to the operator T we always assume that T is defined on the corresponding function space, i.e. $V(\Omega)$. Furthermore, we denote $\sigma_p(T) = \left(\lambda'_{k,T}\right)_{k=1}^{\infty}$ and $1/\sigma_p(T) - 1 = (\lambda_{k,T})_{k=1}^{\infty}$ where $\left(\lambda'_{k,T}\right)_{k=1}^{\infty}$ is sorted in a non-increasing order, $(\lambda_{k,T})_{k=1}^{\infty}$ is sorted in a non-decreasing order and each eigenvalue in both sequences appear according to its multiplicity.

Proposition 45. *Let $k \in \mathbb{N}$ and suppose that $\lambda_{k,T} > 0$. Then*

$$\lambda_{k,T} = \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega \boldsymbol{Du} : \overline{\boldsymbol{Du}} \, d\lambda^n}{\int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{u}}) \, d\mathcal{H}^{n-1}}. \tag{3.18}$$

Proof. Using Theorem 35 we have

$$\lambda'_{k,T} = \max_{D \in (V(\Omega))_k} \min_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle T(\boldsymbol{u}), \boldsymbol{u} \rangle_V}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_V}$$

and hence

$$\frac{1}{\lambda'_{k,T}} = \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_V}{\langle T(\boldsymbol{u}), \boldsymbol{u} \rangle_V}. \tag{3.19}$$

For any $\boldsymbol{u} \in V(\Omega)$ we have

$$\frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_V}{\langle T(\boldsymbol{u}), \boldsymbol{u} \rangle_V} = \frac{\int_\Omega \boldsymbol{Du} : \overline{\boldsymbol{Du}} \, d\lambda^n}{\int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{u}}) \, d\mathcal{H}^{n-1}} + 1. \tag{3.20}$$

Using (3.20) in (3.19) and Lemma 12 we get

$$\frac{1}{\lambda'_{k,T}} - 1 = \lambda_{k,T} = \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega \boldsymbol{Du} : \overline{\boldsymbol{Du}} \, d\lambda^n}{\int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{u}}) \, d\mathcal{H}^{n-1}}.$$

□

The first step towards proving the upper estimate is to reduce the problem to a different problem on an open subset Ω_c of Ω that we will be able to handle. Even though the construction of the subset Ω_c is intuitively clear, it would be quite long and technically difficult to describe it purely mathematically. Therefore, we will demonstrate the construction on a picture. Let $x \in \partial\Omega$. Since $\Omega \in \mathcal{C}^2$ there exists $\Gamma_c \subset \partial\Omega$ such that $x \in \Gamma_c$ and Γ_c is "small" enough so that we can construct a "curved cylinder" $\Omega_c \subset \Omega$ with the upper part of the boundary being Γ_c (see Picture 3.1). For $\boldsymbol{u}\colon \overline{\Omega_c} \to \mathbb{R}^n$, $p\colon \Omega_c \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we now consider

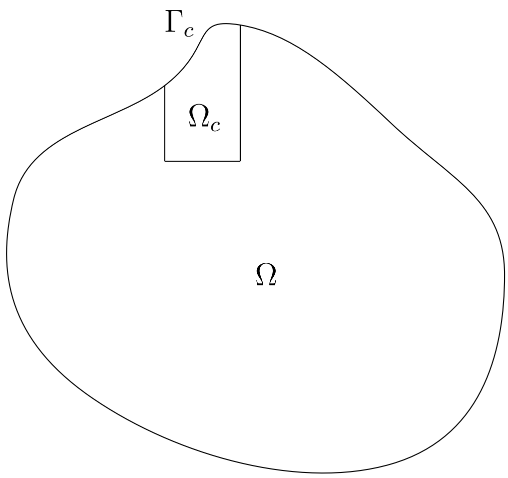

Figure 3.1: Construction of set Ω_c

the following problem

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= 0 && \text{in } \Omega_c, && (3.21)\\
\operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega_c, && (3.22)\\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_c, && (3.23)\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega_c \setminus \Gamma_c, && (3.24)\\
[(\nabla \boldsymbol{u})\, \boldsymbol{n}]_\tau &= \lambda \boldsymbol{u} && \text{on } \Gamma_c. && (3.25)
\end{aligned}
$$

Completely analogous theoretical results that we obtained for the previous auxiliary problems are true also for this setting. We will formulate them without proofs for future reference.

Definition 46. *We define space $V_{\text{cur}}(\Omega_c)$ as follows*

$$V_{\text{cur}}(\Omega_c) = \left\{ \boldsymbol{u} \in \left(W^{1,2}_{\boldsymbol{n},\text{div}}(\Omega_c) \right)^n ;\ \operatorname{Tr}(\boldsymbol{u})|_{\partial\Omega_c \setminus \Gamma_c} = \boldsymbol{0} \right\}.$$

Remark. As for spaces V_2, V_3 and V_c, space $V_{\text{cur}}(\Omega_c)$ is again a Hilbert space, the expression

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V_{\text{cur}}} = \int_{\Omega_c} \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}}\, d\lambda^n$$

defines a scalar product on $V_{\text{cur}}(\Omega_c)$ and the corresponding norm $\|\cdot\|_{V_{\text{cur}}}$ is equivalent to the standard $\|\cdot\|_{W^{1,2}}$ norm.

Definition 47. *Let $\lambda \in \mathbb{R}$ be fixed. We say that $\boldsymbol{u} \in V_{\mathrm{cur}}(\Omega_c)$ is a weak solution to the problem (3.21)-(3.25) if*

$$\int_{\Omega_c} \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}} \, d\lambda^n = \lambda \int_{\Gamma_c} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^{n-1} \tag{3.26}$$

holds for all $\boldsymbol{\varphi} \in V_{\mathrm{cur}}(\Omega_c)$.

Remark. We will only be interested in non-trivial weak solutions hence by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ we immediately get that $\lambda \geq 0$. Moreover, for $\lambda = 0$ we obtain by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ that $\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_{\mathrm{cur}}} = 0$ and hence the only weak solution is the trivial one.

We again define a mapping $B \colon V_{\mathrm{cur}}(\Omega_c) \times V_{\mathrm{cur}}(\Omega_c) \to \mathbb{C}$ by

$$B[\boldsymbol{u},\boldsymbol{v}] = \int_{\Gamma_c} \mathrm{Tr}(\boldsymbol{u}) \cdot \mathrm{Tr}(\overline{\boldsymbol{v}}) \, d\mathcal{H}^{n-1}$$

and get the existence of a unique operator $T_{\mathrm{cur}} \in L(V_{\mathrm{cur}}(\Omega_c))$ satisfying

$$B[\boldsymbol{u},\boldsymbol{v}] = \langle T_{\mathrm{cur}}(\boldsymbol{u}), \boldsymbol{v} \rangle_{V_{\mathrm{cur}}}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V_{\mathrm{cur}}(\Omega_c)$.

Theorem 48. *Operator T_{cur} is compact and self-adjoint. For some fixed $\lambda \in \mathbb{R}, \lambda > 0$, a non-trivial function $\boldsymbol{u}$ is a weak solution to the problem (3.21)-(3.25) if and only if $\boldsymbol{u}$ is an eigenfunction of T_{cur} corresponding to eigenvalue $1/\lambda$. The set of eigenvalues of the problem (3.21)-(3.25) is countably infinite.*

Notation. When we refer to the operator T_{cur} we always assume that T_{cur} is defined on the corresponding function space, i.e. $V_{\mathrm{cur}}(\Omega_c)$. Furthermore, we denote $\sigma_p(T_{\mathrm{cur}}) = \left(\lambda'_{k,T_{\mathrm{cur}}}\right)_{k=1}^{\infty}$ and $1/\sigma_p(T_{\mathrm{cur}}) = (\lambda_{k,T_{\mathrm{cur}}})_{k=1}^{\infty}$ where $\left(\lambda'_{k,T_{\mathrm{cur}}}\right)_{k=1}^{\infty}$ is sorted in a non-increasing order, $(\lambda_{k,T_{\mathrm{cur}}})_{k=1}^{\infty}$ is sorted in a non-decreasing order and each eigenvalue in both sequences appear according to its multiplicity.

The advantage of having the condition that for any $\boldsymbol{u} \in V_{\mathrm{cur}}(\Omega_c)$ it holds $\mathrm{Tr}(\boldsymbol{u})|_{\partial\Omega_c \setminus \Gamma_c} = \boldsymbol{0}$ is that we can extend $\boldsymbol{u}$ by zero to the whole set Ω.

Notation. For any $\boldsymbol{u} \in V_{\mathrm{cur}}(\Omega_c)$ we denote by $E_0(\boldsymbol{u})$ the extension of $\boldsymbol{u}$ by zero to the whole set Ω.

Remark. It clearly holds that for any $\boldsymbol{u} \in V_{\mathrm{cur}}(\Omega_c)$ we have $E_0(\boldsymbol{u}) \in V(\Omega)$.

An analogous version of Corollary 42 holds also for this situation.

Lemma 49. *Let $k \in \mathbb{N}, k \geq 1$. A finitely generated subspace $W_k = \mathrm{span}\{\boldsymbol{u}_1,...,\boldsymbol{u}_k\}$ of space $V_{\mathrm{cur}}(\Omega_c)$ has dimension k if and only if subspace $W'_k = \mathrm{span}\{E_0(\boldsymbol{u}_1),...,E_0(\boldsymbol{u}_k)\}$ of space $V(\Omega)$ has dimension k.*

This allows us to formulate the first estimate where we will use the results and notation introduced after Proposition 45.

Proposition 50. *Let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$ where $n \in \{2,3\}$. Let $x \in \partial\Omega$ and let Ω_c be the corresponding constructed set. Then there exists a constant $C \in \mathbb{R}, C > 0$ such that*

$$\lambda_{k,T} \leq C \lambda_{k,T_{\mathrm{cur}}} \tag{3.27}$$

for all $k \in \mathbb{N}$.

