FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Anna Musilová

Duality for Weak Lebesgue Spaces

Department of Mathematical Analysis

Supervisor of the bachelor thesis: RNDr. Lenka Slavíková, Ph.D.
Study programme: General Mathematics

Prague 2023

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank my supervisor, RNDr. Lenka Slavíková, Ph.D., for all her kindness, patience and time spent on consultations. I would also like to thank my family for always supporting me in everything I do.

Title: Duality for Weak Lebesgue Spaces

Author: Anna Musilová

Department: Department of Mathematical Analysis

Supervisor: RNDr. Lenka Slavíková, Ph.D., Department of Mathematical Analysis

Abstract: When $p \in (0, 1)$, both the dual and the associate space of the weak Lebesgue space $L^{p,\infty}$ contain only the zero function. In this thesis, we study a different method of dualization of the weak Lebesgue space. Then we extend it to more general function spaces.

Keywords: weak Lebesgue spaces, dual space, associate space

Contents

Introduction

Let $p_1, p_2 \in (1, \infty)$ and fix p such that $1/p = 1/p_1 + 1/p_2$. Then p falls into $(1/2, \infty)$. Consider a bilinear operator $T : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, meaning that T is linear in each of its two arguments and

$$\|T(f,g)\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{1}$$

Let $p' \in \mathbb{R}$ be such that $1/p + 1/p' = 1$. When $p \geq 1$, the estimate (1) holds if and only if for every $h \in L^{p'}(\mathbb{R}^n)$

$$\left|\int_{\mathbb{R}^n} T(f,g)(x)h(x)dx\right| \leq c\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|h\|_{L^{p'}(\mathbb{R}^n)}.$$

For this reason, it is useful to study the duality of the space $L^p(\mathbb{R}^n)$. There exist two ways to approach duality - the *dual space* and the *associate space*. When $p \in [1, \infty)$, the associate space of $L^p(\mathbb{R}^n)$ is the space $L^{p'}(\mathbb{R}^n)$ and the dual of $L^p(\mathbb{R}^n)$ is isometric to $L^{p'}(\mathbb{R}^n)$. For $p = \infty$, the associate space of $(L^\infty(\mathbb{R}^n))$ is equal to the space $L^1(\mathbb{R}^n)$ but the dual of $L^\infty(\mathbb{R}^n)$ is not isometric to $L^1(\mathbb{R}^n)$. In this thesis, we will focus on the case when $p \in (1/2, 1)$. Thanks to the Marcinkiewicz interpolation theorem, see, e.g., [2, Theorem 1.3.2], and its bilinear variant formulated in [3], it is sufficient to study the duality of the weak Lebesgue spaces $L^{p,\infty}$ instead.

However, for $0 < p < 1$ both the dual and the associate space of $L^{p,\infty}$ only contain the constant function that is zero almost everywhere. That is why, in this thesis, we will explore a different method of dualization of the weak Lebesgue spaces for $p < 1$, which can be found in [4, Lemma 2.6]. Then we will extend this method to more general function spaces.

This thesis is structured as follows. In the first chapter we start by defining some essential concepts such as the *distribution function* and *nonincreasing rearrangement*, which we then use to introduce the notion of weak Lebesgue spaces. We also introduce the notions of *dual spaces* and *associate spaces*.

The second chapter focuses on the fact that both the dual and the associate space of the weak Lebesgue space $L^{p,\infty}$ for $p < 1$ contain only the zero function. We prove this for the associate space and direct the reader to [1, Theorem 1] for the proof for the dual.

In the third chapter, we recall the Dualization lemma as seen in [4, Lemma 2.6]. Then we extend it to more general function spaces.

1. Preliminaries

In this thesis, we will only consider functions from the measure space $(\mathbb{R}^n, \lambda^n)$, where λ^n denotes the Lebesgue measure on $\mathbb{R}^n$. For conciseness of notation, we shall denote the measure of a set $A \subseteq \mathbb{R}^n$ by $|A|$ instead of $\lambda^n(A)$ and the space $L^p(\mathbb{R}^n)$ by L^p. As is common in the theory L^p spaces, we will consider two functions $f, g \in L^p$ to be equivalent, if $|\{x \in \mathbb{R}^n : f(x) \neq g(x)\}| = 0$.

