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OF MATHEMATICS
AND PHYSICS
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BACHELOR THESIS

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Covering families of triangles by convex sets

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Study programme: Computer Science
Study branch: General Computer Science

Prague 2023

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I would like to thank my supervisor doc. Mgr. Jan Kynčl, Ph.D. for introducing me to this topic and guiding me through it.

Title: Covering families of triangles by convex sets

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Abstract: A convex universal cover of a family $\mathcal{M}$ of sets in the plane is a convex set that contains a congruent copy of every element of $\mathcal{M}$. Park and Cheong conjecture that for every family of triangles with bounded diameter there exists a triangle that is a smallest convex universal cover of this family. We prove this conjecture for

- every family of all triangles with the lengths of their two sides fixed,
- every family of all triangles with the length of a side and the size α of the opposite angle fixed (where α is from an interval $(0, \lambda] \cap [3\pi/7, \pi)$ with λ being approximately 0.396π),
- every finite subfamily of a family of all triangles with the length of a side and the size α of the opposite angle fixed (where $\alpha \geq \pi/2$).

Keywords: triangles, universal cover, convex cover, smallest area

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1. Introduction

Definition 1.1. A *universal cover* of a family $\mathcal{M}$ of sets in the plane is a set that contains a congruent copy of every element of $\mathcal{M}$.

We call a universal cover *smallest* if it has the smallest possible area.

Especially important for us will be *a smallest convex universal cover* which we will often refer to as *SCUC*.

In 1914 Lebesque asked for the smallest universal cover of a family of all sets of diameter one. This problem remains open, with some approximations of the area known [1, 2, 5]. Since then, universal covers have been studied for various families of sets in the plane, for example a set of all curves of unit length (Moser's worm problem) [6].

And while for the more general families we often do not have a better method than an exhaustive computational approximation, for triangles the problem appears to be more approachable. Park and Cheong stated the following conjecture.

Conjecture 1.2. [7, Conjecture 1] *For any family $\mathcal{M}$ of triangles of bounded diameter there is a triangle T that is a smallest convex universal cover of $\mathcal{M}$.*

Several results that prove more specific version of this conjecture are known. Füredi and Wetzel showed that a SCUC of the family of all triangles with given perimeter is a triangle [4], Park and Cheong showed the same for the family of triangles that fit into a unit circle and any family of two triangles [7]. Cheong, Devillers, Glisse and Park showed the same for the family of all triangles that fit into a unit half-disk and the family of all triangles that fit into a unit square [3].

In this thesis we will find a SCUC of

- every family of all triangles with the lengths of their two sides fixed (Theorem 3.2),
- every family of all triangles with the length of a side and the size α of the opposite angle fixed (where α is from an interval $(0, \lambda] \cap [3\pi/7, \pi)$ with λ being the root of expression (Λ)) (Theorem 5.3),
- every finite sufamily of a family of all triangles with the length of a side and the size α of the opposite angle fixed (where $\alpha \geq \pi/2$) (Theorem 5.5).

All of them will be triangles. All three theorems will be proved by finding a SCUC of two elements of the family (Park and Cheong's result [7] will be crucial for that (Lemma 4.4)) and subsequentially showing that the found triangle is also a universal cover of the whole family.

2. Preliminaries

As in this thesis we will be using a term *triangle* a lot, it is neccessary to remark that in this term we also include *degenerate triangles*—triangles with all the vertices lying on a line.

For referring to the area of a set X in the plane we will be using the notation $\mathrm{a}(X)$.

Let us define a few key terms.

Definition 2.1. The *diameter* of a compact set in the plane is the maximum distance of its two points.

The *width* of a compact set S in the plane is the minimum distance of two parallel lines such that every point of S lies either on them or in the region between them.

Observation 2.2. *The diameter of a triangle is its longest edge and the width is its shortest height.*

Observation 2.3. *Both the diameter and the width are monotone in regards to inclusion. In other words, if $A \subseteq B$, the width and the diameter of B are not smaller than those of A.*

Lemma 2.4. *The area of a convex compact set S in the plane of width w and diameter d is at least $dw/2$.*

Proof. Let A and B be any two points from S such that $|AB| = d$ and let h_X be the distance from an arbitrary point X to the line AB. Then there must exist two points C and D from S which lie on the opposite sides of line AB such that $h_C + h_D \geq w$. Since S is convex, both triangles ABC and ABD are subsets of S and therefore the area of S is at least

$$\frac{|AB| \cdot h_C}{2} + \frac{|AB| \cdot h_D}{2} \geq \frac{|AB| \cdot w}{2} = \frac{d \cdot w}{2}.$$

□

3. Triangles with the lengths of their two sides fixed

In this section we will focus on finding a smallest convex universal cover of the family of all triangles with the lengths a and b of their two sides given. We will show that it will be a triangle T_{ab} which can be constructed as follows.

Definition 3.1. With given lengths a and b such that $a \geq b$ let ABX be an isosceles triangle where $|AB| = |BX| = a$ and $|AX| = b$. Then let C be a point on the ray opposite to $\overrightarrow{XB}$ such that $|XC| = b$. Now we define $T_{ab} = \triangle ABC$ (figure 3.1).

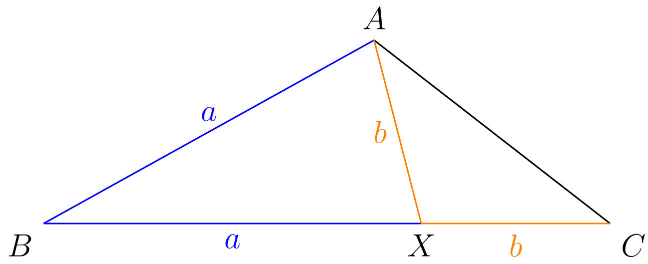


Figure 3.1: Construction of T_{ab}

Now we can formulate a theorem.

Theorem 3.2. *With given lengths a and b such that $a \geq b$ let $\mathcal{M}$ be the family of all triangles with two sides of length a and b. Then T_{ab} is a smallest convex universal cover of the family $\mathcal{M}$.*

Proof. First, we show that no SCUC of $\mathcal{M}$ can have area smaller than the area of T_{ab}.

The family $\mathcal{M}$ includes a segment of length $a + b$ (a degenerate triangle) and also an isosceles triangle with sides of lengths a, a and b. Let h be the height of this isosceles triangle, perpendicular to the side of the length b. Then the diameter of a SCUC is at least $a + b$ and its width is at least h. Therefore, by Lemma 2.4, its area is at least $(a + b)h/2$, which is exactly the area of T_{ab}.

In order to show that T_{ab} is indeed a convex universal cover we must find a congruent copy of each element of $\mathcal{M}$ in T_{ab}. Let KLM be a triangle from $\mathcal{M}$ with $|KL| = a$ and $|KM| = b$. Then we distinguish two cases, depending on the size of the third side LM.

- If $|LM| \geq a$:

  First let us make two observations.

  Observation 3.3. *The height h_K of the triangle KLM is minimal when $|LM| = a + b$ and maximal when $|LM| = a$.*

Observation 3.4. *In T_{ab} we have $|\angle XAC| = (\pi - |\angle CXA|)/2 = |\angle BAX|/2$, and therefore $|\angle BAC| = 3/2|\angle BAX|$. With $\triangle ABX$ being isosceles, the longer the segment XA, the smaller the angle $\angle BAX$. Considering our assumption that $a \geq b$, XA is the longest when $\triangle BAX$ is equilateral and therefore $|\angle BAX| = \pi/3$. That means that $\angle BAC$ is always at least $\pi/2$.*

With these observations we are ready to construct the triangle KLM.

First, let o be the circle with the center B and the radius $|BX|$. We place L in B. Then place M on the segment BC so that the length $|LM|$ is the required length (because of triangle inequality we know that $|LM| \leq a+b = |BC|$). Then we place K onto o so that the $\angle KML$ has the appropriate size and so K lies in the same half-plane defined by the line BC as point A. As we observed in the Observation 3.3 earlier, the height of KLM is smaller than or equal to the height of ABX, which means that K lies on the arc of o between points A and X, which, due to the Observation 3.4 lies in $\triangle ABC$ (Figure 3.2).

We have shown that all three vertices of $\triangle KLM$ indeed lie in $\triangle ABC$.

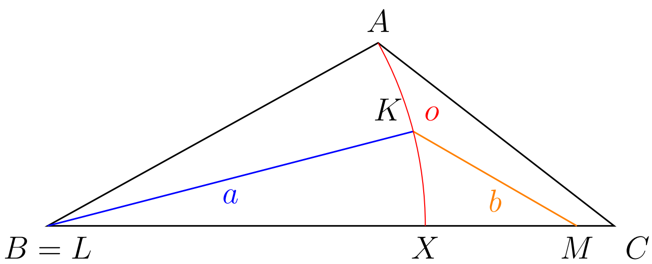


Figure 3.2: KLM in T_{ab} for $|LM| \geq a$

- If $|LM| \leq a$:

Observation 3.5. *The height h_M of triangle KLM is minimal when $|LM| = a - b$ and maximal when $|LM| = a$.*

Now we place M on the line defined by the height h_A of $\triangle ABC$ so that the foot of h_A lies on the foot of h_M (let us denote this point H). Then we place points K and L on the line BC so that $\overrightarrow{LK}$ is of the same orientation as $\overrightarrow{BC}$ (Figure 3.3).

Due to Observation 3.5 M lies on h_A, thus in $\triangle ABC$. The segment HK is the smallest when $M = A$, in which case $|HK| = |HX|$. Also $|HK| \leq |MK| = b$. That implies $|HX| \leq |HK| \leq |HC|$, which means that K lies on the segment XC. Then L lies on the segment BX.

We have again shown that all three vertices of $\triangle KLM$ lie in $\triangle ABC$.

□

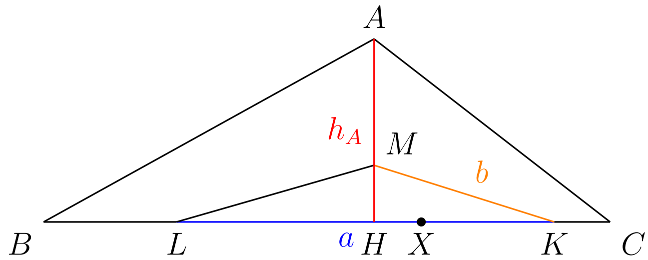


Figure 3.3: KLM in T_{ab} for $|LM| \leq a$

4. Two triangles

Park and Cheong ([7]) have solved the problem of finding a smallest convex universal cover for families of two triangles. This result is extremely important for our own results, especially the following two lemmas.

Definition 4.1. We say that X *fits into* Y if there exists a subset X' of Y such that X' and X are congruent.

Definition 4.2. We say that X *maximally fits into* Y if X fits into Y, but there is no set X' that is similar to X and larger than X that fits into Y.

Lemma 4.3. ([7, Lemma 5]) *If a triangle S maximally fits into a convex polygon T, then there are at least four incidences between vertices of S and edges of T. That is, there are four distinct pairs (p, e), where p is a vertex of S, e is an edge of T, and $p \in e$.*

Lemma 4.4. ([7, Lemma 9]) *Let $\mathcal{M}$ be a family of triangles, and let Z be a convex universal cover for $\mathcal{M}$. Let $S \in \mathcal{M}$, and let S' be the smallest universal cover for $\mathcal{M}$ that is similar to S. If*

$$\frac{\mathrm{a}(S')}{\mathrm{a}(S)} = \left(\frac{\mathrm{a}(Z)}{\mathrm{a}(S)}\right)^2,$$

then Z is a smallest convex universal cover for $\mathcal{M}$.

For completeness, we also provide the proof of this lemma by Park and Cheong (identical to the one in [7]).

Proof. [7] Let $S = \triangle PQR$ and let X be a convex universal cover of $\mathcal{M}$. We can assume $S \subseteq X$. We draw tangents to X that are parallel to to the edges of S, obtaining a triangle $P'Q'R'$ that is similar to $\triangle PQR$ and contains X. That means $\mathrm{a}(\triangle P'Q'R') \geq \mathrm{a}(S')$, therefore by the assumption

$$\frac{\mathrm{a}(\triangle P'Q'R')}{\mathrm{a}(\triangle PQR)} \geq \left(\frac{\mathrm{a}(Z)}{\mathrm{a}(S)}\right)^2,$$

from which we get

$$\frac{|P'Q'|}{|PQ|} \geq \frac{\mathrm{a}(Z)}{\mathrm{a}(S)}.$$

Let U, V, W be points of X on the three edges of $\triangle P'Q'R'$ and then let H be the convex hull of the points P, U, Q, V, R, W. Let K be any point in S and h_u, h_v and h_w be the distances from K to the lines $P'Q'$, $Q'R'$ and $R'P'$ respectively (Figure 4.1).

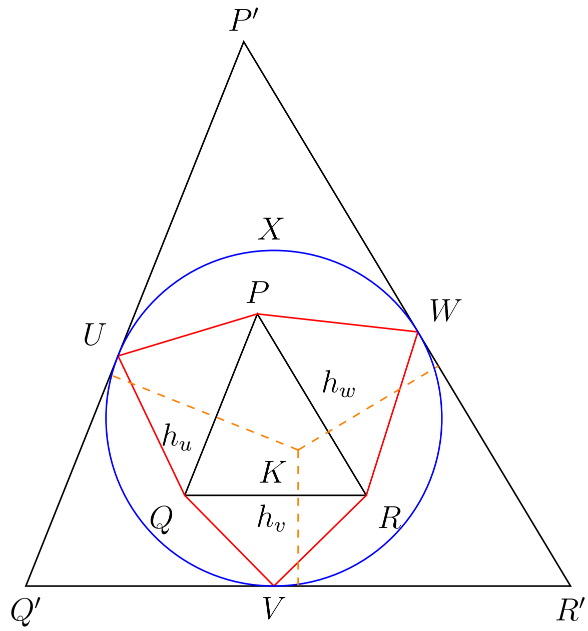


Figure 4.1: Proof of Lemma 4.4

By the convexity of X we know that $H \subseteq X$. Then we have

$$\begin{aligned} a(X) \geq a(H) &= \frac{1}{2}\left(|PQ|h_u + |QR|h_v + |RP|h_w\right) \\ &= \frac{|PQ|}{|P'Q'|} \cdot \frac{1}{2}\left(|P'Q'|h_u + |Q'R'|h_v + |R'P'|h_w\right) \\ = \frac{|PQ|}{|P'Q'|} \cdot a(\triangle P'Q'R') &= \frac{|PQ|}{|P'Q'|} \cdot \left(\frac{|P'Q'|}{|PQ|}\right)^2 \cdot a(\triangle PQR) \\ \frac{|P'Q'|}{|PQ|} \cdot a(S) &\geq \frac{a(Z)}{a(S)} \cdot a(S) = a(Z). \end{aligned}$$

We showed, that the area of any universal cover is larger or equal to Z, which means that Z must be a smallest convex universal cover.

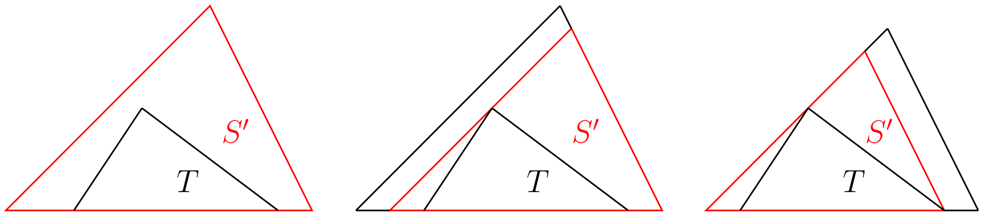
□

4.1 Finding a SCUC of two triangles similar to one of them

In further results we will need to be able to prove that a particular set is a smallest convex universal cover of two triangles. Lemma 4.4 is a very powerful tool for that. However, for using that we need to have SCUC which is similar to one of the triangles.

If S' is the smallest universal cover of two triangles S and T that is similar to S, then T has to maximally fit into S' and so by Lemma 4.3 there has to be an edge of S' that is incident to two vertices of T, thus its edge. That means we have 18 possibilities to consider—one for each combination of a side from S, side of T and their mutual orientation. Note, that many of these configurations might be the same.

