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Tomáš Beránek

Properties of integral operators on Orlicz spaces

Department of Mathematical Analysis

Supervisor of the bachelor thesis: prof. RNDr. Luboš Pick, CSc., DSc.
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Author: Tomáš Beránek

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: Working with function spaces in various branches of mathematical analysis introduces optimality problems, where the question of choosing a function space both accessible and expressive becomes a nontrivial exercise. A good middle ground is provided by Orlicz spaces, parameterized by a single Young function and thus accessible, yet expansive. In this work, we study optimality problems on Sobolev embeddings in the so-called Maz'ya classes of Euclidean domains which are defined through their isoperimetric behavior. In particular, we prove the non-existence of optimal Orlicz spaces in certain Orlicz–Sobolev embeddings in a limiting (critical) state whose pivotal special case is the celebrated embedding of Brezis and Wainger for John domains.

Keywords: integral operator, Orlicz space, optimality, Sobolev embedding.

Contents

Introduction

When tasked with creating a mathematical model of a given scenario, one approach is to interpret the input data and output solutions as measurable functions, which are part of some function spaces, and the assignment of data to a solution as an operator between these spaces. We then obtain the abstract model

$$T : X \to Y, \tag{1}$$

where X and Y are function spaces containing the data and solutions, respectively, and T is a linear operator mapping the data to its solution. For example, in economics, the problem of increasing one's capital in the stock market can be modelled as the relation (1), where X is the behaviour of the market, Y are possible decisions of a trader (purchases or sales), and T maps the current state of the market to the best possible decision. Numerous examples of such modelling in physics, biology and other fields follow naturally.

One of the most important questions when working with such models is what can we say about the quality of the solutions based on the quality of the input data. This is the motivation behind using different function spaces, as we group together functions with the same quality into one space. At this point, it is natural to question whether we may find an optimal space for a given problem. There are two forms of such a task: either we are given data and search for the smallest group of possible solutions (= given space X and finding smallest space Y), or we are given conditions on the solutions and search for the largest amount of data whose solutions satisfy such demands (= given space Y and finding largest space X).

When tasked with solving optimality problems, it is important to remember that we also have to consider the properties of the chosen classes of function spaces in terms of accessibility and expressivity. For instance, Lebesgue L^p spaces are a very well understood and accessible class of spaces (as they are described by a single parameter), though the spaces may be too sparse to provide accurate enough information, and are as such not expressive enough for certain needs. On the other side of the spectrum sit rearrangement-invariant spaces, where an optimal space virtually always exists, however, it is described implicitly, and as such, is practically impossible to work with. A good middle ground is provided by Orlicz spaces, a class of function spaces described by Young functions. As such, these spaces provide both accessibility and expressivity.

Optimality problems are not a new discipline - the earlier results go back into the 2nd half of the 20th century, but it was not until the turn of the millennium that they have seen a boom in interest. As such, the field is now supported by a vast amount of literature, which includes the works Hempel et al. [1970], Adams [1975], Talenti [1976], Brézis and Wainger [1980], Cianchi [1996], Curbera and Ricker [2002], Vybíral [2007], Fontana and Morpurgo [2014], Clavero and Soria [2016], Neves and Opic [2020], and Ho [2021].

This work is, primarily, an extension of the results obtained by [Musil et al., 2022, Chapter 4] to a case that had been left open in that paper. More precisely, we adapt the techniques from the aforementioned paper, where only Lipschitz domains were treated, to the case of considerably less regular domains, in particular those in the so-called Maz'ya classes. We focus on the optimal form of

Sobolev embeddings within rearrangement-invariant spaces, and within Orlicz spaces, on Maz'ya classes of Euclidean domains. Using the general theory we introduce, we prove the nonexistence of certain optimal Orlicz spaces in Orlicz–Sobolev embeddings, namely that there is no largest domain Orlicz space in the embedding

$$W^m L^A(\Omega) \hookrightarrow L^{\infty,q,-1+m(1-\alpha)-\frac{1}{q}}(\Omega)$$

for appropriate values of the parameters involved.

The motivation for studying embeddings into this particular space stems from the fact that it is the optimal (smallest) rearrangement-invariant space which renders the embedding true for every $\Omega \in \mathcal{J}_\alpha$ in the very important case when the corresponding domain Orlicz space L^A is the critical (limiting) Lebesgue space $L^{\frac{1}{m(1-\alpha)}}$. This was first observed in Brézis and Wainger [1980] in connection with the special case $\alpha = \frac{1}{n'}$, that is, $L^{\frac{1}{m(1-\alpha)}} = L^{\frac{n}{m}}$, where n is the dimension of the underlying ambient Euclidean space and m is the order of differentiation. There are various reasons for establishing results involving such Lorentz-type refinements on the target side, perhaps the most notable one being the fact that when such embeddings are considered, then no loss of information occurs under their iterations (for more details, see Cianchi et al. [2015] and the references therein).

The text is structured as follows. In Chapter 1, we present definitions and basic knowledge about the relevant function spaces and the isoperimetric function. In Chapter 2, we collect background results necessary for proofs of our main results, namely the principal alternative and reduction principle for Sobolev embeddings, and the forms of optimal r.i. domain and target spaces. Lastly, in Chapter 3, we present the main results of this work. We first establish a general formula for the fundamental function of an operator-induced space based on the isoperimetric behavior of the underlying domain under certain mild assumptions. Next, we consider a specific situation concerning Maz'ya domains.

1. Preliminaries

1.1 Function spaces

In this section, we recall some definitions and basic facts from the theory of various function spaces. For further details, the standard reference is Bennett and Sharpley [1988].

Let $n \in \mathbb{N}$. In this work, $|E|$ denotes the n-dimensional Lebesgue measure of E for $E \subset \mathbb{R}^n$ measurable.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We assume, without loss of generality, that $|\Omega| = 1$, and define

$$\mathcal{M}(\Omega) = \{f : \Omega \to [-\infty, \infty];\ f \text{ is Lebesgue-measurable in } (0, \infty)\},$$

$$\mathcal{M}_+(\Omega) = \{f \in \mathcal{M}(\Omega) : f \geq 0\},$$

and

$$\mathcal{M}_0(\Omega) = \{f \in \mathcal{M}(\Omega) : f \text{ is finite a.e. in } \Omega\}.$$

The *distribution function* $f_* : (0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}(\Omega)$ is defined as

$$f_*(s) = |\{x \in \Omega : |f(x)| > s\}| \quad \text{for } s \in (0, \infty),$$

and the *decreasing rearrangement* $f^* : [0, 1] \to [0, \infty]$ of a function $f \in \mathcal{M}(\Omega)$ is defined as

$$f^*(t) = \inf \{s \in (0, \infty) : f_*(s) \leq t\} \quad \text{for } t \in [0, 1].$$

The operation $f \to f^*$ is monotone in the sense that for $f_1, f_2 \in \mathcal{M}(\Omega)$,

$$|f_1| \leq |f_2| \quad \text{a.e. in } \Omega \implies f_1^* \leq f_2^* \quad \text{in } [0, 1].$$

The *elementary maximal function* $f^{**} : (0, 1] \to [0, \infty]$ of a function $f \in \mathcal{M}(\Omega)$ is defined as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, \mathrm{d}s \quad \text{for } t \in (0, 1].$$

The operation $f \to f^{**}$ is subadditive in the sense that for $f_1, f_2 \in \mathcal{M}(\Omega)$,

$$(f_1 + f_2)^{**} \leq f_1^{**} + f_2^{**}.$$

The *Hardy–Littlewood inequality* is a classical property of function rearrangements, which asserts that, for $f_1, f_2 \in \mathcal{M}(\Omega)$,

$$\int_\Omega |f_1(x) f_2(x)| \, \mathrm{d}x \leq \int_0^1 f_1^*(t) f_2^*(t) \, \mathrm{d}t. \tag{1.1}$$

A specialization of the inequality states that for every $f \in \mathcal{M}(\Omega)$ and for every $E \subset \Omega$ measurable,

$$\int_E |f(x)| \, \mathrm{d}x \leq \int_0^{|E|} f^*(t) \, \mathrm{d}t.$$

Next, we define the rearrangement-invariant norm. We say that a functional

$$\| \cdot \|_{X(0,1)} : \mathcal{M}_+(0, 1) \to [0, \infty]$$

is a *function norm*, if for all f, g and $\{f_j\}_{j \in \mathbb{N}}$ in $\mathcal{M}_+(0, 1)$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)} = 0 \iff f = 0$ a.e.,
$\|\lambda f\|_{X(0,1)} = \lambda \|f\|_{X(0,1)}$,
$\|f + g\|_{X(0,1)} \leq \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;

(P2) $f \leq g$ a.e. $\implies \|f\|_{X(0,1)} \leq \|g\|_{X(0,1)}$;

(P3) $f_j \nearrow f$ a.e. $\implies \|f_j\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$;

(P4) $\|1\|_{X(0,1)} < \infty$;

(P5) $\int_0^1 f(x)\,\mathrm{d}x \leq c \cdot \|f\|_{X(0,1)}$ for some constant c independent of f.

