FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Mavis Otrubů

Kan extensions and adjoint functors

Department of algebra

Supervisor of the bachelor thesis: doc. Mgr. Jan Šaroch, Ph.D
Study programme: General Mathematics
Study branch: General Mathematics

Prague 2023

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I would like to thank my supervisor Jan Šaroch for helping me understand deeper parts of the category theory and providing me interesting materials for further study. I would also like to thank my boyfriend Maki Krejčí for his support.

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Author: Mavis Otrubů

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Abstract: This thesis is devoted to Kan extensions. First, we provide needed definitions and prove a theorem which gives us an existence condition for a Kan extensions. The proof of this theorem also contains a guide to constructing Kan extensions. The main goal is to present a result which puts Kan extensions and adjoint functors in relation. We also connect this theorem to global Kan extensions. We apply these abstract results in the last chapter, where we formulate and solve a particular problem concerning adjoint functors between the categories of G-sets.

Keywords: Kan extension, Adjoint functor, Limit, Complete category, Category of elements

Contents

Introduction

The category theory was initially introduced by Samuel Eilenberg and Saunders Mac Lane in the 1940'. Its main advantage is that the categorical language can be used in all fields in mathematics. It helps to see mathematics from a higher perspective which can give us interesting connections between different areas in mathematics or help us generalize some mathematical concepts. Some people see category theory as a useful tool but some even build category theory without using set theory and present it as the new foundation of mathematics opposed to the set theory. Another interesting part is that even such an abstract theory has applications outside of mathematics, for example in computer science and even in philosophy, neuroscience or epidemiology. It can capture dynamic systems in a visual way unlike differential equations and it has somewhat simple language which makes it possible for people outside of mathematics to use it.

Daniel Kan made an important contribution to the category theory in his article *Adjoint functors* (Kan [1958]) where he introduced adjoint functors which is one of the most important concepts in the category theory. In this article, he also used constructions which are known today as Kan extensions. Let $F\colon A \longrightarrow C$ and $G\colon A \longrightarrow B$ be functors. Kan extensions answer the question what is the "best" functor $H\colon B \longrightarrow C$ closing the triangle. This is a very universal situation. As Saunders Mac Lane famously said "The notion of Kan extensions subsumes all the other fundamental concepts of category theory" (Mac Lane [1998]). In this thesis, we will differentiate between global Kan extensions and local Kan extensions but we will refer to local Kan extensions just as Kan extensions.

In the first chapter, we will introduce some essential definitions for this thesis. In the second chapter, we will define Kan extensions, state and prove important theorem which gives us the existence condition for Kan extensions. The proof will also be the manual for the construction of Kan extensions. In the third chapter, we will state and prove theorem connecting Kan extensions and adjoint functors, introduce the global Kan extension and put it in context. In the last chapter, we will solve an interesting problem using theorems from this thesis.

1. Basic definitions

In this chapter we will introduce some definitions and theorems which will be needed in proofs.

Definition 1.1 (Category of elements)**.** Consider a functor $F\colon \mathscr{A} \longrightarrow Set$ from a category $\mathscr{A}$ to the category of sets. The category $\mathbf{Elts}(F)$ of *"elements of F"* is defined in the following way.
(1) The objects of $\mathbf{Elts}(F)$ are the pairs (A, a) where A $\in$ obj $\mathscr{A}$ and $a \in F(A)$
(2) The arrow $f\colon (A, a) \longrightarrow (B, b)$ of $\mathbf{Elts}(F)$ is the arrow $f\colon A \longrightarrow B$ of $\mathscr{A}$ such that $Ff(a) = b$.
(3) Composition of arrows of $\mathbf{Elts}(F)$ is induced by composition of arrows of $\mathscr{A}$.

Now let us recall what it means that a functor $F\colon \mathscr{A} \longrightarrow Set$ is representable. One definition is that F is representable if it is naturally equivalent to a functor $\mathscr{A}(A, -)$ for some $A \in \text{obj}(\mathscr{A})$. Another equivalent definition is that F is representable if it has a universal pair. It means that there exist a pair (A, a) where $A \in \mathscr{A}$ and $a \in F(A)$ such that for every $B \in \mathscr{A}$ and $b \in F(B)$ there is a unique morphism $\alpha\colon A \longrightarrow B$ such that $F\alpha(a) = b$. Another characterisation is introduced in the following remark.

Remark. F is representable if and only if $\mathbf{Elts}(F)$ has an initial object.

Proof. First let us assume that F is representable. Let (A, a) be an universal pair. It means that for every $B \in \mathscr{A}$ and $b \in F(B)$ we have a unique morphism $\alpha \in \mathscr{A}(A, B)$ such that $F\alpha(a) = b$. But this means that in the category $\mathbf{Elts}(F)$ we have one and only one morphism $\alpha\colon (A, a) \longrightarrow (B, b)$. Now because B and b were chosen arbitrary we see that (A, a) is the initial object.
Now let us assume that (A, a) is the initial object in $\mathbf{Elts}(F)$. It means that for every $(B, b) \in \mathbf{Elts}(F)$ we have a unique morphism $\alpha\colon (A, a) \longrightarrow (B, b)$. α is a morphism in $\mathbf{Elts}(F)$ so from the second condition in the definition above we have $F\alpha(a) = b$. Because it holds $\forall (B, b) \in \mathbf{Elts}(F)$, we have for every $B \in \mathscr{A}$ and $b \in F(B)$ a unique morphism α such that $F\alpha(a) = b$ and that is exactly the definition of the universal pair for F so F is representable. $\square$

Definition 1.2 (Co-free object)**.** Let $\mathscr{M}$ and $\mathscr{N}$ be categories and $N \in obj(\mathscr{N})$. Let $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ be a functor. A pair (M, f) is a co-free object over N with respect to F if $M \in obj(\mathscr{M})$, $f\colon F(M) \longrightarrow N$ is a morphism in $\mathscr{N}$ and for every $M' \in obj(\mathscr{M})$ and every morphism $g\colon F(M') \longrightarrow N$ there is a unique morphism $h\colon M' \longrightarrow M$ such that $g = f \circ F(h)$.

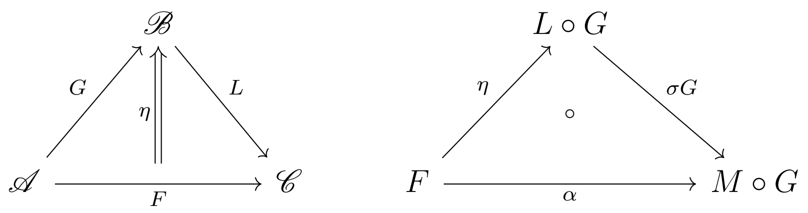
Now we will introduce one of the equivalent definitions of adjoint functors, specifically the one which we will use later.

