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BACHELOR THESIS

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Serre's Conjecture on Projective Modules over Polynomial Rings

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Thank you, doctor Liran Shaul, for your patient help and for sharing with me a small part of your vast knowledge.

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Abstract: This is an expository paper on the Quillen-Suslin Theorem, formerly known as Serre's Conjecture. A self-contained proof of this theorem is presented, followed by a discussion of the related Bass-Quillen Conjecture. The first chapter establishes the necessary theory, building on undergraduate algebra with the essentials of free, projective, and flat modules. The second chapter presents a complete proof of the theorem, dealing with regular rings, stably-free modules, and the related calculus of unimodular rows. The third and final chapter lists partial results surrounding the as yet unresolved Bass-Quillen Conjecture, offering brief explanations and suggestions for further reading.

Keywords: projective, module, commutative algebra, polynomial ring

Contents

Notation

A *ring* is always taken to be a ring with unity. A list of some of the shorthand appearing in this paper follows:

$\mathrm{Mod}{-}R$	the category of right R-modules
$R{-}\mathrm{Mod}$	the category of left R-modules
Ab	the category of abelian groups
$\mathrm{Hom}_R(M, N)$	the group of R-module homomorphisms $M \to N$
$\mathfrak{M}(R)$	finitely generated R-modules (left or right from context)
$\mathfrak{P}(R)$	finitely generated projective R-modules
$R^{(I)}$	direct sum power of a ring R, i.e. $R^{(I)} = \bigoplus_{i \in I} R$
$\operatorname{spec} R$	the set of prime ideals in a commutative ring R
$\max R$	the set of maximal ideals in a commutative ring R
$R\left[S^{-1}\right]$	the localization of a ring R at the multiplicative set S
R_P	$R\left[(R \setminus P)^{-1}\right]$; the localization of a ring R at $P \in \operatorname{spec} R$
$\mathrm{GL}_n(R)$	the group of $n \times n$ invertible matrices with entries in R
$\mathrm{SL}_n(R)$	the subgroup of $\mathrm{GL}_n(R)$ of matrices with determinant 1_R
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$	the naturals, integers, and rationals respectively
$\mathbb{Z}_n$	the ring $\mathbb{Z}/n\mathbb{Z}$

Introduction

Serre's Conjecture, also known as Serre's Problem, or the Quillen-Suslin Theorem since its resolution, comes from the 1955 paper *Faisceaux algébriques cohérents* [1] of Jean-Pierre Serre, in which modern sheaf theory was introduced to the rapidly developing field of Algebraic Geometry. Towards the end of this landmark paper, the author remarks that "one does not know whether finitely generated projective modules over $\mathbf{k}[t_1, ..., t_n]$ [where $\mathbf{k}$ is a field] are free".

This statement was motivated by a theorem (nowadays known as the Serre-Swan Theorem) given in that same paper, which says that vector bundles are like projective modules over commutative rings and that trivial bundles correspond to the free modules.

A standard result in topology says that any topological vector bundle on a *contractible* space is trivial. The space spec $\mathbf{k}[t_1, ..., t_n]$ with the *Zariski* topology (see Section 1.2.3) is just affine n-space $\mathbb{A}^n$ and thus it *ought* to act like a contractible space. If algebraic vector bundles behaved similarly to topological vector bundles, the conjecture would follow:

Serre's Conjecture: Let $\mathbf{k}$ be a field. Any algebraic vector bundle over $\mathbb{A}^n$ is trivial. Any finitely generated projective module over $\mathbf{k}[t_1, ..., t_n]$ is free.

This quickly became known as Serre's *Conjecture*, despite his having never speculated on its plausibility. Interest in projective modules in general and this problem in particular was soon increased by the advent of Homological Algebra and Algebraic K-theory. Serre's Conjecture became one of the most sought-after open problems in Algebra at that time.

Partial results by Serre (1957); Seshadri (1958); Bass (1963[1], 1964); Sharma, Ojanguren, and Srindaran (1971), and Suslin and Vaserstein (1974) were slowly appearing from the start.

The first complete proofs were given in 1976 by Andrey Aleksandrovich Suslin [2] and Daniel Quillen [3] working independently. Thus, the statement was renamed the Quillen-Suslin Theorem. Many clever, shorter proofs have since been presented. One of these, also due to Suslin, is presented in Chapter 2.

However, the story of Serre's Conjecture does not end in 1976. As with many good problems, its resolution has sparked work on various generalisations and analogues. The conjecture also breathed early life into the nascent field of algebraic K-theory. (In fact, the Grothendieck K_0-group is crucial even in this paper, in Section 2.2.)

Chapter 3 is devoted to the best-known of the attempted generalisations—The Bass-Quillen Conjecture—which replaces the field $\mathbf{k}$ with the more general notion of a *regular* ring; and which remains unsolved at the time of writing.

[1]Here, Hyman Bass proved the fact for non-finitely generated projective modules. See Remark 2.2.1 for an idea of why non-finitely-generated modules are easier to deal with.

1. The Basics

We present the theory necessary to give a self-contained proof of Serre's Conjecture in Chapter 2. An acquaintance with localization of commutative rings and basic module theory is assumed.

1.1 Free and Projective Modules

Definition 1.1.1: A subset X of a module $F \in \text{Mod}-R$ is said to form a **free basis** of F if for any map (of sets) $z : X \to M$, $M \in \text{Mod}-R$, there exists a unique R-homomorphism $\alpha : F \to M$ such that $\alpha|_X = z$.

A module F is called **free** if there exists a free basis of F.

We say a free R-module F is **of rank** κ for some cardinal κ, if there exists a free basis of F of cardinality κ. *The rank* of a free module is not well-defined in general—a free module can have two bases of distinct (finite) cardinalities. See Proposition 2.4.2 for more.

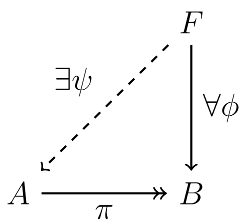
Lemma 1.1.1: *Suppose $F \in \text{Mod}-R$ is free with basis X, then*

i) X is a generating set of F; for any $m \in F$ there exist $r_x \in R$ such that

$$\sum_{x \in X} r_x \cdot x = m$$

(all but finitely many of the r_x being non-zero, so that the sum is well-defined)

ii) F has the factorisation property: If $\pi : A \twoheadrightarrow B$ is onto then for every $\phi : F \to B$ there exists an R-homomorphism $\psi : F \to A$ such that $\phi = \pi \circ \psi$.



Proof. i) Let $F' = \langle X \rangle$ be the submodule generated by X and $\pi : F \twoheadrightarrow F/F'$ the canonical projection. Let $z : X \to F/F', x \mapsto 0$, then both π and the zero homomorphism extend z, from uniqueness in the definition of the free basis X, this implies $\pi = 0$ and hence $F/F' = 0 \Rightarrow F' = F$. X generates F.

ii) Since π is onto, we have $\pi^{-1}(\phi(x)) \neq \emptyset$ for each $x \in X$. By the axiom of choice, we may choose a $\psi(x) \in \pi^{-1}(\phi(x))$ for each X and lift this to a homomorphism $\psi(x) : F \to A$.

We have $\pi(\psi(x)) = \phi(x)$ for all $x \in X$ which, by uniqueness of the lift, yields $\pi \circ \psi = \phi$ as needed. $\square$

Lemma 1.1.2: $F \in \text{Mod}-R$ *is free if and only if $F \simeq R^{(I)}$ for a suitable index set I.*

Proof. (⇐) We want to prove $R^{(I)}$ is free on the basis $X = \{e_i \mid i \in I\}$ where $e_i = (\delta_{j,i})_{j\in I}$ are the "canonical vectors". For a map of sets $z : X \to M$ we may define the homomorphism

$$\begin{aligned} f : R^{(I)} &\to M \\ (r_i)_{i\in I} &\mapsto \sum_{i\in I} r_i \cdot z(e_i) \end{aligned}$$

The sum is well-defined since all but finitely many of the r_i are 0 ($R^{(I)}$ is a direct sum). The elements of $R^{(I)}$ are determined uniquely by their coordinates and we see f is defined to satisfy the minimal requirements for an R-homomorphism for which $f(e_i) = z(e_i)$, so f is well-defined and unique. X forms a free basis.

(⇒) If F is free then it has a free basis X. We claim $F \simeq R^{(X)}$. Take the map

$$\begin{aligned} z : X &\to R^{(X)} \\ x &\mapsto e_x \end{aligned}$$

and let $\alpha : F \to R^{(X)}$ be its lift. We already know $R^{(X)}$ is free so we may extend $z^{-1} : e_x \mapsto x$ to a homomorphism $\alpha' : R^{(X)} \to F$. Then $\alpha' \circ \alpha|_X = id_X$ so, again, $\alpha' \circ \alpha = id_F$ by uniqueness. Similarly $\alpha \circ \alpha' = id_{R^{(X)}}$. So α and α' are mutually inverse isomorphisms. □

Corrolary 1.1.1: *For each module $M \in \mathrm{Mod}{-}R$ there exist a free module F and a homomorphism $\pi : F \twoheadrightarrow M$. That is, any M is a factor of a free module F.*

If $M \in \mathfrak{M}(R)$ (is finitely generated), then F can be chosen to have finite rank.