Definition 1.1. Let X be a vector space. A functional $\|\cdot\| : X \to [0, \infty)$ is called a *norm* on X, if the following are true for every $x, y \in X$:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{R}$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

A functional $\|\cdot\|$ is called a *quasinorm* on X if it satisfies (i) and (ii) and for some constant $C \geq 1$ it holds that

(iii') $\|x + y\| \leq C(\|x\| + \|y\|)$ for every $x, y \in X$.

Notation 1.2. We will denote the set of all measurable functions on $\mathbb{R}^n$ by $\mathcal{M}$.

Definition 1.3. For every $f \in \mathcal{M}$ we define its *distribution function* f_* by

$$f_*(y) = |\{x \in \mathbb{R}^n : |f(x)| > y\}|, \text{ where } y > 0.$$

We define the *nonincreasing rearrangement* f^* of the function f by

$$f^*(t) = \inf\{y \geq 0 : f_*(y) \leq t\}, \text{ for all } t \geq 0.$$

Remark 1.4. *The nonincreasing rearrangement does not satisfy the triangle inequality, only its weaker version. For all functions $f, g \in \mathcal{M}$ and $s, t > 0$ it holds that*

$$(f + g)^*(s + t) \leq f^*(s) + g^*(t).$$

This is proved in [6, Proposition 7.1.13].

Definition 1.5. For $0 < p < \infty$ and $f \in \mathcal{M}$ we define

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t).$$

The *weak Lebesgue space* $L^{p,\infty}$ is the collection of all functions $f \in \mathcal{M}$ such that $\|f\|_{L^{p,\infty}} < \infty$.

Remark 1.6. *Equivalently, we can also express $\|f\|_{L^{p,\infty}}$ by the formula*

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda f_*(\lambda)^{1/p}.$$

It is important to note that the functional $\|\cdot\|_{L^{p,\infty}}$ is not a norm for $p \leq 1$, as it does not satisfy triangle inequality. It is, however, a quasinorm. When $p > 1$, the functional $\|\cdot\|_{L^{p,\infty}}$ itself is also not a norm, but it is equivalent to a norm.

Definition 1.7. Let X be a quasi-normed space. Then the *dual space* of X is the vector space of all continuous linear functionals on X. We shall denote the dual space of X by X^*.

The following definition can be found in [5, Definition 2.15].

Definition 1.8. For a functional $\|\cdot\|_X : \mathcal{M} \to [0,\infty]$, let us set $X = \{f \in \mathcal{M} : \|f\|_X < \infty\}$. Then we shall define the *associate functional* of $\|\cdot\|_X$ by

$$\|f\|_{X'} = \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|d\lambda^n(x) : \, g \in X, \; \|g\|_X \leq 1\right\},$$

and the *associate space* of X as

$$X' = \{f \in \mathcal{M} : \, \|f\|_{X'} < \infty\}.$$

2. Duality for Weak Lebesgue Spaces

In this chapter, we will focus on the dual and associate space for $L^{p,\infty}$ when $p < 1$. First, let us note that although the dual and associate space are similar concepts, they are not equal. Let us demonstrate this on the example of the L^p spaces. When $p \in [1,\infty)$, then the associate space of L^p is the space $L^{p'}$ and the dual of L^p is isometric to $L^{p'}$. However, when $p = \infty$, we have that the associate space of (L^∞) is equal to the space L^1 while the dual of L^∞ is not isometric to L^1. When $p > 1$, the dual space of $L^{p,\infty}$ is equal to the Lorentz space $L^{p',1}$, which is defined as the space of all $f \in \mathcal{M}$ for which $\|f\|_{L^{p',1}} = \|t^{1/p'-1} f^*(t)\|_{L^1(0,\infty)} < \infty$. More information on Lorentz spaces can be found in [6, Chapter 8].

The proof of the following theorem can be found in [1, Theorem 1].

