Report on the master thesis titled "Representation theory of gentle algebras"

Summary of result

The thesis under review focuses on certain homological structures called *torsion classes* arising from the representation theory of the so-called *gentle algebras*. The classification of torsion classes over gentle algebras has been recently completed in a preprint by Chan and Demonet. They found that a torsion classes can be determined by a set of combinatorial generators. Such a set is called a *non-crossing sets of strings*, which serves as *combinatorial* analogues of *triangulations of topological surfaces*. In fact, the examples written in the preprint are presented using a *topological model* of gentle algebras, and relies on the reader to translate between them to the combinatorial model.

The thesis under review aims to address this aspect of Chan-Deomnet's preprint, and at the same time provide hands-on calculations involving a more complex and non-trivial example. More precisely, its content consist of the following.

- Explanation of the translation between the combinatorial and topological model.
- Detailed explanation on the translation as well as the classification result in the case of the path algebra of type $\mathbb{A}_2$, and the Kronecker algebra.
- Calculation of (certain combinatorial generators of) the torsion classes of so-called *Markov algebra*. More precisely, the author determines *all* non-crossing strings of the Markov algebras, and show that how one can associate them with extended real numbers $\mathbb{R} \cup \{\infty\}$ as well as their manifestation in the topological model (a once-punctured torus) of the Markov algebra.
- Along the way, the author also determined some conditions for a pair of these strings to be non-crossing - many of them can be phrased by simply looking at their associated extended real number. This is a significant step towards the full classification of torsion classes, as well as the determination of the poset they form.

Major comments

(1) Quality of exposition: It is very messy; the main reason may be due to the language barrier - which is forgivable, yet there are still a few addressable concerns.

(a) Many wordings should be more precise; too many ambiguities in terminologies used. For instances, (i) the term 'arrow' is often not distinguished from 'inverse', (ii) (non-)uniqueness of 'significant vertex', (iii) meaning of 'extend' in the definition of cooriented string, (iv) 'concatenates' vs 'compose', 'substrings' vs 'components' vs 'parts', (v) the lack of proper definition of the map α, etc.

(b) Many arguments should be arranged into lemmas or propositions. It will also be helpful to explain the strategy before diving into pages of very technical arguments.

(c) Some statements and conditions of claims should be explained more carefully and clearly, e.g., the setup of Lemma 3.1, the setup of J, V in Lemma 5, statement of Lemma 6.1, etc.

(2) Incomplete arguments: Most mathematics seems to be accurate, but there are a few that looks incomplete. The major ones that are of concern are listed as follows.

(i) Lemma 5.3: The proof does not show what is claimed.

(ii) Proof of Lemma 6.1: argument cannot be verified due to unclear exposition.

(iii) Lemma 9 uses Observation 8 without showing the condition of Observation 8 is satisfied. There are also many other incomplete explanations throughout Lemma 9.
(iv) Proof of Lemma 14: few missing arguments
(v) Proof of Lemma 15: many missing arguments and unclear exposition.

Nevertheless, it appears to me that all of the results claimed are correct; perhaps the true reason to some, if not all, of the incompleteness listed above is purely a linguistic one.

(3) Quality of result: The calculation for path algebras (i.e. Section 2) is well-known, but it is nice to verify these classical results as a way to prepare for the (very) technical Section 3.

The results in Section 3 are impressive. While it is perhaps not difficult to see from the topological picture the connection with simple closed curves on the once-punctured torus, providing such a concrete algebraic (combinatorial) explanation is really helpful for data tracking and carrying out further algebraic investigation. It successfully demonstrates that the classification of torsion classes of gentle algebras is far from being a trivial one; there are many mysteries and interesting properties hidden in the poset of torsion classes that could be of interest beyond its algebraic setup - such as geometry and number theory.

It is a pity that the author did not completely finish the full description of the poset of torsion classes, but it seems the author is not very far from completing it. In such a case, I would not be surprised if the final result could be submitted (and accepted) for publication in a well-respected journal.

(4) Potential of tools used and results: In addition to successfully classifying all self-non-crossing strings (generators of torsion classes) – which is already quite an achievement – the thesis also showed how they are intimately related to extended real numbers as well as laminations of the once-punctured torus. Lamination is a well-established concept in differential geometry, but non-integral laminations (such as 'irrational sloped curves') are difficult to handle. The notion of significant vertex introduced in this thesis seems to offer some hint for dealing with this. It will be interesting to see if it can be generalised to a larger class of gentle algebras, and potentially revealling a new algebro-combinatorial tool for studying topological surfaces.

The argument that goes into showing this connection with extended real numbers strongly suggests that there is a strong link with continued fraction expansions. I believe that this thesis will inspire various directions in further research, and will be glad if the author were to polish the thesis into an article for publication.

Verdict

Suggested grade: 2 (Very good)

Summary of reasons: Impressive result and calculation on a very involved algebra. Tools invented even have potential for further developments, from this aspect this is a work of the highest grade. Unfortunately, exposition is rather unsatisfying with a few incompleteness that desire serious fixes, especially in the last two subsections, and for this reason I downgraded to the next-best category.