FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Anna Mlezivová

Representation theory of gentle algebras

Department of Algebra

Supervisor of the master thesis: doc. RNDr. Jan Šťovíček, Ph.D.
Study programme: Mathematical Structures
Study branch: MSPN

Prague 2023

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank my supervisor, doc. RNDr. Jan Šťovíček, Ph.D., for all the time spent on my thesis, carefully reading and choosing a nice topic full of images.

Also, I am grateful to Bc. Radek Olšák for his willingness to listen and discuss my thesis at any time as well as for his mental support.

Last but not least, I would like to thank my family for supporting me during my studies and reminding me that there are other things in the world than my thesis.

Title: Representation theory of gentle algebras

Author: Anna Mlezivová

Department: Department of Algebra

Supervisor: doc. RNDr. Jan Šťovíček, Ph.D., Department of Algebra

Abstract: The object of study of this thesis is a special class of quiver algebras called gentle algebras. To study modules over them, we can use a combinatorial or geometric view. Thanks to Theorem 6.1. in the article Chan and Demonet [2020], we can find the lattice of torsion classes of modules over gentle algebras using string combinatorics. In the thesis, we apply this theorem for a few examples. Especially we derive the lattice of torsion classes of Kronecker algebra, and we do the first steps for finding the lattice for Markov algebra. The emphasis is placed on understanding the relationship with the geometric view.

Keywords: Gentle algebras, torsion classes, string combinatorics, Markov quiver, geometric model

Contents

List of Abbreviations

$\mathbb{K}$	field
$\mathbb{P}_1(\mathbb{K})$	projective line
(S, M)	marked surface
Q	quiver
R	relations
(Q, R)	gentle quiver
Q_0	vertices of quiver Q
Q_1	arrows of quiver Q
$Q_1^{\pm}$	letters of quiver Q
(Q', R')	blossoming of quiver (Q, R)
$s(a)$	source of arrow a
$t(a)$	target of arrow a
$\underline{\gamma}$	string associated to walk γ
$c^+(\gamma, \delta)$	set of positive crossings from γ to δ
$\bigoplus$	direct sum
$\bigotimes$	tensor product
$X(\underline{\gamma})$	string module
$X_M(\underline{\gamma})$	band module
$\mathsf{Fac}(\mathscr{S})$	finite factors of set of strings $\mathscr{S}$
$\mathsf{f.l.}\,\Lambda$	category of finite-dimensional right Λ-modules
$\mathsf{tors}(Q, R)$	set of confined torsion sets in (Q', R')
$\mathsf{maxNC}(Q, R)$	set of maximal non-crossing sets of strings in a blossoming of (Q, R)
$\mathsf{maxNC}_{\mathbb{K}}(Q, R)$	set of maximal parametrized non-crossing sets of strings in a blossoming of (Q, R)
$\mathscr{T}_\Lambda$	isomorphism from $\mathsf{maxNC}_{\mathbb{K}}(Q, R)$ to $\mathsf{tors}(\mathsf{f.l.}\,\Lambda)$ from Theorem 2
A_2	quiver consisting of two vertices and one arrow between them

Introduction

This thesis deals with gentle algebras, especially finding torsion classes of modules over them by string combinatorics and illustrating it in the geometric view. The thesis continues the article written by Chan and Demonet [2020]. It uses the theory stated in this article and brings some applications of it.

Gentle algebras are a class of quiver algebras such that

1. each vertex of the underlying quiver has at most two incoming and two outgoing arrows;

2. each arrow of the underlying quiver can be succeeded/preceded by at most one arrow so that their composition is in relations and by at most one arrow so that their composition is not in relations.

A torsion class is defined as a class of modules closed to factors and extensions.

In Chapter 1, we start with needed definitions and facts. We connect defined notions in combinatorial and geometric view. At the end of this chapter, we state Theorem 2, which is taken from the article Chan and Demonet [2020]. This theorem brings a new way to finding torsion classes of modules over the gentle algebras. To do this, it uses the string combinatorics.

In Chapter 2 we show the power of this theorem and we apply it to a few examples. Concretely in Section 2.1 we derive the lattice of torsion classes of A_2, the quiver consisting of two vertices and one arrow between them. The most difficult example in this section is the Kronecker quiver, consisting of two vertices and two arrows between them in the same direction. This is the subject of Section 2.3. To find torsion classes, we use string combinatorics, but at the same time, we show what happens in the geometric view.

Although lattices of torsion classes for this algebras are well known, the chapter gives another way to find them. Moreover, it helps to understand Theorem 2 and work with string combinatorics.

Finally, the last Chapter 3 aims to find the lattice of torsion sets of the Markov quiver. Compared to the previous examples, this lattice is not yet known. Even if we do not get the description of the whole lattice, we are taking the first important steps that will be easy to follow to obtain it.

The contribution of this thesis is mainly in the not yet shown applications of finding the lattice of torsion classes using string combinatorics. Namely in simple examples, and especially in the first steps to describe the lattice of torsion classes for the Markov algebra. At the same time, emphasis is placed on understanding the relationship with the geometric view, and everything is illustrated with several figures. The whole approach and all results in Chapter 3 are original contributions. Chapter 2 is also mainly author's up to inspiration in Examples 4.12 and 4.13 in Chan and Demonet [2020].

1. Definitions and facts

Firstly we start with a necessary theory about gentle quivers and gentle algebras. Definitions are mostly taken from Chan and Demonet [2020]. Similarly the result 2, which is essential for this thesis, comes from this article. The tiling algebra view is taken from the article by Baur and Simoes [2018]. We assume a basic knowledge of the representation of algebras such as the notions of quiver, quiver algebra and representation. All necessary is listed in the second and third chapter of the book Assem, Simson, and Skowronski [2006].

1.1 Gentle algebras

In this chapter we will use the following notation. Let $\mathbb{K}$ be a field, Q a quiver with a set of vertices Q_0 and a set of arrows Q_1. In general, we denote by Q_k a set of paths of length k. Further, given $a \in Q_1$, we denote its source by $s(a)$ and its target by $t(a)$. We will read paths in Q from left to right.

Definition 1 (Gentle quiver)**.** A *gentle quiver* is a tuple (Q, R) consisting of a finite quiver Q and a set of relations $R \subset Q_2$ such that the following conditions are satisfied.

1. Any $i \in Q_0$ has at most two incoming and two outgoing arrows.
2. For any $a \in Q_1$, there is at most one $b \in Q_1$ such that $t(a) = s(b)$ and $ab \notin R$.
3. For any $a \in Q_1$, there is at most one $b \in Q_1$ such that $t(a) = s(b)$ and $ab \in R$.
4. For any $a \in Q_1$, there is at most one $b \in Q_1$ such that $t(b) = s(a)$ and $ba \notin R$.
5. For any $a \in Q_1$, there is at most one $b \in Q_1$ such that $t(b) = s(a)$ and $ba \in R$.

Example. In Figure 1.1 we can see four examples of gentle quivers. With the first three of them we will work in this thesis.

Definition 2 (Gentle algebra)**.** An algebra A is *gentle* if it admits presentation $A = \mathbb{K}Q/(R)$ where (Q, R) is a gentle quiver.

Now we will introduce so-called tiling algebras. The gentle algebras we will work with in this thesis are also tiling algebras. We will define it similarly as in Baur and Simoes [2018], but not exactly the same because we want to extend the definition to infinite dimensional algebras.

Definition 3. Let S denote an orientable surface, no matter if it has a boundary, with a finite set M of marked points that can be on the boundary of S or not. The pair (S, M) is called a *marked surface*.

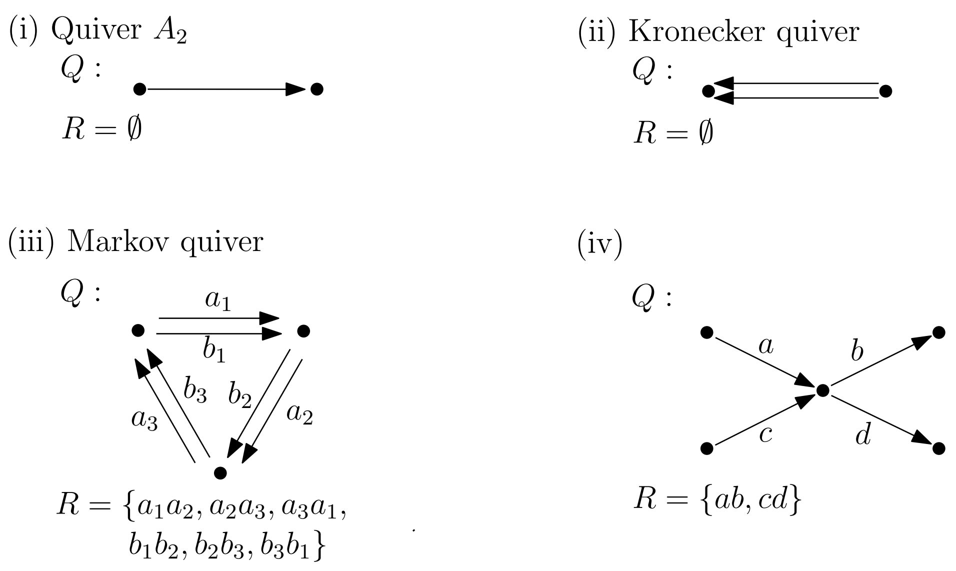


Figure 1.1: Examples of gentle quivers

Definition 4 (Arc)**.** An *arc* in (S, M) is a curve γ in S satisfying the following properties:

1. The endpoints of γ are in M.
2. γ intersects the boundary of S only in its endpoints.
3. γ does not cut out a monogon or a digon.

Arcs are taken up to homotopy relative to their endpoints.

Example. Examples of valid and invalid arcs are shown in Figure 1.2.

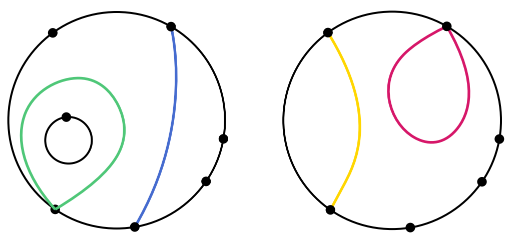


Figure 1.2: Examples of valid and invalid arcs, respectively

Definition 5. A *partial triangulation* is a set P of arcs that do not intersect themselves or each other in the interior of S. A triple (S, M, P) satisfying the conditions above is called *tiling*.

Given a marked point $p \in M$, we can look at all arcs with at least one end in p. All such ends are naturally sorted by the order in which they leave point p. We will look at them in the counter-clockwise order. Thus we can talk about successors and predecessors in this order.

Definition 6. *Tiling algebra* A_P associated to the partial triangulation P of (S, M) is the bound quiver algebra $A_P = \mathbb{K}Q_P/(R_P)$, where (Q_P, R_P) are described as follows:

1. The vertices in $(Q_P)_0$ are in one-to-one correspondence with the arcs in P.
2. There is an arrow a in $(Q_P)_1$ if the arcs corresponding to $s(a)$ and $t(a)$ share an endpoint p_a in M and the second arc is an immediate successor of the first one.
3. Relations R_P consist of paths ab of length two which satisfy one of the following conditions:
   (a) $a = b$, i.e., it forms a loop
   (b) $a \neq b$ and either $p_a \neq p_b$ or we are in one of the two situations presented in Figure 1.3.

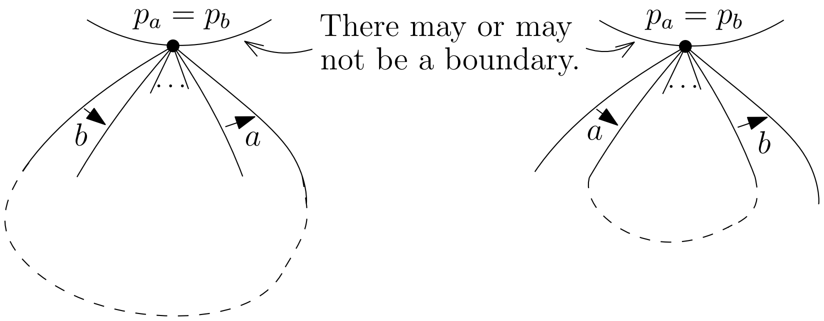


Figure 1.3: Case $ab \in R_P$, $a \neq b$, $p_a = p_b$

Example. In Figure 1.4 we can see tiling algebras for the first two quivers from Figure 1.2.

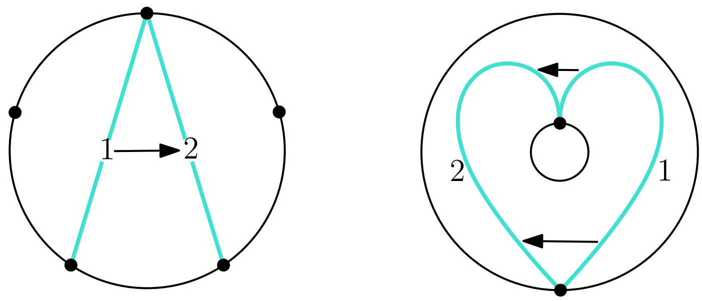


Figure 1.4: Tiling algebras for A_2 and the Kronecker quiver.

If we would have a bit stronger assumptions to a marked surface as in Baur and Simoes [2018] then holds that algebra is a finite dimensional gentle if and only if it is a tiling algebra. It is proven in Theorem 2.10 in this article.

1.2 Blossoming

Let us go back to the definition of gentle quivers. Now we introduce its blossoming.

Definition 7 (Blossoming)**.** A *blossoming* of a gentle quiver (Q, R) is another gentle quiver (Q', R') satisfying the following conditions.

1. $Q_0 \subseteq Q'_0$, $Q_1 \subseteq Q'_1$, and $R' \cap Q_2 = R$.

2. For any $i \in Q_0$, there are exactly two arrows $a \in Q'_1$ such that $s(a) = i$.

3. For any $i \in Q_0$, there are exactly two arrows $a \in Q'_1$ such that $t(a) = i$.

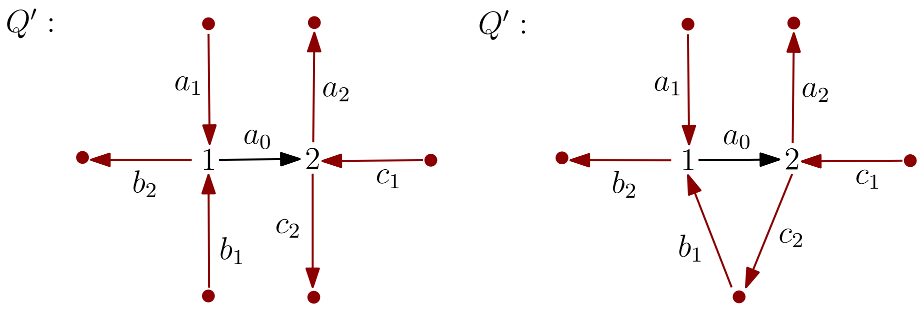
4. For any $a \in Q'_1 \setminus Q_1$, exactly one of $s(a)$ and $t(a)$ is in Q'_0.

5. For any pair of arrows a and b satisfying $t(a) = s(b) \in Q'_0 \setminus Q_0$, $ab \in R'$.

Remark. The point is that we want two incoming and two outgoing arrows for each vertex. A blossoming is such a quiver that we add exactly missing arrows and some new vertices to them so that the new arrows either start or end in a new vertex. This number of new vertices is not unique, so there can be more blossomings for one quiver (see example below). Nevertheless, two different blossomings have the same set of arrows and everything we need from blossomings are only new arrows, i.e., everything we do is independent of the choice of blossoming. The maximum number of vertices of a blossoming is obtained by adding a new vertex for each new arrow; such a blossoming is called the *classical blossoming*.

Example.

- In Figure 1.5, two blossomings for quiver A_2 are shown. The first is the classical blossoming, with the second we will work in Section 2.1. Vertices and arrows that are added in each blossoming are marked in brown.

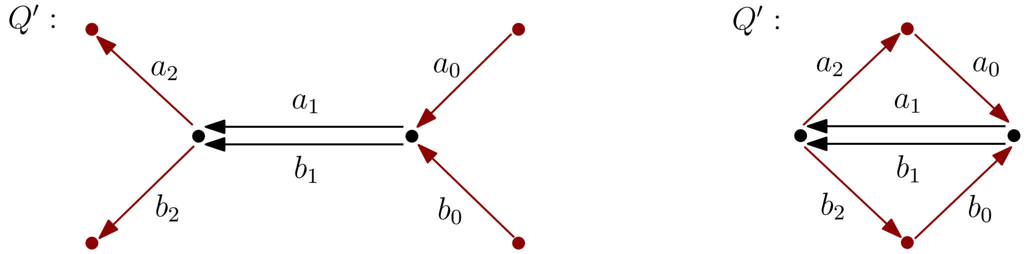


$R' = \{a_1b_2, b_1a_0, a_0c_2, c_1a_2\}$ $\qquad$ $R' = \{a_1b_2, b_1a_0, a_0c_2, c_1a_2, c_2b_1\}$

Figure 1.5: Blossomings of quiver A_2

- In Figure 1.6 we can see two blossomings for the Kronecker quiver. The first is the classical blossoming, with the second we will work in Section 2.3.

- In the Markov quiver, every vertex has two incoming and two outgoing arrows, so its only blossoming is itself.



$R' = \{a_0b_1, b_0a_1, a_1b_2, b_1a_2, \}$

$R' = \{a_0b_1, b_0a_1, a_1b_2, b_1a_2, a_2a_0, b_2b_0\}$

Figure 1.6: Blossomings of the Kronecker quiver

Remark. From the point of view of tiling algebras, we get a blossoming by adding vertices for parts of the boundary, adding arrows between them and arcs which neighbour with them. Indeed, in a tiling algebra every arc has two ends and in each end it has as successor and predecessor either another arc or a part of the boundary.

Example. In Figure 1.7 we see two blossomings from Figure 1.5 for quiver A_2 in the tiling view.

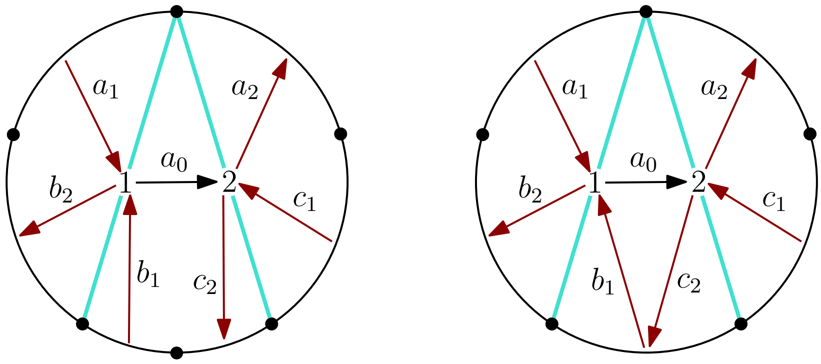


Figure 1.7: Tiling for blossomings of A_2

1.3 Strings

For $a, b \in Q_1$ we will write $ab = 0$ if $ab \in R$ and $ab \neq 0$ if $ab \notin R$.

To each arrow $a \in Q_1$, we associate a formal inverse a^- such that $s(q^-) = t(a)$ and $t(a^-) = s(a)$. Arrows and their formal inverses are called letters, and the set of letters is denoted by $Q_1^{\pm}$.

Let us expand the relations on $Q_1^{\pm}$ in the following way:

1. For $a, b \in Q_1$, $b^- a^- = 0$ if and only if $ab = 0$.

2. If $t(a) = t(b)$ and $a \neq b$, then $ab^- \neq 0$.

3. If $s(a) = s(b)$ and $a \neq b$, then $a^- b \neq 0$.

For every vertex v we have two stationary walks denoted 1_v^+ and 1_v^-, which are mutually inverse.