Definition 1.3 (Adjoint functor)**.** Let $\mathscr{H}$ and $\mathscr{K}$ be categories and $U\colon \mathscr{H} \longrightarrow \mathscr{K}$ be a functor. Then the functor $F\colon \mathscr{K} \longrightarrow \mathscr{H}$ is the left adjoint of U if there is a natural transformation $\epsilon\colon F \circ U \longrightarrow 1_{\mathscr{H}}$ such that the pair $(U(b), \epsilon_b)$ is co-free object over b with respect to F.

2. Kan extension theorem

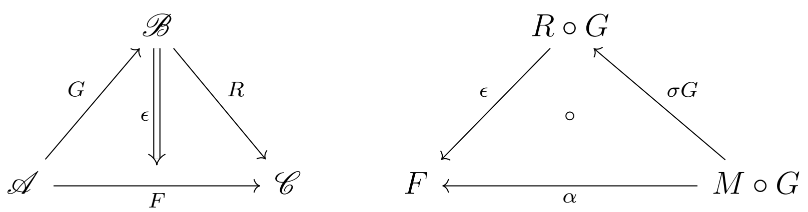
In this chapter we will formulate and prove theorem which gives us an existence condition for Kan extensions. But first, we begin with definitions.

Definition 2.1 (Left Kan extension)**.** Let us consider categories $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ and two functors $F\colon \mathscr{A} \longrightarrow \mathscr{C}$ and $G\colon \mathscr{A} \longrightarrow \mathscr{B}$. The left Kan extension F along G, if it exists, is a pair (L, η) where:
• $L\colon \mathscr{B} \longrightarrow \mathscr{C}$ is a functor
• $\eta\colon F \longrightarrow L \circ G$ is a natural transformation,
as we can see in the left diagram below. (L, η) also has to satisfy the following universal property:
For every functor $M\colon \mathscr{B} \longrightarrow \mathscr{C}$ and a natural transformation $\alpha\colon F \longrightarrow M \circ G$ there is a unique natural transformation $\sigma\colon L \longrightarrow M$ such that $\sigma G \circ \eta = \alpha$, as we can see in the right diagram below. The left Kan extension will be denoted by $Lan_G F$.



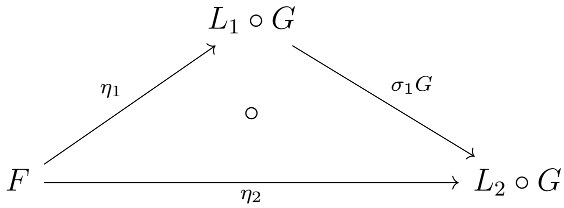
Now we introduce the dual notion to the left Kan extension, the right Kan extension.

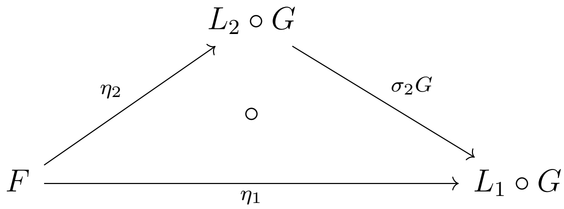
Definition 2.2 (Right Kan extension)**.** Let us consider categories $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ and two functors $F\colon \mathscr{A} \longrightarrow \mathscr{C}$ and $G\colon \mathscr{A} \longrightarrow \mathscr{B}$. The right Kan extension F along G, if it exists, is a pair (R, ϵ) where:
• $R\colon \mathscr{B} \longrightarrow \mathscr{C}$ is a functor
• $\epsilon\colon R \circ G \longrightarrow F$ is a natural transformation,
as we can see in the left diagram below. (R, ϵ) also has to satisfy the following universal property:
For every functor $M\colon \mathscr{B} \longrightarrow \mathscr{C}$ and a natural transformation $\alpha\colon M \circ G \longrightarrow F$ there is a unique natural transformation $\sigma\colon M \longrightarrow R$ such that $\epsilon \circ \sigma G = \alpha$, as we can see in the right diagram below. The right Kan extension will be denoted by $Ran_G F$.



One could ask what will happen if we have two different left Kan extensions F along G. We will answer that in the following remark.

Remark. Let us assume that L_1 and L_2 are both left Kan extensions F along G with the corresponding natural transformations η_1 and η_2 where F and G are the same as in theorem. From the definition we get the natural transformations $\sigma_1 \colon L_1 \longrightarrow L_2$ and $\sigma_2 \colon L_2 \longrightarrow L_1$ such that the following diagrams commute:





We have two equalities:

$$\eta_1 = \sigma_2 G \circ \eta_2$$

$$\eta_2 = \sigma_1 G \circ \eta_1$$

From this we get

$$\eta_1 = \sigma_2 G \circ \sigma_1 G \circ \eta_1$$

and it is equivalent to

$$\sigma_2 G \circ \sigma_1 G = 1_{L_1 G}$$

That mean that for every $A \in \mathscr{A}$ we have ${\sigma_2}_{G(A)} \circ {\sigma_1}_{G(A)} = 1_{(L_1 \circ G)(A)}$ which means that ${\sigma_2}_{G(A)}$ is an inverse of ${\sigma_1}_{G(A)}$ which means that functors $L_1 \circ G$ and $L_2 \circ G$ have to be naturally equivalent.

Now we will introduce Kan extension theorems.

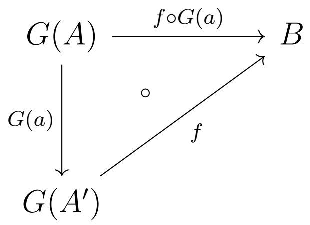
Theorem 2.3 (Left Kan extension theorem)**.** *Consider two functors $G \colon \mathscr{A} \longrightarrow \mathscr{B}$ and $F \colon \mathscr{A} \longrightarrow \mathscr{C}$, with $\mathscr{A}$ small and $\mathscr{C}$ cocomplete. Under these conditions, the left Kan extension F along G exists.*

Proof. We will use the same notation as in the definitions above and we will split this proof into three parts.

Part 1

In this part we will define the functor L on objects and arrows. First, let us fix some $B \in \mathscr{B}$. Consider the category of elements (Definition 1.1) of the contravariant functor $\mathscr{B}(G(-), B) \colon \mathscr{A} \longrightarrow Set$ and denote it by δ_B. δ_B is small since $\mathscr{A}$ is small.
We will show why it has to be contravariant. Consider an arrow $a \colon A \longrightarrow A'$. We need to define the arrow $\mathscr{B}(G(a), B) : \mathscr{B}(G(A'), B) \longrightarrow \mathscr{B}(G(A), B)$. We

will do it the following way: $\mathscr{B}(G(a), B)(f) = f \circ G(a)$, $\forall f \in \mathscr{B}(G(A'), B)$ as we can see in the following picture.



