Posudek diplomové práce

Matematicko-fyzikální fakulta Univerzity Karlovy

Autor práce	Dominik Krasula
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Text posudku:

The master thesis of Dominik Krasula deals with the notion of a Gariel-Roiter measure and its applications to the representation theory of quivers. The notion of a measure refining the composition length of a finite length module was first used by Roiter in 1968 in his celebrated proof of the first Brauer-Thrall Conjecture. The latter says that the notions of finite representation type, and of bounded representation type, coincide for each finite dimensional algebra over any field. Gabriel soon extended the idea to what is now called a (standard) Gabriel-Roiter measure of a finite length module.

For a long time, the use of Gabriel-Roiter measures was restricted to algebras of bounded representation type. In early 2000s, Ringel extended the technology to the unbounded type, and connected it with the Auslander-Reiten theory of indecomposable modules and irreducible maps. An extension of the theory to general Gabriel-Roiter measures, computed from general length functions, possibly different from the composition length, was then performed by Krause. Chen developed the theory and computed the structure of general measures over some hereditary artin algebras of tame representation type.

The master thesis proceeds in the line of Krause's and Chen's work, an provides new results for the case when the underlying graph of the quiver, Q, is a tree - which covers the Dynkin and Euclidean cases (except for $\tilde{A}_n$), but much more - and the representation, M, is *thin* (i.e., all components of its dimension vector are 0 or 1). In his rather restrictive setting, one can work combinatorially, just with the support graphs of representations rather than structural maps. In Section 2.1.2, a combinatorial algorithm is presented for the calculation of the GR-measure $\mu_l(M)$ for an indecomposable thin module M whose support is Q, using only the weighted quiver Q_l of Q. In Theorem 38, the author proves that for any field K, if $M \in \text{mod}-KQ$ is indecomposable, then for any indecomposable filtration $\mathcal{R}$ of M there exists a general length function l on $\text{mod}-KQ$ such that $\mathcal{R}$ is a Gabriel-Roiter filtration for $\mu_l(M)$. The final chapter of the thesis presents a classification of all standard Gabriel-Roiter measures for the algebra $K\tilde{A}_3$ where K is any algebraically closed field, and $\tilde{A}_3$ has the one source and one non-adjacent sink orientation.

The topic of the thesis is demanding and requires a number of technical prerequisites. The thesis contains several new results. However, despite an extensive chapter of preliminaries (taking more than half of the text), the thesis is not easy to read. It is also a pity that the author did not use a spell-checker to correct his English.

Here are my main questions and comments concerning the text:

Parts of the Preliminaries are formulated for general rings and modules. Then one has to distinguish between finitely generated and finite length modules (in particular, on page 5, line 5, finitely generated should be replaced finite length). For general rings (even commutative noetherian), there need not exist any representative set for all indecomposable modules - so I suggest to restrict the definition of indecomposable modules on p.5 to the finite length modules. Similarly, better restrict the definition of uniform modules to the finite length ones: then the formulation 'M is uniform, iff $\mathrm{soc}(M)$ is a simple module' is correct. (It is not for general rings, where the correct definition of uniformity is broader: any two non-zero submodules of M have a non-zero intersection. The submodule structure of uniform modules over general rings may be very complex).

page 8, line -8. You also need Q_1 to be finite for the argument.

p.13, l.-9. It is not true that a finite length module has only finitely many submodules. Better appeal to the Jordan-Hölder Theorem.

p. 14, Definition 5. This is the crucial definition of the lexicographic order $\leq_L$ on the poset $\mathrm{Ch}(S)$ of all finite chains in S where $(S, \leq_S)$ is a poset: $X \leq_L Y$, iff $\min_S(Y \setminus X) \leq_S \min_S(X \setminus Y)$. Then $X \subseteq Y$ implies $X \leq_L Y$, but for $x, y \in S$, we have $\{x\} \leq_L \{y\}$, iff $y \leq_S x$.