Each configuration can be constructed as follows: we take one side of S and one side of T. Then we place T inside a big enough triangle S' similar to S, so T lies inside of S' and the two chosen sides are colinear (Figure 4.2a). That makes two incidences. Then we shrink S' by taking one of the remaining two sides and replacing it with a parallel line that touches T, keeping the new triangle similar to S and adding one incidence (Figure 4.2b). Finally we do the same with the remaining side (Figure 4.2c).



(a) (b) (c)

Figure 4.2: Constructing a SCUC of two triangles

Naturally, the smallest S' out of these 18 is a smallest convex universal cover of the family $\{S, T\}$, similar to S.

5. Triangles with the length of a side and the size of the opposite angle fixed

In this section we will focus on finding a smallest convex universal cover of the family of triangles with the length of one side and the opposite angle fixed. This family can also be described as a set of all the triangles KLM where k is an arc given by a chord LM and K is an arbitrary point on k.

We narrow Conjecture 1.2:

Conjecture 5.1. *With a given length a and a given angle α, let $\mathcal{M}$ be the family of all triangles KLM such that $|LM| = a$ and $|\angle MKL| = \alpha$. Then there exists a triangle T such that T is a smallest convex universal cover of the family $\mathcal{M}$.*

Our result proves the Conjecture 5.1, except for the cases when $\alpha \in (\lambda, 3\pi/7)$, where λ is the only root of the expression

$$\left(-\cos\frac{\pi-\delta}{4} - \sin\frac{\pi-\delta}{4}\cdot\tan\frac{\delta}{2} + \frac{1}{2}\right)^2 + \left(\frac{1}{2\tan\frac{\delta}{2}}\right)^2 - 1 \tag{Λ}$$

for $\delta \in [\pi/3, \pi)$. For reference, $\lambda \approx 0.396\pi$.

This leaves us with a lemma to prove.

Lemma 5.2. *Expression (Λ) has exactly one root for $\delta \in [\pi/3, \pi)$.*

We prove this lemma later on in the next subsection at 5.1.3.

The main result is the following theorem.

Theorem 5.3. *Let λ be the only root of an expression (Λ) for $\delta \in [\pi/3, \pi)$. With a given length a and a given angle $\alpha \in (0, \lambda] \cap [3\pi/7, \pi)$, let $\mathcal{M}$ be the family of all triangles KLM such that $|LM| = a$ and $|\angle MKL| = \alpha$. Then there exists a triangle T such that T is a smallest convex universal cover of the family $\mathcal{M}$.*

We prove this theorem later in Subsection 5.1.

We also state the following conjecture, which is more general version of Conjecture 5.1.

Conjecture 5.4. *With a given length a and a given angle α, let $\mathcal{M}$ be the family of all triangles KLM such that $|LM| = a$ and $|\angle MKL| = \alpha$ and let $\mathcal{L}$ be a subfamily of the family $\mathcal{M}$. Then there exists a triangle T such that T is a smallest convex universal cover of the family $\mathcal{L}$.*

We will prove this conjecture for any finite subfamily with $\alpha \geq \pi/2$, formulating the following theorem.

Theorem 5.5. *With a given length a and a given angle $\alpha \in [\pi/2, \pi)$, let $\mathcal{M}$ be the family of all triangles KLM such that $|LM| = a$ and $|\angle MKL| = \alpha$ and let $\mathcal{L}$ be a finite subfamily of $\mathcal{M}$. Then there exists a triangle T such that T is a smallest convex universal cover of the family $\mathcal{L}$.*

We prove this theorem later in Subsection 5.2.

5.1 Proof of Theorem 5.3

Before we start with the proof itself, we will name three important triangles from the family $\mathcal{M}$:

- V_α is the isosceles triangle with equal sides adjacent to the angle α.
- U_α is the triangle with one of the remaining angles of size $(\pi - \alpha)/4$ (which is half the size of the angles of V_α adjacent to its base).
- P_α is the right-angled triangle. Note that it only exists for $\alpha < \pi/2$.

Now let us make an observation about U_α.

Observation 5.6. *We know the sizes of two angles in U_α and so we can calculate the third, name it β:*

$$\beta = \pi - \alpha - \frac{\pi - \alpha}{4} = \frac{3}{4}(\pi - \alpha).$$

And so if

$$\alpha \geq \frac{3}{7}\pi,$$

then

$$4\alpha \geq 3\pi - 3\alpha$$

and so

$$\alpha \geq \frac{3}{4}(\pi - \alpha) = \beta,$$

which makes α the largest angle in U_α.

On the other hand, when $\alpha \leq 3\pi/7$, then $\alpha \leq \beta$ and so β is the largest angle in U_α.

Also notice that if $\alpha \geq \pi/3$, then $\beta \leq \pi/2$.

For proving Theorem 5.3 it is essential to also prove Lemma 5.2. We shall to that later on, when the context of the expression (Λ) is given.

We distinguish three cases, depending on the size of α.

(a) $\frac{3}{7}\pi \leq \alpha$

(b) $\alpha \leq \frac{1}{3}\pi$

(c) $\frac{1}{3}\pi \leq \alpha \leq \lambda$

5.1.1 Case (a)

In this case $3\pi/7 \leq \alpha$.

We will take SCUC of triangles V_α and U_α and we show that it is also SCUC of the whole family $\mathcal{M}$.

Finding SCUC of $\{U_\alpha, V_\alpha\}$ similar to V_α

For finding SCUC of V_α and U_α we have to find their SCUC similar to V_α. For that we use the process described in Section 4.1, however, because of V_α being isosceles, we only have to consider 9 possibilities for SCUC similar to V_α (the other 9 possibilities created by switching the orientation of V_α will result in the same configurations).

We will denote the vertices of U_α as PQR so that $\alpha = |\angle RPQ|$ and $|\angle PQR| \geq |\angle QRP|$ and the triangle similar to V_α that we fit U_α into will be $V'_\alpha = KLM$ so that $\alpha = |\angle MKL|$. Note that α is the largest angle in $\triangle PQR$, as we observed earlier (Observation 5.6).

Then in Figure 5.1[1] we can see all the 9 configurations.

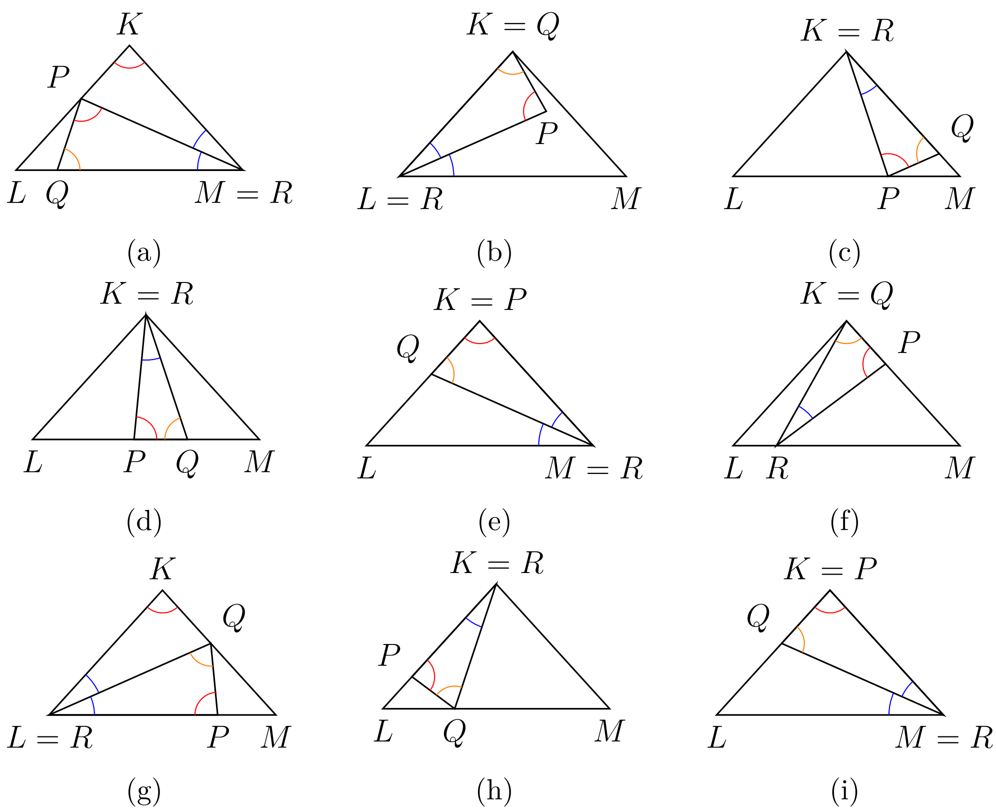


Figure 5.1: Maximally fitting U_α into V_α in case (a)

Now we show that $\triangle KLM$ in the configuration (a) is smaller (not neccesarily strictly) than $\triangle KLM$ in the other configurations.

In (a) $\triangle KPM$ is similar to $\triangle PQR$, which makes $\angle MKP$ the largest angle in $\triangle KPM$ and therefore the side PM is the largest. That means $|KM| \leq |PR|$.

In (b), (d), (f), (h) the segment RQ, which is the longest side of $\triangle PQR$, is between the point K and the base LM of the triangle V'_α. Therefore the leg of the triangle must be at least as long as RQ, which is longer than PR.

In (c) the segment RQ lies on the leg of V'_α and therefore the leg is no smaller than RQ too.

In (e) and (i) (which are the same configuration) $|KM| = |PR|$.

[1]note that for simplicity the pictures are scaled so that the triangle KLM is the same size, even though all the triangles PQR are congruent

In (g) $\triangle KQL$ is congruent with $\triangle PQR$ and so $|PR| = |KL| = |KM|$.

Constructing SCUC of $\{U_\alpha, V_\alpha\}$

We take the configuration (a) and construct points K' and M' on the segments KL and LM respectively so that $\triangle K'LM'$ is congruent wth V_α. Denote $T = \triangle K'LM$ (Figure 5.2).

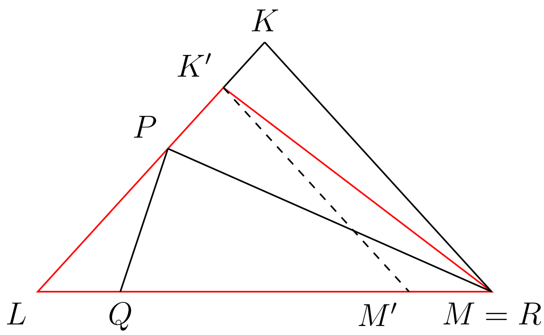


Figure 5.2: Constructing a SCUC of V_α and U_α

We see that

$$\frac{\mathrm{a}(V'_\alpha)}{\mathrm{a}(V_\alpha)} = \frac{\mathrm{a}(\triangle KLM)}{\mathrm{a}(\triangle K'LM')} = \left(\frac{|LM|}{|LM'|}\right)^2 = \left(\frac{\mathrm{a}(\triangle K'LM)}{\mathrm{a}(\triangle K'LM')}\right)^2 = \left(\frac{\mathrm{a}(T)}{\mathrm{a}(V_\alpha)}\right)^2$$

and so, according to Lemma 4.4, T is indeed a smallest convex universal cover of the family $\{V_\alpha, U_\alpha\}$.

Fitting each triangle into T

Now we show that we can fit every element of $\mathcal{M}$ into T, which would mean that T is a SCUC of $\mathcal{M}$.

Let $\triangle S$ be an element of $\mathcal{M}$. Then we construct point X on the circumcircle of $\triangle K'LM'$ so that $\triangle XLM'$ is congruent with S. That is possible, since $|LM'| = a$ and $|\angle M'K'L| = \alpha$ is an inscribed angle with chord LM'. Now we move the triangle XLM' alongside the line LM' so that X lands on the segment $K'L$ (and points L and M' land on the same line). This forms a triangle ABC, also congruent with S (Figure 5.3).

The only thing left to show now is that each vertex of $\triangle ABC$ lies in T.

Let Y be the middle of the smaller arc between K' and L on the curcumcircle of $\triangle K'LM'$. Then $|\angle LM'Y| = |\angle LM'K'|/2$ and therefore $\triangle YLM'$ is congruent with $\triangle PQR$, which means that $|YP| = |M'R|$. Also because of the way we constructed triangle ABC, $|XA| = |M'C|$. Now let X' be the perpendicular foot of X onto the line LK', similarily Y' for the point Y. Notice that triangles $XX'A$ and $YY'P$ are similar. Since Y is in the middle of arc LK', then $|XX'| \leq |YY'|$, which implies $|XA| \leq |YP|$, which implies $|M'C| \leq |M'R|$. From that we see that C lies on segment $M'R$, then also B lies on segment LC and we already know that A lies on segment LK'. Therefore all three vertices lie in T.

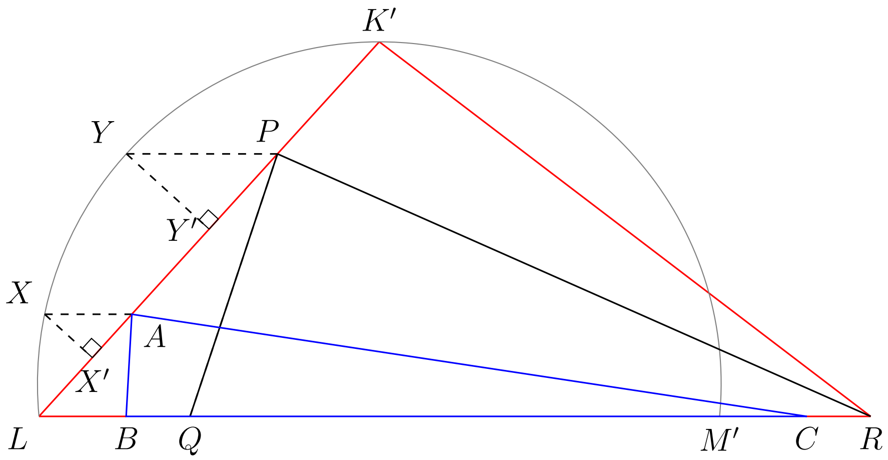


Figure 5.3: Fitting triangles in T

5.1.2 Case (b)

In this case $\alpha \leq \frac{1}{3}\pi$.

Here we take triangles V_α and P_α and find their SCUC.

If V'_α is SCUC of the family $\{V_\alpha, P_\alpha\}$, then, as we observed earlier in Observation 2.3, the diameter of V'_α is at least the diameter of P_α, which is its hypotenuse. And since the diameter of V'_α is its longest side (Observation 2.2), that has to be at least as long as the hypotenuse of P_α. Knowing that, we can construct triangle T, which we will later prove to be a SCUC of the family $\{V_\alpha, P_\alpha\}$.

Constructing SCUC of $\{P_\alpha, V_\alpha\}$

Construct $\triangle PQR$ which is congruent with P_α such that $|\angle RPQ| = \alpha$, $|\angle PQR| = \pi/2$ and $|QR| = a$. Then construct $\triangle KLM$ similar to V_α such that $K = P$, $M = R$ and $Q \in KL$. Then place points L' and M' on the segments KL and KM respectively so that $\triangle KL'M'$ is congruent with V_α. Then $T = \triangle KL'M$ (Figure 5.4).

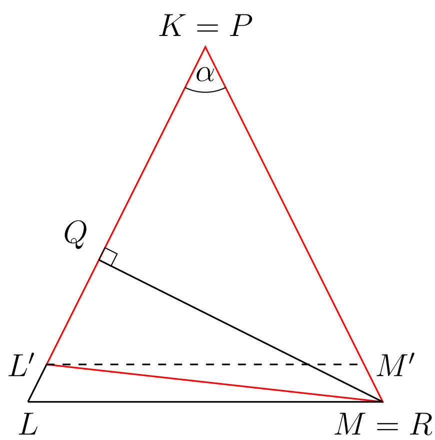


Figure 5.4: Constructing a SCUC of V_α and P_α

As we observed earlier, KLM is SCUC of $\{V_\alpha, P_\alpha\}$, because the longest side KM is the same length as the hypotenuse of $\triangle PQR$ (and also contains copies of both triangles). Denote $V'_\alpha = \triangle KLM$.

Then the following holds:

$$\frac{\mathrm{a}(V'_\alpha)}{\mathrm{a}(V_\alpha)} = \frac{\mathrm{a}(\triangle KLM)}{\mathrm{a}(\triangle KL'M')} = \left(\frac{|MK|}{|M'K|}\right)^2 = \left(\frac{\mathrm{a}(\triangle KL'M)}{\mathrm{a}(\triangle KL'M')}\right)^2 = \left(\frac{\mathrm{a}(T)}{\mathrm{a}(V_\alpha)}\right)^2,$$

and so, according to Lemma 4.4, T is indeed a smallest convex universal cover of the family $\{V_\alpha, P_\alpha\}$.

Fitting each triangle into T

Now we show that we can find a congruent copy of an arbitrary triangle S from $\mathcal{M}$ in the triangle T.

Let X be a point on the circumcircle k of $KL'M'$ so that the triangle $XL'M'$ is congruent with S (which is possible, because $|L'M'| = a$ and the inscribed angle with chord $L'M'$ is of size α). Since K is in the middle of the arc $L'M'$, $|L'X| \leq |L'K|$. We also know that the circumcircles of triangles P_α and V_α are congruent, and so PR is of the same length as the diameter of k, which means that every chord is no greater than $|PR|$. Thus $|XM'| \leq |KR|$. Now we know that we can construct points B and C on segments KL' and KM' respectively so that $|XL'| = |KB|$ and $|XM'| = |KC|$, which makes triangle KBC congruent with S and it is clear that it lies in the triangle $T = KL'M$.

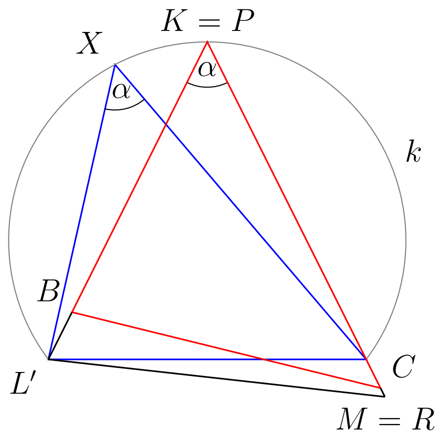


Figure 5.5: Fitting triangles into T

5.1.3 Case (c)

In this case $\pi/3 \leq \alpha \leq \lambda$.

First we will take a SCUC of triangles V_α and U_α and then we will show that it is also a SCUC of the whole family $\mathcal{M}$. For that we first have to find SCUC of $\{U_\alpha, V_\alpha\}$ similar to V_α.

Finding SCUC of $\{U_\alpha, V_\alpha\}$ similar to V_α

As we observed earlier in Observation 5.6, in this case is the angle PQR the largest angle in U_α and so the incidences in some of the 9 configurations change (Figure 5.6). We denote vertices and angles the same way we did in (a).

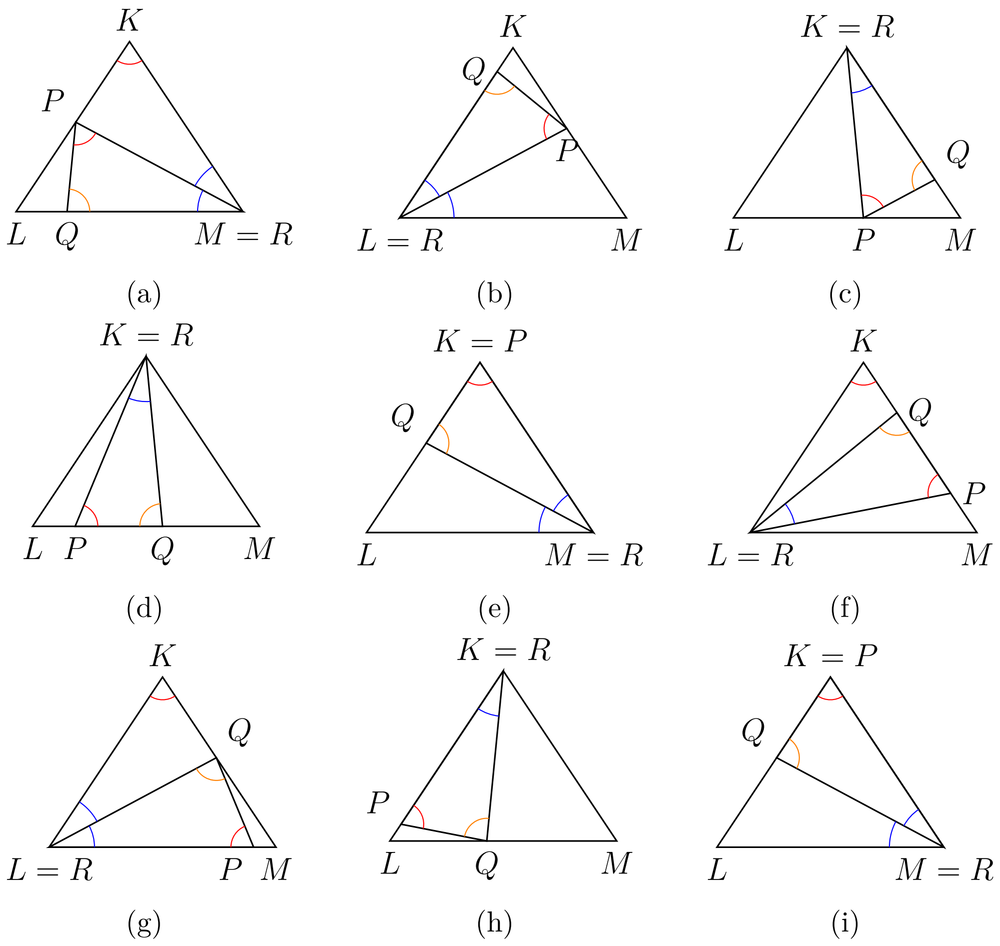


Figure 5.6: Maximally fitting U_α into V_α in case (b)

Now we show that $\triangle KLM$ is the smallest (not neccesarily strictly) in the configuration (g):

In (g) $\triangle PQR$ and $\triangle KQL$ are congruent and so the leg of $\triangle KLM$ is the same length as segment PR.

In (c) and (d) the segment PR lies between vertex K and the base LM, therefore the leg of $\triangle KLM$ has length of at least $|PR|$.

In (e), (h), (i) the segment PR lies on the leg of $\triangle KLM$, therefore the leg has length of at least $|PR|$.

In (a) the triangles PQR and KPR are similar, therefore KR is the largest side of $\triangle KPR$, which means $|KR| \geq |PR|$.

In (b) we see that $|\angle LPK| \geq \alpha = |\angle LKP|$, which means that LK is the longest edge of $\triangle KPL$, implying $|KL| \geq |PR|$.

In (f) $\triangle KLP$ is isosceles and so $|KL| = |PR|$.

Now we have SCUC similar to V_α and we can construct $\triangle T$—a smallest convex universal cover of family $\{V_\alpha, U_\alpha\}$, as we will then show.

Constructing SCUC of $\{U_\alpha, V_\alpha\}$

We take configuration (g) and construct points K' and L' on segments MK and ML respectively so that $\triangle K'L'M$ is congruent with V_α. Now let $T = \triangle K'LM$ (Figure 5.7).

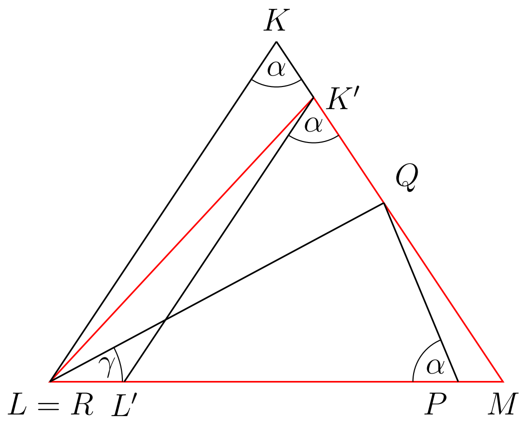


Figure 5.7: Constructing a SCUC of V_α and U_α

We see that

$$\frac{\mathrm{a}(V'_\alpha)}{\mathrm{a}(V_\alpha)} = \frac{\mathrm{a}(\triangle KLM)}{\mathrm{a}(\triangle K'L'M)} = \left(\frac{|LM|}{|L'M|}\right)^2 = \left(\frac{\mathrm{a}(\triangle K'LM)}{\mathrm{a}(\triangle K'L'M)}\right)^2 = \frac{\mathrm{a}(T)}{\mathrm{a}(V_\alpha)},$$

and so, according to Lemma 4.4, T is indeed a smallest convex universal cover of the family $\{V_\alpha, U_\alpha\}$.

