**FACULTY OF MATHEMATICS AND PHYSICS**
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# BACHELOR THESIS

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# Free Boundary Problems

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Study programme: Mathematics
Study branch: General Mathematics

Prague 2023

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To my supervisor, doc. Sebastian Schwarzacher, Dr., and to Giovanni Gravina, Ph.D., I owe my sincerest gratitude. Without their bottomless patience, profound knowledge, and kind mentorship, this thesis would never be possible. The many conversations with these two gentlemen not only brought me an understanding of the subject but also rekindled my love for mathematics.

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Abstract: This thesis deals with the one-phase Bernoulli problem, focusing on the existence and regularity of its solutions. After establishing the necessary preliminary theory on function spaces and convergence in the first chapter, we introduce the one-phase Bernoulli problem in the second chapter, reformulating it as a minimization problem. Then, in the third chapter, we present two illuminating examples of solutions to the problem, which imply that the Lipschitz regularity is optimal. The fourth chapter proves the existence of solutions, employing the direct method of calculus of variations. Finally, the fifth chapter reveals the Lipschitz property of generalized solutions.

Keywords: elliptic partial differential equations, calculus of variations,
free boundary problems

# Contents

# Introduction

In the study of partial differential equations (PDEs), *free boundary problems* constitute a special class of boundary value problems. They are characterized by the presence of an evolving boundary, the determination of which is linked to the solution of the PDE itself. The a priori unknown portion of the boundary is called a *free boundary* and it is subjected to conditioning in addition to standard boundary conditions of a PDE problem. As a result, one seeks to not only describe the solution to the PDE but also determine the free boundary.

These problems emerge in various scientific disciplines, including fluid dynamics, where they can model the interface between two immiscible fluids, in mathematical biology when modeling tumor growth, and also in mathematical finance to name but a few.

A classical example, which is worth mentioning, is the *one-phase Stefan problem* of melting ice, held at 0 degrees Celsius, in contact with its surrounding water. It is one of the simplest possible models describing a phase-change process and its study has greatly benefited the progress of the theory of free boundaries in the last 40 years, and its description may be found in Friedman and Spruck [2011].

This thesis, which is based primarily on the work of Velichkov [2023], focuses on the *one-phase Bernoulli problem*. This problem is of particular pertinence to the study of free boundary regularity theory, as it has inspired significant progress in the field. In the following chapters, we will translate this problem into the language of the calculus of variations, as we prove the solution may be obtained by solving a minimization problem of the functional $\mathcal{F}_\Lambda$. We then continue to prove the existence of minimizers and tackle the question of their regularity. We also give two illuminating examples of minimizers in simplified scenarios.

The first chapter briefs the reader on selected parts of the important theory regarding function spaces, which are relevant to the subsequent sections.

The aim of the second chapter is to introduce the one-phase Bernoulli problem and transform it into a more accessible variational problem of minimizing the functional $\mathcal{F}_\Lambda$. We proceed to study several basic properties of this functional.

In the third chapter, the reader will be presented with two examples of minimizers in simplified scenarios, illustrating the desired properties of regularity.

The main result of the fourth chapter is the proof of the existence of minimizers of the functional $\mathcal{F}_\Lambda$. This is achieved by employing the direct method of calculus of variations.

Finally, in the fifth chapter, we concern ourselves with an interesting question of the regularity of the minimizers. We arrive at the result that the minimizers of $\mathcal{F}_\Lambda$ are Lipschitz continuous.

The aim of this thesis is to provide more detail to the theory described in Velichkov [2023] and thus increase its readability. Our desired objective is to create an introductory text into the free boundary regularity theory, which is accessible to mathematicians of all levels of expertise.

# Notation

$\Delta$ - gradient operator

$\nabla$ - Laplace operator

**div** - divergence operator

$\varnothing$ - empty set

$\omega_d$ - the Lebesgue measure of the unit ball in $\mathbb{R}^d$

$\Omega$ - domain

$\partial\Omega$ - boundary of domain

$\Omega_u$ - the set $\{u > 0\}$ for a real-valued function $u$

$B_r(x)$ - open ball of radius $r$ centered at $x$

$B_r$ - open ball of radius $r$ centered at zero

$\operatorname{supp} u$ - the support of $u$, that is the set $\{u \neq 0\}$

$C_0^\infty(\Omega)$ - class of infinitely differentiable compactly supported functions in $\Omega$

$(u \wedge v)(x)$ - the function $\min(u(x), v(x))$ for real-valued $u, v$

$(u \vee v)(x)$ - the function $\max(u(x), v(x))$ for real-valued $u, v$

$\chi_\Omega$ - the characteristic function on the set $\Omega$

$⨍_\Omega$ - the averaging integral $\frac{1}{|\Omega|}\int_\Omega$

$\mathcal{H}^{d-1}$ - the Hausdorff measure in $d-1$ dimentions

$a.e.$ - almost everywhere

# 1. Preliminaries

## 1.1 Weak derivative and function spaces

### 1.1.1 Lebesgue space

Suppose $1 \le p < \infty$ and $(\Omega, \mathcal{A}, \mu)$ is a measure space. The Lebesgue space $L^p(\Omega)$ consists of equivalence classes of measurable functions $\Omega \to \mathbb{R}$ such that

$$\int |f|^p d\mu < \infty \,.$$

This space is endowed with the $L^p$-norm defined by

$$\|f\|_{L^p} = \|f\|_p := \left( \int |f|^p d\mu \right)^{\frac{1}{p}} \,.$$

In the upcoming chapters, we will rely on the following property of the $L^p$ spaces.

**Lemma 1** ($L^p$ are separable). *If $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a measurable set, then the space $L_p(\Omega)$ is separable.*

We will not give the proof here and refer the reader to Brezis [2010] for its description.

### 1.1.2 Sobolev space

The following definitions and further theory on the topic of Sobolev spaces can be found in Evans [1998](chapter 5, p. 251-307).

**Definition 1** (Weak derivative). *Suppose $\Omega \subset \mathbb{R}^n$ is an open domain and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index. Let $u, v \in L^1_{loc}(\Omega)$. We call $v$ the $\alpha$-th weak partial derivative of $u$, written as $D^\alpha u = v$, on condition that*

$$\int_\Omega u(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x) \varphi(x)\, dx$$

*for every function $\varphi$, which belongs to the space of infinitely differentiable functions with compact support $C_c^\infty(\Omega)$.*

*We denote $D^0 u = D^{(0,\cdots,0)} u = u$.*

*Remark.* Functions $\varphi$ from the definition above are commonly called *test functions.*

*Remark.* The weak $\alpha$-th partial derivative of $u$, if it exists, is defined uniquely up to a set of measure zero. This is not difficult to prove and can be found in Evans [1998].

We define the Sobolev space $H^1(\Omega)$, which is a vital building block in the upcoming chapters. It is a member of the following class of functions.

**Definition 2** (Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain. For a given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define the Sobolev space*

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq |\alpha| \leq k\} ,$$

*where $\alpha$ is a multi-index.*

*We also introduce the space $W_0^{k,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

This means, that $u \in W_0^{k,p}(\Omega)$ if and only if there exists a sequence of functions $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u \in W^{k,p}(\Omega)$

**Definition 3.** *In the special case when $k = 1$ and $p = 2$, we denote $W^{1,2}(\Omega)$ as $H^1(\Omega)$. We endow this space with the norm*

$$\|u\|_{H^1} := \left( \int_\Omega |u|^2 + |\nabla u|^2 \right)^{\frac{1}{2}} ,$$

*for a function $u \in H^1(\Omega)$.*

The space $H^1(\Omega)$ is also sometimes dubbed the Hilbert space for the following reason.

**Theorem 2** (The space $H^1(\Omega)$ is Hilbert)**.** *The Sobolev space $H^1(\Omega)$ with the norm $\|u\|_{H^1}$ is a Hilbert space.*

The proof of this statement is developed in Evans [1998], for example.

The following result neatly describes the relationship between the Lebesgue and the Sobolev spaces.

**Theorem 3** (Rellich-Kondrachov Compactness Theorem)**.** *Let $D \subset \mathbb{R}^n$ be an open set with a smooth boundary. Suppose $1 \leq p < n$ and $p^* = \frac{pn}{n-p}$. Then, $W^{1,p}(D)$ is compactly embedded in $L^q(D)$ for each $1 \leq q < p^*$.*

*In other words, the following is true:*

1. *$W^{1,p}(D) \subset L^q(D)$,*
2. *$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}$ for each $u \in W^{1,p}(D)$ and a constant $C$,*
3. *each bounded sequence in $W^{1,p}(D)$ is precompact in $L^q(D)$.*

*Proof.* See Evans [1998] (section 5.7, theorem 1) for the proof of this theorem. □

## 1.2 Weak(-⋆) convergence and compactness

In this section, we give the definitions of two types of convergence in a linear normed space $X$. Since we cover the theory concerning these modes of convergence only to a slim extent, we refer the interested reader to Royden and Fitzpatrick [2010] for further information.

The upcoming definitions depend heavily on the notion of the dual of a Banach space $(X, \|u\|_X)$, which, as we recall, is the vector space

$$X^* = \{F : X \to \mathbb{C} \text{ continuous and linear}\} .$$

**Definition 4** (Weak convergence). *Let $X$ be a normed linear space. We say a sequence $\{u_n\} \subset X$ converges weakly to $u \in X$ as $n \to \infty$, if the sequence of scalars $\{f(u_n)\}$ converges to $f(u)$ for all $f \in X^*$. We then write $u_n \rightharpoonup u$.*

To distinguish between convergence in the weak sense from the more common $\{u_n\} \to u$, meaning $\lim_{n\to\infty} |u_n - u| = 0$, we will often refer to the latter as *strong convergence* in $X$. Since

$$|f(u_n) - f(u)| = |f(u_n - u)| \le \|f\| \cdot \|f_n - f\| \quad \textit{for all} \quad f \in X^*\,,$$

if a sequence converges in the strong sense, it also converges weakly.

Generally, when dealing with a sequence $u_n$, where the condition on strong convergence of $u_n$ is too strict to request, one may also arrive at meaningful results by only having the information of weak convergence. We will witness this in the case of a minimizing sequence in Chapter 4.

We shall also introduce a type of convergence for a series of functions from the dual space $X^*$.

**Definition 5** (Weak-$\star$ convergence). *Let $X$ be a normed linear space. We say a sequence $\{f_n\} \subset X^*$ converges weakly-$\star$ to $f \in X^*$ as $n \to \infty$, if*

$$\lim_{n\to\infty} f_n(u) = f(u) \quad \textit{for all} \quad u \in X\,.$$

*We then write $u_n \overset{\star}{\rightharpoonup} u$.*

One of the most desirable properties of a given space is compactness. The result for weak–$\star$ topology is usually called the Banach–Alaoglu theorem. In this thesis, we will use an immediate corollary of this theorem for sequences in separable spaces.

**Theorem 4** (Banach–Alaoglu for sequences). *If $X$ is a separable space, then every bounded sequence $\{f_n\}$ in $X^*$ has a subsequence $\{f_{n_k}\}$ which converges weak-$\star$ to a function $f \in X^*$.*

The description of the proofs for both Banach–Alaoglu theorem and its consequence for sequences can be found in Brezis [2010].

## 1.3 Inequalities

In this section, we will declare several important inequalities, which will be leveraged in further chapters.

**Lemma 5** (Poincaré's inequality). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, smooth set and $u \in H_0^1(\Omega)$. Then, there exists a constant $C$ depending only on $d$ and $\Omega$ such that*

$$\|u\|_{L^2(\Omega)} \le \|\nabla u\|_{L^2(\Omega)}\,.$$

The description of this inequality can be found in Evans [1998] (section 5.8., p. 275).

**Lemma 6** (Hölder's inequality)**.** *Assume that* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^{p'}(\mathbb{R}^n)$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$*. Then* $fg \in L^1(\mathbb{R}^n)$ *and*

$$\int |fg| \le \|f\|_p \|g\|_{p'} .$$

**Lemma 7** (Young's inequality)**.** *Assume that* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^{p'}(\mathbb{R}^n)$ *where* $\frac{1}{r} := \frac{1}{p} + \frac{1}{p'} - 1 \ge 0$*. Then, their convolution* $(f * g)(x) := \int f(t)g(x-t)\,dt \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_r \le \|f\|_p \|g\|_{p'} .$$

**Lemma 8** (Jensen's inequality)**.** *Consider a finite-measure set* $\Omega$*, a convex function* $\varphi : \mathbb{R} \to \mathbb{R}$ *and a function* $f \in L^1(\Omega)$*. Then, it holds that*

$$\frac{1}{|\Omega|} \int_\Omega \varphi(f(x))\,dx \ge \varphi\left(\frac{1}{|\Omega|} \int_\Omega f(x)\,dx\right) .$$

For these three inequalities, we refer the reader to Brezis [2010](chapter 4, p. 92-120) where their statements and proofs are fully treated.