Proof. Take $F = R^{(X)}$ where X is a generating set for M (for instance, the set M itself) and define $\pi : F \to M$ by lifting $e_x \mapsto x$. The resulting π is onto, because $\mathrm{Im}\ \pi \subset M$ is a submodule containing the generating set X, hence $\mathrm{Im}\ \pi = M$. □

Definition 1.1.2: Let R be a ring. A sequence $(\alpha_i)_{i\in I}$ of R-module homomorphisms

$$\cdots \xrightarrow{\alpha_{i-1}} M_{i-1} \xrightarrow{\alpha_i} M_i \xrightarrow{\alpha_{i+1}} M_{i+1} \xrightarrow{\alpha_{i+2}} \cdots$$

is **exact** at M_i if $\mathrm{Im}\ \alpha_i = \ker \alpha_{i+1}$; it is an **exact sequence** if it is exact at M_i for each i.

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{\nu} M \xrightarrow{\pi} N \longrightarrow 0, \tag{1.1}$$

the only morphism from or into 0 being the trivial zero morphism. This just means ν is a monomorphism, π is an epimorphism, and $\mathrm{Im}\ \nu = \ker \pi$.

A short exact sequence like the one in (1.1) is **split** if π is a split epimorphism (a retraction), i.e. if there exists a $\iota : N \to M$ such that $\pi \circ \iota = id_N$.

Remark 1.1.1*:* Note that, in (1.1), π is a retraction if and only if ν is a section if and only if $M \simeq K \oplus N$.

Indeed, if $\pi \circ \iota = id_N$, then $\mathrm{Im}\ \iota \simeq N$ is a submodule of M and by exactness at M (and the homomorphism theorem) we have $M/\mathrm{Im}\ \iota \simeq \mathrm{Im}\ \nu \simeq K$ (since ν is injective). The projection $\rho : M \to K$ is a left inverse of ν, so ν is a section. The opposite implication is similar.

If this is the case then obviously Im ι and $\ker\pi =$ Im ν intersect only at 0. The isomorphism $M/\text{Im } \iota \simeq \text{Im } \nu$ implies that Im ι and Im ν span M, hence $M \simeq \text{Im } \nu \oplus \text{Im } \iota \simeq K \oplus N$.

Conversely, if $M \simeq K \oplus N$, then the projections and inclusions of each component are the required retractions and sections respectively.

Definition 1.1.3: Let R, S be rings. A functor $F : \text{Mod}-R \to \text{Mod}-S$ is called **additive** if for any $A, B \in \text{Mod}-R$, the induced map $\text{Hom}_R(A, B) \to \text{Hom}_S(F(A), F(B))$ is a group homomorphism, i.e. if $F(f + g) = F(f) + F(g)$ for any R-homomorphisms $f, g : A \to B$.

An **exact functor** is an additive functor which preserves exact sequences.

Definition 1.1.4: Let R be a ring and $M \in \text{Mod}-R$ a module.

We define a map $\text{Hom}_R(M, -)$ as

$$\begin{aligned} N \in \text{Mod}-R &\mapsto \text{Hom}_R(M, N) \\ \alpha \in \text{Hom}_R(A, B) &\mapsto \text{Hom}_R(M, \alpha) \end{aligned}$$

where

$$\begin{aligned} \text{Hom}_R(M, \alpha) : \text{Hom}_R(M, A) &\to \text{Hom}_R(M, B) \\ \phi &\mapsto \alpha \circ \phi \end{aligned}$$

Noting $(\alpha_1 + \alpha_2) \circ \phi = \alpha_1 \circ \phi + \alpha_2 \circ \phi$, it is obvious that $\text{Hom}(M, \alpha)$ is a morphism of abelian groups and $\text{Hom}(M, -)$ is well-defined. Some important properties follow:

Lemma 1.1.3: *Let R be a ring and $M, A, B, C \in \text{Mod}-R$*

i) $\text{Hom}_R(M, -)$ *is a covariant functor from* $\text{Mod}-R$ *to* Ab*, meaning it satisfies*

$$\begin{aligned} \text{Hom}_R(M, id_A) &= id_{\text{Hom}_R(M,A)} \\ \text{Hom}_R(M, \alpha \circ \beta) &= \text{Hom}_R(M, \alpha) \circ \text{Hom}_R(M, \beta) \end{aligned}$$

ii) $\text{Hom}_R(M, -)$ *is left exact, for any short exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

the sequence

$$0 \longrightarrow \text{Hom}_R(M, A) \xrightarrow{\text{Hom}(M,\alpha)} \text{Hom}_R(M, B) \xrightarrow{\text{Hom}(M,\beta)} \text{Hom}_R(M, C)$$

is also exact.

Proof. i) Take $\alpha : A \to B$, $\beta : B \to C$, then by definition

$$\begin{aligned} \text{Hom}_R(M, id_A)(\phi) &= id_A \circ \phi = \phi \\ \text{Hom}_R(M, \alpha) \circ \text{Hom}_R(M, \beta)(\phi) &= \text{Hom}_R(M, \alpha)(\beta \circ \phi) = \alpha \circ (\beta \circ \phi) = \\ &= (\alpha \circ \beta) \circ \phi = \text{Hom}_R(M, \alpha \circ \beta)(\phi) \end{aligned}$$

ii) Injectivity of $\mathrm{Hom}(M,\alpha)$: Suppose $\mathrm{Hom}(M,\alpha)(\phi) = \alpha \circ \phi = 0$, then $\mathrm{Im}\ \phi \subseteq \ker \alpha$, but α is assumed to be injective, so $\ker \alpha = \{0\} = \mathrm{Im}\ \phi \Rightarrow \phi = 0$. $\mathrm{Hom}(M,\alpha)$ is injective, the sequence is exact at $\mathrm{Hom}_R(P,A)$.

Exactness at $\mathrm{Hom}(P,B)$: By i) we have

$$\mathrm{Hom}(M,\beta) \circ \mathrm{Hom}(M,\alpha) = \mathrm{Hom}(M,\beta \circ \alpha) = \mathrm{Hom}(M,0) = 0,$$

so $\mathrm{Im}\ \mathrm{Hom}(M,\alpha) \subseteq \ker \mathrm{Hom}(M,\beta)$. For the opposite inclusion, suppose

$$\phi \in \ker \mathrm{Hom}(M,\beta) \Rightarrow \beta \circ \phi = 0.$$

Then $\mathrm{Im}\ \phi \subseteq \ker \beta = \mathrm{Im}\ \alpha$. The map α is one-to-one, so it is an isomorphism onto $\mathrm{Im}\ \alpha$ and we have an inverse $\alpha^{-1} : \mathrm{Im}\ \alpha \to A$. The inclusion $\mathrm{Im}\ \phi \subseteq \mathrm{Im}\ \alpha$ means we may compose $\psi = \alpha^{-1} \circ \phi$, then $\psi \in \mathrm{Hom}_R(M,A)$ and $\mathrm{Hom}(M,\alpha)(\psi) = \alpha \circ \psi = \phi$, so $\mathrm{Im}\ \mathrm{Hom}(M,\alpha) \supseteq \ker \mathrm{Hom}(M,\beta)$. □

Remark 1.1.2*:* $\mathrm{Hom}_R(M,-)$ is not exact in general: Take e.g. $R = \mathbb{Z}, M = \mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ and the short exact sequence

$$0 \longrightarrow 5 \cdot \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_5 \longrightarrow 0$$

then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_5,\mathbb{Z}) = 0$ since any $f : \mathbb{Z}_5 \to \mathbb{Z}$ satisfies $5 \cdot f([k]) = f([5k]) = 0 \Rightarrow f([k]) = 0$ ($\mathbb{Z}$ is an integral domain). But $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_5,\mathbb{Z}_5)$ contains at least the zero morphism and the identity (in fact $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_5,\mathbb{Z}_5)) \simeq \mathbb{Z}_5$) so there can be no epimorphism from $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_5,\mathbb{Z})$ onto $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_5,\mathbb{Z}_5)$.

The **contravariant** Hom**-functor**, denoted $\mathrm{Hom}_R(-,M)$, is defined similarly. These functors are compatible with taking direct sums:

Proposition 1.1.1: *Let M_i $(i \in I)$ and N be left R-modules, then*

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \simeq \prod_{i\in I} \mathrm{Hom}_R(M_i,N) \tag{1.2}$$

$$\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \simeq \prod_{i\in I} \mathrm{Hom}_R(N,M_i) \tag{1.3}$$

Proof. We shall prove 1.2, the other proof is similar. Take the canonical inclusion and projection homomorphisms

$$\begin{aligned}\nu_i : &M_i \to \bigoplus_{i\in I} M_i \\ &m \mapsto (m_j)_{j\in I}, \text{ where } m_j = m \text{ if } j = i \text{ and otherwise } m_j = 0 \\ \pi_i : &\bigoplus_{i\in I} M_i \to M_i \\ &(m_j)_{j\in I} \mapsto m_i\end{aligned}$$

Then we have

$$\pi_i \circ \nu_j = \begin{cases} id_{M_j} & \text{for } i = j \\ 0 & \text{otherwise}\end{cases}$$

We may define a map

$$\Phi : \mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \to \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$$
$$\phi \mapsto (\phi \circ \nu_i)_{i\in I},$$

this is obviously a group homomorphism. Suppose $\Phi(\phi) = (\phi \circ \nu_i)_{i\in I} = 0$, then $\phi(\nu_i(m_i)) = 0$ for every $i \in I$ and $m_i \in M_i$, but these elements generate $\bigoplus_{i\in I} M_i$, so ϕ is determined by its values at these points, hence $\phi = 0$, Φ is injective.