If, in addition, the property

(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)} \iff f^* = g^*$,

holds, we call the functional $\|\cdot\|_{X(0,1)}$ a *rearrangement-invariant function norm.* For any such rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, we define the functional $\|\cdot\|_{X'(0,1)}$ as

$$\|f\|_{X'(0,1)} = \sup\Big\{ \int_0^1 f(t)g(t)\,\mathrm{d}t : g \in \mathcal{M}_+(0,1), \|g\|_{X(0,1)} \leq 1 \Big\}$$

for $f \in \mathcal{M}_+(0,1)$. The functional $\|\cdot\|_{X'(0,1)}$ is then also a rearrangement-invariant function norm, see [Bennett and Sharpley, 1988, Chapter 1, Theorem 2.2], and it is called the *associate function norm* of $\|\cdot\|_{X(0,1)}$ and, by [Bennett and Sharpley, 1988, Chapter 1, Theorem 2.7] it holds that $\|\cdot\|_{X''(0,1)} = \|\cdot\|_{X(0,1)}$. We say that $\|\cdot\|_{X(0,1)}$ is a *rearrangement-invariant function quasinorm*, if it satisfies the conditions (P2), (P3), (P4) and (P6), and (Q1), a weaker version of (P1), where

(Q1) $\|f\|_{X(0,1)} = 0 \iff f = 0$ a.e.,
$\|\lambda f\|_{X(0,1)} = \lambda \|f\|_{X(0,1)}$,
$\exists\, c \in (0,\infty)$ such that $\|f + g\|_{X(0,1)} \leq c \cdot (\|f\|_{X(0,1)} + \|g\|_{X(0,1)})$,

for all $f, g \in \mathcal{M}_+(0,1)$ and every $\lambda \geq 0$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, we define the functional $\|\cdot\|_{X(\Omega)}$ as

$$\|f\|_{X(\Omega)} = \|f^*\|_{X(0,1)} \quad \text{for } f \in \mathcal{M}(\Omega),$$

and we call the set

$$X(\Omega) = \{f \in \mathcal{M}(\Omega) : \|f\|_{X(\Omega)} < \infty\}$$

a *rearrangement-invariant space.* Furthermore, the space $X(0,1)$ is called the *representation space* of $X(\Omega)$, and we define the *associate space* $X'(\Omega)$ of $X(\Omega)$ as

$$X'(\Omega) = \{f \in \mathcal{M}(\Omega) : \|f\|_{X'(\Omega)} < \infty\}.$$

Then, the *Hölder inequality*

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \leq \|f\|_{X(\Omega)} \|g\|_{X'(\Omega)}$$

holds for every $f \in X(\Omega)$ and $g \in X'(\Omega)$. For any rearrangement-invariant spaces $X(\Omega)$ and $Y(\Omega)$, the continuous embedding of $X(\Omega)$ into $Y(\Omega)$ is denoted by $X(\Omega) \hookrightarrow Y(\Omega)$ and means that there exists a constant $c > 0$ such that for any $f \in X(\Omega)$, it holds that $f \in Y(\Omega)$ and $\|f\|_{Y(\Omega)} \leq c \cdot \|f\|_{X(\Omega)}$. By [Bennett and Sharpley, 1988, Chapter 1, Proposition 2.10] it holds that

$$X(\Omega) \hookrightarrow Y(\Omega) \iff Y'(\Omega) \hookrightarrow X'(\Omega),$$

and by [Bennett and Sharpley, 1988, Chapter 1, Theorem 1.8], it holds that

$$X(\Omega) \subset Y(\Omega) \implies X(\Omega) \hookrightarrow Y(\Omega).$$

Note that the functional $\|\cdot\|_{X(\Omega)}$ may be also defined if its corresponding functional $\|\cdot\|_{X(0,1)}$ is only a rearrangement-invariant quasinorm, however, some of the properties listed here for the case where $\|\cdot\|_{X(0,1)}$ is a rearrangement-invariant norm then do not necessarily hold.

Occasionally, when no confusion can arise, we will, for simplicity's sake, omit the underlying domain in the notation, more precisely, we will write X in place of $X(\Omega)$ or $X(0,1)$, etc.

For given rearrangement-invariant spaces X and Y, we denote the boundedness of an operator T from X to Y by $T : X \to Y$.

Let $\lambda > 0$. For any $f \in \mathcal{M}(0,1)$, the *dilation operator* E_λ is defined as

$$E_\lambda f(t) = \begin{cases} f(\frac{t}{\lambda}) & \text{if } 0 < t \leq \lambda \\ 0 & \text{if } \lambda < t \leq 1. \end{cases}$$

Such operator is bounded on any rearrangement-invariant space $X(0,1)$, with norm smaller than or equal to $\max\{1, \frac{1}{\lambda}\}$.

We introduce, for $\gamma \in (0,1)$, the operator

$$T_\gamma g(t) = t^{-\gamma} \sup_{s \in [t,1]} s^\gamma g^*(s), \quad g \in \mathcal{M}(0,1),\ t \in (0,1). \tag{1.2}$$

Given any $f_1, f_2 \in \mathcal{M}_+(0,1)$ such that

$$\int_0^t f_1(s)\,\mathrm{d}s \leq \int_0^t f_2(s)\,\mathrm{d}s \quad \text{for every } t \in (0,1),$$

by *Hardy's lemma* the inequality

$$\int_0^1 f_1(s)h(s)\,\mathrm{d}s \leq \int_0^1 f_2(s)h(s)\,\mathrm{d}s$$

holds for every non-increasing function $h : (0,1) \to [0,\infty]$. Consequently, the *Hardy–Littlewood–Pólya principle*, which asserts that if $g_1, g_2 \in \mathcal{M}(\Omega)$ satisfy

$$\int_0^t g_1^*(s)\,\mathrm{d}s \leq \int_0^t g_2^*(s)\,\mathrm{d}s \quad \text{for every } t \in (0,1),$$

then

$$\|g_1\|_{X(\Omega)} \leq \|g_2\|_{X(\Omega)},$$

holds for every rearrangement-invariant space $X(\Omega)$.

The *fundamental function* $\varphi_X : [0,1] \to [0,1]$ of a rearrangement-invariant space $X(\Omega)$ is defined as

$$\varphi_X(t) = \|\chi_E\|_{X(\Omega)} \quad \text{for } t \in [0,1],$$

where $E \subset \Omega$ is measurable and such that $|E| = t$. Thanks to the rearrangement invariance of $\|\cdot\|_{X(\Omega)}$, the function φ_X is well defined. We define the *fundamental level* as the collection of all rearrangement-invariant spaces, which share the same fundamental function.

We say that a function $\varphi : [0,\infty) \to [0,\infty)$ is *quasiconcave*, if it is positive, non-decreasing, and the function $\frac{t}{\varphi(t)} : (0,\infty) \to (0,\infty)$ is non-decreasing. Recall that by [Bennett and Sharpley, 1988, Chapter 2 Corollary 5.3] for any rearrangement-invariant space $X(\Omega)$, its fundamental function φ_X is quasiconcave. Furthermore, by [Bennett and Sharpley, 1988, Chapter 2 Proposition 5.10] it holds that for any quasiconcave function $\varphi : [0,\infty) \to [0,\infty)$ there exists a concave function $\overline{\varphi} : [0,\infty) \to [0,\infty)$ such that for every $t \in [0,\infty)$, the inequality $\frac{1}{2}\overline{\varphi}(t) \le \varphi(t) \le \overline{\varphi}(t)$ holds.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a quasiconcave function, let $\overline{\varphi} : [0,\infty) \to [0,\infty)$ be a concave function such that $\frac{1}{2}\overline{\varphi}(t) \le \varphi(t) \le \overline{\varphi}(t)$ for every $t \in [0,\infty)$. We then define the functionals

$$\|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(t)\, \mathrm{d}\overline{\varphi}(t), \quad f \in \mathcal{M}(\Omega),$$

and

$$\|f\|_{M_\varphi} = \sup_{t\in(0,\infty)} \varphi(t) f^{**}(t), \quad f \in \mathcal{M}(\Omega).$$

By [Bennett and Sharpley, 1988, Chapter 2, Theorem 5.13], these functionals are rearrangement-invariant function norms, and as such, we define the corresponding rearrangement-invariant spaces Λ_φ, and M_φ. Both of these spaces have the same fundamental function equal to φ. Furthermore, the space Λ_φ is the smallest rearrangement-invariant space with the fundamental function φ, while M_φ is the largest rearrangement-invariant space with the fundamental function φ.

Given a rearrangement-invariant space X, we define the corresponding *Lorentz space* $\Lambda(X) = \Lambda_{\varphi_X}$ and *Marcinkiewicz space* $M(X) = M_{\varphi_X}$. Let us now recall the Lorentz–Marcinkiewicz sandwich

$$\Lambda(X) \hookrightarrow X \hookrightarrow M(X). \tag{1.3}$$

Note that the norm of both embeddings equals 1.