# 2. One-phase Bernoulli problem

In this chapter, we will introduce a problem of great significance to the study of free boundary regularity theory, the one-phase Bernoulli problem. As one of Bernoulli's problems, its origin lies in the description of free surfaces in ideal fluids.

The contents of this chapter are based on the first chapter of [Velichkov, 2023].

## 2.1 Problem formulation

Let $D$ be a smooth bounded open set in $\mathbb{R}^d$ and $\Lambda > 0$ a given constant. Bernoulli's free boundary problem asks to find a domain $\Omega \subset D$ and a function $u : \Omega \to \mathbb{R}$, such that

$$\begin{cases} \Delta u = 0 \in \Omega\,; \\ u = g \quad \text{on} \quad \partial\Omega \cap \partial D\,; \\ u = 0 \quad \text{and} \quad |\nabla u| = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega \cap D\,, \end{cases} \tag{2.1}$$

where $g : \partial D \to \mathbb{R}$ is a given non-negative function.

As we may observe, the solution to 2.1 depends both on the choice of the set $D$ and the boundary value function $g$. For this reason, we are usually not able to find the domain $\Omega$ and the function $u$ explicitly, outside of some very special cases. In fact, even the question of the existence of the pair $(\Omega, u)$, solving the problem 2.1, presents a challenge.

In their response to this challenge, H. W. Alt and L. A. Caffarelli formulated in their seminal paper Alt and Caffarelli [1980] the following associated variational problem:

*We seek to minimize the functional*

$$u \to \mathcal{F}_\Lambda(u, D) = \int_D |\nabla u(x)|^2\, dx + \Lambda\, |\{u > 0\} \cap D| \tag{2.2}$$

*among all functions $u : D \to \mathbb{R}$ such that*

$$u \in H^1(D) \quad \text{and} \quad u = g \text{ on } \partial D.$$

In other words, we are looking for a function $u$ that would satisfy the following definition.

**Definition 6** (Minimizer of $\mathcal{F}_\Lambda$). *Suppose $D \subset \mathbb{R}^d$ is a bounded open set. The function $u : D \to \mathbb{R}$ is said to be a minimizer of the functional $\mathcal{F}_\Lambda$ in $D$, if $u \in H^1(D)$ is non-negative in $D$ and*

$$\mathcal{F}_\Lambda(u, D) \le \mathcal{F}_\Lambda(v, D) \quad \text{for every} \quad v \in H^1(D) \quad \text{such that} \quad u - v \in H^1_0(D).$$

We continue to demonstrate the connection between the minimization problem and the one-phase Bernoulli problem 2.1, which allows us to shift our focus from solving the former directly to a more accessible variational problem.

**Proposition 1.** *Let $u : D \to \mathbb{R}$ be a minimizing function of $\mathcal{F}_\Lambda$, which is of the class $C^2(\overline{D})$. We consider the domain $\Omega_u := \{u > 0\}$ and the pertaining free boundary $\partial\Omega_u \cap D$. Then, the couple $(u, \Omega_u)$ is a solution to the free boundary problem 2.1.*

We organize the proof in three steps.

(i) The fulfillment of the conditions

$$u = g \text{ on } \partial\Omega \cap \partial D \quad \text{and} \quad u = 0 \text{ on } \partial\Omega \cap D$$

comes directly from the construction of the couple $(u, \Omega_u)$.

(ii) To show harmonicity, we will formulate the following lemma.

**Lemma 9** (Minimizer is harmonic)**.** *Let $D \subset \mathbb{R}^d$ be a bounded open set. Then, any minimizing function $u$ of $\mathcal{F}_\Lambda$ in $D$, which is a $C^2(\overline{D})$ function, is harmonic in the open set $\Omega_u = \{u > 0\}$.*

*Proof of Lemma.* Suppose $u$ is a minimizer of $\mathcal{F}_\Lambda$ as above. Consider an arbitrary function $\varphi \in H_0^1(\Omega_u)$ and let $\varepsilon > 0$. Since $\varphi$ is zero on the boundary $\partial\Omega_u$, we know that

$$u + \varepsilon\varphi \in H^1(D) \quad \text{and} \quad u + \varepsilon\varphi = g \text{ on } \partial D.$$

We notice that for an $\varepsilon$ very close to zero

$$\{u + \varepsilon\varphi > 0\} = \{u > 0\}.$$

Thus, we can focus only on the Dirichlet energy part of the functional.

The minimality of $u$ provides

$$\begin{aligned}
\frac{1}{2\varepsilon}\int_{\Omega_u} |\nabla u|^2 \, dx \le \frac{1}{2\varepsilon}\int_{\Omega_u} &|\nabla(u + \varepsilon\varphi)|^2 \, dx\,, \\
0 \le \frac{1}{2\varepsilon}\int_{\Omega_u} &|\nabla(u + \varepsilon\varphi)|^2 - |\nabla u|^2 \, dx\,, \\
0 \le \frac{1}{2\varepsilon}\int_{\Omega_u} &|\nabla u|^2 - 2\varepsilon\,\nabla u \cdot \nabla\varphi + \varepsilon^2 |\nabla\varphi|^2 - |\nabla u|^2 \, dx\,, \\
0 \le \int_{\Omega_u} &\nabla u \cdot \nabla\varphi + \frac{\varepsilon}{2}|\nabla\varphi|^2 \, dx\,.
\end{aligned}$$

We pass to the limit $\varepsilon \to 0$

$$0 \le \int_{\Omega_u} \nabla u \cdot \nabla\varphi \, dx\,.$$

This holds for both $\varphi$ and $-\varphi$, hence

$$0 = \int_{\Omega_u} \nabla u \cdot \nabla\varphi \, dx\,.$$

By partial integration for $\varphi$, we have

$$\begin{aligned}
0 &= \int_{\partial\Omega_u} \partial_\nu u\,\varphi - \int_{\Omega_u} \Delta u\,\varphi \\
0 &= -\int_{\Omega_u} \Delta u\,\varphi\,.
\end{aligned}$$

Because $\varphi$ is auxiliary, this means that

$$0 = \Delta u \quad \text{a.e. on} \quad \Omega_u\,.$$

□

(iii) Lastly, we address the overdetermined condition on the free boundary. Let $\varepsilon > 0$ and consider any smooth vector field $\zeta : D \to \mathbb{R}^d$ with compact support in $D$. Now, the function

$$\rho_\varepsilon(x) = x + \varepsilon\,\zeta(x)$$

is a diffeomorphism on $D$ and, by the nature of $\zeta$, the function

$$u_\varepsilon = u \circ \rho_\varepsilon^{-1}$$

is well-defined and belongs to $H^1(D)$.

As shown in Velichkov [2023] (lemma 9.5, p.82), the function $\varepsilon \to \mathcal{F}_\Lambda(u_\varepsilon, D)$ is differentiable and for smooth enough $u$ and $\partial\Omega_u$ can be computed as follows:

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathcal{F}_\Lambda(u_\varepsilon, D) = \int_{\partial\Omega_u} (-|\nabla u|^2 + \Lambda)\zeta \cdot \nu d\mathcal{H}^{d-1}\,,$$

where $\nu$ is the exterior normal to $\partial\Omega_u$.

Owing to the optimality of $u$, we get the identity

$$0 = \int_{\partial\Omega_u} (-|\nabla u|^2 + \Lambda)\zeta \cdot \nu d\mathcal{H}^{d-1}$$

and since $\zeta$ is arbitrary, we deduce that

$$|\nabla u| = \sqrt{\Lambda} \quad on \quad \partial\Omega_u \cap D\,.$$

Considering all this, we may conclude that by minimizing the functional $\mathcal{F}_\Lambda$, we obtain the pair $(u, \Omega)$ solving the one-phase Bernoulli problem 2.1.

## 2.2 Properties of the functional $\mathcal{F}_\Lambda$

We proceed to discuss several properties of the functional $\mathcal{F}_\Lambda$, which will prove helpful in later sections.

**Lemma 10** (Scaling). *Let $\Omega \subset \mathbb{R}^d$ be an open domain and $u \in H^1(\Omega)$.*

*(i) Consider $x_0 \in \mathbb{R}^d, r > 0$ and define*

$$u_{x_0,r}(x) := \frac{1}{r}u(x_0 + rx) \quad \text{and} \quad \Omega_{x_0,r} := \left\{x \in \mathbb{R}^d \,:\, x = \frac{y - x_0}{r} \text{ for } y \in \Omega\right\}.$$

*Then, $u_{x_0,r}(x) \in H^1(\Omega_{x_0,r})$ and also*

$$\mathcal{F}_\Lambda(u_{x_0,r}, \Omega_{x_0,r}) = r^{-d}\mathcal{F}_\Lambda(u, \Omega)\,.$$

*(ii) The identity $\mathcal{F}_{t^2\Lambda}(tu, \Omega) = t^2\mathcal{F}_\Lambda(u, \Omega)$ holds for every $t > 0$.*

*Proof.* (i) Take a function $u_{x_0,r}$ as defined above. First, we show that $u_{x_0,r}$ is an element of $H^1(\Omega_{x_0,r})$. By the shape of $\Omega_{x_0,r}$ and $u_{x_0,r}$, and by using the substitution $\varphi(y) := \frac{y-x_0}{r}$, $y \in \Omega$ we compute

$$\int_{\Omega_{x_0,r}} |\nabla u_{x_0,r}(x)|^2 \, dx = \int_{\{\frac{y-x_0}{r};\, y\in\Omega\}} \left|\nabla\left(\frac{1}{r}u(x_0+rx)\right)\right|^2 dx =$$

$$= \int_\Omega \left|\nabla\left(\frac{1}{r}u(y)\right)\right|^2 |\det(\varphi'(y))|\, dy\,.$$

In the right-hand term, $u \in H^1(\Omega)$ provides finiteness

$$\int_\Omega \left|\nabla\left(\frac{1}{r}u(y)\right)\right|^2 |\det(\varphi'(y))|\, dy = \int_\Omega \left|\nabla\left(\frac{1}{r}u(y)\right)\right|^2 \frac{1}{r^d}\, dy =$$

$$= \frac{1}{r^d}\int_\Omega |\nabla u(y)|^2 \, dy < \infty\,.$$

The last identity above and the fact that

$$\begin{aligned}\Lambda\left|\{u_{x_0,r} > 0\} \cap \Omega_{x_0,r}\right| &= \int_{\Omega_{x_0,r}\smallsetminus\{u_{x_0,r}\le 0\}} 1\, dx \\ &= \lambda \int_\Omega \chi_{\{\frac{1}{r}u(y)>0\}}\, |\det(\varphi'(y))|\, dy \\ &= \Lambda\frac{1}{r^d}\left|\Omega \cap \{u(y)>0\}\right|\,,\end{aligned}$$

results in

$$\begin{aligned}\mathcal{F}_\Lambda(u_{x_0,r},\Omega_{x_0,r}) &= \int_{\Omega_{x_0,r}} |\nabla u_{x_0,r}(x)|^2\, dx + \Lambda\left|\{u_{x_0,r}>0\}\cap\Omega_{x_0,r}\right| = \\ &= r^{-d}\mathcal{F}_\Lambda(u,\Omega)\,.\end{aligned}$$

(ii) A computation gives

$$\mathcal{F}_{t^2\Lambda}(t\,u,\Omega) = \int_\Omega |t\,\nabla u(x)|^2\, dx + t^2\Lambda\left|\{u>0\}\cap\Omega\right| = t^2\mathcal{F}_\Lambda(t\,u,\Omega)\,.$$

□

**Lemma 11** (Truncation)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open domain and $u \in H^1(\Omega)$. Then*

$$\mathcal{F}_\Lambda(u,\Omega) - \mathcal{F}_\Lambda(0\vee u,\Omega) = \int_{\Omega\smallsetminus\{u>0\}} |\nabla u|^2\, dx\,.$$

*In addition, the identity*

$$\mathcal{F}_\Lambda(u,\Omega) - \mathcal{F}_\Lambda(u\wedge t,\Omega) = \int_{\Omega\smallsetminus\{u<t\}} |\nabla u|^2\, dx$$

*holds for every $t \ge 0$.*

*Proof.* The proof follows immediately by the definition of $\mathcal{F}$ and the identities

$$\nabla(u\wedge t) = \chi_{\{u<t\}}\nabla u \quad \text{and} \quad \nabla(0\vee u) = \chi_{\{u>0\}}\nabla u\,.$$