Let $(\phi_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$ and define

$$\left(\sum \phi_i \circ \pi_i\right)((m_j)_{j\in I}) := \sum_{i\in I} \phi_i(\pi_i((m_j)_{j\in I})) = \sum_{i\in I} \phi_i(m_i)$$

the sum is well-defined since only finitely many summands are non-zero. From the fact that π_i and ϕ_i are homomorphisms, we see $\sum \phi_i \circ \pi_i \in \mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right)$ and of course

$$\Phi\left(\sum \phi_i \circ \pi_i\right) = \left(\sum_{i\in I} \phi_i \circ \pi_i \circ \nu_j\right)_{j\in I} = (\phi_j)_{j\in I}$$

Φ is surjective. □

Lemma 1.1.4: *Suppose* $0 \to A \to B \xrightarrow{\phi} C \to 0$ *is a short exact sequence of modules. Then this sequence is split if and only if the map*

$$\mathrm{Hom}_R(C, \phi) : \mathrm{Hom}_R(C, B) \to \mathrm{Hom}_R(C, C)$$
$$\psi \mapsto \phi \circ \psi$$

is surjective.

Proof. Suppose the sequence splits and take $\varepsilon \in \mathrm{Hom}_R(C, C)$. There exists $\iota \in \mathrm{Hom}_R(C, B)$ such that $\phi \circ \iota = id_C$. Taking $\iota \circ \varepsilon \in \mathrm{Hom}_R(C, B)$ we get $\mathrm{Hom}_R(C, \phi)(\iota \circ \varepsilon) = \varepsilon$.

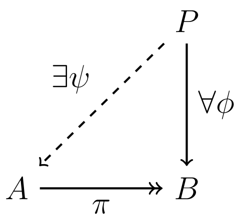
Suppose the map $\mathrm{Hom}_R(C, \phi)$ is surjective. Then there exists $\iota \in \mathrm{Hom}_R(C, B)$ such that $\mathrm{Hom}_R(C, \phi)(\iota) = \phi \circ \iota = id_C$. The sequence is split. □

There are a few ways to define a projective module, we present four common definitions, each of which will be useful going forward.

Definition 1.1.5: A module $P \in \mathrm{Mod}-R$ is called **projective** if it has any of the following equivalent properties:

i) P is a direct summand of a free module, i.e. there exist modules $F, Q \in \mathrm{Mod}-R$ such that F is free and $F = P \oplus Q$.

ii) P has the factorisation property: If $\pi : A \twoheadrightarrow B$ is onto then for every $\phi : P \to B$ there exists a map $\psi : P \to A$ such that $\phi = \pi \circ \psi$.



iii) The covariant functor $\mathrm{Hom}_R(P, -)$ is exact, that is, for any exact sequence

$$0 \longrightarrow A \xrightarrow{\nu} B \xrightarrow{\pi} C \longrightarrow 0$$

the image

$$0 \longrightarrow \mathrm{Hom}_R(P, A) \xrightarrow{\mathrm{Hom}(P,\nu)} \mathrm{Hom}_R(P, B) \xrightarrow{\mathrm{Hom}(P,\pi)} \mathrm{Hom}_R(P, C) \longrightarrow 0$$

is also exact.

iv) Any epimorphism $\alpha : A \twoheadrightarrow P$ splits, i.e. there exists a map $\iota : P \hookrightarrow B$ such that $\alpha \circ \iota = id_P$.

Note that free modules are a special case of projective modules. We have claimed that the four properties in Definition 1.1.5 are equivalent, the proof of this fact follows:

Proof. i) $\Rightarrow$ ii) We have $F = P \oplus Q$ with F free. Let $\rho : F \to P$ be the natural projection onto P and note that $\rho|_P = \rho \circ \iota = id_P$, where $\iota : P \subseteq F$. Then $\phi \circ \rho : F \to B$ factorises through π because F is free, i.e. there is a map $\psi' : F \to A$ such that

$$\pi \circ \psi' = \phi \circ \rho \Longrightarrow \pi \circ \psi' \circ \iota = \phi \circ \rho \circ \iota$$

The sought-after map is then $\psi = \psi' \circ \iota$.

ii) $\Rightarrow$ iii) $\mathrm{Hom}_R(P, -)$ is always left exact (Lemma 1.1.3), so it remains only to check that $\mathrm{Hom}_R(P, \pi)$ is onto, but that is precisely what ii) states, since in that setting we have $\mathrm{Hom}(P, \pi)(\psi) = \phi$ for any ψ in the co-domain.

iii) $\Rightarrow$ iv) Applying $\mathrm{Hom}_R(P, -)$ to the sequence

$$0 \longrightarrow \ker\alpha \xrightarrow{\subseteq} B \xrightarrow{\alpha} P \longrightarrow 0$$

we get from iii) that $\mathrm{Hom}(P, \alpha) : \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, P)$ is onto so, specially, some $\iota \in \mathrm{Hom}_R(P, B)$ is sent to the identity $id_P \in \mathrm{Hom}_R(P, P)$, hence $\beta \circ \iota = id_P$.

iv) $\Rightarrow$ i) By Corrolary 1.1.1 there exists a free module F and a map $\pi : F \twoheadrightarrow P$. This yields a short exact sequence

$$0 \longrightarrow \ker\pi \xrightarrow{\subseteq} F \xrightarrow{\pi} P \longrightarrow 0$$

which splits by iv), so $F = \ker\pi \oplus P$. □

1.2 Tensor Products

A more complete treatment of tensor products can be found e.g. in Dummit and Foote [4]. In this paper, we present a slightly different approach and deal with *bimodules* in more detail.

Definition 1.2.1: Let R be a ring, $G \in \mathrm{Ab}$, $M \in \mathrm{Mod}{-}R$ and $N \in R{-}\mathrm{Mod}$. A map $\phi : M \times N \to G$ is called **R-balanced** if

$$\begin{aligned} \phi(m + m', n) &= \phi(m, n) + \phi(m', n) \\ \phi(m, n + n') &= \phi(m, n) + \phi(m, n') \\ \phi(m \cdot r, n) &= \phi(m, r \cdot n) \end{aligned}$$

for all $m, m' \in M, n, n' \in N, r \in R$.

The **tensor product** of M by N is an abelian group $M \otimes_R N$ equipped with an R-balanced map

$$\otimes : M \times N \to M \otimes_R N$$

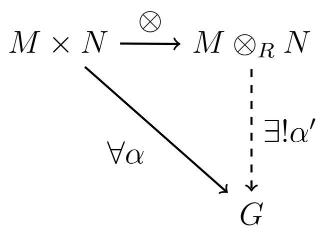
satisfying the following so-called **universal property of the tensor product**:

Given any abelian group G and R-balanced map $\alpha : M \times N \to G$, there exists a unique

$$\alpha' : M \otimes_R N \to G$$

such that

$$\alpha' \circ \otimes = \alpha$$



The following proposition gives an idea of what the group $M \otimes_R N$ looks like:

Proposition 1.2.1: *The elements of $M \otimes_R N$ can be expressed (non-uniquely) as finite sums*

$$\sum m_i \otimes n_i$$

where $m_i \in M, n_i \in N$ and we denote $\otimes(m, n) = m \otimes n$ (so-called simple tensors).

Proof. Let $L = \langle m \otimes n \mid (m, n) \in M \times N \rangle$ the subgroup of $M \otimes_R N$ generated by the simple tensors (the image of $\otimes$). We want to prove that $L = M \otimes_R N$, so let $O = M \otimes_R N/L$ and $\pi : M \otimes_R N \to O$ the projection map.

Taking $\alpha = 0 : M \otimes_R N \to O$ to be the zero morphism, we get $\pi \circ \otimes = 0$ and $0 \circ \otimes = 0$, so $\pi = 0$ by the uniqueness required in the universal property. Hence, $O = \{0\}$ which means $L = M \otimes_R N$ as needed. $\square$

Remark 1.2.1*:* With this result in mind, we can construct the tensor product $M \otimes_R N$ by taking the free abelian group (the free $\mathbb{Z}$-module) F with a basis consisting of the symbols

$$\{m \otimes n \mid (m, n) \in M \times N\}$$

F consist precisely of finite sums $\sum m_i \otimes n_i$, it remains to force the relations in Definition 1.2.1 on $\otimes$ to make it R-balanced. Take K to be the subgroup of F generated by elements of the form

$$m \otimes n + m' \otimes n - (m + m') \otimes n$$
$$m \otimes n + m \otimes n' - m \otimes (n + n')$$
$$m \cdot r \otimes n - m \otimes r \cdot n,$$

then $\otimes$ acts as an R-balanced map $\otimes : M \times N \to F/K$. Using Proposition 1.2.1, it turns out that $M \otimes_R N \simeq F/K$.

Importantly, this means that $M \otimes_R N$ is uniquely (up to isomorphism) determined by the universal property of the tensor product.

Definition 1.2.2: Let R, S be rings. An abelian group M is an (S, R)**-bimodule** if M is a left S-module as well as a right R-module and it satisfies the additional relation $(sm)r = s(mr)$ for any $s \in S, m \in M, r \in R$.

Remark 1.2.2*:* If R is a commutative ring, then any R-module (left or right) is naturally an (R, R)-bimodule. Indeed, if e.g. $M \in \mathrm{Mod}{-}R$ is a right R-module, then we can define a *left* action of R on M as $r \cdot m = mr$. Commutativity then gives $s \cdot (r \cdot m) = (mr)s = m(rs) = m(sr) = (sr) \cdot m$, the other R-module axioms carry over directly. So M is a left R-module under this action and $(s \cdot m)r = msr = (mr)s = s \cdot (mr)$ makes it an (R, R)-bimodule.

Lemma 1.2.1: *If M is an (S, R)-bimodule and $N \in R{-}\mathrm{Mod}$ is a left R-module, then $M \otimes_R N$ has a natural S-module structure. In this case the universal property can be refined:*

If G is a left S-module and $\alpha : M \times N \to G$ is R-balanced with the additional property $s \cdot \alpha(m, n) = \alpha(s \cdot m, n), \forall s \in S, m \in M, n \in N$, then the lift $\alpha' : M \otimes_R N \to G$ is a homomorphism of left S-modules.

