Evaluation of a Master Thesis by the Opponent

Title: Symmetric terms
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Summary of the contents

In finite algebras, the existence of terms satisfying a system of non-trivial identities of height one is equivalent to several more specific equational conditions (such as the existence of weak near unanimity terms, or cyclic terms). In a recent, unpublished proof, Barto, Brady, Pinsker and Zhuk further showed the equivalence to the existence of so called *2-WNU terms* (of infinitely many arities).

In this Master thesis, Martin Boroš, reproves their result. Chapter 1 gives some general background from universal algebra. Chapter 2 investigates the existence of k-WNU terms in finite affine algebras via the study of *symmetric affine subspaces*. Chapter 3 then uses Brady's *stability concepts* to lift the result from affine to general algebras, thus proving the main result of the thesis, Corollary 56.

Evaluation of the thesis

The topic of the thesis is of an appropriate level of difficulty for a Master thesis: Besides basic knowledge of universal algebra, it required to understand recent developments in the field (stability, minimal Taylor algebras) and to prove some non-trivial results in linear algebra.

The student's own contributions mainly lie in proving the linear algebra results in Chapter 2, while otherwise following the outline of the BBPZ proof. This is also correctly stated as such in the Introduction (and, although the BBPZ proof is still unpublished, I am familiar enough with it to confirm it).

The mathematical quality of the thesis is overall good, giving me the impression that the student understands the material. Nevertheless, I found several mistakes and unclarities - some of them in crucial proof parts. Although they might be attributed to a poor or rushed writing process, I don't think that they should be ignored in the evaluation. Similar things can be said about some formal aspects of the thesis. This is quite unfortunate, since otherwise this could have been an excellent thesis.

Here are some examples:

- Some important notions (e.g. *clone*, $\binom{[n]}{k}$) are already used before they are formally introduced in the text. Others are never properly defined (*subclone*). Some are named in an ambiguous way (e.g. *symmetric space* can have various different meanings depending on the context; *constant* is used for *constant tuple* without ever specifying this). This is confusing.
- p7: Proposition 26 is not true! (at least not using the stated definition of "reduct")
- p9, proof of Lemma 30: it is not clear to me how $\mathbf{C}' \prec \mathbf{C}_i \cap \mathbf{C}_n = \mathbf{B}'$ follows from Definition 27 (note the difference to Brady's original axioms!)
- p9, proof of Lemma 31: "we have that $\pi_i^{-1}(\mathbf{C}_i) \cap \pi_j^{-1}(\mathbf{C}_j)$ for all i, j.": is non-empty?
- p11, Lemma 34: This statement is hard to parse. For an affine space $a+V$, the vector space V is usually called its *associated* vector space. Using this terminology could have helped throughout the thesis.

- p12: I did not manage to understand the proof of Proposition 35 ($\subseteq$), beyond the initial construction of b. Why "doesn't is matter" that repeating the construction changes a on other coordinates? Why can we only repeat it until $2k-1$ elements are left?
  "These elements satisfy that if $x \in [n]$ is not among them, then there does not exist $I \in \binom{[n]}{k}$ containing x such that $a_I \neq 0$" - what is x here, why is this sentence a triple negation? How is I chosen in the following formula? Or is it universally quantified?...
- Corollary 39 is exactly the same statement as Proposition 38!
- p20: Corollary 46, p odd: the second set is clearly not a vector space; by the condition $a_\emptyset = \frac{n}{2}$ it does not contain 0!
- p24: Proof of Proposition 50: Why is $\Rightarrow$ "clear"?
- p24: Proof of Proposition 51: What is σ?
  "We can continue like this until we get what we want": this should be formally proved by induction
- p25: Proof of Lemma 53: What are the sets X_i? Are they universally quantified?

Conclusion

In conclusion I recommend to recognize this thesis for a defense. The suggested grading will be communicated directly to the committee.

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