It easily follows that $\leq_L$ is a total order on $\mathrm{Ch}(S)$ if $\leq_S$ is a total order, but not otherwise. *Don't you thus have to add the assumption of T being totally ordered to your further applications, notably to the statement of Theorem 8 ?* (The original version of this theorem in [14] has $T = \mathbb{N}$ with its natural total order.)

p.15, l.-2. The definition of equivalent functions should be moved from the claim of Theorem 8 up, ahead of the *Example* on top of p. 15, where this notion of equivalence is first used.

p.16, l.9. Better use *recursive formula* instead of *recursive definition.* (The notion has already been defined in Definition 7.).

p.16, l.-12. Here, the author suddenly starts to use an alternative notation for the length of a module M, namely $|M|$ rather than $l(M)$. I suggest to stick to the l notation throughout the text to avoid confusion (as on p.37, l.-6, where the same symbol in one formula has two meanings: it denotes both the cardinality and the composition length).

p.17, l.2. *submodule* rather than *ideal.*

p.17, l.16. I rather than N.

p.19, l.4. The subscript l is missing from μ (2 times).

p.19, l.11. (GR2) is trivial, as $l(M)$ is the last term of $\mu_l(M)$.

p.19, Proposition 16. $X \subseteq \bigoplus_{i \leq n} Y_i$ rather than $X \leq \oplus Y_i$ ($\leq$ has other meanings in the text than the submodule relation).

p.20, l.7. $T_1 \oplus T_2$ rather than $T_1 \cup T_2$.

p.20, Corollary 18. First, the module C must be from mod$-R$ for $\mu_l(C)$ to make sense. The final sentence should read: *Then for any length function l on mod$-R$, the GR-measure μ_l on mod$-R$ is bounded by $\mu_l(C)$.*

p.20, Example 19. Note that the (infinite dimensional) path algebra in this case is just the polynomial algebra $K[x]$.

p.21, l.13. ... finitely many *indecomposable* projective modules ...

p.21, l.18. ... submodules *of P* are ...

p.21, l.-12. ... that the *standard* GR-measure ...

p.22, l.4. ... *a module with* the maximal GR-measure ...

p.22, l.13. A rather than R (2 times).

p.23, introduction of Section 1.6. It is not difficult to express the AR-translation τ as a functor for any hereditary artin algebra R (which is the setting for all path algebras of quivers). Namely, $\tau(-) = \mathrm{D}(\mathrm{Ext}^1_R(-, R))$ where D is the standard duality.

p.25, l.1. ... the reflection σ_i ...

p.26, Proposition 26. The P(i) and P(j) are *indecomposable* projective KQ-modules.

p.28, l.13. For a *representation infinite* artin algebra A, ...

p.29, l.5. r rather than R.

p.34, l.2. There is a missing vertex 6 in the second quiver.

p.35, Theorem 38. One could perhaps add here that in the given setting, $R_i = R(Q_i)$ $(i \leq m)$ for suitable subquivers Q_i of Q.

p.35, l.-11. l_m rather than l_l.

p.36, l.9. Explain in more detail the following step: ‘ we set their new length to be their l_i-length divided by some constant big enough such that ...’

p.39. Is there any conjectural criterion in order for two length functions l and k that induce the same ordering of simple modules to yield equivalent GR-measures μ_l and μ_k ?

p. 45, l.-1 Give a reference for the ‘other methods’.

Formal comments.

The thesis uses a weird numbering. Lemmas, Propositions, Examples, Theorems, etc. (but not Definitions) have a common numbering irrespective of the sections or chapters in which they appear (e.g., Theorem 1 appears in Section 1.2.1, there are no Theorems numbered 2, 3 or 4, but there is Theorem 5 appearing in Section 1.2.3). Definitions enjoy a separate numbering, again irrespective of the section or chapter in which they appear. However, there are also unnumbered *Examples* and *Remarks* scattered through the text.

The presentation of references is unusual: For most books, the publisher, the place of issue, and sometimes even the year of publication, are missing, while the journal references include even the months of publication (albeit given by their Czech abbreviations, such as dub. or zář.).

Práci doporučuji k obhajobě.

V Praze dne 26. 8. 2023

Podpis: