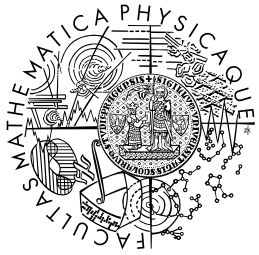

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AND PHYSICS
Charles University

MASTER THESIS

Martin Boroš

Symmetric terms

Department of Algebra

Supervisor of the master thesis: doc. Mgr. Libor Barto, Ph.D.
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Author: Martin Boroš

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Abstract: In this thesis, we study symmetric relations and affine symmetric subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. The thesis is divided into three parts. In the first part, we provide the basic definitions and facts that are needed in the rest of the thesis. In the second part, we study symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. We will provide a full characterization of symmetric vector subspaces when $k = 2$. Using that result, we will give a characterization of when every symmetric affine subspace contains a constant for $k = 2$. In the third part, we study the symmetric relations of an algebra. We will prove that under some assumptions an algebra has a k-WNU term operation.

Keywords: universal algebra symmetric terms symmetric relations symmetric subspaces

Contents

Introduction

The thesis is divided into three chapters. In the first chapter, we introduce basic notions and facts that will be used in the rest of the thesis.

In the second chapter we study the symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. Here "symmetric" means that for every permutation on $[n]$ the affine subspace is invariant under the naturally induced permutation on $\mathbb{Z}_p^{\binom{[n]}{k}}$. First, we begin the chapter by stating some general facts about symmetric affine subspaces. Then, we study when do symmetric affine subspaces contain a constant. We ask whether all symmetric affine subspaces contain a constant and answer this question in the case $k = 2$ in the third section.

The last chapter contains results concerning symmetric relations in a more general setting of Taylor algebras. We apply the results to prove that under some conditions a Taylor algebra has a k-WNU term operation of arity n.

Theorem 47 in Chapter 2 was obtained in an unpublished work of L. Barto, Z. Brady, M. Pinsker, and D. Zhuk. The presented proof as well as all the other results in this chapter are original. Chapter 3 applies Brady's technique of stable subalgebras in a standard way. The specific results, however, were not yet published. Lemma 53 appears to be original.

1. Preliminaries

The definitions in the first two sections of this chapter are taken from Clifford Bergman's book Universal Algebra: Fundamentals and Selected Topics Bergman [2011]. This chapter contains all the basic definitions and facts that we will need in the rest of the thesis. First, we will discuss algebras, clones, then Taylor algebras and finally we will introduce a notion called stability concept and prove some of its basic properties.

1.1 Algebras

In this section, we will define algebras and related notions, such as subalgebras, products, homomorphisms and congruences. Algebras are the main objects of study in Universal Algebra.

Let A, B be sets, we denote by A^B the set of all functions from A to B. If n is a positive integer then we define $A^n = A^{\{1,\dots,n\}}$ which is the set of all n-tuples of elements of A. We also define $A^0 = \{\emptyset\}$.

For any A and n as above, we call a function $A^n \to A$ an *n-ary operation* on A. The number n is called the *arity* of the operation. Operations of arity 0, 1, 2, 3 are also called *nullary*, *unary*, *binary*, and *ternary*, respectively. Notice that a nullary operation is a function $c : \{\emptyset\} \to A$, such a function is completely determined by the value $c(\emptyset)$, so it can be identified with an element of A. Such functions are also called *constants*.

Definition 1. *Any set of symbols $\Sigma = \{f, g, h, \dots\}$ such that to every symbol $f \in \Sigma$, there is an assigned arity $n_f \in \mathbb{N}_0$, is called a* signature.

Definition 2. *An* algebra $\boldsymbol{A}$ *of signature Σ is a pair $\boldsymbol{A} = (A, (f^{\boldsymbol{A}})_{f\in\Sigma})$, where A is a nonempty set, and $f^{\boldsymbol{A}}$ is an operation on A of arity n_f. The set A is called the* universe *or the* underlying set *of the algebra $\boldsymbol{A}$. The operations $(f^{\boldsymbol{A}})_{f\in\Sigma}$ are called* basic operations *of $\boldsymbol{A}$.*

A subset $X \subseteq A$ is called a subuniverse *of $\boldsymbol{A}$, if for all $f \in \Sigma$, and any elements $x_1, \dots, x_{n_f} \in X$, we have $f^{\boldsymbol{A}}(x_1, \dots, x_{n_f}) \in X$. We denote this by $X \leq \boldsymbol{A}$.*

An algebra $\boldsymbol{B} = (B, (f^{\boldsymbol{B}})_{f\in\Sigma})$ of signature Σ is called a subalgebra *of $\boldsymbol{A}$, if B is a subuniverse of $\boldsymbol{A}$ and $f^{\boldsymbol{B}} = f^{\boldsymbol{A}} \restriction B^{n_f}$, for all $f \in \Sigma$. We denote this by $\boldsymbol{B} \leq \boldsymbol{A}$.*

Examples of algebras include groups, modules, rings, monoids, lattices, semilattices and many others.

Definition 3. *Let $\boldsymbol{A} = (A, (f^{\boldsymbol{A}})_{f\in\Sigma})$ be an algebra of signature Σ and $X \subseteq A$. We say that X* generates *$\boldsymbol{A}$, if the only subuniverse of $\boldsymbol{A}$ that contains X is A.*

Definition 4. *Let I be a nonempty set, $\boldsymbol{A}_i$ be algebras of signature Σ, for all $i \in I$. Then we define the* product *$\boldsymbol{A} = \prod_{i\in I} \boldsymbol{A}_i$ of the algebras to be the algebra of signature Σ with the universe $\prod_{i\in I} A_i$, for all $f \in \Sigma, g_1, \dots, g_{n_f} \in \prod_{i\in I} A_i$ we define the operation $f^{\boldsymbol{A}}$ as $f^{\boldsymbol{A}}(g_1, \dots, g_{n_f})(i) = f^{\boldsymbol{A}_i}(g_1(i), \dots, g_{n_f}(i))$. For an algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}^I$ the product $\prod_{i\in I} \boldsymbol{A}$.*

With notation as above, suppose that the subset $X \subseteq \prod_{i\in I} A_i$ generates $\prod_{i\in I} \mathbf{A}_i$. Define a map $\pi_j : \prod_{i\in I} A_i \to A_j$ as $\pi_j(h) = h(j)$. Then $\pi_j(X)$ generates $\mathbf{A}_j$, for any $j \in I$. If it did not, then there exists a subuniverse $U \leq \mathbf{A}_j$ such that $\pi_j(X) \subseteq U \subsetneq A_j$. But then $\prod_{i\in I} X_i$ is a proper subuniverse of $\prod_{i\in I} \mathbf{A}_i$ containing X, where $X_i = A_i$ for $j \neq i \in I$, and $X_j = U$. That is a contradiction with the fact that X generates $\prod_{i\in I} \mathbf{A}_i$.

Definition 5. *Let $\boldsymbol{A}, \boldsymbol{B}$ be algebras of the same signature. A map $f : A \to B$ is called a* homomorphism*, if for every k-ary operation symbol g, and $x_1, \ldots, x_k \in A$, we have*

$$f(g^{\boldsymbol{A}}(x_1, \ldots, x_k)) = g^{\boldsymbol{B}}(f(x_1), \ldots, f(x_k)).$$

We denote this by writing $f : \boldsymbol{A} \to \boldsymbol{B}$.

A bijective homomorphism is called an isomorphism*. If $\boldsymbol{A} = \boldsymbol{B}$ and f is an isomorphism, then it is also called an* automorphism*.*

It can be easily checked that, if $\mathbf{A}_i$ are algebras of the same signature for all $i \in I$, then $\pi_j : \prod_{i\in I} A_i \to A_j$ is a homomorphism, for any $j \in I$. It is also easy to see that composition of homomorphisms is a homomorphism, and that the inverse of an isomorphism is also an isomorphism.

Definition 6. *Let $\boldsymbol{A}$ be an algebra of signature Σ. A* congruence θ *on $\boldsymbol{A}$ is an equivalence relation on A such that for all $f \in \Sigma$*

$$x_1 \ \theta \ y_1, \ldots, x_{n_f} \ \theta \ y_{n_f} \implies f^{\boldsymbol{A}}(x_1, \ldots, x_{n_f}) \ \theta \ f^{\boldsymbol{A}}(y_1, \ldots, y_{n_f}).$$

The congruences $A \times A$, and $\{(a, a) \in A \times A | a \in A\}$ are called trivial*.*

The notion of a congruence generalizes the notions of normal subgroups in the case of groups, and ideals in the case of rings.

Definition 7. *An algebra is* simple *if it has only trivial congruences.*

This definition corresponds to the definition of a simple group.

Definition 8. *Let A be a set, and f be an n-ary operation on A. We say that f is* idempotent *if $f(a, \ldots, a) = a$ for all $a \in A$. We say that an algebra $\boldsymbol{A}$ is* idempotent *if all its basic operations are idempotent.*

Definition 9. *Let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ be algebras, and $R \subseteq A_1 \times \cdots \times A_n$. We say that the relation R is* subdirect *in $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$, and write $R \leq_{sd} \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$, if $R \leq \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$, and $\pi_i(R) = A_i$, for all $i \in \{1, \ldots, n\}$, where $\pi_i(x_1, \ldots, x_n) = x_i$.*

Next, we introduce the notion of a term.

Definition 10. *Let X be a set of variable symbols and Σ a signature. The set T of* terms over the set of variables X and signature Σ *is defined as the smallest set satisfying the following conditions:*

- *every variable is a term, i.e. $X \subseteq T$,*
- *if $n \in \mathbb{N}_0$, $t_1, \ldots, t_n$ are terms, and $f \in \Sigma$ is an n-ary operation symbol, then $f(t_1, \ldots, t_n) \in T$.*

Example. If we take $\Sigma = \{\cdot, ^{-1}, 1\}$, $X = \{x, y, z\}$ then $(y \cdot x^{-1}) \cdot (z^{-1} \cdot 1)$ is a term.

We can use terms to obtain operations.

Definition 11. *Let $t \in T$ be a term over the set of variables $X = \{x_1, \ldots, x_k\}$ and signature Σ. Let $\boldsymbol{A} = (A, (f^{\boldsymbol{A}})_{f \in \Sigma})$ be an algebra in the signature Σ. Then we define a k-ary* term operation $t^{\boldsymbol{A}} : A^k \to A$ *in the natural way:*

- *if $t = x_i$, then $t^{\boldsymbol{A}}(a_1, \ldots, a_n) = a_i$,*
- *if $t = f(t_1, \ldots, t_n)$, where $f \in \Sigma$ is n-ary, then*

$$t^{\boldsymbol{A}}(a_1, \ldots, a_n) = f^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(a_1, \ldots, a_n), \ldots, t_n^{\boldsymbol{A}}(a_1, \ldots, a_n)).$$

A polynomial operation $p : A^m \to A$ *on $\boldsymbol{A}$ is an operation given by a term in which we can use elements of A as nullary operations.*

Term operations are of interest to us because clones are sets of term operations.