□

**Lemma 12** (Comparison)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be an open domain and* $u, v \in H^1(\Omega)$*. The following holds*

$$\mathcal{F}_\Lambda(u \vee v, \Omega) + \mathcal{F}_\Lambda(u \wedge v, \Omega) = \mathcal{F}_\Lambda(u, \Omega) + \mathcal{F}_\Lambda(v, \Omega)\,.$$

*Proof.* We proceed by straightforward computation, integrating non-negative functions:

$$\begin{aligned}
&\mathcal{F}_\Lambda(u \vee v, \Omega) + \mathcal{F}_\Lambda(u \wedge v, \Omega) = \\
&\quad = \int_\Omega |\nabla(u \vee v)|^2 \, dx + \Lambda \left|\{u \vee v > 0\}\right| + \int_\Omega |\nabla(u \wedge v)|^2 \, dx + \Lambda \left|\{u \wedge v > 0\}\right| \\
&\quad = \int_{\{u \geq v\}} |\nabla u|^2 \, dx + \int_{\{u < v\}} |\nabla v|^2 \, dx + \Lambda \left|\{u > 0\} \cup \{v > 0\}\right| \\
&\qquad + \int_{\{u \geq v\}} |\nabla v|^2 \, dx + \int_{\{u < v\}} |\nabla u|^2 \, dx + \Lambda \left|\{u > 0\} \cap \{v > 0\}\right| \\
&\quad = \int_\Omega |\nabla(u)|^2 \, dx + \Lambda \left|\{u > 0\}\right| + \int_\Omega |\nabla(v)|^2 \, dx + \Lambda \left|\{v > 0\}\right| \\
&\quad = \mathcal{F}_\Lambda(u, \Omega) + \mathcal{F}_\Lambda(v, \Omega)\,.
\end{aligned}$$

□

# 3. Examples of solutions

As the theory of the one-phase Bernoulli problem is complex in nature, we may find it useful to explore some simple scenarios and examples of solutions. In this chapter, we will examine two examples of local minimizers of $\mathcal{F}_\Lambda$, which can be expressed explicitly, and develop intuition behind them.

The contents of this chapter are based mainly on the work of Velichkov [2023] (chapter 2, p. 21-26) and are organized as follows. In the first section, we observe that the so-called half-plane solutions are minimizers of $\mathcal{F}_\Lambda$. The second section investigates the changes to energy caused by symmetrization and finds a connection between the class of radial functions and the minimizers.

## 3.1 Half-plane solutions

Let $\mu \in \mathbb{R}^d$ be a unit vector and $\Lambda$ be positive. We introduce the so-called half-plane solution

$$h_\nu(x) = \sqrt{\Lambda}\,\sup\{0, x\cdot\nu\}\,. \tag{3.1}$$

Functions of the above structure are useful concepts in the free boundary regularity theory, for example, they are used in proving that the free boundary is $C^{1,\alpha}$ regular. We will not delve further into this statement, one may find it disclosed in Velichkov [2023].

*Remark.* Notice, that the function $h_\nu(x)$, in the set where it is positive, has the slope $|\nabla h_\nu(x)| = \sqrt{\Lambda}$ and also $\Delta h_\nu(x) = 0$.

Our aim is to show that the functions of the form $h_\nu(x)$ are global minimizers. First, we establish the definitions of local and global minimizers.

**Definition 7** (Local and global minimizer). *Let $D$ be an open set in $\mathbb{R}^d$. Then,*

*- a non-negative function $u : D \to \mathbb{R}^+$ is said to be a local minimizer of $\mathcal{F}_\Lambda$ in $D$, if $u \in H^1_{loc}(D)$ and for any bounded open set $\Omega$, satisfying $\overline{\Omega} \in D$, it meets the condition:*

$$\mathcal{F}_\Lambda(u,\Omega) \le \mathcal{F}_\Lambda(v,\Omega) \quad \textit{for every} \quad v \in H^1_{loc}(D) \quad \textit{such that} \quad u - v \in H^1_0(D)\,,$$

*- a function $u : \mathbb{R}^d \to \mathbb{R}$ is called a global minimizer of $\mathcal{F}_\Lambda$ if $u$ is a local minimizer of $\mathcal{F}_\Lambda$ in $\mathbb{R}^d$.*

Note, that the definition of a minimizer we gave previously in 6 coincides with the definition of a global minimizer.

The following proposition is the main result of this section.

**Proposition 2** (The half-plane solutions are local minimizers). *Let $\nu \in \mathbb{R}^n$ be a unit vector. Then the function $h_\nu(x)$, as defined in 3.1, is a local minimizer of $\mathcal{F}_\Lambda$.*

In order to prove Proposition 2, we require the assistance of the following two lemmas.

**Lemma 13.** *Consider a function $f : \mathbb{R} \to \mathbb{R}^+$, meeting the condition*

$$f(\alpha) = 0 \quad \textit{for some} \quad \alpha < 0 .$$

*Then,*

$$\begin{aligned} f(0) = \int_\alpha^0 f'(t)\,dt &\overset{(1)}{\leq} |\alpha|^{1/2} \left( \int_\alpha^0 |f'(t)|^2 \, dt \right)^{1/2} \\ &\overset{(2)}{\leq} \frac{1}{2} \left( |\{f \neq 0\} \cap \{t \leq 0\}| + \int_\alpha^0 |f'(t)|^2 \, dt \right) . \end{aligned}$$

*Proof.* We may assume without loss, that $\alpha$ is the *"last negative root"* of $f$, that is $f > 0$ on the open interval $(\alpha, 0)$. Then, the first inequality (1) is an immediate consequence of Hölder's inequality 6. The second inequality (2) follows by Young's inequality 7, together with the fact that, by our choice of $\alpha$,

$$|\alpha| \leq |\{f \neq 0\} \cap \{t < 0\}| .$$

□

**Lemma 14.** *Let $D$ be either an open, smooth, bounded set in $\mathbb{R}^n$ or $D = \mathbb{R}^n$. For a given point $x_0 \in \mathbb{R}^d$ and a unit vector $\nu \in \mathbb{R}^d$, consider the function*

$$v(x) := h_\nu(x - x_0) = \sqrt{\Lambda} \sup\{0, (x - x_0) \cdot \nu\} . \tag{3.2}$$

*Then, provided $u \in H^1(D)$ is a non-negative function, such that*

$$u = 0 \ \textit{on} \ \partial D \cap \{v = 0\} ,$$

*it holds that*

$$\mathcal{F}_\Lambda(u \wedge v, D) \leq \mathcal{F}_\Lambda(u, D) . \tag{3.3}$$

*satisfying the equality if and only if $u = u \wedge v$.*

*Proof.* Without loss of generality, we may assume that $x_0 = 0$ and $\nu = e_d$.

Let us divide the domain $D$ into the following two parts:

$$D_+ = D \cap \{x_d > 0\} \quad \text{and} \quad D_- = D \cap \{x_d \leq 0\} ,$$

where $x_d$ denotes the value on the $d$-th coordinate of the point $x \in \mathbb{R}^d$. We calculate

$$\begin{aligned} &\mathcal{F}_\Lambda(u, D) - \mathcal{F}_\Lambda(u \wedge v, D) = \\ &= \int_D |\nabla u|^2 \, dx - \int_D |\nabla(u \wedge v)|^2 \, dx + \Lambda |\{u > 0\} \cap D| - \Lambda |\{u \wedge v > 0\} \cap D| \\ &= \int_{D_-} |\nabla u|^2 \, dx + \Lambda |\{u > 0\} \cap D_-| + \int_{D_+ \cap \{u > \sqrt{\Lambda} x_d\}} \left( |\nabla u|^2 - |\nabla v|^2 \right) dx. \end{aligned}$$

Employing the inequality in Lemma 13, we receive

$$\int_{\{x_d < 0\}} |\nabla u|^2 \, dx + \Lambda |\{u > 0\} \cap \{x_d < 0\}| \geq 2\sqrt{\Lambda} \int_{\{x_d = 0\}} u \, d\mathcal{H}^{d-1} ,$$

where equality is satisfied if and only if $u = 0$ in $\{x_d < 0\}$.

Moreover, we have

$$\int_{D_+\cap\left\{u>\sqrt{\Lambda}\,x_d\right\}} |\nabla u|^2 - |\nabla v|^2 \, dx = \int_{D_+\cap\left\{u>\sqrt{\Lambda}\,x_d\right\}} |\nabla(u-v)|^2 + 2\nabla v \cdot \nabla(u-v)\, dx\,.$$

Owing to the harmonicity of $v = \sqrt{\Lambda}\,x_d$ on $D_+$, we get

$$\int_{D_+\cap\left\{u>\sqrt{\Lambda}\,x_d\right\}} \nabla v \cdot \nabla(u-v)\, dx = -\sqrt{\Lambda}\int_{\{x_d=0\}} u\, d\mathcal{H}^{d-1}\,.$$

We may conclude, that

$$\mathcal{F}_\Lambda(u,D) - \mathcal{F}_\Lambda(u\wedge v, D) \ge \int_{D_+\cap\left\{u>\sqrt{\Lambda}\,x_d\right\}} |\nabla(u-v)|^2 \ge 0\,,$$

where the last inequality is an equality if and only if $u \le v$ on $\mathbb{R}^d$. □

We are now in a position to prove the Proposition 2.

*Proof of Proposition 2.* We may assume without loss that $\nu = e_d$ and put

$$h(x) := h_{e_d}(x) = \sqrt{\Lambda}\,\sup\{0, x_d\}\,.$$

For a given $R > 0$, let $u \in H^1_{loc}(\mathbb{R}^d)$ be a non-negative function such that $(u-h) \in H^1_0(B_R)$. We shall prove that $\mathcal{F}_\Lambda(h, B_R) \le \mathcal{F}_\Lambda(u, B_R)$, which will give us the desired conclusion.

As demonstrated in Lemma 14

$$\mathcal{F}_\Lambda(u\wedge h, B_R) \le \mathcal{F}_\Lambda(u, B_R)\,.$$

and so we may assume that $u \le h$.

Because $h$ is harmonic in $\{x_d > 0\}$, we derive that

$$\begin{aligned}\mathcal{F}_\Lambda(u,B_R) - \mathcal{F}_\Lambda(h,B_R) &= \int_{B_R\cap\{x_d>0\}} |\nabla(u-h)|^2 \, dx - \Lambda\,|\{x_d>0\}\cap\{u=0\}| \\ &= \int_{B_R\cap\{x_d>0\}\cap\{u>0\}} |\nabla(u-h)|^2 \, dx\end{aligned}$$

Where the second equality follows from the fact

$$|\nabla(u-h)| = |\nabla h| = \sqrt{\Lambda} \quad \text{on} \quad u = 0\,.$$

□

## 3.2 Radial solutions

In this section, we aim to comprehend two examples of local minimizers of the functional $\mathcal{F}_\Lambda$, which are radial functions. We show the connection between the minimizers of the functional and the class of radial functions and explore, how symmetrization changes energy values.

### 3.2.1 Symmetrization and Pólya–Szegő inequality

The following result is our main motivation.

**Proposition 3** (Energy of symmetrized functions). *Assume $u : \mathbb{R}^d \to \mathbb{R}$ is a function in $H^1(\mathbb{R}^d)$, then*

$$\mathcal{F}_1(u^{\#}, \mathbb{R}^n) \leq \mathcal{F}_1(u, \mathbb{R}^n)\,, \tag{3.4}$$

*where $u^{\#}$ denotes the Schwarz symmetrization of $u$.*

We refer the reader to Talenti [1994], which is our main source for this subsection, for a deeper dive into the theory of Schwartz's symmetrization.

**Definition 8** (Schwartz's symmetrization). *Let $\Omega$ be a measurable subset of $\mathbb{R}^d$ and suppose the function $u : \Omega \to \mathbb{R}$ vanishes at infinity, meaning all the sets where $u > 0$ have a finite measure:*

$$|\{x \in \Omega : |u(x)| > t\}| < \infty$$

*for every non-negative $t$. We shall call these sets the positive level sets of $u$.*

*The distribution function of $u$ is defined as the measure of its positive level sets*

$$\mu_u(t) := |\{x \in \Omega : |u(x)| > t\}| \quad \textit{for} \quad t \geq 0\,.$$

*We then define the decreasing rearrangement of $u$ as*

$$u^*(s) := \sup\{t \geq 0 : \mu_u(t) \geq s\}\,.$$

*If $\Omega^{\#}$ is an open ball in $\mathbb{R}^n$, centered at 0 and with the same measure as $\Omega$, then the Schwartz symmetrization of $u$ is defined as*

$$u^{\#}(x) = u^*(\omega_n |x|^n)\,; \quad x \in \Omega^{\#}.$$

We formulate several properties of the functions $\mu_u(t)$ and $u^{\#}$, which are straightforward results.