Definition 8 (Walk and string)**.** A *walk* γ in a quiver is a stationary walk or a reduced sequence (possibly infinite) of consecutive letters which avoids relations, i.e. $\gamma = \cdots a_i a_{i+1} \cdots$ with indexing set $I \subseteq \mathbb{Z}$, $a_j \in Q_1^{\pm}, t(a_i) = s(a_{i+1}), a_i \neq a_{i+1}^-, a_i a_{i+1} \neq 0 \; \forall i \in I$. An *inverse of a walk* γ is a walk γ^- such that it is a sequence of inverses of letters γ in the opposite direction. That is $\gamma^- = \cdots a_i^- a_{i-1}^- \cdots$.

For a walk γ in Q, we denote by $\underline{\gamma}$ the pair $\{\gamma, \gamma^-\}$, and we call this the associated *string*. Obviously, $\underline{\gamma} = \underline{\gamma^-}$. The string associated to a stationary walk for vertex v we will denote 1_v.

In this chapter, we will follow the notation of strings with underlining. In next chapters, we will work with strings and no walks, so for simplicity, we will write strings without underlining. It means that we will denote them by walks which determine them.

In the following definition we will work with a walk in a blossoming (Q', R') of a quiver (Q, R). Of course, the endpoints of such a walk can lie in $Q_0' \setminus Q_0$. On the other hand, midpoints must lie in Q_0 because for any pair of arrows a and b satisfying $t(a) = s(b) \in Q_0' \setminus Q_0$ holds that $ab \in R'$. Hence, for a midpoint in $Q_0' \setminus Q_0$ we would not have a valid walk.

Definition 9 (Confined, infinite, and periodic walks)**.** Let γ be a walk in a blossoming (Q', R') of a quiver (Q, R).

1. γ is *left-infinite* if

   (a) it is *left-unbounded*, i.e. the indexing set of γ has no finite lower bound;

   (b) or it is left-bounded $\gamma = a_i a_{i+1} a_{i+2} \cdots$ with $s(a_i) \in Q_0' \setminus Q_0$.

   Similarly, a walk is *right-infinite* if it is either right-unbounded or right-bounded with the target of the last arrow in $Q_0' \setminus Q_0$.

2. γ is *infinite* if it is both left-infinite and right-infinite.

3. γ is *periodic* if it is unbounded in both ends, and there is some $r \in \mathbb{Z}$ such that the i-th letter in γ is equal to the $(i \pm r)$-th letters in γ for all $i \in \mathbb{Z}$.

4. γ is *left-confined* (respectively, *right-confined*) if it is not left-infinite (respectively, right-infinite).

5. γ is *confined* if it is left-confined and right-confined; in particular, it consists of only finitely many letters of Q.

We also say that a string is confined (resp. infinite, resp. periodic) if so is its underlying walk. We will call a periodic string which not consists of all arrows in the same direction a *band*.

Let us look at strings in the tiling (S, M, P). The letter, that is an arrow or its inverse in a quiver, corresponds to ordered pair of neighbouring arcs in the tiling. A walk can be viewed as some oriented curve on S and string as the corresponding non-oriented curve. For confined strings, endpoints of the corresponding curve lie on some arc in P. If an infinite string is unbounded, then the corresponding curve has no ends, or its ends lie in the blossoming, that is, ends of the corresponding curve lie on the boundary of S. Periodic strings can be viewed as closed curves. Curves corresponding to confined string $\underline{a_0}$ and infinite string $\underline{c_2^- a_2}$ in the notation of Figure 1.7 are shown in Figure 1.8.

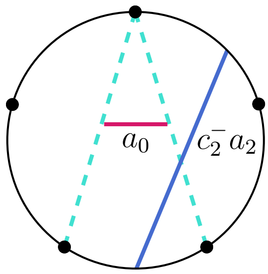


Figure 1.8: Curves corresponding to strings

Definition 10 (Concatenation)**.** Consider two walks, γ and δ. Let γ be right-confined, δ left-confined, $t(\gamma) = s(\delta)$ and holds that $ab \neq 0$, $a \neq b^-$ for a the last letter of γ and b the first letter of δ. The *concatenation* of γ and δ as words gives rise to a well-defined walk, denoted by $\gamma\delta$.

For γ confined we will denote:

- γ^m the confined walk given by concatenating γ with itself m times;
- $\gamma^\infty = \gamma\gamma\gamma\cdots$, the right-infinite walk;
- ${}^\infty\gamma = \cdots\gamma\gamma\gamma$, a left-infinite walk;
- ${}^\infty\gamma^\infty = ({}^\infty\gamma)(\gamma^\infty) = \cdots\gamma\gamma\cdots$, an infinite walk.

Note that because of the definition of gentle algebra and blossoming, each confined string γ of length at least one can be succeeded by a unique arrow, a unique inverse arrow and also preceded by a unique arrow and a unique inverse arrow.

Definition 11 (Crossing)**.** Consider two walks, γ and δ. A *positive crossing* from γ to δ is a pair of decompositions of the form $\gamma = \gamma_1 a^- \omega a' \gamma_2$, $\delta = \delta_1 b \omega b'^- \delta$ where a, a', b, b' are arrows and $\gamma_1, \gamma_2, \delta_1, \delta_2$ are subwalks. It is illustrated in Figure 1.9.

We will say there is a positive crossing from γ to δ at ω. We denote positive crossing by

$$\begin{array}{rr} \gamma = & \gamma_1 a^- \\ \delta = & \delta_1 b \end{array} \Big\rangle \omega \Big\langle \begin{array}{l} a'\gamma_2 \\ b'^-\delta \end{array} .$$

The set of positive crossings from γ to δ is denoted $c^+(\gamma, \delta)$. Note that there is an immediate identification $c^+(\gamma, \delta) = c^+(\gamma^-, \delta^-)$. For two strings $\underline{\gamma}$ and $\underline{\delta}$, we write $c^+(\underline{\gamma}, \underline{\delta}) = c^+(\gamma, \delta) \cup c^+(\gamma, \delta^-)$. We say that $\underline{\gamma}$ and $\underline{\delta}$ are *crossing* if $c^+(\underline{\gamma}, \underline{\delta}) \neq \emptyset$ or $c^+(\underline{\delta}, \underline{\gamma}) \neq \emptyset$.

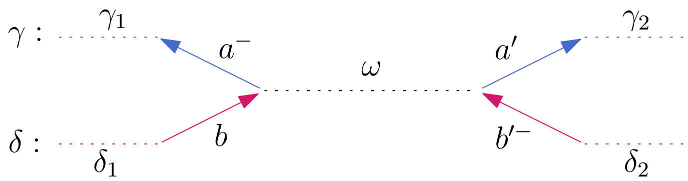


Figure 1.9: Illustration of a crossing

We do not prove that crossings in strings correspond to topological crossings of corresponding curves in tilings. It is not important for this thesis because we use the tiling view only for illustration.

In this thesis, we will work mainly with infinite non-self-crossing strings. Corresponding curves are called *accordions*. Now we will define accordions precisely, and we will show examples of what accordions can look like.

Definition 12 (Accordion)**.** An accordion on tiling (S, M, P) is a curve $\gamma : I \to S$ (possibly closed) satisfying the following conditions:

1. I is any connected subset of $\mathbb{R}$. Examples of accordions for different types of intervals are shown in Figure 1.10.

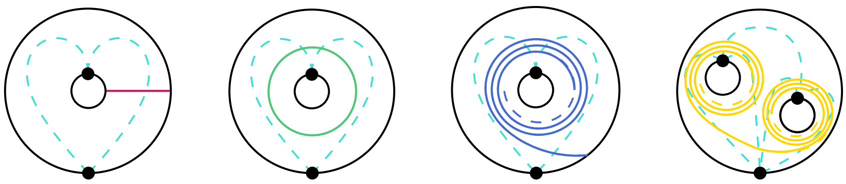


Figure 1.10: Examples of accordions

2. γ has no self-intersection.
3. γ does not intersect any vertex of M.
4. Each bounded end must lie on the boundary of S.
5. γ is not homotopically equivalent to the part of boundary. Also it is not homotopically trivial if it is a closed curve.
6. A curve γ that rotates along a simple closed curve δ infinitely (in both of ends) is identified to δ. An example of this situation is illustrated in Figure 1.11.

Accordions are taken up to homotopy relative to boundary segments between marked points. We mean that if a curve has an endpoint, we can move this endpoint along the boundary segment between marked points. On the other hand, accordions with ends in different boundary segments are necessarily different.

Definition 13 (Non-crossing set)**.** A set $\mathscr{S}$ of strings in (Q', R') is called *non-crossing* if for any $\underline{\gamma}, \underline{\delta} \in \mathscr{S}$, $c^+(\underline{\gamma}, \underline{\delta}) = \emptyset$. In particular, it consists of non-self-crossing strings. A set of strings is called *maximal non-crossing* if

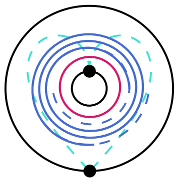


Figure 1.11: Example of a curve infinite rotating along closed curve

- It consists only of infinite strings.
- It is non-crossing.
- It is a maximal set with respect to the above properties.

We know that a set of arrows in a blossoming is always the same, so maximal non-crossing sets of strings do not depend on a choice of blossoming. Hence, we can denote by $\mathsf{maxNC}(Q, R)$ the set of all maximal non-crossing sets of strings in a blossoming of (Q, R).

Definition 14 (Lamination)**.** A maximal collection of non-intersecting accordions is called *lamination.*

In other words, maximal non-crossing sets of strings are in the tiling view laminations.

Note that according to Definition 11, a string determined by a walk consisting only of arrows does not cross with any string. Hence, such infinite strings always belong to each maximal non-crossing set.

In the tiling view, a sequence of arrows of the same direction without relation are viewed as curves around some marked point. For example, all such strings of A_2 are shown in Figure 1.12.

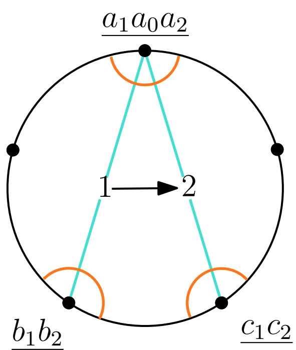


Figure 1.12: Strings consisting only from arrows for A_2

1.4 Modules

In this section we will introduce needed notions of string and band modules. This part is taken from section 5.2 in Chan and Demonet [2020]. From now on, we will fix a gentle quiver (Q, R) and a blossoming (Q', R'). We denote by Λ the gentle algebra associated to (Q, R) and $\mathsf{f.l.}\,\Lambda$ the category of finite-dimensional right Λ-modules.

Definition 15 (String module)**.** Let $\gamma = \cdots a_i a_{i+1} \cdots$ be a walk in (Q, R) with indexing interval $I \subseteq \mathbb{Z}$. By a position in γ we mean a number i in the set I, or in the set $I \cup \{\max(\mathrm{I}) + 1\}$ if I is bounded above (because the last arrow ends in some vertex).

For a position i, we denote by $v_i \in Q_0$ the corresponding vertex. Consider the vector space $\bigoplus_i \mathbb{K}x_i$, where i goes through all positions i. We define an action of Λ on it as follows:

- for $v \in Q_0$, $x_i e_v = x_i \delta_{v,v_i}$, where $\delta_{v,v_i} = 1$ if $v = v_i$ and 0 otherwise, where e_v denoted the stationary path at v;
- for $q \in Q_1$ and a position i,

$$x_i q = \begin{cases} x_{i+1} & \text{if } q = a_i, \\ x_{i-1} & \text{if } q = a_i^-, \\ 0 & \text{else.} \end{cases}$$

The resulting Λ-module is isomorphic to the one defined by γ^-, and so it makes sense called such a module *string module*. We will denote it by $X(\underline{\gamma})$.

Note that if γ is a stationary walk, then $X(\underline{\gamma})$ is just the corresponding simple module. Clearly, $X(\underline{\gamma})$ is finite-dimensional if and only if $\underline{\gamma}$ is bounded.

Now let γ be a periodic walk that contains at least one arrow and one inverse of an arrow in its letters. A subwalk δ of γ is said to be the *primitive period* of γ if $\gamma = {}^{\infty}\delta^{\infty}$ and there is no other subwalk ϵ of γ such that $\epsilon^r = \delta$ for some $r > 1$. We fix such a γ with an indexing set I, and δ its primitive period with an indexing set $J \subset I$.

Definition 16 (Band module)**.** Let σ be the operation on γ given by shifting the letters to the right by $|J|$ places. Abusing notation, we also denoted by σ the induced *period-shifting* automorphism

$$\begin{aligned} \sigma : X(\underline{\gamma}) &\to X(\underline{\gamma}) \\ x_i &\mapsto x_{i-|J|}. \end{aligned}$$

This gives $X(\underline{\gamma})$ the structure of a $\mathbb{K}[T, T^{-1}]$-Λ-bimodule where T acts as σ. Therefore, for a finite-dimensional $\mathbb{K}[T, T^{-1}]$-module M, we can define a Λ-module

$$X_M(\underline{\gamma}) = M \otimes_{\mathbb{K}[T,T^{-1}]} X(\underline{\gamma}).$$

If M is indecomposable, we call $X_M(\underline{\gamma})$ a *band module.*

We will describe the action of Λ on a band module $X_M(\underline{\gamma})$ more explicitly. By construction, $X_M(\underline{\gamma})$ is free of rank $|J|$ over $\mathbb{K}[T, T^{-1}]$ with *canonical basis* $\{x_j\}_{j \in J}$, i.e. we have $X_M(\underline{\gamma}) \cong \bigoplus_{j \in J} M x_j$ as a vector space. Then Λ-module structure is given by

- for $v \in Q_0$ and $m \in M$, $mx_i e_v = mx_i \delta_{v,v_i}$;

- for $q \in Q_1$, $i \in J$ and $m \in M$,

$$mx_i q = \begin{cases} mx_{i+1} & \text{if } i < \max J \text{ and } q = a_i, \\ mx_{i-1} & \text{if } i > \min J \text{ and } q = a_{i-1}^-, \\ (m \cdot T)x_{\min J} & \text{if } i = \max J \text{ and } q = a_i, \\ (m \cdot T^{-1})x_{\max J} & \text{if } i = \min J \text{ and } q = a_{i-1}^-, \\ 0 & \text{else.} \end{cases}$$

Note that $X_M(\underline{\gamma}) \cong X_{\iota(M)}(\underline{\gamma}^-)$, where ι is the $\mathbb{K}[T, T^{-1}]$-automorphism on M given by swapping the action of T and T^{-1}. In particular, $X_M(\underline{\gamma}) \cong X_{M'}(\underline{\gamma}')$ if and only if

1. $M \cong M'$ or $M \cong \iota(M')$, and

2. γ' can be obtained by γ via shifting letters, or reversing the whole walk, or both.

Let us look at finite dimensional $\mathbb{K}[T, T^{-1}]$-modules in the case when $\mathbb{K}$ is algebraically closed. They are representations of the quiver which is one vertex with a loop. Indecomposable are those of the form $M_{n,k} = (\mathbb{K}^n, J_{n,k})$, where $J_{n,k}$ is a Jordan block with dimension $n \in \mathbb{N}$ and eigenvalue $k \in \mathbb{K} \setminus 0$.

At the end of this section, we define a torsion class of modules.

Definition 17 (Torsion class of modules)**.** The full subcategory $\mathscr{T}$ of $\mathsf{f.l.}\,\Lambda$ is called a *torsion class* if it is closed under factors and extensions. In other words, for any short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ in $\mathsf{f.l.}\,\Lambda$ holds that if M is an element of $\mathscr{T}$, also N is, and if L, N are elements, then also M is.

1.5 Bijection between sets of strings and torsion classes

Now we will go back to strings and according to sections 4.1, 4.2, 4.7, 6.1 in Chan and Demonet [2020], we will introduce notions needed for stating the main results of this article.

Definition 18 (Factor and extension of strings)**.** 1. For a string $\underline{\gamma}$ in (Q', R') a *factor of* $\underline{\gamma}$ is any string $\underline{\omega}$ such that $\gamma = \omega$ or there exists a decomposition $\gamma = \gamma_1 a^- \omega a' \gamma_2$ or $\gamma = \gamma_1 a^- \omega$ or $\gamma = \omega a' \gamma_2$, with $a, a' \in Q_1'$. A factor is illustrated in Figure 1.13.

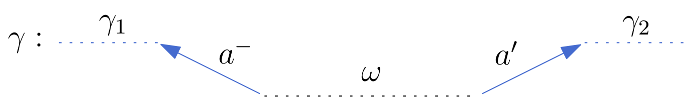


Figure 1.13: Illustration of a factor

2. Let $\underline{\gamma}$ and $\underline{\delta}$ be strings and a be an arrow in Q_1 such that $\underline{\gamma a \delta}$ is also a string. We say that $\underline{\gamma a \delta}$ is an *extension of* $\underline{\delta}$ *by* $\underline{\gamma}$.

Remark. Let us look at the relation of notions factor and extension for modules and strings. In notation from Definition 18.1, there are arrows towards $\underline{\gamma}_1, \underline{\gamma}_2$, so $X(\underline{\gamma}_1), X(\underline{\gamma}_2)$ are submodules of $X(\underline{\gamma})$. That is, $X(\underline{\omega}) = X(\underline{\gamma})/(X(\underline{\gamma}_1) \bigoplus X(\underline{\gamma}_1))$ is a factor module of $X(\gamma)$. On the other hand, there are factor modules of $X(\gamma)$ which are not string modules of some factor of γ.

Similarly, with the notation from Definition 18.2, $X(\underline{\delta})$ is a submodule and $X(\underline{\gamma})$ a factor module of $X(\underline{\gamma a \delta})$. Moreover, there is a short exact sequence $0 \to X(\underline{\delta}) \to X(\underline{\gamma a \delta}) \to X(\underline{\gamma}) \to 0$.

Definition 19 (Torsion set of strings)**.** A set $\mathscr{T}$ of strings in (Q', R') is called a *torsion set* of (Q, R) if it is closed under factors and extensions of strings. If, moreover, $\mathscr{T}$ consists only of confined strings, we say that $\mathscr{T}$ is a *confined torsion set.* We denote by $\mathsf{tors}(Q, R)$ the set of confined torsion sets in (Q', R').

Let $\mathscr{S}$ be a set of strings. We denote by $\mathsf{fin}(\mathscr{S})$ the confined part of $\mathscr{S}$, i.e. the set of all confined strings in $\mathscr{S}$. We define

$$\mathsf{T}^\infty(\mathscr{S}) = \bigcap_{\text{torsion set } \mathscr{T} \supseteq \mathscr{S}} \mathscr{T} \quad \text{and} \quad \mathsf{T}(\mathscr{S}) = \mathsf{fin}(\mathsf{T}^\infty(\mathscr{S})).$$

Consider now a confined torsion set $\mathscr{T} \in \mathsf{tors}(Q, R)$ and define

$$\begin{aligned} L(\mathscr{T}) &= \{\underline{\gamma} \text{ string in } (Q', R') \mid \mathsf{fin}\{\text{all factors of } \underline{\gamma}\} \subseteq \mathscr{T}\} \\ \text{and } G(\mathscr{T}) &= \{\underline{\gamma} \in L(\mathscr{T}) \text{ infinite} \mid c^+(\underline{\delta}, \underline{\gamma}) = \emptyset \text{ for all } \underline{\delta} \in L(\mathscr{T})\}. \end{aligned}$$

Further, for a set of strings $\mathscr{S}$, we define

$$\mathsf{Fac}^\infty(\mathscr{S}) = \{\text{factors of } \underline{\gamma} \mid \underline{\gamma} \in \mathscr{S}\} \quad \text{and} \quad \mathsf{Fac}(\mathscr{S}) = \mathsf{fin}(\mathsf{Fac}^\infty(\mathscr{S})).$$

Now we can state the first important result of Chan and Demonet [2020].