Theorem 2.1. *For $0 < p < 1$ we have $(L^{p,\infty})^* = \{0\}$.*

Proposition 2.2. *For every $f \in \mathcal{M}$ and $p \in (0,\infty)$ we can equivalently express*

$$\|f\|_{(L^{p,\infty})'} = \sup\left\{\int_0^\infty f^*(s)g^*(s)ds : g \in L^{p,\infty}, \|g\|_{L^{p,\infty}} \leq 1\right\}.$$

Proof. This proof will be inspired by [6, Proposition 7.6.5], where it is proven for rearrangement-invariant Banach function norms. However, we cannot apply it directly because, for $p \in (0,1)$, $\|\cdot\|_{L^{p,\infty}}$ is not a norm, so we will have to change the main argument of the proof to suit our situation.
From [6, Theorem 7.3.8], we know that $(\mathbb{R}^n, \lambda^n)$ is a resonant measure space, meaning that for every $f, g \in \mathcal{M}$ we have

$$\int_0^\infty f^*(s)g^*(s)ds = \sup\left\{\int_{\mathbb{R}^n} |f(x)\tilde{g}(x)| d\lambda^n(x) : \tilde{g} \in \mathcal{M}, \tilde{g}_* = g_*\right\}.$$

Thus we have

$$\begin{aligned}
&\sup\left\{\int_0^\infty f^*(s)g^*(s)ds : g \in L^{p,\infty}, \|g\|_{L^{p,\infty}} \leq 1\right\} \\
&\qquad = \sup\left\{\sup\left\{\int_{\mathbb{R}^n} |f(x)\tilde{g}(x)| d\lambda^n(x) : \tilde{g} \in \mathcal{M}, \tilde{g}_* = g_*\right\} : \|g\|_{L^{p,\infty}} \leq 1\right\}
\end{aligned}$$

When $\tilde{g}_* = g_*$, we have that $\|\tilde{g}\|_{L^{p,\infty}} = \|g\|_{L^{p,\infty}}$, so the previous expression is equal to

$$\sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)| d\lambda^n(x) : \|g\|_{L^{p,\infty}} \leq 1\right\} = \|f\|_{(L^{p,\infty})'},$$

which gives us the desired equality.

□

The following theorem will be based on [6, Theorem 9.6.1 (i)], where it is proven for Generalized Lorentz-Zygmund spaces, which are a generalization of the weak Lebesgue spaces.

Theorem 2.3. *For $0 < p < 1$ we have $(L^{p,\infty})' = \{0\}$.*

Proof. First let us set $g_t = \chi_{(0,\sqrt[n]{t})^n}$ for $t \in (0,1)$ and notice that $g_t^* = \chi_{(0,t)}$. Then

$$\begin{aligned}
\|g_t\|_{L^{p,\infty}} &= \sup_{s>0} s^{1/p} g_t^*(s) \\
&= \sup_{s>0} s^{1/p} \chi_{(0,t)}(s) \\
&= \sup_{t>s>0} s^{1/p} \\
&= t^{1/p}.
\end{aligned}$$

Now let f be a measurable function such that $f \not\equiv 0$. We will show that then $f \notin (L^{p,\infty})'$. Notice that there exist two positive constants ε and δ such that $f^*(s) \geq \delta$ for all $s \in (0,\varepsilon)$. That is because for every $f \not\equiv 0$ we have $|\{x \in \mathbb{R}^n : f(x) \neq 0\}| > 0$, and since f^* is nonnegative and nonincreasing, there must exist some interval where $f^* > 0$. We can choose an arbitrary ε in this interval and set $\delta = f^*(\varepsilon)$. Then we can calculate

$$\begin{aligned}
\|f\|_{(L^{p,\infty})'} &= \sup_{\|g\|_{L^{p,\infty}} \leq 1} \int_0^\infty f^*(s) g^*(s) ds \\
&\geq \sup_{0<t<\varepsilon} \int_0^\infty f^*(s) \frac{g_t^*(s)}{\|g_t\|_{L^{p,\infty}}} ds \\
&\geq \sup_{0<t<\varepsilon} \delta \int_0^t t^{-1/p} ds \\
&= \sup_{0<t<\varepsilon} \delta t^{-1/p} \int_0^t ds \\
&= \sup_{0<t<\varepsilon} \delta t^{1-1/p} \\
&= \infty.
\end{aligned}$$

Therefore $(L^{p,\infty})' = \{0\}$.

$\square$

3. Duality for More General Function Spaces

In this chapter, we first recall the Dualization lemma for the weak Lebesgue spaces, found in [4, Lemma 2.6]. Then we extend it to more general function spaces.

Definition 3.1. We say that a function $\varphi : (0, \infty) \to (0, \infty)$ satisfies the Δ_2-*condition*, if there exists a constant $k > 0$ such that for every $t > 0$ it holds that $\varphi(2t) \le k\varphi(t)$.

Notation 3.2. By writing $A \lesssim B$, where A and B are some expressions containing a function φ which satisfies the Δ_2-condition with a constant k, we mean that there exists a constant $c > 0$, which can only depend on k, such that $A \le c \cdot B$. If both $A \lesssim B$ and $B \lesssim A$ hold, then we write $A \simeq B$.

Definition 3.3. For $f, g \in \mathcal{M}$ we define $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$.

Notation 3.4. For $p \in (0, \infty)$ we set $p' \in \mathbb{R}$ to be such that $1/p + 1/p' = 1$. Notice that, for $p \in (0, 1)$, p' is negative.

Theorem 3.5 (Dualization lemma for $L^{p,\infty}$)**.** *Let $f \in \mathcal{M}$, $p \in (0, 1]$ and $A > 0$. Then the following are equivalent:*

(i) $\|f\|_{L^{p,\infty}} \lesssim A$;

(ii) for every set $E \subseteq \mathbb{R}^n$, such that $0 < |E| < \infty$, there exists a subset $E' \subseteq E$ such that $|E| \simeq |E'|$ and $|\langle f, \chi_{E'} \rangle| \lesssim A|E|^{1/p'}$.

In this case, we are using Notation 3.2 for the function $\varphi(t) = t^{1/p}$ and a constant c which can only be dependent on p.

Notation 3.6. For a function $\varphi : (0, \infty) \to (0, \infty)$ and $f \in \mathcal{M}$, let us define

$$\|f\|_{\varphi} = \sup_{t>0} f^*(t)\varphi(t).$$

In this section, let us fix a strictly increasing continuous function $\varphi : (0, \infty) \to (0, \infty)$ and consider the space of all functions $f \in \mathcal{M}$ for which $\|f\|_{\varphi} < \infty$.

First, let us show an alternative formula for $\|f\|_{\varphi}$, which contains the distribution function of f instead of its nonincreasing rearrangement.

Proposition 3.7. *For every $f \in \mathcal{M}$ we have*

$$\sup_{t>0} f^*(t)\varphi(t) = \sup_{s>0} \varphi(f_*(s))s.$$

Proof. First we will show that the left-hand side is greater than or equal to the right-hand side. Let $s > 0$ be such that $f_*(s) > 0$ and let $\varepsilon \in (0, f_*(s))$. Then

$$f^*(f_*(s) - \varepsilon) > s,$$

so we get

$$\sup_{t>0} f^*(t)\varphi(t) \geq f^*(f_*(s) - \varepsilon)\varphi(f_*(s) - \varepsilon) > \varphi(f_*(s) - \varepsilon)s.$$

Now, by sending $\varepsilon \to 0$ and taking the supremum over all $s > 0$ we obtain the desired inequality.
For the opposite inequality, let $t > 0$ be such that $f^*(t) > 0$ and let $\varepsilon \in (0, f^*(t))$. Then

$$f_*(f^*(t) - \varepsilon) > t,$$

thus we get

$$\begin{aligned}\sup_{s>0} \varphi(f_*(s))s &\geq (f^*(t) - \varepsilon)\varphi(f_*(f^*(t) - \varepsilon)) \\ &\geq (f^*(t) - \varepsilon)\varphi(t).\end{aligned}$$

Again, by sending $\varepsilon \to 0$ and taking the supremum over all $t > 0$ we get the desired inequality and the proof is finished.

□

The previous proof for the special case when $p \in (0, \infty)$ and $\varphi(t) = t^{1/p}$ can be found in [2, Proposition 1.4.5 (16)].

Now we will prove a generalization of the Dualization lemma for $\| \cdot \|_\varphi$.

Theorem 3.8 (Dualization lemma)**.** *Let $A > 0$ and $f \in \mathcal{M}$. Then the following are equivalent:*

(i) $\|f\|_\varphi \lesssim A$;

(ii) for every set $E \subseteq \mathbb{R}^n$, such that $0 < |E| < \infty$, there exists a subset $E' \subseteq E$ such that $|E| \simeq |E'|$ and $|\langle f, \chi_{E'} \rangle| \lesssim A\frac{|E|}{\varphi(|E|)}$.

The proof will be based on the proof of the Dualization lemma for $L^{p,\infty}$, as found in [4, Lemma 2.6]. The main difference between the two proofs is that we will need to use the fact that φ satisfies the Δ_2-condition.

Proof. (i) $\Rightarrow$ (ii) Let us fix $E \subseteq \mathbb{R}^n$ and let $C = k^2$, where $k > 0$ is such that $\varphi(2t) \leq k\varphi(t)$. From (i) we have $\|f\|_\varphi \lesssim A < \infty$, therefore there exists a constant $d > 0$ such that $\|f\|_\varphi \leq dA$. Let us denote $B = dA$ and set $\Omega = \{x \in \mathbb{R}^n : |f(x)| > CB\frac{1}{\varphi(|E|)}\}$. Then

$$\begin{aligned}\|f\|_\varphi &= \sup_{t>0} t\varphi(f_*(t)) \\ &\geq \frac{CB}{\varphi(|E|)}\varphi(|\Omega|).\end{aligned}$$

So

$$\begin{aligned}\varphi(|\Omega|) &\leq \frac{\|f\|_\varphi \varphi(|E|)}{CB} \\ &\lesssim \frac{\varphi(|E|)}{C}.\end{aligned}$$

Since φ satisfies the Δ_2-condition, we get

$$\begin{aligned}\varphi(|\Omega|) &\lesssim \frac{\varphi(|E|)}{C} \\ &\leq \frac{k\varphi\left(\frac{|E|}{2}\right)}{C} \\ &\leq \frac{k^2\varphi\left(\frac{|E|}{4}\right)}{C} \\ &= \varphi\left(\frac{|E|}{4}\right).\end{aligned}$$

Now since φ is a strictly increasing function, we have

$$|\Omega| \leq \frac{|E|}{4}.$$

Now let us set $E' = E \setminus \Omega$. Then $|E| \simeq |E'|$, since

$$|E'| = |E \setminus \Omega| \geq \frac{3}{4}|E|$$

and

$$|E| \geq |E \setminus \Omega| = |E'|.$$

Now we have

$$\begin{aligned}|\langle f, \chi_{E'}\rangle| &\leq \int_{E'} |f(x)|dx \\ &\leq |E'|CB\frac{1}{\varphi(|E|)} \\ &\lesssim B\frac{|E|}{\varphi(|E|)} \\ &= dA\frac{|E|}{\varphi(|E|)} \\ &\lesssim A\frac{|E|}{\varphi(|E|)},\end{aligned}$$

which is what we wanted to prove.

(ii) $\Rightarrow$ (i) Let $t > 0$ and set $E = \{x \in \mathbb{R}^n : f(x) > t\}$. From (ii) we have the existence of $E' \subseteq E$, $|E| \simeq |E'|$ for which $|\langle f, \chi_{E'}\rangle| \lesssim A\frac{|E|}{\varphi(|E|)}$.
We also have

$$\begin{aligned}|\langle f, \chi_{E'}\rangle| &= \int_{E'} f \\ &\geq |E'|t \simeq |E|t.\end{aligned}$$

By the combination of these two we get

$$|E|t \lesssim A\frac{|E|}{\varphi(|E|)},$$

and thus

$$\varphi(|E|)t \lesssim A.$$

Analogously we can proceed for the set $F = \{x \in \mathbb{R}^n : f(x) < -t\}$ and get that $\varphi(|F|)t \lesssim A$.
By the definition of E and F, it holds that $|E \cup F| = f_*(t)$. The sets E and F are clearly disjoint, therefore $|E \cup F| = |E| + |F|$, and we get

$$\begin{aligned}\varphi(f_*(t))t &= \varphi(|E| + |F|)t \\ &\leq \varphi(2 \cdot \max\{|E|, |F|\})t,\end{aligned}$$

since φ is strictly increasing. Now from the Δ_2-condition we have

$$\begin{aligned}\varphi(2 \cdot \max\{|E|, |F|\})t &\leq k\varphi(\max\{|E|, |F|\})t \\ &\lesssim kA \\ &\lesssim A,\end{aligned}$$

so we obtain

$$\varphi(f_*(t))t \lesssim A.$$

Since the previous inequality holds for all $t > 0$, we get that

$$\|f\|_\varphi = \sup_{t>0} \varphi(f_*(t))t \lesssim A$$

as desired.

□

Now, we will generalize the previous theorem to the case when $p \in (0, \infty)$, $\alpha \in \mathbb{R}$ and $\varphi(t) = t^{1/p}(1 + |\log t|)^\alpha$. We will show that, even though φ is not strictly increasing when $\alpha \notin [-1/p, 1/p]$, the previous theorem still holds for $\|f\|_\varphi$ with this function φ. The theorem for this specific φ looks as follows.