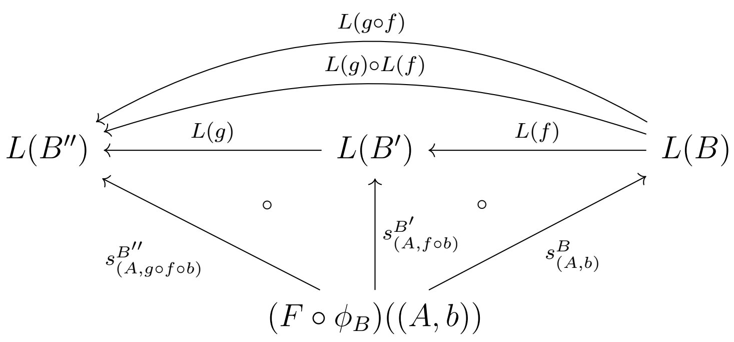
From this picture it is also easy to see that if we try to do it the other way there would be no clear method to define a map from $\mathscr{B}(G(A), B)$ to $\mathscr{B}(G(A'), B)$.

Now we write $\phi_B \colon \delta_B \longrightarrow \mathscr{A}$ for the corresponding forgetful functor defined in the way that $\phi_B((A, a)) = A$, $\forall (A, a) \in \delta_B$. Let us denote the colimit of the diagram $(F \circ \phi_B) \colon \delta_B \longrightarrow \mathscr{C}$ by $(L(B), (s^B_{(A,b)})_{(A,b) \in \delta_B})$ where $(s^B_{(A,b)})_{(A,b) \in \delta_B} :$ $(F \circ \phi_B)((A, b)) \longrightarrow L(B)$ are the components of the natural transformation which witnesses the colimit. And this is how we define L on objects.

Now we need to define define L on arrows. Given a morphism $f \colon B \longrightarrow B'$ and an object $(A, b) \in \delta_B$, the pair $(A, f \circ b)$ is an object in $\delta_{B'}$. Let us observe that the morphism $a \colon (A, b) \longrightarrow (A', b')$ in δ_B immediately gives rise to the morphism $a \colon (A, f \circ b) \longrightarrow (A', f \circ b')$ in $\delta_{B'}$. Another important thing is that $(F \circ \phi_B)((A, b))$ and $(F \circ \phi_{B'})((A, f \circ b))$ are the same objects in $\mathscr{C}$. Together this gives us that $(L(B'), (s^{B'}_{(A,f \circ b)})_{(A,b) \in \delta_B})$ is a cocone on $F \circ \phi_B$. Because $L(B)$ is a colimit and $L(B')$ is a cocone we get the unique factorisation $L(f) \colon L(B) \longrightarrow L(B')$ such that $L(f) \circ s^B_{(A,b)} = s^{B'}_{(A,f \circ b)}$ $\forall (A, b) \in \delta_B$ and this is how we define L on arrows.

Part 2

In this part we will show that L is a functor. Equality $L(1_B) = 1_{L(B)}$ clearly holds because the equality $1_{L(B)} \circ s^B_{(A,b)} = s^B_{(A,b)}$ holds $\forall (A, b) \in \delta_B$. For the composition suppose morphisms $f \colon B \longrightarrow B'$ and $g \colon B' \longrightarrow B''$ and a fixed object $(A, b) \in \delta_B$. This gives us objects $(A, f \circ b) \in \delta_{B'}$ and $(A, g \circ f \circ b) \in \delta_{B''}$. The morphism $L(g \circ f)$ is defined as the unique morphism such that equality $L(g \circ f) \circ s^B_{(A,b)} = s^{B''}_{(A,g \circ f \circ b)}$ holds. We also have morphisms $L(f)$ and $L(g)$ such that the equalities $L(f) \circ s^B_{(A,b)} = s^{B'}_{(A,f \circ b)}$ and $L(g) \circ s^{B'}_{(A,f \circ b)} = s^{B''}_{(A,g \circ f \circ b)}$ hold as is captured in the diagram below.



Now we have three equalities:

$$
\begin{aligned}
L(f) \circ s^{B}_{(A,b)} &= s^{B'}_{(A,f\circ b)} \\
L(g) \circ s^{B'}_{(A,f\circ b)} &= s^{B''}_{(A,g\circ f\circ b)} \\
L(g\circ f) \circ s^{B}_{(A,b)} &= s^{B''}_{(A,g\circ f\circ b)}
\end{aligned}
$$

If we combine the first two equalities we get:

$$
\begin{aligned}
L(g)\circ L(f) \circ s^{B}_{(A,b)} &= s^{B''}_{(A,g\circ f\circ b)} \\
L(g\circ f) \circ s^{B}_{(A,b)} &= s^{B''}_{(A,g\circ f\circ b)}
\end{aligned}
$$

This holds $\forall (A,b) \in \delta_B$ so the morphism $L(g)\circ L(f)$ also satisfy the condition for the unique morphism from the colimit $L(B)$ to the cocone $L(B'')$. This means that $L(g)\circ L(f) = L(g\circ f)$ so L is indeed a functor.

Part 3

In this part we will define the natural transformation η and show that for any functor $H\colon \mathscr{B} \longrightarrow \mathscr{C}$ and a natural transformation $\alpha\colon F \longrightarrow H\circ G$ there is a natural transformation $\sigma\colon L \longrightarrow H$ such that the triangle from definition commutes.

To define η, we must construct a morphism $\eta_A\colon F(A) \longrightarrow (L\circ G)(A)$ for each object $A \in \mathscr{A}$. It suffices to choose $\eta_A = s^{G(A)}_{(A,1_{G(A)})}$. Let us prove the naturality of η. Given a morphism $a\colon A \longrightarrow A'$, we need to show that the following square commutes.

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\eta_A = s^{G(A)}_{(A,1_{G(A)})}} & (L\circ G)(A) \\
{\scriptstyle F(a)}\downarrow & \circ & \downarrow{\scriptstyle (L\circ G)(a)} \\
F(A') & \xrightarrow[\eta_{A'} = s^{G(A')}_{(A',1_{G(A')})}]{} & (L\circ G)(A')
\end{array}
$$

We have

$$
(L\circ G)(a)\circ s^{G(A)}_{(A,1_{G(A)})} = s^{G(A')}_{(A,G(A))} = s^{G(A')}_{(A',1_{G(A')})}\circ F(a)
$$

where the first equality holds by the definition of $(L\circ G)(a)$ and the second equality holds because $a\colon (A, G(a)) \longrightarrow (A', 1_{G(A')})$ is a morphism of $\delta_{G(A')}$. So η is indeed a natural transformation.