Proposition 12. *Let $\boldsymbol{A}, \boldsymbol{B}$ be algebras of the same signature Σ. Let t be a term over the set of variables $X = \{x_1, \ldots, x_k\}$ and signature Σ. If $f : \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then*

$$f(t^{\boldsymbol{A}}(a_1, \ldots, a_k)) = t^{\boldsymbol{B}}(f(a_1), \ldots, f(a_k))$$

for all $a_1, \ldots, a_k \in A$.

Proof. We prove this by induction on the complexity of the term t. If t is just a variable the the equality obviously holds. Otherwise let $g \in \Sigma$, and $t = g(t_1, \ldots, t_{n_g})$, where $t_1, \ldots, t_{n_g}$ are terms for which the equality already holds. Then

$$f(t^{\mathbf{A}}(a_1, \ldots, a_k)) = f(g^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_k), \ldots, t_{n_g}^{\mathbf{A}}(a_1, \ldots, a_k))) =$$

$$= g^{\mathbf{B}}(f(t_1^{\mathbf{A}}(a_1, \ldots, a_k)), \ldots, f(t_{n_g}^{\mathbf{A}}(a_1, \ldots, a_k))) =$$

$$= g^{\mathbf{B}}(t_1^{\mathbf{B}}(f(a_1), \ldots, f(a_k)), \ldots, t_{n_g}^{\mathbf{B}}(f(a_1), \ldots, f(a_k))) = t^{\mathbf{B}}(f(a_1), \ldots, f(a_k)).$$

Which is what we wanted to prove.

□

Proposition 13. *Let $\boldsymbol{A}$ be an algebra that is generated by the set $X \subseteq A$. Then*

$$A = \{t(x_1, \ldots, x_k) | k \in \mathbb{N} \wedge x_1, \ldots, x_k \in X \text{ and } t \text{ is a } k\text{-ary term operation}\}.$$

Moreover if $X = \{x_1, \ldots, x_k\}$, then

$$A = \{t(x_1, \ldots, x_k) | t \text{ is a } k\text{-ary term operation}\}.$$

Proof. This follows from the fact that X generates $\mathbf{A}$ and the set on the right is obviously a subuniverse containing X, since composition of term oprations is again a term operation.

□

1.2 Clones

In this section we will define the notion of a clone. This is a central notion in Universal algebra.

Definition 14. *Let A be a set, and C a set of operations on A of arity at least one. We say that C is a* clone*, if C contains all the projections, and is closed with repect to the generalized composition: If $g \in C$ is k-ary, $f_1, \ldots, f_k \in C$ are n-ary, then the operation*

$$g(f_1, \ldots, f_k)(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n))$$

must be in C. We denote by C_n all the n-ary operations in C, for $n \in \mathbb{N}$.

Definition 15. *Let A be a set and C, D be two clones on A. We say that D is a* subclone *of C if $D \subseteq C$.*

Definition 16. *Let $\boldsymbol{A}$ be an algebra, we define $\mathrm{Clo}(\boldsymbol{A})$ to be the set of all term operations on $\boldsymbol{A}$ of arity at least one.*

It is obvious, from the definition of a term, that $\mathrm{Clo}(\mathbf{A})$ is a clone. Next, we define identities.

Definition 17. *Let $\boldsymbol{A}$ be an algebra and $t, s \in \mathrm{Clo}_n(\boldsymbol{A}) = \mathrm{Clo}(\boldsymbol{A}) \cap A^{A^n}$. We say that $\boldsymbol{A}$ satisfies the identity*

$$t \approx s$$

if for all $x \in A^n$ we have that

$$t(x) = s(x).$$

1.3 Taylor algebras

In this section we will present an important class of algebras called Taylor algebras and prove some of its properties. Taylor algebras will be used in the third chapter. First we will introduce the notion of a Taylor operation. With that we will define Taylor algebras. Afterwards we will prove some properties of Taylor algebras. Taylor algebras will be used in the third chapter.

Definition 18. *Let $\boldsymbol{A}$ be an algebra. We say that $t \in \mathrm{Clo}(\boldsymbol{A})$ is a Taylor operation if the following identities hold*

$$t(x, *, \ldots, *) \approx t(y, *, \ldots, *),$$

$$\vdots$$

$$t(*, \ldots, *, x) \approx t(*, \ldots, *, y),$$

where the positions marked by $*$ stand for arbitrary variables.

Definition 19. *An idempotent finite algebra $\boldsymbol{A}$ is said to be* Taylor *if there exists $t \in \mathrm{Clo}(\boldsymbol{A})$ such that t is a Taylor operation.*

Definition 20. *An algebra $\boldsymbol{A}$ is an* affine module *if $\mathrm{Clo}(\boldsymbol{A})$ is the set of all idempotent term operation of some R-module for some ring R.*

Definition 21. *An algebra* $\boldsymbol{A}$ *is* abelian *if for every term operation* t *of arity* $n+1$ *it holds that*

$$\forall x, y \in A \forall u, v \in A^n : t(x, u) = t(x, v) \Rightarrow t(y, u) = t(y, v).$$

Theorem 22. *A Taylor algebra is abelian if and only if it is an affine module.*

Proof. The theorem follows from results in Hobby and McKenzie [1988]. It is stated as Theorem 2.4 in Barto et al. [2023].

□

Definition 23. *An algebra* $\boldsymbol{A}$ *is called* minimal Taylor *if it is Taylor and every proper subclone of* $\mathrm{Clo}(\boldsymbol{A})$ *does not contain a Taylor operation.*

Let $\mathbf{A}$ be an algebra, then by $\mathrm{Con}(\mathbf{A})$ we denote the set of all congruences on $\mathbf{A}$.

Proposition 24. *If an algebra* $\boldsymbol{A}$ *is minimal Taylor, then* $\boldsymbol{A}/\theta$ *is also minimal Taylor for all* $\theta \in \mathrm{Con}(\boldsymbol{A})$.

Proof. This follows from Proposition 5.4. in Barto et al. [2023].

□

The following proposition is a crucial ingredient of the proof of Theorem 55, which is one of the most important results of this thesis.

Proposition 25. *If* $\boldsymbol{A}$ *is a simple minimal Taylor abelian algebra, then* $\mathrm{Clo}(\boldsymbol{A}) = \mathrm{Clo}(A; x-y+z)$*, where* $+, -$ *are abelian group operations on* A *and* $|A|$ *is a prime number.*

Proof. Algebra $\mathbf{A}$ being Taylor abelian implies that it is an affine module by Theorem 22. From the assumption that it is minimal Taylor, it follows that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(A; x - y + z)$, because $x - y + z$ is a Taylor operation. The group is simple because $\mathbf{A}$ is simple. Indeed, if there existed a nontrivial congruence $\theta \in \mathrm{Con}(A; +, -, 0)$, then it would we closed under all $t \in \mathrm{Clo}(A; x - y + z) = \mathrm{Clo}(\mathbf{A})$, so it would be a congruence in $\mathbf{A}$. Therefore $(A; +, -, 0)$ is isomorphic to $\mathbb{Z}_p$ for some prime number p. This proves the proposition.

□

Remark. Let $a + V$ be an affine subspace of $\mathbb{Z}_p^n$. Then this occurs if and only if

$$a + V \leq (\mathbb{Z}_p; x - y + z)^n.$$

Indeed this is true because an affine space is closed under all affine combinations and $\mathrm{Clo}(\mathbb{Z}_p; x - y + z)$ is a set of all affine combinations.

Let $\mathbf{A} = (A, (f^{\mathbf{A}})_{f \in \Sigma})$ and $\mathbf{B} = (B, (f^{\mathbf{B}})_{f \in \Pi})$ be two algebras. We say that $\mathbf{A}$ is a reduct of $\mathbf{B}$ if $A = B$, $\Sigma \subseteq \Pi$ and $f^{\mathbf{A}} = f^{\mathbf{B}}$ for all $f \in \Sigma$.

Proposition 26. *Every Taylor algebra has a minimal Taylor reduct.*

Proof. It is stated as Proposition 5.2. in Barto et al. [2023].

□

1.4 Stability concept

In this section we introduce the notion of a stability concept and prove some of its properties. Stability concept will be used in the third chapter. The definition of stability concept is taken from Brady [2022].

Let K be a class of algebras in the same signature, then by $H(K)$ we denote the class of all algebras which are isomorphic to a quotient of an algebra from K. By $S(K)$ we denote the class of all algebras which are isomorphic to a subalgebra of an algebra from K. By $P_{fin}(K)$ we denote the class of all algebras which are isomorphic to a finite product of algebras from K.

Definition 27. *Let $\boldsymbol{A}$ be a finite idempotent algebra, $V = HSP_{fin}(\boldsymbol{A})$. We say that a binary relation $\prec$ on V is a* stability concept *on V if it satisfies the following*

- *If $\boldsymbol{C} \prec \boldsymbol{B}$ then $\boldsymbol{C} \leq \boldsymbol{B}$*
- *If $\boldsymbol{D} \prec \boldsymbol{C} \prec \boldsymbol{B}$ then $\boldsymbol{D} \prec \boldsymbol{B}$*
- *If $\boldsymbol{C}, \boldsymbol{D} \prec \boldsymbol{B}$ and $C \cap D \neq \emptyset$ then $\boldsymbol{C} \cap \boldsymbol{D} \prec \boldsymbol{B}$*
- *If $f : \boldsymbol{B} \to \boldsymbol{C}$ is a surjective homomorphism then*
  - *if $\boldsymbol{D} \prec \boldsymbol{B}$ then $f(\boldsymbol{D}) \prec \boldsymbol{C}$*
  - *if $\boldsymbol{E} \prec \boldsymbol{C}$ then $f^{-1}(\boldsymbol{E}) \prec \boldsymbol{B}$*
- *If $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E} \prec \boldsymbol{B}$ are such that $C \cap D \neq \emptyset$, $C \cap E \neq \emptyset$ and $D \cap E \neq \emptyset$, then $C \cap D \cap E \neq \emptyset$.*
- *If $\boldsymbol{B} \in V$, $|B| \gneq 1$, then either*
  - *there is some $\boldsymbol{C} \prec \boldsymbol{B}$ such that $\boldsymbol{C} \neq \boldsymbol{B}$ or*
  - *there is some proper congruence $\theta \in \mathrm{Con}(\boldsymbol{B})$ such that $\boldsymbol{B}/\theta$ is an abelian algebra.*

We say that a subalgebra $\boldsymbol{C} \leq \boldsymbol{B}$ is stable if $\boldsymbol{C} \prec \boldsymbol{B}$.