**Lemma 15.** *The function $\mu_u(t)$ is decreasing, right-continuous and defined in $(0, \infty)$. It holds that if we put*

$$\mu_u(t-) := |\{x \in \Omega : |u(x)| \geq t\}|\,,$$

*then*

$$\mu_u(t-) - \mu_u(t) = |\{x \in \Omega : |u(x)| = t\}|\,.$$

**Lemma 16.** *The function $u^*$ is a decreasing, right continuous function defined in $(0, \infty)$ . The level sets*

$$\{s \geq 0 : u^*(s) > t\} = [0, \mu_u(t))$$

*for all non-negative $t$. In other words*

$$\mu_u(t) = |\{s \geq 0 : u^*(s) > t\}|\,.$$

*Proof.* By the definition of $u^*$, if $s$ is on the left-hand side, then t must be such that $\mu_u(t) > s$, therefore $\{s \geq 0 : u^*(s) > t\} \subseteq [0, \mu_u(t))$ .

Similarly, by the definition and because $\mu_u(t)$ is decreasing, if $s < \mu_u(t)$ then belongs to the left hand side of $\{s \geq 0 : u^*(s) > t\} \supseteq [0, \mu_u(t))$ . ☐

**Lemma 17.** *If $s$ is non-negative, then*

$$\mu_u(u^*(s)) \leq s\,.$$

*Additionally, for such $s$ that $0 \leq s < |\mathrm{supp}(u)|$, we have*

$$\mu_u(u^*(s)-) \geq s\,.$$

*Proof.* As established previously in Lemma 16,

$$\mu_u(u^*(s)) = |\{\xi \geq 0 : u^*(\xi) > u^*(s)\}|\,.$$

Because $u^*$ is monotonically decreasing, we deduce

$$\{\xi \geq 0 : u^*(\xi) > u^*(s)\} \subseteq [0, s)\,,$$

leading to the first conclusion.

Next, taking the left limit of the function $\mu_u$, we obtain

$$\mu_u(u^*(s)-) = |\{\xi \geq 0 : u^*(\xi) \geq u^*(s)\}|\,.$$

Given that $u^*(s)$ has a positive value only if $0 \leq s < |\mathrm{supp}(u)|$ , and as $u^*$ decreases monotonically, we may establish the inclusion

$$\{\xi \geq 0 : u^*(\xi) \geq u^*(s)\} \supseteq [0, s]\,.$$

The second conclusion follows. ☐

The Proposition 3 is a direct consequence of the following theorem, together with the fact that the Schwarz symmetrization maintains the measure of level sets. Notice, that the theorem is formulated only for Lipschitz continuous functions. That is, however, sufficient, as the Lipschitz functions are dense in $H^1(\mathbb{R}^d)$, and so the estimate is true for all $H^1(\mathbb{R}^d)$ functions.

**Theorem 18** (Pólya–Szegő inequality)**.** *Let $1 \leq p < \infty$ and consider a Lipschitz continuous function $u$ which vanishes at infinity. That is, for all non-negative $t$*

$$|\{x \in \mathbb{R}^n : |u(x)| > t\}| < \infty\,.$$

*Under these conditions, we may establish the following inequality:*

$$\int_{\mathbb{R}^n} \left|\nabla u^{\#}\right|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx\,.$$

*Proof.* As $u^{\#}$ is a radial function, with the use of polar coordinates and by the definition of $u^*$, we arrive at the following result

$$\int_{\mathbb{R}^n} \left|\nabla u^{\#}\right|^p \, dx = \int_0^\infty \left(-\frac{du^*}{ds}(s)\, n\, \omega_n^{1/n}\, s^{\frac{n-1}{n}}\right)^p ds\,.$$

On the other hand

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \int_0^\infty \left( \frac{d}{ds} \int_{\{x\in\mathbb{R}^n : |u(x)|>u^*(s)\}} |\nabla u|^p \, dx \right) ds \,,$$

as the integral

$$\int_{\{x\in\mathbb{R}^n : |u(x)|>u^*(s)\}} |\nabla u|^p \, dx$$

monotonically increases from 0 to the integral $\int_{\mathbb{R}^n} |\nabla u|^p \, dx$ as $s$ goes from 0 to $\infty$.

The proof can be readily obtained from the following lemma. □

**Lemma 19.** *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ be an open set. Suppose $u$ is a Lipschitz continuous function in $\Omega$ that decays at $\infty$. Then*

*(i) $u^*$ is locally absolutely continuous,*

*(ii) the inequality*

$$\frac{d}{ds} \int_{\{x\in\mathbb{R}^n : |u(x)|>u^*(s)\}} |\nabla u|^p \, dx \geq \left( -\frac{du^*}{ds}(s) \, n \, \omega_n^{1/n} \, s^{\frac{n-1}{n}} \right)^p \tag{3.5}$$

*holds for almost every positive $s$.*

We construct the proof in several steps, proving each before moving on to the next. We will leverage some important theorems during this process, which we will briefly state hereby for the reader.

(Coarea formula). *Suppose $u : \mathbb{R}^d \supset \Omega \to \mathbb{R}$ is a Lipschitz continuous function. Let $f$ be a non-negative measurable function on $\Omega$, then*

$$\int_{\mathbb{R}^d} f(x) \, |\nabla u(x)| \, dx = \int_0^\infty \int_{x\in\mathbb{R}^d : |u(x)|=t} f(x) d\mathcal{H}^{d-1} \, dx \,.$$

*A proof of this formula may be found in Federer [2014]*

(Isoperimetric Theorem). *If $\Omega$ is a measurable subset of $\mathbb{R}^d$ and its measure is finite, then*

$$\mathcal{H}^{d-1} \, |\partial\Omega| \geq \omega_d^{1/d} \, (|\Omega|)^{1-1/d} \,.$$

*We refer to Burago et al. [2013], where his theorem is treated.*

We shall now dive into the main proof.

*Proof of Lemma 19.* (1) The inequalities

$$\int_{\{x\in\mathbb{R}^n : u^*(a)>|u(x)|>u^*(b)\}} |\nabla u| \, dx \geq n \, \omega_n^{1/n} \, \left(u^*(a) - u^*(b)\right) \tag{3.6}$$

and

$$|\{x \in \mathbb{R}^n : u^*(a) > |u(x)| > u^*(b)\}| \leq b - a \tag{3.7}$$

hold if $|supp(u)| > b > a \geq 0$.

*Proof of (1).*

$$\int_{\{x\in\mathbb{R}^n: u^*(a)>|u(x)|>u^*(b)\}} |\nabla u|\, dx$$
$$= \quad \textit{(by the Coarea formula)}$$
$$\int_{u^*(a)}^{u^*(b)} \left(\mathcal{H}^{n-1}\left|\{x\in\mathbb{R}^n : |u(x)|=t\}\right|\right) dt$$
$$\geq \quad \textit{(by the Isoperimetric theorem)}$$
$$\int_{u^*(a)}^{u^*(b)} n\,\omega_n^{1/n}\left(\left|\{x\in\mathbb{R}^n : |u(x)|\geq t\}\right|\right)^{\frac{n-1}{n}} dt$$
$$\geq \quad \textit{(the integrand is a monotone function)}$$
$$n\,\omega_n^{1/n}\left(\left|\{x\in\mathbb{R}^n : |u(x)|\geq u^*(a)\}\right|\right)^{\frac{n-1}{n}} (u^*(a)-u^*(b))$$
$$\geq \quad \textit{(by Lemma 17)}$$
$$n\,\omega_n^{1/n}\, a^{\frac{n-1}{n}} (u^*(a)-u^*(b))\,.$$

Now, by the definition of $u^*$ and the properties of $\mu_u$ we have

$$\left|\{x\in\mathbb{R}^n : u^*(a)>|u(x)|>u^*(b)\}\right| = \mu_u(u^*(b)) - \mu_u(u^*(a)-)\,,$$

By Lemma 17 follows

$$\mu_u(u^*(b))\leq b \quad \text{and} \quad \mu_u(u^*(a)-)\geq a\,.$$

We continue with the use of the properties of $u^*$. Because by Lemma 16, we see that

$$\operatorname{supp}(u^*) = [0, |\operatorname{supp}(u)|]$$

The previous inequalities show the following

$$(b-a)\operatorname{esupp}|\nabla u| \geq n\,\omega_n^{1/n}\, a^{\frac{n-1}{n}} (u^*(a)-u^*(b))\,,$$

when $|\operatorname{supp}(u)| > b > a \geq 0$. Hence

$$\operatorname{esupp}|\nabla u| \geq -n\,\omega_n^{1/n}\, s^{\frac{n-1}{n}} \frac{du^*}{ds}(s)$$

for almost every non-negative s. ☐

(2) For almost all $s\geq 0$ the following inequality holds

$$\frac{d}{ds}\int_{\{x\in\mathbb{R}^n: |u(x)|>u^*(s)\}} |\nabla u|\; dx \geq -n\,\omega_n^{1/n}\, s^{\frac{n-1}{n}} \frac{du^*}{ds}(s)\,. \tag{3.8}$$

*Proof of (2).* For $0\leq s<|\operatorname{supp}(u)|$ we have

$$\frac{d}{ds}\int_{\{x\in\mathbb{R}^n: |u(x)|>u^*(s)\}} |\nabla u|\; dx = \lim_{h\to 0^+}\frac{1}{h}\int_{\{x\in\mathbb{R}^n: u^*(s)\geq|u(x)|>u^*(s+h)\}} |\nabla u|\; dx\,. \tag{3.9}$$

Previously, we saw that the right-hand side of the inequality is greater than or equal to

$$n\,\omega_n^{1/n}\, s^{\frac{n-1}{n}} (u^*(s)-u^*(s+h))\,. \tag{3.10}$$

We have thus shown the proof of 3.5 in the case of $p=1$ for almost all $s\geq 0$. ☐

(3) We desire to prove 3.5 for all $1 \leq p < \infty$. Without loss of generality, we can suppose that $u^*$ is strictly decreasing in a neighborhood of $s$ (otherwise there is nothing to prove).

We claim that

$$h = |\{x \in \mathbb{R}^n : u^*(s) \geq |u(x)| > u^*(s+h)\}| \tag{3.11}$$

for small enough positive $h$.

*Proof of (3).* Take the distribution function $\mu_u$ of $u$. Then

$$h = \mu_u(u*(s+h)) - \mu_u(u^*(s)) . \tag{3.12}$$

By Lemma 15, we know that

$$\mu_u(t-) - \mu_u(t) = |\{\xi \geq 0 : u^*(\xi) = t\}| .$$

By the properties of $\mu_u$ in Lemma 17:

$$\mu_u(u*(\xi)) \leq \xi \leq \mu_u(u*(\xi)-) ,$$

if $0 \leq \xi < |\text{supp}(u)|$. The claim follows. □

Now

$$\frac{1}{h}\int_{\{x\in\mathbb{R}^n: u^*(s)\geq|u(x)|>u^*(s+h)\}} |\nabla u|^p \, dx$$
$$\geq \quad \textit{(by Jensen's inequality 8)}$$
$$\left(\frac{1}{h}\int_{\{x\in\mathbb{R}^n: u^*(s)\geq|u(x)|>u^*(s+h)\}} |\nabla u| \, dx\right)^p .$$

Consequently

$$\frac{d}{ds}\int_{\{x\in\mathbb{R}^n:|u(x)|>u^*(s)\}} |\nabla u|^p \, dx \geq \left(\frac{d}{ds}\int_{\{x\in\mathbb{R}^n:|u(x)|>u^*(s)\}} |\nabla u| \, dx\right)^p .$$

The proof is now complete. □

### 3.2.2 Radial minimizers

Let $B_r$ be the open ball in $\mathbb{R}^d$ centered at zero with its radius set to $r > 0$. We consider the following minimization problem in the exterior domain $\mathbb{R}^d \setminus \overline{B_r}$

$$min\left\{\int_{\mathbb{R}^d} |\nabla(u)|^2 \, dx + |\{u > 0\}| \, : \, u \in H^1(\mathbb{R}^d); \, u = 1 \text{ in } B_r\right\} . \tag{3.13}$$

The interior version of the problem reads as

$$min\left\{\int_{B_r} |\nabla(u)|^2 \, dx + |\{u > 0\}| \, : \, u \in H^1(B_r); \, u = 1 \text{ in } \partial B_r\right\} . \tag{3.14}$$

We will prove that the problems 3.13 and 3.14 admit unique solutions, which can be explicitly calculated.