Theorem 3.9. *Let $f \in \mathcal{M}$, $p \in (0, \infty)$, $\alpha \in \mathbb{R}$ and $A > 0$. Then the following are equivalent:*

(i) $\|f\|_\varphi \lesssim A$;

(ii) for every set $E \subseteq \mathbb{R}^n$, such that $0 < |E| < \infty$, there exists a subset $E' \subseteq E$, for which $|E| \simeq |E'|$ and $|\langle f, \chi_{E'}\rangle| \lesssim A|E|^{1/p'}(1 + |\log |E||)^{-\alpha}$.

In this case, we are using Notation 3.2 for the function $\varphi(t) = t^{1/p}(1+|\log t|)^\alpha$ and a constant c which can depend on p and α.
To prove this theorem, we first need to define what it means that two functions are equivalent. Then we will prove that our function $\varphi(t) = t^{1/p}(1 + |\log t|)^\alpha$ is equivalent to some strictly increasing continuous function $\psi : (0, \infty) \to (0, \infty)$, which satisfies the Δ_2-condition. Finally, by considering Theorem 3.8 for $\|f\|_\psi$, we will be able to prove this theorem.

Definition 3.10. We say, that a function $\varphi : (0, \infty) \to (0, \infty)$ is *equivalent* to the function $\psi : (0, \infty) \to (0, \infty)$, if there exist constants $c, d > 0$ such that $c\psi(t) \leq \varphi(t) \leq d\psi(t)$ for every $t > 0$.

Remark 3.11. *The equivalence of functions is symmetric, meaning that if a function φ is equivalent to a function ψ, then ψ is equivalent to φ.*

Proposition 3.12. *There exists a strictly increasing continuous function $\psi : (0, \infty) \to (0, \infty)$ which satisfies the Δ_2-condition and is equivalent to the function $\varphi(t) = t^{1/p}(1 + |\log t|)^\alpha$.*

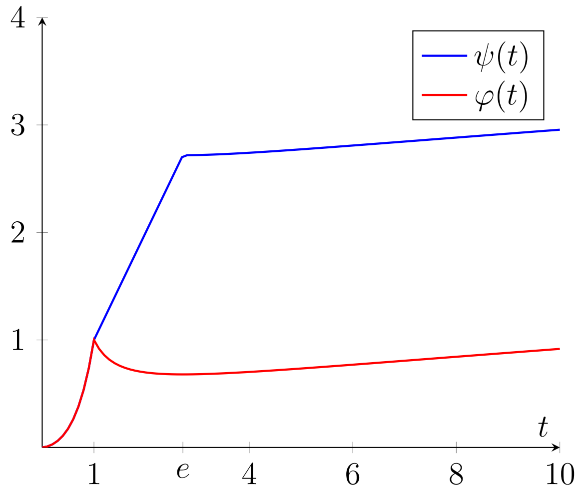
Proof. First we need to find the intervals where φ is not strictly increasing. The derivative of the function φ for $t > 0, t \neq 1$ is

$$\begin{aligned}\varphi'(t) &= \frac{1}{p} t^{1/p-1}(1 + |\log t|)^\alpha + t^{1/p-1}\alpha(1 + |\log t|)^{\alpha-1} \frac{\log t}{|\log t|} \\ &= t^{1/p-1}(1 + |\log t|)^{\alpha-1} \left(\frac{1}{p}(1 + |\log t|) + \alpha \frac{\log t}{|\log t|} \right).\end{aligned}$$

For every $t > 0$, the function $t^{1/p-1}(1 + |\log t|)^{\alpha-1}$ is positive, therefore we only need to analyze for which $t > 0$ is $1/p(1 + |\log t|) + \alpha \frac{\log t}{|\log t|}$ positive and for which it is negative. Then we will be able to find a strictly increasing function ψ that is equivalent with φ.
We will now consider three different cases depending on the value of α with respect to the value of p.
The first case is when $\alpha \in (-\infty, -1/p)$. Then the function φ is increasing for $t \in (0, 1)$ and for $t \in (e^{-\alpha p-1}, \infty)$ and it is decreasing for $t \in (1, e^{-\alpha p-1})$. Therefore we can define the function ψ as follows.

$$\psi(t) = \begin{cases} \varphi(t) & t < 1, \\ t & 1 \leq t \leq e^{-\alpha p-1}, \\ \varphi(t) - \varphi(e^{-\alpha p-1}) + e^{-\alpha p-1} & e^{-\alpha p-1} < t. \end{cases}$$

Then ψ is a strictly increasing positive continuous function, as desired. For better visualization of how ψ looks in comparison with φ, the following graph shows the functions φ and ψ for the particular case when $\alpha = -2$ and $p = 1$.



It holds that $\varphi(t) \leq \psi(t)$ for every $t > 0$, so to show that ψ is equivalent to φ we only need to find a constant c such that $c\psi(t) \leq \varphi(t)$ for every $t > 0$. Equivalently, we need a constant c such that $c \leq \frac{\varphi(t)}{\psi(t)}$ for every $t > 0$.

The minimum of $\frac{\varphi(t)}{\psi(t)}$ occurs at $t = e^{-\alpha p - 1}$, so for $c = \frac{\varphi(e^{-\alpha p-1})}{\psi(e^{-\alpha p-1})}$ it holds that $c\psi(t) \leq \varphi(t)$ for every $t > 0$. Therefore for $\alpha \in (-\infty, -1/p)$ the functions φ and ψ are equivalent.

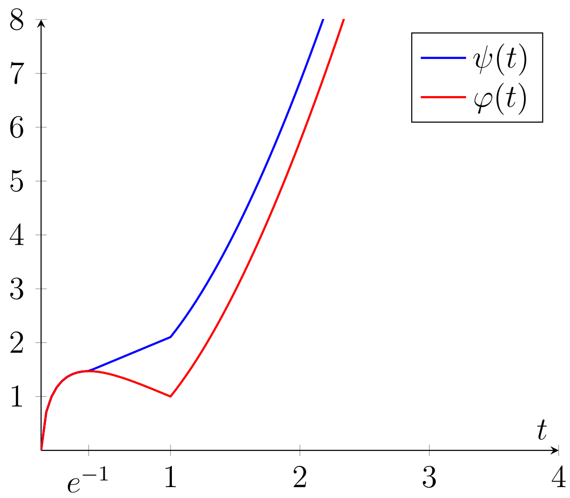
Next, for $\alpha \in [-1/p, 1/p]$ the function φ is increasing for all $t > 0$, therefore we can set $\psi(t) = \varphi(t)$ for every $t > 0$.

Finally, when $\alpha \in (1/p, \infty)$, then φ is increasing for $t \in (0, e^{1-\alpha p})$ and $t \in (1, \infty)$ and it is decreasing for $t \in (e^{1-\alpha p}, 1)$. Therefore we can define the function ψ as follows.

$$\psi(t) = \begin{cases} \varphi(t) & t < e^{1-\alpha p}, \\ t + \varphi(e^{1-\alpha p}) - e^{1-\alpha p} & e^{1-\alpha p} \leq t \leq 1, \\ \varphi(t) + \varphi(e^{1-\alpha p}) - e^{1-\alpha p} & 1 < t. \end{cases}$$

Then ψ is a strictly increasing positive continuous function as desired.

For better visualization of how ψ looks in comparison with φ, we provide a graph showing the functions φ and ψ for the particular case when $\alpha = 2, p = 1$.



It holds that $\varphi(t) \leq \psi(t)$ for every $t > 0$, so we only need to find a constant c such that $c\psi(t) \leq \varphi(t)$ for every $t > 0$. Equivalently, we need a constant c such that $c \leq \frac{\varphi(t)}{\psi(t)}$ for every $t > 0$. The minimum of $\frac{\varphi(t)}{\psi(t)}$ occurs at $t = 1$, so it is sufficient to set $c = \frac{\varphi(1)}{\psi(1)}$. Therefore the functions φ and ψ are equivalent.

Now we only need to show that the function ψ satisfies the Δ_2-condition. First we shall show that the function φ satisfies the Δ_2-condition, and then, from the equivalence of ψ and φ, we will get that ψ satisfies it as well. So we want to find a constant $k > 0$ such that for every $t > 0$ it holds that $\varphi(2t) \leq k\varphi(t)$.