Throughout this work, we use the convention that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$. Let $p \in [1,\infty]$. We define the *Lebesgue norm* $\|\cdot\|_{L^p(0,1)}$ as

$$\|f\|_{L^p(0,1)} = \begin{cases} \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{1/p} & p \in [1,\infty) \\ \operatorname*{esssup}_{x\in(0,1)} |f(x)| & p = \infty \end{cases}$$

for $f \in \mathcal{M}(0,1)$. We then define the *Lebesgue space* $L^p(0,1)$ as

$$L^p(0,1) = \left\{ f : f \in \mathcal{M}(0,1),\ \|f\|_{L^p(0,1)} < \infty \right\}.$$

Let us recall the Lebesgue sandwich for any rearrangement-invariant space $X(\Omega)$

$$L^\infty(\Omega) \hookrightarrow X(\Omega) \hookrightarrow L^1(\Omega).$$

Let $1 \le p, q \le \infty$. The functionals $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{(p,q)}(0,1)}$ are respectively defined as

$$\|f\|_{L^{p,q}(0,1)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,1)} \quad \text{and}$$

$$\|f\|_{L^{(p,q)}(0,1)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\|_{L^q(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. Let us recall that if $1 < p \le \infty$,

$$L^{p,q}(\Omega) = L^{(p,q)}(\Omega),$$

and if one of the conditions

(L1) $1 < p < \infty,\ 1 \le q \le \infty$,

(L2) $p = q = 1$,

(L3) $p = q = \infty$,

is met, then $\|\cdot\|_{L^{p,q}(0,1)}$ is equivalent to a rearrangement-invariant function norm. Then, the corresponding rearrangement-invariant function space $L^{p,q}(\Omega)$ is called a *Lorentz space*.

Let $p \in [1, \infty]$. Then $L^p(\Omega) = L^{p,p}(\Omega)$, and

$$1 \le r \le s \le \infty \implies L^{p,r}(\Omega) \hookrightarrow L^{p,s}(\Omega),$$

where equality of the spaces is attained if and only if $r = s$. Note that by equality of two rearrangement-invariant spaces X and Y, we mean that X and Y coincide in set-theoretical sense, and, moreover, that their norms are equivalent in the sense that there exists a constant $c > 0$ such that

$$c^{-1} \cdot \|f\|_X \le \|f\|_Y \le c \cdot \|f\|_X$$

for every $f \in X$.

Let $1 \le p, q \le \infty$, $\alpha \in \mathbb{R}$. The functionals $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ and $\|\cdot\|_{L^{(p,q;\alpha)}(0,1)}$ are defined as

$$\|f\|_{L^{p,q,\alpha}(0,1)} = \|t^{\frac{1}{p}-\frac{1}{q}} \log^\alpha\left(\tfrac{2}{t}\right) f^*(t)\|_{L^q(0,1)} \quad \text{and}$$

$$\|f\|_{L^{(p,q,\alpha)}(0,1)} = \|t^{\frac{1}{p}-\frac{1}{q}} \log^\alpha\left(\tfrac{2}{t}\right) f^{**}(t)\|_{L^q(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. If one of the conditions

(Z1) $1 < p < \infty$, $1 \le q \le \infty$, $\alpha \in \mathbb{R}$,

(Z2) $p = 1,\ q = 1,\ \alpha \ge 0$,

(Z3) $p = \infty,\ q = \infty,\ \alpha \le 0$,

(Z4) $p = \infty,\ 1 \le q < \infty,\ \alpha + \frac{1}{q} < 0$, is met, then $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to a rearrangement-invariant function norm. Then the corresponding rearrangement-invariant function space $L^{p,q;\alpha}(\Omega)$ is called a *Lorentz–Zygmund* space.

We say that $A : [0,\infty) \to [0,\infty]$ is a *Young function*, if it is a convex non-constant left-continuous function such that $A(0) = 0$. Let us recall that any such function may be written in the integral form

$$A(t) = \int_0^t a(s)\mathrm{d}s \quad \text{for } t \geq 0,$$

where $a : [0,\infty) \to [0,\infty]$ is a non-decreasing, left-continuous function, which is not identically 0 or ∞.

The *Luxemburg function norm* is defined as

$$\|f\|_{L^A(0,1)} = \inf\left\{\lambda > 0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right)\mathrm{d}s \leq 1\right\} \quad \text{for } f \in \mathcal{M}_+(\Omega),$$

the *Orlicz space* $L^A(\Omega)$ is defined as the rearrangement-invariant space associated with the Luxemburg function norm. Then, for some $p \in [1,\infty)$ and $A(t) = t^p$, $L^A(\Omega) = L^p(\Omega)$; and for $B(t) = \infty \cdot \chi_{(1,\infty)}(t)$, $L^B(\Omega) = L^\infty(\Omega)$.

Let A and B be Young functions. We say that A dominates B near infinity if there exist constants $c > 0$ and $t_0 > 0$ such that

$$B(t) \leq A(ct) \quad \text{for } t \geq t_0.$$

We say that A and B are equivalent near infinity if they dominate each other near infinity. Furthermore, it holds that

$$L^A(\Omega) \hookrightarrow L^B(\Omega) \iff A \text{ dominates } B \text{ near infinity.}$$

We denote certain Orlicz spaces without explicitly defining the corresponding Young functions. The Orlicz space associated with a Young function equivalent near infinity to $t^p \log^\alpha t$, where $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$, is denoted by $L^p \log^\alpha L(\Omega)$, and the Orlicz space associated with a Young function equivalent near infinity to e^{t^β}, where $\beta > 0$, is denoted by $\exp L^\beta(\Omega)$.

In certain cases, the classes of Lorentz-Zygmund and Orlicz spaces overlap. If $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$, then $L^{p,p;\alpha}(\Omega) = L^p \log^{p\alpha} L(\Omega)$. Additionally, if $\beta > 0$, then $L^{\infty,\infty;-\beta}(\Omega) = \exp L^{\frac{1}{\beta}}(\Omega)$.

It is a fact that for every fundamental level of rearrangement-invariant spaces, there exists a unique Orlicz space with the same fundamental function. For a rearrangement-invariant space X, this *fundamental Orlicz space* is denoted by $L(X)$.

For certain classes of function spaces, their fundamental functions are known. By Opic and Pick [1999], it holds that

$$\varphi_{L^p}(t) \approx \begin{cases} t^{\frac{1}{p}} & p < \infty \\ 1 & p = \infty, \end{cases}$$

$$\varphi_{L^{p,q}} \approx \varphi_{L^p} \quad \forall p, q,$$

and

$$\varphi_{L^{p,q;\alpha}} \approx \begin{cases} t^{\frac{1}{p}} \log^\alpha \frac{2}{t} & \text{if conditions (Z1) or (Z2) are met} \\ (\log \frac{2}{t})^{\alpha + \frac{1}{q}} & \text{if conditions (Z3) or (Z4) are met.} \end{cases}$$

Furthermore, by [Pick et al., 2013, Example 7.9.4 (iv), p. 260], it holds that

$$\varphi_{L^A}(t) = \frac{1}{A^{-1}(\frac{1}{t})}.$$

Let h_1, $h_2 \in \mathcal{M}(\Omega)$. The fact that there exists a positive constant c such that $h_1(x) \le c \cdot h_2(x)$, or $h_1(x) \ge c \cdot h_2(x)$ for any $x \in \Omega$ is denoted by $h_1 \lesssim h_2$ or $h_1 \gtrsim h_2$, respectively. If both inequalities $h_1 \lesssim h_2$ and $h_2 \gtrsim h_1$ hold, we write $h_1 \approx h_2$.

Let A, $B : (0, \infty) \to (0, \infty)$. By $A(t) \approx B(t)$ for $t \gg 1$, we denote that there exist constants c, $t_0 > 1$ such that $c^{-1} \cdot A(t) \le B(t) \le c \cdot A(t)$ for every $t > t_0$.

We say that $A : (0, \infty) \to (0, \infty)$ satisfies the Δ_2 condition, if it is non-decreasing and the inequality $A(2t) \lesssim A(t)$ holds for every $t \in (0, \infty)$. Then, by Pick et al. [2013, Theorem 4.7.3] it holds that

$$f \in L^A \iff \int A(f) < \infty \quad \text{for any } f \in \mathcal{M}_+(\Omega).$$

1.2 Isoperimetric functions and Sobolev spaces

In this section, we shall define some basic terms and recall simple facts concerning the isoperimetric function and Sobolev spaces.