Proof. For each $s \in S$ define

$$\alpha_s : M \times N \to M \otimes_R N$$
$$(m, n) \mapsto sm \otimes n$$

And extend uniquely to group homomorphisms $s \cdot - : M \otimes_R N \to M \otimes_R N$, noting that α_s is R-balanced:

$$\begin{aligned} \alpha_s(m + m', n) &= s(m + m') \otimes n = (sm + sm') \otimes n = sm \otimes n + sm' \otimes n = \\ &= \alpha_s(m, n) + \alpha_s(m', n) \\ \alpha_s(m, n + n') &= m \otimes (n + n') = m \otimes n + m \otimes n' = \alpha_s(m, n) + \alpha_s(m, n') \\ \alpha_s(mr, n) &= s(mr) \otimes n = (sm)r \otimes n = sm \otimes rn = \alpha_s(m, rn) \end{aligned}$$

where we have used distributivity of $s \in S$ over M, properties of $\otimes$, and also, importantly, the extra relation required of an (S, R)-bimodule.

The properties required for $s \cdot -$ to make $M \otimes_R N$ into an S-module are obvious on simple tensors. They hold on all of $M \otimes_R N$ by uniqueness.

G is an abelian group, so any R-balanced $\alpha : M \times N \to G$ lifts to a unique $\alpha' : M \otimes_R N \to G$, where $\alpha' \circ \otimes = \alpha$. With the extra assumption, we get (on simple tensors)

$$\alpha'(s \cdot (m \otimes n)) = \alpha'(sm \otimes n) = \alpha(sm, n) = s\alpha(m, n) = s \cdot \alpha'(m \otimes n)$$

since α' is already additive, this extends to all of $M \otimes_R N$ (Proposition 1.2.1). So α' is an S-module homomorphism. □

Remark 1.2.3*:* This could have been done analogously for N an (R, S)-bimodule, making $M \otimes_R N$ into a right S-module.

By the previous remark, if R is commutative, then the Lemma says that $M \otimes_R N$ is a left and a right R-module (in fact an (R, R)-bimodule).

Lemma 1.2.2: *Let* $M, M' \in \mathrm{Mod}{-}R, N, N' \in R{-}\mathrm{Mod}$ *and* $\phi : M \to M'$*, resp.* $\psi : N \to N'$ *be homomorphisms of right and left* R*-modules respectively.*

i) There exists a unique group homomorphism $\phi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$ *such that for any* $m \in M, n \in N$

$$(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$$

ii) If M, M' *are* (S, R)*-bimodules and* ϕ *is a left* S*-homomorphism as well as a right* R*-homomorphism, then* $\phi \otimes \psi$ *is also an* S*-homomorphism.*

iii) If, additionally, $\phi' : M' \to M''$ *and* $\psi' : N' \to N''$ *are as above, then* $(\phi \otimes \psi) \circ (\phi' \otimes \psi') = (\phi \circ \phi') \otimes (\psi \circ \psi')$.

Proof. i) Define a map

$$\begin{aligned} \alpha : M \times N &\to M' \otimes_R N' \\ (m, n) &\mapsto \phi(m) \otimes \psi(n) \end{aligned}$$

This is obviously R-balanced since ϕ, ψ are additive and $\otimes$ is R-balanced. By the universal property, α lifts uniquely to a group homomorphism $\phi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$, such that $(\phi \otimes \psi)(m \otimes n) = \alpha(m, n) = \phi(m) \otimes \psi(n)$.

ii) $M \otimes N$ and $M' \otimes N'$ are left S-modules by Lemma 1.2.1. Take an arbitrary $s \in S$, then

$$s \cdot \alpha(m, n) = s \cdot (\phi(m) \otimes \psi(n)) = \phi(sm) \otimes \psi(n),$$

so Lemma 1.2.1 applies, making $\phi \otimes \psi$ into an S-homomorphism.

iii) Considering the R-balanced map

$$\begin{aligned} \beta : M \times N &\to M'' \otimes_R N'' \\ (m, n) &\mapsto (\phi \circ \phi')(m) \otimes (\psi \circ \psi')(n) \end{aligned}$$

we see that both $(\phi \otimes \psi) \circ (\phi' \otimes \psi')$ and $(\phi \circ \phi') \otimes (\psi \circ \psi')$ extend β to $M \otimes_R N$, they are therefore equal by uniqueness in the universal property. □

Definition 1.2.3: S is said to be a commutative associative R-algebra, if S is a commutative ring, S is an R-module, and the module action of R on S is compatible with ring multiplication in S, i.e. $r \cdot (ss') = (r \cdot s)s'$ for any $r \in R$ and $s, s' \in S$.

Remark 1.2.4*:* If R is a commutative ring and S is a commutative associative R-algebra, then S is an (S,R)-bimodule and also an (S,S)-, (R,S)-, or (R,R)-bimodule. If $M \in R-\mathrm{Mod}$, then $S \otimes_R M \in S-\mathrm{Mod}$.

The tensor product is compatible with direct sums in the following sense:

Theorem 1.2.1: *Let I be an arbitrary index set and $M, M_i \in \mathrm{Mod}-R$ and $N, N_i \in R-\mathrm{Mod}$ for each $i \in I$, then*

$$\bigoplus_{i\in I} M_i \otimes_R N \simeq \bigoplus_{i\in I}(M_i \otimes_R N)$$
$$M \otimes_R \bigoplus_{i\in I} N_i \simeq \bigoplus_{i\in I}(M \otimes_R N_i)$$

as abelian groups. If M, M_i $(i \in I)$ are (S,R)-bimodules, then the above are isomorphisms of left S-modules.

Proof. We will prove the first statement, the second is similar. Define

$$\alpha : \bigoplus_{i\in I} M_i \times N \to \bigoplus_{i\in I}(M_i \otimes_R N)$$
$$((m_i)_{i\in I}, n) \mapsto (m \otimes n)_{i\in I}$$

This map is well-defined since elements of $\oplus M_i \times N$ are expressed uniquely as these tuples. The map α is also R-balanced:

$$\begin{aligned}\alpha\Big((m_i)_{i\in I} + (m_i')_{i\in I}, n\Big) &= ((m_i + m_i') \otimes n)_{i\in I} = \\ &= (m_i \otimes n)_{i\in I} + (m_i' \otimes n)_{i\in I} = \\ &= \alpha\Big((m_i)_{i\in I}, n\Big) + \alpha\Big((m_i')_{i\in I}, n\Big)\end{aligned}$$

$$\begin{aligned}\alpha\Big((m_i)_{i\in I}, n + n'\Big) &= (m_i \otimes (n + n'))_{i\in I} = \\ &= (m_i \otimes n)_{i\in I} + (m_i \otimes n')_{i\in I} = \\ &= \alpha\Big((m_i)_{i\in I}, n\Big) + \alpha\Big((m_i)_{i\in I}, n'\Big)\end{aligned}$$

$$\begin{aligned}\alpha\Big((m_i \cdot r)_{i\in I}, n\Big) &= (m_i \cdot r \otimes n)_{i\in I} = (m_i \otimes rn)_{i\in I} = \\ &= \alpha\Big((m_i)_{i\in I}, r \cdot n\Big)\end{aligned}$$

So there exists a (unique)

$$\alpha' : \bigoplus_{i\in I} M_i \otimes_R N \to \bigoplus_{i\in I}(M_i \otimes_R N)$$

such that $\alpha' \circ \otimes = \alpha$.

In the other direction, take the embeddings $\nu_j : M_j \to \bigoplus_{i\in I} M_i$, mapping M_j to the jth component, and define $\beta_j = \nu_j \otimes id_N$.

Once more making use of the fact that $\otimes$ is R-balanced, we may define

$$\beta : \bigoplus_{i\in I}(M_i \otimes_R N) \to \bigoplus_{i\in I} M_i \otimes_R N$$
$$(\mathbf{m}_i) \mapsto \sum_{i\in I} \beta_i(\mathbf{m}_i)$$

This is well-defined because, firstly, only finitely many $\mathbf{m}_i$ are non-zero meaning the sum is actually finite and, secondly, the elements of $\oplus(M_i \otimes_R N)$ are uniquely determined as these sequences of tensors.

On simple tensors, we have by definition $\beta(m_i \otimes n)_{i\in I} = (\sum_{i\in I} \nu_i(m_i)) \otimes n = (m_i)_{i\in I} \otimes n$. So $\alpha' \circ \beta$ and $\beta \circ \alpha'$ are seen to act as the identity on simple tensors and hence are the identity on the whole groups by uniqueness in the universal property. Hence, α' and β are mutually inverse isomorphisms.