Fitting each triangle into T

Now we show that we can find a congruent copy of an arbitrary triangle $S \in \mathcal{M}$ in T.

We will distinguish three cases, depending on the size of the smallest angle in S, which we will denote ω. Also let $|\angle QRP| = \gamma$.

(A) $\omega \leq \gamma$

(B) $\gamma \leq \omega \leq |\angle K'RM|$

(C) $|\angle K'RM| \leq \omega \leq 2\gamma$

Case (A)

Here we assume $\omega \leq \gamma$.

We know that $\alpha \geq \pi/3$ and therefore $\gamma \leq (\pi - \pi/3)/4 = \pi/6$. Then $|\angle RQM| = \pi - |\angle QRM| - |\angle RMQ| = \pi - 3\gamma \geq \pi/2$.

Let k be the arc of a circle with the center R and radius $|RQ|$ defined by the angle QRM. Since $|\angle RQM| \geq \pi/2$, the whole arc k lies on one side of the line $K'M$, thus in T.

Now we can place a point Q' on k so that $|\angle MRQ'| = \omega$. Then we place a point P' on the line RM so that $|\angle RP'Q'| = \alpha$. Since $|\angle RP'Q'| \geq |\angle RMQ|$, point P' must lie on the segment RM.

Triangle $P'Q'R$ is congruent with S and all of its vertices lie in T (Figure 5.8).

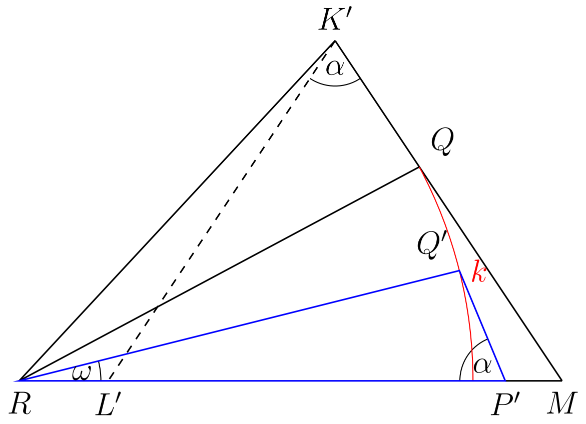


Figure 5.8: Fitting triangles into T (A)

Case (B)

Here we assume $\gamma \leq \omega \leq |\angle K'RM|$.

Let Q_1 be a point on the line $K'M$ such that $|RQ_1| = |RQ|$ and $Q_1 \neq Q$ (unless when $\alpha = \pi/3$ and $|\angle RQK'| = 3\gamma = \pi/2$, in which case let $Q_1 = Q$) and let P_1 be also a point on $K'M$ such that $|\angle RP_1Q_1| = \alpha$ (Figure 5.9).

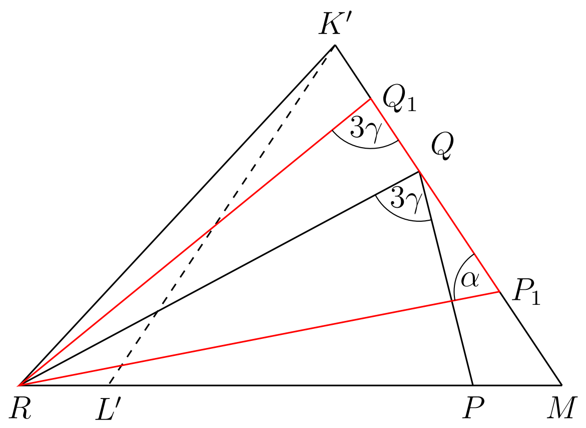


Figure 5.9: Constructing P_1Q_1R

We see that $|\angle RQK'| = \pi - \alpha - \gamma = 3\gamma$ and therefore if $\alpha \geq \pi/3$, then the angle RQK' is not obtuse. That means, that Q_1 lies on the ray $\overrightarrow{QK'}$.

Triangle Q_1RQ is an isosceles triangle, thus $|\angle QQ_1R| = |\angle Q_1QR| = 3\gamma$. With that we can see that $\triangle P_1Q_1R$ is congruent with $\triangle PQR$.

We will show that Q_1 lies in T for $\alpha \in [\pi/3, \lambda]$. For that we will use a bit of analytic geometry.

Without loss of generality, let $a = 1$. Then let the midpoint of $L'M$ be the origin and

$$L' = \left(-\frac{1}{2}, 0\right),$$
$$M = \left(\frac{1}{2}, 0\right),$$
$$K' = \left(0, \frac{1}{2\tan\frac{\alpha}{2}}\right).$$

Since we know the size of the angle QRM and that $|QR| = 1$, the y coordinate of the point Q is $\sin((\pi - \alpha)/4)$ and as it lies on the line $K'M$, we can calculate the x coordinate too:

$$Q = \left(-\sin\frac{\pi - \alpha}{4} \cdot \tan\frac{\alpha}{2} + \frac{1}{2}, \sin\frac{\pi - \alpha}{4}\right).$$

Finally we can deduce the x coordinate of R from Q and so

$$R = \left(-\cos\frac{\pi - \alpha}{4} - \sin\frac{\pi - \alpha}{4} \cdot \tan\frac{\alpha}{2} + \frac{1}{2}, 0\right).$$

If Q_1 is to lie on the segment $K'M$, it must hold that $|K'R| \geq |Q_1R|$. When expressing the length of $K'R$ we get the following inequality.

$$|K'R| = \left(-\cos\frac{\pi - \alpha}{4} - \sin\frac{\pi - \alpha}{4} \cdot \tan\frac{\alpha}{2} + \frac{1}{2}\right)^2 + \left(\frac{1}{2\tan\frac{\alpha}{2}}\right)^2 \geq 1 = |Q_1R|.$$

Now that we can see where the expression (Λ) comes from, we can prove Lemma 5.2.

Proof of Lemma 5.2. We see that (Λ) being equal to 0 is equivalent to $|K'R|$ being equal to 1. That is equivalent to $|K'R| = |Q_1R|$, which is equivalent to $K' = Q_1$. We shall show, that the lengths of QQ_1 and QK' are equal only for one value of α.

Let us define two functions $f(\delta)$ and $g(\delta)$ as the lengths of QQ_1 and QK' for when $\alpha = \delta$ ($f(\delta) = |QQ_1|$ and $g(\delta) = |QK'|$). We show, that f is strictly increasing while g is stricly decreasing.

- Triangle RQQ_1 is an isosceles triangle with the angles adjacent to its base the size of 3γ. The length of its legs is constant ($|RQ| = |RQ_1| = 1$) and so the smaller the angle by the base, the longer the base. And since 3γ is strictly decreasing, the length of QQ_1 is strictly increasing.
- By increasing α the height of $\triangle K'L'M$ is decreasing and so is $|K'M|$. In addition to that we see that the x coordinate of Q is also strictly decreasing, as both

  $$\sin\frac{\pi - \alpha}{4} \qquad \text{and} \qquad \tan\frac{\alpha}{2}$$

  are strictly increasing (on our interval). Therefore the length of $K'Q$ is strictly decreasing too.

For $\alpha = \pi/3$ is $|\angle RQK'| = \pi/2$ and so the triangle RQQ_1 is a degenerate triangle where $Q = Q_1$. Therefore $f(\pi/3) \leq g(\pi/3)$.

On the other hand for $\alpha = 3\pi/7$ it holds that $|\angle QRQ_1| = \gamma \geq |\angle QRK'|$, thus clearly $f(3\pi/7) \geq g(3\pi/7)$

Now from the intermediate value theorem we see that for α in between the values of $\pi/3$ and $3\pi/7$ it it holds exactly once that $f(\alpha) = g(\alpha)$.

□

Because of the monotonicity of both functions from the proof we also see that for $\alpha \leq \lambda$ it holds that $f(\alpha) \leq g(\alpha)$ and so the point Q_1 lies on the segment $K'Q$. Otherwise $f(\alpha) \geq g(\alpha)$.

Now let us distinguish two cases depending on the size of ω.