Theorem 28. *Let $\boldsymbol{A}$ be a minimal Taylor algebra. Then there exists a stability concept on $HSP_{fin}(\boldsymbol{A})$.*

Proof. This follows from Theorem 3.15.18. in Brady [2022]. □

Next we prove some basic properties of a stability concept. The following three lemmas are taken from Brady [2022]. Let $\mathbf{C} \leq \mathbf{A}$ and $R \leq_{sd} \mathbf{A} \times \mathbf{B}$, then by $\mathbf{C} + R$ we mean the set $\{b \in \mathbf{B} | \exists a \in C : (a, b) \in R\}$. Let $n \in \mathbb{N}$, by $[n]$ we denote the set $\{1, \ldots, n\}$.

Lemma 29. *Let $\prec$ be a stability concept on $HSP_{fin}(\boldsymbol{A})$, $\boldsymbol{C} \prec \boldsymbol{B}$, $R \leq_{sd} \boldsymbol{A} \times \boldsymbol{B}$ be a subdirect relation. Then $\boldsymbol{C} + R \prec \boldsymbol{B}$.*

Proof. Let $\pi_1 : \mathbf{R} \to \mathbf{A}$, $\pi_2 : \mathbf{R} \to \mathbf{B}$ be the restriction of the projection homomorphisms to $\mathbf{R}$. Then we have that $\pi_1^{-1}(C) \prec \mathbf{R}$ because of the fourth axiom from the definition of a stability concept. And now we get that $\mathbf{C} + R = \pi_2(\pi_1^{-1}(C)) \prec \mathbf{B}$ by the same axiom.

□

Lemma 30. *Let $\prec$ be a stability concept on $HSP_{fin}(\boldsymbol{A})$, $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n \prec \boldsymbol{B} \in HSP_{fin}(\boldsymbol{A})$, $\boldsymbol{C}_i \cap \boldsymbol{C}_j \neq \emptyset$ for all $i, j \in [n]$. Then $\bigcap_{i=1}^n \boldsymbol{C}_i \neq \emptyset$.*

Proof. We will prove this by induction on n. For $n = 3$ it follows from the definition of stability concept. Let $n \geq 4$. Set $\mathbf{B}' = \mathbf{C}_n$ and $\mathbf{C}'_i = \mathbf{C}_i \cap \mathbf{C}_n$ for $i < n$. Then from the definition of stability concept we have that $\mathbf{C}'_i \cap \mathbf{C}'_j = \mathbf{C}_i \cap \mathbf{C}_j \cap \mathbf{C}_n \neq \emptyset$ for all $i, j < n$. Again from the definition of stability concept we get $\mathbf{C}'_i = \mathbf{C}_i \cap \mathbf{C}_n \prec \mathbf{C}_n = \mathbf{B}'$ for all $i < n$. Now we can apply the induction hypothesis to get $\bigcap_{i=1}^n \mathbf{C}_i = \bigcap_{i=1}^{n-1} \mathbf{C}'_i \neq \emptyset$.

□

Let $R \leq \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$, then by R_{ij} we denote the set

$$\{(x, y) \in \mathbf{B}_i \times \mathbf{B}_j | \exists a \in R : a_i = x \wedge a_j = y\}.$$

Lemma 31. *Let $\prec$ be a stability concept on $HSP_{fin}(\boldsymbol{A})$, $R \leq_{sd} \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_n$, $\boldsymbol{C}_i \prec \boldsymbol{B}_i$ for all $i \in [n]$ and*

$$R_{ij} \cap (\boldsymbol{C}_i \times \boldsymbol{C}_j) \neq \emptyset$$

for all $i, j \in [n]$. Then $R \cap \prod_{i=1}^n \boldsymbol{C}_i \neq \emptyset$.

Proof. From the definition of stability concept we get that $\pi_i^{-1}(\mathbf{C}_i) \prec R$. From our assumptions we have that $\pi_i^{-1}(\mathbf{C}_i) \cap \pi_j^{-1}(\mathbf{C}_j)$ for all $i, j \in [n]$. Therefore Lemma 30 implies that $\bigcap_{i=1}^n \pi_i^{-1}(\mathbf{C}_i) \neq \emptyset$, which proves what we wanted.

□

2. Symmetric Affine Subspaces

Definition 32. *$R \subseteq \mathbb{Z}_p^{\binom{[n]}{k}}$ is symmetric if for all permutations $\pi : [n] \to [n]$ and all $a \in R$ it holds that $b \in R$, where $b_{\pi(I)} = a_I$, for all $I \in \binom{[n]}{k}$. We denote b by $\pi(a)$. By S_n we denote the set of all permutations $\pi : [n] \to [n]$.*

In this section, we are going to study the symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. First we are going to present general facts about symmetric affine subspaces. Then we are going to reduce the question of whether every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ contains a constant to a question about the symmetric vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. After that, we are going to characterize all symmetric vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}$ and using that result, we will characterize all the symmetric vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{2}}$. Lastly, we are going to characterize, when every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{2}}$ contains a constant.

Let $a \in \mathbb{Z}_p^{\binom{[n]}{k}}$ and $K \subseteq [n]$, then we define

$$a_K := \sum_{K \subseteq I \in \binom{[n]}{k}} a_I,$$

where by $\binom{[n]}{k}$ we mean the set of all subsets of $[n]$ of size k.

Let $\mathrm{Sym}(n, k, p)$ be the set of all symmetric vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. Let $V \in \mathrm{Sym}(n, k, p)$, then we define

$$S(V) = \{W \in \mathrm{Sym}(n, k, p) | W \subseteq V, \dim(V) = \dim(W) + 1\},$$

$$E(V) = \{W \in \mathrm{Sym}(n, k, p) | V \subseteq W, \dim(W) = \dim(V) + 1\}.$$

Let $X \subseteq \mathbb{Z}_p^{\binom{[n]}{k}}$, then by $\langle X \rangle_S$ we denote the subspace generated by $\{\pi(x) | x \in X \wedge \pi \in S_n\}$. We say that $\langle X \rangle_S$ is *symmetrically generated* by X.

2.1 General facts

Lemma 33. *Let $a + V$ be a symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$, then V is a symmetric (vector) subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$.*

Proof. Let $x \in V$ and $\pi \in S_n$. Then $\pi(a + x) = \pi(a) + \pi(x) \in a + V$. So there exists $w \in V$ such that

$$\pi(a) + \pi(x) = a + w.$$

This implies that

$$\pi(a) - a + \pi(x) - w \in V.$$

But since $a, \pi(a) \in a + V$, it follows that $\pi(a) - a \in V$. And from this follows that $\pi(x) - w \in V$. Therefore

$$\pi(x) \in V$$

because $w \in V$. This is what we wanted to prove.

□

Let $W \leq V$ be vector spaces, then by $\frac{V}{W}$ we denote the factor space of V by W.

Lemma 34. *Let V be a symmetric subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$. Then τ defined by*

$$\tau(a+V) = \langle a \rangle_S + V$$

is a mapping from the set of symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$ with the space of vectors V which do not contain 0 to the set $E(V)$. Moreover $\ker(\tau) = \{(a+V, b+V) | a, b \notin V, \exists i \in \mathbb{Z}_p \setminus \{0\} : b = ia\}$.

Proof. The map τ is well defined because $a+V = b+V$ implies that $a-b \in V$. Hence $a = b + (a-b) \in \langle b \rangle_S + V$, and since $\langle b \rangle_S + V$ is symmetric, we have that $\langle a \rangle_S + V \subseteq \langle b \rangle_S + V$. The opposite inclusion follows similarly.

Because $a+V$ is symmetric, we have that $a - \pi(a) \in V$ for all permutations π. This means that

$$\dim(\frac{\langle a \rangle_S + V}{V}) = 1,$$

in other words $\langle a \rangle_S + V \in E(V)$.

If $b = ia$, then $\langle b \rangle_S + V = \langle a \rangle_S + V$, so $(a+V, b+V) \in \ker(\tau)$.

Let $(a+V, b+V) \in \ker(\tau)$, then $\langle b \rangle_S + V = \langle a \rangle_S + V$. So $b \in \langle a \rangle_S + V$. Therefore

$$b + V \in \frac{\langle a \rangle_S + V}{V},$$

which is generated by $a+V$. Therefore $b+V = ia+V$ for some $i \in \mathbb{Z}_p \setminus \{0\}$. It follows that $b - ia \in V$, which implies that the affine spaces $ia+V$ and $b+V$ are equal.

□

The following proposition is needed to prove Proposition 36. Let $n \geq 2k$ and $B_i = \{i, i+1\}$ for $i = 1, 3, 5, \ldots, 2k-1$. Define $\lambda_k \in \mathbb{Z}_p^{\binom{[n]}{k}}$ as follows: $\lambda_k(\{b_1, \ldots, b_{2k-1}\}) = (-1)^{b_1 + \cdots + b_{2k-1}}$, where $b_i \in B_i$, otherwise $\lambda_k(I) = 0$ for all other coordinates $I \in \binom{[n]}{k}$. Here, for clarity, we use the notation $\lambda_k(I)$ instead of $(\lambda_k)_I$.

We denote

$$\Lambda_k = \{a \in \mathbb{Z}_p^{\binom{[n]}{k}} | \forall I \subseteq [n] : |I| \leq k-1 \Rightarrow a_I = 0\},$$

$$\Sigma_k = \{a \in \mathbb{Z}_p^{\binom{[n]}{k}} | a_\emptyset = 0\}.$$

Proposition 35. *If $n \geq 2k$, then $\Lambda_k = \langle \lambda_k \rangle_S$.*

Proof. We prove this by induction on k. For $k = 1$, we have that $\Lambda_1 = \Sigma_1$ and $\lambda_1 = (-1, 1, 0, \ldots, 0)$. So we just have to prove that Σ_1 is symmetrically generated by $(-1, 1, 0, \ldots, 0)$. Indeed, if we take any vector $v = (v_1, \ldots, v_n) \in \mathbb{Z}_p^{\binom{[n]}{1}}$, then

by adding multiples of permutations of the vector $(-1,1,0,\dots,0)$ to v, we can get the vector $(v_1+\cdots+v_n,0,\dots,0)$. Therefore if $v\in\Sigma_1$, then this vector is zero.