**Proposition 4** (Optimal exterior solution)**.** *For every $r > 0$, there exists a unique solution $u_r$ to the problem 3.13 in $\mathbb{R}^d$. The solution can be expressed as*

$$u_r(x) = \begin{cases} 1 & in \quad B_r\,, \\ h_r(x) & in \quad B_R \setminus B_r\,, \\ 0 & in \quad \mathbb{R}^d \setminus B_R\,, \end{cases}$$

*where the radius $R > r$ is determined by $r$ and the dimension $d$, and $h_r$ is a radial function, specifically given by*

$$h_r(x) = \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}} \quad if \quad d \geq 3 \qquad and \qquad h_r(x) = \frac{\ln|x| - \ln R}{\ln r - \ln R} \quad if \quad d = 2\,.$$

*Proof.* By Proposition 3, taking Schwarz symmetrization $u^{\#}$ of every function $u$ gives that $\mathcal{F}_1(u^{\#}, \mathbb{R}^n) \leq \mathcal{F}_1(u, \mathbb{R}^n)$. Thus, there is a minimizer of $\mathcal{F}_1$ which is a radial function. First, we seek to minimize $\mathcal{F}_1$ in the class of radial functions.

Let $d \geq 3$. For every $0 < r < R$, consider the function

$$u_{r,R}(x) = \begin{cases} 1, & \textit{if } |x| \leq r, \\ \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}, & \textit{if } r < |x| < R, \\ 0, & \textit{if } |x| \geq R. \end{cases}$$

Function $u_{r,R}$ is the unique harmonic function in the set $B_R \setminus B_r$. Therefore, every minimizer of $\mathcal{F}_1$ among the radial functions must take the form of a function $u_{r,R}$, for some $0 < r < R$.

Now, taking the functional

$$\mathcal{F}_1(u_{r,R}, \mathbb{R}^n) = \int_{B_R \setminus B_r} \left| \nabla \left( \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}} \right) \right|^2 dx + |B_R| = \frac{d(d-2)}{r^{2-d} - R^{2-d}} \omega_d + \omega_d R^d\,,$$

we wish to find the optimal radius $R$.

Considering the function $f(R) := \frac{d(d-2)}{r^{2-d} - R^{2-d}} + R^d$ we see that it is strictly convex, since

$$\begin{aligned}(f(R))'' &= \left( -d(d-2)^2 \frac{R^{1-d}}{(r^{2-d} - R^{2-d})^2} + dR^{d-1} \right)' = \\ &= 2d(d-2)^2 \frac{R^{2-2d}}{(r^{2-d} - R^{2-d})^3} + d(d-1)R^{d-2}\end{aligned}$$

is greater than zero for $d \geq 3$ if $R > r > 0$. Moreover

$$\lim_{R \to r^+} f(R) = \lim_{R \to \infty} f(R) = +\infty \tag{3.15}$$

and so $f(R)$ has a unique minimum $R_* > r$.

Let $d = 2$. For every $0 < r < R$, we select the function

$$u_{r,R}(x) = \begin{cases} 1, & \textit{if } |x| \leq r, \\ \frac{\ln|x| - \ln R}{\ln r - \ln R}, & \textit{if } r < |x| < R, \\ 0, & \textit{if } |x| \geq R. \end{cases}$$

which is the unique harmonic function in $B_R \setminus B_r$.

We take the energy

$$\mathcal{F}_1(u_{r,R}, \mathbb{R}^n) = \int_{B_R \setminus B_r} \left| \nabla \left( \frac{\ln|x| - \ln R}{\ln r - \ln R} \right) \right|^2 dx + |B_R| = \frac{2}{\ln(R/r)}\pi + \pi R^2 .$$

As in the case of $d \geq 3$, there exists a unique minimizer of the calculated energy $R_*$.

Finally, we show that the functions $u_{r,R_*}$ are the unique minimizers of $\mathcal{F}_1$ among all admissible functions.

Assume $u$ is a minimizer of $\mathcal{F}_1$ which is not a radial function. By 3, the symmetrized function $u^\#$ is also a minimizer. As a radial function, $u^\#$ has the form $u^\# = u_{r,R_*}$ in its corresponding dimension $d$ and we note that in particular $|\{u > 0\}| = |B_{R_*}|$.

Now, with $u$ and $u^\#$ both optimal, by Lemma 12 are the functions $v := u \wedge u^\#$ and $V := u \vee u^\#$ also minimizes of $\mathcal{F}_1$. Since $u$ is not radial, we have either

$$|\{v > 0\}| \neq |B_{R_*}| \quad \text{or} \quad |\{V > 0\}| \neq |B_{R_*}| .$$

Because their respective symmetrized functions $v^\#$ and $V^\#$ are optimal as well by 3, we necessarily have the identity $v^\# = u^\# = V^\#$. In particular

$$|\{v > 0\}| = |\{V > 0\}| = |B_{R_*}| ,$$

which is a contradiction.

In conclusion, the functions $u_{r,R_*}$ are the unique minimizers of $\mathcal{F}_1$. □

Let us investigate the relationship between $R(r)$ and $r$ a little further. We make several observations, based on the previous Proposition 4.

*Corollary.* The following statements are true for $d \geq 2$.

(i) The radius $R(r)$ from 4 is a continuous function of $r$, such that

$$r < R < r + 1 .$$

(ii) It holds that

$$\lim_{r \to 0} R(r) = 0 .$$

(iii) The gradient of the radial function $h_r$ is of the form

$$|\nabla h_r(x)| = \left( \frac{|x|}{R} \right)^{1-d} .$$

*Proof of corollary.* (i) We remember the result given in Lemma 14, which implies that the optimal slope of the solution $u_r$ to 3.14 is 1. Therefore, we may conclude that $u_r$ has bounded support. Specifically, $u = 0$ outside of the set $\overline{B_r} + B_1$.

(ii) Let $d \geq 3$. Consider the optimal radius $R_*$, minimizing the function $f(R)$. Since $R_*$ is minimum, we have the identity

$$\begin{aligned} 0 &= f'(R_*) \\ 0 &= -d(d-2)^2 \frac{R_*^{1-d}}{(r^{2-d} - R_*^{2-d})^2} + dR_*^{d-1} \\ (d-2)^2 R_*^{1-d} &= R_*^{d-1}(r^{2-d} - R_*^{2-d})^2 \\ d-2 &= R_*^{d-1}(r^{2-d} - R_*^{2-d})\,. \end{aligned}$$

With this, we obtain a very nice, continuously differentiable implicit function

$$\mathcal{I}(R,r) = R^{d-1} - Rr^{d-2} - (2-d)r^{d-2} \tag{3.16}$$

in $\mathbb{R}^+ \times \mathbb{R}^+$.

(iii) We are now interested in the behavior of $R(r)$ as $r \to 0$. By the Implicit function theorem (see Rudin [1964]), $R$ can be presented as a single-variable function of $r$ of the class $C^1$, that is $R = \gamma(r)$.

We notice that since $\mathcal{I}(0,0) = 0$, we have $\gamma(0) = 0$. Remember the definition of the first derivation of a single-variable function in the point $r = 0$:

$$\begin{aligned} \frac{d\gamma}{dr}(0) &= \lim_{h\to 0} \frac{\gamma(0+h) - \gamma(0)}{h} \\ &= \lim_{h\to 0} \frac{\gamma(h)}{h}\,. \end{aligned}$$

On the other hand, we may differentiate the implicit function:

$$\frac{d\gamma}{dr} = -\frac{\frac{\partial f}{\partial r}}{\frac{\partial f}{\partial R}} = \frac{r^{d-3}(R(d-2) - (d-2)^2)}{(d-1)R^{d-2} - r^{d-2}}\,,$$

where we treat the resulting function as a function of the variable $r$.

Combining the identities, we get

$$0 = \frac{d\gamma}{dr}(0) = \lim_{h\to 0} \frac{\gamma(h)}{h} = \lim_{r\to 0} \frac{R(r)}{r}$$

which implies, that $R(r) \to 0$ as $r$ approaches zero. (All limits in 0 were taken from the right.)

Let $d = 2$ and consider the optimal radius $R_*$. By the same process as for $d \geq 3$, we obtain the identity

$$R_*(\ln r - \ln R_*) = 1 \tag{3.17}$$

and the associated implicit function

$$\mathcal{I}(R,r) = r\mathrm{e}^{1/R} - R\,,$$

which is continuously differentiable in $\mathbb{R}^+ \times \mathbb{R}^+$.

Since we necessarily have that $\mathcal{I}(R,r) = 0$, even for those $r$ which are close to zero, the function $R\mathrm{e}^{1/R}$ must also go to zero. This means, that the decay of $R(r)$ must overpower the growing term $\mathrm{e}^{1/R}$. That is, $R(r) \to 0$ as $r \to 0$.

(iv) Let us now calculate the norm of the gradient of $h_r$. We describe the steps for $d \geq 3$, as in two dimensions the process is analogous.

First, we find the partial derivatives:

$$\begin{aligned}\frac{\partial h_r}{\partial x_i} &= \frac{\partial h_r}{\partial |x|}\frac{\partial |x|}{\partial x_i} \\ &= \frac{1}{r^{2-d} - R^{2-d}}(2-d)|x|^{1-d}\frac{x_i}{|x|} \\ &= \frac{1}{r^{2-d} - R^{2-d}}(2-d)\frac{x_i}{|x|^d}\,.\end{aligned}$$

Then, the euclidian norm of $\nabla h_r = \left(\frac{\partial h_r}{\partial x_1}, \dots, \frac{\partial h_r}{\partial x_d}\right)$:

$$|\nabla h_r| = \frac{1}{r^{2-d} - R^{2-d}}(2-d)\frac{1}{|x|^{d-1}} = \left(\frac{|x|}{R}\right)^{1-d}\,,$$

where in the second equality we utilized the identity 3.16.

□

Now, we move to the interior version of the problem.

**Proposition 5** (Optimal interior solution)**.** *For every $R > 0$, there exists a dimensional constant $C_d > 0$ such that for every $R > C_d$ there is a unique solution $u_R$ to the problem 3.14. Moreover, the solution can be expressed as*

$$u_R(x) = \begin{cases} 1 & in \quad \partial B_R\,, \\ h_R(x) & in \quad B_R \setminus B_r\,, \\ 0 & in \quad B_R\,, \end{cases} \tag{3.18}$$

*where $h_R$ is a radially symmetric function, specifically given by*

$$h_R(x) = \frac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}} \quad if \quad d \geq 3 \qquad and \qquad h_R(x) = \frac{\ln|x| - \ln r}{\ln R - \ln r} \quad if \quad d = 2\,.$$

*Proof.* Once again, we notice that by Proposition 3, for every function $u$ there is a radially symmetric function $u^\#$ with lower energy. Let us consider a function $v = 1 - u$ and its Schwarz symmetrization $v^\#$. We now redefine the function $u^\#$ as $u^\# = 1 - v^\#$ and take the energy functional

$$\begin{aligned}\mathcal{F}_1(u^\#, B_R) &= \int_{B_R} \left|\nabla u^\#\right|^2\,dx + |B_R| + \left|\{u^\# > 0\} \cap B_R\right| \\ &= \left|\nabla v^\#\right|^2\,dx + |B_R| + \left|\{v^\# < 1\} \cap B_R\right| \\ &\leq |\nabla v|^2\,dx + |B_R| + |\{v < 1\} \cap B_R| \\ &\leq |\nabla u|^2\,dx + |B_R| + |\{u < 1\} \cap B_R| \\ &= \mathcal{F}_1(u, B_R)\,.\end{aligned}$$

Therefore, we know there exists a radially symmetric function $u^\#$ which minimizes $\mathcal{F}_1$. Because by 9, $u^\#$ is harmonic in $\{u^\# > 0\}$, it has necessarily the form of the unique harmonic function $u_R$ in $B_R \setminus B_r$, defined by 3.18. That is, $u^\# = u_R$ for some $0 < r < R$.

Let $d \geq 3$. Now, for a given $r \in (0,R)$, we calculate the functional

$$\begin{aligned}\mathcal{F}_1(u_R, B_R) &= \int_{B_R \setminus B_r} \left| \nabla \left( \frac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}} \right) \right|^2 dx + |B_R \setminus B_r| \\ &= \frac{d(d-2)}{r^{2-d} - R^{2-d}} \omega_d + \omega_d (R^d - r^d)\,.\end{aligned}$$

Consider the part of the energy, which is dependent on $r$, as the function

$$f(r) := \frac{d(d-2)}{r^{2-d} - R^{2-d}} - r^d\,.$$

As we may observe,

$$\lim_{r \to 0+} f(r) = 0 \qquad \text{and} \qquad \lim_{r \to R} f(r) = +\infty.$$

Furthermore, if we take $R$ large enough and $r = \frac{R}{2}$, then we find that $f(R/2) < 0$. Now, we calculate

$$f'(r) = \frac{d(d-2)^2 r^{1-d}}{(r^{2-d} - R^{2-d})^2} - d r^{d-1}\,.$$

It is easy to check, that

$$f'(r) = 0 \qquad \Longleftrightarrow \qquad g(r) := (d-2) - r + r^{d-1} R^{2-d} = 0\,,$$

therefore we will focus on the function $g(r)$.

Upon rearranging the equation

$$(d-2) - r + r^{d-1} R^{2-d} = 0$$

$$r^{d-1} = \frac{r - (d-2)}{R^{2-d}}$$

we see that on the right-hand side of the equality is a linear function. The equation $g(r)$, by its polynomial nature, has at most $d-1$ zeroes. However, since the term $r^{d-1}$ can meet the linear function at most twice, the number of possible solutions of $g(r) = 0$ is reduced to two.

Now, if we take again a large enough $R$, then

$$g(d-1) < 0 \qquad \text{and} \qquad g(R-2) < 0\,,$$

and we also have

$$g(0) = g(R) = d - 2 > 0\,.$$

Therefore, the equation $g(r) = 0$ has exactly two solutions

$$r_1 \in (0, d-1) \qquad \text{and} \qquad r_2 \in (R-2, R).$$

Let the $m_d$ be the minimum of $f$ in the interval $[0, d-1]$. Then, for a large enough $R$, we have

$$f(R-2) = (R-2)^{d-2} \left( \frac{d(d-2)}{1 - (1 - \frac{2}{R})^{d-2} - (R-2)^2} \right) < m_d$$

Hence, there is a unique $r \in (0, R)$ that minimizes $f$ and therefore also $\mathcal{F}_\Lambda$ in $(0, R)$. Moreover, we know that $R - 2 < r < R$.

Let now $d = 2$. For every $r \in (0, R)$, consider the function $u_R$ as given by 3.18. Let us take the energy functional

$$\mathcal{F}_1(u_R, B_R) = \int_{B_R \setminus B_r} \left| \nabla \left( \frac{\ln|x| - \ln r}{\ln R - \ln r} \right) \right|^2 dx + |B_R \setminus B_r| = \frac{2\pi}{\ln R - \ln r} + \pi(R^2 - r^2)$$

Similarly to before, we set

$$f(r) := \frac{2}{\ln R - \ln r} - r^2 \quad with \quad f'(r) = \frac{2}{r(\ln R - \ln r)^2} - 2r\,.$$

Next, we take

$$g(r) := 1 - r(\ln R - \ln r)\,.$$

As above, $g(r) = 0$ can have at most two solutions in the interval $(0, R)$.

For a large enough $R$ we then have

$$g(1) = 1 - \ln R < 0 \quad \text{and} \quad g(R-2) = 1 - (R-2)\ln 1 - \frac{2}{R-2} < 0\,,$$

meanwhile

$$g(0) = g(R) = 1\,.$$

Consequently, the two zeroes of $g$ are in the intervals $(0, 1)$ and $(R - 2, R)$, respectively,

Let us, again, take a large enough R, then we have

$$f(R-2) = \frac{2}{\ln\left(1 + \frac{2}{R-2}\right)} - (R-2)^2 < -1 < f(1).$$

Thus, for a large enough $R$ there is a unique $r$ that minimizes $f$ in $(0, R)$ and $R - 2 < r < R$.

The uniqueness of the solution is proven analogically as in Proposition 3.14.

□

By almost the exact same process as in Proposition 4, we may deduce the magnitude of $\nabla h_R$.

*Corollary.* For $d \geq 2$, the gradient of the radial function $h_R$ is of the form

$$|\nabla h_R(x)| = \left( \frac{|x|}{r} \right)^{1-d}\,.$$

# 4. Existence of solutions

In this chapter, we will prove that there exist local minimizers of the functional $\mathcal{F}_\Lambda$. We then move on to discuss several qualitative properties of these functions.

### 4.0.1 Existence by direct method

The main result of this chapter is the following proposition.

**Proposition 6** (Existence)**.** *Let $\Lambda > 0$ and $D \subset \mathbb{R}^d$ be a bounded open set. Consider a fixed boundary value function $g \in H^1(D)$ such that $g$ is non-negative in $D$. Then, there exists a solution to the variational problem*

$$\min\{\mathcal{F}_\Lambda(u, D) : u \in H^1(D),\, u - g \in H^1_0(D)\}\,. \tag{4.1}$$

As we aim to prove the existence of a solution to the minimization problem 4.1, we follow the structure of the *direct method of calculus of variations.* The method consists of three steps:

(i) We find a minimizing sequence $(u_n)$ along which $\mathcal{F}_\Lambda$ converges to its infimum on $D$.

(ii) We show that $(u_n)$ admits a subsequence $(u_{n_k})$ which converges in $H^1(D)$ an element $u$.

(iii) We prove that $u$ is indeed a minimizer of $\mathcal{F}_\Lambda$.

The third step (iii) depends on a specific property of the functional $\mathcal{F}_\Lambda$, called sequential lower weak semi-continuity with respect to weak convergence. We continue to show this property.

**Lemma 20** (Lower weak semi-continuity)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain. For any sequence of non-negative functions $\{u_k\}_{k\in\mathbb{N}}$ in $H^1(\Omega)$, converging weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$, $1 \le p < \infty$ to a function $u \in H^1(\Omega)$, it holds that*

$$\mathcal{F}_\Lambda(u, \Omega) \le \liminf_{k\to\infty} \mathcal{F}_\Lambda(u_k, \Omega) \tag{4.2}$$

We base the proof of Lemma 20 on the work of Arama and Leoni [2012](theorem 2.2). In the second half of the proof we rely heavily on a corollary of the well-known Hahn-Banach theorem, which states:

*If $X$ is a normed space and $x \in X$, there exists $\rho \in X^*$ such that*

$$\rho(x_0) = \|x_0\| \quad \textit{and} \quad |\rho(x)| \le \|x\| \textit{ for all } x \in X\,. \tag{4.3}$$

This statement together with the Hahn-Banach theorem can be found in Rudin [1991] (chapter 3, p. 56 - 59).

*Proof.* Consider a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ conditioned as above. Taking the energy functional of $u_n$

$$\mathcal{F}_\Lambda(u_n, D) = \int_D (|\nabla u_n(x)|^2 + \chi_{\{u_n>0\}})dx\,,$$

we note that the sequence $\{\chi_{\{u_n>0\}}\}_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. Because $L^1(\Omega)$ is a separable function space, by Banach–Alaoglu (4), we may find a function $\xi \in L^\infty(\Omega)$, $0 \le \xi \le 1$ and extract a subsequence $\{u_k\}$ along which convergence occurs:

$$\chi_{\{u_k>0\}} \overset{\star}{\rightharpoonup} \xi \quad \text{in} \quad L^\infty(\Omega)\,.$$

We show that $\xi \ge \chi_{\{u>0\}}$. Having $u_k \to u$ in $L^1_{loc}(\Omega)$ and $\chi_{\{u_k>0\}} \overset{\star}{\rightharpoonup} \xi$ in $L^\infty(\Omega)$, we pass to the limit $k \to \infty$ in the identity

$$\int_\Omega u_k(1-\chi_{\{u_k>0\}})dx = 0\,,$$

which yields

$$\int_\Omega u_k(1-\xi)dx = 0 \text{ for all } r>0\,.$$

Since $u_k \ge 0$ and $0 \le \xi \le 1$, we arrive at the conclusion

$$u_k(1-\xi) = 0 \quad \text{a.e. in } \Omega\,.$$

Hence

$$\xi = 1 \quad \text{a.e. in the set } \{u_k>0\}$$

and thus

$$\chi_{\{u_k>0\}} \overset{\star}{\rightharpoonup} \xi \ge \chi_{\{u>0\}}\,.$$

The proof will follow by lower semi-continuity of the $H^1$ norm (with respect to the weak $H^1$ convergence).

We define closed balls $\mathcal{B}(\alpha) := \{u \in H1(\Omega) : \|u\|_{H^1} \le \alpha\}$. We see that the norm is lower semi-continuous if $\mathcal{B}(r)$ are weakly closed for every $\alpha > 0$.

Let $\{u_n\}$ be a sequence in the set $\mathcal{B}(\alpha)$, $\alpha > 0$ weakly converging to $u \in H^1(\Omega)$. Then, by 4.3, there exists a linear functional $\rho \in (H^1(\Omega))^*$ such that

$$\rho(u) = \|u\|_{H^1} \quad \text{and} \quad |\rho(u_n)| \le \|u_n\|_{H^1} \text{ for all } u_n \in \mathcal{B}(\alpha)\,.$$

Due to weak convergence,

$$\|u\|_{H^1} = \rho(u) = \lim_{n\to\infty} \rho(u_n) \le \liminf_{n\to\infty} \|u_n\|_{H^1} \le \alpha\,.$$

Therefore

$$\begin{aligned}\mathcal{F}_\Lambda(u,D) &= \int_D \left(|\nabla u(x)|^2 + \chi_{\{u>0\}}\right) dx \le \\ &\le \int_D \left(|\nabla u(x)|^2 + \xi\right) dx \le \\ &\le \liminf_{n\to\infty} \int_D |\nabla u_n(x)|^2\, dx + \lim_{n\to\infty} \int_D \chi_{\{u_n>0\}}\, dx = \liminf_{n\to\infty} \mathcal{F}_\Lambda(u_n, D)\,.\end{aligned}$$

□

Now, we are equipped to prove the existence of a solution to 4.1.

*Proof of Proposition (6).* (i) The functional $\mathcal{F}_\Lambda(u,D)$ is non-negative, i.e. bounded from below by 0 on $D$. Therefore, it has an infimum greater than $-\infty$ and by the definition, we may consider a sequence $\{u_n\} \in H^1(D)$ where

$$\lim_{n\to\infty} \mathcal{F}_\Lambda(u_n, D) = \inf\{\mathcal{F}_\Lambda(u, D) : u \in H^1(D),\, u_n - g \in H^1_0(D)\} > -\infty\,. \quad (4.4)$$

We call this sequence a *minimizing sequence*.

Without loss of generality, we may assume that

$$\mathcal{F}_\Lambda(u_n, D) \leq \mathcal{F}_\Lambda(g, D) \leq \infty \quad (4.5)$$

We also assume by 11, that $u_n \geq 0$ for each $n \in \mathbb{N}$.

(ii) We wish to show the functions $u_n$ are uniformly bounded in $H^1(D)$ By definition of the $H^1$ norm, we have the identity

$$\|u_n\|_{H^1} = \|\nabla u_n\|_{L^2} + \|u_n\|_{L^2}\,. \quad (4.6)$$

We notice, that the first term of 4.6 comes from the shape of the functional and is therefore bounded by $\mathcal{F}_\Lambda(g, D)$, according to 4.5.

Now we want to gain control over the second term as well. Because $u_n - g = 0$ on the boundary of $D$, we have by Poincaré's inequality 5

$$\|u_n - g\|_{L^2} \leq C(d, D)\|\nabla u_n - \nabla g\|_{L^2}$$

where $C(d, D)$ is independent of the function $u_n$. Therefore $u_n$ is bounded in $H^1(D)$.

Because $H^1(D)$ is compactly embedded in $L^p(D)$, $1 \leq p < \infty$ by Rellich-Kondrachov 3, the minimizing sequence contains a subsequence $\{u_{n_k}\}$ which converges weakly in $H^1(D)$ and strongly in $L^p(D)$ to a function $u \in H^1(D)$.

(iii) Now we have a sequence $\{u_{n_k}\}$ in accordance with the conditions of 20, which gives us weak lower semicontinuity

$$\mathcal{F}_\Lambda(u, D) \leq \liminf_{n\to\infty} \mathcal{F}_\Lambda(u_n, D)\,,$$

and so, $u$ is a solution to 4.1.

□

### 4.0.2 Properties of solution

**Definition 9** (Subarmonic function). *Let $D \subset \mathbb{R}^d$ be an open set. A continuous function $u$ on $D$ is said to be subharmonic, if for any ball $B_r(x) \subset D$ and for all functions $\psi$ harmonic in $B_r(x)$, it holds that*

$$\text{if } u \leq \psi \text{ on } \partial B_r(x) \quad \text{then} \quad u \leq \psi \text{ in } B_r(x)\,.$$

**Lemma 21** (The minimizers are subharmonic functions I.). *Suppose $D \subset \mathbb{R}^d$ is a bounded open set and the function $u \in H^1(D)$ is a minimizer of $\mathcal{F}_\Lambda$ in $D$. Then $u$ $u$ is non-negative, and subharmonic in $D$ (or, in other words, $\Delta u \geq 0$) in the sense of distributions:*

*For every $\varphi \in C^\infty_c(D)$, such that $\varphi \geq 0$ on $D$, holds the identity*

$$\int_D \nabla u \cdot \nabla \varphi \, dx \leq 0\,.$$

*Proof.* The fact that $u \geq 0$ stems straight from the Lemma 11.