Theorem 1. *Let (Q, R) be a gentle quiver.*

1. *The set $\mathsf{maxNC}(Q, R)$ of all maximal non-crossing sets of strings is in one-to-one correspondence with the set $\mathsf{tors}(Q, R)$ of confined torsion sets given by*

$$\begin{aligned} \mathsf{maxNC}(Q, R) &\longleftrightarrow \mathsf{tors}(Q, R) \\ \mathscr{S} &\longrightarrow \mathsf{T}(\mathscr{S}) \\ G(\mathscr{T}) &\longleftarrow \mathscr{T} \end{aligned}$$

2. *Let $\geq$ be the partial order on $\mathsf{maxNC}(Q, R)$ induced by the bijection in 1. Then we have $\mathscr{S} \geq \mathscr{S}'$ if and only if $c^+(\mathscr{S}', \mathscr{S}) = \emptyset$.*

Proof. See Theorem 4.11. in Chan and Demonet [2020]. □

Before stating the second important result we need to introduce some more terminology.

For a set $\mathscr{S}$ of strings, we denote by $\mathscr{S}^p$ the subset of bands in $\mathscr{S}$.

Definition 20 (Parametrization)**.** A (B-)*parametrized* set of strings is a pair $(\mathscr{S}, \boldsymbol{\lambda})$ consisting of a set $\mathscr{S}$ of strings and a (B-)*parametrization* map $\boldsymbol{\lambda} : \mathscr{S}^p \to 2^B$, where 2^B is the power set of B. Denote by $\mathsf{maxNC}_B(Q, R)$ the set of parametrized maximal sets of non-crossing (infinite) strings, i.e.

$$\mathsf{maxNC}_B(Q, R) = \{(\mathscr{S}, \boldsymbol{\lambda}) \text{ } B\text{-parametrized set } \mid \mathscr{S} \in \mathsf{maxNC}(Q, R)\}.$$

We define a relation $\geq$ on $\mathsf{maxNC}_B(Q, R)$ given by $(\mathscr{S}, \boldsymbol{\lambda}) \geq (\mathscr{S}', \boldsymbol{\lambda}')$ if $\mathscr{S} \geq \mathscr{S}'$ (i.e. $c^+(\underline{\gamma}, \underline{\delta}) = \emptyset$ for all $\underline{\gamma} \in \mathscr{S}'$ and $\underline{\delta} \in \mathscr{S}$) and for any $\underline{\gamma} \in \mathscr{S}^p \cap \mathscr{S}'^p$, we have $\boldsymbol{\lambda}(\underline{\gamma}) \supseteq \boldsymbol{\lambda}'(\underline{\gamma})$.

Let us denote by $B = \mathsf{irr}\,\mathbb{K}[T, T^{-1}]$ a set of representative of isoclasses of simple $\mathbb{K}[T, T^{-1}]$-modules in the category $\operatorname{mod} \mathbb{K}[T, T^{-1}]$ of finitely generated (left) $\mathbb{K}[T, T^{-1}]$-modules. For a parametrized set $(\mathscr{S}, \boldsymbol{\lambda} : \mathscr{S}^p \to 2^B)$ of infinite strings, we define

$$\begin{aligned} X(\mathscr{S}) &= \{X(\underline{\gamma}) \mid \underline{\gamma} \in \mathscr{S}\}, \\ X_{\boldsymbol{\lambda}}(\mathscr{S}^p) &= \{X_M(\underline{\gamma}) \mid \underline{\gamma} \in \mathscr{S}^p, M \in \boldsymbol{\lambda}(\underline{\gamma})\}. \end{aligned}$$

We will denote

$$\mathsf{maxNC}_{\mathbb{K}}(Q, R) = \mathsf{maxNC}_{\mathsf{irr}\,\mathbb{K}[T,T^{-1}]}(Q, R).$$

Theorem 2. *There is a complete lattice isomorphism between* $\mathsf{maxNC}_{\mathbb{K}}(Q, R)$*, the lattice of maximal parametrized non-crossing sets of infinite strings, and* $\mathsf{T}(\mathsf{f.l.}\,\Lambda)$*, the lattice of torsion classes in* $\mathsf{f.l.}\,\Lambda$*, via*

$$\begin{aligned} \mathscr{T}_\Lambda : \mathsf{maxNC}_{\mathbb{K}}(Q, R) &\xrightarrow{\sim} \mathsf{tors}(\mathsf{f.l.}\,\Lambda) \\ (\mathscr{S}, \boldsymbol{\lambda}) &\mapsto \mathscr{T}_\Lambda(\mathscr{S}, \boldsymbol{\lambda}) = \mathsf{T}(X(\mathsf{Fac}(\mathscr{S})) \cup X_{\boldsymbol{\lambda}}(\mathscr{S}^p). \end{aligned}$$

Proof. See Theorem 6.1. in Chan and Demonet [2020]. □

2. Examples

Now we will illustrate Theorem 2 with a few examples – quiver A_2 in Section 2.1, the Kronecker quiver in Section 2.3 and in short we will look at quivers A_n consisting of n arrows and no relations in Section 2.2. Lattices of maximal non-crossing sets of strings for A_2 and the Kronecker quiver are (almost without explanation) shown also in Chan and Demonet [2020] in Examples 4.12. and 4.13, respectively.

From this chapter on, we will write strings without underlying. So by writing γ or γ^- we mean string $\underset{\sim}{\gamma}$.

2.1 The simplest algebra

We will start with the quiver algebra of quiver A_2. For this algebra, it is easy to find all torsion classes without our theory. Therefore, we determine the lattice of torsion classes in this straightforward way, and then we show that using Theorem 2 we obtain the same result.

Torsion classes are from definition determined by indecomposable modules. We have up to isomorphism precisely three indecomposable modules over our algebra. They are $\mathbb{K} \to 0, 0 \to \mathbb{K}, \mathbb{K} \xrightarrow{1} \mathbb{K}$.

There is always an empty torsion class and a torsion class containing all modules. Obviously, each simple module generates a torsion class containing no other indecomposable module. This gives us torsion classes $\mathsf{T}(\mathbb{K} \to 0)$ and $\mathsf{T}(0 \to \mathbb{K})$.

Torsion class $\mathsf{T}(\mathbb{K} \xrightarrow{1} \mathbb{K})$ contains $\mathbb{K} \to 0$ because it is a factor by $0 \to \mathbb{K}$. On the other hand, this torsion class does not contain $0 \to \mathbb{K}$. It means that we have found the fifth torsion class.

There is no other torsion class – $\mathbb{K} \xrightarrow{1} \mathbb{K}$ is an extension of the other two indecomposable modules. Together we get the lattice of torsion classes 2.1.

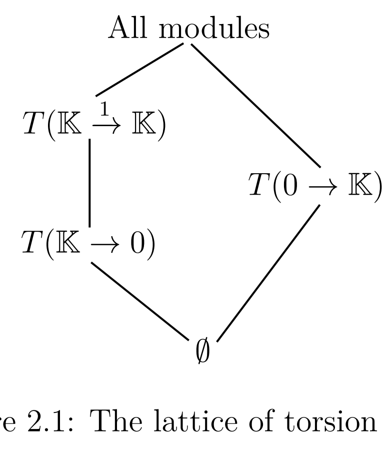


Figure 2.1: The lattice of torsion classes

Now we want to reach the same result using Theorem 2. There is no band, so the application of the theorem is relatively easy. Torsion classes are generated by string modules determined by factors of a maximal non-crossing sets of infinite strings.

Without strings consisting of all arrows in the same direction there are exactly five different infinite non-self-crossing strings. These strings are $a_1 a_0 c_1^-$, $a_1 b_1^-$, $b_2^- a_0 a_2$, $b_2^- a_0 c_1^-$, $c_2^- a_2$. Corresponding accordions are drawn in Figure 2.2, each in a different colour. The remaining procedure is to look at possible maximal non-crossing sets and their lattice. Then we will find all factors of these sets, use Theorem 2 and compare the result with the lattice in Figure 2.1.

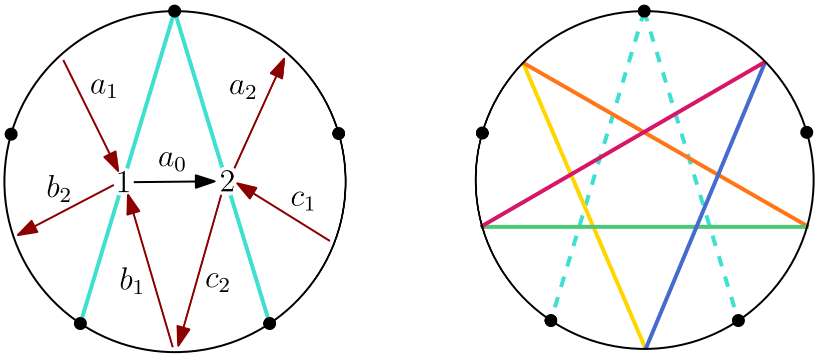


Figure 2.2: Maximal infinite strings

Let us notice that there are exactly five maximal non-crossing sets. Each of them consists of a pair of strings from the previous paragraph, which have a common point in the blossoming. They are shown in Figure 2.3.

Mutual crossings of all infinite non-self-crossing strings are listed in Table 2.1. The entry in the ith row and jth column says if there is a positive crossing from some string in the ith set to some string in the jth set. If there is, the entry is a substring at which they cross.

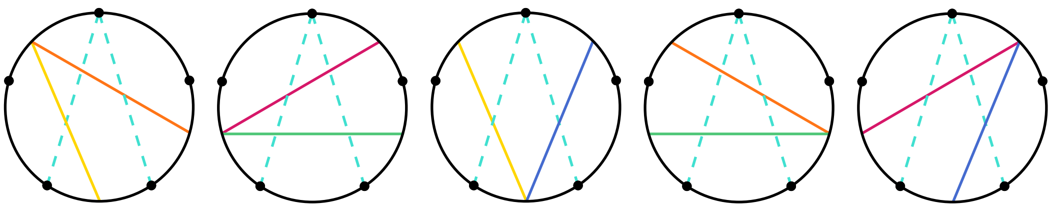


Figure 2.3: All maximal non-crossing sets of infinite strings

Hence, according to Theorem 1, the lattice of laminations looks like in Figure 2.4.

Finite factors of infinite non-self-crossing strings are as follows:

$$\mathsf{Fac}(a_1 a_0 c_1^-) = \emptyset$$
$$\mathsf{Fac}(a_1 b_1^-) = \emptyset$$
$$\mathsf{Fac}(b_2^- a_0 c_1^-) = \{1_1\}$$
$$\mathsf{Fac}(b_2^- a_0 a_2)c = \{1_1, a_0\}$$
$$\mathsf{Fac}(c_2^- a_2) = \{1_2\}.$$

These factors give string modules

$$X(1_1) = \mathbb{K} \to 0, X(1_2) = 0 \to \mathbb{K}, X(a_0) = \mathbb{K} \xrightarrow{1} \mathbb{K}.$$

Table 2.1: Table of positive crossings

	$a_1a_0c_1^-$, $a_1b_1^-$	$b_2^-a_0a_2$, $b_2^-a_0c_1^-$	$a_1b_1^-$, $c_2^-a_2$	$a_1a_0c_1^-$, $b_2^-a_0c_1^-$	$b_2^-a_0a_2$, $c_2^-a_2$
$a_1a_0c_1^-$, $a_1b_1^-$	-	-	-	-	-
$b_2^-a_0a_2$, $b_2^-a_0c_1^-$	a_0, 1_1,	-	1_1	a_0	-
$a_1b_1^-$, $c_2^-a_2$	1_2	1_2	-	1_2	-
$a_1a_0c_1^-$, $b_2^-a_0c_1^-$	1_1	-	1_1	-	-
$b_2^-a_0a_2$, $c_2^-a_2$	a_0, 1_1, 1_2	1_2	1_1	a_0, 1_2	-

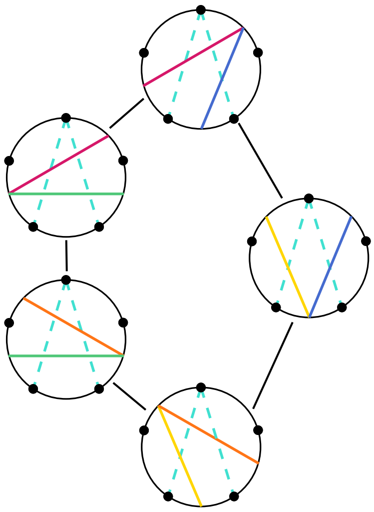


Figure 2.4: The lattice of laminations

Hence, from Theorem 2 we get:

$$\begin{aligned}
\mathscr{T}_\Lambda(a_1a_0c_1^-, a_1b_1^-) &= \mathsf{T}(\emptyset) = \emptyset \\
\mathscr{T}_\Lambda(a_1b_1^-, c_2^-a_2) &= \mathsf{T}(X(1_2)) = \mathsf{T}(0 \to \mathbb{K}) \\
\mathscr{T}_\Lambda(a_1a_0c_1^-, b_2^-a_0c_1^-) &= \mathsf{T}(X(1_1)) = \mathsf{T}(\mathbb{K} \to 0) \\
\mathscr{T}_\Lambda(b_2^-a_0a_2, b_2^-a_0c_1^-) &= \mathsf{T}(\{X(1_1), X(a_0)\}) = \mathsf{T}(\{\mathbb{K} \to 0, \mathbb{K} \xrightarrow{1} \mathbb{K}\}) = \mathsf{T}(\mathbb{K} \xrightarrow{1} \mathbb{K}) \\
\mathscr{T}_\Lambda(b_2^-a_0a_2, c_2^-a_2) &= \mathsf{T}(\{X(1_1), X(1_2), X(a_0)\}) = \mathsf{T}(\{0 \to \mathbb{K}, \mathbb{K} \to 0, \mathbb{K} \xrightarrow{1} \mathbb{K}\}) \\
&= \text{All modules} .
\end{aligned}$$

We can see that $\mathscr{T}_\Lambda$ maps lattice 2.4 exactly to lattice 2.1.

2.2 A_n

Further examples of gentle quivers consist of path going through $n \in \mathbb{N}$ vertices without any relation. In other words, they are of the form

$$1 \to 2 \to 3 \to \cdots \to n{-}1 \to n.$$

It is usual to denote these quivers A_n.

The lattice of torsion classes of modules over these quiver algebras can be found in the same way using maximal strings as in the example in previous

Subsection 2.1. We will not show the search of the lattice for these analogous cases, but we will calculate how many torsion classes these algebras have.

We take a tiling consisting of a disk, $n+3$ marked points on the boundary, and n arcs between them, all starting at one point. This tiling is shown in Figure 2.5. Let us note that its associated quiver is exactly A_n.

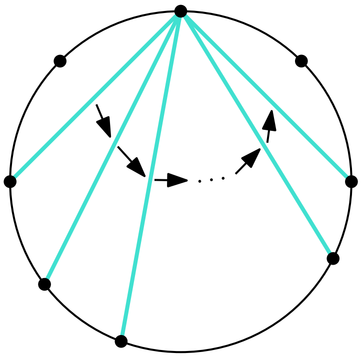


Figure 2.5: Tiling for A_n

Every accordion has to start and end in one of the $n+3$ segments of the disk boundary. We will add a new point to each of these parts. These points form some $(n+3)$-gon. Every diagonal of this $(n+3)$-gon gives us a different accordion. On the other hand, every accordion is homotopic to one of the diagonals of this $(n+3)$-gon.

It implies that the laminations correspond to the maximal sets of non-crossing diagonals, that is triangulations of the $(n+3)$-gon. It is well-known that the number of triangulations of an m-gon is equal to the Catalan number C_{m-2}, where $C_m = \frac{1}{m+1}\binom{2m}{m}$. Hence, the number of laminations of the tiling algebra corresponding to A_n is equal to C_{n+1}.

There is no band, so the lattice of accordions is isomorphic to the lattice of torsion classes according to Theorem 2. Thus, the number of torsion classes of A_n is equal to C_{n+1}.

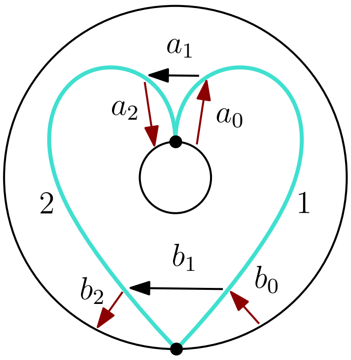
2.3 Kronecker algebra

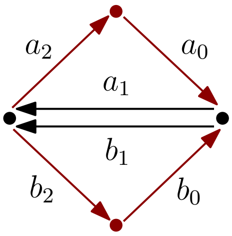
Let us show a slightly more complicated example. Compared to the previous examples, there is an infinite number of infinite non-self-crossing strings. Moreover, one of them is a band. Hence, in the application of Theorem 2 we must work with a band module. The lattice of laminations is infinite and not isomorphic to the lattice of torsion classes.

In this section $\mathbb{K}$ is an algebraically closed field, (Q, R) denotes the Kronecker quiver and (Q', R') its blossoming. We denote vertices and arrows as in Figure 2.6.

2.3.1 Non-self-crossing strings

We need to find all infinite non-self-crossing strings. We start with infinite strings which are bounded, e.g., they start and end in vertices in $Q'_0 \setminus Q_0$, which is just a two-element set.





$R' = \{a_0 b_1, b_0 a_1, a_1 b_2, b_1 a_2, a_2 a_0, b_2 b_0\}$

Figure 2.6: Kronecker algebra with its tiling

Because of relations, we have only four types of bounded infinite strings. They are $a_0(a_1 b_1^-)^k b_0^-$, $b_2^-(b_1^- a_1)^k a_2$, $a_0(a_1 b_1^-)^k a_1 a_2$, $b_2^-(b_1^- a_1)^k b_1^- b_0^-$, $k \in \mathbb{N}_0$. Strings of the second two types are self-crossing for $k > 0$ because they have self-crossings at a_1 and b_1, respectively. For $k = 0$ we get in the second two types strings $a_0 a_1 a_2$ and $b_0 b_1 b_2$; that is, only strings consisting of arrows with the same orientation.

On the other hand, we will show that the strings of the first two types are non-self-crossing. We denote them

$$\alpha_k = a_0(a_1 b_1^-)^k b_0^-, \beta_k = b_2^-(b_1^- a_1)^k a_2.$$

Of course, string $(a_1 b_1^-)^k$ is non-self-crossing. Suppose that α_k is self-crossing and it crosses itself at some substring ω. Thus, a_0 or b_0^- must neighbour ω. Because $(a_1 b_1^-)^k$ consists only of two alternating arrows, both a_0 and b_0^- neighbour ω, otherwise, there would not be enough different arrows to get a crossing. They are both oriented towards ω, so ω must be the whole $(a_1 b_1^-)^k$, which is not possible.

Now we look at these types of strings from the view of accordions. The cases for $k = 1$ are shown, in the order in which they are listed above, in Figure 2.7. For general k, the first two types correspond to spirals that start at the outer boundary, end at the inner boundary, turn around the middle k-times, and the first turn clockwise and the second counter-clockwise.

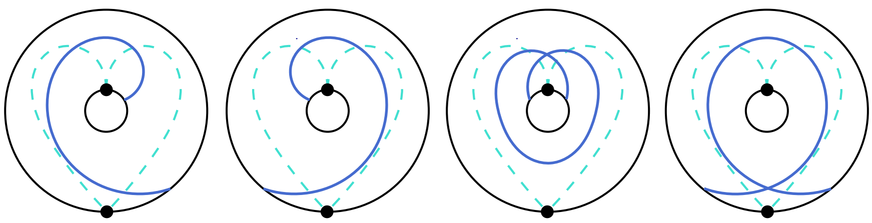


Figure 2.7: Different types of infinite strings for $k = 1$

Let us note that the only two-sided unbounded string is ${}^\infty(a_1 b_1^-)^\infty$, we denote it by γ. We also have four different one-side bounded infinite strings. We can start at one of the two vertices in $Q_0' \setminus Q_0$ and continue by one of the two arrows. Then we have only one way how to continue with an infinite cycle. Namely, we have strings

$$\alpha_\infty^o = {}^\infty(a_1 b_1^-) b_0^-, \beta_\infty^o = b_2^- b_1^- (a_1 b_1^-)^\infty, \alpha_\infty^i = a_0(a_1 b_1^-)^\infty, \beta_\infty^i = {}^\infty(a_1 b_1^-) a_1 a_2.$$

Of course, they are non-self-crossing.