Remark. As we shall see later in this Section, we will often need Ω_c and Γ_c to be chosen "sufficiently small" so that some estimates or claims hold. Since it would be inconvenient to always specify that we will rather use a convention that sets Ω_c and Γ_c are always chosen in such a way that the required estimates and claims hold. Moreover, we will always choose Γ_c with the largest possible Hausdorff measure and then choose the set Ω_c with the largest possible Lebesgue measure so that the mentioned convention holds. This will ensure that $\lambda_{k,T_{\text{cur}}}$ depends merely on Ω and not Ω_c.

Proof. Let $k \in \mathbb{N}$. Combining Theorem 35, Theorem 48 and (3.2) we get that

$$\lambda_{k,T_{\text{cur}}} = \min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_{\text{cur}}}}{\langle T_{\text{cur}}(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_{\text{cur}}}}. \tag{3.28}$$

Using (3.18), the definition of E_0 and the fact that $E_0(V_{\text{cur}}(\Omega_c)) \subset V(\Omega)$ we obtain

$$\begin{aligned}
\lambda_{k,T} = \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} & \frac{\int_\Omega |\boldsymbol{D}\boldsymbol{u}|^2 \, d\lambda^n}{\int_{\partial\Omega} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&\le \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{C \int_\Omega |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{\partial\Omega} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&\le \min_{D \in (E_0(V_{\text{cur}}(\Omega_c)))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{C \int_\Omega |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{\partial\Omega} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&\overset{\text{(L49)}}{=} \min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{C \int_\Omega |\nabla E_0(\boldsymbol{u})|^2 \, d\lambda^n}{\int_{\partial\Omega} |\mathrm{Tr}(E_0(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1}} \\
&\overset{\text{(def. } E_0)}{=} \min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{C \int_{\Omega_c} |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{\Gamma_c} |\mathrm{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&= C \min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{V_{\text{cur}}}}{\langle T_{\text{cur}}(\boldsymbol{u}), \boldsymbol{u} \rangle_{V_{\text{cur}}}}.
\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29) we indeed obtain (3.27) for all $k \in \mathbb{N}$.

$\square$

Proposition 50 tell us that in order to obtain upper estimate for eigenvalues of the problem (2.1)-(2.4) on Ω it is enough to obtain upper estimate for eigenvalues of the problem (3.21)-(3.25) on Ω_c. We will essentially do this by using an appropriate transformation of variables and then deriving estimates in a suitable form so that we can use the results from Section 3.1. We will further refer to the notation introduced earlier in this Section.

Notation. Let $A = (a_{ij})$ be $n \times n$ matrix. We denote

$$\mathrm{Trace}(A) = \sum_{i=1}^{n} a_{i,i}.$$

Notation. For $n \in \mathbb{N}$ we denote by $\boldsymbol{e}_n$ the n-th canonical vector in $\mathbb{R}^n$. Moreover, for $\boldsymbol{x} \in \mathbb{R}^n$ we denote $\boldsymbol{x} = (\boldsymbol{x}_0, x_n)$ where $\boldsymbol{x}_0 \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Let Ω be fixed. Suppose that $\boldsymbol{y} = (\boldsymbol{y}', y_n) \in \partial\Omega$ is such that $\boldsymbol{n}(\boldsymbol{y}) = \boldsymbol{e}_n$. Let $r \in \mathbb{R}, r > 0$ and suppose that $B(\boldsymbol{y}', r) \subset \mathbb{R}^{n-1}$. Let $a \colon B(\boldsymbol{y}', r) \to \mathbb{R}$ be such that $a \in \mathcal{C}^2\left(\overline{B(\boldsymbol{y}',r)}\right)$ and suppose that $\Gamma_c = \{(\boldsymbol{y}_0, y_n) \in \mathbb{R}^n, \boldsymbol{y}_0 \in B(\boldsymbol{y}', r);\ y_n = a(\boldsymbol{y}_0)\}$. Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^n$ such that $\overline{\mathcal{O}} \subset \overline{B(\boldsymbol{y}', r)} \times (-\infty, 0]$ and that function $\boldsymbol{\phi}\colon \overline{\mathcal{O}} \to \mathbb{R}^n$ defined by

$$\boldsymbol{\phi}(\boldsymbol{y}_0, y_n) = \begin{pmatrix} \boldsymbol{y}_0 \\ y_n + a(\boldsymbol{y}_0) \end{pmatrix} \tag{3.30}$$

satisfies $\boldsymbol{\phi}(\mathcal{O}) = \Omega_c$.

Notation. We denote by Σ_o the set $B(\boldsymbol{y}', r) \times \{0\}$.

It follows that $\boldsymbol{\phi}(\Sigma_o) = \Gamma_c$, $\boldsymbol{\phi}(\partial\mathcal{O} \setminus \Sigma_o) = \partial\Omega_c \setminus \Gamma_c$ and the inverse function $\boldsymbol{\phi}^{-1}\colon \overline{\Omega_c} \to \overline{\mathcal{O}}$ for $(\boldsymbol{x}_0, x_n) \in \overline{\Omega_c}$ has the form

$$\boldsymbol{\phi}^{-1}(\boldsymbol{x}_0, x_n) = \begin{pmatrix} \boldsymbol{x}_0 \\ x_n - a(\boldsymbol{x}_0) \end{pmatrix}.$$

Furthermore, the following identities hold for $\boldsymbol{y} \in \overline{\mathcal{O}}$

$$\nabla\boldsymbol{\phi}(\boldsymbol{y}) = \begin{pmatrix} \mathrm{Id}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} \\ \nabla a(\boldsymbol{y}_0) & 1 \end{pmatrix} \tag{3.31}$$

where $\mathrm{Id}_{(n-1)\times(n-1)}$ stands for a $(n-1)\times(n-1)$ identity matrix and $\mathbf{0}_{(n-1)\times 1}$ stands for a column of $(n-1)$ zeros. Consequently we obtain

$$(\nabla\boldsymbol{\phi}(\boldsymbol{y}))^\top = \begin{pmatrix} \mathrm{Id}_{(n-1)\times(n-1)} & (\nabla a(\boldsymbol{y}_0))^\top \\ \mathbf{0}_{1\times(n-1)} & 1 \end{pmatrix} \tag{3.32}$$

and

$$(\nabla\boldsymbol{\phi}(\boldsymbol{y}))^{-1} = \begin{pmatrix} \mathrm{Id}_{(n-1)\times(n-1)} & \mathbf{0}_{(n-1)\times 1} \\ -\nabla a(\boldsymbol{y}_0) & 1 \end{pmatrix}. \tag{3.33}$$

Remark. From (3.31) follows that for any $\boldsymbol{y} \in \mathcal{O}$ it holds

$$\det(\nabla\boldsymbol{\phi}(\boldsymbol{y})) = 1. \tag{3.34}$$

Since $\boldsymbol{\phi}$ is a bijection it follows that $\boldsymbol{\phi}$ is a diffeomorphism.

Definition 51. *For $\boldsymbol{u} \in (W^{1,2}(\Omega_c))^n$ we define a corresponding function $\boldsymbol{v}\colon \mathcal{O} \to \mathbb{R}^n$ by*

$$\boldsymbol{v}(\boldsymbol{y}) = (\nabla\boldsymbol{\phi}(\boldsymbol{y}))^{-1}\, \boldsymbol{u} \circ \boldsymbol{\phi}(\boldsymbol{y}) \tag{3.35}$$

for all $\boldsymbol{y} \in \mathcal{O}$. We denote $F(\boldsymbol{u}) = \boldsymbol{v}$.

The main advantage why we introduce this definition is covered in the following Lemma.

Lemma 52. *Let* $\boldsymbol{u} \in (W^{1,2}(\Omega_c))^n$. *Then the corresponding function* $\boldsymbol{v}$ *defined by (3.35) lies in* $(W^{1,2}(\mathcal{O}))^n$. *For any* $\boldsymbol{y} \in \Sigma_o$ *it holds*

$$\boldsymbol{u} \circ \boldsymbol{\phi}(\boldsymbol{y}) \cdot \boldsymbol{n} \circ \boldsymbol{\phi}(\boldsymbol{y}) = c(\boldsymbol{y})\, \boldsymbol{v}(\boldsymbol{y}) \cdot \boldsymbol{e}_n \tag{3.36}$$

in the trace sense for some non-zero function $c \colon B(\boldsymbol{y}', r) \to \mathbb{R}$. *For any* $\boldsymbol{x} \in \Omega_c$ *and* $\boldsymbol{y} = \boldsymbol{\phi}^{-1}(\boldsymbol{x})$ *it holds*

$$\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) = \operatorname{Trace}\left(\nabla\boldsymbol{\phi}(\boldsymbol{y})\nabla\boldsymbol{v}(\boldsymbol{y})\,(\nabla\boldsymbol{\phi}(\boldsymbol{y}))^{-1}\right). \tag{3.37}$$

Proof. The first property is clear by construction. Concerning (3.36), for any $\boldsymbol{y} \in \Sigma_o$ we have

$$\begin{aligned}\boldsymbol{u} \circ \boldsymbol{\phi}(\boldsymbol{y}) \cdot \boldsymbol{n} \circ \boldsymbol{\phi}(\boldsymbol{y}) &\overset{(3.35)}{=} \nabla\boldsymbol{\phi}(\boldsymbol{y})\boldsymbol{v}(\boldsymbol{y}) \cdot \boldsymbol{n} \circ \boldsymbol{\phi}(\boldsymbol{y}) \\ &= (\nabla\boldsymbol{\phi}(\boldsymbol{y})\boldsymbol{v}(\boldsymbol{y}))^\top \boldsymbol{n} \circ \boldsymbol{\phi}(\boldsymbol{y}) = \boldsymbol{v}(\boldsymbol{y})^\top (\nabla\boldsymbol{\phi}(\boldsymbol{y}))^\top \boldsymbol{n} \circ \boldsymbol{\phi}(\boldsymbol{y})\end{aligned} \tag{3.38}$$

in the trace sense. The first $(n-1)$ columns of the matrix $\nabla\boldsymbol{\phi}(\boldsymbol{y})$ span the tangent space at point $\boldsymbol{\phi}(\boldsymbol{y})$ and thus combining (3.32) and (3.38) we obtain

$$\boldsymbol{v}(\boldsymbol{y})^\top (\nabla\boldsymbol{\phi}(\boldsymbol{y}))^\top \boldsymbol{n} \circ \boldsymbol{\phi}(\boldsymbol{y}) = c(\boldsymbol{y})\, \boldsymbol{v}(\boldsymbol{y}) \cdot \boldsymbol{e}_n \tag{3.39}$$

where $c(\boldsymbol{y}) = \boldsymbol{n}_n(\boldsymbol{y})$ and thus for sufficiently small r function c is non-zero on $B(\boldsymbol{y}', r)$. Thus (3.36) follows. Concerning (3.37), we get by using (3.35) that

$$\boldsymbol{u}(\boldsymbol{x}) = \nabla\boldsymbol{\phi} \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, \boldsymbol{v} \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x}) = (\nabla\boldsymbol{\phi}\, \boldsymbol{v}) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x}).$$

for all $\boldsymbol{x} \in \Omega_c$. Thus for $j \in \mathbb{N}, j \leq n$ the j-th coordinate is

$$\boldsymbol{u}_j(\boldsymbol{x}) = \sum_{i=1}^{n} ((\partial_i\boldsymbol{\phi}_j)\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})$$

Hence for all $\boldsymbol{x} \in \Omega_c$ we have

$$\begin{aligned}\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) &= \sum_{j=1}^{n} \partial_j \boldsymbol{u}_j(\boldsymbol{x}) = \sum_{j=1}^{n} \partial_j \left(\sum_{i=1}^{n} ((\partial_i\boldsymbol{\phi}_j)\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\right) \\ &= \sum_{i,j=1}^{n} \sum_{k=1}^{n} \partial_k ((\partial_i\boldsymbol{\phi}_j)\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, \partial_j \boldsymbol{\phi}_k^{-1}(\boldsymbol{x}) \\ &= \sum_{i,j=1}^{n} \sum_{k=1}^{n} ((\partial_i\boldsymbol{\phi}_j)\,(\partial_k\boldsymbol{v}_i)) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, \partial_j \boldsymbol{\phi}_k^{-1}(\boldsymbol{x})\end{aligned} \tag{3.40}$$

since by using (3.31) we know that for $i,j \in \mathbb{N}, j \leq n-1, i \leq n$ it holds $\partial_i \boldsymbol{\phi}_j = l$ where $l \in \{0,1\}$ and thus for $k \in \mathbb{N}, k \leq n$ it holds $\partial_k\partial_i\boldsymbol{\phi}_j = 0$. Moreover, for $j = n$, $i \in \mathbb{N}, i \leq n$ and $\boldsymbol{x} \in \Omega_c$ we have

$$\begin{aligned}\partial_n \left(((\partial_i\boldsymbol{\phi}_n)\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}\right)(\boldsymbol{x}) &= \sum_{k=1}^{n} \partial_k ((\partial_i\boldsymbol{\phi}_n)\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, \partial_n \boldsymbol{\phi}_k^{-1}(\boldsymbol{x}) \\ = \sum_{k=1}^{n-1} ((\partial_k\partial_i\boldsymbol{\phi}_j)\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, &\underbrace{\partial_n \boldsymbol{\phi}_k^{-1}(\boldsymbol{x})}_{=0} + (\underbrace{(\partial_n\partial_i\boldsymbol{\phi}_n)}_{=0}\, \boldsymbol{v}_i) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, \partial_n \boldsymbol{\phi}_n^{-1}(\boldsymbol{x}) \\ &+ \sum_{k=1}^{n} ((\partial_i\boldsymbol{\phi}_n)\,(\partial_k\boldsymbol{v}_i)) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\, \partial_n \boldsymbol{\phi}_k^{-1}(\boldsymbol{x})\end{aligned}$$

since function a depends only on the first $(n-1)$ variables and thus $\partial_n \partial_i \boldsymbol{\phi}_n = 0$. Formula (3.40) for $\boldsymbol{x} \in \Omega_c$ can be rewritten as

$$\begin{aligned}\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) &= \sum_{i,j,k=1}^{n} \left((\partial_i \boldsymbol{\phi}_j) (\partial_k \boldsymbol{v}_i) \right) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x}) \, \partial_j \boldsymbol{\phi}_k^{-1}(\boldsymbol{x}) \\ &= \operatorname{Trace} \left((\nabla \boldsymbol{\phi} \nabla \boldsymbol{v}) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x}) \, \nabla \boldsymbol{\phi}^{-1}(\boldsymbol{x}) \right) = \operatorname{Trace} \left(\nabla \boldsymbol{\phi} \nabla \boldsymbol{v} \, (\nabla \boldsymbol{\phi})^{-1} \right) \circ \boldsymbol{\phi}^{-1}(\boldsymbol{x})\end{aligned}$$

which implies (3.37).

□

Remark. If $\nabla \boldsymbol{\phi}(\boldsymbol{y}) = \mathrm{Id}_{n\times n}$ for all $y \in \mathcal{O}$, formula (3.37) would reduce to the form $\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) = \operatorname{div} \boldsymbol{v}(\boldsymbol{y})$.

From Lemma 52 we know that for any $\boldsymbol{u} \in V_{\mathrm{cur}(\Omega_c)}$ condition $\mathrm{Tr}(\boldsymbol{u})|_{\Gamma_c} \cdot \boldsymbol{n} = 0$ transforms to condition $\mathrm{Tr}(F(\boldsymbol{u}))|_{\Sigma_o} \cdot \boldsymbol{e}_n = 0$. By construction it also follows that condition $\mathrm{Tr}(\boldsymbol{u})|_{\partial\Omega_c \setminus \Gamma_c} = \boldsymbol{0}$ transforms to condition $\mathrm{Tr}(F(\boldsymbol{u}))|_{\partial\mathcal{O} \setminus \Sigma_o} = \boldsymbol{0}$ and vice versa. However, since $\nabla \boldsymbol{\phi} \neq \mathrm{Id}_{n\times n}$, the divergence free condition is thus the only condition that is not preserved and that is a crucial thing that we have to deal with.

The first step towards obtaining the desired upper estimate will be contained in the following Proposition.

Definition 53. *We define space $V^*_{\mathrm{cur}}(\mathcal{O})$ as follows*

$$\begin{aligned}V^*_{\mathrm{cur}}(\mathcal{O}) = \{\boldsymbol{v} \in \left(W^{1,2}_{\boldsymbol{n}}(\mathcal{O}) \right)^n ; \ &\mathrm{Tr}(\boldsymbol{v})|_{\partial\mathcal{O} \setminus \Sigma_o} = \boldsymbol{0}, \\ &\operatorname{Trace} \left(\nabla \boldsymbol{\phi}(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) \, (\nabla \boldsymbol{\phi}(\boldsymbol{y}))^{-1} \right) = 0\}.\end{aligned}$$

Remark. As for all the previous considered spaces, $V^*_{\mathrm{cur}}(\mathcal{O})$ is again a Hilbert space, the expression

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V^*_{\mathrm{cur}}} = \int_{\mathcal{O}} \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}} \, d\lambda^n$$

defines a scalar product on $V^*_{\mathrm{cur}}(\mathcal{O})$ and the corresponding norm $\|\cdot\|_{V^*_{\mathrm{cur}}}$ is equivalent to the standard $\|\cdot\|_{W^{1,2}}$ norm.

Corollary 54. *Lemma 52 implies that $F\left(V_{\mathrm{cur}}(\Omega_c)\right) = V^*_{\mathrm{cur}}(\mathcal{O})$. Moreover, F is a bijection between $V_{\mathrm{cur}}(\Omega_c)$ and $V^*_{\mathrm{cur}}(\mathcal{O})$.*

Notation. We denote by $\boldsymbol{\phi}_0$ the mapping $\boldsymbol{\phi}_0 \colon B(\boldsymbol{y}', r) \to \Gamma_c$ defined by $\boldsymbol{\phi}_0(\boldsymbol{y}_0) = \boldsymbol{\phi}(\boldsymbol{y}_0, 0)$.

Notation. For $\boldsymbol{y}_0 \in B(\boldsymbol{y}', r)$ we denote

$$\operatorname{vol}\left(\nabla \boldsymbol{\phi}_0(\boldsymbol{y}_0) \right) = \sqrt{\det\left((\nabla \boldsymbol{\phi}_0(\boldsymbol{y}_0)^\top) \, \nabla \boldsymbol{\phi}_0(\boldsymbol{y}_0) \right)}.$$

Remark. Since $\Omega \in \mathcal{C}^2$, we can choose r to be sufficiently small so that for any $\boldsymbol{y}_0 \in B(\boldsymbol{y}', r)$ holds $\operatorname{vol}\left(\nabla \boldsymbol{\phi}_0(\boldsymbol{y}_0) \right) \geq 1/2$.

Proposition 55. *Let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$ where $n \in \{2,3\}$. Let $\boldsymbol{x} \in \partial\Omega$ be such that $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{e}_n$ and suppose that Ω_c and $\mathcal{O}$ are the corresponding constructed sets. Then there exists a constant $C(\Omega) \in \mathbb{R}, C(\Omega) > 0$ depending on Ω such that*

$$\lambda_{k,T_{\mathrm{cur}}} \leq C(\Omega) \min_{D \in (V^*_{\mathrm{cur}}(\mathcal{O}))_k} \max_{\boldsymbol{v} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \tag{3.41}$$

for all $k \in \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$. From (3.28) we know that

$$\lambda_{k,T_{\text{cur}}} = \min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\Omega_c} |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{\Gamma_c} |\text{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}}.$$

Firstly, we will estimate from above the numerator in (3.28). Due to the fact that $\boldsymbol{\phi}$ is a diffeomorphism and due to relation (3.34) we obtain by using the Change of Variables Theorem (see, [12], Section 3.3.3, Theorem 2, p. 99) that

$$\int_{\Omega_c} |\nabla \boldsymbol{u}(\boldsymbol{x})|^2 \, d\lambda^n(\boldsymbol{x}) = \int_{\mathcal{O}} |(\nabla \boldsymbol{u}) \circ \boldsymbol{\phi}(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y}). \tag{3.42}$$

From (3.35) follows that

$$\nabla \boldsymbol{u} \circ \boldsymbol{\phi}(\boldsymbol{y}) = \nabla \left(\nabla \boldsymbol{\phi} F(\boldsymbol{u}) \right) (\boldsymbol{y}) \left(\nabla \boldsymbol{\phi}(\boldsymbol{y}) \right)^{-1} \tag{3.43}$$

for any $\boldsymbol{y} \in \mathcal{O}$. Using (3.43) in (3.42) and using basic properties of Frobenius norm yields

$$\int_{\mathcal{O}} |(\nabla \boldsymbol{u}) \circ \boldsymbol{\phi}(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y}) \leq \int_{\mathcal{O}} |\nabla \left(\nabla \boldsymbol{\phi} F(\boldsymbol{u}) \right) (\boldsymbol{y})|^2 |(\nabla \boldsymbol{\phi}(\boldsymbol{y}))^{-1}|^2 \, d\lambda^n(\boldsymbol{y}). \tag{3.44}$$

Since each first and each second partial derivative of $\boldsymbol{\phi}$ is continuous on $\overline{\mathcal{O}}$ we can estimate them by a constant depending on Ω. Right-hand side of (3.44) can hence be estimated

$$\begin{aligned} &\int_{\mathcal{O}} |\nabla \left(\nabla \boldsymbol{\phi} F(\boldsymbol{u}) \right) (\boldsymbol{y})|^2 |(\nabla \boldsymbol{\phi}(\boldsymbol{y}))^{-1}|^2 \, d\lambda^n(\boldsymbol{y}) \\ &\qquad \leq c(\Omega) \int_{\mathcal{O}} |\nabla F(\boldsymbol{u})(\boldsymbol{y})|^2 + |F(\boldsymbol{u})(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y}). \end{aligned} \tag{3.45}$$

Using Poincaré's inequality (see [11], Section 5.8.1., Theorem 1 for $\mathcal{C}^1$ domains, however, can be easily extended to Lipschitz domains by using Theorem 6.1., p. 102 from [5]) for the right-hand side in (3.45) we get

$$\begin{aligned} &c(\Omega) \int_{\mathcal{O}} |\nabla F(\boldsymbol{u})(\boldsymbol{y})|^2 + |F(\boldsymbol{u})(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y}) \\ &\qquad \leq d(\Omega) \int_{\mathcal{O}} |\nabla F(\boldsymbol{u})(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y}) = d(\Omega) \int_{\mathcal{O}} |\nabla \boldsymbol{v}(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y}) \end{aligned} \tag{3.46}$$

for some constants $c(\Omega), d(\Omega) \in \mathbb{R}, c(\Omega), d(\Omega) > 0$ depending on Ω. Now, we will estimate from below the denominator in (3.28). Using Change of Variables Theorem (see, [12], Section 3.3.3, Theorem 2, p. 99) we obtain

$$\int_{\Gamma_c} |\text{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1} = \int_{\Sigma_o} |\text{Tr}(\boldsymbol{u} \circ \boldsymbol{\phi})|^2 \underbrace{\text{vol}\left(\nabla \boldsymbol{\phi}_0\right)}_{\geq 1/2} \, d\mathcal{H}^{n-1}. \tag{3.47}$$

Using (3.35) for the right-hand side in (3.47) we get

$$\int_{\Sigma_o} |\text{Tr}(\boldsymbol{u} \circ \boldsymbol{\phi})|^2 \, d\mathcal{H}^{n-1} = \int_{\Sigma_o} |\nabla \boldsymbol{\phi} \, \text{Tr}(F(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1}$$

and it clearly follows that

$$\int_{\Sigma_o} |\nabla \boldsymbol{\phi} \, \text{Tr}(F(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1} \geq \int_{\Sigma_o} |\text{Tr}(F(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1}. \tag{3.48}$$

Combining (3.44)-(3.48) we obtain

$$\lambda_{k,T_{\text{cur}}} \le C(\Omega) \min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\mathcal{O}} |\nabla F(\boldsymbol{u})|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(F(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1}}. \tag{3.49}$$

for some constant $C(\Omega) \in \mathbb{R}, C(\Omega) > 0$ depending on Ω. Finally, Corollary 54 yields

$$\min_{D \in (V_{\text{cur}}(\Omega_c))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\mathcal{O}} |\nabla F(\boldsymbol{u})(\boldsymbol{y})|^2 \, d\lambda^n(\boldsymbol{y})}{\int_{\Sigma_o} |\mathrm{Tr}(F(\boldsymbol{u}))|^2 \, d\mathcal{H}^{n-1}} = \min_{D \in (V^*_{\text{cur}}(\mathcal{O}))_k} \max_{\boldsymbol{v} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}}. \tag{3.50}$$

Combining relations (3.49) and (3.50) finishes the proof.

□

With the knowledge of Proposition 55 we can now move on to the second step towards obtaining the desired upper estimate.

Definition 56. *We define spaces $\tilde{V}_{\text{cur}}(\mathcal{O})$ and $V'_{\text{cur}}(\mathcal{O})$ as follows*

$$\tilde{V}_{\text{cur}}(\mathcal{O}) = \left\{ \boldsymbol{v} \in \left(W^{1,2}_{\boldsymbol{n},\text{div}}(\mathcal{O}) \right)^n ; \, \mathrm{Tr}(\boldsymbol{v})|_{\partial\mathcal{O} \setminus \Sigma_o} = \boldsymbol{0} \right\},$$
$$V'_{\text{cur}}(\mathcal{O}) = \left\{ \boldsymbol{v} \in \left(W^{1,2}_{\boldsymbol{n}}(\mathcal{O}) \right)^n ; \, \mathrm{Tr}(\boldsymbol{v})|_{\partial\mathcal{O} \setminus \Sigma_o} = \boldsymbol{0} \right\}.$$

Remark. As for all the previous considered spaces, $\tilde{V}_{\text{cur}}(\mathcal{O})$ is again a Hilbert space, the expression

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\tilde{V}_{\text{cur}}} = \int_{\mathcal{O}} \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}} \, d\lambda^n$$

defines a scalar product on $\tilde{V}_{\text{cur}}(\mathcal{O})$ and the corresponding norm $\|\cdot\|_{\tilde{V}_{\text{cur}}}$ is equivalent to the standard $\|\cdot\|_{W^{1,2}}$ norm. Analogously for space $V'_{\text{cur}}(\mathcal{O})$.

As for space $V_{\text{cur}}(\Omega_c)$, space $\tilde{V}_{\text{cur}}(\mathcal{O})$ is a space of weak solutions to the problem (3.21)-(3.25) if we replace Ω_c with $\mathcal{O}$ and Γ_c with Σ_o. We could again define corresponding mapping B and obtain a unique operator $\tilde{T}_{\text{cur}} \in L(\tilde{V}_{\text{cur}}(\mathcal{O}))$ satisfying

$$B[\boldsymbol{u}, \boldsymbol{v}] = \left\langle \tilde{T}_{\text{cur}}(\boldsymbol{u}), \boldsymbol{v} \right\rangle_{\tilde{V}_{\text{cur}}}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \tilde{V}_{\text{cur}}(\mathcal{O})$. Completely analogous theoretical results as for T_{cur} are true also for this setting. We will formulate them without proofs for future reference.

Theorem 57. *Operator $\tilde{T}_{\text{cur}}$ is compact and self-adjoint. For some fixed $\lambda \in \mathbb{R}, \lambda > 0$, a non-trivial function $\boldsymbol{u}$ is a weak solution to the problem (3.21)-(3.25) with Ω_c replaced by $\mathcal{O}$ and Γ_c replaced by Σ_o if and only if $\boldsymbol{u}$ is an eigenfunction of $\tilde{T}_{\text{cur}}$ corresponding to eigenvalue $1/\lambda$. The set of eigenvalues of the problem (3.21)-(3.25) with $\Omega_c = \mathcal{O}$ and $\Gamma_c = \Sigma_o$ is countably infinite.*

Notation. When we refer to the operator $\tilde{T}_{\text{cur}}$ we always assume that $\tilde{T}_{\text{cur}}$ is defined on the corresponding function space, i.e. $\tilde{V}_{\text{cur}}(\mathcal{O})$. Furthermore, we denote

$\sigma_p(\tilde{T}_{\text{cur}}) = \left(\lambda'_{k,\tilde{T}_{\text{cur}}}\right)_{k=1}^{\infty}$ and $1/\sigma_p(\tilde{T}_{\text{cur}}) = \left(\lambda_{k,\tilde{T}_{\text{cur}}}\right)_{k=1}^{\infty}$ where $\left(\lambda'_{k,\tilde{T}_{\text{cur}}}\right)_{k=1}^{\infty}$ is sorted in a non-increasing order, $\left(\lambda_{k,\tilde{T}_{\text{cur}}}\right)_{k=1}^{\infty}$ is sorted in a non-decreasing order and each eigenvalue in both sequences appear according to its multiplicity.

Our approach will now be the following. We will prove that there exists $c \in \mathbb{R}, c > 0$ such that for any $k \in \mathbb{N}$ and any k-dimensional subspace $\tilde{W}$ of $\tilde{V}_{\text{cur}}(\mathcal{O})$ there exists a k-dimensional subspace W of $V^*_{\text{cur}}(\mathcal{O})$ such that

$$\max_{\boldsymbol{v}\in W} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\text{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \leq c \max_{\boldsymbol{v}\in \tilde{W}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\text{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}}. \tag{3.51}$$

In order to do that, we will introduce an operator that is similar to Bogovski operator mentioned in Remark after Theorem 14. First of all, we will proof auxiliary Lemmata.

Notation. For $\boldsymbol{y} \in \mathcal{O}$ we denote by $N(\boldsymbol{y})$ the matrix satisfying $\nabla \boldsymbol{\phi}(\boldsymbol{y}) = \text{Id}_{n\times n} + N(\boldsymbol{y})$ and by $N^-(\boldsymbol{y})$ the matrix satisfying $(\nabla \boldsymbol{\phi}(\boldsymbol{y}))^- = \text{Id}_{n\times n} + N^-(\boldsymbol{y})$.

Lemma 58. *Let $\boldsymbol{v} \in V'_{\text{cur}}(\mathcal{O})$. Then*

$$\begin{aligned} \text{Trace}\left(\nabla \boldsymbol{\phi}(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) \left(\nabla \boldsymbol{\phi}(\boldsymbol{y})\right)^{-1}\right) &= \operatorname{div} \boldsymbol{v}(\boldsymbol{y}) \\ &+ \text{Trace}\left(\nabla \boldsymbol{v}(\boldsymbol{y}) N^-(\boldsymbol{y}) + N(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) + N(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) N^-(\boldsymbol{y})\right). \end{aligned} \tag{3.52}$$

for all $y \in \mathcal{O}$.

Proof.

$$\begin{aligned} &\text{Trace}\left(\nabla \boldsymbol{\phi}(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) \left(\nabla \boldsymbol{\phi}(\boldsymbol{y})\right)^{-1}\right) \\ &= \text{Trace}\left(\left(\text{Id}_{n\times n} + N(\boldsymbol{y})\right) \nabla \boldsymbol{v}(\boldsymbol{y}) \left(\text{Id}_{n\times n} + N^-(\boldsymbol{y})\right)\right) \\ &= \text{Trace}\left(\nabla \boldsymbol{v}(\boldsymbol{y})\right) \\ &+ \text{Trace}\left(\nabla \boldsymbol{v}(\boldsymbol{y}) N^-(\boldsymbol{y}) + N(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) + N(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) N^-(\boldsymbol{y})\right). \end{aligned}$$

Since $\text{Trace}\left(\nabla \boldsymbol{v}(\boldsymbol{y})\right) = \operatorname{div} \boldsymbol{v}(\boldsymbol{y})$, claim follows.

□

Lemma 59. *Let $\boldsymbol{v} \in V'_{\text{cur}}(\mathcal{O})$. Then*

$$\int_{\mathcal{O}} \text{Trace}\left(\nabla \boldsymbol{v}(\boldsymbol{y}) N^-(\boldsymbol{y}) + N(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) + N(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) N^-(\boldsymbol{y})\right) d\lambda^n = 0. \tag{3.53}$$

Proof. Firstly, using Gauss's theorem (see Theorem 6) we get

$$\int_{\mathcal{O}} \operatorname{div} \boldsymbol{v}(\boldsymbol{y}) = \int_{\Sigma_o} \text{Tr}(\boldsymbol{v})(\boldsymbol{y}) \cdot \boldsymbol{e_n} = 0$$

since $\boldsymbol{v} \in V'_{\text{cur}}(\mathcal{O})$. From Lemma 58 it follows that it remains to show

$$\int_{\mathcal{O}} \text{Trace}\left(\nabla \boldsymbol{\phi}(\boldsymbol{y}) \nabla \boldsymbol{v}(\boldsymbol{y}) \left(\nabla \boldsymbol{\phi}(\boldsymbol{y})\right)^{-1}\right) d\lambda^n = 0.$$

Using Change of Variables Theorem (see, [12], Section 3.3.3, Theorem 2, p. 99) and Gauss's theorem we get that

$$\begin{aligned}
&\int_{\mathcal{O}} \operatorname{Trace}\left(\nabla\boldsymbol{\phi}(\boldsymbol{y})\nabla\boldsymbol{v}(\boldsymbol{y})\,(\nabla\boldsymbol{\phi}(\boldsymbol{y}))^{-1}\right)\,d\lambda^n \\
&= \int_{\Omega_c} \operatorname{Trace}\left(\nabla\boldsymbol{\phi}\nabla\boldsymbol{v}\,(\nabla\boldsymbol{\phi})^{-1}\right)\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\,d\lambda^n = \int_{\Omega_c} \operatorname{div}\left((\nabla\boldsymbol{\phi}\,\boldsymbol{v})\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)\,d\lambda^n \\
&= \int_{\Gamma_c} (\nabla\boldsymbol{\phi}\,\boldsymbol{v})\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\,d\mathcal{H}^{n-1} = \int_{\Gamma_c} \left((\nabla\boldsymbol{\phi}\,\boldsymbol{v})\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\boldsymbol{n}(\boldsymbol{x})\,d\mathcal{H}^{n-1} \\
&\qquad = \int_{\Gamma_c} \left(\boldsymbol{v}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\left(\nabla\boldsymbol{\phi}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\boldsymbol{n}(\boldsymbol{x})\,d\mathcal{H}^{n-1}. \qquad (3.54)
\end{aligned}$$

Similarly as in (3.39) we obtain

$$\int_{\Gamma_c} \left(\boldsymbol{v}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\left(\nabla\boldsymbol{\phi}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\boldsymbol{n}(\boldsymbol{x})\,d\mathcal{H}^{n-1} = \int_{\Gamma_c} c\left(\boldsymbol{v}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\boldsymbol{e}_n\,d\mathcal{H}^{n-1}$$

for some non-zero function c. Since $\boldsymbol{v}\in V'_{\mathrm{cur}}(\mathcal{O})$ it holds $\left(\boldsymbol{v}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\boldsymbol{e}_n = 0$ for any $\boldsymbol{x}\in\Gamma_c$ and hence

$$\int_{\Gamma_c} \left(\boldsymbol{v}\circ\boldsymbol{\phi}^{-1}(\boldsymbol{x})\right)^{\top}\boldsymbol{e}_n\,d\mathcal{H}^{n-1} = 0. \qquad (3.55)$$

Combining (3.54) and (3.55) completes the proof.

□

We will now construct bounded linear operator $\operatorname{Bog}^{\top}\colon \tilde{L}^2(\mathcal{O})\to\left(W^{1,2}_{\mathbf{0}}(\mathcal{O})\right)^n$ such that

$$\begin{aligned}
\operatorname{Trace}\left(\nabla\boldsymbol{\phi}(\boldsymbol{y})\nabla\operatorname{Bog}^{\top}(f)(\boldsymbol{y})\,(\nabla\boldsymbol{\phi}(\boldsymbol{y}))^{-1}\right) &= f \quad \text{in } \mathcal{O}, \\
\operatorname{Bog}^{\top}(f) &= \mathbf{0} \quad \text{on } \partial\mathcal{O}
\end{aligned} \qquad (3.56)$$

for any $f\in\tilde{L}^2(\mathcal{O})$. The crucial thing in (3.56) is that $\operatorname{Tr}\left(\operatorname{Bog}^{\top}(f)\right)=\mathbf{0}$.

Definition 60. *Let $f\in\tilde{L}^2(\mathcal{O})$ and let* Bog *denote Bogovski operator. We define operator $M_f\colon V'_{\mathrm{cur}}(\mathcal{O})\to V'_{\mathrm{cur}}(\mathcal{O})$ by*

$$M_f(\boldsymbol{w}) = \operatorname{Bog}\left(f - \operatorname{Trace}\left(\nabla\boldsymbol{w}N^{-} + N\nabla\boldsymbol{w} + N\nabla\boldsymbol{w}N^{-}\right)\right) \qquad (3.57)$$

for $\boldsymbol{w}\in V'_{\mathrm{cur}}(\mathcal{O})$.

Remark. Operator M_f is well defined by Lemma 59 and Theorem 14.

The operator norm of operator Bog, i.e. $\|\operatorname{Bog}\|_{\tilde{L}^2(\mathcal{O})\to\left(W^{1,2}_{\mathbf{0}}(\mathcal{O})\right)^n}$ obviously depends on the set $\mathcal{O}$. However, by our construction of sets $\mathcal{O}$, we can estimate it from above by a constant β depending merely on Ω. Leaving out technical details, it follows from a Lemma from Galdi (see [7], Lemma III.3.1, p. 162). Thus

$$\|\operatorname{Bog}\|_{\tilde{L}^2(\mathcal{O})\to\left(W^{1,2}_{\mathbf{0}}(\mathcal{O})\right)^n} \le \beta(\Omega). \qquad (3.58)$$

Definition 61. *Let (P,ϱ) be a metric space and $f: P\to P$. We say that f is a contraction if there exists $\gamma\in\mathbb{R},\gamma\in[0,1)$ such that for all $x,y\in P$ holds $\varrho(f(x),f(y))\le\gamma\varrho(x,y)$.*

Notation. Let $A = (a(\boldsymbol{x})_{i,j})$ be $n \times n$ matrix where $a_{i,j} \colon \Omega \to \mathbb{R}$ for some $\Omega \subset \mathbb{R}^n$. We denote $\|A\|_\infty = \sup\{|a_{i,j}(\boldsymbol{x})|\,;\, i,j \in \mathbb{N}, \boldsymbol{x} \in \Omega\}$.