Let n, Ω be as in section 1.1. Let $n' = \frac{n}{n-1}$. We define the *perimeter* $P(E, \Omega)$, of a Lebesgue-measurable set $E \subset \Omega$ as

$$P(E, \Omega) = \int_{\Omega \cap \partial^M E} \mathrm{d}\mathcal{H}^{n-1}(x),$$

where $\partial^M E$ denotes the essential boundary of E (for details see Maz'ya [2011]. We then define the *isoperimetric function* $I_\Omega : [0, 1] \to [0, \infty]$ of Ω as

$$I_\Omega(t) = \begin{cases} \inf\{P(E, \Omega) : E \subset \Omega,\ t \le |E| \le \frac{1}{2}\} & t \in [0, \frac{1}{2}], \\ I_\Omega(1 - t) & t \in (\frac{1}{2}, 1]. \end{cases}$$

Note that $I_\Omega(t)$ is finite for $t \in [0, \frac{1}{2})$ (for the detailed proof, see [Cianchi et al., 2015, Chapter 4]). Also, by [Cianchi et al., 2015, Proposition 4.1], there exists a constant $c > 0$ such that

$$I_\Omega(t) \le c \cdot t^{\frac{1}{n'}} \quad \text{near zero.}$$

Thus, the best possible behavior of the isoperimetric function at 0 is $I_\Omega(t) \approx t^{\frac{1}{n'}}$. What this means is that, essentially, $I_\Omega(t)$ cannot decay more slowly than $t^{\frac{1}{n'}}$ as $t \to 0$, independently of Ω.

We will say that Ω has a *Lipschitz boundary*, or simply is a *Lipschitz domain*, if at each point of the boundary of Ω, the boundary is locally the graph of a Lipschitz function.

We shall call Ω a *John domain*, if there exists $c \in (0, 1)$ and $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve parametrized by arclength $\xi : [0, r] \to \Omega$, such that $\xi(0) = x$, $\xi(r) = x_0$ and

$$\operatorname{dist}(\xi(s), \partial\Omega) \ge c \cdot s \quad \text{for } s \in [0, r].$$

An important link between John domains and the theory of isoperimetric functions is that if Ω is a John domain, then $I_\Omega(t) \approx t^{\frac{1}{n'}}$.

Let $\alpha \in [\frac{1}{n'}, 1]$. We define the *Maz'ya class* of Euclidean domains X as

$$\mathcal{J}_\alpha = \Big\{ X : I_X(t) \geq c \cdot t^\alpha \quad \text{for some constant } c > 0 \text{ and } t \in [0, \tfrac{1}{2}] \Big\}.$$

By definition, every Lipschitz domain is a John domain. Furthermore, we observe that $\mathcal{J}_\alpha \subseteq \mathcal{J}_\beta$ for $\alpha \leq \beta$.

Let $m \in \mathbb{N}$, let $X(\Omega)$ be a rearrangement-invariant function space. The *m-th order Sobolev space* $W^m X(\Omega)$ is defined as

$$W^m X(\Omega) = \{ u : u \text{ is } m\text{-times weakly differentiable in } \Omega,$$
$$|\nabla^k u| \in X(\Omega) \text{ for } k = 0, \dots, m\},$$

and the *m-th order Sobolev space* $V^m X(\Omega)$ is defined as

$$V^m X(\Omega) = \{ u : u \text{ is } m\text{-times weakly differentiable in } \Omega,$$
$$|\nabla^m u| \in X(\Omega)\}.$$

Assume now that

$$\int_0^1 \frac{1}{I_\Omega(s)} \, \mathrm{d}s < \infty. \tag{1.4}$$

Then, by [Cianchi et al., 2015, Proposition 4.5]

$$W^m X(\Omega) = V^m X(\Omega)$$

in set-theoretical sense with their norms equivalent. If we only consider a weaker form of (1.4), namely that there exists a positive constant c such that

$$I_\Omega(t) \geq c \cdot t \quad \text{for } t \in [0, \tfrac{1}{2}],$$

it then holds by [Cianchi et al., 2015, Chapter 4, Corollary 4.3, Proposition 4.4] that $V^m X(\Omega) \hookrightarrow V^m L^1(\Omega)$, $V^m X(\Omega) \subset V^k L^1(\Omega)$ for every $k = 0, \dots, m-1$, and furthermore for any $Y(\Omega)$ rearrangement-invariant space, $V^m X(\Omega) \hookrightarrow Y(\Omega)$ if and only if there exists a positive constant c such that $\|u\|_{Y(\Omega)} \leq c \cdot \|\nabla^m u\|_{X(\Omega)}$ for all $u \in V^m_\perp X(\Omega)$, where

$$V^m_\perp X(\Omega) = \Big\{ u \in V^m X(\Omega) : \int_\Omega \nabla^k u(x) \, \mathrm{d}x = 0 \quad \text{for } k = 0, \dots, m-1 \Big\}.$$

In the case where $X(\Omega) = L^A(\Omega)$ is an Orlicz space, we define an *Orlicz–Sobolev space* as $W^m L^A(\Omega) = W^m X(\Omega)$.

We say that $X(\Omega)$ is the optimal (largest) rearrangement-invariant domain space in embedding

$$W^m X(\Omega) \hookrightarrow Y(\Omega), \tag{1.5}$$

if $X(\Omega)$ is a rearrangement-invariant space, embedding (1.5) holds, and if (1.5) holds with $X(\Omega)$ replaced by a rearrangement-invariant space $Z(\Omega)$, then embedding $Z(\Omega) \hookrightarrow X(\Omega)$ holds. Similarly, we say that $Y(\Omega)$ is the optimal (smallest) r.i. target space in embedding (1.5), if $Y(\Omega)$ is a rearrangement-invariant space, embedding (1.5) holds, and if embedding (1.5) holds with $Y(\Omega)$ replaced by a rearrangement-invariant space $V(\Omega)$, then embedding $Y(\Omega) \hookrightarrow V(\Omega)$ holds.

2. Background results

Let $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be an open set, and let $X(\Omega)$ be a rearrangement-invariant space. For the sake of brevity, we shall refer to rearrangement-invariant spaces as r.i. spaces from this point onwards. In this work, we consider Sobolev spaces $W^m X(\Omega)$ together with their norm defined as

$$\|u\|_{W^m X(\Omega)} = \sum_{k=0}^{m} \|\nabla^k u\|_{X(\Omega)}$$

for $m \in \mathbb{N}$. Furthermore, we consider Sobolev embeddings of the form

$$W^m X(\Omega) \hookrightarrow Y(\Omega), \tag{2.1}$$

where $Y(\Omega)$ is an r.i. space. We restrict ourselves to such sets Ω which fulfil the property

$$\inf_{t\in(0,1)} \frac{I_\Omega(t)}{t} > 0$$

and classes of such sets. Furthermore, by Cianchi et al. [2015], it is known that given such an r.i. space $Y(\Omega)$, the optimal r.i. domain space always exists and can be explicitly described. Namely, such optimal r.i. space $X(\Omega)$ obeys

$$\|u\|_{X(\Omega)} = \sup_h \left\| \int_t^1 \frac{h(s)}{I_\Omega(s)}\, \mathrm{d}s \right\|_{Y(0,1)} \qquad \text{for } u \in \mathcal{M}(\Omega), \tag{2.2}$$

where the supremum is taken over all $h \in \mathcal{M}_+(0,1)$ such that $h^* = u^*$.

We first examine the possible approaches to the reduction of Sobolev embeddings to significantly simpler one-dimensional inequalities for Maz'ya domains. Such problems have already been examined and solved, and as such, for our purposes, it suffices to use the reduction principle stated and proven in [Cianchi et al., 2015, Theorem 6.4], which follows. For the proof, see the original paper.

Theorem 2.1 (reduction principle for Maz'ya domains)**.** *Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$ and $\alpha \in [\frac{1}{n'}, 1)$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be r.i. function norms. Let $c > 0$ such that*

$$\left\| \int_t^1 f(s) s^{-1+m(1-\alpha)}\, \mathrm{d}s \right\|_{Y(0,1)} \leq c \cdot \|f\|_{X(0,1)} \tag{2.3}$$

for any $f \in X(0,1)$ nonnegative. Then, the Sobolev embedding (2.1) *holds for every $\Omega \in \mathcal{J}_\alpha$. Conversely, if the Sobolev embedding* (2.1) *holds for every $\Omega \in \mathcal{J}_\alpha$, then the inequality* (2.3) *holds.*

Note that we have omitted the case $\alpha = 1$. While our results may be, after some modification, applied to such case, it remains rather technical and is beyond the scope of this work.

As a consequence of Theorem 2.1, we can identify the optimal r.i. target space Y associated with a given domain space X in the Sobolev embedding (2.1) for any $\Omega \in \mathcal{J}_\alpha$. This is also a known result, for the proof see [Cianchi et al., 2015, Theorem 6.5].

Theorem 2.2 (optimal r.i. target for Maz'ya domains)**.** *Let n, m, α and $\|\cdot\|_{X(0,1)}$ be as in Theorem 2.1. Define the functional $\|\cdot\|_{X'_{m,\alpha}(0,1)}$ as*

$$\|\cdot\|_{X'_{m,\alpha}(0,1)} = \left\| t^{-1+m(1-\alpha)} \int_0^t f^*(s)\,\mathrm{d}s \right\|_{X'(0,1)}$$

where $\|\cdot\|_{X'(0,1)}$ denotes the associate function norm to $\|\cdot\|_{X(0,1)}$. Then, the functional $\|\cdot\|_{X'_{m,\alpha}(0,1)}$ is an r.i. function norm, whose associate norm $\|\cdot\|_{X_{m,\alpha}(0,1)}$ satisfies

$$W^m X(\Omega) \hookrightarrow X_{m,\alpha}(\Omega) \tag{2.4}$$

for every $\Omega \in \mathcal{J}_\alpha$, and for some constant c depending on Ω, m, $X(\Omega)$ and $Y(\Omega)$

$$\|u\|_{X_{m,\alpha}(\Omega)} \le c \cdot \|\nabla^m u\|_{X(\Omega)} \tag{2.5}$$

for every $\Omega \in \mathcal{J}_\alpha$ and every $u \in V_\perp^m X(\Omega)$. Furthermore, the function norm $\|\cdot\|_{X(0,1)}$ is optimal in (2.4) and (2.5) among all r.i. norms, as Ω ranges in $\mathcal{J}_\alpha$.