Now, consider a given non-negative function $\varphi \in C_c^\infty(D)$. Suppose that $t \geq 0$ and $v = u - t\varphi$. Then when have that $(v \vee 0) \leq u$. We may integrate on the support of $\varphi$

$$\begin{aligned}\mathcal{F}_\Lambda(u, D) &= \int_D |\nabla u|^2 \, dx + \Lambda \left|\{u > 0\} \cap D\right| \\ &\leq \int_D |\nabla(v \vee 0)|^2 \, dx + \Lambda \left|\{(v \vee 0) > 0\} \cap D\right| \\ &\leq \int_D |\nabla v|^2 \, dx + \Lambda \left|\{u > 0\} \cap D\right| .\end{aligned}$$

By this reasoning, we have that

$$\begin{aligned}\int_D |\nabla u|^2 \, dx &\leq \int_D |\nabla(u - t\varphi)|^2 \, dx \\ &= \int_D |\nabla u|^2 \, dx - 2t \int_D \nabla u \cdot \nabla \varphi \, dx + t^2 \int_D |\nabla \varphi|^2 \, dx ,\end{aligned}$$

The claim immediately follows by taking the right derivative at $t = 0$. □

It is an interesting result to show, that we may weaken the conditions on the optimality of $u$, and still receive subharmonicity for those non-negative functions, which are harmonic on the set where they are strictly positive. We formulate the precise statement in the following Lemma.

**Lemma 22** (The minimizers are subharmonic functions II.)**.** *Suppose the set $D \subset \mathbb{R}^d$ is bounded and open and the non-negative function $u \in H^1(D)$ is harmonic in the set $\Omega_u := \{u > 0\}$. That is*

$$\int_D |\nabla u|^2 \, dx \leq \int_D |\nabla v|^2 \, dx \quad \textit{for every} \quad v \in H^1(D)$$

$$\textit{such that} \quad u - v \in H_0^1(D) \quad \textit{and} \quad v = 0 \quad \textit{on} \quad D \setminus \Omega_u .$$

*Then, $u$ is subharmonic on $D$, $\Delta u \geq 0$, on $D$ in the sense of distributions.*

*Proof.* Consider a given non-negative function $\varphi \in C_c^\infty(D)$ and let $p_\varepsilon : \mathbb{R} \to \mathbb{R}$ be given by

$$p_\varepsilon(x) = \begin{cases} 0 & \text{if } x \leq \frac{\varepsilon}{2}, \\ \frac{2x}{\varepsilon} - 1 & \text{if } \frac{\varepsilon}{2} \leq x \leq \varepsilon, \\ 1 & \text{if } x \geq \varepsilon . \end{cases}$$

Now, let us consider the function $v := u - t p_\varepsilon(u)\varphi$, which is of the class $H^1(D)$ and we may compare it to $u$ on an energy level. Then, for small enough $t \geq 0$, it is true that

$$\{u > 0\} = \{v > 0\} \quad \text{and} \quad \int_D |\nabla u|^2 \, dx \leq \int_D |\nabla v|^2 \, dx .$$

Also,

$$\begin{aligned}\int_D |\nabla v|^2 \, dx &= \int_D |\nabla(u - t p_\varepsilon(u)\varphi)|^2 \, dx \\ &= \int_D |\nabla u|^2 - 2t \nabla u \cdot \nabla(p_\varepsilon(u)\varphi) + t^2 |\nabla(p_\varepsilon(u)\varphi)|^2\end{aligned}$$

Therefore,

$$\int_D \nabla u \cdot \nabla(p_\varepsilon(u)\varphi) \le \frac{t}{2}\int_D |\nabla(p_\varepsilon(u)\varphi)|^2$$

and by passing $t \to 0$, we get

$$\begin{aligned} 0 \ge \int_D \nabla u \cdot \nabla(p_\varepsilon(u)\varphi) &= \int_D \nabla u \cdot (p'_\varepsilon(u)\varphi\nabla u + p_\varepsilon(u)\nabla\varphi) \\ &= \int_D p'_\varepsilon(u)\varphi|\nabla u|^2 + p_\varepsilon(u)\nabla u\cdot\nabla\varphi \\ &\ge \int_D p_\varepsilon(u)\nabla u\cdot\nabla\varphi\,. \end{aligned}$$

since $p_\varepsilon(u)$ is increasing and, as we notice, differentiable almost everywhere, which is sufficient in our case.

Notice that $p_\varepsilon(u)$ converges to the characteristic function $\chi_{\{u>0\}}$ as $\varepsilon \to 0$, since:

(i) if $x \in \{u = 0\}$, then $p_\varepsilon(u(x)) = p_\varepsilon(0) = 0$,

(ii) if $x \in \{u > 0\}$, then there is $\varepsilon_0 > 0$ such that $u(x) \ge \varepsilon_0$ and then for all $\varepsilon \le \varepsilon_0$ we have $p_\varepsilon(u(x)) = 1$.

Now, let us prescribe a function $g := |\nabla u\cdot\nabla\varphi| \ge |p_\varepsilon(u)\nabla u\cdot\nabla\varphi|$. Since it holds that

$$\begin{aligned} \int_D g\,dx = \int_D |\nabla u\cdot\nabla\varphi|\,dx &\le \int |\nabla u||\nabla\varphi|\,dx \\ &\le \|\nabla\varphi\|_{L^\infty}\int_D |\nabla u|\,dx \le \\ &\le \|\nabla\varphi\|_{L^\infty}\left(\int_D |\nabla u|^2\,dx\right)^{1/2}\sqrt{|D|} \le \infty \end{aligned}$$

by Hölder inequality 6. Now, we may conclude by Lebesgue's dominated convergence theorem (see Rudin [1964]), that

$$\int_D \nabla u\cdot\nabla\varphi\,dx \le 0\,.$$

The proof is completed. □

**Lemma 23.** *Let $D \subset \mathbb{R}^d$ be an open set and $u \in H^1(D)$ be a subharmonic function. Then take any $x_0 \in D$. Then, the functions*

$$r \mapsto f(r) := \fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1} \quad \text{and} \quad r \mapsto g(r) := \fint_{B_r(x_0)} u\,dx\,.$$

*are non-decreasing. We obtain two properties of $u$ as a consequence:*
*- $u$ is locally bounded, $u \in L^\infty_{loc}(D)$,*
*- for $u$ there exists a function $u : D \to \mathbb{R}$*

$$u = \lim_{r\to 0}\frac{1}{|B_r|}\int_{B_r(x_0)} u(x)\,dx \quad \text{for every} \quad x_0 \in D\,. \tag{4.7}$$

*such that $u \ge 0$ and $u = u$ almost everywhere in $D$*

*Proof.* We show the monotonicity of the function $f(r)$, as the case of $g(r)$ is analogical. Notice, that by substituting $y := \frac{x-x_0}{r}$, we obtain

$$\frac{1}{d^{d-1}} \int_{\partial B_r(x_0)} u(x)\, d\mathcal{H}^{d-1} = \frac{1}{d^{d-1}} \int_{\partial B_1(0)} u(x_0 + ry) r^{d-1}\, d\mathcal{H}^{d-1}.$$

We wish to examine the derivative of $f$:

$$\begin{aligned} f'(r) &= \frac{d}{dr} \int_{\partial B_1(0)} u(x_0 + ry) r^{d-1}\, d\mathcal{H}^{d-1} \\ &= \int_{\partial B_1(0)} \nabla u(x_0 + ry) \cdot y\, d\mathcal{H}^{d-1} \\ &\overset{(1)}{=} \int_{\partial B_1(0)} \partial_\nu u(x_0 + ry)\, d\mathcal{H}^{d-1} \\ &\overset{(2)}{=} \frac{1}{r^{d-1}} \int_{\partial B_r(x_0)} \partial_\nu u(x)\, d\mathcal{H}^{d-1} \\ &= \frac{1}{r^{d-1}} \int_{B_r(x_0)} \operatorname{div}(\nabla u) \\ &= \frac{1}{r^{d-1}} \int_{B_r(x_0)} \Delta u \geq 0 \end{aligned}$$

The equality (1) follows by the fact that, in our case, $y$ serves as the unit normal vector to $\partial B_1(0)$. Step (2) then applies the Divergence theorem (see Evans [1998]) □

# 5. Lipschitz continuity of the minimizers

In this chapter, we examine the regularity property of minimizers. We will prove that the local minimizers of $\mathcal{F}_\Lambda$ are locally Lipschitz continuous functions. The contents of this chapter are based on Velichkov [2023] (chapter 3, p.29-31).

Our aim is to prove the following result.

**Proposition 7.** *Let $D$ be an open set in $\mathbb{R}^n$ and $u \in H^1_{loc}(D)$ be a non-negative function. Suppose that $u$ is a local minimizer of $\mathcal{F}_\Lambda$ in $D$. Then, $u$ is locally Lipschitz continuous in $D$, i.e. for every $x \in \mathbb{R}^n$ there exists a neighborhood $U$ of $x$ and a constant $L$, such that*

$$|u(x) - u(y)| \leq L|x-y| \quad \textit{for all} \quad y \in U\,.$$

The proof of Proposition 7, which we shall present, was originally proposed by Alt and Cafarelli in Alt and Caffarelli [1980], and divided into two steps by Velichkov in Velichkov [2023]. In the main proof, the Lipschitz continuity of $u$ will come as a consequence of an estimate on the growth of the function $u$ at the free boundary. The following Lemma 24 holds the precise statement.

**Lemma 24.** *Let $u \in H^1(D)$ be a non-negative function under the following conditions*

1. *$u$ is harmonic in the interior of the set $\Omega_u := \{u > 0\}$,*
2. *$u$ satisfies the inequality:*

$$\frac{1}{\mathcal{H}^{d-1}|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} u d\mathcal{H}^{d-1} \leq Cr \quad \textit{for every} \quad 0 < r < r_0 \quad \textit{and every} \quad x_0 \in \partial\Omega_u\,, \tag{5.1}$$

   *uniformly with constants $r_0 > 0$ and $C$ depending on the distance to the boundary $\partial D$.*

*Then, the gradient of $u$ can be estimated as*

$$\|\nabla u\|_{L^\infty(D_\delta} \leq C_d\left(C + \frac{\|u\|_{L^1(D_{\delta/2})}}{\delta^{d+1}}\right) \quad \textit{for every } 0 < \delta < r_0,$$

*where $C_d$ is a dimensional constant and, for all $r > 0$, we put*

$$D_r := \{x \in D \,:\, dist(x, \partial D) > r\}\,.$$

*Therefore, the set $\Omega_u$ is open and the function $u$ is locally Lipschitz continuous in $D$.*

*Remark.* For simplicity, we will be writing the averaging integral $\frac{1}{|\Omega|}\int_\Omega$ as $⨍_\Omega$ from here onward.

*Proof.* We choose the point $x_0$ from different parts of the domain and reason accordingly.

(i) Suppose $x_0 \in \partial\Omega_u \cap D$. In the estimate 5.1, we may pass to the limit $r \to 0$, which gives us $u(x_0) = 0$ and so $x_0 \notin \Omega_u$. That implies $\Omega_u \cap \partial\Omega_u =$ and $\Omega_u$ is an open set.

(ii) Let $x_0 \in D_\delta$ such that $\operatorname{dist}(x, \partial\Omega_u) \geq \frac{\delta}{4}$, then either $u = 0$ or $u$ is harmonic in the ball $B_{\delta/4}$. The gradient of a harmonic function in an open set can be locally estimated, as shown in Evans [1998](chapter 2, p.29), by the (unsurprisingly) so-called gradient estimate

$$\|\nabla u(x_0)\| \leq \frac{C_d}{\delta^{d+1}} \int_{B_\delta(x_0)} u\, dx\,,$$

where $C_d$ is a constant depending only on the dimension of the space.

(iii) Let $x_0 \in D_\delta$ such that $\operatorname{dist}(x, \partial\Omega_u) < \frac{\delta}{4}$. Suppose the distance to the boundary is realized by a point $y_0 \in \partial\Omega_u$, which simply means that $|x_0 - y_0| = \operatorname{dist}(x_0, \partial\Omega_u) = \min\{|x_0 - y| \,:\, y \in \partial\Omega_u\}$. Let us denote this distance as $r$.