Each of the following lines is equivalent to the line below itself.

$$\begin{aligned} \varphi(2t) = (2t)^{1/p}(1 + |\log(2t)|)^\alpha &\leq kt^{1/p}(1 + |\log(t)|)^\alpha = k\varphi(t), \\ 2^{1/p}(1 + |\log(2t)|)^\alpha &\leq k(1 + |\log(t)|)^\alpha, \\ 2^{1/p}(1 + |\log(t) + \log(2)|)^\alpha &\leq k(1 + |\log(t)|)^\alpha, \\ \frac{2^{1/p}(1 + |\log(t) + \log(2)|)^\alpha}{(1 + |\log(t)|)^\alpha} &\leq k. \end{aligned}$$

We want this to hold for every $t > 0$. For $\alpha \geq 0$, the supremum of the left hand side occurs at $t = 1$ and we have that

$$\sup_{t>0} \left(\frac{2^{1/p}(1 + |\log(t) + \log(2)|)^{\alpha}}{(1 + |\log(t)|)^{\alpha}} \right) = 2^{1/p}(1 + \log(2))^{\alpha}.$$

For $\alpha < 0$, the supremum of the left hand side occurs at $t = 1/2$ and is equal to $2^{1/p}(1 + \log(2))^{-\alpha}$.
Therefore picking $k = 2^{1/p}(1 + \log(2))^{|\alpha|}$ gives us the desired inequality for every $\alpha \in \mathbb{R}$.
We have shown that φ satisfies the Δ_2-condition. From the equivalence of φ and ψ, we have the existence of $c > 0$ such that $c\psi(t) \leq \varphi(t) \leq \psi(t)$ for every $t > 0$. Putting this all together, for every $t > 0$ we get that

$$\psi(2t) \leq \frac{1}{c}\varphi(2t) \leq \frac{1}{c}k\varphi(t) \leq \frac{1}{c}k\psi(t),$$

so ψ satisfies the Δ_2-condition as well.

□

Proof of Theorem 3.9. From the equivalence of φ and ψ, we have that $\|f\|_{\varphi} \lesssim A$ if and only if $\|f\|_{\psi} \lesssim A$. So (i) from Theorem 3.8 holds for $\|f\|_{\varphi}$ if and only if it also holds for $\|f\|_{\psi}$. The function ψ is a positive strictly increasing function satisfying the Δ_2-condition, therefore Theorem 3.8 holds for $\|f\|_{\psi}$. Now, let us fix a set $E \subseteq \mathbb{R}^n$ and let E' be the subset of E from Theorem 3.8 for $\|f\|_{\psi}$, meaning that $|E'| \simeq |E|$ and $|\langle f, \chi_{E'} \rangle| \lesssim A\frac{|E|}{\psi(|E|)}$.
Now we will show that (ii) from Theorem 3.8 holds for ψ if and only if it also holds for φ. If it holds for ψ, then from the equivalence of φ and ψ we have

$$\begin{aligned}
|\langle f, \chi_{E'} \rangle| &\lesssim A\frac{|E|}{\psi(|E|)} \\
&\leq A\frac{|E|}{\varphi(|E|)} \\
&= A\frac{|E|}{|E|^{1/p}(1 + |\log|E||)^{\alpha}} \\
&= A|E|^{1/p'}(1 + |\log|E|)^{-\alpha},
\end{aligned}$$

so it also holds for φ. For the converse, if (ii) from Theorem 3.8 holds for φ, then from the equivalence of φ and ψ we have

$$\begin{aligned}
|\langle f, \chi_{E'} \rangle| &\lesssim A\frac{|E|}{\varphi(|E|)} \\
&\lesssim Ac\frac{|E|}{\varphi(|E|)} \\
&\leq A\frac{|E|}{\psi(|E|)},
\end{aligned}$$

so it also holds for ψ. This completes the proof. □

Bibliography

[1] M. Cwikel. On the conjugates of some function spaces. *Studia Mathematica*, 45(1):49–55, 1973.

[2] L. Grafakos. *Classical Fourier Analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[3] S. Janson. On interpolation of multi-linear operators. In M. Cwikel, J. Peetre, Y. Sagher, and H. Wallin, editors, *Function Spaces and Applications*, pages 290–302, Berlin, Heidelberg, 1988. Springer Berlin Heidelberg.

[4] C. Muscalu and W. Schlag. *Classical and Multilinear Harmonic Analysis, Volume 2*, volume 138 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[5] A. Nekvinda and D. Peša. On the properties of quasi-banach function spaces, 2020.

[6] L. Pick, A. Kufner, O. John, and S. Fučík. *Function Spaces*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications*. De Gruyter, Berlin, second revised and extended edition, 2013.