Again, we will show accordions that correspond to these strings. We can imagine the only two-sided unbounded string as a closed curve around the middle. The one-side unbounded strings correspond to spirals which can start at outer or inner boundary and rotate clockwise or counter-clockwise. These accordions are illustrated in Figure 2.8.

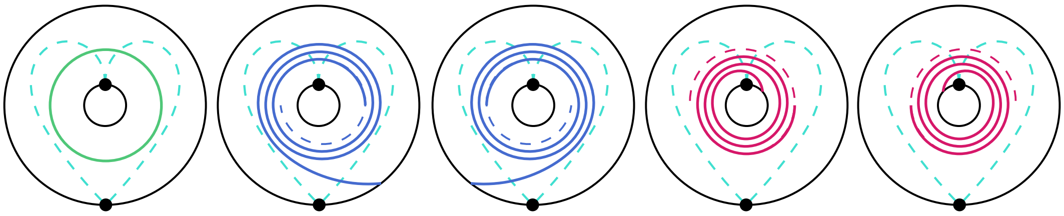


Figure 2.8: Unbounded strings $\gamma, \alpha^o_\infty, \beta^o_\infty, \alpha^i_\infty, \beta^i_\infty$ respectively

2.3.2 Maximal non-crossing sets

In the next step, we will find all maximal non-crossing sets of infinite strings or, in the tiling algebra view, all laminations and their lattice.

First, let us notice that there are positive crossings from β_l to α_k for all $l, k \in \mathbb{N}$:

$$\begin{array}{rr} \beta_l = & b_2^- \\ \alpha_k = & a_0(a_1 b_1^-)^{k-1} a_1 \end{array} \Bigg\rangle b_1^- \Bigg\langle \begin{array}{l} a_1(b_1^- a_1)^{l-1} a_2 \\ b_0^- \end{array} .$$

Also, there is a positive crossing from β_0 to all $\alpha_k, k \in \mathbb{N}$ and from all $\beta_k, k \in \mathbb{N}$ to α_0 at 1_2, respective 1_1. Strings α_0, β_0 are obviously non-crossing.

On the other hand, there is no positive crossing from α_k to $\beta_l, k, l \in \mathbb{N}_0$. Their substrings of the form $(a_1 b_1^-)^n$ are non-crossing. Hence, the crossing would have to use arrows from thr blossoming. These arrows are oriented from the middle for β_l and to the middle for α_k. Therefore the definition of positive crossing cannot be satisfied.

Further, we have crossing from β_l to all unbounded strings, these crossings are listed in 2.1. In a similar way, we have crossings from all unbounded strings to α_k. Again, from the same reason as above there are no converse crossings.

$$\begin{array}{rr} \beta_l = & b_2^- \\ \gamma = & {}^\infty(b_1^- a_1) \end{array} \Bigg\rangle (b_1^- a_1)^l \Bigg\langle \begin{array}{l} a_2 \\ (b_1^- a_1)^\infty \end{array} \qquad \begin{array}{rr} \beta_l = & b_2^- \\ \alpha^i_\infty = & a_0 a_1 \end{array} \Bigg\rangle (b_1^- a_1)^l \Bigg\langle \begin{array}{l} a_2 \\ (b_1^- a_1)^\infty \end{array}$$

$$\begin{array}{rr} \beta_l = & b_2^- \\ \alpha^o_\infty = & {}^\infty(b_1^- a_1) \end{array} \Bigg\rangle (b_1^- a_1)^l \Bigg\langle \begin{array}{l} a_2 \\ b_1^- b_0^- \end{array} \qquad \begin{array}{rr} \beta_l = & b_2^- \\ \beta^i_\infty = & {}^\infty(b_1^- a_1) \end{array} \Bigg\rangle (b_1^- a_1)^l \Bigg\langle \begin{array}{l} a_2 \\ b_1^- a_1 a_2 \end{array} \tag{2.1}$$

$$\begin{array}{rr} \beta_l = & b_2^- \\ \beta^o_\infty = & b_2^- b_1^- a_1 \end{array} \Bigg\rangle (b_1^- a_1)^l \Bigg\langle \begin{array}{l} a_2 \\ (b_1^- a_1)^\infty \end{array}$$

The previous implies that the only candidates for forming a non-crossing set with β_l are β_k's. Actually, there is a positive crossing from β_l to all $\beta_k, k \geq l + 2$:

$$\begin{array}{rr} \beta_l = & b_2^- \\ \beta_k = & b_2^- b_1^- a_1 \end{array} \Bigg\rangle (b_1^- a_1)^l \Bigg\langle \begin{array}{l} a_2 \\ (b_1^- a_1)^{k-l-1} a_2 \end{array} .$$

Again, there are no converse crossings. Let us note that strings β_k, β_{k+1} make a non-crossing set. Moreover, this non-crossing set has to be maximal.

Similarly, α_l forms non-crossing sets only with α_{l-1} and α_{l+1}. In contrast to β_l there is a positive crossing in the opposite direction, from all $\alpha_k, k \geq l+2$ to α_l:

$$\begin{array}{rr} \alpha_k = & a_2 a_1 b_1^- \\ \alpha_l = & a_0 \end{array} \Big\rangle (a_1 b_1^-)^l \Big\langle \begin{array}{l} (a_1 b_1^-)^{k-l-1} b_0^- \\ b_0^- \end{array} .$$

It remains to find non-crossing sets composed of unbounded strings. First, let us notice that γ form a non-crossing set with each one-sided unbounded strings because from one side there are the same, so there is no space to crossing. The same reason holds for pairs $\alpha_\infty^o, \beta_\infty^i$ and $\beta_\infty^o, \alpha_\infty^i$. Non-crossing are also pairs $\alpha_\infty^o, \alpha_\infty^i$ and $\beta_\infty^o, \beta_\infty^i$, because both strings in each pair consist only of arrows to the middle, respectively arrow from the middle, and the cycle.

The rest two pairs $\alpha_\infty^o, \beta_\infty^o$ and $\alpha_\infty^i, \beta_\infty^i$ have only crossings from β's to α's, similarly as in the case for α_k and β_l.

From this subsection we can conclude that we have exactly four types of maximal-non crossing sets:

1. $\{\alpha_k, \alpha_{k+1}\}, k \in \mathbb{N}_0$
2. $\{\beta_k, \beta_{k+1}\}, k \in \mathbb{N}_0$
3. $\{\alpha_0, \beta_0\}$
4. $\{\gamma, \delta_\infty^o, \epsilon_\infty^i\}, \delta, \epsilon \in \{\alpha, \beta\}$.

Moreover, we have reasoned the partial order of these sets, where we take partial order as in Theorem 1. Hence, we can draw the lattice of corresponding laminations. It is illustrated in Figure 2.9. In the figure our four types of torsion classes from the previous paragraph correspond to the right lower part, the right upper part, the only lamination on the left, and to the four laminations right in the middle, respectively.

Another possibility to get this lattice would be not to use the string combinatorics view and find the intersections of accordions.

2.3.3 Torsion classes

The last step to find the lattice of torsion classes of the Kronecker algebra is to use the isomorphism from Theorem 2.

We start with finding factors of our strings. We will work in the order in which we listed the four types of torsion classes in the end of Subsection 2.3.2.

1. From the directions of the arrows follows that the factors of the string $\alpha_k = a_0(a_1 b_1^-)^k b_0^-$ are exactly substrings between some occurrence of b_1^- and a_1.

$$\mathsf{Fac}(\alpha_k) = \mathsf{Fac}(a_0(a_1 b_1^-)^k b_0^-) = \{1_1, a_1 b_1^-, \ldots, (a_1 b_1^-)^{k-2}\}$$
$$\mathsf{Fac}(\{\alpha_k, \alpha_{k+1}\}) = \{1_1, a_1 b_1^-, \ldots, (a_1 b_1^-)^{k-1}\}$$

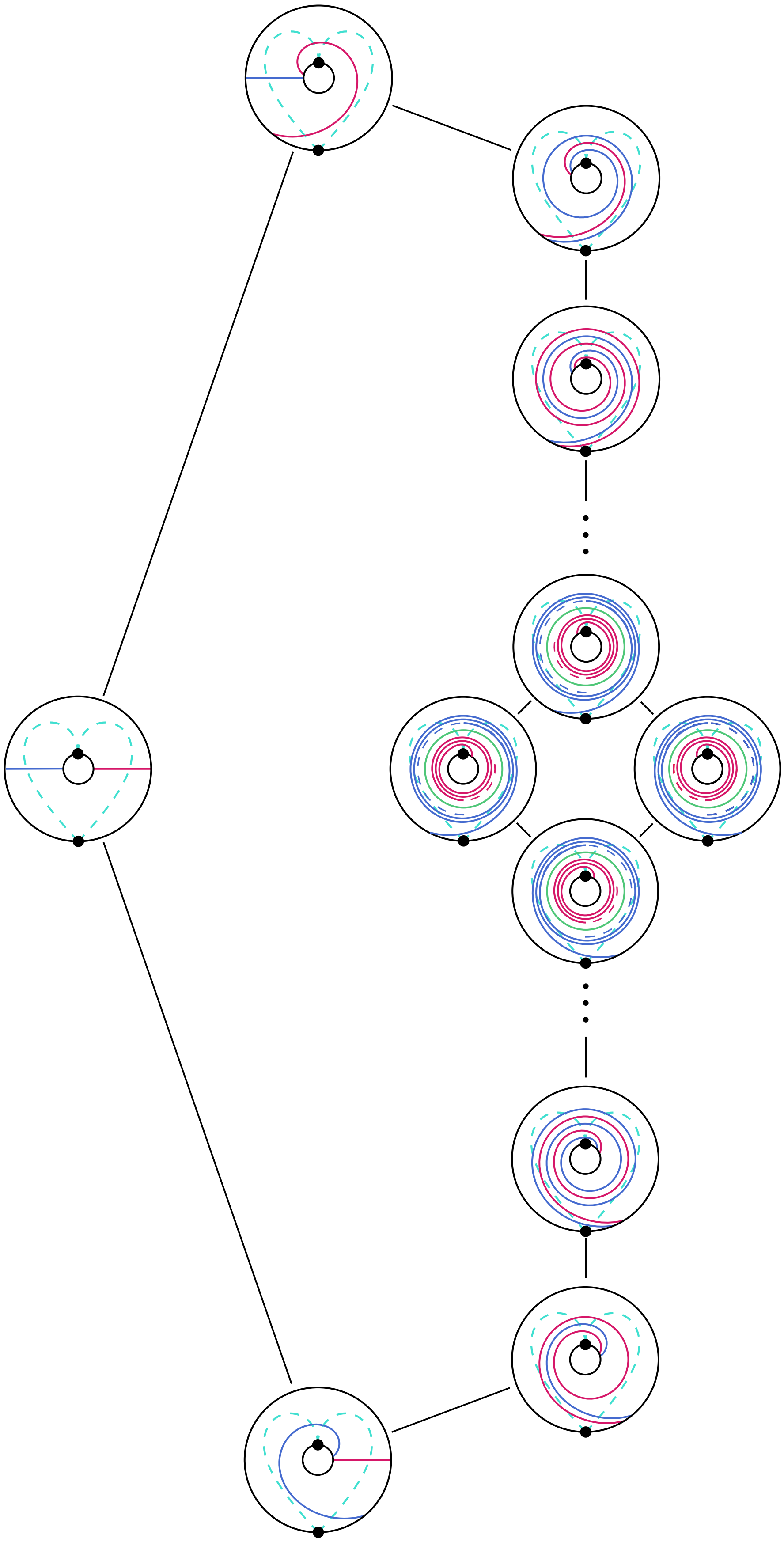


Figure 2.9: The lattice of all laminations for Kronecker algebra

2. The situation for $\beta_k = b_2^-(b_1^- a_1)^k a_2, k \in \mathbb{N}$ is a bit more complicated. If we do not use arrows from the blossoming, we get the same factors as for α_{k-1}. If we use from the blossoming only the first arrow b_2^-, factors are exactly substrings succeeding by a_1. That is, we get factors of the form $(b_1^- a_1)^n b_1^-, n < k$. On the other hand, if we use from the blossoming only a_2, we get factors of the form $a_1(b_1^- a_1)^n, n < k$. If we use both arrows from the blossoming, we add as a factor $(b_1^- a_1)^k$. Together we get for $k \in \mathbb{N}$:

$$\begin{aligned}\mathsf{Fac}(\beta_k) =& \{1_1, a_1 b_1^-, \dots, (a_1 b_1^-)^{k-1}, b_1^-, \dots, (b_1^- a_1)^{k-1} b_1^-,\\ & a_1, \dots, a_1(b_1^- a_1)^{k-1}, (b_1^- a_1)^k\}.\\ \mathsf{Fac}(\{\beta_k, \beta_{k+1}\}) =& \{1_1, a_1 b_1^-, \dots, (a_1 b_1^-)^k, b_1^-, \dots, (b_1^- a_1)^k b_1^-,\\ & a_1, \dots, a_1(b_1^- a_1)^k, (b_1^- a_1)^k, (b_1^- a_1)^{k+1}\}\\ \mathsf{Fac}(\{\beta_0, \beta_1\}) =& \{1_2, 1_1, b_1^-, a_1, b_1^- a_1\}.\end{aligned}$$

3. Factors of $\{\alpha_0, \beta_0\}$ are $\mathsf{Fac}(\{\alpha_0, \beta_0\}) = \{1_2\}$.

4. It remains to find factors of unbounded strings. All of them contain the infinite cycle consisting of alternating b_1^- and a_1. Hence, they all have factors $1_1, (a_1 b_1^-)^n, n \in \mathbb{N}$. Of course, γ has no other factors. Further, $\alpha_\infty^o, \alpha_\infty^i$ contains without the infinite cycle only one arrow with direction to the middle. It means that they do not add any other factor. On the other hand, β_∞^o starts with an arrow b_2^-, so it adds factors $(b_1^- a_1)^n b_1^-, n \in \mathbb{N}_0$. Similarly, β_∞^i adds factors $a_1(b_1^- a_1)^n, n \in \mathbb{N}_0$.

According to Definition 15 we can find string modules corresponding to the factors. We denote

$$I_0 = X_{1_1} = 0 \leftleftarrows \mathbb{K} \qquad P_i = X_{(b_1^- a_1)^i} = \mathbb{K}^{i+1} \underset{(0,id)^T}{\overset{(id,0)^T}{\leftleftarrows}} \mathbb{K}^i, i \in \mathbb{N}$$

$$I_i = X_{(a_1 b_1^-)^i} = \mathbb{K}^i \underset{(0,id)}{\overset{(id,0)}{\leftleftarrows}} \mathbb{K}^{i+1}, i \in \mathbb{N} \qquad A_i = X_{a_1(b_1^- a_1)^i} = \mathbb{K}^{i+1} \underset{J_{i+1,0}}{\overset{id}{\leftleftarrows}} \mathbb{K}^{i+1}, i \in \mathbb{N}_0$$

$$P_0 = X_{1_2} = \mathbb{K} \leftleftarrows 0 \qquad B_i = X_{b_1^-(a_1 b_1^-))^i} = \mathbb{K}^{i+1} \underset{id}{\overset{J_{i+1,0}^T}{\leftleftarrows}} \mathbb{K}^{i+1}, i \in \mathbb{N}_0.$$

Let us notice that we have short exact sequences $0 \to A_0 \to A_{i+1} \to A_i \to 0$ and $0 \to B_0 \to B_{i+1} \to B_i \to 0$ for all $i \in \mathbb{N}_0$. Hence, if a torsion class contains A_0, respective B_0, it contains all A_i, respective B_i. Similarly, among the modules above there are some other relations that would reduce the numbers of generators of torsion classes, but we will not deal with that.

Now we know almost the whole lattice of torsion classes of the Kronecker algebra. It remains to look to the four maximal non-crossing sets of strings containing the band.

For each of these four sets $\mathscr{S}$ the set $\mathscr{S}^p = \{\gamma\}$. By Theorem 2 we have the lattice isomorphism

$$(\mathscr{S}, \boldsymbol{\lambda}) \mapsto \mathsf{T}(X(\mathsf{Fac}(\mathscr{S})) \cup X_{\boldsymbol{\lambda}}(\gamma)).$$

The map $\boldsymbol{\lambda} : \{\gamma\} \to 2^B$ only chooses a subset of the set B, recall B is the set of non-isomorphic simple modules over $\mathbb{K}[T, T^{-1}]$. Further, $X_{\boldsymbol{\lambda}}(\gamma)=\{X_M(\gamma) \mid M \in \boldsymbol{\lambda}(\gamma)\}$.

From the paragraph below Definition 16 we know that indecomposable modules over $\mathbb{K}[T, T^{-1}]$ are of the form $M_{n,k} = (\mathbb{K}^n, J_{n,k})$, where $J_{n,k}$ is a Jordan block with dimension $n \in \mathbb{N}$ and eigenvalue $k \in \mathbb{K} \setminus \emptyset$. The simple ones are exactly $M_{1,k}$ because we can embed $M_{1,k}$ to each $M_{n,k}$, and on the other hand, there is no homomorphism between $M_{1,k}$ and $M_{1,k'}$ for different k, k'.

Hence, from the definition of band modules, we get for the simple module $M_{1,k}$ and the band γ the band module $X_M(\gamma) = \mathbb{K} \underset{k}{\overset{1}{\leftleftarrows}} \mathbb{K}$.

Together we get that all torsion classes obtained from the set of maximal strings containing the band have the following generators:

- All of them have generators I_i, $i \in \mathbb{N}$
- Torsion classes obtained from the sets $\mathscr{S}$ containing β^i_∞ add a generator $\mathbb{K} \underset{0}{\overset{1}{\leftleftarrows}} \mathbb{K}$. Similarly, torsion classes obtained from the $\mathscr{S}$ containing β^o_∞ add a generator $\mathbb{K} \underset{1}{\overset{0}{\leftleftarrows}} \mathbb{K}$.
- The torsion class $\mathscr{T}_\Lambda(\mathscr{S}, \boldsymbol{\lambda})$ also has generators $\{\mathbb{K} \underset{k}{\overset{1}{\leftleftarrows}} \mathbb{K} | M_{1,k} \in \boldsymbol{\lambda}(\gamma)\}$.

The last two points we can reformulate so that depending on $(\mathscr{S}, \boldsymbol{\lambda})$ we add as generators some $R_{(\alpha,\beta)} = \mathbb{K} \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \mathbb{K}, (\alpha, \beta) \in \mathbb{P}_1(\mathbb{K})$, where $\mathbb{P}_1(\mathbb{K})$ denotes the projective line.

Moreover, $(\mathscr{S}, \boldsymbol{\lambda}) \geq (\mathscr{S}', \boldsymbol{\lambda}')$ if and only if $\mathscr{S} \geq \mathscr{S}'$ in the order of maximal non-crossing sets of strings and $\boldsymbol{\lambda}(\gamma) \supseteq \boldsymbol{\lambda}'(\gamma)$. It means that the part of the lattice obtained from $(\mathscr{S}, \boldsymbol{\lambda})$ with $\mathscr{S}$ containing the band is isomorphic to the lattice of all subsets of $\mathbb{P}_1(\mathbb{K})$.

In this subsection, we have given all the observations needed to complete the goal of this section – finding the lattice of torsion classes of Kronecker algebra. The lattice is shown in Figure 2.10. We recall that some elements of the lattice can be written in a form with fewer generators, but it is not our goal.

Again, it is possible to find the lattice of torsion classes without this theory, but contrary to the previous example it is not easy. Representations of Kronecker quiver are discussed in Section 9.3. in Krause [2010]. The whole lattice of torsion classes is shown in Figure 2 in Thomas [2021].