Now consider a functor $H\colon \mathscr{B} \longrightarrow \mathscr{C}$ and a natural transformation $\alpha\colon F \longrightarrow H\circ G$. To construct σ let us fix an object $B \in \mathscr{B}$. For each object $(A,b) \in \delta_B$ we have the following composite.

$$
(F\circ \phi_B)((A,b)) = F(A) \xrightarrow{\quad \alpha_A \quad} (H\circ G)(A) \xrightarrow{\quad H(b) \quad} H(B)
$$

Let us prove that $(H(B), (H(b)\circ \alpha_A)_{(A,b)\in\delta_B})$ is a cocone on $F\circ\phi_B$. Given a morphism $a : (A,b) \longrightarrow (A',b')$ of δ_B we need to show that the following diagram commutes.

$$\begin{array}{ccccc}
(F \circ \phi_B)((A, b)) = F(A) & \xrightarrow{\alpha_A} & (H \circ G)(A) & \xrightarrow{H(b)} & H(B) \\
\Big\downarrow {\scriptstyle F(a)} & \circ & & \nearrow {\scriptstyle H(b')} & \\
& & (H \circ G)(A') & & \\
& \nearrow {\scriptstyle \alpha_{A'}} & & & \\
(F \circ \phi_B)((A', b)) = F(A') & & & &
\end{array}$$

We have

$$\begin{aligned}
H(b') \circ \alpha_{A'} \circ F(a) &= H(b') \circ (H \circ G)(a) \circ \alpha_A \\
&= H(b' \circ G(a)) \circ \alpha_A \\
&= H(b) \circ \alpha_A
\end{aligned}$$

where the first equality holds by naturality of α and the third equality holds by definition of the morphism a of δ_B. So $(H(B), (H(b) \circ \alpha_A)_{(A,b) \in \delta_B})$ is a cocone on the $F \circ \phi_B$. This gives us a unique factorisation $\sigma_B \colon LB \longrightarrow H(B)$ through the colimit $L(B)$ yielding $\sigma_B \circ s^B_{(A,b)} = H(b) \circ \alpha_A$. To prove the naturality of σ, consider a morphism $f \colon B \longrightarrow B'$. We need to show that the following square commutes.

$$\begin{array}{ccc}
L(B) & \xrightarrow{\sigma_B} & H(B) \\
{\scriptstyle L(f)} \Big\downarrow & \circ & \Big\downarrow {\scriptstyle H(f)} \\
L(B') & \xrightarrow[\sigma_{B'}]{} & H(B')
\end{array}$$

It suffices to show it on the colimit injections. For each $s^B_{(A,b)}$ we get

$$\begin{aligned}
H(f) \circ \sigma_B \circ s^B_{(A,b)} &= H(f) \circ H(b) \circ \alpha_A \\
&= H(f \circ b) \circ \alpha_A \\
&= \sigma_{B'} \circ s^{B'}_{(A, f \circ b)} \\
&= \sigma_{B'} \circ L(f) \circ s^B_{(A,b)}
\end{aligned}$$

where the first equality holds by the definition of σ_B, the third holds by the definition of $\sigma_{B'}$ and the fourth holds by the definition of $L(f)$. Together, we got that σ is a natural transformation.

It suffices to show that the condition $\sigma G \circ \eta = \alpha$ holds on objects. We get $\sigma_{GA} \circ \eta_A = \alpha_A$ which is just a relation

$$\sigma_{G(A)} \circ s^{G(A)}_{(A, 1_{G(A)})} = \alpha_A = H(1_{G(A)}) \circ \alpha_A.$$

But this holds just from the definition of $\sigma_{G(A)}$. □

The skeleton of this proof is from Borceux [1994], specifically from pages 123 and 124. Now because of the duality principle we immediately get the following theorem.

Theorem 2.4 (Right Kan extension theorem)**.** *Consider two functors* $G \colon \mathscr{A} \longrightarrow \mathscr{B}$ *and* $F : \mathscr{A} \longrightarrow \mathscr{C}$*, with* $\mathscr{A}$ *small and* $\mathscr{C}$ *complete. Under these conditions, the right Kan extension* F *along* G *exists.*

3. Relation of adjoint functors to Kan extensions

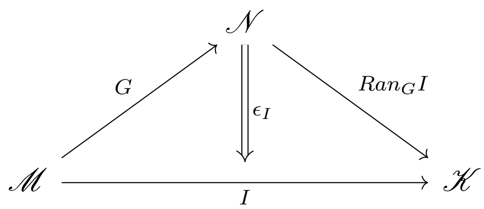
In this chapter, we will introduce a theorem which connects adjoint functors and Kan extensions in some way and introduce the concept of the general Kan extension.

Theorem 3.1 (Adjoint functors of functors between categories of functors)**.** *Let $\mathscr{M}$ and $\mathscr{N}$ be small categories and let $\mathscr{K}$ be a complete category. Let $G\colon \mathscr{M} \longrightarrow \mathscr{N}$ be a functor. It gives rise to the functor $F\colon \mathscr{K}^{\mathscr{N}} \longrightarrow \mathscr{K}^{\mathscr{M}}$ such that $F(B) = B \circ G$ for every $B \in \operatorname{obj}\left(\mathscr{K}^{\mathscr{N}}\right)$ and $F(\tau) = \tau G$ for every $\tau \in \operatorname{mor}\left(\mathscr{K}^{\mathscr{N}}\right)$. Then F has a right adjoint $U\colon \mathscr{K}^{\mathscr{M}} \longrightarrow \mathscr{K}^{\mathscr{N}}$.*

Proof. We will split this proof into two parts. In the first part we will define the functor U and in the second part we will show that F is the left adjoint of U which is equivalent to U being a right adjoint to F.

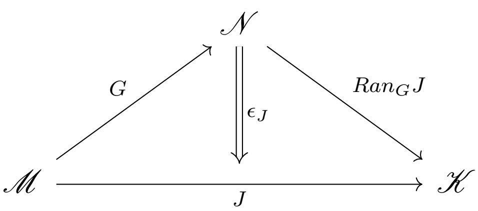
Part 1

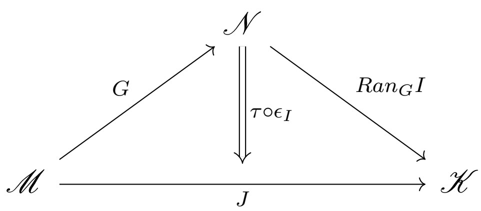
First, we will define U on objects. Let $I \in \operatorname{obj}\left(\mathscr{K}^{\mathscr{M}}\right)$. We know that $\mathscr{M}$ is small and $\mathscr{K}$ is complete. Theorem 2.4 gives us existence of the right Kan extension I along G as is captured in the following diagram:



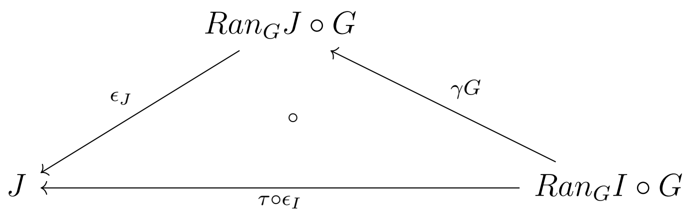
We put $U(I) = Ran_G I$. And this is how we define U on objects.

Now we will define U on morphisms. Let $J \in \operatorname{obj}\left(\mathscr{K}^{\mathscr{M}}\right)$ and let $\tau\colon I \longrightarrow J$ be a morphism. We again get the right Kan extension J along G. We already have the functor $Ran_G I\colon \mathscr{N} \longrightarrow \mathscr{K}$ and τ gives rise to the natural transformation $\tau \circ \epsilon_I\colon Ran_G I \circ G \longrightarrow J$ as is captured in the following diagrams:



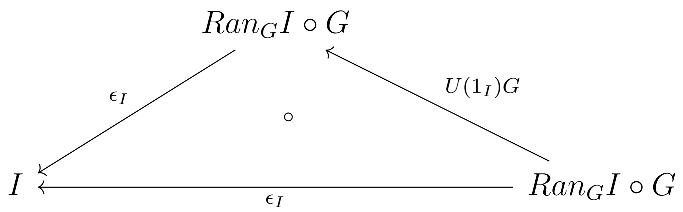


Now from definition of the right Kan extension J along G we get a unique natural transformation $\gamma\colon Ran_G I \longrightarrow Ran_G J$ such that the following diagram commutes:



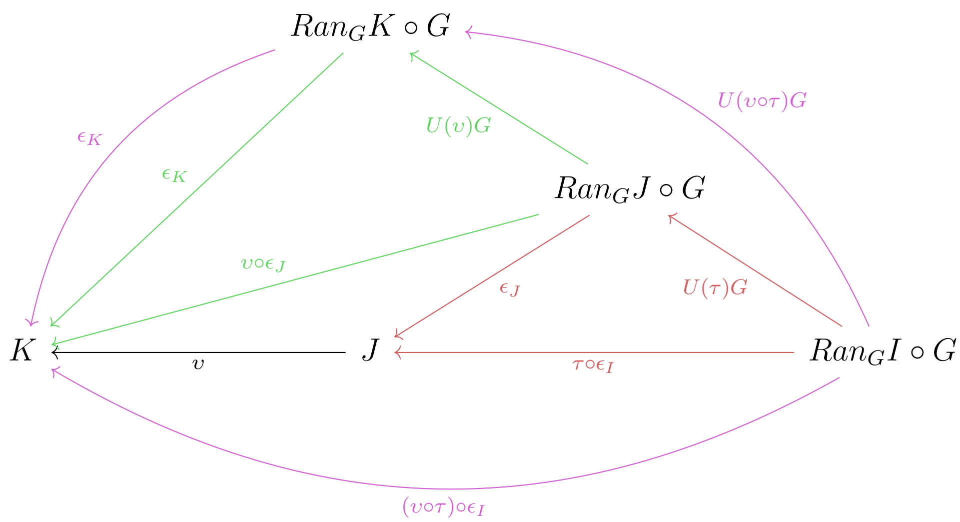
Because $Ran_G I = U(I)$ and $Ran_G J = U(J)$, we can put $U(\tau) = \gamma$. And this is how we define U on morphisms.

Now we need to show that U is a functor. First, let us show that $U(1_I) = 1_{U(I)}$ for all $I \in \operatorname{obj}\left(\mathscr{K}^{\mathscr{M}}\right)$. From the definition of U on morphisms we get that $U(1_I)\colon Ran_G I \longrightarrow Ran_G I$ is the unique natural transformation such that the following diagram commutes:



In other words $\epsilon_I = \epsilon_I \circ U(1_I)G$. But $\epsilon_I = \epsilon_I \circ 1_{Ran_G I}G$ also holds so from the uniqueness of the identity morphism we get $U(1_I) = 1_{Ran_G I} = 1_{U(I)}$ and because I was chosen arbitrary this holds for all objects in $\mathscr{K}^{\mathscr{M}}$.

Now let us assume two morphsims $\tau\colon I \longrightarrow J$ and $\upsilon\colon J \longrightarrow K$ and let ϵ_I, ϵ_J and ϵ_K be the natural transformations from definitions of $U(I)$, $U(J)$ and $U(K)$. From definitions of $U(\tau)$, $U(\upsilon)$ and $U(\upsilon \circ \tau)$ we get the three commuting diagrams which we will merge into one:



The red diagram comes from the definition of $U(\tau)$ and commutes, the green diagram comes from the definition of $U(\upsilon)$ and commutes and the purple diagram comes from the definition of $U(\upsilon \circ \tau)$ and commutes. From the red diagram we get

$$\epsilon_J \circ U(\tau)G = \tau \circ \epsilon_I.$$

We see from the diagram that we can compose it with υ from left and we get

$$\upsilon \circ \epsilon_J \circ U(\tau)G = \upsilon \circ \tau \circ \epsilon_I.$$

Now from the green diagram we get

$$\epsilon_K \circ U(\upsilon)G = \upsilon \circ \epsilon_J.$$

From these two equalities we can conclude

$$\epsilon_K \circ U(\upsilon)G \circ U(\tau)G = \upsilon \circ \tau \circ \epsilon_I.$$

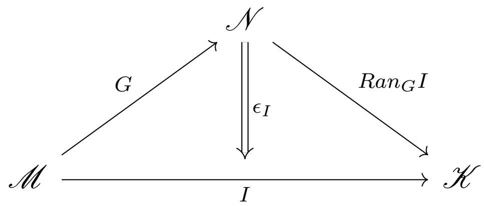
But from the purple diagram we see that

$$\epsilon_K \circ U(\upsilon \circ \tau)G = \upsilon \circ \tau \circ \epsilon_I$$

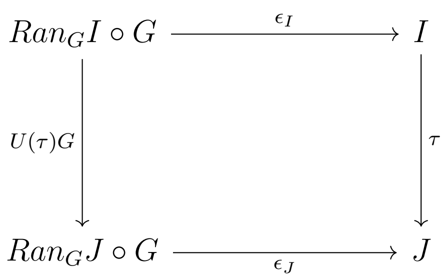
and from uniqueness of $U(\upsilon \circ \tau)$ we get $U(\upsilon \circ \tau) = U(\upsilon) \circ U(\tau)$. Because τ and υ were chosen arbitrary we see that U is indeed a functor.

Part 2

We will show that F is the left adjoint of U by finding a counit $\epsilon\colon FU \longrightarrow 1_{\mathscr{K}^{\mathscr{M}}}$. For $I \in \text{obj}\left(\mathscr{K}^{\mathscr{M}}\right)$ we see that $FU(I) = Ran_G I \circ G$ so we get the candidate for a component of ϵ directly from the definition of $U(I)$ which is $\epsilon_I\colon Ran_G I \circ G \longrightarrow I$ as we can see in the following diagram:

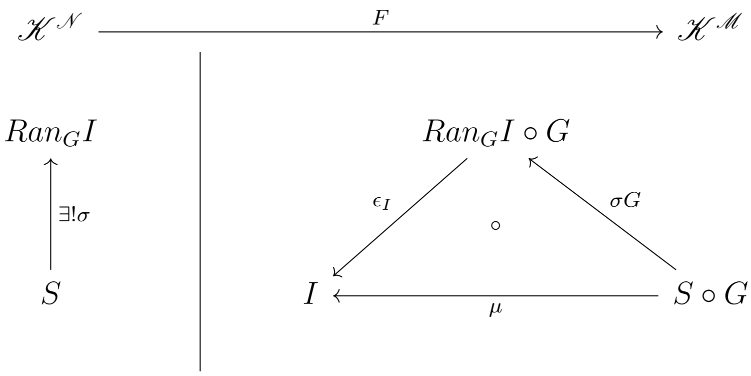


Now we verify that $\epsilon = \left(\epsilon_I; I \in obj\left(\mathscr{K}^{\mathscr{M}}\right)\right)$ is a natural transformation. Let us assume the morphism $\tau\colon I \longrightarrow J$. We see that $FU(\tau) = U(\tau)G$. We need to show that the following diagram commutes:



But this follows directly from the definition of $U(\tau)$.

Now we need to show that $(U(I), \epsilon_I)$ is a co-free object over I with respect to F for all $I \in \text{obj}\left(\mathscr{K}^{\mathscr{M}}\right)$. This means that for every $S \in obj\left(\mathscr{K}^{\mathscr{N}}\right)$ and for every morphism $\mu\colon F(S) \longrightarrow I$ there is a unique morphism $\sigma\colon S \longrightarrow U(I)$ such that $\mu = F(\sigma) \circ \epsilon_I$. We see that $U(I) = Ran_G I$, $FU(I) = Ran_G I \circ G$, $F(\sigma) = \sigma G$ and $F(S) = S \circ G$ from definition of F and U. We capture our situation in the following picture:



In other words, for every functor $S\colon \mathscr{N} \longrightarrow \mathscr{K}$ and a natural transformation $\mu\colon S \circ G \longrightarrow I$ there is a unique natural transformation $\sigma\colon S \longrightarrow Ran_G I$ such that $\mu = \epsilon_I \circ \sigma G$. But the existence of such unique natural transformation comes directly from $Ran_G I$ being the right Kan extension. So $(U(I), \epsilon_I)$ is a co-free object over I with respect to F and because I was chosen arbitrary it holds for all $I \in \text{obj}\left(\mathscr{K}^{\mathscr{M}}\right)$ so ϵ is a counit and thus F is the left adjoint of U. □

In the following remark, we will define the global Kan extensions and show interesting connection to the previous theorem.

Remark. Let $\mathscr{K}^{\mathscr{M}}$ and $\mathscr{K}^{\mathscr{N}}$ be categories of functors and $G\colon \mathscr{M} \longrightarrow \mathscr{N}$ as in Theorem 3.1. Assume the functor $F\colon \mathscr{K}^{\mathscr{N}} \longrightarrow \mathscr{K}^{\mathscr{M}}$ defined in the same fashion as in Theorem 3.1. The right adjoint U of F is called *the global right Kan extension.* From the last section of the previous theorem we know that components of the counit have needed properties for being the natural transformation from definition of the right Kan extension for the corresponding functors. That is, if $\epsilon = \left(\epsilon_I; I \in \mathscr{K}^{\mathscr{M}}\right)$ is the counit of the adjunction than $U(I) = Ran_G I$ with the corresponding natural transformation ϵ_I. This means that if we put a functor into the global right Kan extension we get the local Kan extension. From duality principle, the same holds for *the global left Kan extension* which is the left adjoint of F and we get the needed natural transformations from the unit of the adjunction. We can see that it is in some way the opposite implication than the one in Theorem 3.1.

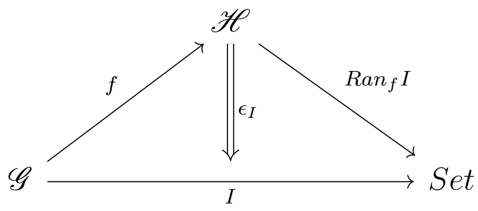
4. Example

In this chapter, we will formulate a problem and answer it using the previous theorems. First we need to recall what is the category G-set for some group G. Objects of such category are sets together with a left group action. This means that two same sets with different left group actions are different objects in this category. Morphisms in this category are such set functions that preserve respective left group actions. For example let us have a set A with the left group action $g \cdot -$ for all $g \in G$ and a set B with the left group action $g \star -$ for all $g \in G$. A morphism in our category is such function $f \colon A \longrightarrow B$ that $f(g \cdot a) = g \star f(a)$ for all $g \in G$ and $a \in A$.

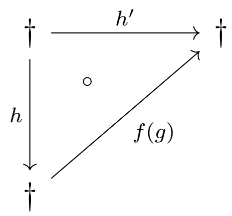
Problem *Let G and H be groups and $f \colon G \longrightarrow H$ a group homomorphism. Assume a functor $F \colon H$-set $\longrightarrow G$-set defined such as $F(A) = A$, $\forall A \in \mathrm{obj}(H$-set) and the G-set structure is induced by f in the way that $g \cdot a = f(g) \cdot a$, $\forall a \in A$ and $g \in G$. Find the right adjoint $U \colon G$-set $\longrightarrow H$-set of F.*

Solution. Let us represent G and H as one object categories with the object $\star$ for G and $\dagger$ for H and with morphisms being the elements of the corresponding group. In this representation, the composition of morphisms works like the group multiplication which will be important for the solution of this problem. We name these categories $\mathscr{G}$ and $\mathscr{H}$. Now we can represent the category G-set as the category of functors from $\mathscr{G}$ to *Set*. For example, let us have $A \in \mathrm{obj}(G$-set). We define the corresponding functor $I \colon \mathscr{G} \longrightarrow Set$ in the way that $I(\star) = A$ and $I(g) = g \cdot -$, for all morphisms $g \in \mathrm{mor}(\mathscr{G})$. We represent the category H-set in the same fashion. Now, because *Set* is complete, we can use the construction of the right adjoint from Theorem 3.1.