Because $u$ is harmonic in $B_r(x_0)$, we may again estimate the gradient as

$$\|\nabla u(x_0)\| \leq \frac{C_d}{r^{d+1}} \int_{B_r(x_0)} u\, dx \leq \frac{C_d}{r^{d+1}} \int_{B_{2r}(y_0)} u\, dx \leq C_d C\,,$$

where the second inequality holds, because $u$ is positive and the balls are nested, $B_r(x_0) \subset B_{2r}(y_0)$. In the last inequality, we use the estimate 5.1 together with the fact that, by the use of polar coordinates,

$$\int_{B_{2r}(y_0)} u\, dx = \int_0^{2r} \int_{\partial B_t(y_0)} u\, d\mathcal{H}^{d-1}\, dt\,.$$

□

With Lemma 24 in mind, we now embark on proving the estimate 5.1. We structure the original arguments by Alt and Caffarelli in three comprehensive steps as follows:

(1) In Lemma 25, we compare the energy $\mathcal{F}_\Lambda(u, B_r(x_0))$ of the function $u$ in the ball $B_r(x_0)$ with the energy of its harmonic extension $h$ in $B_r(x_0)$ and receive

$$\int_{B_r(x_0)} |\nabla(u - h)|^2\, dx \leq \Lambda\, |\{u = 0\} \cap B_r(x_0)|\,. \tag{5.2}$$

(2) In Lemma 26 we estimate from below the left-hand side of the inequality above for $u \in H^1(D)$

$$\frac{1}{r^2}\, |\{u = 0\} \cap B_r(x_0)| \left( \fint_{\partial B_r(x_0)} u\, d\mathcal{H}^{d-1} \right)^2 \leq C_d \int_{B_r(x_0)} |\nabla(u - h)|^2\, dx \tag{5.3}$$

(3) If $x_0 \in \partial\Omega_u$ then $|\{u = 0\} \cap B_r(x_0)| \neq 0$. Combining the previous two inequalities, we obtain

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u\, d\mathcal{H}^{d-1} \leq \sqrt{C_d \Lambda}\,.$$

Let us move on to the proof itself.

**Lemma 25.** *Suppose $u \in H^1_{loc}(D)$ is a local minimizer of $\mathcal{F}_\Lambda$ in the open set $D \subset \mathbb{R}^d$. Consider the harmonic replacement of $u$ in $B_r$, that is the function $h \in H^1(D)$ which is harmonic in $B_r$ and also $h = u$ on $\partial B_r$. Then it holds that*

$$\int_{B_r(x_0)} |\nabla(u-h)|^2\,dx = \int_{B_r(x_0)} |\nabla u|^2 - |\nabla h|^2\,dx \le \Lambda\,|\{u=0\} \cap B_r(x_0)|\,.$$

*Proof.* We analyze the integral furthest to the left

$$\int_{B_r(x_0)} |\nabla(u-h)|^2\,dx = \int_{B_r(x_0)} |\nabla u|^2 - 2\nabla u\cdot\nabla h + |\nabla h|^2)\,dx \tag{5.4}$$

Now, we the integral

$$0 = \int_{B_r(x_0)} \Delta h(h-u)\,dx = \int_{\partial B_r(x_0)} (h-u)\nabla h\cdot\nu - \int_{B_r(x_0)} \nabla h\cdot\nabla(h-u)\,,$$

where the first identity is caused simply by the harmonicity of $h$, and in the second equality we employ the Divergence theorem (see Evans [1998]), with $\nu$ being the normal vector to the boundary.

Now, since on the boundary we have $h-u=0$, it necessarily holds that

$$\int_{B_r(x_0)} \nabla h\cdot\nabla(h-u) = 0$$

and therefore

$$\int_{B_r(x_0)} \nabla h\cdot\nabla u = \int_{B_r(x_0)} \nabla h\cdot\nabla h\,. \tag{5.5}$$

The first equality of 25 follows. Now, since $u$ is optimal, we have that

$$\int_{B_r(x_0)} |\nabla u|^2\,dx + \Lambda\,|\{u>0\}\cap B_r(x_0)| \le \int_{B_r(x_0)} |\nabla h|^2\,dx + \Lambda\,|B_r(x_0)|\,.$$

The claim follows immediately. □

**Lemma 26.** *For each non-negative $u \in H^1(B_r)$ we may estimate the following:*

$$\frac{1}{r^2}\,|\{u=0\}\cap B_r|\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2 \le C_d \int_{B_r} |\nabla(u-h)|^2\,dx\,.$$

*Here, $C_d$ is a dimensional constant and $h$ is the harmonic replacement of $u$, that is a harmonic function in $B_r$ with $h = u$ on $\partial B_r$,*

*Proof.* Consider the function $vn \in H^1(B_r)$ which is a solution to the minimizing problem

$$\min\{\int_{B_r} |\nabla v|^2\,dx \,:\, u - v \in H^1_0(B_r),\, v \ge u\}.$$

Then, $v$ is harmonic on set $\{v > u\}$, and we claim it is also super-harmonic on $B_r$. We prove this by variation of minimization:

Let $\varphi \in C^\infty_c(B_r)$ be a non-negative test function and $t > 0$. Then by optimality of $v$, we have

$$\begin{aligned}\int_{B_r} |\nabla v|^2\,dx &\le \int_{B_r} |\nabla(v+t\varphi)|^2\,dx \\ &= \int_{B_r} \left(|\nabla v|^2 + 2t\,\nabla v\cdot\nabla\varphi + t^2|\nabla\varphi|^2\right)dx\,,\end{aligned}$$

therefore, by taking derivative at $t = 0$ we get super-harmonicity

$$0 \leq \int_{B_r} \nabla v \cdot \nabla \varphi \, dx \, .$$

Let us rescale the functions $u, v$ in the following way:
For every $|t| \leq \frac{1}{2}$, we set the functions

$$u_t(x) := u(x + (r - |x|)t) \quad \text{and} \quad v_t(x) := v(x + (r - |x|)t) \, .$$

The rescaled functions $u_t, u_v$ still belong to $H^1(B_r)$, and their gradients are controlled from above and below by the gradients of their respective functions $u, v$. Now we define a peculiar set $S_t$ as

$$S_t := \{\xi \in \mathbb{R}^d \, : \, |\xi| = 1 \quad \text{and} \quad \{\rho \, : \, \frac{r}{8} \leq \rho \leq r, \, u_t(\rho\xi) = 0\} \neq \varnothing\} \, .$$

For almost all $\xi \in \partial B_1$, and therefore for almost all $\xi \in S_t$, are the functions

$$\rho \mapsto \nabla u_t(\rho\xi) \quad \text{and} \quad \rho \mapsto \nabla v_t(\rho\xi)$$

square integrable, since the gradients of $u_t, v_t$ are $L^2$ functions. For those $\xi$, we may consider an equation

$$((u_t(\rho_2\xi) - v_t(\rho_2\xi)) - (u_t(\rho_1\xi) - v_t(\rho_1\xi))) = \int_{\rho_1}^{\rho_2} \xi \cdot \nabla(u_t(\rho\xi) - v_t(\rho\xi)) \, d\rho \tag{5.6}$$

and suppose it holds for $\rho_1, \rho_2 \in [0, r]$.

Moreover, if we now define for the suitable $\xi \in S_t$

$$r_\xi := \inf\{\rho \, : \, \frac{r}{8} \leq \rho \leq r, \, u_t(\rho\xi) = 0\} \, ,$$

by plugging the pair $r_\xi, r$ into 5.6 and the fact that $h = u$ on $\partial B_r$, we obtain the estimate

$$v_t(r_\xi\xi) = \int_{r_\xi}^{r} \xi \cdot \nabla(v_t(\rho\xi) - u_t(\rho\xi)) \, d\rho \leq \sqrt{r - r_\xi} \left( \int_{r_\xi}^{r} |\nabla(v_t(\rho\xi) - u_t(\rho\xi)|^2 \, d\rho \right)^{1/2} .$$

Now, suppose that $h$ is the harmonic function such that $h = u (= v)$ on $\partial B_r$. Since v is super-harmonic on $B_r$, we have that

$$\begin{aligned} v(x) \geq h(x) &= \frac{r^2 - |x|^2}{d\omega_d r} \int_{\partial B_r} \frac{u(y)}{|x - y|^d} \, d\mathcal{H}^{d-1}(y) \\ &\geq \frac{(r - |x|)(r + |x|)}{d\omega_d r} \int_{\partial B_r} \frac{u(y)}{r^d} \, d\mathcal{H}^{d-1}(y) \\ &\geq c_d \frac{(r - |x|)}{r} \frac{1}{r^{d-1}} \int_{\partial B_r} u(y) \, d\mathcal{H}^{d-1}(y) = c_d \frac{(r - |x|)}{r} \fint_{\partial B_r} u \, d\mathcal{H}^{d-1} \, , \end{aligned}$$

where in the first equality we applied Poisson's formula for the harmonic function $h$. Taking

$$x = (r - r_\xi)t + r_x\xi$$

we have

$$v_t(r_\xi\xi) = v((r - r_\xi)t + r_x\xi) \geq \frac{c_d}{2} \frac{(r - r_\xi)}{r} \fint_{\partial B_r} u \, d\mathcal{H}^{d-1} = \fint_{\partial B_r} u_t \, d\mathcal{H}^{d-1} \, .$$

By combining the two acquired inequalities, we have

$$\frac{r-r_\xi}{r^2}\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2 \le C_d\int_{r_\xi}^r |\nabla(v_t(\rho\xi)-u_t(\rho\xi)|^2\,d\rho\,.$$

We follow by integrating over $\xi\in S_t\subset\partial B_1$, obtaining

$$\int_{S_t}\frac{r-r_\xi}{r^2}\,d\xi\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2 \le C_d\int_{\partial B_1}\int_{r_\xi}^r |\nabla(v_t(\rho\xi)-u_t(\rho\xi)|^2\,d\rho\,d\xi\,,$$

and, by the bounds $\frac{r}{8}\le r_\xi\le r$, we get

$$\begin{aligned}\frac{1}{r^2}\left|\{u=0\}\cap B_r\setminus B_{\frac{r}{4}}(rt)\right|\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2 &\le C_d\int_{B_r}|\nabla(v_t-u_t)|^2\,dx\\ &\le C_d\int_{B_r}|\nabla(v-u)|^2\,dx\,.\end{aligned}$$

Now, we integrate over $t\le\frac{1}{2}$ and conclude that

$$\frac{1}{r^2}\left|\{u=0\}\cap B_r\right|\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2 \le C_d\int_{B_r}|\nabla(v-u)|^2\,dx\,.$$

□

**Lemma 27.** *Let $u\in H^1_{loc}(D)$ be a local minimizer of $\mathcal{F}_\Lambda$ in the open set $D\subset\mathbb{R}^d$. Then for each ball $\overline{B_r}(x_0)\subset D$ we have*

$$\left|\{u=0\}\cap B_r(x_0)\right|\left(\sqrt{C_d\Lambda}-\frac{1}{r}\fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1}\right)\ge 0\,.$$

*Specifically for $x_0\in\partial\Omega_u$, this means that $|\{u=0\}\cap B_r(x_0)|>0$ and so*

$$\frac{1}{r}\fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1}\le\sqrt{C_d\Lambda}\,.$$

*Proof.* Suppose that $x_0=0$ and let $h\in H^1(B_r)$ be a harmonic function such that $h=u$ on $\partial B_r$. Now, combining the results of both Lemma 25 and Lemma 26, we immediately receive the inequality

$$\frac{1}{r^2}\left|\{u=0\}\cap B_r\right|\left(\fint_{\partial B_r} u\,d\mathcal{H}^{d-1}\right)^2\le C_d\Lambda\left|\{u=0\}\cap B_r\right|\,.$$

□

# Conclusion

In this thesis, we have discussed an engaging topic of the one-phase Bernoulli problem, which may serve as a pleasant introduction to the free boundary regularity theory. We transformed this problem into a minimization problem of the functional $\mathcal{F}_\Lambda$, which allows us to gain a better grasp at understanding its solutions. Afterward, we studied two examples of minimizing functions, which can be explicitly expressed, and which suggest that the Lipschitz property is optimal. We then described the proof of the existence of generalized solutions to the minimization problem. Finally, in the last chapter, we showed the solutions are Lipschitz regular. This was the most intriguing result of this thesis.

Working with the mathematical text of Velichkov [2023], which discusses this absorbing topic, proved challenging in places. The proofs often assume a high level of prerequisite knowledge and mathematical intuition of their reader, while a bachelor student would dearly welcome a more detailed approach. For example, in the case of radial solutions in Chapter 3, the original text declared one more interesting property of the solutions, which, unfortunately, proved difficult to understand without any added arguments. Therefore, we did not elaborate on it.

In conclusion, the subject of the one-phase Bernoulli problem and the free boundary regularity theory offers many more interesting topics for further study. We highly encourage the reader to delve further into this branch of mathematics, where they might encounter its surprising beauty.

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