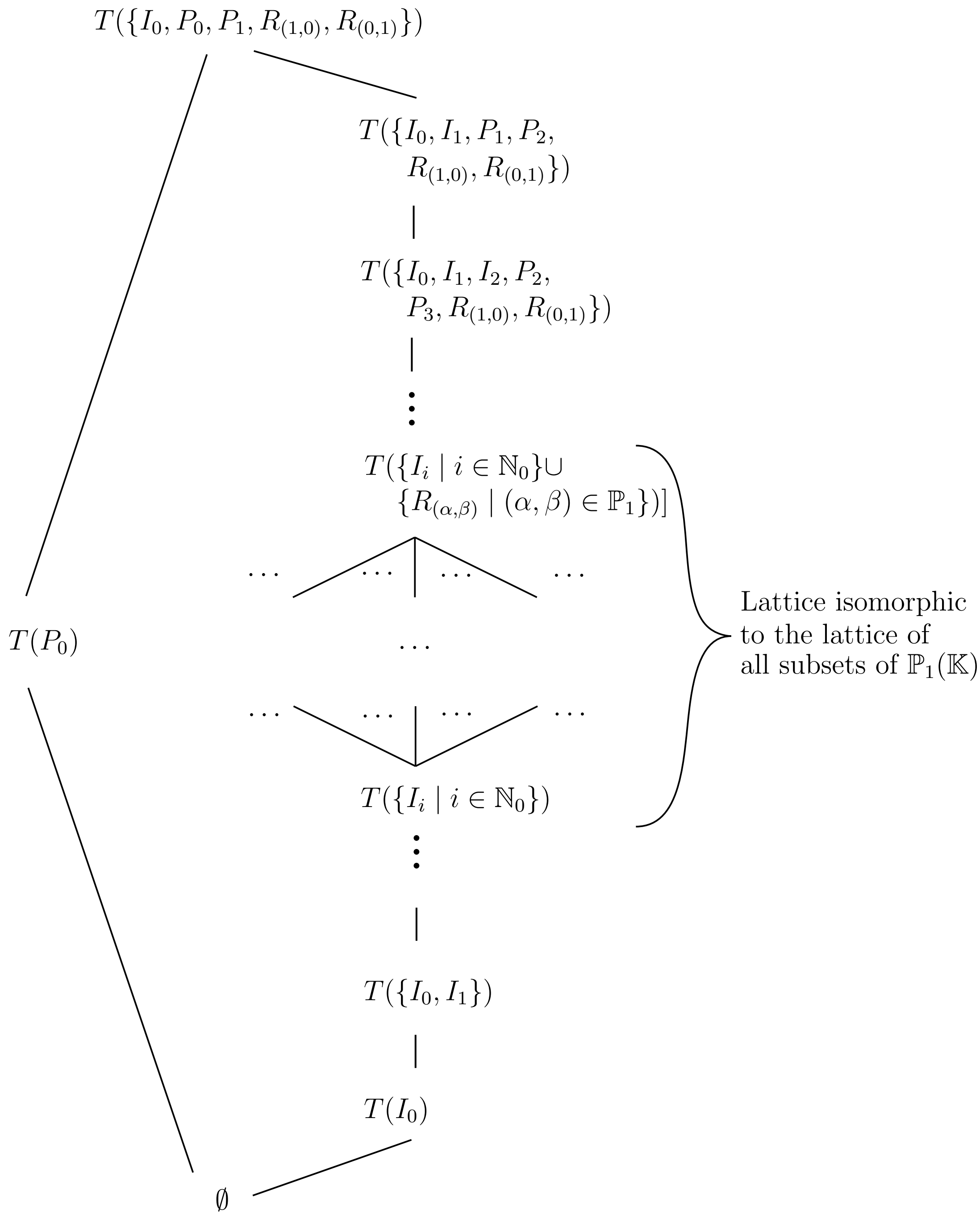


Figure 2.10: The lattice of torsion classes of the Kronecker algebra

3. Markov algebra

In this chapter, we will look at a much more complicated example. The chapter aims to find a lattice of torsion classes of gentle algebra called Markov algebra, the underlying quiver in Figure 3.1 where relations are formed by pairs of succeeding arrows of the same colour. It is the same quiver as in Example (iii) below Definition 1.

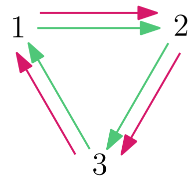


Figure 3.1: Markov quiver

The inspiration for this chapter was the unpublished notes of Aaron Chan. Helpful were especially figures and ideas how could some parts of the wanted lattice look. On the other hand, the whole approach and all results in this chapter are original contributions.

As in examples in the previous chapter, there exists a tiling corresponding to the Markov quiver. Just take a torus with one marked point and triangulation as in Figure 3.2.

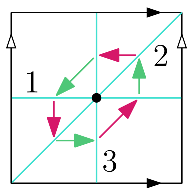


Figure 3.2: Corresponding marked surface

First, let us clarify the notation that we will use in the whole section. A string is by Definition 8 a pair containing a walk (reduced sequence of arrows which avoids relations) and the inverse of this walk.

We can specify each arrow in the Markov quiver in Figure 3.1 by a colour (green or pink) and the source and target. Hence, we can write walks as symbols of arrows in green or pink colour and between them numbers 1, 2, or 3 for vertices. For example, there is a walk $1{\to}2{\to}3{\to}1{\leftarrow}3$. Because of relations, arrows of green and pink colour must alternate in each walk. Further, we can write an inverse of an arrow as an arrow in the opposite direction with switched source and target, i.e. the inverse of $1{\to}2$ is arrow $2{\leftarrow}1$.

For simplicity, we will write strings as walks that determine them. So "string $1{\to}2{\to}3{\to}1{\leftarrow}3$" formally means the pair $\{1{\to}2{\to}3{\to}1{\leftarrow}3, 3{\to}1{\to}3{\to}2{\leftarrow}1\}$ and also we can mention this pair as "string $3{\to}1{\to}3{\to}2{\leftarrow}1$".

Especially when we search crossings it is important to note that the word "string" also means the inverse of the mentioned sequence. Let us again give an example. There is indeed a crossing between strings $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \leftarrow 3$ and $2 \leftarrow 1 \leftarrow 3 \rightarrow 1$ at $1 \leftarrow 3$, although there is no sequence $1 \leftarrow 3$ in the chosen walk determining the first string.

Note that even if we are talking about some substring, e.g. $2 \rightarrow 3 \rightarrow 1$, we do not fix the orientation of the whole string. That is we still consider a sequence of arrows as a string not as a walk.

If we, for strings, use notions as first, last, preceding, succeeding and so on, we refer to the walk which is listed. Hence, by the "first arrow of string $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \leftarrow 3$" we mean arrow $1 \rightarrow 2$.

3.1 Non-self-crossing strings

First, let us see what an infinite non-self-crossing string might look like in the Markov quiver. This quiver is a blossoming of itself, so each infinite string is a two-sided infinite sequence of arrows.

In this section, s will denote an infinite non-self-crossing string and X, Y, Z will denote the three vertices of the Markov quiver so that there are arrows in the direction $X \rightarrow Y, Y \rightarrow Z, Z \rightarrow X$.

Definition 21 (Cooriented substring)**.** Let s be an infinite non-self-crossing string in the Markov quiver. We call *cooriented* its substring that consists of consecutive arrows of the same orientation and cannot be extended to a larger one.

Lemma 3 (Forms of cooriented substrings)**.**

1. *Let p be a substring of string s, which does not contain any arrow which is contained in a cooriented substring of s of length greater than two. Then there is a vertex Y such that each arrow of p starts or ends in Y.*

2. *A non-zero finite cooriented substring p of s cannot have a length other than 1, 2, 4, or 5.*

3. *A string s cannot contain more than one cooriented substrings of length greater than two. The only exception may be s containing two one-side-infinite cooriented substrings of opposite directions.*

Proof. 1. Suppose p does not contain two consecutive arrows of the same orientation. In that case, p is only an vertex, or it must consist of alternating arrows between two vertices X and Y, therefore from alternating arrows $X \rightarrow Y$ and $Y \leftarrow X$.

Otherwise, p contains a substring of the form $X \rightarrow Y \rightarrow Z$. Thus, in s, we have $\leftarrow X \rightarrow Y \rightarrow Z \leftarrow$. We will show that each arrow of p starts or ends in Y. In the Markov quiver, an arrow $X \rightarrow Z$ does not exist. We claim that there cannot be an arrow $Z \rightarrow X$ in p. If there were, it could not succeed an arrow $X \leftarrow$ and could not precede $\leftarrow Z$ thanks to non-self-crossing property at Z and X respectively. Hence, the only option is a substring $Y \rightarrow Z \rightarrow X \rightarrow Y$ in s which is not possible because from assumption there are not three consecutive arrows.

2. Our quiver is symmetric, so it is sufficient to disprove that the length of p may differ from 1, 2, 4, and 5 only for p starting with $1 \to 2$. Thanks to the relations, cooriented substrings of fixed length with a given starting arrow are uniquely determined.

First, assume that p has length three, so it is of the form $1 \to 2 \to 3 \to 1$. The arrow preceding p is in the opposite direction to arrows in p, so the only reasonable possibility is $2 \leftarrow 1$, and similarly, the arrow succeeding the substring has to be $1 \leftarrow 3$. Thus, we get substring $2 \leftarrow 1 \to 2 \to 3 \to 1 \leftarrow 3$ which has a crossing at 1.

In p, all six arrows alternate regularly. Thus if p had a length divisible by three, p would start and end at the same vertex. It means that, by the same argument as above, no such p exists.

Now suppose that p has a finite length greater than six. Thus, the same arrows as the first and last in p occur at least once more in p. We claim that there is a self-crossing with a common part between the nearest occurrence of the first and last arrow, including them (if the first and last are the same, it is sufficient to take only this arrow). We denote this part by r. Then the string s has a substring of the form $(\leftarrow p \leftarrow) = (\leftarrow r \to \ldots \to r \leftarrow)$. For better understanding there is an illustration of an example of such self-crossing in Figure 3.3. We can see that we exactly have found a self-crossing at r.

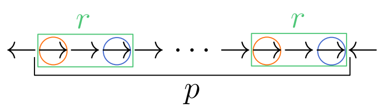


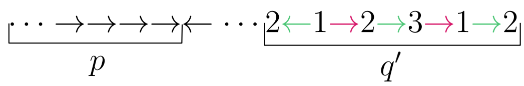
Figure 3.3: Illustration of a self-crossing of p

3. From the second part of the lemma, we know that s cannot contain a cooriented substring of length greater than two for another length than 4, 5, or infinity.

Consider p and q two cooriented substrings of s of length 4, 5, or infinity such that there is no any other cooriented substring of length 4, 5, or infinity between them in s. Without loss of generality, we can suppose that the right one q from the left starts with the arrow $1 \to 2$. That is, including the preceding arrow, s has a substring $q' = 2 \leftarrow 1 \to 2 \to 3 \to 1 \to 2$.

We distinguish two cases:

- Case 1 – Substring p has the same orientation of arrows as q.
  Then we get the following substring.

  We see that p does not end in vertex 1; otherwise, we would get a crossing in this vertex.

  The substring of s starting immediately right of p and ending immediately left of q, containing arrow $2 \leftarrow 1$, satisfies the assumption of the first part of the lemma. Hence, every second vertex of this substring is 2, or every second vertex is 1. In both cases, each arrow in this substring immediately right of 2 is green, immediately left of 2 is pink, and vice versa for 1.

The previous two paragraphs imply the following. If p ends in the vertex 2, then there is a substring of s of the form $p' = 1\rightarrow 2\rightarrow 3\rightarrow 1\rightarrow 2\leftarrow$. Hence, there is a crossing between p' and q' at $1\rightarrow 2$. Similarly, for p ending in 3 we get crossing between q' and $p' = 2\rightarrow 3\rightarrow 1\rightarrow 2\rightarrow 3\leftarrow$ at $1\rightarrow 2\rightarrow 3$. Note that the colours in string p' must be as we have stated. It holds because we know from the orientation of arrow $3\leftarrow$ that its source must be 2, and the arrow immediately left of 2 is pink.

- Case 2 – String p has a different orientation than q.

$$\underbrace{\cdots\leftarrow\leftarrow\leftarrow\leftarrow}_{p}\rightarrow\cdots\underbrace{2\leftarrow 1\rightarrow 2\rightarrow 3\rightarrow 1\rightarrow 2}_{q'}$$

Again, we use the observation about the colours of arrows between p and q.

If p ends in the vertex 1, there is the substring of s of the form $p_1 = 2\leftarrow 1\leftarrow 3\leftarrow 2\leftarrow 1\rightarrow$. If p ends in the vertex 2, we get $p_2 = 3\leftarrow 2\leftarrow 1\leftarrow 3\leftarrow 2\rightarrow$. Finally, if p ends in the vertex 3, we get $p_3 = 1\leftarrow 3\leftarrow 2\leftarrow 1\leftarrow 3\rightarrow$. We cannot switch colours in p_3 because its last arrow $3\rightarrow$ must have a target 1 and the arrow immediately left of 1 is green.

If q is of length four, adding the succeeding arrow after q, we get the substring of s of the form $q_4 = 1\rightarrow 2\rightarrow 3\rightarrow 1\rightarrow 2\leftarrow$. Similarly, if q is of length five, we get the string $q_5 = 1\rightarrow 2\rightarrow 3\rightarrow 1\rightarrow 2\rightarrow 3\leftarrow$.

Note that there is a crossing between p_1 and q_4 with at $1\rightarrow 2$ and between p_1 and q_5 at $1\rightarrow 2\rightarrow 3$. For p_2, we get a crossing at 2 with q_4 and at $2\rightarrow 3$ with q_5. Finally, there is a crossing of p_3 and q_4 at $3\rightarrow 1\rightarrow 2$ and p_3 and q_5 at 3.

In the same way, there are crossings between p of the length 4 or 5 and q of arbitrary length greater than three. Together we have proven that there are no two cooriented substrings of length greater than two except two cooriented substrings of infinite length and opposite orientation.

□

We have narrowed down a possible form of infinite non-self-crossing strings. According to cooriented strings of length greater than two, we can sort them into four groups:

1. No cooriented substring of length greater than two
2. One infinite cooriented substring
3. One cooriented substring of length four or five
4. Two infinite cooriented substrings of different orientation

In all cases, for any infinite non-self-crossing string s there exists a vertex such that each arrow outside the cooriented strings of length greater than two contains this vertex. Let us call this vertex *significant*.

It is a bit stronger statement than in the first part of Lemma 3, where we claim that such a vertex exists for each substring outside cooriented strings of a length greater than two. We will show it holds and also when the significant vertex is unique.

- First, let the string s contain a cooriented substring of length two, that is there exists its substring of the form $Y \leftarrow X \rightarrow Y \rightarrow Z \leftarrow Y$. Thus, X and Z cannot be significant. According to the proof of Lemma 3, there is no arrow between Z and X in any non-cooriented part. Hence, Y is significant.

- Let the string s does not contain a cooriented substring of length two. It means that each non-cooriented part consists of two alternating vertices. Strings of type 1, 2, and 4 have only one non-cooriented part, which has two significant vertices. Strings of type 3 have a cooriented part and two non-cooriented infinite parts, each having two significant vertices. Note that, it has to be the same two vertices in both parts because if in one part were significant X, Y and in the second Y, Z, we would get substrings $X \rightarrow Y \leftarrow X$ and $Z \leftarrow Y \rightarrow Z$ with a crossing at Y. Thus, if s does not contain a cooriented substring of length two there exist two significant vertices.

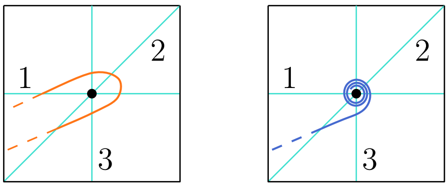


Figure 3.4: Examples of cooriented strings of length five and infinite length

Let us look at the cooriented substrings in the tiling algebra view. Each cooriented substring corresponds to a part of the accordion that rotates around the puncture. In the second part of the lemma, we proved that if a curve rotates around a puncture once, it must rotate around it infinitely. Figure 3.4 shows illustrations of cooriented parts in the tiling view.

Now we want to add more conditions for non-self-crossing. Moreover, we want find conditions such that all strings that satisfy them are non-self-crossing.

3.1.1 Type 1

For simplicity, let us first focus on strings without cooriented substrings of length greater than two. All lemmas in this subsection also hold for strings of other types, but conclusions after these lemmas are specific to type 1.

Let us denote $o = Y \leftarrow X \rightarrow Y$ and $i = Y \rightarrow Z \leftarrow Y$. From the first part of Lemma 3 follows that s is a composition of o's and i's. However, not every such composition gives us a non-self-crossing string, as we will see soon. Note that we have not forgotten walks composed of i^-'s and o^-'s because the inverse of such walk is composed of i's and o's.

Our procedure will be to gradually enlarge non-self-crossing strings from which the original string is composed.

Lemma 4. *It holds that s contains at most one of the substrings o^2, i^2. Furthermore, if s contains the substring $oi^n o, n \in \mathbb{N}$, it does not contain the substring i^{n+2}. Similarly, if s contains the substring $io^n i, n \in \mathbb{N}$, it does not contain the substring o^{n+2}.*

Proof. Let us notice that o^2 and i^2 have a crossing at Y.

Let s contain the substring $oi^n o, n \in \mathbb{N}$. Especially, it contains also the smaller substring $\rightarrow i^n \leftarrow$. In contrast, i^{n+2} contains the substring $\leftarrow i^n \rightarrow$. Hence, there is a crossing between $oi^n o$ and i^{n+2}. In the same way, we can find a crossing between $io^n i$ and o^{n+2}. □

We know that the string s is composed of i's and o's. Now we can specify the form of s more precisely. From Lemma 4, we deduce that at least one of i, o is contained in s in at most the first power. Let us denote it J_1 and the other by V_1.

Further, we know from the lemma that if n_1 is the smallest power such that $J_1 V_1^{n_1} J_1$ is a substring of s, then there is no substring V_1^m for $m \neq n_1, n_1{+}1$. In other words, s is a composition of parts of the form $J_1 V_1^{n_1}$ and $J_1 V_1^{n_1+1}$ or $s = {}^\infty V_1^\infty$ or $s = {}^\infty V_1 J_1 V_1^\infty$.

Strings composed of the components $J_1 V_1^{n_1}$ and $J_1 V_1^{n_1+1}$ still can be self-crossing. For further specification we use the following claim.

Lemma 5. *Let V, J be strings such that one of the strings JV and VJ is of the form $p \leftarrow q_1 \leftarrow r$ and the other of the form $p \rightarrow q_2 \rightarrow r$, where p, q_1, q_2, r are some (possibly empty) strings. Then the following holds:*

1. *The strings VJV^nJV^nJV and $JV^{n+1}JV^{n+1}J, n \in \mathbb{N}$, have a crossing.*
2. *There is a crossing between $VJV^n(JV^{n+1})^m JV^nJV$ and $(JV^{n+1})^{m+2}J$ and also between $JV^{n+1}(JV^n)^m JV^{n+1}J$ and $V(JV^n)^{m+2}JV$.*
3. *The strings JV^n and JV^{n+1} also satisfy the condition on the form of compositions. Thus, one of the compositions JV^nJV^{n+1} and $JV^{n+1}JV^n$ has the form $p' \leftarrow q_1' \leftarrow r'$ and second has the form $p' \rightarrow q_2' \rightarrow r'$, where p', q_1', q_2', r' are some strings.*

Proof. 1. We claim that rV^nJV^np is the common part of the crossing. It is, of course, a substring of both strings and from the assumption on the form of JV and VJ, we see that we have indeed found a crossing.

2. Note that both pairs are of the forms $VJtJV, JVtVJ$ for an appropriate string t. Thus, by assumption, we have a crossing.

3. Just set $p' = JV^np, r' = rJV^n, q_1' = q_1, q_2' = q_2$. □

We have defined V_1, J_1, which satisfy the assumptions of Lemma 5. We proceed by induction. If V_i, J_i are defined, they satisfy the assumptions of the lemma, and s is composed of $J_iV_i^{n_i}$ and $J_iV_i^{n_i+1}$. Then we know from the first part of the lemma that at least one of $J_iV_i^{n_i}$ and $J_iV_i^{n_i+1}$ is contained in s in at most the first power. Let us denote such a one by J_{i+1} and the second one by V_{i+1}.