Let us consider the following diagram:



Here I is the functor representing some $A \in \mathrm{obj}(G$-set). According to Theorem 3.1, $Ran_f I = U(I)$, so we need to find $Ran_f I$. We will use the construction from Theorem 2.3 modified for the right Kan extension. Let δ be the category of elements of the functor $\mathscr{H}(\dagger, f(-))$. Now because $f(\star) = \dagger$ objects of δ are pairs $(\star, h)$ for all morphisms $h \in \mathrm{mor}(\mathscr{H})$. A morphism from $(\star, h)$ to $(\star, h')$ is a morphism g from $\mathscr{G}$ such that $\mathscr{H}(\dagger, f(g))(h) = h'$. Equivalently, we need the following diagram to commute:



In other words, the morphism we are looking for is such g that $f(g) \circ h = h'$. We write ϕ for the forgetful functor from the category δ to the category $\mathscr{G}$. To define $U(I)$ we will find the limit of the functor $I \circ \phi \colon \delta \longrightarrow Set$. We will use the Maranda construction.

We define the discrete categories δ_0 and δ_m in the way that $\mathrm{obj}(\delta_0) = \mathrm{obj}(\delta)$ and $\mathrm{obj}(\delta_m) = \mathrm{mor}(\delta)$. Assume the functor $codom\colon \delta_m \longrightarrow \delta_0$. Let $N\colon \delta_0 \longrightarrow Set$ be the functor such that $N((\star, h)) = (I \circ \phi)((\star, h)) = A$ for all $(\star, h) \in \delta_0$. We need to find the limit of N and $codom \circ N$. Now because $N((\star, h)) = A$ for all $(\star, h) \in \delta_0$ and because δ_0 has the same number of objects as H has elements we see that lim $N = \big(A^H; (\pi_h, h \in H)\big)$ where π_h is the h-th coordinate projection to A. To find the limit of $codom \circ N$ we observe that every codomain of a morphism can be represented by the morphism itself and every morphism in δ can be indexed by some $g \in G$ and $h \in H$ because there is one and only one h' such that the relation $f(g) \circ h = h'$ holds. From this we get that lim $codom \circ N = \big(A^{G\times H}, (\pi_{g,h}; g \in G, h \in H)\big)$ where $\pi_{g,h}$ is the g, h-th coordinate projection to A.

The first limit gives us two cones of the diagram $codom \circ N\colon \delta_m \longrightarrow Set$. These cones are $\big(A^H, (\pi_{f(g)h}; g \in G, h \in H)\big)$ and $\big(A^H, ((I \circ \phi)(g) \circ \pi_h; g \in G, h \in H)\big)$. From these we get two unique morphisms $\alpha, \beta\colon A^H \longrightarrow A^{G\times H}$ such that $\pi_{f(g)h} = \pi_{g,h} \circ \alpha$ and $(I \circ \phi)(g) \circ \pi_h = \pi_{g,h} \circ \beta$ for all $g \in G$ and $h \in H$.

To define α and β let us consider some $\mathbf{a} \in A^H$. Let us start with α. We need the equality $\alpha(\mathbf{a})_{g,h} = \mathbf{a}_{f(g)h}$ to hold for all $h \in H$ and $g \in G$ so we put $\forall h \in H$ $\alpha(\mathbf{a})_{g,h'} = \mathbf{a}_h$ for all $g \in G$ and $h' \in H$ such that $h = f(g)h'$ and this is sufficient to define α. To define β let us observe that $((I \circ \phi)(g) \circ \pi_h)(\mathbf{a}) = g \cdot \mathbf{a}_h$. In this case we put $\beta(\mathbf{a})_{g,h} = g \cdot \mathbf{a}_h$ for all $g \in G$ and $h \in H$ and this is sufficient to define β.

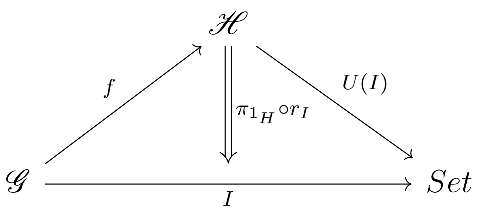
The last step in the Maranda construction is to find the equaliser of α and β. To find it, we separate such $\mathbf{a} \in A^H$ that $\mathbf{a}_{f(g)h} = g \cdot \mathbf{a}_h$ for all $g \in G$ and $h \in H$. Those are exactly the $\mathbf{a} \in A^H$ such that $\alpha(\mathbf{a}) = \beta(\mathbf{a})$. To do so, let us look at the elements of A^H like at the functions from the set H to the set A. For example, let s be such function. We put $s(h) = \mathbf{a}_h$ for all $h \in H$ and by this we get the wanted representation. A function representing $\mathbf{a}$ such that $\alpha(\mathbf{a}) = \beta(\mathbf{a})$ must satisfy the following equality: $s(f(g)h) = g \cdot s(h)$. We will describe such functions using G-sets.

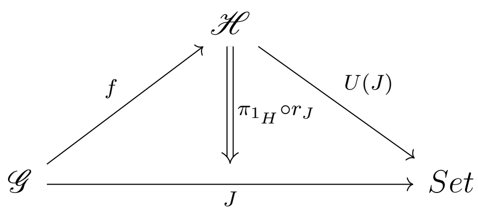
Let us assume the H-set H where the action of H is defined in the way that $h \cdot h' = hh'$. Now let us consider the set of functions $G\text{-set}(F(H), A)$. From definition we see that for every $s \in G\text{-set}(F(H), A)$, $h \in H$ and $g \in G$ the equality $g \cdot s(h) = s(g \cdot h) = s(f(g)h)$ must hold and also the set $G\text{-set}(F(H), A)$ is the set of all functions such that the previous equality holds. But this is exactly the condition we wanted. In other words, for every $\mathbf{a}$ such that $\alpha(\mathbf{a}) = \beta(\mathbf{a})$ there is one and only one function $s \in G\text{-set}(F(H), A)$ such that $\mathbf{a}_h = s(h)$ for all $h \in H$. We define the morphism $r_I\colon G\text{-set}(F(H), A) \longrightarrow A^H$ in the way that for all $s \in G\text{-set}(F(H), A)$ $r_I(s) = \mathbf{a}$ such that $\mathbf{a}_h = s(h)$ for all $h \in H$. It is clear that r_I equalises α and β. We want to show that $(G\text{-set}(F(H), A), (r_I, \alpha \circ r_I))$ is the equaliser.