We shall now use Theorem 2.2 to show the optimal target r.i. space for certain critical r.i. spaces.

Example. Let $\alpha \in [\frac{1}{n'}, 1)$. Then, for any $\Omega \in \mathcal{J}_\alpha$ and for any $q \in (1,\infty]$, given the critical spaces $W^m L^{\frac{1}{m(1-\alpha)},q}(\Omega)$, we obtain the embedding

$$W^m L^{\frac{1}{m(1-\alpha)},q}(\Omega) \hookrightarrow L^{\infty,q;-1+(1-\alpha)m-\frac{1}{q}}(\Omega),$$

where $L^{\infty,q;-1+(1-\alpha)m-\frac{1}{q}}(\Omega)$ is the smallest (= optimal) space with this property for any $\Omega \in \mathcal{J}_\alpha$. In the case $q = 1$, we obtain the embedding

$$W^m L^{\frac{1}{m(1-\alpha)},1}(\Omega) \hookrightarrow L^\infty(\Omega),$$

where $L^\infty(\Omega)$ is the optimal target space. In the case $q = \infty$, we obtain the embedding

$$W^m L^{\frac{1}{m(1-\alpha)},\infty}(\Omega) \hookrightarrow \exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega),$$

where $\exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega)$ is the optimal target space. In the case $q = \frac{1}{m(1-\alpha)}$, we obtain the embedding

$$W^m L^{\frac{1}{m(1-\alpha)}}(\Omega) \hookrightarrow L^{\infty,\frac{1}{m(1-\alpha)};-1}(\Omega),$$

where $L^{\infty,\frac{1}{m(1-\alpha)};-1}(\Omega)$ is the optimal target space, and which recovers, as its special case for $\alpha = \frac{1}{n'}$, the result of Brézis and Wainger [1980].

The question remains whether we can say anything about the optimality of r.i. domains, given a target r.i. space. This problem has also been extensively studied, and as such, for our purposes, it suffices to modify a known result by [Kerman and Pick, 2006, Theorem 3.3].

Theorem 2.3 (optimal r.i. domain space)**.** *Let m, $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$. Let $Y(0,1)$ be an r.i. space such that*

$$Y(0,1) \hookrightarrow L^{\frac{1}{1-m(1-\alpha)}}(0,1).$$

Define the functional $\|\cdot\|_{X(0,1)}$ *as*

$$\|f\|_{X(0,1)} = \sup_{h\sim f} \left\| \int_t^1 s^{-1+m(1-\alpha)} h(s)\,\mathrm{d}s \right\|_{Y(0,1)} \quad \text{for } f \in \mathcal{M}(0,1),$$

where the supremum is extended over all $h \in \mathcal{M}_+(0,1)$ *such that* $h^* = f^*$, *and the set*

$$X(0,1) = \Big\{ f \in \mathcal{M}(0,1) : \|f\|_{X(0,1)} < \infty \Big\}.$$

Then, the embedding

$$W^m X(\Omega) \hookrightarrow Y(\Omega)$$

holds for every $\Omega \in \mathcal{J}_\alpha$. *Furthermore,* $X(0,1)$ *is the largest r.i. space with this property.*

The proof of Theorem 2.3 is a simple modification of the proof in the aforementioned paper and therefore is omitted.

We shall now discuss the fundamental Orlicz spaces of r.i. spaces. The following theorem has been established and proven by Musil et al. [2022].

Theorem 2.4 (the principal alternative)**.** (i) *Let* Y *be an r.i. space and* $L(Y)$ *its fundamental Orlicz space. Then either* $Y \subset L(Y)$ *and* $L(Y)$ *is the smallest Orlicz space containing* Y, *or* $Y \not\subset L(Y)$ *and no smallest Orlicz space containing* Y *exists.*

(ii) *Let* X *be an r.i. space and* $L(X)$ *its fundamental Orlicz space. Then either* $L(X) \subset X$ *and* $L(X)$ *is the largest Orlicz space contained in space* X, *or* $L(X) \not\subset X$ *and no largest Orlicz space contained in* X *exists.*

Next, we specify the principal alternative to Sobolev embeddings. The following theorem is an adjustment to Musil et al. [2022, Theorem 4.1], its proof is a simple modification of the proof in the original paper and therefore is omitted.

Theorem 2.5 (principal alternative for Sobolev embeddings)**.** *Let* m, $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$, *and let* $\|\cdot\|_{X(0,1)}$, $\|\cdot\|_{Y(0,1)}$ *be r.i. norms.*

(i) *If* X *is the largest among all r.i. spaces rendering embedding* (2.1) *true for every* $\Omega \in \mathcal{J}_\alpha$, *then either* $L(X(0,1)) \subset X(0,1)$ *and* $L^A = L(X)$ *is the largest Orlicz space such that*

$$W^m L^A(\Omega) \hookrightarrow Y(\Omega)$$

holds for every $\Omega \in \mathcal{J}_\alpha$, *or no such largest Orlicz space exists.*

(ii) *If* Y *is the smallest among all r.i. spaces rendering embedding* (2.1) *true for every* $\Omega \in \mathcal{J}_\alpha$, *then either* $Y(0,1) \subset L(Y(0,1))$ *and* $L^B = L(Y)$ *is the smallest Orlicz space such that*

$$W^m X(\Omega) \hookrightarrow L^B(\Omega)$$

holds for every $\Omega \in \mathcal{J}_\alpha$, *or no such smallest Orlicz space exists.*

3. Main results

First, we shall present a general result concerning the fundamental function of an operator-induced space. The theorem and the proof are an extension to [Musil et al., 2022, Theorem 4.2], where Lipschitz domains are considered instead.

Theorem 3.1 (fundamental function of an operator-induced space)**.**
Let $I : [0,1] \to [0,\infty]$. Let $Y(0,1)$ be an r.i. space such that

$$\lim_{t\to 0^+} \varphi_Y(t) = 0, \tag{3.1}$$

and

$$\int_0^t \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s \lesssim t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^s \frac{\mathrm{d}\tau}{I(\tau)}, \quad t \in (0,1). \tag{3.2}$$

Let $X(0,1)$ be defined by

$$\|f\|_{X(0,1)} = \sup_h \left\| \int_\tau^1 \frac{h(s)}{I(s)}\mathrm{d}s \right\|_{Y(0,1)} \quad \text{for } f \in \mathcal{M}(0,1),$$

where the supremum is extended over all $h \in \mathcal{M}_+(0,1)$ such that $h^* = f^*$. Then $X(0,1)$ is an r.i. space, and one has

$$\varphi_X(t) \approx t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^s \frac{\mathrm{d}\tau}{I(\tau)} \quad \text{for } t \in (0,1).$$

Proof. The fact that X is an r.i. space follows from Theorem 2.3. First, we will examine the lower bound of φ_X. Fix $t \in (0,1)$. By the boundedness of the dilation operator on Y, we get

$$\varphi_X(t) = \|\chi_{[0,t)}\|_{X(0,1)} \geq \left\| \int_\tau^1 \frac{\chi_{(0,t)}(s)}{I(s)}\mathrm{d}s \right\|_{Y(0,1)} \geq \left\| \chi_{(0,\frac{t}{2})}(\tau) \int_\tau^t \frac{\mathrm{d}s}{I(s)} \right\|_{Y(0,1)}$$

$$\geq \left\| \chi_{(0,\frac{t}{2})}(\tau) \int_{\frac{t}{2}}^t \frac{\mathrm{d}s}{I(s)} \right\|_{Y(0,1)} = \varphi_Y(\tfrac{t}{2}) \int_{\frac{t}{2}}^t \frac{\mathrm{d}s}{I(s)},$$

showing that

$$\frac{\varphi_X(t)}{t} \geq \frac{\varphi_Y(\frac{t}{2})}{t} \int_{\frac{t}{2}}^t \frac{\mathrm{d}s}{I(s)} \quad \text{for every } t \in (0,1).$$

Since the function $t \mapsto \frac{\varphi_X(t)}{t}$ is nonincreasing, it follows that

$$\frac{\varphi_X(t)}{t} \geq \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^s \frac{\mathrm{d}\tau}{I(\tau)} \quad \text{for every } t \in (0,1).$$

As such, we have obtained the lower bound of φ_X, as

$$\varphi_X(t) \geq t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^s \frac{\mathrm{d}\tau}{I(\tau)} \quad \text{for every } t \in (0,1). \tag{3.3}$$