Further, if n_{i+1} is the smallest power such that $J_{i+1}V_{i+1}^{n_{i+1}}J_{i+1}$ is a substring of s, then there is no substring V_{i+1}^m for $m \neq n_{i+1}+1$. This is true because of the second part of the lemma. Hence, s is a composition of parts of the form $J_{i+1}V_{i+1}^{n_{i+1}}$ and $J_{i+1}V_{i+1}^{n_{i+1}+1}$ or $s = {}^\infty V_{i+1}^\infty$ or $s = {}^\infty V_{i+1}J_{i+1}V_{i+1}^\infty$. Moreover, V_{i+1} and J_{i+1} satisfy the assumptions of the lemma thanks to the third part.

Together we get that infinite strings of type 1 are of the form ${}^\infty(V_i)^\infty$ or ${}^\infty(V_i)J_i(V_i)^\infty$ for some i or any substring of s is not contained in s to infinite power and for any $i \in \mathbb{N}$ the string s is composed of both $J_iV_i^{n_i}$ and $J_iV_i^{n_i+1}$.

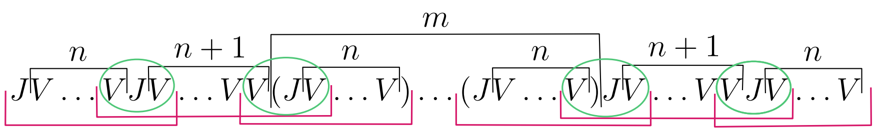
On the other hand, we will show that every string constructed in this way from o, i is non-self-crossing.

Lemma 6. *Let J, V be as follows: We cannot embed VJV into $V^kJV^l, k, l \in \mathbb{N}$ in other than the obvious way. Moreover, VJV^nJV and $VJV^{n+1}JV$ are such a pair of strings that they do not cross each other and are non-self-crossing. Then:*

1. *Strings in each of the pairs* $JV^nJV^{n+1}(JV^n)^mJV^{n+1}JV^n$, $JV^nJV^{n+1}(JV^n)^{m+1}JV^{n+1}JV^n$ *and* $JV^{n+1}JV^n(JV^{n+1})^mJV^nJV^{n+1}$, $JV^{n+1}JV^n(JV^{n+1})^{m+1}JV^nJV^{n+1})$ *are non-self-crossing and mutually non-crossing.*

2. *We cannot embed the string $JV^nJV^{n+1}JV^n$ into $(JV^n)^{k'}JV^{n+1}(JV^n)^{l'}$ and $JV^{n+1}JV^nJV^{n+1}$ into $(JV^{n+1})^{k'}JV^n(JV^{n+1})^{l'}$ in other than the obvious way for all $k', l' \in \mathbb{N}$.*

Proof. From assumption, VJV^nJV and $VJV^{n+1}JV$ are non-self-crossing and mutually non-crossing. Hence each potential (self)crossing between strings from 1. has to contain the substring VJV. Below are marked in pink parts of string $JV^nJV^{n+1}(JV^n)^mJV^{n+1}JV^n$, which are non-self-crossing and mutually non-crossing from the assumption. The remaining three strings would be marked similarly.



Since we cannot embed VJV nontrivially, it is precisely that VJV, that we see in the notation. However, there is no way for two substrings in the same pair and of the same length containing VJV to start and end differently. Hence, there is no crossing.

The second part immediately follows from the assumption on embedding VJV. ☐

First, let us notice that V_1 and J_1 as i and o satisfy the assumption of Lemma 6. By induction, we get from the lemma that $J_iV_i^{n_i}J_i$ and $J_iV_i^{n_i+1}J_i$ are non-self-crossing and do not cross each other for every $n_i \in \mathbb{N}$. Thus, all infinite strings constructed in the mentioned way are non-self-crossing. Especially $V_i^{n_i}$ is non-self-crossing for every n_i, therefore ${}^\infty(V_i)^\infty$ is also non-self-crossing. Similarly ${}^\infty(V_i)J_i(V_i)^\infty$ must be non-self-crossing.

This subsection concludes that we know all non-self-crossing infinite strings without a cooriented substring of length greater than two, and we can classify them into three types:

1.A Periodic strings, i.e., bands, in the previous notation ${}^\infty(V_{i+1})^\infty$.

1.B Two sides periodic strings with one change, ${}^\infty(V_i)J_i(V_i)^\infty$

1.C Strings which have no substrings in the infinite power.

Example. For a better understanding, let us give two examples of corresponding accordions on torus to strings of types 1.A and 1.B in Figure 3.5. They are ${}^\infty(3\rightarrow1\leftarrow3\leftarrow2\rightarrow)^\infty$ and ${}^\infty(1\leftarrow3\rightarrow)\rightarrow2\leftarrow(1\leftarrow3\rightarrow)^\infty$ respectively.

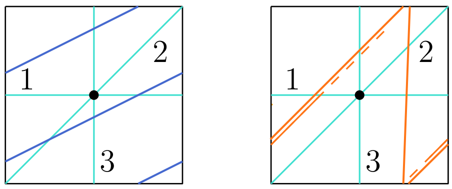


Figure 3.5: Corresponding accordions

3.1.2 Bijections between strings and numbers

In this subsection, we will show another perspective on strings of type 1 using the knowledge from the previous subsection. We start with strings of type 1.A. It holds that they are in a bijection with $\mathbb{Q}_\infty$, and in the tiling view, they are closed curves with corresponding rational slopes. We will inductively define a map α from a set of strings to rational numbers with infinity. Since each string s of type 1.A is periodic with a unique primitive period p of the form $p = J_iV_i^n$ (using notation introduced in the previous section), for simplicity we will sometimes write $\alpha(s)$ as $\alpha(p)$. We identify $\infty = -\infty = \frac{1}{0} = \frac{-1}{0}$.

First let $\alpha(1 \rightarrow 2 \leftarrow) = \frac{1}{0} = \frac{-1}{0}, \alpha(2 \rightarrow 3 \leftarrow) = \frac{0}{1}$ and $\alpha(3 \rightarrow 1 \leftarrow) = \frac{1}{1}$. Thus we have defined α for all strings of type 1.A. that do not have a unique significant vertex.

Next, suppose we have a string s of type 1.A with the significant vertex Y. We can look at string ${}^\infty(1 \rightarrow 2 \leftarrow)^\infty$ in two ways – as the string with significant vertex 1 and as the string with significant vertex 2. In the first case, we consider the fraction in its reduced form for $\alpha(1 \rightarrow 2 \leftarrow) = \infty$ to be the fraction $\frac{1}{0}$ and in the second case $\frac{-1}{0}$. We do not distinguish our view of strings $(2 \rightarrow 3 \leftarrow)$, $(3 \rightarrow 1 \leftarrow)$ by their significant vertices. We will use a construction of s by J_i, V_i described in the previous subsection. Strings J_1, V_1 are just i, o from the previous section, so we have already defined $\alpha(J_1)$ and $\alpha(V_1)$.

Further, if the image of α is defined for strings A, B, we will define it for AB.

Let $\alpha(A) = \dfrac{a}{b}, \alpha(B) = \dfrac{c}{d}, a, c \in \mathbb{Z}, b, d \in \mathbb{N}_0, \dfrac{a}{b}$ and $\dfrac{c}{d}$ are in reduced form then $\alpha(AB) = \dfrac{a+c}{b+d}$.

Using this inductive definition we have defined all $\alpha(J_i V_i^n)$ for all n.

We will study this definition more closely. For that, the following observation will be helpful.

Observation 7. *Assume that $a, c \in \mathbb{Z}$, $b, d \in \mathbb{N}$, $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.*

Proof. We get

$$\begin{gathered} ad < bc \\ ad + ab < bc + ab \text{ and } ad + cd < bc + cd \\ a(d+b) < b(a+c) \text{ and } d(a+c) < c(b+d) \\ \frac{a}{b} < \frac{a+c}{b+d} \text{ and } \frac{a+c}{b+d} < \frac{c}{d}. \end{gathered}$$

□

To work with $\frac{1}{0}$ and $\frac{-1}{0}$ we note that $\frac{a-1}{b+0} < \frac{a}{b} < \frac{a+1}{b+0}$.

By repeated application of Observation 7, we get that for strings A, B, the rational number $\alpha(AB^n)$ lies between $\alpha(A)$ and $\alpha(B)$. Let $\alpha(A) < \alpha(B)$. For $n < m$ we have $\alpha(AB^n) < \alpha(AB^m)$ and conversely $\alpha(BA^n) > \alpha(BA^m)$. Thus $(\alpha(A), \ldots, \alpha(BA^3), \alpha(BA^2), \alpha(AB), \alpha(AB^2), \alpha(AB^3), \ldots, \alpha(B))$ is a division of the interval $(\alpha(A), \alpha(B))$.

In the beginning, when α is defined only for three strings without significant vertex, we have a division of real numbers $(-\infty, 0, 1, \infty)$. In each following step, we divide intervals finer than in the previous step. Thus, in our definition of $\alpha(s)$, we gradually narrow the intervals in which it can lie, and in a finite step, $\alpha(s)$ will lie on the boundary of some two intervals.

Observation 8. *Let $\frac{c}{d} < \frac{a}{b}$ and $ad - bc = 1$. Then also for $\frac{c'}{d'} = \frac{(n-1)a+c}{(n-1)b+d}$, $\frac{a'}{b'} = \frac{na+c}{nb+d}$, $n \in \mathbb{N}$ holds $a'd' - b'c' = 1$. In particular, all neighbouring images in interval division from the previous paragraph by α have this property.*

Proof. We have

$$\begin{aligned} a'd' - b'c' &= (na+c)((n-1)b+d) - (nb+d)((n-1)a+c) \\ &= nad + (n-1)cb - (n-1)ad - ncb = ad - cb = 1. \end{aligned}$$

The second part of observation follows from induction. □

Note that if $\frac{a}{b}, \frac{c}{d}$ are in reduced form, then $\frac{a+b}{c+d}$ is in reduced form. It holds from Observation 8 because if $\frac{a'}{b'} = \frac{ka''}{kb''}$, then $k(a''d' - b''c') = 1$.

Lemma 9. *The map α is a bijection. Moreover, the string mapped to the rational number q in the tiling algebra view corresponds to an accordion which is the line with the slope q.*

By a line with the slope q on the torus we mean line with the slope q in the square view of the torus which does not pass through the marked point. Such a line exists for each slope q, and all such lines are homotopic.

Proof. The map α is injective because we have different sequences of intervals that determine q for different strings.

On the other hand, let $\frac{p}{q} \in \mathbb{Q}_\infty$ lie in interval $(\frac{c}{d}, \frac{a}{b})$, where $\frac{c}{d}$ and $\frac{a}{b}$ are of the form $\alpha(J_i V_i^n), \alpha(J_i V_i^{n+1})$. We will suppose these fractions are positive. For negative, the proof is similar. Then

$$\begin{aligned} aq - bp \geq 1 \text{ and } pd - cq \geq 1 \\ (c+d)(aq-bp) \geq c+d \text{ and } (a+b)(pd-cq) \geq a+b \\ (c+d)(aq-bp) + (a+b)(pd-cq) \geq a+b+c+d \\ p+q = (p+q)(ad-cb) \geq a+b+c+d \end{aligned}$$

where the equality in the last line holds from Observation 8. By narrowing intervals, sum $a+b+c+d$ grows, so there is a finite step such that in this step there is no interval in our subdivision that contains $\frac{p}{q}$ strictly inside. Hence, $\frac{p}{q}$ must be an endpoint of some interval. Thus, $\frac{p}{q} = \alpha(J_i V_i^n)$ for some appropriate J_i, V_i, n.

For the second part of the proof, we claim that q indicates the ratio between the number of passes of p through vertex 1 and passes through vertex 3. In the other words, the ratio between the number of crossing arc 1 and arc 3 by the corresponding accordion. A number q is negative if and only if the significant vertex is 2, respectively, each second arc crossed by the line with slope q is 2.

First, we note that the line of slope $q = \frac{\pm a}{b}, a, b \in \mathbb{N}_0$ is a closed non-self crossing curve, and it crosses a times the arc 1 and b times the arc 3.

On the other hand, $\alpha(1 \to 2 \leftarrow) = \frac{1}{0} = \frac{-1}{0}$ and $1 \to 2 \leftarrow$ contains once vertex 1 and none vertex 3. For $p = 2 \to 3 \leftarrow$ it holds $\alpha(p) = \frac{0}{1}$ and p contains once vertex 3 and none vertex 1. For $p = 3 \to 1 \leftarrow$ we have $p = \frac{1}{1}$ and it contains vertices 1 and 3 once.

Further, let the string A pass a times through vertex 1 and b times through vertex 3, and let B pass c times through vertex 1 and d times through vertex 3. Then the composition AB passes $(a+c)$ times through vertex 1 and $(b+d)$ times through vertex 3. By induction, we get that s, which is mapped to $\frac{a}{b}$, contains a times vertex 1 and b times vertex 3.

For a string s with a period p holds that $\alpha(p) < 0$ if and only if p is composed of strings for which $\alpha(s) = \frac{-1}{0}$ and $\frac{0}{1}$. These strings are exactly $1 \to 2 \leftarrow$ and $2 \to 3 \leftarrow$. It follows that $\alpha(p) < 0$ if and only if the significant vertex is 2. On the other hand, we notice that every other arc that the line crosses is the diagonal arc 2 if and only if the slope of this line is negative.

A closed curve of slope q is definitely non-self-crossing and determines a periodic string, therefore a string of type 1.A. Thanks to the arcs it crosses, it must be a string with a period p satisfying $\alpha(p) = q$. Since α is a bijection, it holds that strings of types 1.A correspond to closed curves with slope from $\mathbb{Q}_\infty$ up to homotopy. □

From now on, we will denote a string of type 1.A with period p, $\alpha(p) = q$ by ${}^{\infty}q^{\infty}$.

Example. In Figure 3.6 we illustrate accordions ${}^{\infty}\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{-2}{3}^{\infty}$, ${}^{\infty}\frac{3}{2}^{\infty}$, ${}^{\infty}\frac{-3}{2}^{\infty}$ respectively. We can check that periods of the corresponding strings are
${}^{\infty}(3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow)^{\infty}$, ${}^{\infty}(2\leftarrow 1\rightarrow 2\rightarrow 3\leftarrow 2\leftarrow 1\leftarrow 2\rightarrow 3\leftarrow 2\rightarrow 3\leftarrow)^{\infty}$,
${}^{\infty}(1\rightarrow 2\leftarrow 1\leftarrow 3\rightarrow 1\leftarrow 3\rightarrow)^{\infty}$, and ${}^{\infty}(2\rightarrow 3\leftarrow 2\leftarrow 1\rightarrow 2\rightarrow 3\leftarrow 2\leftarrow 1\rightarrow 2\leftarrow 1\rightarrow)^{\infty}$.

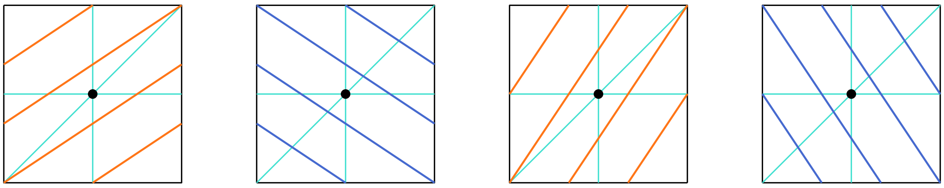


Figure 3.6: Accordions ${}^{\infty}\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{-2}{3}^{\infty}$, ${}^{\infty}\frac{3}{2}^{\infty}$, ${}^{\infty}\frac{-3}{2}^{\infty}$

Strings of type 1.B are of the form ${}^{\infty}(V_i)J_i(V_i)^{\infty}$. Recall that $V_i = J_{i-1}V_{i-1}^{n_{i-1}}$. So we can use the bijection α between such V_i and $\mathbb{Q}_{\infty}$. Moreover, as we know from Lemmas 5 and 6, there are exactly two options for J_i, they are $J_{i-1}V_{i-1}^{n_{i-1}+1}$ and $J_{i-1}V_{i-1}^{n_{i-1}-1}$ (if $n_{i-1} = 1$ we replace the second of them by $J_{i-1}^2V_{i-1}$).

In terms of the interval division described above, $\alpha(J_{i-1})$ and $\alpha(V_{i-1})$ are endpoints of an interval obtained in step $i-1$. If we look at the finer division in step i, there is a triple of adjacent endpoints such that the middle is an image of V_i and on the sides are images of the possible J_i.

Further, we will denote strings of type 1.B by ${}^{\infty}q{+}q^{\infty}$ and ${}^{\infty}q{-}q^{\infty}$, $q \in \mathbb{Q}_{\infty}$. Where $\alpha(V_i) = q$ and $+$, respectively $-$, corresponds to the option of J_i which α maps to number grater, respective less than q. Hence, we can write each string of type 1.B in only one way as ${}^{\infty}q{+}q^{\infty}$ or ${}^{\infty}q{-}q^{\infty}$.

Example. In Figure 3.7 we illustrate ${}^{\infty}\frac{2}{3}{-}\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{2}{3}{+}\frac{2}{3}^{\infty}$. The corresponding strings are ${}^{\infty}(3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow)3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow(3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow)^{\infty}$ and
${}^{\infty}(3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow)3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow(3\leftarrow 2\rightarrow 3\rightarrow 1\leftarrow 3\rightarrow 1\leftarrow)^{\infty}$

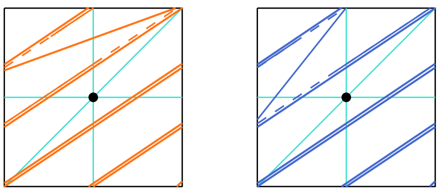


Figure 3.7: Accordions ${}^{\infty}\frac{2}{3}{-}\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{2}{3}{+}\frac{2}{3}^{\infty}$

What remains are strings s of type 1.C. We extend the map α to strings of this type so that if s is composed of substrings $J_iV_i^{n_i}$ and $J_iV_i^{n_i+1}$, then $\alpha(s)$ lies between images of these strings. We mentioned that this is true for s of type 1.A.

Strings of type 1.C are composed of both $J_iV_i^{n_i}$ and $J_iV_i^{n_i+1}$ at each step i. These substrings thus determine an infinite sequence of finer intervals such that $\alpha(s)$ lies in them. Hence, we get a bijection between strings of types 1.C and $\mathbb{R} \setminus \mathbb{Q}$. For two different strings, we have a different sequence of intervals that converges to different real numbers. On the other hand, for each real number r we can find a sequence of intervals from which we can construct a string s satisfying $\alpha(s) = r$.

The tiling view is similar to that of strings of types 1.A. We can imagine s of type 1.C as a line with slope $\alpha(s)$.

From now on, we will denote strings of type 1.C by ${}^{\infty}r^{\infty}, r \in \mathbb{R} \setminus \mathbb{Q}$ according to an image by extended α.

We know that each non-cooriented part of a non-self-crossing infinite string either contains some substring of the form $J_i V_i^n$ in an infinite power or it contains an infinite sequence of J_i, V_i. So in the same way as for strings of type 1 we can assign a number in $\mathbb{R}_\infty$ to each non-cooreinted part. Of course it will be surjective but non-injective map.

3.1.3 Type 2

Now move to infinite non-self-crossing strings with exactly one infinite cooriented part. In this subsection, we denote by s strings of type 2.

For the cooriented part, we have two options for orientation and six options for the last arrow. It might seem that we have also two options on which side of the string is a cooriented part. However, let us note that

$$^{\infty}(\rightarrow X \rightarrow Y \rightarrow Z \rightarrow X \rightarrow Y \rightarrow Z) = (Z \leftarrow Y \leftarrow X \leftarrow Z \leftarrow Y \leftarrow X \leftarrow)^{\infty}$$

and similarly for the other colours and orientations. Hence, we have exactly 12 possible cooriented one-side infinite strings.

Lemma 10. *If the significant vertex of s is unique, it must be the second to last vertex of the cooriented part of s.*

Proof. Let us suppose that the end of the cooriented part together with the succeeding arrow is $Z \rightarrow X \rightarrow Y \rightarrow Z \leftarrow Y$. Obviously, the significant vertex cannot be X because we have an arrow $Z \leftarrow Y$ in the non-cooriented part.