Lemma 62. *Set Γ_c can be chosen in such a way that operator M_f is a contraction.*

Remark. Lemma 62 is independent of the choice of $f \in \tilde{L}^2(\mathcal{O})$.

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in V'_{\mathrm{cur}}(\mathcal{O})$ and denote $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. Since Bog is linear we get

$$\begin{aligned}
&\|M_f(\boldsymbol{u}) - M_f(\boldsymbol{v})\|_{V'_{\mathrm{cur}}(\mathcal{O})} \\
&\qquad = \|\mathrm{Bog}\left(\mathrm{Trace}\left(\nabla \boldsymbol{w} N^- + N\nabla \boldsymbol{w} + N\nabla \boldsymbol{w} N^-\right)\right)\|_{V'_{\mathrm{cur}}(\mathcal{O})} \\
&\qquad \overset{(3.58)}{\leq} \beta(\Omega)\|\mathrm{Trace}\left(\nabla \boldsymbol{w} N^- + N\nabla \boldsymbol{w} + N\nabla \boldsymbol{w} N^-\right)\|_{L^2(\mathcal{O})} \\
&\qquad \leq \beta(\Omega) C(n) \max\left\{\|N\|_\infty, \|N^-\|_\infty, \|N\|_\infty \|N^-\|_\infty\right\} \|\nabla \boldsymbol{w}\|_{L^2(\mathcal{O})} \qquad (3.59)
\end{aligned}$$

where $C(n) \in \mathbb{R}, C(n) > 0$ is some constant depending on n. Let $\varepsilon \in \mathbb{R}, \varepsilon \in (0,1)$. Since $\Omega \in \mathcal{C}^2$, we can choose Γ_c in such a way that

$$\beta(\Omega) C(n) \max\left\{\|N\|_\infty, \|N^-\|_\infty, \|N\|_\infty \|N^-\|_\infty\right\} < \varepsilon. \qquad (3.60)$$

Thus (3.59) and (3.60) give that

$$\|M_f(\boldsymbol{u}) - M_f(\boldsymbol{v})\|_{V'_{\mathrm{cur}}(\mathcal{O})} \leq \varepsilon \|\boldsymbol{u} - \boldsymbol{v}\|_{V'_{\mathrm{cur}}(\mathcal{O})}$$

and hence M_f is a contraction.

□

Remark. Further we will always assume that Γ_c was chosen in such a way that (3.60) hold.

Remark. Banach Fixed-Point Theorem (see [13], Theorem 2.1, p.7) ensures that for each $f \in \tilde{L}^2(\mathcal{O})$ operator M_f has exactly one fixed point.

Definition 63. *For $f \in \tilde{L}^2(\mathcal{O})$ we define operator $\mathrm{Bog}^\top \colon \tilde{L}^2(\mathcal{O}) \to \left(W^{1,2}_{\mathbf{0}}(\mathcal{O})\right)^n$ by*

$$\mathrm{Bog}^\top(f) = \boldsymbol{w} \qquad (3.61)$$

where $\boldsymbol{w}$ is fixed point of operator M_f.

Corollary 64. *Operator $\mathrm{Bog}^\top$ is linear and bounded. Moreover, it holds*

$$\|\mathrm{Bog}^\top\|_{\tilde{L}^2(\mathcal{O}) \to \left(W^{1,2}_{\mathbf{0}}(\mathcal{O})\right)^n} \leq 2\beta(\Omega). \qquad (3.62)$$

Proof. Let $f, g \in \tilde{L}^2(\mathcal{O})$. Then there exist corresponding fixed points $\boldsymbol{w}_1$ of operator M_f and $\boldsymbol{w}_2$ of operator M_g. Denote $\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ and $h = f + g$. Using (3.57) and linearity of operator Bog we get

$$\boldsymbol{w} = \mathrm{Bog}\left(h - \mathrm{Trace}\left(\nabla \boldsymbol{w} N^- + N\nabla \boldsymbol{w} + N\nabla \boldsymbol{w} N^-\right)\right).$$

Thus $\boldsymbol{w}$ is fixed point of operator M_h and hence

$$\mathrm{Bog}^\top(f+g) = \mathrm{Bog}^\top(h) = \boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2 = \mathrm{Bog}^\top(f) + \mathrm{Bog}^\top(g).$$

Scalar multiplication would be proved similarly. Therefore, $\mathrm{Bog}^\top$ is indeed linear. Furthermore, let $f \in \tilde{L}^2(\mathcal{O})$ and let $\boldsymbol{w} = \mathrm{Bog}^\top(f)$ be corresponding fixed point of M_f. Using similar estimates as in (3.59) we get

$$\|\boldsymbol{w}\|_{\left(W_{\mathbf{0}}^{1,2}(\mathcal{O})\right)^n} \leq \beta(\Omega)\left(\|f\|_{\tilde{L}^2(\mathcal{O})} + \varepsilon\|\boldsymbol{w}\|_{\left(W_{\mathbf{0}}^{1,2}(\mathcal{O})\right)^n}\right)$$

for some small enough $\varepsilon \in \mathbb{R}, \varepsilon > 0$. Suppose further that $\varepsilon < 1/(2\beta(\Omega))$. Then we obtain

$$\|\boldsymbol{w}\|_{\left(W_{\mathbf{0}}^{1,2}(\mathcal{O})\right)^n} \leq \frac{\beta(\Omega)}{1-\varepsilon\beta(\Omega)}\|f\|_{L^2(\mathcal{O})}.$$

and thus (3.62) follows.

□

Remark. Further we will always assume that Γ_c was chosen in such a way that (3.60) and (3.62) hold.

Corollary 65. *Let $f \in \tilde{L}^2(\mathcal{O})$. Then relations (3.56) are indeed satisfied for* $\mathrm{Bog}^\top(f)$.

Proof. Denote $\boldsymbol{w} = \mathrm{Bog}^\top(f)$. The fact that $\boldsymbol{w} \in \left(W_{\mathbf{0}}^{1,2}(\mathcal{O})\right)^n$ follows from the fact $\boldsymbol{w}$ is fixed point of M_f and from properties of Bogovski operator. Moreover, we get

$$\operatorname{div}\left(M_f(\boldsymbol{w})\right) = \operatorname{div}\boldsymbol{w} = f - \mathrm{Trace}\left(\nabla\boldsymbol{w}N^- + N\nabla\boldsymbol{w} + N\nabla\boldsymbol{w}N^-\right).$$

Using (3.52) we obtain

$$\mathrm{Trace}\left(\nabla\boldsymbol{\phi}(\boldsymbol{y})\nabla\mathrm{Bog}^\top(f)(\boldsymbol{y})\left(\nabla\boldsymbol{\phi}(\boldsymbol{y})\right)^{-1}\right) = f$$

in $\mathcal{O}$. Hence the claim follows.

□

Now we are going to use operator $\mathrm{Bog}^\top$ to define another operator that will finally allow as to prove the desired estimate (3.51).

Lemma 66. *Let $\boldsymbol{v} \in \tilde{V}_{\mathrm{cur}}(\mathcal{O})$. Denote*

$$L(\boldsymbol{v}) = \boldsymbol{v} - \mathrm{Bog}^\top\left(\mathrm{Trace}\left(\nabla\boldsymbol{v}N^- + N\nabla\boldsymbol{v} + N\nabla\boldsymbol{v}N^-\right)\right).$$

Then

$$\mathrm{Tr}(L(\boldsymbol{v})) = \mathrm{Tr}(\boldsymbol{v}) \tag{3.63}$$

on $\partial\mathcal{O}$ and

$$\mathrm{Trace}\left(\nabla\boldsymbol{\phi}\nabla L(\boldsymbol{v})\left(\nabla\boldsymbol{\phi}\right)^{-1}\right) = 0 \tag{3.64}$$

in $\mathcal{O}$.

Proof. Relation for $L(\boldsymbol{v})$ is well defined by Lemma 59. Relation (3.63) follows from Corollary 65 and the second condition in (3.56). Relation (3.64) follows from Corollary 65 and the first condition in (3.56) since we know that $\boldsymbol{v} \in \tilde{V}_{\mathrm{cur}}(\mathcal{O})$ and thus

$$\mathrm{Trace}\left(\nabla\boldsymbol{\phi}\nabla\boldsymbol{v}\left(\nabla\boldsymbol{\phi}\right)^{-1}\right) = \mathrm{Trace}\left(\nabla\boldsymbol{v}N^- + N\nabla\boldsymbol{v} + N\nabla\boldsymbol{v}N^-\right)$$

in $\mathcal{O}$ by Lemma 58.

□

Definition 67. *We define linear operator* $L\colon \tilde{V}_{\mathrm{cur}}(\mathcal{O}) \to V^*_{\mathrm{cur}}(\mathcal{O})$ *by*

$$L(\boldsymbol{v}) = \boldsymbol{v} - \mathrm{Bog}^\top \left(\mathrm{Trace}\left(\nabla \boldsymbol{v} N^- + N \nabla \boldsymbol{v} + N \nabla \boldsymbol{v} N^-\right)\right) \tag{3.65}$$

for $\boldsymbol{v} \in \tilde{V}_{\mathrm{cur}}(\mathcal{O})$.

Remark. Operator L indeed maps to $V^*_{\mathrm{cur}}(\mathcal{O})$ by Lemma 66 and Definition 53 and is linear since $\mathrm{Bog}^\top$ is linear.