Let us focus on the upper bound of φ_X. Note that φ_Y is differentiable a.e. and for any measurable f, one has

$$\|f\|_{\Lambda(Y)} = \varphi_Y(0_+)\|f\|_\infty + \int_0^1 f^*\varphi_Y'.$$

Therefore, by fundamental embedding (1.3), (3.1) and Fubini's theorem, we obtain that for any $h \geq 0$, it holds that

$$\begin{aligned}
\left\|\int_\tau^1 \frac{h(s)}{I(s)}\mathrm{d}s\right\|_{Y(0,1)} &\lesssim \left\|\int_\tau^1 \frac{h(s)}{I(s)}\mathrm{d}s\right\|_{\Lambda(Y(0,1))} = \int_0^1 \int_\tau^1 \frac{h(s)}{I(s)}\,\mathrm{d}s\,\varphi_Y'(\tau)\,\mathrm{d}\tau \\
&= \int_0^1 \frac{h(s)}{I(s)} \int_0^s \varphi_Y'(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^1 \frac{h(s)}{I(s)}\varphi_Y(s)\,\mathrm{d}s \\
&\leq \int_0^1 h(s)\frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s\,.
\end{aligned}$$

Hence, by the definition of φ_X and by the Hardy-Littlewood inequality (1.1), it holds for any $t \in (0,1)$, that

$$\begin{aligned}
\varphi_X(t) = \|\chi_{[0,t)}\|_{X(0,1)} &= \sup_{h^*=\chi_{[0,t)}} \left\|\int_\tau^1 \frac{h(s)}{I(s)}\,\mathrm{d}s\right\|_{Y(0,1)} \\
&\leq \sup_{h^*=\chi_{[0,t)}} \int_0^1 h(s) \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s \\
&\leq \sup_{h^*=\chi_{[0,t)}} \int_0^1 h^*(s) \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s \\
&= \int_0^t \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s\,.
\end{aligned}$$

These inequalities give us an upper bound of φ_X, as

$$\varphi_X(t) \leq \int_0^t \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s \quad \text{for every } t \in (0,1). \tag{3.4}$$

Thus, by combining the lower bound (3.3) and the upper bound (3.4) of φ_X for $t \in (0,1)$, we obtain the sandwich

$$t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^s \frac{\mathrm{d}\tau}{I(\tau)} \lesssim \varphi_X(t) \lesssim \int_0^t \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{I(\tau)}\,\mathrm{d}s\,.$$

Finally, applying the presupposed inequality (3.2) on the sandwich above, we get the desired result

$$\varphi_X(t) \approx t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^s \frac{\mathrm{d}\tau}{I(\tau)} \quad \text{for } t \in (0,1).$$

□

We shall now use the result obtained in the preceding theorem to obtain important information about fundamental functions of optimal domain r.i. spaces in Sobolev embeddings corresponding to given target spaces sharing the same fundamental level. The theorem and proof are again an extension of [Musil et al., 2022, Corollary 4.3].

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^n$ with isoperimetric profile I_Ω. Suppose that Y_1, Y_2 are r.i. spaces over Ω on the same fundamental level, i.e. $\varphi_{Y_1} \approx \varphi_{Y_2}$, and* (3.1) *and* (3.2) *hold for both Y_1 and Y_2. Let X_j be the optimal r.i. domain spaces in the embedding*

$$W^m X_j(\Omega) \hookrightarrow Y_j(\Omega),$$

for $j = 1, 2$. Then X_1, X_2 are also on the same fundamental level, i.e. $\varphi_{X_1} \approx \varphi_{X_2}$.

Proof. The proof follows immediately from the formula of the norm in the optimal r.i. domain space (2.2) and by Theorem 3.1. □

Although the results of Theorem 3.1 are applicable to any isoperimetric function I_Ω, the presuppositions of the theorem, namely (3.2), aren't particularly flexible, as there possibly exist classes of isoperimetric functions for which the presupposition is needlessly strong, or even entirely unnecessary. We aim to prove, that for Maz'ya classes of domains, Theorem 3.1 may be applied to them directly, omitting the need for presupposition (3.2) entirely. To prove this, we must find a work-around, so we do not need to consider every domain in $\mathcal{J}_\alpha$.

The necessary condition of embedding (2.1) for some $\alpha \in [\frac{1}{n'}, 1)$ tells us, that the embedding holds for every $\Omega \in \mathcal{J}_\alpha$. We focus, in particular, on the worst domain with such property, denoted Ω_α. To describe this domain, we use [Cianchi et al., 2015, Proposition 10.1 (i)], which follows. In the statement, ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

Proposition 3.3. *Let $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$. Define $\eta_\alpha : [0, \frac{1}{1-\alpha}] \to [0, \infty)$ as*

$$\eta_\alpha(t) = \omega_{n-1}^{-\frac{1}{n-1}} (1 - (1-\alpha)t)^{\frac{\alpha}{(1-\alpha)(n-1)}} \quad \text{for } t \in [0, \tfrac{1}{1-\alpha}].$$

Let Ω_α be the Euclidean domain in $\mathbb{R}^n$ given by

$$\Omega_\alpha = \left\{ (x, x_n) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1},\ 0 < x_n < \tfrac{1}{1-\alpha},\ |x| < \eta_\alpha(x_n) \right\}.$$

Then $|\Omega_\alpha| = 1$, and

$$I_{\Omega_\alpha}(t) \approx t^\alpha \quad \text{for } t \in [0, \tfrac{1}{2}].$$

The best case scenario happens when $\alpha = \frac{1}{n'}$ and we are dealing with John domains. For example, if $n = 2$ and $\alpha = \frac{1}{2}$, then the function η_α takes the form

$$\eta_{\frac{1}{2}}(t) = \tfrac{1}{2}(1 - \tfrac{1}{2}t)$$

and, as a result,

$$\Omega_\alpha = \{x_1 \in (0, 2),\ |x_2| < \tfrac{1}{2}(1 - \tfrac{1}{2}x_1)\}.$$

In other words, Ω_α takes the form of a triangle with verteces (denoted by $[x_1, x_2]$) $[0, \frac{1}{2}]$, $[0, -\frac{1}{2}]$, and $[2, 0]$. We may easily verify that $|\Omega_\alpha| = 1$. The domains start to get worse as we choose $\alpha > \frac{1}{2}$. It is on these domains Ω_α that we consider the following Corollary.

Corollary 3.4. *Let $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$. Define Ω_α as in Proposition 3.3. Then, for $I_{\Omega_\alpha}(t)$, Theorem 3.1 and Corollary 3.2 are applicable without the restriction* (3.2)*.*

Proof. Let $\alpha \in [\frac{1}{n'}, 1)$. Returning to the proof of Theorem 3.1, in the case of I_{Ω_α}, the lower bound (3.3) contains a numerable integral. Thus, it holds that for some constant $c > 0$

$$\varphi_X(t) \geq t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \int_{\frac{s}{2}}^{s} \frac{\mathrm{d}\tau}{\tau^\alpha} = t \sup_{s\in(t,1)} \frac{\varphi_Y(\frac{s}{2})}{s} \frac{1}{1-\alpha} \left(s^{1-\alpha} - (\tfrac{s}{2})^{1-\alpha}\right)$$

$$= t \sup_{s\in(t,1)} \varphi_Y(\tfrac{s}{2}) s^{-\alpha} c\,.$$

Therefore,

$$\varphi_X(t) \gtrsim t \sup_{s\in(t,1)} \varphi_Y(\tfrac{s}{2}) s^{-\alpha}.$$

Moreover, it follows from (3.4) that

$$\varphi_X(t) \lesssim \int_0^t \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{\tau^\alpha} \,\mathrm{d}s\,.$$

As such, to prove the Corollary, it is sufficient to prove that for any $t \in (0, 1)$

$$t \sup_{s\in(t,1)} \varphi_Y(\tfrac{s}{2}) s^{-\alpha} \gtrsim \int_0^t \sup_{\tau\in(s,1)} \frac{\varphi_Y(\tau)}{\tau^\alpha} \,\mathrm{d}s\,. \tag{3.5}$$

Define

$$\psi(t) := t \sup_{s\in(t,1)} \varphi_Y(\tfrac{s}{2}) s^{-\alpha} \quad t \in (0, 1).$$

Then ψ is non-decreasing, since

$$\psi(t) = t \sup_{s\in(t,1)} \varphi_Y(\tfrac{s}{2}) s^{-\alpha} = \sup_{s\in(0,1)} \varphi_Y(\tfrac{s}{2}) \min\left\{t^\alpha, ts^{-\alpha}\right\}. \tag{3.6}$$

It also follows from equality (3.6) that for any $\sigma \in (0, 1)$ and any $k \in \mathbb{N}$, one has $\psi(t\sigma^k) \leq \sigma^{\alpha k}\psi(t)$ for $t \in (0, 1)$ and therefore

$$\int_0^t \frac{\psi(s)}{s} \,\mathrm{d}s = \sum_{k=0}^{\infty} \int_{t\sigma^{k+1}}^{t\sigma^k} \frac{\psi(s)}{s} \,\mathrm{d}s \leq \sum_{k=0}^{\infty} \psi(t\sigma^k) \int_{t\sigma^{k+1}}^{t\sigma^k} \frac{\mathrm{d}s}{s}$$

$$\leq \psi(t) \log\frac{1}{\sigma} \sum_{k=0}^{\infty} \sigma^{\alpha k} = \psi(t) \frac{\log\frac{1}{\sigma}}{1-\sigma^\alpha}.$$

Hence,

$$\psi(t) \gtrsim \int_0^t \frac{\psi(s)}{s} \mathrm{d}s,$$

and by definition of ψ, we obtain the desired result (3.5). ☐

Combining all the results of this chapter so far yields the following theorem, which is an extension of [Musil et al., 2022, Theorem 4.7].