- $\gamma \leq \omega \leq |\angle MRQ_1|$:

  Let l be the circumcircle of the triangle P_1Q_1R and P' a point on its arc between P_1 and RM so that $|\angle Q_1RP'| = \omega$. $\triangle P'Q_1R$ is congruent with S and since $|\angle RMQ_1| \leq |\angle RP'Q_1|$, P' lies in T and so do the other two vertices (Figure 5.10).

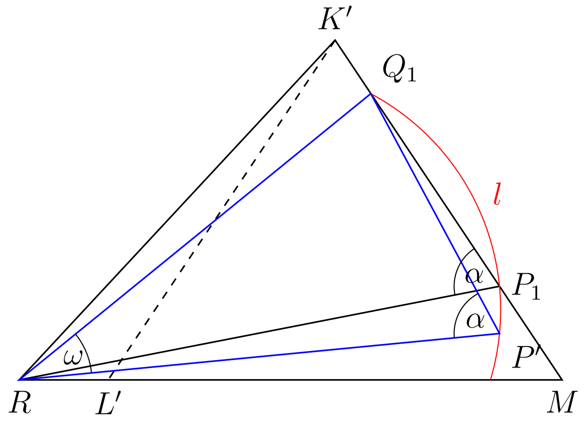


Figure 5.10: Fitting triangles into T (B1)

- $|\angle MRQ_1| \leq \omega \leq |\angle MRK'|$:

  Let m be a circle with center R and radius a and Q' a point on its arc between Q_1 and RK' so that $|\angle Q'RM| = \omega$. Then let P' be a point on RM such that $|\angle RP'Q'| = \alpha$ (Figure 5.11).

  Now $\triangle P'Q'R$ is congruent with S and since $|\angle RQ_1K'| \geq \pi/2$, the arc (including Q') is in T. So is P', because $|\angle RP'Q'| \geq |\angle RMK'|$.

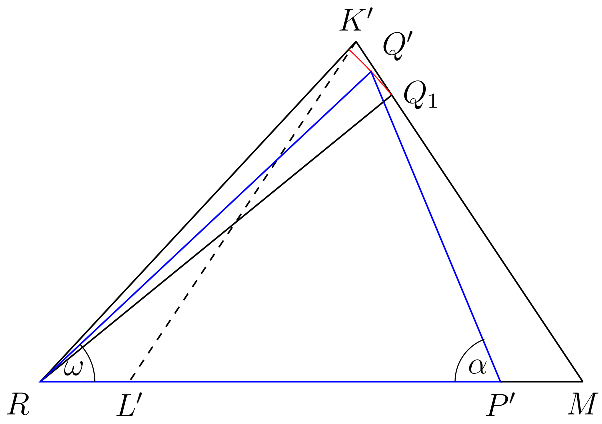


Figure 5.11: Fitting triangles into T (B2)

Case (C)

Here we assume $|\angle K'RM| \leq \omega \leq 2\gamma$.

V_α has the largest height opposite to the angle α out of all the triangles in $\mathcal{M}$. Therefore we can place a point P' on the height $K'V$ so that $|P'V|$ equals the height of S. Then we place R' on the ray $\overrightarrow{VR}$ and Q' on the ray $\overrightarrow{VM}$ so that $\triangle P'Q'R'$ is congruent with S. Now as $|\angle K'RM| \leq |\angle P'R'Q'|$ and $|\angle K'MR| \leq |\angle P'Q'R'$ both Q' and R' lie on the segment RM.

We have found $\triangle P'Q'R'$ that lies in T and is congruent with S.

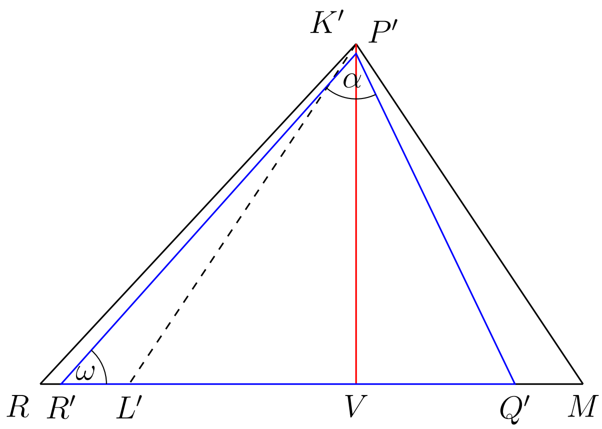


Figure 5.12: Fitting triangles into T (C)

□

5.2 Proof of Theorem 5.5

First of all, if the family $\mathcal{L}$ consists only of one triangle, it is clear that our theorem holds. So further we assume, that there are at least two triangles in $\mathcal{L}$.

This proof will be very similar to the proof of case (a) (5.1.1) of Theorem 5.3. However, we can not choose the same triangles as before, as they might not be in the family $\mathcal{L}$. We will take a different pair of triangles.

First, we take the triangle V, which is the triangle with the largest height (perpendicular to the base of length a). Then let k be its circumcircle. We know, that a copy of every triangle from the family $\mathcal{M}$ can be constructed as a triangle RLM, where R is a point on the longer arc of k between the points L and M. Since KLM is the higest triangle from $\mathcal{L}$, a copy of every triangle from $\mathcal{L}$ can then be constructed as a triangle PLM, where P is a point on the shorter arc of k between K and L. Let d be the distance between P and the line KL. Then let U be the triangle from $\mathcal{L}$ with maximal d

We will construct the SCUC of U and V and then show, that it is also a SCUC of the whole family $\mathcal{L}$.

Finding SCUC of $\{U, V\}$ similar to V

Let V' be the SCUC of U and V similar to V. Then let $V' = \triangle KLM$ so that $\alpha = |\angle MKL| \geq |\angle KLM| \geq |\angle LMK|$ and $U = \triangle PQR$ so that $\alpha = |\angle RPQ| \geq |\angle PQR| \geq |\angle QRP|$. From the construction of U it is also clear that $|\angle QRP| \leq |\angle LMK|$ and $|\angle PQR| \geq |\angle KLM|$.

Now we consider the 18 possible configurations of maximally fitting U in V', as described in 4.1 (Figure 5.13).

We will show, that V' is the smallest (not neccesarily strictly) in the configuration (a).

For referring to points, segments and triangles in a particular configuration we will be using subscript. For example triangle ABC in configuration a would be $\triangle ABC_{(\mathrm{a})}$.

In (a) the segment $PR_{(\mathrm{a})}$ is the longest side of $\triangle KPR_{(\mathrm{a})}$ and therefore $|KM_{(\mathrm{a})}| \leq |PR|$.

In (b), (c), (d), (e), (f), (h), (i), (k), (l), (m), (n), (o), (q), (r) there is either segment PR or QR leading from the vertex K to the opposite side LM. That means, that in each of these configurations $|KM| \geq |PR|$.

The triangles $QLM_{(\mathrm{p})}$ and $\triangle PLM_{(\mathrm{a})}$ are similar. However, the side $PM_{(\mathrm{a})}$ is smaller than the side $QR_{(\mathrm{p})}$ and therefore also $|LM_{(\mathrm{a})}| \leq |LM_{(\mathrm{p})}|$.

Similar argument can be applied for the pair of similar triangles $QLM_{(\mathrm{g})}$ and $PLM_{(\mathrm{j})}$ and that leads to $|LM_{(\mathrm{j})}| \leq |LM_{(\mathrm{g})}|$.

Triangles $PQL_{(\mathrm{a})}$ and $PQM_{(\mathrm{j})}$ have congruent sides (PQ) and the same angle adjacent to it ($|\angle PQL_{(\mathrm{a})}| = |PQM_{(\mathrm{j})}|$). Then since $|\angle QPM_{(\mathrm{j})}| \geq |\angle QPL_{(\mathrm{a})}|$ we see that $|QM_{(\mathrm{j})}| \geq |QL_{(\mathrm{a})}|$. And then

$$|LM_{(\mathrm{a})}| = |QL_{(\mathrm{a})}| + |QR_{(\mathrm{a})}| \leq |QM_{(\mathrm{j})}| + |QR_{(\mathrm{j})}| = |LM_{(\mathrm{j})}|$$

and so the triangle V' is smaller in (a) than in (j) and therefore also than in (g).

Now that have the SCUC similar to V, we can construct the actual SCUC.

Constructing SCUC of $\{U, V\}$

We take the configuration (a) and construct points K' and M' so that they lie on the segments KL and LM respectively and $K'LM'$ is congruent to V. Since V has larger height than U, point K' will lie on the segment KP. Then let $T = \triangle K'LM$ (Figure 5.14).

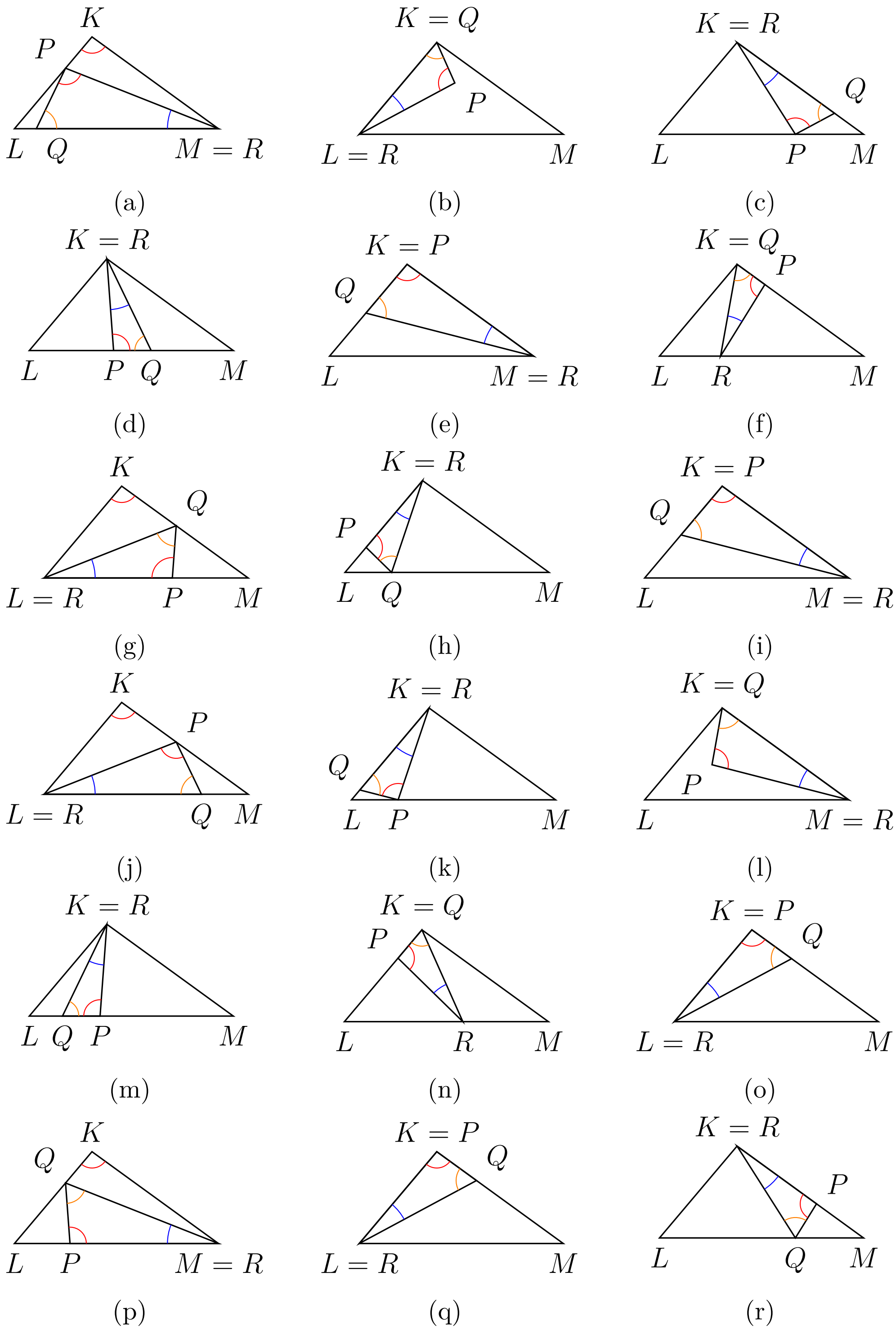


Figure 5.13: Maximally fitting U into V

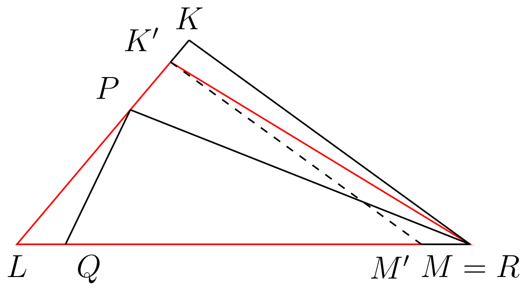


Figure 5.14: Constructing SCUC of V and U

We see that

$$\frac{\mathrm{a}(V')}{\mathrm{a}(V)} = \frac{\mathrm{a}(\triangle KLM)}{\mathrm{a}(\triangle K'LM')} = \left(\frac{|LM|}{|LM'|}\right)^2 = \left(\frac{\mathrm{a}(\triangle K'LM)}{\mathrm{a}(\triangle K'LM')}\right)^2 = \left(\frac{\mathrm{a}(T)}{\mathrm{a}(V)}\right)^2$$

and then, according to Lemma 4.4, T is a smallest convex universal cover of the family $\{V, U\}$.

Fitting each triangle into T

Now we will find a congruent copy of an arbitrary triangle S from $\mathcal{L}$ in T in order to show, that T is a SCUC of the family $\mathcal{L}$. We will do that the same way as in Section 5.1.1.

We construct point X on the arc of the circumcircle k of $\triangle K'LM'$ between the points K' and L so that $\triangle XLM'$ is congruent with S. Then we move $\triangle XLM'$ along LR so that X lands on the point A on $K'L$, creating another congruent copy of S—triangle ABC.

Let Y be a point on the arc of k between K' and L such that $\triangle YLM'$ is congruent with $\triangle PQR$.

Let Y' and X' be the perpendicular feet from X and Y onto $K'L$ respectively. Then triangles $YY'P$ and $XX'A$ are similar. By the definition of U, $|YY'| \geq |XX'|$ and therefore $|YP| \geq |XA|$, which implies $|M'R| \geq |M'C|$ and so C lies inside T. So do the other two vertices of $\triangle ABC$, which means we have found a congruent copy of S in T (Figure 5.15).

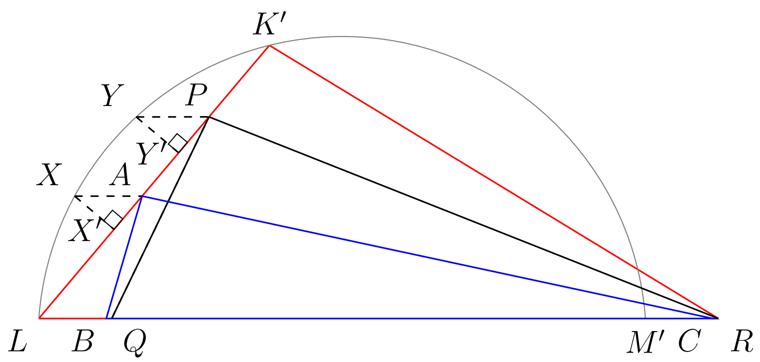


Figure 5.15: Fitting triangles into T

□

6. Conclusion

In this thesis we have found a smallest convex universal cover of certain families of triangles, which can, hopefully, get us closer to proving Conjecture 1.2.

However, we have also left a few obvious questions to be answered.

For Conjecture 5.1 to be proved it is left to find a SCUC for when $\alpha \in (\lambda, 2\pi/7)$. For that, our approach of finding a SCUC of two elements seems to come short and probably a different method is needed.

For Conjecture 5.4 to be proved we would have to extend Theorem 5.5 to infinite subsets and also an acute angle α. In our proof of Theorem 5.5 we relied heavily on knowing the longest side of each triangle in $\mathcal{L}$. That ensured that we knew the ordering of angles of our triangles U and V and finding their SCUC similar to V was rather simple (Section 5.2). This is not the case for an acute value of α, which would lead to many different cases. We also relied on the subset being finite when we picked a triangle with the maximal height. This would not be possible with infinite subsets. However, we believe that choosing supremum from the set would lead to a solution.

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