Lemma 68. *Let* $\boldsymbol{v} \in \tilde{V}_{\mathrm{cur}}(\mathcal{O})$. *Then the following estimate hold*

$$\|L(\boldsymbol{v})\|_{\tilde{V}_{\mathrm{cur}}(\mathcal{O})} \leq 3\|\boldsymbol{v}\|_{V^*_{\mathrm{cur}}(\mathcal{O})}. \tag{3.66}$$

Proof. Using (3.62) and similar estimates as in (3.59) and (3.60) and convention mentioned in Remark after Lemma 62 we indeed obtain

$$\|L(\boldsymbol{v})\|_{\tilde{V}_{\mathrm{cur}(\mathcal{O})}} \leq 3\|\boldsymbol{v}\|_{V^*_{\mathrm{cur}}(\mathcal{O})}.$$

□

Lemma 69. *Operator* L *is injective.*

Proof. Suppose that $\boldsymbol{u}, \boldsymbol{v} \in \tilde{V}_{\mathrm{cur}}(\mathcal{O})$ are such that $L(\boldsymbol{u}) = L(\boldsymbol{v})$. Denote $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and denote

$$h = \mathrm{Trace}\left(\nabla \boldsymbol{w} N^- + N \nabla \boldsymbol{w} + N \nabla \boldsymbol{w} N^-\right).$$

Using (3.65) we obtain

$$\mathbf{0} = L(\boldsymbol{u}) - L(\boldsymbol{v}) = \boldsymbol{w} - \mathrm{Bog}^\top(h).$$

Thus by definition of $\mathrm{Bog}^\top$, $\boldsymbol{w}$ is fixed point of operator M_h. By definition of M_h we get

$$\boldsymbol{w} = M_h(\boldsymbol{w}) = \mathrm{Bog}(\mathbf{0}) = \mathbf{0}.$$

Hence $\boldsymbol{u} = \boldsymbol{v}$ which completes the proof.

□

Finally, we prove the desired estimate (3.51).

Lemma 70. *Let* $k \in \mathbb{N}$ *and let* $\tilde{W}$ *be a* k*-dimensional subspace of* $\tilde{V}_{\mathrm{cur}}(\mathcal{O})$. *Then there exists a* k*-dimensional subspace* W *of* $V^*_{\mathrm{cur}}(\mathcal{O})$ *such that*

$$\max_{v \in W} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \leq 3 \max_{v \in \tilde{W}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}}. \tag{3.67}$$

Proof. We define $W = L(\tilde{W})$. Using Lemma 69 we get that W a k-dimensional subspace of $V^*_{\mathrm{cur}}(\mathcal{O})$. Using (3.63) and (3.66) we obtain

$$\begin{aligned} \max_{v \in W} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} &= \max_{v \in \tilde{W}} \frac{\int_{\mathcal{O}} |\nabla L(\boldsymbol{v})|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(L(\boldsymbol{v}))|^2 \, d\mathcal{H}^{n-1}} \\ &\leq 3 \max_{v \in \tilde{W}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}}. \end{aligned}$$

Thus we indeed proved (3.68).

□

Proposition 71. *It holds*

$$\min_{D\in(V^*_{\mathrm{cur}}(\mathcal{O}))_k} \max_{\boldsymbol{v}\in D\setminus\{\boldsymbol{0}\}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \leq 3 \min_{D\in\left(\tilde{V}_{\mathrm{cur}}(\mathcal{O})\right)_k} \max_{\boldsymbol{v}\in D\setminus\{\boldsymbol{0}\}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \tag{3.68}$$

for all $k \in \mathbb{N}$.

Proof. Claim follows directly from Lemma 70.

□

With knowledge of Proposition 71 we can now move on to the last step towards obtaining the desired upper estimate. We would now like to once again reduce this remaining problem in such a way that we could use the results from Section 3.1. For the constructed and appropriate set $\mathcal{O}$ we construct a cylindrical subset $\mathcal{O}_c$ in the following way

$$\mathcal{O}_c = \mathcal{O} \cap \{(\boldsymbol{x}_0, x_n) \in \mathbb{R}^n;\ x_n > \gamma\}$$

where $\gamma \in \mathbb{R}$,

$$\gamma = -\frac{3}{4} \inf\{\lambda^1 (\{z_n \in \mathbb{R}; (\boldsymbol{z}_0, z_n) \in \mathcal{O}\})\,;\ \boldsymbol{z}_0 \in B(\boldsymbol{y}', r)\}$$

(see paragraph above (3.30) for $B(\boldsymbol{y}', r)$). Thus the only change is in the lower disk part of the boundary (see Picture 3.2 where the blue line is $S_{l,\mathcal{O}_c}$).

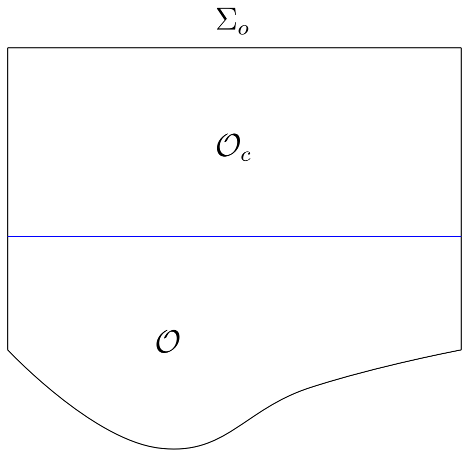

Figure 3.2: Construction of set $\mathcal{O}_c$

Notation. For any $\boldsymbol{u} \in V_c(\mathcal{O}_c)$ (see Definition 37) we denote by $\mathcal{E}_0$ the extension of $\boldsymbol{u}$ by zero to the whole set $\mathcal{O}$.

An analogous version of Lemma 49 holds also for this situation.

Lemma 72. *Let $k \in \mathbb{N}, k \geq 1$. A finitely generated subspace $W_k = \mathrm{span}\{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k\}$ of space $V_c(\mathcal{O}_c)$ has dimension k if and only if subspace $W'_k = \mathrm{span}\{\mathcal{E}_0(\boldsymbol{u}_1),\ldots,\mathcal{E}_0(\boldsymbol{u}_k)\}$ of space $\tilde{V}_{\mathrm{cur}}(\mathcal{O})$ has dimension k.*

Proposition 73. *The inequality*

$$\lambda_{k,\tilde{T}_{\mathrm{cur}}} \leq \lambda_{k,T_c}.$$

holds for all $k \in \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$. Combining Theorem 35, Theorem 57 and (3.2) we get that

$$\lambda_{k,\tilde{T}_{\mathrm{cur}}} = \min_{D \in \left(\tilde{V}_{\mathrm{cur}}(\mathcal{O})\right)_k} \max_{\boldsymbol{v} \in D \setminus \{\mathbf{0}\}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}}$$

Using the definition of $\mathcal{E}_0$ and the fact that $\mathcal{E}_0(V_c(\mathcal{O}_c)) \subset \tilde{V}_{\mathrm{cur}}(\mathcal{O})$ we obtain

$$\begin{aligned}
\lambda_{k,\tilde{T}_{\mathrm{cur}}} \leq & \min_{D \in (\mathcal{E}_0(V_c(\mathcal{O}_c)))_k} \max_{\boldsymbol{v} \in D \setminus \{\mathbf{0}\}} \frac{\int_{\mathcal{O}} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \\
& \overset{(\mathrm{L72})}{=} \min_{D \in (V_c(\mathcal{O}_c))_k} \max_{\boldsymbol{v} \in D \setminus \{\mathbf{0}\}} \frac{\int_{\mathcal{O}} |\nabla \mathcal{E}_0(\boldsymbol{v})|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\mathcal{E}_0(\boldsymbol{v}))|^2 \, d\mathcal{H}^{n-1}} \\
& \overset{(\mathrm{def.}\ \mathcal{E}_0)}{=} \min_{D \in (V_c(\mathcal{O}_c))_k} \max_{\boldsymbol{v} \in D \setminus \{\mathbf{0}\}} \frac{\int_{\mathcal{O}_c} |\nabla \boldsymbol{v}|^2 \, d\lambda^n}{\int_{\Sigma_o} |\mathrm{Tr}(\boldsymbol{v})|^2 \, d\mathcal{H}^{n-1}} \\
& = \min_{D \in (V_c(\mathcal{O}_c))_k} \max_{\boldsymbol{v} \in D \setminus \{\mathbf{0}\}} \frac{\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{V_c(\mathcal{O}_c)}}{\langle T_c(\boldsymbol{v}), \boldsymbol{v} \rangle_{V_c(\mathcal{O}_c)}} = \lambda_{k,T_c} \qquad (3.69)
\end{aligned}$$

and the claim follows.

□

Thus we are finally able to formulate the general Theorem that we wanted to prove.

Theorem 74. *Let* Ω *be a bounded* $\mathcal{C}^2$ *domain in* $\mathbb{R}^n$ *where* $n \in \{2{,}3\}$. *Then there exists a constant* $C(\Omega, n) \in \mathbb{R}, C(\Omega) > 0$ *depending on* Ω *and dimension* n *such that*

$$\lambda_{k,T} \leq C(\Omega) k^{1/(n-1)} \qquad (3.70)$$

for all $k \in \mathbb{N}$.

Proof. Proof consists of applying and combining each particular result that we proved so far. Step by step by applying Proposition 50, Proposition 55, Proposition 71, Proposition 73, Proposition 43, Proposition 26 and Proposition 34 we indeed obtain estimate (3.70).

□

Theorem 75. *Let* Ω *be a bounded* $\mathcal{C}^2$ *domain in* $\mathbb{R}^n$ *where* $n \in \{2{,}3\}$. *Then there exists a constant* $C(\Omega, n) \in \mathbb{R}, C(\Omega) > 0$ *depending on* Ω *and dimension* n *such that*

$$\limsup_{k \to \infty} \frac{\lambda_{k,T}}{k^{1/(n-1)}} \leq C(\Omega, n). \qquad (3.71)$$

Proof. Theorem follows immediately from Theorem 74.