Theorem 3.5 (nonexistence of an optimal Orlicz space)**.** *Let $m, n \in \mathbb{N}$, let $\alpha \in [\frac{1}{n'}, 1)$, and let $Y(\Omega)$ be an r.i. space. Assume there is no largest Orlicz space L^A such that*

$$W^m L^A(\Omega) \hookrightarrow M(Y)(\Omega) \tag{3.7}$$

for every Ω in $\mathcal{J}_\alpha$. Then, there is no largest Orlicz space L^A such that

$$W^m L^A(\Omega) \hookrightarrow Y(\Omega) \tag{3.8}$$

for every Ω in $\mathcal{J}_\alpha$.

Proof. We know that $Y(\Omega)$ and $M(Y)(\Omega)$ share the same fundamental function. We denote by $X_Y(\Omega)$ and $X_{M(Y)}(\Omega)$ the largest domain r.i. spaces in embeddings

$$W^m X_Y(\Omega) \hookrightarrow Y(\Omega)$$

and

$$W^m X_{M(Y)}(\Omega) \hookrightarrow M(Y)(\Omega),$$

respectively. Then, by Corollary 3.2 combined with Corollary 3.4, we obtain that $\varphi_{X_Y} = \varphi_{M_{X_Y}}$. Owing to the assumption, there is no largest Orlicz space $L^A(\Omega)$ in the Sobolev embedding (3.7). Therefore, by Theorem 2.5, it follows that $L_{X_{M(Y)}}(\Omega) \not\hookrightarrow X_{M(Y)}(\Omega)$. Since it holds that $Y(\Omega) \hookrightarrow M(Y)(\Omega)$, it is also true that $X_Y(\Omega) \hookrightarrow X_{M(Y)}(\Omega)$. Consequently, $L_{X_{M(Y)}}(\Omega) \not\hookrightarrow X_Y(\Omega)$. But, since $\varphi_{X_Y} = \varphi_{M_{X_Y}}$, we have $L_{X_{M(Y)}}(\Omega) = L_{X_Y}(\Omega)$. Altogether, $L_{X_Y}(\Omega) \not\hookrightarrow X_Y(\Omega)$, whence, using Theorem 2.5 once again, there is no largest Orlicz space $L^A(\Omega)$ in the Sobolev embedding (3.8). □

We shall now apply Theorem 3.5 to show that there is no optimal Orlicz domain space $L^A(\Omega)$ in the embedding

$$W^m L^A(\Omega) \hookrightarrow L^{\infty, q, -1+(1-\alpha)m-\frac{1}{q}}(\Omega),$$

thereby solving the open problem mentioned in the introduction. First, we shall prove three Lemmas, which we will later use in the proof. The first Lemma concerns the r.i. norm of the characteristic function.

Lemma 3.6. *Let $\zeta > 0$, $a \in (0,1)$ and $\|\cdot\|_{X(0,1)}$ be an r.i. norm. Then*

$$\|\chi_{(0,a)}(t)(a^\zeta - t^\zeta)\|_{X(0,1)} \approx a^\zeta \|\chi_{(0,a)}(t)\|_{X(0,1)}.$$

Proof. The inequality

$$\|\chi_{(0,a)}(t)(a^\zeta - t^\zeta)\|_{X(0,1)} \lesssim a^\zeta \|\chi_{(0,a)}(t)\|_{X(0,1)}$$

is trivial. Conversely, we observe that

$$\begin{aligned}
\|\chi_{(0,a)}(t)(a^\zeta - t^\zeta)\|_{X(0,1)} &\geq \|\chi_{(0,\frac{a}{2})}(t)(a^\zeta - t^\zeta)\|_{X(0,1)} \\
&\geq \|\chi_{(0,\frac{a}{2})}(t)(a^\zeta - (\tfrac{a}{2})^\zeta)\|_{X(0,1)} \\
&= a^\zeta(1 - (\tfrac{1}{2})^\zeta)\|\chi_{(0,\frac{a}{2})}\|_{X(0,1)} \\
&\gtrsim a^\zeta \|\chi_{(0,a)}(t)\|_{X(0,1)},
\end{aligned}$$

since, by the triangle inequality and by the rearrangement-invariance of $\|\cdot\|_{X(0,1)}$,

$$\|\chi_{(0,a)}\|_{X(0,1)} = \|\chi_{(0,\frac{a}{2}]} + \chi_{(\frac{a}{2},a)}\|_{X(0,1)} \leq \|\chi_{(0,\frac{a}{2})}\|_{X(0,1)} + \|\chi_{(\frac{a}{2},a)}\|_{X(0,1)}$$
$$\leq 2\|\chi_{(0,\frac{a}{2})}\|_{X(0,1)}.$$

Altogether, one has

$$\|\chi_{(0,a)}(t)(a^\zeta - t^\zeta)\|_{X(0,1)} \approx a^\zeta \|\chi_{(0,a)}(t)\|_{X(0,1)},$$

with constants of equivalence depending only on ζ (even independent of $\|\cdot\|_{X(0,1)}$). □

The second Lemma concerns the inverse of a function, which will later appear in the proof.

Lemma 3.7. *Let $p > 1$, $\gamma \in \mathbb{R}$. Define $A(t) = t^p \log^\gamma t$ for $t > 1$ sufficiently large. Then $A^{-1}(t) \approx t^{\frac{1}{p}} \log^{-\frac{\gamma}{p}} t$ for large t.*

Proof. Since A is increasing, there exists a function B such that $B(t) = A^{-1}(t)$ for every $t \in (0, \infty)$. We shall prove that $B(t) = t^{\frac{1}{p}} \log^{-\frac{\gamma}{p}} t$. As such, it suffices to prove that $A(B(t)) \approx t$. We have

$$A(B(t)) = B(t)^p \log^\gamma(B(t)) = (t^{\frac{1}{p}} \log^{-\frac{\gamma}{p}} t)^p \cdot \log^\gamma(t^{\frac{1}{p}} \log^{-\frac{\gamma}{p}}) =$$
$$= t \cdot \log^{-\gamma} t \cdot \log^\gamma(t^{\frac{1}{p}} \log^{-\frac{\gamma}{p}}),$$

Therefore, it is sufficient to prove that $\log(t^{\frac{1}{p}} \cdot \log^{-\gamma} p) \approx \log t$ for $t \gg 1$. We obtain that

$$\lim_{t\to\infty} \frac{\log(t^{\frac{1}{p}} \cdot \log^{\frac{-\gamma}{p}} t)}{\log t} = \lim_{t\to\infty} \frac{\frac{1}{p}\log t - \frac{\gamma}{p}\log\log t}{\log t} = \tfrac{1}{p} \in (0,\infty).$$

Hence, $B \approx A^{-1}$. □

The third Lemma concerns the boundedness of the operator T_γ introduced in (1.2).

Lemma 3.8. *Let $\gamma \in (0,1)$. Then*

$$T_\gamma : L^{1,1;1-\gamma}(0,1) \to L^{1,1;1-\gamma}(0,1).$$

Proof. This is a particular case of [Gogatishvili et al., 2006, Theorem 3.2], applied to $p = q = 1$, and $\varphi(t) = t^\gamma$, $v(t) = w(t) = (\log \frac{2}{t})^{1-\gamma}$ for $t \in (0,1)$. □

Theorem 3.9. *Let $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$, $m \in \mathbb{N}$ such that $m < \frac{1}{1-\alpha}$, and let $q \in [\frac{1}{1-m(1-\alpha)}, \infty]$. Then there is no largest Orlicz space L^A such that*

$$W^m L^A(\Omega) \hookrightarrow L^{\infty,q,-1+(1-\alpha)m-\frac{1}{q}}(\Omega) \tag{3.9}$$

for every Ω in $\mathcal{J}_\alpha$.