($\supseteq$) It is enough to show $\lambda_k\in\Lambda_k$ because Λ_k is symmetric. Let $I\subseteq[n]$ be such that $|I|\leq k-1$. If $I\nsubseteq B_1\cup\cdots\cup B_{2k-1}$ or there exists $i\in\{1,3,\dots,2k-1\}$ such that $B_i\subseteq I$, then we clearly have that $\lambda_k(I)=0$. If for all $i\in\{1,3,\dots,2k-1\}$ we have that $|B_i\cap I|\leq 1$, then $\lambda_k(I)=0$ follows from induction hypothesis, because $\lambda_k(I)=\pm\lambda_{k-|I|}(\emptyset)$, this fact is clear from the definition of λ_k.

($\subseteq$) Let $0\neq a\in\Lambda_k$. We will prove that $a\in\langle\lambda_k\rangle_S$ by adding multiples of elements of $\langle\lambda_k\rangle_S$ to a and in the end getting zero. This will prove that $a\in\langle\lambda_k\rangle_S$. Let $x\in[n]$ so that there exists $I\in\binom{n}{k}$ such that $x\in I$ and $a_I\neq 0$. Without loss of generality, we can assume that $x=2k$. Then we define $b\in\mathbb{Z}_p^{\binom{[n]}{k-1}}$ as follows. Let $J\in\binom{n}{k}$ be such that $x\in J$, then we define $b_{J\setminus\{x\}}=a_J$, and as 0 on all the other coordinates. If we perform this procedure on λ_k, we will get λ_{k-1}. This is important, because if we can reduce b to zero by adding to it elements of $\langle\lambda_{k-1}\rangle_S$, then we can add elements of $\langle\lambda_k\rangle_S$ to a so the $a_I=0$ for all $I\in\binom{[n]}{k}$ which contain x. We claim that $b\in\langle\lambda_{k-1}\rangle_S$. If $K\subseteq[n]$ is such that there exists $I\in\binom{[n]}{k}$ which contains x and $K\subseteq I$, then $b_K=a_{K\cup\{x\}}=0$. For all other K, we have that $b_K=0$ from the definition of b. It follows that $b\in\Lambda_{k-1}=\langle\lambda_{k-1}\rangle_S$. Therefore by adding multiples of elements of $\langle\lambda_k\rangle_S$, we can make sure that $a_I=0$ for all I containing x. This can change a on other coordinates, but this does not matter to us. We can repeat this process until we are left with only $2k-1$ elements of $[n]$ that we did not yet consider. These elements satisfy that if $x\in[n]$ is not among them, then there does not exist $I\in\binom{[n]}{k}$ containing x such that $a_I\neq 0$.

Let $J\in\binom{[n]}{2k-1}$ be the set of these elements, and let $I\in\binom{J}{k}$. Then we have that

$$0=\sum_{i=0}^{k-1}(-1)^i\sum_{K\in\binom{J}{i}}a_K=a_I.$$

The first equality holds because $a\in\Lambda_k$ and all the sets K in the sum are of size at most $k-1$. The second equality holds because a_I is contained as a summand only in $a_\emptyset$. While if we take any other $L\in\binom{J}{k}$, then $(J\setminus I)\cap L\neq\emptyset$. Let $l=|(J\setminus I)\cap L|$, then a_L appears in the sum

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}=0$$

times. So we have proved that a can be reduced to 0 by adding to it elements of $\langle\lambda_k\rangle_S$, which means that $a\in\langle\lambda_k\rangle_S$.

□

We say that a subset $X\subseteq\mathbb{Z}_p^{\binom{[n]}{k-1}}\times\cdots\times\mathbb{Z}_p^{\binom{[n]}{1}}\times\mathbb{Z}_p^{\binom{[n]}{0}}$ is symmetric, if for all $a\in X$, we have that $b\in X$, where $\pi_i(b)=\sigma(\pi_i(a))$ for any $\sigma\in S_n$ and π_i is the projection on the i-th coordinate for $i\in\{k-1,\dots,0\}$.

The following proposition is crucial in order to reduce the study of symmetric

vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$ to the study of symmetric vector subspaces of

$$\mathbb{Z}_p^{\binom{[n]}{k-1}} \times \cdots \times \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}.$$

This reduction is achieved via a linear map

$$\varphi_k : \mathbb{Z}_p^{\binom{[n]}{k}} \to \mathbb{Z}_p^{\binom{[n]}{k-1}} \times \cdots \times \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}$$

defined as follows

$$\varphi_k(a) = ((a_K)_{K\in\binom{[n]}{k-1}}, (a_K)_{K\in\binom{[n]}{k-2}}, \ldots, (a_K)_{K\in\binom{[n]}{1}}, a_\emptyset).$$

It is obvious that $\mathrm{Ker}(\varphi_k) = \Lambda_k$.

Let $V \leq \mathbb{Z}_p^n$, then we define

$$V^\perp = \{x \in \mathbb{Z}_p^n | \forall v \in V : x \cdot v = 0\},$$

where by $x \cdot v$ we mean the sum

$$\sum_{i=0}^{n} x_i v_i.$$

From Linear algebra, we know that if $V \leq W$, then $W^\perp \leq V^\perp$, $(V^\perp)^\perp = V$ and $\dim(V^\perp) = n - \dim(V)$.

Proposition 36. *Let $n \geq 2k$ and $V \in \mathrm{Sym}(n,k,p)$. Then either $\Lambda_k \subseteq V$ or $V \subseteq \Lambda_k^\perp$.*

Proof. If $V \subseteq \Lambda_k^\perp$, then we are done. Suppose $V \nsubseteq \Lambda_k^\perp$, then there exists $a \in V$ such that $a \cdot \pi(\lambda_k) \neq 0$ by Proposition 35. Without loss of generality we may suppose that $\pi = id$ and $a \cdot \pi(\lambda_k) = 1$. If it did not equal 1, then we could just divide $a \cdot \pi(\lambda_k)$ to get 1. We claim that

$$x := \sum_{(i_1,\ldots,i_k)\in\{0,1\}^k} (-1)^{i_1+\cdots+i_k} (12)^{i_1}(34)^{i_2}\ldots(2k-1,2k)^{i_k}(a) \in V$$

is equal to λ_k or $-\lambda_k$, where by (12) etc. we mean the transposition which switches 1 and 2. Let $I \in \binom{n}{k}$, then

$$x_I = \sum_{(i_1,\ldots,i_k)\in\{0,1\}^k} (-1)^{i_1+\cdots+i_k} a_{(12)^{i_1}(34)^{i_2}\ldots(2k-1,2k)^{i_k}(I)}.$$

If there exists $i \in \{1,3,\ldots,2k-1\}$ such that $B_i \cap I = \emptyset$ or $B_i \subseteq I$, then one of the transpositions fixes I. Without loss of generality, let that transposition be (12). The for any $i_2,\ldots,i_k$ we have that

$$(12)^0(34)^{i_2}\ldots(2k-1,2k)^{i_k}(I) = (12)^1(34)^{i_2}\ldots(2k-1,2k)^{i_k}(I).$$

But the corresponding two summands have opposite signs, therefore $x_I = 0$. Now let $|B_i \cap I| = 1$ for all $i \in \{1,3,\ldots,2k-1\}$. Then

$$(12)^{i_1}(34)^{i_2}\ldots(2k-1,2k)^{i_k}(I) \in \binom{[n]}{k}$$

give us all the coordinates $J \in \binom{[n]}{k}$ such that $\lambda_k(J) \neq 0$. Moreover $\lambda_k(\alpha(J)) = -\lambda_k(J)$, where α is one of the permutations $(12), (34), \ldots, (2k-1, 2k)$. Since

$$a \cdot \lambda_k = \sum_{\{b_1, b_3, \ldots, b_{2k-1}\}} (-1)^{b_1 + \cdots + b_{2k-1}} a_{\{b_1, \ldots, b_{2k-1}\}},$$

where the sum is taken over the sets $\{b_1, b_3, \ldots, b_{2k-1}\}$ such that $b_i \in B_i$ for all $i \in \{1, 3, \ldots, 2k-1\}$. It follows that $x_I = \pm\lambda_k(I)(a \cdot \lambda_k) = \pm\lambda_k(I)$. This proves that $x = \pm\lambda_k$. Therefore $\Lambda_k \subseteq V$ by Proposition 35.

□

2.2 Constants

The following lemma will allow us to reduce the question of whether symmetric affine subspaces contain a constant to a question about symmetric vector subspaces.

Denote

$$E_k = \langle (1, \ldots, 1) \rangle \leq \mathbb{Z}_p^{\binom{[n]}{k}}.$$

Lemma 37. *A symmetric affine subspace $a + V$ of the vector space $\mathbb{Z}_p^{\binom{[n]}{k}}$ such that $E_k \nsubseteq V$ contains a nonzero constant if and only if $\langle a \rangle_S + V$ contains a nonzero constant.*

Proof. ($\Rightarrow$) This is true since $a + V \subseteq \langle a \rangle_S + V$.

($\Leftarrow$) From the assumption and Lemma 34 we have that $\langle a \rangle_S + V = E_k \oplus V$. Since $a \in \langle a \rangle_S + V = E_k \oplus V$, we have that $a = i + v$ for some $i \in E_k, v \in V$. Hence $i = a - v \in a + V$. If $i = 0$, then $a = v \in V$. Therefore $a + V = V$ contains a nonzero constant and so $E_k \subseteq V$, which is a contradiction.

□

The following proposition gives us a way to check that every symmetric affine subspace contains a constant just by looking at $\mathrm{Sym}(n, k, p)$.

Proposition 38. *Every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ with the space of vectors V contains a constant tuple if the following conditions are true.*

1. *If $E_k \subseteq V$, then $|E(V)| = 0$,*
2. *if $E_k \nsubseteq V$, then $|E(V)| = 1$.*

Proof. Let $a + V$ be a symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ which does not contain a constant. Then V is a symmetric vector subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ by Lemma 33. Since $a + V$ does not contain a constant, in particular we have that $a + V \neq V$. If $E_k \subseteq V$, then $|E(V)| = 0$ by our assumption. But $\tau(a+V) \in E(V)$ by Lemma 34, which is a contradiction. If $E_k \nsubseteq V$, then $|E(V)| = 1$ by our assumption. Since $E_k \oplus V \in E(V)$ we have that $E(V) = \{E_k \oplus V\}$. Therefore $\langle a \rangle_S + V = E_k \oplus V$ because $\tau(a + V) \in E(V)$ by Lemma 34. It follows that $\langle a \rangle_S + V = E_k \oplus V$

contains a constant nonzero tuple. By Lemma 37, $a+V$ also contains a constant nonzero tuple, which is a contradiction.