□

4. Lower estimate of eigenvalues on general domains

So far we have only discussed the upper estimate of eigenvalues of the problem (2.1)-(2.4) and the reason was that the lower estimate is a consequence of the work of Sandgren [1] where (as mentioned in the Introduction) they studied Steklov problem, i.e. for $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ bounded $\mathcal{C}^2$ domain, $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^n$ and $\mu \in \mathbb{R}$ they considered

$$-\Delta \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{4.1}$$

$$(\nabla \boldsymbol{u})\, \boldsymbol{n} = \mu \boldsymbol{u} \qquad \text{on } \partial\Omega \tag{4.2}$$

and determined asymptotic behaviour of the eigenvalue sequence $(\mu_k)_{k=1}^{\infty}$ of the problem (4.1)-(4.2). More precisely, they proved that there exists a constant $C_{\text{Stek}}(\Omega, n) \in \mathbb{R}, C_{\text{Stek}}(\Omega, n) > 0$ depending on Ω and dimension n such that

$$\mu_k = C_{\text{Stek}}(\Omega, n) k^{1/(n-1)} + o(k^{1/(n-1)}). \tag{4.3}$$

We will now briefly comment on theoretical results regarding problem (4.1)-(4.2) since it will be very similar to the results obtained for problem (2.1)-(2.4). If not stated otherwise, we will assume throughout this Section that $n \in \mathbb{N}$ and Ω is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$

Notation. Let $H^1(\Omega)$ denote the space $(W^{1,2}(\Omega))^n$.

Remark. $H^1(\Omega)$ is a Hilbert space with the following inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{H^1(\Omega)} = \int_{\Omega} \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{v}} \, d\lambda^n + \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{v}}) \, d\mathcal{H}^{n-1},$$

where $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)$. By $\|\cdot\|_{H^1(\Omega)}$ we denote corresponding norm. Moreover, $\|\cdot\|_{H^1(\Omega)}$ is equivalent to the standard $\|\cdot\|_{(W^{1,2}(\Omega))^n}$.

Definition 76. *Let $\mu \in \mathbb{R}$ be fixed. We say that $\boldsymbol{u} \in H^1(\Omega)$ is a weak solution to the problem (4.1)-(4.2) if*

$$\int_{\Omega} \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{\varphi}} \, d\lambda^n = \mu \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{\varphi}}) \, d\mathcal{H}^{n-1} \tag{4.4}$$

holds for all $\boldsymbol{\varphi} \in H^1(\Omega)$.

Remark. We will only be interested in non-trivial weak solutions hence by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ in (4.4) we immediately get that $\mu \geq 0$.

We could now once again define a mapping $B\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$

$$B[\boldsymbol{u}, \boldsymbol{v}] = \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{v}}) \, d\mathcal{H}^{n-1}.$$

and obtain a unique operator $T_{\text{Stek}} \in L(H^1(\Omega))$ satisfying

$$B[\boldsymbol{u}, \boldsymbol{v}] = \langle T_{\text{Stek}}(\boldsymbol{u}), \boldsymbol{v} \rangle_{H^1(\Omega)}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)$. Then we would obtain completely analogous results as in Lemma 12 and Theorem 13. We will summarize that in the following Theorem without proof.

Theorem 77. *Operator T_{Stek} is compact and self-adjoint. For some fixed $\mu \in \mathbb{R}, \mu > 0$, a non-trivial function $\boldsymbol{u}$ is an eigenfunction of T_{Stek} corresponding to eigenvalue μ if and only if $\boldsymbol{u}$ is a weak solution to the problem (4.1)-(4.2) with $1/\mu - 1$ instead of μ. The set of eigenvalues of the problem (4.1)-(4.2) is countably infinite.*

Notation. When we refer to the operator T_{Stek} we always assume that T_{Stek} is defined on the corresponding function space, i.e. $H^1(\Omega)$. Furthermore, we denote $\sigma_p(T_{\text{Stek}}) = \left(\lambda'_{k,T_{\text{Stek}}}\right)_{k=1}^{\infty}$ and $1/\sigma_p(T_{\text{Stek}}) - 1 = (\lambda_{k,T_{\text{Stek}}})_{k=1}^{\infty}$ where $\left(\lambda'_{k,T_{\text{Stek}}}\right)_{k=1}^{\infty}$ is sorted in a non-increasing order, $(\lambda_{k,T_{\text{Stek}}})_{k=1}^{\infty}$ is sorted in a non-decreasing order and each eigenvalue in both sequences is counted with its multiplicity.

Proposition 78. *Let $k \in \mathbb{N}$ and suppose that $\lambda_{k,T_{\text{Stek}}} > 0$. Then*

$$\lambda_{k,T_{\text{Stek}}} = \min_{D \in (H^1(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega \nabla \boldsymbol{u} : \overline{\nabla \boldsymbol{u}} \, d\lambda^n}{\int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{u}}) \, d\mathcal{H}^{n-1}}. \tag{4.5}$$

Proof. Proof is completely analogous to proof of Proposition 45.

□

We now have all the tools ready to prove the desired lower estimate for eigenvalues of the problem (2.1)-(2.4).

Theorem 79. *Let $n \in \mathbb{N}$ and let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$. Then there exists constant $C_{\text{Stek}}(\Omega, n) \in \mathbb{R}, C_{\text{Stek}}(\Omega, n) > 0$ depending on Ω and dimension n such that*

$$\liminf_{k \to \infty} \frac{\lambda_{k,T}}{k^{1/(n-1)}} \geq C_{\text{Stek}}(\Omega, n). \tag{4.6}$$

Proof. Let $k \in \mathbb{N}$. Using Proposition 45 and Korn's inequality (see [6], Proposition 3.13., p.271) we obtain

$$\begin{aligned}
\lambda_{k,T} + 1 = \frac{1}{\lambda'_{k,T}} &= \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega \boldsymbol{D u} : \overline{\boldsymbol{D u}} \, d\lambda^n + \int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{u}}) \, d\mathcal{H}^{n-1}}{\int_{\partial\Omega} \operatorname{Tr}(\boldsymbol{u}) \cdot \operatorname{Tr}(\overline{\boldsymbol{u}}) \, d\mathcal{H}^{n-1}} \\
&\geq C_{\text{Korn}} \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\|\boldsymbol{u}\|^2_{(W^{1,2}(\Omega))^n}}{\int_{\partial\Omega} |\operatorname{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&\geq C_{\text{Korn}} \min_{D \in (V(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{\partial\Omega} |\operatorname{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} \\
&\overset{(V(\Omega) \subset H^1(\Omega))}{\geq} C_{\text{Korn}} \min_{D \in (H^1(\Omega))_k} \max_{\boldsymbol{u} \in D \setminus \{\boldsymbol{0}\}} \frac{\int_\Omega |\nabla \boldsymbol{u}|^2 \, d\lambda^n}{\int_{\partial\Omega} |\operatorname{Tr}(\boldsymbol{u})|^2 \, d\mathcal{H}^{n-1}} = C_{\text{Korn}} \lambda_{k,T_{\text{Stek}}}.
\end{aligned} \tag{4.7}$$

Dividing both sides of the inequality (4.7) by $k^{1/(n-1)}$ then taking $\liminf_{k \to \infty}$ and finally using (4.3) we indeed obtain inequality (4.6).

□

5. Asymptotic behaviour of eigenvalues on general domains

By combining the results from Chapter 2, Chapter 3 and Chapter 4 we are able to summarize our results into one Theorem about the asymptotic behaviour of the eigenvalue sequence of the problem (2.1)-(2.4) and confirm our hypothesis from the Introduction.

Theorem 80. *Let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^n$ where $n \in \{2,3\}$. Then there exist constants $c_{\text{Stokes}}, C_{\text{Stokes}}(\Omega, n) \in \mathbb{R}, c_{\text{Stokes}}, C_{\text{Stokes}}(\Omega, n) > 0$ depending on domain Ω and dimension n such that*

$$c_{\text{Stokes}}(\Omega, n) \leq \liminf_{k\to\infty} \frac{\lambda_k}{k^{1/(n-1)}} \leq \limsup_{k\to\infty} \frac{\lambda_k}{k^{1/(n-1)}} \leq C_{\text{Stokes}}(\Omega, n). \tag{5.1}$$

Proof. Theorem follows from Theorem 75 and Theorem 79.

□

Conclusion

Our goal in this thesis was to modify techniques used in [1] and [2] in order to determine asymptotic behaviour of the eigenvalue sequence of the problem (2.1)-(2.4) on bounded $\mathcal{C}^2$ domains. The lower bound of the asymptotic growth of the eigenvalue sequence was a consequence of the results from [1] (see Theorem 79), however, the upper bound was, as far as we know, not known. The goal was achieved, however, due to technical difficulties, only in dimension two and three. Nevertheless, in these cases, our hypothesis about asymptotic growth turned out to be true as summarized in Theorem 80.

Firstly, we introduced some theoretical results yielding that the eigenvalue sequence of the problem (2.1)-(2.4) is corresponding to an eigenvalue sequence of a certain compact and self-adjoint operator. This is particularly useful since we have an explicit formula for the k-th eigenvalue of the considered operator (see Theorem 35).

Next, we introduced auxiliary problems on simple domains for which we were able to find all non-trivial weak solutions explicitly and consequently we determined the asymptotic behaviour of the eigenvalue sequences of these problems precisely.

Finally, by using Theorem 35, we showed that eigenvalues of the problem (2.1)-(2.4) can be estimated from above by eigenvalues of several different, auxiliary problems. Eventually, we were capable of estimating the eigenvalues of these auxiliary problems by eigenvalues of problems for which we had proved the precise asymptotic behaviour and thus yielding the main result of this thesis, i.e. Theorem 80.

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