If the significant vertex is Z, then the only valid non-cooriented part is ${}^{\infty}(Z \leftarrow Y \rightarrow)^{\infty}$, because with the first occurrence of X we get a substring of s of the form $Z \leftarrow Y \rightarrow Z \rightarrow X$, which gives a crossing with the mentioned end of the cooriented string at $Y \rightarrow Z$.

The proof is the same for the opposite orientation or colours. □

For the non-cooriented part, we can use our knowledge from the previous subsection. Next, the question is which crossings will be added by composition with the cooriented part.

Without loss of generality, let us suppose that the string s has the cooriented part ${}^{\infty}(\rightarrow X \rightarrow Y \rightarrow Z \rightarrow X \rightarrow Y \rightarrow Z)$.

We add the last arrow of the cooriented part to the non-cooriented part so that this part consists of $o = Y \leftarrow X \rightarrow Y$ and $i = Y \rightarrow Z \leftarrow Y$, let us denote this extended non-cooriented part by s'. Then s' starts with i. We know from Lemma 4 that at least one of i and o is contained in s' in at most first power. Depending on that, we can distinguish two cases:

- First, suppose that i is not contained in power greater than one. Lemma 4 implies $s' = io^\infty$, or there exists n such that s contains $io^n i$ and contains no substring of the form $io^m i$ for $m \neq n, n+1$. In the second case, s' starts with io^n or io^{n+1}. We claim that it cannot start with io^{n+1}. If it starts with $io^{n+1}i$, from the choice of n, the string s' contains $io^n i$. Then we get substrings $Z{\to}X{\to}io^{n+1} = Z{\to}X{\to}io^n{\leftarrow}X{\to}Y$ and $oio^n i = {\leftarrow}X{\to}io^n{\to}Z{\leftarrow}Y$, which have the crossing at $X{\to}io^n$.

  On the contrary, we will prove that s' and $\cdots \to\to\to io^n i$ are non-crossing. For contradiction, suppose that there is a crossing between them at p. This p must contain i because both substrings $\cdots \to\to\to i$ and $io^n i$ do not cross with s'. Anywhere in s', a substring containing i continues to the right with $o^n i$ or $o^{n+1} i$. In the first case, p cannot be succeeded by different arrows in our two strings. In the second case, the first different arrow is $\to$ in string $\cdots \to\to\to io^n i$, so there is no crossing. For a better idea, what is described in this paragraph is illustrated in Figure 3.8.

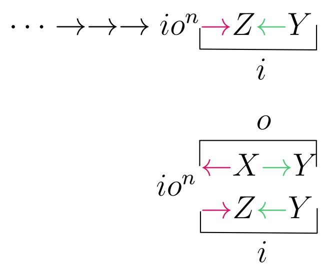


Figure 3.8: Starting with io^n

  Using a similar notation as in the previous section, we have $J_1 = i, V_1 = o$. We have reasoned that $s' = J_1 V_1^\infty$ or s' is composed of strings of the form $J_1 V_1^n$ and $J_1 V_1^{n+1}, n \in \mathbb{N}$, and it starts with the first of them.

- Now, let o not be contained in power greater than one. Then $s' = i^\infty$ or there exists n such that s contains $i^n o$ and no substring of the form $oi^m o$ for $m \neq n, n+1$. Let us discuss the second case. Lemma 4 implies that s' cannot start with $i^m o, m \geq n+2$. We show that s' cannot start with $i^m o$ where m is less than some l such that s' contains $oi^l o$. If it started with such $i^m o$, we would get a crossing between $Z{\to}X{\to}i^m o = Z{\to}X{\to}i^m{\leftarrow}X{\to}Y$ and $oi^{m+1} = Y{\leftarrow}X{\to}i^m{\to}Z{\leftarrow}Y$.

  On the contrary, we will show that s' and $\cdots \to\to\to i^{n+1}o$ are non-crossing. Suppose there is a crossing at p between them. Because of the non-crossing of $\cdots \to\to\to i$ and $i^{n+1}o$ with s', p must contain first i in $\cdots \to\to\to i^{n+1}o$. Anywhere in s', i is followed from the right by at most n other i's and then by o. Hence, the only way how to obtain different arrows after p is that i succeeded it in $\cdots \to\to\to i^{n+1}o$ and o in s'. Thus the first different arrow of $\cdots \to\to\to i^{n+1}o$ and s' is $\to$ and $\leftarrow$ respectively. Hence, we do not get a crossing.

  Together we have that $J_1 = o, V_1 = i$, $s' = V_i^\infty$ or s' is composed of strings of the form $V_1^{n_1+1}J_1, V_1^{n_1}J_1$ and moreover starts with the first of them.

Let us note that if, on the contrary, the cooriented string has direction outwards s', the first substrings are exactly the other than those described above. It holds because the opposite direction of the cooriented part requires the opposite succeeding arrow after the common part to avoid relations. We illustrated the difference between composing with differently oriented cooriented parts in Figure 3.9.

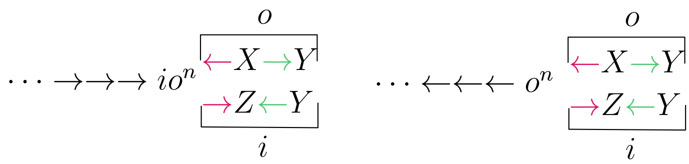


Figure 3.9: Composing with otherwise oriented cooriented parts

As in the previous subsection, we want to continue with greater J_i and V_i. We will define them only a bit differently than there.

We have defined J_1, V_1. Now we will define J_{i+1}, V_{i+1} inductively from J_i, V_i. Let s' be composed of either $J_i V_i^{n_i}$ and $J_i V_i^{n_i+1}$, or $V_i^{n_i} J_i$ and $V_i^{n_i+1} J_i$. From Lemma 5 at least one of the pair is contained in s' in at most the first power. Let it be J_{i+1} and the second of them be V_{i+1}. Then the only possible form of s' is one of:

- $s' = J_{i+1}(V_{i+1})^\infty, s' = {}^\infty(V_{i+1})^\infty$ (depending on the starting part)
- s' is composed of $J_{i+1}V_{i+1}^n$ and $J_{i+1}V_{i+1}^{n+1}$ or $V_{i+1}^n J_{i+1}$ and $V_{i+1}^{n+1}J_{i+1}$, $n \in \mathbb{N}$.

It is almost implied by Lemma 5. It only remains to prove that if s' starts with V_i and it contains $J_{i+1}V_{i+1}^n J$ and $J_{i+1}V_{i+1}^{n+1}J_{i+1}$, it cannot start with $V_{i+1}^m J_{i+1}$ for $m \leq n$. Further, we will prove that if the second case above occurs, it is uniquely determined which of the above components occurs first at s'.

Lemma 11. *Let the cooriented part be oriented towards s'. If the first different arrow of J_iV_i and V_iJ_i from the left is $\rightarrow$ for J_iV_i and $\leftarrow$ for V_iJ_i, the first component of s' cannot be V_i. Otherwise, the first component cannot be J_i. If the cooriented part is oriented outwards s', the reverse is true.*

Proof. We will prove it by induction for cooriented part oriented towards s', the proof for the orientation outwards s' is similar.

First, note that it holds for J_1, V_1. We know that the first arrow of io is $\rightarrow$, the first arrow of oi is $\leftarrow$, and s' cannot start with o.

Before the inductive step, we notice that because s' starts with i, J_2 and V_2 end with $o = Y{\leftarrow}X{\rightarrow}Y$. Hence, for all $i > 1$, substrings J_i, V_i end also with o.

Now suppose that the lemma holds for J_i, V_i. In the inductive step we distinguish two cases.

- First, let s' start with J_i, so it is composed of $J_iV_i^n$ and $J_iV_i^{n+1}$. From assumption the first different arrow from the left of $J_iV_i^nJ_iV_i$ and $J_iV_i^{n+1}J_i$ is $\rightarrow$ for the first and $\leftarrow$ for the second. In other words, we have their substrings $J_iV_i^nq \rightarrow$ and $J_iV_i^nq \leftarrow$, respectively. Hence, s' cannot start with $J_iV_i^{n+1}$, because we would get a crossing at $X{\rightarrow}J_iV_i^nq$. The whole s would be

$$\cdots \rightarrow\rightarrow Z{\rightarrow}X{\rightarrow}J_iV_i^nq \leftarrow \cdots Y{\leftarrow}X{\rightarrow}J_iV_i^nq \rightarrow \cdots ,$$

where the right occurrence of $J_iV_i^n$ in the figure above is preceded by $Y{\leftarrow}X{\rightarrow}$ because the previous substring ends with o.

If $J_{i+1} = J_iV_i^n$ and $V_{i+1} = J_iV_i^{n+1}$, the first different arrow of $J_{i+1}V_{i+1}$ and $V_{i+1}J_{i+1}$ is $\rightarrow$ for the first and $\leftarrow$ on the rightfor the second and we proved s' cannot start with V_{i+1}. Similarly if $J_{i+1} = J_iV_i^{n+1}$ and $V_{i+1} = J_iV_i^n$, the first different arrow of $J_{i+1}V_{i+1}$ and $V_{i+1}J_{i+1}$ is $\leftarrow$ for the first and $\rightarrow$ for the second and we proved s' cannot start with J_{i+1}. Thus for this case the inductive step holds.

- Now let s' start with V_i, so it is composed of $V_i^nJ_i$, $V_i^{n+1}J_i$ and theoretically at the start of s' can also be $V_i^mJ_i, m < n$. We will prove that s' cannot start differently than with V_i^{n+1}. Suppose that s' starts with $V_i^mJ_i, m < n+1$. The first different arrow from the left of $V_i^mJ_iV_i$ and $V_i^{n+1}J_i$ is $\leftarrow$ and $\rightarrow$ respectively from assumption. In other words, we have their substrings $V_i^mq \leftarrow$ and $V_i^mq \rightarrow$ respectively. Hence, s' cannot start with $V_i^mJ_i$, because we would get a crossing at $X{\rightarrow}V_i^mq$. The whole s would be

$$\cdots \rightarrow\rightarrow Z{\rightarrow}X{\rightarrow}V_i^mq \leftarrow \cdots Y{\leftarrow}X{\rightarrow}V_i^mq \rightarrow \cdots .$$

  If $J_{i+1} = V_i^nJ_i$ and $V_{i+1} = V_i^{n+1}J_i$, the first different arrow of $J_{i+1}V_{i+1}$ and $V_{i+1}J_{i+1}$ is $\leftarrow$ for the first and $\rightarrow$ for the second and we proved s' cannot start with J_{i+1}. Similarly if $J_{i+1} = V_i^{n+1}J_i$ and $J_{i+1} = V_i^nJ_i$, the first different arrow of $J_{i+1}V_{i+1}$ and $V_{i+1}J_{i+1}$ is $\rightarrow$ for the first and $\leftarrow$ for the second and we proved s' cannot start with V_{i+1}. Thus for this case the inductive step holds too.

We have proved the inductive step in both cases, so by induction the lemma holds.

□

Corollary. The starting part of s' cannot be different from $J_iV_i^{n_i}$ or $V_i^{n_i+1}J_i$.

Proof. It is a direct consequence of the proof of Lemma 11. □

To classify all strings of type 2, it remains to prove that all strings characterized above Lemma 11 and in the second of those cases, starting as in Lemma 11, are non-self-crossing. Of course, its cooriented and non-cooriented parts are non-self-crossing and mutually non-crossing from Lemma 6. Hence, it remains to prove that there cannot be a self-crossing at the border between cooriented and non-cooriented parts. Hence, it is enough to prove the following.

Lemma 12. *Let the cooriented part be oriented towards s'. If the first different arrow of J_iV_i and V_iJ_i from the left is $\rightarrow$ for J_iV_i and $\leftarrow$ for V_iJ_i, then $\cdots \rightarrow\rightarrow\rightarrow J_i$ is non-crossing with s'. Otherwise, $\cdots \rightarrow\rightarrow\rightarrow V_i$ and s' are non-crossing. If the cooriented part is oriented outwards s', the reverse is true.*

Proof. Again, we will prove it only for cooriented part oriented towards s'. We will use the following claim:

Claim. Let p be a substring of the form $p = ABC$, where $A, B, C \in \{J_{i+1}, V_{i+1}\}$. Any way we embed p to s', B must map exactly to some of B, which we see in the notation of s'.

Proof. From Lemma 6 we know that there is only one way how to embed $V_iJ_iV_i$ to $V_i^2J_iV_i^2$. Hence, anywhere we find $ABC = ...V_iV_iJ_iV_i^nJ_iV_iV_i\cdots$ in s' we see B in the notation of s' exactly as B. $\square$

We will prove the lemma by induction. First, we show that it holds for $i = 1, 2$. The first different arrow of io and oi is $\rightarrow$ and $\leftarrow$ respectively, and we know that s' and $\cdots \rightarrow\rightarrow\rightarrow i$ are non-crossing. Further, as we have shown at the beginning of this section, s' and $\cdots \rightarrow\rightarrow\rightarrow io^n$ are non-crossing, and the first different arrow of io^nio^{n+1} and $io^{n+1}io^n$ is the same as the first different arrow of io and oi and it is $\rightarrow$ and $\leftarrow$ respectively. Also, s' and $\cdots \rightarrow\rightarrow\rightarrow i^{n+1}o$ are non-crossing and again, the first different arrow of $i^{n+1}oi^no$ and $i^noi^{n+1}o$ is $\rightarrow$ and $\leftarrow$ respectively.

Assume that the lemma holds for all $j \leq i+1$. In the inductive step, we will show that it also holds for $i+2$. We divide it into two cases:

- Let the first different arrow of J_iV_i and V_iJ_i, which is also the first different arrow for $J_iV_i^{n_i}J_iV_i^{n_i+1}$ and $J_iV_i^{n_i+1}J_iV_i^{n_i}$, be $\rightarrow$ and $\leftarrow$ respectively. So by assumption, s' and $\cdots \rightarrow\rightarrow\rightarrow J_iV_i^{n_i}$ are non-crossing.

  Let W is the string such that we want to prove $\cdots \rightarrow\rightarrow\rightarrow W$ is non-crossing with s'. According to the first different arrows and 11, W is one of the strings $J_iV_i^{n_i}(J_iV_i^{n_i+1})^{n_{i+1}}$ and $(J_iV_i^{n_i})^{n_{i+1}+1}J_iV_i^{n_i+1}$. For contradiction, suppose that there is a crossing between W and s' at some string p. This p must contain $J_iV_i^{n_i}$ because s' is non-self-crossing and it does not cross with $\cdots \rightarrow\rightarrow\rightarrow J_iV_i^{n_i}$. From the claim, wherever in s' we find p we see in the notation of s' the substring $V_i^{n_i-1}$. We will distinguish two cases according to whether J_i or V_i precedes this substring.

  - Let the component J_i precede substring $V_i^{n_i-1}$. That is, we have a string
    $$\rightarrow\rightarrow\rightarrow J_iV_i^{n_i}\cdots J_iV_i^{n_i-1}\cdots .$$
    For W the following holds. From the form of components of s', substring $J_iV_i^{n_i-1}$ anywhere in s' must be succeeded by V_i. And further if $W = J_iV_i^{n_i}(J_iV_i^{n_i+1})^{n_{i+1}}$, the substring from the previous sentence can either continue by V_iJ_i or its continuation has to look as W because of the characterization of Lemma 11. It is shown below.

    $$W : (J_i\overbrace{V_i\ldots V_i}^{n_i})(J_i\overbrace{V_i\ldots V_iV_i}^{n_i+1})^{n_{i+1}}$$

    $$\begin{array}{r}\text{Any part of } s' \\ \text{starting by } J_iV_i^{n_i}\end{array} : (J_iV_i\ldots V_i)\begin{array}{l}V_iJ_i\ldots \\ (J_iV_i\ldots V_iV_i)^{n_{i+1}} = W\end{array}$$

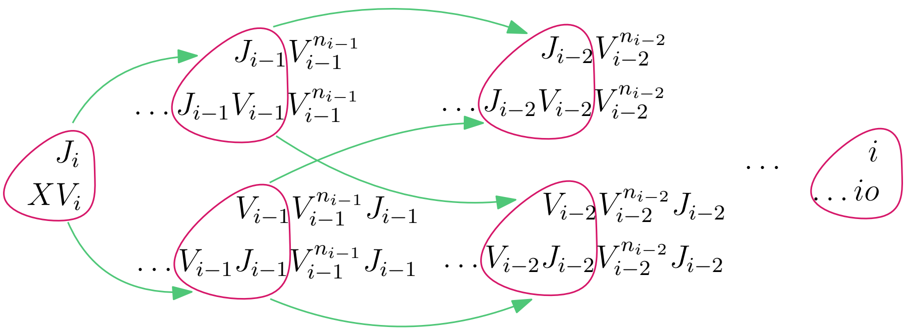
    The case for $W = (J_iV_i^{n_i})^{n_{i+1}+1}J_iV_i^{n_i+1}$ is similar, it is shown below.

$$W : \underbrace{(J_i\overbrace{V_i \ldots V_i}^{n_i})^k (J_i\overbrace{V_i \ldots V_i}^{n_i})^l}_{k+l = n_{i+1}+1} (J_i\overbrace{V_i \ldots V_i V_i}^{n_i+1})$$

Any part of s' starting by $J_iV_i^{n_i}$: $(J_i\overbrace{V_i \ldots V_i}^{n_i})^k V_iJ_i \ldots$
$(J_iV_i \ldots V_i)^l (J_iV_i \ldots V_iV_i) = W$

From the assumption, the first different arrows of J_iV_i and V_iJ_i are $\rightarrow$ and $\leftarrow$, respectively. That is, arrow succeeding the first occurrence of p, can be only $\rightarrow$ from the illustrations above. All arrows preceding this occurence of p are also $\rightarrow$ for, so we cannot have a crossing.

– Let the component V_i precede substring $V_i^{n_i-1}$. We denote the component preceding this V_i (by the component is meant J_i or V_i) by X. Hence, J_i is a substring of X_iV_i from the right (it means that X_iV_i ends by J_i), because of the form of p. Substring J_i can be of the form $J_{i-1}V_{i-1}^{n_{i-1}}$, then $V_i = J_{i-1}V_{i-1}^{n_{i-1}+1}$ or $J_i = V_{i-1}^{n_{i-1}+1}J_{i-1}$, then $V_i = V_{i-1}^{n_{i-1}}J_{i-1}$ because of the Corollary of Lemma 11. Hence, J_{i-1} is a substring from the right of $J_{i-1}V_{i-1}$, or V_{i-1} is a substring from the right of $V_{i-1}J_{i-1}$. Again, we know its forms as compositions of J_{i-2} and V_{i-2}, so we can continue in the same way before we get i is from the right substring of io, which is impossible. This reduction process is illustrated below.



- Let the first different arrow of J_iV_i and V_iJ_i be $\leftarrow$ and $\rightarrow$ respectively. Therefore, by assumption, s' and $\cdots \rightarrow\rightarrow\rightarrow V_i^{n_i+1}J_i$ are non-crossing. The further procedure is the same as in the previous case. In short, the potential crossing must be at the substring containing $V_i^{n_i+1}J_i$, so from the claim, we look to parts of s' of the form $V_i^{n_i}$. We note that if an occurrence of this substring in s' is preceded by V_i, it must continue as the first component, and the first different arrow can be only $\rightarrow$ for the first component. Thus we have no crossing.

  If an occurrence of $V_i^{n_i}$ is preceded by J_i, we cannot get a crossing. From the recursive argument, it is impossible for $V_iV_i^{n_i}$ be from the right the substring of $XJ_iV_i^{n_i}$ (X can be V_i or J_i) because i is not from the right substring of io.

Together, the inductive step, hence also the whole lemma, is proved.

□

Collecting results from Lemmas 11 and 12 we get the following corollary.

Corollary. A valid sequence of J_i, V_i for a fixed cooriented part uniquely determines a string of type 2.