□

Corollary 39. *Every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ contains a constant tuple if the following conditions are true for every $V \in \mathrm{Sym}(n,k,p)$.*

1. *If $E_k \subseteq V$, then $|E(V)| = 0$,*
2. *if $E_k \nsubseteq V$, then $|E(V)| = 1$.*

Proof. This follows immediately from Proposition 38.

□

The following lemma will help to further reduce the work of studying the spaces in $\mathrm{Sym}(n,k,p)$ to only studying $V \in \mathrm{Sym}(n,k,p)$ which contain Λ_k.

Lemma 40. *Let $V \in \mathrm{Sym}(n,k,p)$. Then $W \in E(V)$ if and only if $W^\perp \in S(V^\perp)$. In particular, $|E(V)| = |S(V^\perp)|$ for all $V \in \mathrm{Sym}(n,k,p)$.*

Proof. If $W \in E(V)$, then we have that $V \subseteq W$ and $\dim(W) = \dim(V) + 1$. It follows that $W^\perp \subseteq V^\perp$ and $\dim(V^\perp) = \dim(W^\perp) + 1$, which means that $W^\perp \in S(V^\perp)$.

If $W^\perp \in S(V^\perp)$, then $W^\perp \subseteq V^\perp$ and $\dim(V^\perp) = \dim(W^\perp) + 1$. It follows that $V \subseteq W$ and $\dim(W) = \dim(V) + 1$, which means that $W \in E(V)$.

□

The following lemma assures us that for n large enough the spaces Λ_k and $\Lambda_k^\perp$ are sufficiently separated.

Lemma 41. *There exists $n_0 \in \mathbb{N}$ such that $\dim(\Lambda_k) \geq \dim(\Lambda_k^\perp)+2$ for all $n \geq n_0$.*

Proof. We have $\dim(\Lambda_k) \geq \dim(\Lambda_k^\perp) + 2$ if and only if $\dim(\Lambda_k) \geq \frac{1}{2}\binom{n}{k} + 1$ because $\dim(\Lambda_k^\perp) = \binom{n}{k} - \dim(\Lambda_k)$. We also have that

$$\dim(\Lambda_k) \geq \binom{n}{k} - 1 - n - \binom{n}{2} - \cdots - \binom{n}{k-1}$$

because Λ_k is defined by $1 + n + \binom{n}{2} + \cdots + \binom{n}{k-1}$ equations. So we want that

$$\binom{n}{k} - 1 - n - \binom{n}{2} - \cdots - \binom{n}{k-1} \geq \frac{1}{2}\binom{n}{k} + 1$$

which is equivalent to

$$\frac{1}{2}\binom{n}{k} - 1 - n - \binom{n}{2} - \cdots - \binom{n}{k-1} - 1 \geq 0.$$

The limit of the expression on the left is infinity and that is what we wanted to prove.

□

Let L_k be the set of all symmetric vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$ containing Λ_k. Let $V \in L_k$, then we define $\sigma(V) = S(V) \cap L_k$ and $\epsilon(V) = E(V) \cap L_k$.

Lemma 42. *If n is such that $n \geq 2k$ and $\dim(\Lambda_k) \geq \dim(\Lambda_k^\perp) + 2$, then for all $V \in L_k$ it holds that $\epsilon(V) = E(V)$ and $\sigma(V) = S(V)$.*

Proof. The fact that $\epsilon(V) = E(V)$ is clear. If $W \in S(V)$, then $\dim(W) + 1 = \dim(V) \geq \dim(\Lambda_k) \geq \dim(\Lambda_k^\perp) + 2$ by our assumption. Therefore $\dim(W) \gneq \dim(\Lambda_k^\perp)$, so we cannot have that $W \subseteq \Lambda_k^\perp$. It follows that $\Lambda_k \subseteq W$ by Proposition 36, which means that $W \in \sigma(V)$.

□

The following proposition gives us a convenient way to check that every symmetric affine subspace contains constant just by looking at L_k.

Proposition 43. *Let $n \in \mathbb{N}$ be such that $n \geq 2k$ and $\dim(\Lambda_k) \geq \dim(\Lambda_k^\perp) + 2$. Then every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ contains a constant tuple if the following conditions are true for every $V \in L_k$.*

1. *If $E_k \subseteq V$, then $|\epsilon(V)| = 0$,*
2. *if $E_k \nsubseteq V$, then $|\epsilon(V)| = 1$,*
3. *if $V \subseteq \Sigma_k$, then $|\sigma(V)| = 0$,*
4. *if $V \nsubseteq \Sigma_k$, then $|\sigma(V)| = 1$.*

Proof. For the sake of contradiction, let $a + V$ be a symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ such that it does not contain a constant. Then V is a symmetric vector subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ by Lemma 33.

If $V \in L_k$ and $E_k \subseteq V$, then $|\epsilon(V)| = 0$ by our assumption. Then, we have that $|\epsilon(V)| = |E(V)| = 0$ by Lemma 42.

If $V \in L_k$ and $E_k \nsubseteq V$, then $|\epsilon(V)| = 1$ by our assumption. Then, we have that $|\epsilon(V)| = |E(V)| = 1$ by Lemma 42. So if $V \in L_k$, this proves that $a + V$ contains a constant by Proposition 38, which is a contradiction.

Now let $V \notin L_k$. Then $V^\perp \in L_k$ by Proposition 36. If $E_k \subseteq V$, then $V^\perp \subseteq \Sigma_k$ and $|\sigma(V^\perp)| = 0$ by our assumption. Therefore $|E(V)| = |S(V^\perp)| = |\sigma(V^\perp)| = 0$ by Lemma 42 and Lemma 40. If $E_k \nsubseteq V$, then $V^\perp \nsubseteq \Sigma_k$ and $|\sigma(V^\perp)| = 1$ by our assumption. Therefore $|E(V)| = |S(V^\perp)| = |\sigma(V^\perp)| = 1$ by Lemma 42 and Lemma 40. This again proves that $a + V$ contains a constant by Proposition 38, which is again a contradiction.

□

2.3 Case $k = 2$

Now we prove a lemma which allows us to characterize symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{2}}$. Recall that we have a linear map $\varphi : \mathbb{Z}_p^{\binom{[n]}{2}} \to \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}$ given by

$$\varphi(a) = ((a_{\{1\}}, \ldots, a_{\{n\}}), a_\emptyset).$$

It is clear from the definition that $\mathrm{Ker}(\varphi) = \Lambda_2$.

Lemma 44. $\mathrm{Im}(\varphi) = \{((a_1, \ldots, a_n), b) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | \sum_{i=1}^n a_i = 2b\}$

Proof. ($\subseteq$) This obviously holds because in the equation $\sum_{i=1}^n a_{\{i\}} = 2a_\emptyset$, we have every coordinate $a_{\{k,l\}}$ twice on each side.

($\supseteq$) Let $n = 3$. If $p = 2$, then we have the equation

$$a_1 + a_2 + a_3 = 0.$$

It follows that $|\{a_1, a_2, a_3\} \cap \{1\}|$ is even. Let $|\{a_1, a_2, a_3\} \cap \{1\}| = 0$, then $\varphi(0) = 0$ and $\varphi((1, 1, 1)) = ((0, 0, 0), 1)$. Let $|\{a_1, a_2, a_3\} \cap \{1\}| = 2$, without loss of generality suppose that $a_1 = a_2 = 1$ and $a_3 = 0$. Then $\varphi(1, 0, 0) = ((1, 1, 0), 1)$, where 1 in $(1, 0, 0)$ is at the coordinate $\{1, 2\}$. And $\varphi(0, 1, 1) = ((1, 1, 0), 0)$, where 1 in $(1, 0, 0)$ is again at the coordinate $\{1, 2\}$.

Let $p \geq 3$. We have that $\varphi(x, y, z) = ((a_1, a_2, a_3), b)$, where x is at coordinate $\{1, 2\}$, y at $\{1, 3\}$ and z at $\{2, 3\}$, if and only if the following equations hold

$$x + y = a_1,$$
$$x + z = a_2,$$
$$y + z = a_3,$$
$$x + y + z = b.$$

These equations have a solution

$$x = \frac{a_1 + a_2 - a_3}{2},$$
$$y = \frac{a_1 - a_2 + a_3}{2},$$
$$z = \frac{-a_1 + a_2 + a_3}{2}.$$

Now let $n \geq 4$. We have the equation $\sum_{i=1}^n a_i = 2b$. It follows that $(a_1 - a_4 - a_5 - \cdots - a_n) + a_2 + a_3 = 2(b - a_4 - a_5 - \cdots - a_n)$. So from the case $n = 3$, we now get that there exists $x \in \mathbb{Z}_p^{\binom{[3]}{2}}$ such that

$$(a_1 - a_4 - a_5 - \cdots - a_n) = x_{\{1\}},$$
$$a_2 = x_{\{2\}},$$
$$a_3 = x_{\{3\}},$$
$$b - a_4 - a_5 - \cdots - a_n = x_\emptyset.$$

Now by setting $a_{\{1,2\}} = x_{\{1,2\}}$, $a_{\{1,3\}} = x_{\{1,3\}}$, $a_{\{2,3\}} = x_{\{2,3\}}$, $a_{\{1,i\}} = a_i$, for $i = 4, \ldots, n$, and 0 at all other coordinates we get what we wanted.

□

Remark. From the previous lemma it follows that $n = \dim(\mathrm{Im}(\varphi)) = \dim(\mathbb{Z}_p^{\binom{[n]}{2}}) - \dim(\Lambda_2)$. This implies that $\dim(\Lambda_2) = \binom{n}{2} - n$. Therefore if we want the inequality

$$\dim(\Lambda_2) \geq \dim(\Lambda_2^{\perp}) + 2$$

to be satisfied, we want

$$\binom{n}{2} - n \geq n + 2$$

to be satisfied. So in this case we can choose $n_0 = 6$ in Lemma 41.

In the following, by a symmetric subspace of $\mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}$ we mean a subspace V such that for all $x \in V$, we have that $y \in V$, where $\pi_i(y) = \sigma(\pi_i(x))$ for all $i = 0, 1$ and $\sigma \in S_n$. By π_i we mean the projection

$$\pi_i : \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} \to \mathbb{Z}_p^{\binom{[n]}{i}}$$

on the i-th coordinate. In other words, we just permute x coordinatewise.

Theorem 45. *If V is a symmetric subspace of $\mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}$, then it is one of the spaces in the following list*

1. 0,
2. $0 \times \mathbb{Z}_p^{\binom{[n]}{0}} = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | \forall i \in [n] : v_i = 0\}$,
3. $\langle((1, \ldots, 1), m)\rangle = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | v_1 = v_2 = \cdots = v_n \wedge v_1 = mc\}$, *for $m \in \mathbb{Z}_p$,*
4. $\langle(1, \ldots, 1)\rangle \times \mathbb{Z}_p^{\binom{[n]}{0}} = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | \forall i, j \in [n] : v_i = v_j\}$,
5. $\Sigma_1 \times 0 = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | c = 0 \wedge \sum_{i=0}^{n} v_i = 0\}$,
6. $\mathbb{Z}_p^{\binom{[n]}{1}} \times 0 = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | c = 0\}$,
7. $(\Sigma_1 \times 0) \oplus \langle((m, 0, \ldots, 0), 1)\rangle = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | v_1 + v_2 + \cdots + v_n = mc\}$, *for $m \in \mathbb{Z}_p \setminus \{0\}$,*
8. $\Sigma_1 \times \mathbb{Z}_p^{\binom{[n]}{0}} = \{(v, c) \in \mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}} | \sum_{i=0}^{n} v_i = 0\}$,
9. $\mathbb{Z}_p^{\binom{[n]}{1}} \times \mathbb{Z}_p^{\binom{[n]}{0}}$.

Proof. Let V be any symmetric subspace. Let $(v, c) \in V$ be any element. If v is not constant, then we can switch the two unequal coordinates and substract the result from (v, c), which gives us that $\Sigma_1 \times 0 \subseteq V$. This follows from the fact that Σ_1 is symmetrically generated by $(-1, 1, 0, \ldots, 0)$ as we have proven above. Therefore $\dim(V)$ is $n - 1$, n, or $n + 1$. If $\dim(V) = n - 1$, then $\Sigma_1 \times 0 = V$ because $\dim(\Sigma_1 \times 0) = n - 1$. If $\dim(V) = n$, then there exists $(v, c) \in V \setminus (\Sigma_1 \times 0)$ and

$$V = (\Sigma_1 \times 0) \oplus \langle(v, c)\rangle.$$

If $c = 0$, then $V = \mathbb{Z}_p^{\binom{[n]}{1}} \times 0$. If $c \neq 0$ and $\sum v_i = 0$, then $V = \Sigma_1 \times \mathbb{Z}_p^{\binom{[n]}{0}}$. Otherwise

$$V = (\Sigma_1 \times 0) \oplus \langle (v, c) \rangle,$$

where $c \neq 0$ and $\sum_{i=1}^n v_i \neq 0$. Dividing by c, we get that $V = (\Sigma_1 \times 0) \oplus \langle (v, 1) \rangle$ and $\sum_{i=1}^n v_i \neq 0$.

Suppose that $v, u \in \mathbb{Z}_p^{\binom{[n]}{1}}$ are such that

$$\sum_{i=1}^n v_i = \sum_{i=1}^n u_i.$$

Then $v - u \in \Sigma_1$. It follows that in this case we have

$$V = (\Sigma_1 \times 0) \oplus \langle (v, 1) \rangle = (\Sigma_1 \times 0) \oplus \langle (u, 1) \rangle.$$

Therefore

$$V = (\Sigma_1 \times 0) \oplus \langle ((i, 0, \ldots, 0), 1) \rangle,$$

for some $i \in \mathbb{Z}_p \setminus \{0\}$. This space is symmetric because as we have shown Σ_1 is symmetrically generated by $(-1, 1, 0, \ldots, 0)$ so we can move i to any other coordinate.

If for all $(v, c) \in V$ we have that v is constant, then $V \subseteq \langle (1, \ldots, 1) \rangle \times \mathbb{Z}_p^{\binom{[n]}{0}}$. Therefore $\dim(V) \leq 2$. If $\dim(V) = 2$, then $V = \langle (1, \ldots, 1) \rangle \times \mathbb{Z}_p^{\binom{[n]}{0}}$. If $\dim(V) = 1$, then $V = \langle (v, c) \rangle$ for $(v, c) \neq 0$. If $v = 0$, then $V = 0 \times \mathbb{Z}_p^{\binom{[n]}{0}}$. If $v \neq 0$, then $V = \langle ((1, \ldots, 1), i) \rangle$ for some $i \in \mathbb{Z}_p$. If $\dim(V) = 0$, then $V = 0$. □

In the next corollary we provide a full classification of symmetric vector subspaces of $\mathbb{Z}_p^{\binom{[n]}{2}}$. We list only the spaces which contain Λ_2. The rest is obtained as the orthogonal spaces to the spaces in the list.

Corollary 46. *Let $n \geq 4$. The symmetric subspaces of $\mathbb{Z}_p^{\binom{[n]}{2}}$ are the following and their orthogonal spaces.*

If $p = 2$ and n is even:

- Λ_2
- $\{a \in \mathbb{Z}_p^{\binom{[n]}{2}} | \forall i \in [n] : a_{\{i\}} = 0\} = \varphi^{-1}(0 \times \mathbb{Z}_p^{\binom{[n]}{0}})$
- $\{a \in \mathbb{Z}_p^{\binom{[n]}{2}} | a_\emptyset = 0 \wedge \forall i, j \in [n] : a_{\{i\}} = a_{\{j\}}\} = \varphi^{-1}(\langle ((1, \ldots, 1), 0) \rangle)$
- $\{a \in \mathbb{Z}_p^{\binom{[n]}{2}} | \forall i, j \in [n] : a_\emptyset = a_{\{i\}} = a_{\{j\}}\} = \varphi^{-1}(\langle ((1, \ldots, 1), 1) \rangle)$
- $\{a \in \mathbb{Z}_p^{\binom{[n]}{2}} | \forall i, j \in [n] : a_{\{i\}} = a_{\{j\}}\} = \varphi^{-1}(\langle (1, \ldots, 1) \rangle \times \mathbb{Z}_p^{\binom{[n]}{0}})$
- $\Sigma_2 = \varphi^{-1}(\Sigma_1 \times 0)$
- $\mathbb{Z}_p^{\binom{[n]}{2}}$

If $p = 2$ and n is odd:

- Λ_2
- $\{a \in \mathbb{Z}_p^{\binom{[n]}{2}} | \forall i \in [n] : a_{\{i\}} = 0\} = \varphi^{-1}(0 \times \mathbb{Z}_p^{\binom{[n]}{0}})$
- $\Sigma_2 = \varphi^{-1}(\Sigma_1 \times 0)$
- $\mathbb{Z}_p^{\binom{[n]}{2}}$

If p is odd:

- Λ_2
- $\{a \in \mathbb{Z}_p^{\binom{[n]}{2}} | a_\emptyset = \frac{n}{2} \wedge \forall i, j \in [n] : a_{\{i\}} = a_{\{j\}}\} = \varphi^{-1}(\langle((1, \ldots, 1), \frac{n}{2})\rangle)$
- $\Sigma_2 = \varphi^{-1}(\Sigma_1 \times 0)$
- $\mathbb{Z}_p^{\binom{[n]}{2}}$

Proof. This follows directly from Lemma 44 and Theorem 45. Since $\mathrm{Ker}(\varphi) = \Lambda_2$, φ induces an isomophism

$$\mathbb{Z}_p^{\binom{[n]}{2}}/\Lambda_2 \to \mathrm{Im}(\varphi).$$

This tells us that the spaces in L_2 are in a bijection with the symmetric subspaces of $\mathrm{Im}(\varphi)$. So we just look at the spaces in the list in the statement of Theorem 45, which are contained in $\mathrm{Im}(\varphi)$ and take their preimages.

□

The following theorem gives a characterization of when all the symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{2}}$ contain a constant. This is one of the most important results of this thesis.

Theorem 47. *Let $n \geq 6$, p be a prime number. Then p does not divide $n\binom{n}{2}$ if and only if every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{2}}$ contains a constant tuple.*

Proof. This follows directly from Proposition 43 and Corollary 46. As we have already seen, $n \geq 6$ implies that $\dim(\Lambda_2) \geq \dim(\Lambda_2^\perp) + 2$, so Proposition 43 applies.

($\Rightarrow$) If $p = 2$, then all the symmetric subspaces contained in $\mathrm{Im}(\varphi)$ by Corollary 46 are

$$0, 0 \times \mathbb{Z}_p^{\binom{[n]}{0}}, \Sigma_1 \times 0, \mathrm{Im}(\varphi).$$

Therefore

$$L_2 = \{\varphi^{-1}(0), \varphi^{-1}(0 \times \mathbb{Z}_p^{\binom{[n]}{0}}), \varphi^{-1}(\Sigma_1 \times 0), \varphi^{-1}(\mathrm{Im}(\varphi))\}.$$

It is clear that if $V, W \in L_2$ and $V \subseteq W$, then

$$\dim(\frac{W}{V}) = \dim(\frac{\varphi(W)}{\varphi(V)}).$$

The only $V \in L_2$ such that $E_2 \subseteq V$ are $\varphi^{-1}(0 \times \mathbb{Z}_p^{\binom{[n]}{0}})$ and $\varphi^{-1}(\mathrm{Im}(\varphi))$. In both cases we clearly have that $|\epsilon(V)| = 0$. It follows that the spaces $V \in L_2$ such that $E_2 \nsubseteq V$ are $\varphi^{-1}(0)$ and $\varphi^{-1}(\Sigma_1 \times 0)$. In both cases we have that $|\epsilon(V)| = 1$. The spaces $V \in L_2$ such that $V \subseteq \Sigma_2$ are $\varphi^{-1}(0)$ and $\varphi^{-1}(\Sigma_1 \times 0) = \Sigma_2$. In both cases we have that $|\sigma(V)| = 0$. The spaces $V \in L_2$ such that $V \nsubseteq \Sigma_2$ are $\varphi^{-1}(0 \times \mathbb{Z}_p^{\binom{[n]}{0}})$ and $\varphi^{-1}(\mathrm{Im}(\varphi))$. In both cases we have that $|\sigma(V)| = 1$. This proves the result for $p = 2$.