Let us consider some set K and some morphism $r'\colon K \longrightarrow A^H$ such that $\alpha \circ r' = \beta \circ r'$. We will find the morphism $d\colon K \longrightarrow G\text{-set}(F(H), A)$ such that $r' = r_I \circ d$. For every $k \in K$, $r'(k) = \mathbf{a}$ such that $\alpha(\mathbf{a}) = \beta(\mathbf{a})$. But we know that for such $\mathbf{a}$ there is $s_k \in G\text{-set}(F(H), A)$ such that $\mathbf{a}_h = s_k(h)$ for all $h \in H$. To define d, we simply put $d(k) = s_k$, for all $k \in K$ and the equality $r' = r_I \circ d$ easily holds. Because K and r' were chosen arbitrary we see that $(G\text{-set}(F(H), A), (r_I, \alpha \circ r))$ is the equaliser of α and β and from the Maranda construction it follows that $(G\text{-set}(F(H), A), (\pi_h \circ r_I; h \in H)$ is the limit of the diagram $I \circ \phi\colon \delta \longrightarrow Set$ and thus $Ran_f I(\dagger) = G\text{-set}(F(H), A)$.

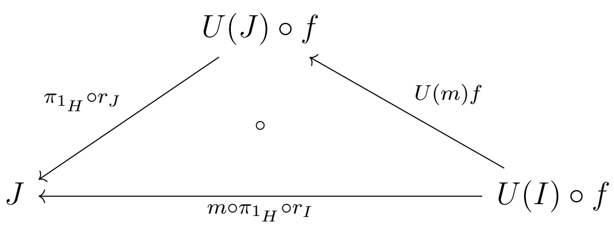
Now we define $Ran_f I$ on morphisms. We will again use the construction from Theorem 2.3 modified for the right Kan extension. Every $h \in \text{mor}(\mathscr{H})$ gives rise to the cone $(G\text{-set}(F(H), A), (\pi_{h'h} \circ r_I; h' \in H))$. Here we use that we can also view morphisms from the category $\mathscr{H}$ as elements of the group H. We will find the unique morphism $Ran_f I(h)\colon G\text{-set}(F(H), A) \longrightarrow G\text{-set}(F(H), A)$ such that $\pi_{h'h} \circ r_I = \pi_{h'} \circ r_I \circ Ran_f I(h)$ for all $h' \in H$. Let $s \in G\text{-set}(F(H), A)$ and $h' \in H$. We see that $(\pi_{h'h} \circ r_I)(s) = s(h'h)$ and $(\pi_{h'} \circ r_I)(s) = s(h')$. Let us define $Ran_f I(h)$ on the elements. We put $((Ran_f I(h))(s))(h') = s(h'h)$ for all $s \in G\text{-set}(F(H), A)$ and $h' \in H$ and with this definition the equality $\pi_{h'h} \circ r_I = \pi_{h'} \circ r_I \circ Ran_f I(h)$ easily holds. Because this all holds for every $h \in \text{mor}(\mathscr{H})$, $Ran_f I$ is sufficiently defined on arrows and thus we completely defined U on objects.

Now we define U on morphisms. To do so, we need to know specifically for every $I\colon \mathscr{G} \longrightarrow Set$ what is the natural transformation ϵ_I from the definition of the right Kan extension $Ran_f I$. According to the construction from Theorem 2.3, it suffices to put $\epsilon_I = \pi_{1_H} \circ r_I$ where 1_H is the unit of the group H. Now we use the construction from Theorem 3.1. Let us consider two functors $I, J\colon G \longrightarrow Set$ such that $I(\star) = A$ and $J(\star) = B$. Assume a natural transformation m from I to J that is a morphism $m\colon A \longrightarrow B$. From definition of U on objects we get the following pictures:





Now m gives rise to the natural transformation $m \circ \pi_{1_H} \circ r_I\colon U(I) \circ f \longrightarrow J$ and from the definition of the right Kan extension and the construction from Theorem 3.1 we get the following commutative diagram:



That is there exist the unique natural transformation $U(m)\colon U(I) \longrightarrow U(J)$ such that $m \circ \pi_{1_H} \circ r_I = \pi_{1_H} \circ r_J \circ U(m)f$. Now because G has only one object this equality reduces to $m \circ \pi_{1_H} \circ r_I = \pi_{1_H} \circ r_J \circ U(m)$. Now let us show that $U(m) = G\text{-set}(F(H), m)$. We get

$$m \circ \pi_{1_H} \circ r_I = \pi_{1_H} \circ r_J \circ G\text{-set}(F(H), m).$$

But this is the equality of two functions from $G\text{-set}(F(H), A)$ to B so we can verify the equality on elements. Let $p \in G\text{-set}(F(H), A)$. We get

$$(m \circ \pi_{1_H} \circ r_I)(p) = (\pi_{1_H} \circ r_J \circ G\text{-set}(F(H), m))(p)$$

and this is equivalent to

$$m \circ p(1_H) = m \circ p(1_H).$$

The last equality holds and because p was chosen arbitrary we have that $U(m) = G\text{-set}(F(H), m)$.

We have successfully defined U on objects and morphisms and because we followed the construction from Theorem 3.1 we immediately have that U is the right adjoint of F. Now let us translate our result to the language of standard group actions. For a $A \in G$-set we have $U(A) = G\text{-set}(F(H), A)$ where H-set structure is induced by the relation $(h \cdot s)(h') = s(h'h)$ for all $s \in G\text{-set}(F(H), A)$ and $h, h' \in H$. For a morphism $m \in \text{mor}(G\text{-set})$ we have $U(m) = G\text{-set}(F(H), m)$.

Conclusion

First, we familiarised a reader with the category of elements and presented our version of definition of adjoint functors. In the second chapter we, with the help of Borceux [1994], proved theorem which gives us the existence condition for left Kan extensions and its construction using colimits and we introduced the dual version of this theorem. In the third chapter, we introduced and proved a theorem which puts in relation adjoint functors and Kan extensions and we showed its connection to global Kan extensions. In the last chapter, we solved an interesting problem using everything from this thesis, namely a construction of a Kan extension from the proof of Theorem 2.3 and the method of finding a right adjoint from Theorem 3.1.

It would be also interesting to study relation of Kan extensions and ends and coends following, for example, Fosco [2021] but unfortunately there was no time and space for that in this thesis.

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