In Subsection 3.1.2 we introduced a bijective map α which maps a string with the period of the form $J_iV_i^{n_i}$ to $\mathbb{Q}_\infty$. Also, we can take α as a bijection between strings of the form $J_iV_i^n$ and $\mathbb{Q}_\infty$ and similarly as a bijection between strings of the form $V_i^nJ_i$ and $\mathbb{Q}_\infty$. We will use it in the following lemma.

Lemma 13. *Let s contain both substrings $J_iV_i^{n_i}J_i$ and $J_iV_i^{n_i+1}J_i$. If the cooriented part is pointing outwards s', then the first component of s' of $V_i^{n_i}J_i$ and $V_i^{n_i+1}J_i$ is that one which corresponds to the smaller number. Otherwise, if the cooriented part is pointing towards s', the first one is that which corresponds to the bigger number.*

Proof. Let A, B be the strings $V_i^{n_i}J_i$ and $V_i^{n+1}J_i$ so that $\alpha(A) < \alpha(B)$, where $\infty = -\infty < 0 < 1 < \infty$. Then we claim that their first different arrow from the left is $\rightarrow$ for BA and $\leftarrow$ for AB.

We prove it by induction. For $i = Y \rightarrow X \leftarrow Y, o = Y \leftarrow Z \rightarrow Y$ it holds that $\alpha(o) < \alpha(i)$ and o, i differ by $\rightarrow$ for o and $\leftarrow$ for i. Further, suppose that $\alpha(J_i) < \alpha(V_i)$ and from inductive assumption, the first different arrow of J_i and V_i is $\rightarrow$ for J_i and $\leftarrow$ for V_i. Then, $\alpha(V_i^{n_i}J_i) < \alpha(V_i^{n_i+1}J_i)$ and the first different arrows for $V_i^{n_i+1}J_i$ and $V_i^{n_i}J_i$ are the same as for V_i, J_i, thus, as we claimed. The proof for $\alpha(J_i) < \alpha(V_i)$ is similar.

Hence, by Lemmas 11 and 12 the proof is done. □

Also, for strings of type 2 we will introduce a new notation according to used J_i, V_i similarly as in Subsection 3.1.2.

- There is no substring in an infinite power. That is s contains both $J_iV_i^{n_i}J_i$ and $J_iV_i^{n_i+1}J_i$ at each step i. So we get an infinite sequence of J_i, V_i, which is bijective to $\mathbb{R} \setminus \mathbb{Q}$ as in the previous subsection. For a fixed cooriented string, in each step i, the first component is uniquely determined. Hence, for a sequence corresponding to $r \in \mathbb{R} \setminus \mathbb{Q}$ we get four strings of type 2 – the string with the cooriented part oriented towards s' and ending by the green arrow, oriented towards s' and ending by the pink arrow, oriented towards s' and ending by the green arrow, oriented towards s' and ending by the pink arrow. We denote them by $^\infty r\langle, \rangle r^\infty, \langle r^\infty, ^\infty r\rangle$ respectively.

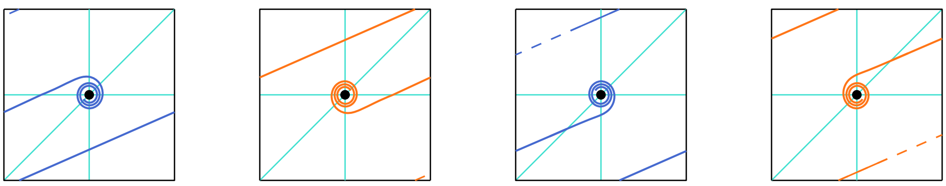


Figure 3.10: Accordions $^\infty r\langle, \rangle r^\infty, \langle r^\infty, ^\infty r\rangle$ respectively.

- The non-cooriented part is equal to $(J_iV_i^{n_i})^{\infty}$ or $(V_i^{n_i}J_i)^{\infty}$. For $\alpha(J_iV_i^{n_i}) = q, q \in \mathbb{Q}_{\infty}$ we get according to the choice of the cooriented part four strings. We denote them, according to the same rule as in the previous case, by $^{\infty}q\langle, \rangle q^{\infty}, ^{\infty}q\rangle, \langle q^{\infty}$.
- The non-cooriented part is equal to $J_iV_i^{\infty}, \alpha(V_i) = q, q \in \mathbb{Q}_{\infty}$. According to Lemmas 11 and 12 there is exactly one possible J_i for each V_i. We get, depending on the cooriented part, four strings $^{\infty}q{+}\langle, \rangle{+}q^{\infty}, ^{\infty}q{-}\rangle, \langle{-}q^{\infty}$.

Example. What four different strings of type 2 belonging to the same $r \in \mathbb{R} \setminus \mathbb{Q}$ look like is shown in Figure 3.10. In Figure 3.11 we can see an example of two different strings belonging to the same number with the same cooriented part.

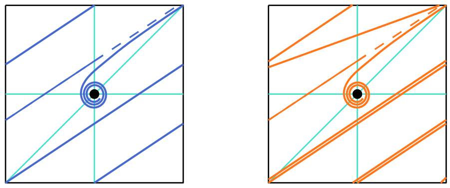


Figure 3.11: Accordions $^{\infty}\frac{2}{3}\rangle$ and $^{\infty}\frac{2}{3}{-}\rangle$

3.1.4 Type 3

Firstly, let s denote an infinite non-crossing string with a cooriented part of length four. This case is rather rare. Without loss of generality, we can assume that the cooriented part of the string with preceding and succeeding arrows is $Z{\leftarrow}Y{\rightarrow}Z{\rightarrow}X{\rightarrow}Y{\rightarrow}Z{\leftarrow}Y$.

From Lemma 10, the significant vertex for the right non-cooriented side is Y, and in the same way, for the left non-cooriented side it is Z. Because of the uniqueness of a significant vertex, one part must be only alternating Y and Z. Without loss of generality we assume that part is the right part. If the left part contains X, then s contains substring $X{\leftarrow}(Z{\leftarrow}Y{\rightarrow}Z)^n{\rightarrow}X$ for some n. Hence we have a self-crossing because on the right side we can find the substring $Y{\rightarrow}(Z{\leftarrow}Y{\rightarrow}Z)^n{\leftarrow}Y$ for arbitrary n. Hence, both non-cooriented parts of s consist of alternating Z and Y.

Now, let s contain a cooriented part of length five. We can assume that this part with the succeeding and preceding arrow is $Y{\leftarrow}X{\rightarrow}Y{\rightarrow}Z{\rightarrow}X{\rightarrow}Y{\rightarrow}Z{\leftarrow}Y$. We note that the significant vertex of both parts is the same, it is Y.

We described the non-self-crossing of the cooriented part together right side in the previous subsection. We used only three arrows of the cooriented string everywhere, so now it is precisely the same. Similarly, we know the possible form of a non-self-crossing substring consisting of the left and cooriented parts. It remains to find out how we can give these three parts together.

From Section 3.1.2 we have an assignment of numbers $\mathbb{R}_{\infty}$ to non-cooriented parts. We will work with it in the following lemma.

Lemma 14. *Two non-cooriented parts with different assignment of numbers in $\mathbb{R}_\infty$ are crossing.*

Proof. The assignment of numbers is determined either by an infinite sequence of J_i, V_i (in the non-rational case) or by an infinite period $V_{j+1} = J_j V_j^{n_j}$ for some j (in the rational case), we will denote $J_{j+1} = \emptyset$. In other words in the rational case the assignment is determined by a finite sequence (J_i, V_i).

We take two non-cooriented parts p, p' with different assignments. That is, they are determined by different finite or infinite sequences $(J_i, V_i), (J'_i, V'_i)$. If p and p' have a different significant vertex, they are crossing. Otherwise, there exists the last i such that $J_i = J'i$ and $V_i = V'_i$. We note that $J_i \neq \emptyset$ because p and p' would correspond to the same number. If any of the pair J_{i+1}, V_{i+1} contains V_i in a power which differs by at least two from the power of V_i in any of the pair J'_{i+1}, V'_{i+1}, then p and p' are crossing according to 5. The only different possibility is that $J_{i+1}, V_{i+1}, J'_{i+1}, V'_{i+1} \in \{J_i V_i^n, J_i V_i^{n+1}, \emptyset\}$. Then either $V_{i+1} \neq V'_{i+1}$, or one of J_{i+1} is $\emptyset$ and the second is not. In both cases p and p' are crossing from Lemma 5.1 respective 5.2. □

We get that strings of type 3 consist of one cooriented part of length five (in three particular cases of length four), which is connected with two non-cooriented parts assigned to the same number. That is, from the end of subsection 3.1.3 there are for each irrational number two candidates to string of type 3, which differs only by switching colours of all arrows. For rational numbers there are up to colours two possibilities for each one-side-infinite non-cooriented part. Hence there are eight candidates for strings of type 3 for each rational number. In the rest of this subsection we will show that exactly six of eight candidates for every rational number and both candidates for every irrational are non-self-crossing.

We will denote these candidates in the similar way as strings of type 2, which is described in the end of subsection 3.1.3. Hence, we have ${}^\infty r\rangle r^\infty$ and ${}^\infty r\langle r^\infty$ for $r \in \mathbb{R}$, where the first is for the string with the cooriented part ending by green arrows and the second for the string with the cooriented part ending by pink arrows. Similarly, we denote candidates for rational numbers by ${}^\infty q\rangle q^\infty$, ${}^\infty q\langle q^\infty$, ${}^\infty q{-}\rangle q^\infty$, ${}^\infty q{+}\langle q^\infty$, ${}^\infty q\rangle{+}q^\infty$, ${}^\infty q\langle{-}q^\infty$, ${}^\infty q{-}\rangle{+}q^\infty$, ${}^\infty q{+}\langle{-}q^\infty$

Lemma 15. *Strings ${}^\infty q{-}\rangle{+}q^\infty$, ${}^\infty q{+}\langle{-}q^\infty$ are self-crossing, the rest of the candidates are non-self-crossing.*

Proof. First we note that strings ${}^\infty q{-}\rangle + q^\infty$, ${}^\infty q{+}\langle{-}q^\infty$ are self-crossing because they contain substrings denoted by $+$ and $-$. We recall that one of $+$ and $-$ is of the form $J_i V_i^{n_i-1}$ or $V_i^{n_i-1} J_i$ and second of the form $J_i V_i^{n_i+1}$ or $V_i^{n_i+1} J_i$. Alternatively, they are $J_i^2 V_i$ or $V_i^2 J_i$ and $J_i V_i^2$ or $V_i^2 J_i$ if $n_i = 1$. Hence, together with the neighbourhood of these substrings, we get a crossing.

Now we will show that the rest of the candidates are non-self-crossing. Both non-cooriented parts of each candidate are non-crossing, among other things, both are substrings of the same string of type 1. Also, they are non-crossing with the cooriented part. It remains to look at substrings that contain parts of both non-cooriented and cooriented part. If such substring p contains at least three following arrows with the same orientation, there is no substring which p can cross.

Further, we can consider p as a walk that starts in the cooriented part and ends in a non-cooriented. We realize that each other occurrence of this walk p has also start closer to the cooriented part than the end. It holds thanks to colours of arrows. Next to each significant vertex from the direction of the cooriented part are arrows of the same colours, thanks to the alternating colours of the arrows. For a better understanding, see Figure 3.12 for significant vertex Y.

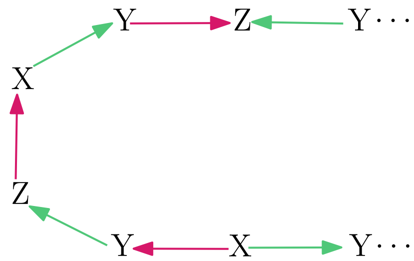


Figure 3.12: Colours of arrows of the cooriented part

Hence, no other occurrence of p reaches both the cooriented and a non-cooriented part. The last option to the crossing is an occurrence of p in the second non-cooriented part. Nevertheless, it cannot give a crossing because of the similar structure of both non-cooriented parts. When we find another occurrence of p from Lemma 12 it is succeeding by the same arrow, or if both occurrences are succeeding by the different arrow, the arrow preceding p has the same direction as the arrow succeeding p, so there is no crossing. □

Example. Figure 3.13 shows three accordions corresponding to three different strings of type 3 which are assigned to $\frac{2}{3}$. They are ${}^{\infty}\frac{2}{3}\langle-\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{2}{3}\langle\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{2}{3}+\langle\frac{2}{3}^{\infty}$ respectively. If we write these strings down arrow by arrow we get

$$^{\infty}(2\to 3\to 1\leftarrow 3\to 1\leftarrow 3\leftarrow)2\leftarrow 1(\leftarrow 3\leftarrow 2\to 3\to 1)(\leftarrow 3\leftarrow 2\to 3\to 1\leftarrow 3\leftarrow 1)^{\infty},$$

$$^{\infty}(2\to 3\to 1\leftarrow 3\to 1\leftarrow 3\leftarrow)2\leftarrow 1(\leftarrow 3\leftarrow 2\to 3\to 1\leftarrow 3\leftarrow 1)^{\infty},$$

$$^{\infty}(2\to 3\to 1\leftarrow 3\to 1\leftarrow 3\leftarrow)(2\to 3\to 1\leftarrow 3\to 1\leftarrow 3\to 1\leftarrow 3\leftarrow)2\leftarrow 1(\leftarrow 3\leftarrow 2\to 3\to 1\leftarrow 3\leftarrow 1)^{\infty}.$$

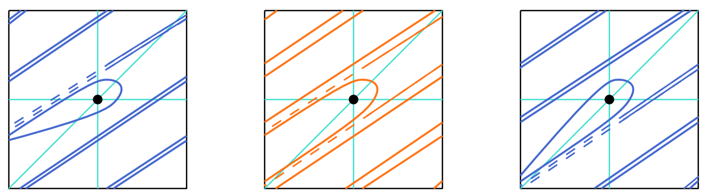


Figure 3.13: Accordions ${}^{\infty}\frac{2}{3}\langle-\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{2}{3}\langle\frac{2}{3}^{\infty}$, ${}^{\infty}\frac{2}{3}+\langle\frac{2}{3}^{\infty}$

3.2 Observations about the lattice

In the previous section we have found all infinite non-self-crossing strings of type 1, 2 and 3. We recall that for each irrational number we have exactly seven infinite non-self-crossing strings with it assigned. Concretely

- one of type 1: ${}^{\infty}r^{\infty}$,
- four of type 2: ${}^{\infty}r\langle, \rangle r^{\infty}, \langle r^{\infty}, {}^{\infty}r\rangle$,
- two of type 3: ${}^{\infty}r\rangle r^{\infty}, {}^{\infty}r\langle r^{\infty}$.

For each number in $\mathbb{Q}_{\infty}$ we have 17 non-self-crossing infinite strings. They are

- three of type 1: ${}^{\infty}q^{\infty}, {}^{\infty}q{+}q^{\infty}, {}^{\infty}q{-}q^{\infty}$,
- eight of type 2: ${}^{\infty}q\langle, \rangle q^{\infty}, {}^{\infty}q\rangle, \langle q^{\infty}, {}^{\infty}q{+}\langle, \rangle{+}q^{\infty}, {}^{\infty}q{-}\rangle, \langle{-}q^{\infty}$,
- six of type 3: ${}^{\infty}q\rangle q^{\infty}$, ${}^{\infty}q\langle q^{\infty}$, ${}^{\infty}q{-}\rangle q^{\infty}$, ${}^{\infty}q{+}\langle q^{\infty}$, ${}^{\infty}q\rangle{+}q^{\infty}$, ${}^{\infty}q\langle{-}q^{\infty}$.

From Lemma 14 we know that two strings that are assigned to different numbers are crossing. Now we will prove a stronger statement for strings that have numbers from different intervals of $(-\infty, 0), (0, 1), (1, \infty)$ assigned to them.

Lemma 16. *Let s, s' be two infinite non-self-crossing strings with unique and different significant vertices X, Y. Then there is both a positive crossing from s to s' and a positive crossing from s' to s.*

Proof. String s contains a substring of the form $X \leftarrow Z \rightarrow X$ and a string s' substring of the form $Y \rightarrow Z \rightarrow$, so we have found a positive crossing from s to s'.

On the other hand, we will find a positive crossing from s' to s with the common part $X{\rightarrow}Y$. From Lemma 4 and assumption we have that there exists a vertex Z such that the string $Z{\rightarrow}X{\rightarrow}Y$ is the substring of s. The vertex X is significant, so we get the substring of s $Z{\rightarrow}X{\rightarrow}Y{\leftarrow}X$. In a similar way we get a substring of s' $Y{\leftarrow}X{\rightarrow}Y{\rightarrow}$; therefore the lemma is proven. □

In this section we denote maximal sets of non-self-crossing infinite strings over the Markov algebra by maxNC. Similarly we will denote maximal sets of these strings of type 1, 2 and 3 by maxNC3.

In other words Lemma 16 says that two strings which have been assigned numbers from different intervals $(-\infty, 0), (0, 1), (1, \infty)$ are crossing in both directions. Hence, two elements of maxNC containing them are incomparable in partial order from Theorem 1.

Let us summarize our knowledge about the lattice we are looking for into Figure 3.14. In the figure there is a round bubble for each number in $\mathbb{R}_{\infty}$. It contains elements of maxNC3 consisting of strings with this number assigned. That is, each element of maxNC3 is located in one round bubble. We do not know about comparability inside. Also we do not know the comparability between elements inside pink bubbles and between elements of pink and orange bubbles. Whatever is in a different pink bubble is incomparable.

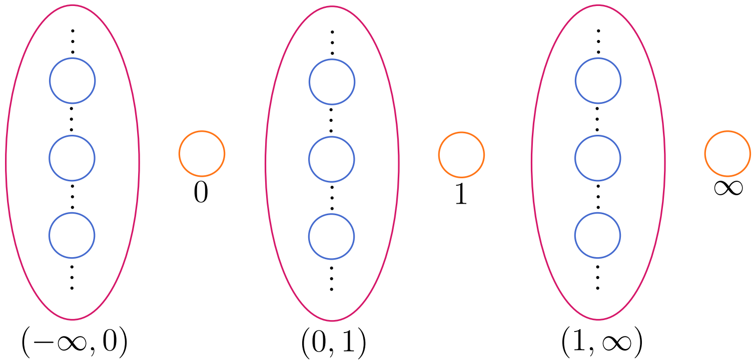


Figure 3.14: Scheme of the lattice

Note that adding strings of type 4 can just change the lattice in the following ways. Every element of maxNC3 could split into more elements of maxNC or we could get completely new elements of maxNC consisting only of strings of type 4. Adding strings can create incomparability, but it cannot cancel it.

What remains to show is the following. We want to explore the structure of round bubbles, that is find positive crossings between strings which have the same number assigned. Further, we need to specify comparability between elements of different round bubbles in the same pink bubble and also with elements of orange bubbles and elements across all the orange bubbles. In the end, we need to add strings of type 4. Then we will gain the lattice maxNC and with the help of Theorem 2 also the lattice of torsion classes.

Conclusion

In this thesis we introduced gentle algebras. We showed a way to find the lattice of torsion classes over them in a few examples. To do it we used string combinatorics and followed the theory in Chan and Demonet [2020]. At the same time, we tried to present our results in a geometrical view.

The large part of the thesis consists of chapter 3, where we took the first steps to find the lattice of torsion classes of the Markov algebra. We found all infinite non-self-crossing strings of types 1, 2 and 3. Also, we stated some properties about their crossings. On the other hand, a lot of work remains to be done to achieve our goal, the classification of torsion classes of the Markov algebra. Concretely we need to find all strings of type 4 and determine remaining positive crossings between strings of all types.

Bibliography

Ibrahim Assem, Daniel Simson, and Andrzej Skowronski. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006. ISBN 9780521584234.

Karin Baur and Raquel Coelho Simoes. A geometric model for the module category of a gentle algebra, 2018.

Aaron Chan and Laurent Demonet. Classifying torsion classes of gentle algebras, 2020.

Henning Krause. Representations of quivers via reflection functors, 2010.

Hugh Thomas. An introduction to the lattice of torsion classes, 2021.

List of Figures