Now let p be odd. All the symmetric subspaces contained in $\mathrm{Im}(\varphi)$ by Corollary 46 are

$$0, \langle((1, \ldots, 1), \frac{n}{2})\rangle, \Sigma_1 \times 0, \mathrm{Im}(\varphi).$$

Therefore

$$L_2 = \{\varphi^{-1}(0), \varphi^{-1}(\langle((1, \ldots, 1), \frac{n}{2})\rangle), \varphi^{-1}(\Sigma_1 \times 0), \varphi^{-1}(\mathrm{Im}(\varphi))\}.$$

The only $V \in L_2$ such that $E_2 \subseteq V$ are $\varphi^{-1}(\langle((1, \ldots, 1), \frac{n}{2})\rangle)$ and $\varphi^{-1}(\mathrm{Im}(\varphi))$. In both cases we clearly have that $|\epsilon(V)| = 0$. It follows that the spaces $V \in L_2$ such that $E_2 \nsubseteq V$ are $\varphi^{-1}(0)$ and $\varphi^{-1}(\Sigma_1 \times 0)$. In both cases we have that $|\epsilon(V)| = 1$. The spaces $V \in L_2$ such that $V \subseteq \Sigma_2$ are $\varphi^{-1}(0)$ and $\varphi^{-1}(\Sigma_1 \times 0) = \Sigma_2$. In both cases we have that $|\sigma(V)| = 0$. The spaces $V \in L_2$ such that $V \nsubseteq \Sigma_2$ are $\varphi^{-1}(\langle((1, \ldots, 1), \frac{n}{2})\rangle)$ and $\varphi^{-1}(\mathrm{Im}(\varphi))$. In both cases we have that $|\sigma(V)| = 1$. So the result follows from Proposition 43.

($\Leftarrow$) Let p divide $\binom{n}{2}$. Then $E_2 \subseteq \Sigma_2$ and $(1, 0, \ldots, 0) + \Sigma_2$ is a symmetric affine subspace which does not contain a constant. Indeed $v = (1, 0, \ldots, 0)$ is not in Σ_2 because $v_\emptyset = 1$. Further, we have that $\pi(v)_\emptyset = v_\emptyset$, therefore $v - \pi(v) \in \Sigma_2$. This tells us that $v + \Sigma_2$ is a symmetric affine subspace that does not include a constant because $v + \Sigma_2 \neq \Sigma_2$ and $E_2 \subseteq \Sigma_2$. That is a contradiction.

Let p divide n, then

$$\langle((1, \ldots, 1), 0)\rangle \subseteq \Sigma_1 \times 0 \subseteq \mathrm{Im}(\varphi).$$

Therefore

$$U = \varphi^{-1}(\langle((1, \ldots, 1), 0)\rangle) \subseteq \Sigma_2.$$

In other words, we have that

$$E_2 \subseteq U^\perp.$$

Define

$$(\chi_K)_I = \begin{cases} 1 & K \subseteq I \\ 0 & K \nsubseteq I, \end{cases}$$

for any $K \subseteq [n]$ and $I \in \binom{[n]}{2}$. Then we have that $\chi_K \cdot a = a_K$ for all $a \in \mathbb{Z}_p^{\binom{[n]}{2}}$. Therefore $\chi_{\{1\}} \in \Lambda_2^\perp$ but $\chi_{\{1\}} \notin U^\perp$ because

$$U = \{x \in \mathbb{Z}_p^{\binom{[n]}{2}} | x_\emptyset = 0 \wedge x_{\{1\}} = x_{\{2\}} = \cdots = x_{\{n\}}\}.$$

Therefore the affine space $\chi_{\{1\}} + U^\perp \neq U^\perp$. Moreover, we have that $\pi(\chi_{\{1\}}) = \chi_{\pi(\{1\})}$. Therefore $\chi_{\{1\}} - \pi(\chi_{\{1\}}) \in U^\perp$ for all $\pi \in S_n$. So the affine subspace $\chi_{\{1\}} + U^\perp$ is symmetric but does not contain a constant since $E_2 \subseteq U^\perp$.

□

3. Symmetric Operations and Relations

In this section, we are going to study symmetric relations and relate them to the existence of a k-WNU term operation of arity n. We are going to use results of the previous chapter to prove that under some conditions, a Taylor algebra has a 2-WNU term operation of arity n. First, we begin with the definition of a symmetric relation. After that, we will define a k-WNU term operation.

Definition 48. *$R \subseteq \boldsymbol{A}^{\binom{[n]}{k}}$ is symmetric if for all permutations $\pi : [n] \to [n]$ and all $a \in R$ it holds that $b \in R$, where $b_{\pi(I)} = a_I$, for all $I \in \binom{[n]}{k}$. We denote b by $\pi(a)$. $R \subseteq (\boldsymbol{A}^{\binom{[n]}{k}})^{A^2}$ is called* symmetric*, if for all $x \in R$, we have that $y \in R$, where $(y)_{(a,b)} = \pi((x)_{(a,b)})$ for all $(a, b) \in A^2$. We also denote $y = \pi(x)$.*

Definition 49. *An algebra $\boldsymbol{A}$ is said to have a k-WNU term operation of arity n if there exists $t \in \mathrm{Clo}(\boldsymbol{A})$ such that for all $(a, b) \in A^2$ and for all $x, y \in \{a, b\}^n$ we have that $t(x) = t(y)$, whenever $|\{i : x_i = a\}| = |\{i : y_i = a\}| = k$.*

Now we define elements, which will allow us to relate the existence of a k-WNU term operation to the existence of a constant in the algebra generated by these elements. Let $x_1, \ldots, x_n \in (\mathbf{A}^{\binom{[n]}{k}})^{A^2}$ be such that

$$((x_i)_{(a,b)})_M = \begin{cases} a & i \in M \\ b & i \notin M \end{cases}$$

for all $(a, b) \in A^2$ and $M \in \binom{[n]}{k}$. Then we have that $\sigma(x_i) = x_{\sigma(i)}$ for all $i \in [n]$ and $\sigma \in S_n$. Indeed

$$\sigma(x_i)_{(a,b)} = \sigma((x_i)_{(a,b)}).$$

So we need to prove that

$$\sigma((x_i)_{(a,b)}) = (x_{\sigma(i)})_{(a,b)}.$$

This is true becasue

$$(\sigma((x_i)_{(a,b)}))_M = ((x_i)_{(a,b)})_{\sigma^{-1}(M)}$$

and we have that

$$((x_{\sigma(i)})_{(a,b)}))_M = \begin{cases} a & \sigma(i) \in M \\ b & \sigma(i) \notin M \end{cases}.$$

Since we have that $\sigma(i) \in M$ if and only if $i \in \sigma^{-1}(M)$, the result follows.

An element $x \in A^n$ is said to be *constant* if all its coordinates are equal. An element $x \in (\mathbf{A}^{\binom{[n]}{k}})^{A^2}$ is called *constant* if $x_{(a,b)}$ is constant for all $(a, b) \in A^2$.

In the following proposition we relate the existence of a k-WNU term operation to the existence of a constant in a certain relation.

Proposition 50. *An algebra $\boldsymbol{A}$ has a k-WNU term operation if and only if the subalgebra of $(\boldsymbol{A}^{\binom{[n]}{k}})^{A^2}$ generated by $x_1, \ldots, x_n$ contains an element c such that $(c)_{(a,b)}$ is constant for all $(a, b) \in A^2$.*

Proof. ($\Rightarrow$) This direction is clear.

($\Leftarrow$) Let $\mathbf{B}$ be the subalgebra generated by $x_1, \dots, x_n$ containing a constant $c \in B$. Then we know by Proposition 13 that $B = \{t(x_1, \dots, x_n) | t \in \mathrm{Clo}_n(\mathbf{A})\}$. This says that there exists a term operation t of arity n such that for all $(a, b) \in A^2$ and for all $x, y \in \{a, b\}^n$ we have that $t(x) = t(y)$, whenever $|\{i : x_i = a\}| = |\{i : y_i = a\}| = k$. And this means that t is a k-WNU term operation.

□

In the following proposition, we prove that if all symmetric subalgebras contain a constant, then the subalgebra generated by $x_1, \dots, x_n$ contains a constant. This is equivalent to the existence of a k-WNU term operation on $\mathbf{A}$ by the previous proposition.

Proposition 51. *Let $\boldsymbol{A}$ be a finite algebra, such that every symmetric subalgebra $\boldsymbol{C} \leq \boldsymbol{A}^{\binom{[n]}{k}}$ contains a constant. Then the subalgebra $\boldsymbol{B}$ of $(\boldsymbol{A}^{\binom{[n]}{k}})^{A^2}$ generated by $x_1, \dots, x_n$ contains an element c such that $(c)_{(a,b)}$ is constant for all $(a, b) \in A^2$.*

Proof. Let $\pi_{(a,b)} : (\mathbf{A}^{\binom{[n]}{k}})^{A^2} \to \mathbf{A}^{\binom{[n]}{k}}$ be the projectin on the coordinate (a, b). We claim that the projection $\pi_{(a,b)}(\mathbf{B})$ is symmetric. This is true because as we said for all $i \in [n]$ we have that $\sigma(x_i) = x_{\sigma(i)}$. Then, since these generators are symmetric, we have that $\mathbf{B}$ is symmetric, therefore $\pi_{(a,b)}(\mathbf{B})$ is symmetric. Since $\pi_{(a,b)}(\mathbf{B})$ is just a projection of $\mathbf{B}$ and $\pi_{(a,b)}(\mathbf{B})$ contains a constant tuple by our assumption for all $(a, b) \in A^2$, there is an element $z \in \mathbf{B}$ such that $\pi_{(a,b)}(z)$ is constant. Since $\mathbf{A}$ is idempotent, $C = \{x \in (\mathbf{A}^{\binom{[n]}{k}})^{A^2} | \pi_{(a,b)}(x) = \pi_{(a,b)}(z)\}$ is a subuniverse of $(\mathbf{A}^{\binom{[n]}{k}})^{A^2}$. Let $(c, d) \in A^2$ be such that $\pi_{(c,d)}(z)$ is not constant. Since $\mathbf{B}$ is symmetric, we have that $\mathbf{C}$ is symmetric. Therefore $\pi_{(c,d)}(\mathbf{C})$ is symmetric and it contains a constant by our assumption. So now we get $y \in \mathbf{B}$ such that $\pi_{(c,d)}(y)$ is constant and $\pi_{(a,b)}(y)$ is constant. We can continue like this until we get what we want.

□

The following lemma is needed to prove Lemma 53. If X is a set, then by $\binom{X}{k}$ we denote the set of all subsets of X of size k.

Lemma 52. *Let $n, l \geq 2k + 1$, X be a set of size n, $M, N \in \binom{X}{k}$, $(p_1, \dots, p_l) \in \{0, \dots, k-1\}^l$. Then there exist sets*

$$M = M_0, M_1, M_2, \dots, M_l = N$$

which are all in $\binom{X}{k}$ and $|M_{i-1} \cap M_i| = p_i$ for all $i \in \{1, \dots, l\}$.

Proof. We prove this by induction on k. If $k = 1$, then the statement is clear. Let $k > 1$. Without loss of generality let $X = [n]$. If $p_i < k - 1$, we set $p'_i = p_i$. If $p_i = k - 1$, we set $p'_i = k - 2$.

We can find sets M_1 and M_2 such that $|M \cap M_1| = p_1$, $|M_1 \cap M_2| = p_2$, $M_2 \neq N$ and $M_2 \cap N \neq \emptyset$. Indeed, first suppose that $M = N$, then we can find M_1 such that $|M \cap M_1| = p_1$. Secondly, we find p_2 elements in M_1 and add them to M_2, if needed we find an element in $M = N$ and add it to M_2. Then we find

the remaining elements in $[n] \setminus (M_1 \cup N)$ and add them to M_2. Now suppose that $M \cap N = \emptyset$. Then we will find M_1 such that $|M \cap M_1| = p_1$ and $M_1 \cap N \neq \emptyset$. Then we find M_2 such that $|M_2 \cap M_1| = p_2$ and $M_2 \cap N \neq \emptyset$ and $M_2 \neq N$. Therefore, without loss of generality, we may suppose that $M \neq N$ and $M \cap N \neq \emptyset$, because we apply the induction hypothesis to a set of size $k-1$, so we can decrease l by 2 if we need to.

It follows that, without loss of generality, we may find $1 \in M \cap N$ and $2 \in X \setminus (M \cup N)$. These elements exists because $M \cap N \neq \emptyset$. Now we use induction hypothesis on $M \cap \{3, \ldots, n\}, N \cap \{3, \ldots, n\}$, the set $X = \{3, \ldots, n\}$ and $(p'_1, \ldots, p'_n)$. So we get a sequence of subsets of X

$$M \cap \{3, \ldots, n\} = M_0, M_1, M_2, \ldots, M_l = N \cap \{3, \ldots, n\},$$

such that $|M_{i-1} \cap M_i| = p'_i$ for all $i \in \{1, \ldots, n\}$.

Now we add 1 and 2 in the obtained sets to get what we want. First we add 1 in M_0. If $p'_1 \neq p_1$, then we add 1 in M_1, otherwise we add 2 in M_1. Further if $p'_2 \neq p_2$, then we add the element in $M_1 \cap \{1, 2\}$ to M_2, otherwise we add the other element. Now, we continue doing the same with the other sets. If, after completing this procedure, we end up with N, then we are done. Otherwise, we choose $1 \in M \setminus N$ and $2 \in N \setminus M$. These elements exists because $M \cap N \neq \emptyset$. Now we add 2 in M_l instead of 1. This will give us what we want.

□

Let $I, J, K, L \in \binom{[n]}{k}$ be such that $|I \cap J| = |K \cap L|$. Let $R \subseteq A^{\binom{[n]}{k}}$ be a symmetric relation, and $(x, y) \in R_{I,J}$. Then there exists $a \in R$ such that $a_I = x$ and $a_J = y$. There also exists a permutation $\pi \in S_n$ such that $\pi(I) = K$, $\pi(J) = L$ and $\pi(I \cap J) = K \cap L$. It follows that $b = \pi(a) \in R$, and that $b_K = a_I = x$, $b_L = a_J = y$. Therefore $(x, y) \in R_{K,L}$. So we have proved that if $|I \cap J| = |K \cap L|$, then $R_{I,J} = R_{K,L}$.

The following lemma is crucial in order to prove Theorem 55.

Lemma 53. *Let $\prec$ be a stability concept on $HSP_{fin}(\boldsymbol{A})$, $\boldsymbol{C} \prec \boldsymbol{B}$ such that $\boldsymbol{C} \neq \boldsymbol{B}$, $R \leq \boldsymbol{B}^{\binom{[n]}{k}}$ be a symmetric subdirect relation for $n \geq 2k+1$. Then there exist sets $I_1, J_1, \ldots, I_t, J_t$ such that $D = C + R_{I_1,J_1} + \cdots + R_{I_t,J_t} \neq B$ and for any $I, J \in \binom{[n]}{k}$ we have that $(\boldsymbol{D} + R_{I,J}) \cap \boldsymbol{D} \neq \emptyset$. Moreover we have that $\boldsymbol{D} \prec \boldsymbol{B}$.*

Proof. If there does not exist $I, J \in \binom{[n]}{k}$ such that $(\mathbf{C} + R_{I,J}) \cap \mathbf{C} = \emptyset$, then we are done. Otherwise set $I_1 = I$ and $J_1 = J$. In particular we have that $\mathbf{C} + R_{I,J} \neq B$. Now we can continue doing the above with $\mathbf{C} + R_{I,J}$. Lemma 52 and the discussion above assures us that the procedure will end after at most $2k+1$ steps. Indeed, suppose that

$$D = C + R_{I_1,J_1} + \cdots + R_{I_t,J_t}.$$

First notice that

$$C + R_{X_1,X_m} \subseteq C + R_{X_1,X_2} + \cdots + R_{X_{m-1},X_m}.$$

This follows directly from the definition. Now let $M, N \in \binom{[n]}{k}$ be any sets and define $p_i = |I_i \cap J_i|$ for all $i \in \{1, \ldots, t\}$. From Lemma 52, we obtain subsets of

$[n]$ of size k

$$M = M_1, M_2, \ldots, M_t, M_{t+1} = N,$$

such that $|M_i \cap M_{i+1}| = p_i$ for all $i \in \{1, \ldots, t\}$. Then from the discussion above, we have that $R_{I_i,J_i} = R_{M_i,M_{i+1}}$. Hence

$$D = C + R_{M_1,M_2} + \cdots + R_{M_t,M_{t+1}}.$$

Therefore $C + R_{M,N} \subseteq D$. In particular, this means that $D + R_{I,J}$ cannot be disjoint from D, for all $I, J \in \binom{[n]}{k}$, because they both contain $C + R_{M,N}$ for all $M, N \in \binom{[n]}{k}$.

The fact that $\mathbf{D} \prec \mathbf{B}$, follows from Lemma 29.

□

Definition 54. *We call a triple $(n, k, p) \in \mathbb{N}^3$, where p is a prime number and $k \leq n$,* suitable*, if every symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$ contains a constant.*

The next theorem is one of the most important results of this thesis.

Theorem 55. *Let $n \in \mathbb{N}$ be such that $n \geq 2k + 1$. If an algebra $\boldsymbol{A}$ is Taylor and (n, k, p) is suitable for all prime numbers p such that $p \leq |A|$. Then $\boldsymbol{A}$ has a k-WNU term operation of arity n.*

Proof. Without loss of generality, we may assume that $\mathbf{A}$ is minimal Taylor, otherwise we can pass to a minimal Taylor reduct by Proposition 26. By Theorem 28, there exists a stability concept $\prec$ on $HSP_{fin}(\mathbf{A})$. To prove that there exists a k-WNU term operation of arity n we just have to prove that every symmetric subalgebra of $\mathbf{A}^{\binom{[n]}{k}}$ contains a constant by Lemma 51 and 50.

We prove the this by induction on $|A|$, the case $|A| = 1$ is clear. So let $R \leq \mathbf{A}^{\binom{[n]}{k}}$ be a symmetric relation. Without loss of generality, we may suppose that R is a subdirect relation. Because if it isn't, we can take $R \cap R_I^{\binom{[n]}{k}} \leq R_I^{\binom{[n]}{k}}$ for any coordinate I and then use induction hypothesis on the algebra $R_I^{\binom{[n]}{k}}$ and relation $R \cap R_I^{\binom{[n]}{k}}$ which is again symmetric. ($R_I = R_J$ for all $I, J \in \binom{[n]}{2}$ because R is symmetric.)

(Case 1) There exists $\mathbf{B} \prec \mathbf{A}$ such that $\mathbf{B} \neq \mathbf{A}$.

Then we may find $\mathbf{C} \prec \mathbf{A}$ such that $(\mathbf{C} + R_{I,J}) \cap \mathbf{C} \neq \emptyset$ for all $I, J \in \binom{[n]}{k}$ by Lemma 53.

Notice that $R_{I,J} \cap (\mathbf{C} \times \mathbf{C}) \neq \emptyset$ if and only if $(\mathbf{C} + R_{I,J}) \cap \mathbf{C} \neq \emptyset$. So by Lemma 31, $R \cap \mathbf{C}^{\binom{[n]}{k}} \neq \emptyset$. And now we can use the induction hypothesis on $R \cap \mathbf{C}^{\binom{[n]}{k}} \leq \mathbf{C}^{\binom{[n]}{k}}$.

(Case 2) There is some proper congruence $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{A}/\theta$ is an abelian algebra.

Then, since $\mathbf{A}$ is finite, we can find a maximal congruence $\eta/\theta \in \mathrm{Con}(\mathbf{A}/\theta)$. Then

$$(\mathbf{A}/\theta)/(\eta/\theta) \simeq \mathbf{A}/\eta$$

is abelian simple minimal Taylor. The fact that it is simple follows from the fact that η/θ is maximal, it is abelian because quotients of abelian algebras are abelian and it is minimal Taylor by Proposition 24. This means that $\mathrm{Clo}(\mathbf{A}/\eta) = \mathrm{Clo}(A/\eta; x-y+z)$, where $+, -$ are abelian group operations, and $|A/\eta|$ is a prime number by Proposition 25. From the remark after Proposition 25, it follows that we can consider R/η to be a symmetric affine subspace of $\mathbb{Z}_p^{\binom{[n]}{k}}$, where by R/η we mean the set of all equivalence classes of the relation where two elements of R are related if all their coordinated are related by η. Since (n, k, p) is suitable for all $p \leq |A|$, we have that R/η contains a constant. The quotient R/η contains a constant, it follows that there exists a tuple $a \in R$ such that all its coordinates are in one equivalence class $[c]_\eta$. Because $\mathbf{A}$ is idempotent, $[c]_\eta$ is a subuniverse of $\mathbf{A}$. So we can restrict R to $[c]_\eta$, the restriction is again symmetric, and use induction hypothesis.

□

Corollary 56. *If an algebra $\boldsymbol{A}$ is finite Taylor and $n \in \mathbb{N}$ satisfy the following:*

1. *$n \geq 6$,*
2. *if p is a prime number such that $p \leq |A|$, then p does not divide $n\binom{n}{2}$,*

then $\boldsymbol{A}$ has a 2-WNU term operation of arity n.

Proof. This follows directly from Theorem 55 and Theorem 47.

□

Conclusion

In this thesis we studied symmetric relations and symmetric affine subspaces of $\mathbb{Z}_p^{\binom{[n]}{k}}$. For $k = 2$ we have obtained satisfying answers. In the other cases the problem is more complicated and we have not been able to solve it.

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