Proof. By [Opic and Pick, 1999, Lemma 3.7], the spaces $L^{\infty,q,-1+(1-\alpha)m-\frac{1}{q}}$ and $\exp L^{\frac{1}{1-m(1-\alpha)}}$ share the same fundamental function φ, where

$$\varphi(t) = \frac{1}{(\log \frac{2}{t})^{1-m(1-\alpha)}} \quad \text{for } t \in (0,1),$$

and it holds that $L^{\infty,q,-1+(1-\alpha)m-\frac{1}{q}} \hookrightarrow \exp L^{\frac{1}{1-m(1-\alpha)}}$. Moreover, by the aforementioned Lemma, it holds that $M(Z) = \exp L^{\frac{1}{1-m(1-\alpha)}}$ for any r.i. space Z with the fundamental function φ, and in fact, $L(Z) = \exp L^{\frac{1}{1-m(1-\alpha)}}$.

We aim to prove that there is no largest Orlicz space L^A such that the embedding (3.9) holds for every Ω in $\mathcal{J}_\alpha$. Therefore, by Theorem 3.5, it suffices to prove that there is no largest Orlicz space L^A such that the embedding

$$W^m L^A(\Omega) \hookrightarrow \exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega) \tag{3.10}$$

holds for every Ω in $\mathcal{J}_\alpha$. By Theorem 2.5, it is therefore enough to prove that if X is the optimal domain r.i. space in

$$W^m X(\Omega) \hookrightarrow \exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega)$$

for every Ω in $\mathcal{J}_\alpha$, then $L(X(0,1)) \not\subset X(0,1)$.

First, we find the optimal domain space $X(\Omega)$. We use Theorem 2.3 and obtain the formula

$$\|f\|_{X(0,1)} = \sup_{h\sim f} \left\| \int_t^1 s^{-1+m(1-\alpha)} h(s)\,\mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \qquad \text{for } f \in \mathcal{M}(0,1).$$

By Lemma 3.8, applied to $\gamma = m(1-\alpha)$, the operator $T_{m(1-\alpha)}$ is bounded on $L^{1,1;1-m(1-\alpha)}$. Since $(L^{1,1;1-m(1-\alpha)}(0,1))' = \exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)$, it follows from [Edmunds et al., 2020, Theorem 4.7], that

$$\|f\|_{X(0,1)} \approx \left\| \int_t^1 s^{-1+m(1-\alpha)} f^*(s)\,\mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)}.$$

As such, we will with no loss of information suppose that

$$\|f\|_{X(0,1)} = \left\| \int_t^1 s^{-1+m(1-\alpha)} f^*(s)\,\mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)}.$$

Next, we describe $L(X)$. For that purpose, we need to find φ_X so we can extrapolate its Young function A. Let $a \in (0,1)$. Then, as $\chi_{(0,a)} = \chi^*_{(0,a)}$ and by Lemma 3.6, we get

$$\begin{aligned}
\varphi_X(a) = \|\chi_{(0,a)}\|_{X(0,1)} &= \left\| \int_t^1 s^{-1+m(1-\alpha)} \chi_{(0,a)}(s)\,\mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \\
&= \left\| \chi_{(0,a)}(t) \int_t^a s^{-1+m(1-\alpha)} \mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \\
&\approx \left\| \chi_{(0,a)}(t)(a^{m(1-\alpha)} - t^{m(1-\alpha)}) \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \\
&\approx \left\| \chi_{(0,a)} \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \cdot a^{m(1-\alpha)} = \frac{1}{(\log \frac{2}{a})^{1-m(1-\alpha)}} \cdot a^{m(1-\alpha)}.
\end{aligned}$$

We therefore know that

$$\varphi_X(a) \approx \frac{a^{m(1-\alpha)}}{(\log\frac{2}{a})^{1-m(1-\alpha)}} \quad \text{for } a \in (0,1). \tag{3.11}$$

Let $A(t) = t^p \log^\gamma t$ for $t > 1$ sufficiently large, $p > 1$ and $\gamma \in \mathbb{R}$. Then, by Lemma 3.7, it holds that $A^{-1}(t) = t^{\frac{1}{p}} \log^{-\frac{\gamma}{p}} t$. We know that $\|\chi_{(0,a)}\|_{L^A} \approx \frac{1}{A^{-1}(\frac{1}{a})}$ for a in $(0,1)$. As such, we obtain

$$\|\chi_{(0,a)}\|_{L^p \log L^\gamma(0,1)} \approx \frac{1}{(\frac{1}{a})^{\frac{1}{p}}(\log\frac{2}{a})^{-\frac{\gamma}{p}}} = \frac{a^{\frac{1}{p}}}{(\log\frac{2}{a})^{-\frac{\gamma}{p}}}.$$

Applying this result to (3.11) tells us that $p = \frac{1}{m(1-\alpha)}$, and $\gamma = 1 - \frac{1}{m(1-\alpha)}$. Hence,

$$A(t) = t^{\frac{1}{m(1-\alpha)}} (\log t)^{1-\frac{1}{m(1-\alpha)}} \quad \text{for } t \gg 1,$$

and $L(X) = L^{\frac{1}{m(1-\alpha)}} \log L^{1-\frac{1}{m(1-\alpha)}}$.

Finally, we need to prove that $L(X) \not\subset X$. We know X is optimal in the embedding $W^m X \hookrightarrow \exp L^{\frac{1}{1-m(1-\alpha)}}$, and consequently, it suffices to prove that $W^m L(X) \not\hookrightarrow \exp L^{\frac{1}{1-m(1-\alpha)}}$. Then, by Theorem 2.1, it is enough to prove that the inequality

$$\left\| \int_t^1 s^{-1+m(1-\alpha)} f(s)\, \mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \lesssim \|f\|_{L(X(0,1))}$$

does not hold. Let $f(t) = t^{-m(1-\alpha)}(\log\frac{2}{t})^\beta$, where β is anywhere in the interval $(-m(1-\alpha), 1-2m(1-\alpha))$. Owing to the assumption, the interval is non-empty. First, we prove that $f \in L(X)$, that is, $\|f\|_{L(X(0,1))} < \infty$. Since $A(t)$ satisfies the Δ_2 condition, it holds that $f \in L^A \iff \int_0^1 A(f) < \infty$. Furthermore,

$$\begin{aligned} \int_0^1 A(f) < \infty &\iff \int_0^1 s^{-1}(\log\tfrac{2}{s})^{\beta\frac{1}{m(1-\alpha)}+1-\frac{1}{m(1-\alpha)}}\, \mathrm{d}s < \infty \\ &\iff \beta < 1 - 2m(1-\alpha). \end{aligned}$$

Therefore, our choice of β guarantees that $f \in L(X)$. Moreover, we know that

$$\|g\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}(0,1)} \approx \sup_{t\in(0,1)} \frac{g^*(t)}{(\log\frac{2}{t})^{1-m(1-\alpha)}} \quad \text{for } g \in \mathcal{M}(0,1),$$

thus it is sufficient to prove that

$$\sup_{t\in(0,1)} \frac{\int_t^1 s^{-1+m(1-\alpha)} f(s)\, \mathrm{d}s}{(\log\frac{2}{t})^{1-m(1-\alpha)}} = \infty. \tag{3.12}$$

Suppose the equality (3.12) holds. We solve the integral and obtain

$$\begin{aligned} \infty &= \sup_{t\in(0,1)} \frac{\int_t^1 s^{-1+m(1-\alpha)} \cdot s^{-m(1-\alpha)}(\log\frac{2}{s})^\beta\, \mathrm{d}s}{(\log\frac{2}{t})^{1-m(1-\alpha)}} = \sup_{t\in(0,1)} \frac{\int_t^1 \frac{(\log\frac{2}{s})^\beta}{s}\, \mathrm{d}s}{(\log\frac{2}{t})^{1-m(1-\alpha)}} \\ &\approx \sup_{t\in(0,1)} \frac{(\log\frac{2}{t})^{\beta+1}}{(\log\frac{2}{t})^{1-m(1-\alpha)}} = \sup_{t\in(0,1)} (\log\tfrac{2}{t})^{\beta+m(1-\alpha)}, \end{aligned}$$

so the equality holds if $\beta > -m(1-\alpha)$. Hence, once again, our choice of β yields (3.12). Therefore, we have found a function $f \in L(X)$ such that

$$\left\| \int_t^1 s^{-1+m(1-\alpha)} f(s)\, \mathrm{d}s \right\|_{\exp L^{\frac{1}{1-m(1-\alpha)}}} = \infty.$$

Consequently, by Theorem 2.1 $L(X) \not\subset X$. By Theorem 2.5, there is no largest Orlicz space which would render the embedding (3.10) true for every Ω in $\mathcal{J}_\alpha$. Finally, by Theorem 3.5, there is no largest Orlicz space which would render the embedding (3.9) true for every Ω in $\mathcal{J}_\alpha$. □

Remark 3.10. In this work, we have focused primarily on Maz'ya classes of Euclidean domains. We are aware that there are plenty of open questions worth pursuing which we leave open. Pivotal examples are the case $\alpha = 1$ in Maz'ya domains, Gaussian–Sobolev embeddings, embeddings on domains endowed with Frostman-Ahlfors measures, or embeddings on probability spaces. The reason we do not consider these cases is that the corresponding integral operators get too complicated (e.g. they take the form, at least for higher-order embeddings, of kernel-type operators). Hence such considerations reach beyond the scope of this text. We plan to study them in our following work.

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