If M_i are (S, R)-bimodules, then

$$s \cdot \alpha\Big(((m_i)_{i\in I}, n)\Big) = s \cdot (m \otimes n)_{i\in I} = (sm_i \otimes n)_{i\in I} = \alpha\Big((s(m_i)_{i\in I}, n)\Big)$$

and Lemma 1.2.1 says that α' is an isomorphism of S-modules. □

Corrolary 1.2.1: *Let S be a commutative associative R-algebra. If F is a free R-module, then $S \otimes_R F$ is a free S-module. If P is a projective R-module, then $S \otimes_R P$ is a projective S-module. If, additionally, $P \in \mathfrak{P}(R)$ then $S \otimes_R P \in \mathfrak{P}(S)$.*

Proof. We have $S \otimes_R R \simeq S$ by the isomorphism lifted from the R-balanced map $(s, r) \mapsto sr$. Suppose $F \simeq R^{(I)}$, then by Theorem 1.2.1 $S \otimes_R F \simeq (S \otimes R)^{(I)} \simeq S^{(I)}$ and these are S-module isomorphisms by Lemma 1.2.1. If $F \simeq P \oplus Q$, then the same theorem and lemma give $S \otimes_R F \simeq (S \otimes_R P) \oplus (S \otimes Q)$ and the second statement follows from the first. For the third statement, the argument is repeated with free modules of finite rank. □

1.2.1 Flatness

Definition 1.2.4: Let $M \in \text{Mod}-R$, then the map $M \otimes_R -$ is defined as $A \mapsto M \otimes_R A$ for $A \in R-\text{Mod}$ and $\alpha \mapsto (id_M \otimes \alpha : M \otimes_R A \to M \otimes_R B)$ for $\alpha : A \to B$.

Remark 1.2.5*:* We see $M \otimes_R -$ is well-defined by the uniqueness of the tensor product as deduced from Proposition 1.2.1 and the uniqueness of $id_M \otimes_R \alpha$ in item i) of Lemma 1.2.2.

Lemma 1.2.3: *Let R be a ring and $M \in \text{Mod}-R$:*

i) $M \otimes_R -$ is a covariant functor from $R-\text{Mod}$ to Ab.

ii) $M \otimes_R -$ is right-exact.

iii) If M is an (S, R)-bimodule, then $M \otimes_R -$ is a right-exact covariant functor from $R-\text{Mod}$ to $S-\text{Mod}$.

Proof. i) We have

$$M \otimes (\phi \circ \psi) = (M \otimes \phi) \circ (M \otimes \psi)$$

directly from item iii) in Lemma 1.2.2. And $id_M \otimes id_A$ acts like the identity on simple tensors, so $id_{M\otimes_R A} = id_M \otimes id_A$ by uniqueness.

ii) Take a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

we want prove that

$$M \otimes_R A \xrightarrow{id\otimes\alpha} M \otimes_R B \xrightarrow{id\otimes\beta} M \otimes_R C \longrightarrow 0$$

is exact.

To prove exactness at $M \otimes_R C$, we must show that $id \otimes \beta$ is onto. Take $\sum m_i \otimes c_i \in M \otimes_R C$, since β is onto C, we may find b_i such that $\beta(b_i) = c_i$, but then

$$(id \otimes \beta)\left(\sum m_i \otimes b_i\right) = \sum id(m_i) \otimes \beta(b_i) = \sum m_i \otimes c_i$$

To check exactness at $M \otimes_R B$, let $D = \text{Im }(id \otimes \alpha)$. We have $D \subseteq \ker id \otimes \beta$ from the composition $(id \otimes \beta) \circ id \otimes \alpha = id \otimes (\beta \circ \alpha) = id \otimes 0 = 0$. For the opposite inclusion, take

$$\begin{aligned} \beta' : (M \otimes_R B)/D &\to M \otimes_R C \\ \mathbf{m} + D &\mapsto (id \otimes \beta)(\mathbf{m}) \end{aligned}$$

the homomorphism induced by $id \otimes \beta$ in the homomorphism theorem. If it turns out that β' is an isomorphism, then $\ker(id \otimes \beta) = D$ and we are done.

Define $\gamma : M \times C \to (M \otimes B)/D$ by $\gamma(m, c) = m \otimes b + D$ where b is such that $\beta(b) = c$. This is well-defined, because m is unchanged and if we suppose that both b and b' are preimages of c, we get $\beta(b - b') = 0 \Rightarrow b - b' \in \ker \beta = \text{Im } \alpha \Rightarrow m \otimes (b - b') \in \text{Im }(id \otimes \alpha) = D \Rightarrow m \otimes b - m \otimes b' \in D \Rightarrow m \otimes b + D = m \otimes b' + D$. The map γ also inherits the properties of an R-balanced map from $\otimes$, so it can be lifted to a group homomorphism $\gamma' : M \otimes C \to (M \otimes B)/D$.

The maps β' and γ' act as mutual inverses on simple tensors, by definition, they are therefore mutually inverse homomorphisms by uniqueness, β' is a isomorphism as needed.

iii) $M \otimes A$ is an S-module by Lemma 1.2.1 and the maps $id \otimes \alpha$ are S-homomorphisms by item iii) in Lemma 1.2.2. The rest is carried over from i) and ii). $\square$

Definition 1.2.5: A right R-module M is called **flat** if $M \otimes_R -$ is exact.

Remark 1.2.6*:*

1. In view of Lemma 1.2.3, it is sufficient to prove $id \otimes \alpha$ is injective for any injective $\alpha : A \hookrightarrow B$ in order to show that M is flat.

2. Not all modules are flat. Take e.g. $R = \mathbb{Z}$, $M = \mathbb{Z}_5$ and the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Then $\mathbb{Z}_5 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ since for any $q \in \mathbb{Q}$ and $[k] \in \mathbb{Z}_5$ we have $k \otimes q = [k \cdot 5] \otimes \frac{q}{5} = 0$. But $\mathbb{Z}_5 \otimes_{\mathbb{Z}} \mathbb{Z} \simeq \mathbb{Z}_5$ since we can use Proposition 1.2.1 to rewrite an arbitrary element

$$\sum [k_i] \otimes l_i = \sum [k_i l_i] \otimes 1 = \left[\sum k_i l_i\right] \otimes 1 \longleftrightarrow \left[\sum k_i l_i\right]$$

So $\mathbb{Z}_5 \otimes \iota = 0$ is not injective.

Proposition 1.2.2:

i) Let $M_i, i \in I$ be right R-modules. Then $\bigoplus_{i \in I} M_i$ is flat if and only if each M_i is flat.

ii) Free modules are flat.

iii) Projective modules are flat.

Proof. i) Let A, B be left R-modules and $\alpha : A \hookrightarrow B$ an injective R-module homomorphism. By Theorem 1.2.1, there is an isomorphism

$$\beta : \left(\bigoplus_{i \in I} M_i \right) \otimes_R B \to \bigoplus_{i \in I} (M_i \otimes_R B)$$

where for simple tensors $\beta((m_i)_{i \in I} \otimes n) = (m_i \otimes n)_{i \in I}$.

Denote $M = \bigoplus_{i \in I} M_i$, $\alpha^* = id_M \otimes \alpha$, and $\alpha_i^* = id_{M_i} \otimes \alpha$ for each $i \in I$. Then

$$\beta \circ \alpha^* \Big((m_i)_{i \in I} \otimes n \Big) = \beta \Big((m_i)_{i \in I} \otimes \alpha(n) \Big) = (m_i \otimes \alpha(n)) = (\alpha_i^*(m_i \otimes n))_{i \in I}$$

Extending this to general tensor, we see $\beta \circ \alpha^* = (\alpha_i^*)_{i \in I}$ and it is obvious that α^* is injective if and only if every α_i^* is injective.

ii) Take the map $R \times A \to A$, $(r, a) \mapsto ra$. This is easily seen to be R-balanced by the axioms of modules for A and thus extends (uniquely) to a group homomorphism

$$\iota_A : R \otimes_R A \to A$$

ι_A is obviously onto A. To prove injectivity, take $0 = \iota_A \left(\sum r_j \otimes a_j \right) = \sum r_j a_j$ which implies

$$\sum r_j \otimes a_j = \sum 1 \otimes (r_j a_j) = 1 \otimes \left(\sum r_j a_j \right) = 1 \otimes 0 = 0$$

We have $\iota_B \circ (id_R \otimes \alpha) = \alpha$ so $id_R \otimes \alpha$ is injective if α is injective. Hence R is flat and by i) so is any free module.

iii) Follows from i) and ii) since a projective module is a direct summand of a free module. □

Remark 1.2.7*:* The proofs of i) and ii) give us a bit more: The way that α^* decomposes into components in i) taken along with the argument for ii) means that $R^{(I)} \otimes_R A \to R^{(I)} \otimes_R B \to R^{(I)} \otimes_R C$ is exact if *and only if* $A \to B \to C$ is. $R^{(I)}$ is said to be *faithfully flat*.

An important special case is that of $R[t]$. Forgetting the multiplicative structure, this is a free R module on the basis $1, t, t^2, \ldots$. It can be regarded as an $(R[t], R)$-bimodule and Remark 1.2.4 and Proposition 1.2.2.ii) together tell us $R[t]$ is faithfully flat, i.e. $R[t] \otimes_R - : \text{Mod}-R \to \text{Mod}-R[t]$ is an exact functor, with exactness also being preserved in the opposite direction as above.

1.2.2 Localization

For the rest of this chapter, let R be a commutative ring.

If $U \subset R$ is a multiplicatively closed set, we denote the localization of R with respect to U by $R[U^{-1}]$. The localized ring $R[U^{-1}]$ consists of equivalence classes of elements $\frac{r}{u}$ where $r \in R, u \in U$ under the relation $\sim$ defined by

$$\frac{r}{u} \sim \frac{r'}{u'} \iff \exists t \in U : t(ru' - r'u) = 0.$$

A similar definition can be applied to modules: If M is an R-module, then $M[U^{-1}]$ is obtained by canonically adding the elements $\frac{m}{u}$ where $m \in M$, $u \in U$, the result is an $R[U^{-1}]$-module.

For our purpose, the best way to define the localization of a module is using the tensor product.

Note that $R[U^{-1}]$ is an associative R-algebra: R acts on $R[U^{-1}]$ naturally by $r \cdot \frac{p}{q} = \frac{rp}{q}$ and associativity is checked easily $\left(r \cdot \frac{p}{q}\right) \frac{p'}{q'} = \frac{rpp'}{qq'} = r \cdot \left(\frac{p}{q}\frac{p'}{q'}\right)$.

We recall Remark 1.2.4 in order to define the localization of a module

Definition 1.2.6: Let R be a commutative ring, $U \subset R$ a multiplicatively closed set, and $M \in R-\mathrm{Mod}$. Then the **localization** of M with respect to U is the $R[U^{-1}]$-module $M[U^{-1}] = R[U^{-1}] \otimes_R M$.

Remark 1.2.8*:* It is important to note that all elements of $M[U^{-1}]$ reduce to simple tensors:

$$\frac{r}{u} \otimes m + \frac{r'}{u'} \otimes m' = \frac{1}{uu'} \otimes u'rm + \frac{1}{uu'} \otimes ur'm' = \frac{1}{uu'} \otimes (ur'm + ur'm')$$

and apply Proposition 1.2.1 and induction.

With this in mind, we may denote $\frac{1}{u} \otimes m$ by $\frac{m}{u}$ to get the localization in a more intuitive form.

This definition also agrees with the definition of $R[U^{-1}]$ in the sense that $R[U^{-1}] \simeq R[U^{-1}] \otimes_R R$ since, in general, if M is an R-module, then $M \otimes_R R \simeq M$ by the isomorphism $m \otimes r \mapsto mr$.

Lemma 1.2.4: *Let F, G be left-exact (additive) contravariant functors from* $\mathrm{Mod}-R$ *to* Ab $(= \mathrm{Mod}-\mathbb{Z})$ *and let $\alpha : F \to G$ be a natural transformation. Suppose $M \in \mathrm{Mod}-R$ is finitely presented, i.e. there exist $n, m \in \mathbb{N}$ such that $R^m \xrightarrow{\phi} R^n \xrightarrow{\psi} M \longrightarrow 0$ is exact.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & F(M) & \xrightarrow{F(\psi)} & F(R^n) & \xrightarrow{F(\phi)} & F(R^n) \\
 & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_{R^n}} & & \downarrow{\scriptstyle \alpha_{R^m}} \\
0 & \longrightarrow & G(M) & \xrightarrow[G(\psi)]{} & G(R^n) & \xrightarrow[G(\phi)]{} & G(R^n)
\end{array}$$

If α_{R^m} is an isomorphism for free modules, then α_M is also an isomorphism.

Proof. To prove α_M is injective, suppose $x \in \ker \alpha_M$. Since the diagram commutes (by the definition of natural transformations),

$$\Big(\alpha_{R^n} \circ F(\psi)\Big)(x) = \Big(G(\psi) \circ \alpha_M\Big)(x) = 0.$$

Since α_{R^n} is an isomorphism, $F(\psi)(x) = 0$. But $F(\psi)$ is injective by exactness, so that $x = 0$.

To prove surjectivity, take $y \in G(M)$ and find $\overline{y} \in F(R^n)$ such that

$$\alpha_{R^n}(\overline{y}) = G(\psi)(y).$$

By exactness $0 = G(\phi) \circ G(\psi)(y) = G(\phi) \circ \alpha_{R^n}(\overline{y})$ and by commutativity also

$$\alpha_{R^m} \circ F(\phi)(\overline{y}) = G(\phi) \circ \alpha_{R^n}(\overline{y}) = 0.$$

But α_{R^m} is an isomorphism, so $F(\phi)(\overline{y}) = 0$ and hence $\overline{y} \in \ker F(\phi) = \operatorname{Im} F(\psi)$ so that $\overline{y} = F(\psi)(x)$ for some $x \in F(M)$. Again, by commutativity

$$\alpha_{R^n} \circ F(\psi)(x) = G(\psi) \circ \alpha_M(x) = G(\psi)(y).$$

Finally, since $G(\psi)$ is injective, this gives $\alpha_M(x) = y$, proving α_M is surjective. □

Remark 1.2.9*:* If M is only supposed to be finitely generated, α_M is at least still a monomorphism, by the same proof.

Localization is compatible with the Hom functor in the following sense:

Proposition 1.2.3: *Let S be a commutative R-algebra, let $M, N \in R{-}\mathrm{Mod}$, then there exists a unique S-module homomorphism*

$$\alpha_M : S \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$$

such that $\alpha_M(1 \otimes \phi) = id_S \otimes \phi$.

If S is a flat R-module and M is finitely presented, then α_M is an isomorphism.

Proof. First, $\operatorname{Hom}_R(M, N)$ has an R-module structure by the action $r \cdot \phi : m \mapsto r \cdot \phi(m)$. Using this and Remark 1.2.4 we see $S \otimes_R \operatorname{Hom}_R(M, N)$ and $\operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$ are indeed S-modules.

We require α_M to be an S-homomorphism so

$$\alpha_M(s \otimes \phi) = s \cdot \alpha_M(1 \otimes \phi) = id_S \otimes (s \cdot \phi),$$

if α_M exists at all. The map $\alpha : S \times \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N), (s, \phi) \mapsto id_S \otimes s \cdot \phi$ is easily seen to be R-balanced with $r \cdot \alpha(s, \phi) = \alpha(rs, \phi)$, so, by Lemma 1.2.1, α induces a unique α_M as required.

For the second statement, suppose M is finitely presented and S is flat. The proof that α_M is an isomorphism is done in several steps:

- If $M = R$, we have $\operatorname{Hom}_R(R, N) \simeq N$ since a $\phi : R \to N$ is uniquely determined by $\phi(1) \in N$, the isomorphism is given as $f : \phi \mapsto \phi(1)$. As seen above, $S \otimes_R R \simeq S$. Therefore also $\operatorname{Hom}_S(S \otimes_R R, S \otimes_R N) \simeq S \otimes N$ by the isomorphism $g : \psi \mapsto \psi(1 \otimes 1)$.

  Since $S \otimes_R -$ is a functor, it preserves isomorphisms, so $id_S \otimes f$ is also an isomorphism. Then $g^{-1} \circ (id_S \otimes f)$ is an isomorphism and $\alpha_M = g^{-1} \circ (id_S \otimes f)$ by the uniqueness proved above, since

  $$g^{-1} \circ (id_S \otimes f)(1 \otimes \phi) = g^{-1}(1 \otimes \phi(1)) = id_S \otimes \phi$$

- Now suppose $M = R^n$ is a free module of finite rank. By Proposition 1.1.1 and Theorem 1.2.1, we have the isomorphisms

  $$f : S \otimes_R \operatorname{Hom}_R(R^n, N) \simeq \Big(S \otimes_R \operatorname{Hom}_R(R, N)\Big)^n$$

  $$g : \operatorname{Hom}_S(S \otimes_R R^n, S \otimes_R N) \simeq \Big(\operatorname{Hom}_S(S \otimes_R R, S \otimes_R N)\Big)^n$$

The constructions of these as seen in the respective lemmas allow us to check that

$$g^{-1} \circ \left(\bigoplus_{i=1}^{n} \alpha_R \right) \circ f(1 \otimes \phi) = g^{-1} \left(\bigoplus_{i=1}^{n} \alpha_R(1 \otimes (\phi \circ \nu_i)) \right) =$$

$$= g^{-1}(id \otimes (\phi \circ \nu_i))_{i \in I} = id \otimes \phi$$

and therefore $\alpha_R^n = g^{-1} \circ (\bigoplus_{i=1}^{n} \alpha_R) \circ f$ is also an isomorphism, because each α_R is.

Finally, we claim $\alpha = \{\alpha_A : A \in \text{Mod}-R\}$ is a natural transformation

$$S \otimes_R \text{Hom}_R(-, N) \longrightarrow \text{Hom}_S(S \otimes_R -, S \otimes_R N)$$

Take $A, B \in \text{Mod}-R$ and $f : A \to B$, we have for each $\phi \in \text{Hom}_R(B, N)$

$$\alpha_A \circ S \otimes_R \text{Hom}_R(f, N)(1 \otimes \phi) = \alpha_A(1 \otimes (\phi \circ f)) = id_S \otimes (\phi \circ f)$$

$$\text{Hom}_S(S \otimes_R f, S \otimes_R N) \circ \alpha_B(1 \otimes \phi) = (id_S \otimes \phi) \circ (id_S \otimes f) = id_S \otimes (\phi \circ f)$$

this extends to general tensors by linearity.

By assumption, $S \otimes_R -$ is an exact covariant and $\text{Hom}_R(-, N), \text{Hom}_S(-, S \otimes N)$ are left-exact contravariant functors. Hence, both compositions

$$S \otimes_R \text{Hom}_R(-, N) \text{ and } \text{Hom}_S(S \otimes_R -, S \otimes_R N)$$

are left-exact contravariant functors.

We have also proved α_{R^n} is an isomorphism for free modules, so Lemma 1.2.4 applies, proving α_M is an isomorphism for M finitely presented. □

Proposition 1.2.4: *If R is a commutative ring and $U \subset R$ is a multiplicative set, then $R[U^{-1}]$ is flat.*

Proof. Let $\alpha : A \hookrightarrow B$ be an injective R-module homomorphism. Then

$$id \otimes \alpha : A[U^{-1}] \to B[U^{-1}]$$
$$\frac{a}{u} \mapsto \frac{\alpha(a)}{u}$$

a series of calculations proves injectivity directly:

$$\frac{\alpha(a)}{u} = 0 \left(= \frac{0}{u'} \right) \Rightarrow \exists u' \in U : u' \cdot \alpha(a) = 0 \Rightarrow u'a \in \ker \alpha$$

$$\Rightarrow u'a = 0 \Rightarrow 0 = \frac{u}{uu'} \cdot 0 = \frac{uu'a}{uu'} = \frac{a}{u}$$

□

The two previous results combine to give us a handy isomorphism:

Corrolary 1.2.2: *If U is a multiplicative set in R, $M, N \in \text{Mod}-R$, and M is finitely presented then,*

$$\text{Hom}_R(M, N)[U^{-1}] \simeq \text{Hom}_{R[U^{-1}]}(M[U^{-1}], N[U^{-1}])$$

as S-modules.

For a prime ideal P in R, $R \setminus P$ is a multiplicatively closed set and we denote $R_P = R\left[(R \setminus P)^{-1}\right]$ the **localization of R at P**. Similarly for M_P where M is an R-module. Lastly, we may write $\phi_P = id_{R_P} \otimes \phi$.

We shall denote spec R the set of prime ideals and max R the set of maximal ideals of a commutative ring R.

1.2.3 The Local-global Principle

The following is not needed for the proof in Chapter 2. It is included, firstly, to lend some credence to the algebro-geometric motivation for Serre's Conjecture described in the introduction and, secondly, to aid in discussing the Bass-Quillen Conjecture in Chapter 3.

Definition 1.2.7: We may define a topology on spec R, called the **Zariski topology**, by the basis of open sets

$$\mathcal{B} = \Big\{D(x) \mid x \in R\Big\} \text{ where } D(x) = \Big\{P \in \text{spec } R \mid x \notin P\Big\}.$$

The space spec R is quasi-compact (i.e. not necessarily Hausdorff but having the finite subcovering property): Suppose for some index set I and $x_\alpha \in R$ that

$$\bigcup_{\alpha \in I} D(x_\alpha) = \text{spec } R,$$

then the set $\{x_\alpha\}$ is not contained in any prime (nor maximal) ideal, therefore it generates R as an ideal. A finite subset therefore already generates R (it is enough for 1 to be a linear combination of x_α).

Definition 1.2.8: A commutative ring R is called **local** if it has a unique maximal ideal.

Remark 1.2.10*:*

1. A local ring R can be denoted (R, P) to emphasize that P is the unique maximal ideal in R.
2. R is local if and only if the non-units of R form an ideal: Suppose (R, P) is local and $r \in R \setminus P$ is not a unit, then (using the Zorn Lemma) there exists a maximal ideal M containing r, hence $P \neq M$, a contradiction.

   Conversely, if the non-units form an ideal P, then any proper ideal is contained in P because a proper ideal cannot contain a unit, so P is the only maximal ideal in R.

Proposition 1.2.5: *If R is any commutative ring and $P \in$ spec R, then (R_P, P_P) is local.*

Proof. Firstly, $P_P = R_P \otimes_R P$ is an ideal in R_P, because it is obviously a subset (it consist of those fractions, which have an element of P in the numerator) and it is an R_P-module by definition.

Suppose $\frac{r}{u} \in R_P \setminus P_P$, then $r \notin P \Rightarrow r \in U \Rightarrow \frac{u}{r} \in R_P$, so $\frac{r}{u}$ is a unit. On the other hand, P_P is a proper ideal, because $1_{R_P} \in P_P$ would mean

$$\frac{r}{u} = 1 \Leftrightarrow \exists v \in U : v(r - u) = 0 \Rightarrow vr = vu \in P,$$

a contradiction, since P is a prime ideal and $v, u \notin P$.

P_P is an ideal of R_P consisting precisely of the non-units in R_P, so it is the unique maximal ideal in R_P. □

This somewhat justifies the name of *localization*. Next, we will see two examples of how information about localizations of a ring or module can yield information about the original—the *local-global principle*.

Proposition 1.2.6: *Let R be a ring, $M, N \in \mathrm{Mod}{-}R$, and $\phi : M \to N$ an R-homomorphism, then*

i) For a fixed $m \in M$, $m = 0$ if and only if $\frac{m}{1} = 0$ in $M_\mathfrak{m}$ for every $\mathfrak{m} \in \max R$.

ii) $M = 0$ if and only if $M_\mathfrak{m} = 0$ for every $\mathfrak{m} \in \max R$.

iii) ϕ is a mono-, epi-, or isomorphism if and only if $\phi_\mathfrak{m} = id_{R_I} \otimes \phi$ is a mono-, epi-, or isomorphism (respectively) for every $\mathfrak{m} \in \max R$.

Proof. i) As in the proof of Proposition 1.2.4, we have $\frac{m}{1} = 0 \Leftrightarrow \exists u \in R \setminus \mathfrak{m} : mu = 0 \Leftrightarrow \mathrm{ann}(m) \nsubseteq \mathfrak{m}$. Then $\frac{m}{1} = 0$ in every $M_\mathfrak{m}$ if and only if $\mathrm{ann}(m) \nsubseteq \mathfrak{m}$ is not contained in any maximal ideal if and only if $\mathrm{ann}(m) = R$ (if $\mathrm{ann}(m)$ were a proper ideal, there would exist a maximal ideal containing it, by Zorn's lemma). Of course, $\mathrm{ann}(m) = R \iff m = 0$.

ii) $M = 0$ if and only if $m = 0$ for every $m \in M$. By i), this is if and only if $\frac{m}{1} = 0$ for every $\mathfrak{m} \in \max R$ and every $m \in M$ and this is equivalent to saying $M_\mathfrak{m} = 0$ for every $\mathfrak{m} \in \max R$

iii) Localization, that is the functor $R_\mathfrak{m} \otimes_R -$ (exact by Proposition 1.2.4), preserves exact sequences, therefore $(\ker \phi)_\mathfrak{m} = \ker \phi_\mathfrak{m}$. Applying ii), we get the statement for monomorphisms.

Similarly $(\mathrm{Im}\ \phi)_\mathfrak{m} = \mathrm{Im}\ \phi_\mathfrak{m}$. Also, the short exact sequence

$$0 \to \mathrm{Im}\ \phi \to N \to N/\mathrm{Im}\ \phi \to 0$$

induces a short exact sequence

$$0 \to (\mathrm{Im}\ \phi)_\mathfrak{m} \to N_\mathfrak{m} \to (N/\mathrm{Im}\ \phi)_\mathfrak{m} \to 0$$

which then gives $(N/\mathrm{Im}\ \phi)_\mathfrak{m} \simeq N_\mathfrak{m}/(\mathrm{Im}\ \phi)_\mathfrak{m} \simeq N_\mathfrak{m}/\mathrm{Im}\ \phi_\mathfrak{m}$ by the isomorphism theorem. And we see $\phi_\mathfrak{m}$ is an epimorphism $\forall \mathfrak{m} \in \max R$ if and only if $\forall \mathfrak{m} \in \max R : N_\mathfrak{m}/\mathrm{Im}\ \phi_\mathfrak{m} \simeq (N/\mathrm{Im}\ \phi)_\mathfrak{m} = 0$, which, by ii), is equivalent to $N/\mathrm{Im}\ \phi = 0$. This gives the statement for epimorphisms.

The statement for isomorphisms now follows immediately. □

Next, we recall the definition of the Jacobson radical and Nakayama's Lemma (see e.g. Eisenbud [5] for details):

Definition 1.2.9: If M is a (left) R-module, then the **Jacobson radical** of M, denoted $\mathrm{Rad}(M)$ is the intersection of all maximal (left) R-submodules of M.

Lemma 1.2.5 (Nakayama)**:** *If $M \neq 0$ is finitely generated, then*

$$M \cdot \mathrm{Rad}(R) \subseteq \mathrm{Rad}(M) \subsetneq M$$

Remark 1.2.11*:* For a local ring (R, P), obviously $\mathrm{Rad}(R) = P$.

Also note the contrapositive of this statement of Nakayama's lemma: If $M\mathrm{Rad}(R) = M$ and $M \in \mathfrak{M}(R)$, then $M = 0$.

Theorem 1.2.2: *Let R be a commutative ring and M an R-module.*

i) If (R, P) is local and $M \in \mathfrak{P}(R)$, then M is free.

ii) If M is finitely presented, then M is projective if and only if M_P is free over R_P for every $P \in \max R$ (and this is iff M_P is free over R_P for every $P \in \text{spec } R$).

Proof. i) Let $m_1, ..., m_n$ be a minimal set of generators for M. The images of $m_1, ..., m_n$ generate M/PM. Suppose

$$\sum_{i=1}^{n} \alpha_i \cdot m_i \in PM,$$

for some $\alpha_i \in R$. Then there exist $p_i \in P$ such that

$$\sum_{i=1}^{n} \alpha_i \cdot m_i = \sum_{i=1}^{n} p_i \cdot m_i \Longrightarrow \sum_{i=1}^{n} (\alpha_i - p_i) \cdot m_i = 0 \Longrightarrow (p_j - \alpha_j) \cdot m_j = \sum_{i \neq j} (\alpha_i - p_i) \cdot m_i$$

If $p_j - \alpha_j$ were invertible, this would imply $m_j \in \langle m_1, ..., m_{j-1}, m_{j+1}, ..., m_n \rangle$ contradicting the minimality condition. Since (R, P) is local, this means $p_j - \alpha_j$ and hence also α_j are contained in P. So the images of $m_1, ..., m_n$ form a basis of the R/P vector space $M/PM = R/P \otimes M$, hence $\dim M/PM = n$. By Theorem 1.2.1, $R^n/PR^n = R/P \otimes R^n = (R/P)^n$ is also an R/P-vector space of dimension n and so $M/PM \simeq R^n/PR^n$.

M is generated by n elements, so there exists an R-module homomorphism $\phi : R^n \twoheadrightarrow M$. M is also projective, so

$$R^n \simeq M \oplus K$$

where $K = \ker \phi$. Taking $R/P \otimes -$ on both sides, using Theorem 1.2.1 and noting that functors preserve isomorphism, we get the R/P-module isomorphism

$$R^n/PR^n \simeq M/PM \oplus K/PK \simeq R^n/PR^n \oplus K/PK$$

this means $K/PK = 0$ and by Nakayama's lemma $K = \ker \phi = 0$. So ϕ is an isomorphism and $M \simeq R^n$ is free.

ii) ($\Rightarrow$) Let P be a prime ideal of R and $R^n \simeq M \oplus K$. The tensor product preserves direct sums so taking $R_P \otimes -$ on both sides, we get

$$(R_P)^n \simeq M_P \oplus K_P$$

This implies M_P is a finitely generated (f.g.) projective R_P-module. But R_P is local by Proposition 1.2.5, so M_P is free by i).

($\Leftarrow$) Now suppose M_P is free over R_P for any maximal ideal P of R and take the short exact sequence $0 \to K \to F \xrightarrow{\phi} M \to 0$ (this exists by Corrolary 1.1.1). We wish to show that this sequence splits, we achieve this using Lemma 1.1.4, by proving that the map $\phi^* = \text{Hom}_R(M, \phi)$ is an epimorphism. Because M_P is free, the localized sequence (using flatness of $R_P \otimes -$)

$$0 \longrightarrow K_P \longrightarrow F_P \xrightarrow{\phi_P} M_P \longrightarrow 0$$

is split for any P. By Lemma 1.1.4, $(\phi_P)^* = \mathrm{Hom}_{R_P}(M_P, \phi_P)$ is surjective.

M is finitely presented, so Corrolary 1.2.2 applies. Let α, β be the R_P-module isomorphisms given by this Corrolary, so that

$$\alpha : \mathrm{Hom}_R(M, F)_P \simeq \mathrm{Hom}_{R_P}(M_P, F_P)$$
$$\beta : \mathrm{Hom}_R(M, M)_P \simeq \mathrm{Hom}_{R_P}(M_P, M_P)$$

Where $\alpha(1 \otimes \tau) = \tau_P$ and $\beta(1 \otimes \psi) = \psi_P$. Then

$$\beta^{-1} \circ (\phi_P)^* \circ \alpha : \mathrm{Hom}_R(M, F)_P \twoheadrightarrow \mathrm{Hom}_R(M, M)_P$$
$$1 \otimes \tau \mapsto \beta^{-1}(\phi_P \otimes \tau_P) = 1 \otimes \phi \circ \tau$$

Hence $\beta^{-1} \circ (\phi_P)^* \circ \alpha = (\phi^*)_P$ is the localization of the map ϕ^* at P.

We have shown that $(\phi^*)_P : \mathrm{Hom}_R(M, F)_P \twoheadrightarrow \mathrm{Hom}_R(M, M)_P$ is an epimorphism for all maximal ideals P of R. By Proposition 1.2.6, Φ is an epimorphism. M is projective by Lemma 1.1.4. □

Remark 1.2.12*:* Note that any $P \in \mathfrak{P}(R)$ is finitely presented. Indeed, if $R^k = P \oplus Q$, then $R^k \xrightarrow{\pi_Q} R^k \xrightarrow{\pi_P} P \to 0$ is a free presentation, where $\pi_Q : R^k \to R^k$ is the projection of R^k onto $Q \subset R^k$.

Remark 1.2.13*:* The statement that projective modules over a local ring are free is true even for modules that are not finitely generated. This theorem is due to Kaplansky, the proof is omitted here. [8] (pp. 9-11)

2. The Proof of Serre's Conjecture

Serre's Conjecture: Let $\mathbf{k}$ be a field and $R = \mathbf{k}[t_1, ..., t_n]$. Any finitely generated projective R-module (any $P \in \mathfrak{P}(R)$) is free.

The proof will be done in three stages: First, we will show that $\mathbf{k}[x_1, ..., x_n]$ is a *regular* ring (see Definition 2.1.1), which will be presented as a special case of a more general result (Swan's Theorem 2.1.1). A discussion of the *Grothendieck* K_0 *group* will then allow us to prove that any projective module over $\mathbf{k}[x_1, ..., x_n]$ is *stably free* (see Definition 2.2.1). This will be followed by a proof of the fact that $\mathbf{k}[x_1, ..., x_n]$ is *Hermite*, which just means that every stably free $\mathbf{k}[x_1, ..., x_n]$-module is actually free.

The proof follows Lam [6], expanding on it with e.g. Lemma 2.3.1, Lemma 2.3.2, and Lemma 2.4.3. The following notation is taken from there.

Definition 2.0.1: Let R be a subring of S. An $M \in \text{Mod}-S$ is **extended from** R if there exists an $M_0 \in \text{Mod}-R$ such that $M \simeq S \otimes_R M_0$.

We have been using the symbol $\mathfrak{M}(R)$ to denote finitely generated R-modules. If R is a subring of a ring S, we write $M \in \mathfrak{M}^R(S)$ to mean M is finitely generated and *extended from* R. Similarly, we might write $\mathfrak{P}^R(S)$ in the projective case.

Our main interest lies in the case $S = R[t]$, which has at least one convenient property:

Lemma 2.0.1: *If* $M \in \mathfrak{M}^R(R[t])$*, then the module* M_0 *from which* M *is extended (i.e. satisfying* $M \simeq R[t] \otimes_R M_0$*) is determined uniquely by* M *up to isomorphism, and is given by* $M_0 \simeq M/tM$*.*

$M \in \mathfrak{P}(R[t])$ *if and only if* $M_0 \in \mathfrak{P}(R)$*.*

Proof. Suppose $M \simeq R[t] \otimes_R M_0$. A simple tensor $(r_n t^n + \cdots + r_0) \otimes m$ ($m \in M_0$) can be rewritten as $\sum_{i=0}^{n} t^i \otimes r_n m$. Therefore, in a general element of $R[t] \otimes_R M_0$, we may group together all the terms $t^i \otimes m$ with the same i and add them together using the distributivity of $\otimes$ as in

$$t^i \otimes m + t^i \otimes m' = t^i \otimes (m + m').$$

The result is that elements of $R[t] \otimes_R M_0$ can be thought of as polynomials over M_0. M_0 is thus contained in M as the R-submodule of constant polynomials and $M_0 \simeq M/tM$.

The latter statement follows from Proposition 1.2.2 and Remark 1.2.7. □

2.1 Left Regular Rings

Definition 2.1.1: A ring R is called **left regular** if it is left noetherian and, given any (left) $M \in \mathfrak{M}(R)$, there exists a **finite projective resolution** of M, i.e. an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where $P_i \in \mathfrak{P}(R)$ and n depends on M.

The aim of this section will be Swan's theorem, which states that *left regularity* of a ring R carries over to $R[t]$. To do this, we introduce two basic constructions from Homological Algebra (see Rotman [7]) and a lemma of Swan.

Remark 2.1.1*:*

1. If R is left noetherian, then any $M \in \mathfrak{M}(R)$ admits an (infinite) projective resolution: By Corrolary 1.1.1, there exists a free module R^{k_1} (obviously f.g. projective) such that $R^{k_1} \xrightarrow{\pi} M \longrightarrow 0$ is exact. Since R is noetherian, so is R^{k_1}, hence $\ker \pi \subset R^{k_1}$ is finitely generated and, again, there exists a map $R^{k_2} \to R^{k_1}$ which is onto $\ker \pi$, making $R^{k_2} \to R^{k_1} \to M \to 0$ exact. This proceeds recursively.

2. The 0 module is free with finite basis $\emptyset$ (and hence f.g. projective). Since $\cdots \to 0 \to 0$ is always exact, we can think of a finite exact sequence as an infinite one with all but finitely many terms equal to 0.

3. A commutative noetherian ring R is regular if and only if $R_{\mathfrak{m}}$ is regular for each $\mathfrak{m} \in \max R$ (compare this with Definition 3.1.1) [6] (p. 81).

   The "$\Rightarrow$" direction is obvious since localization at $\mathfrak{m}$ preserves exact sequences and projectivity as we have seen. The opposite implication is more complicated, involving the quasi-compactness property of spec R discussed in Section 1.2.3.

Definition 2.1.2: If $X_\bullet$ and $Y_\bullet$ are the following exact sequences,

$$\cdots \longrightarrow X_3 \xrightarrow{\partial_3^1} X_2 \xrightarrow{\partial_2^1} X_1 \xrightarrow{\partial_1^1} X_0$$

$$\cdots \longrightarrow Y_3 \xrightarrow{\partial_3^2} Y_2 \xrightarrow{\partial_2^2} Y_1 \xrightarrow{\partial_1^2} Y_0$$

then a **map of chain complexes** $g_\bullet$ is a sequence of R-module homomorphisms $g_i, i = 0, 1, 2...$ making the following diagram commute

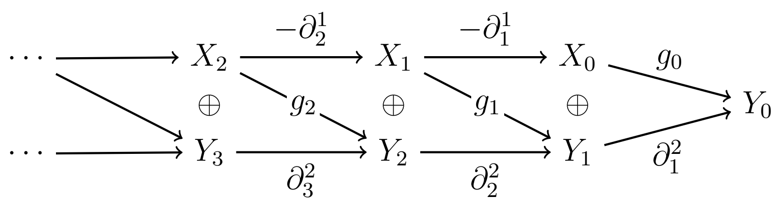
$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X_3 & \xrightarrow{\partial_3^1} & X_2 & \xrightarrow{\partial_2^1} & X_1 & \xrightarrow{\partial_1^1} & X_0 \\
 & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\
\cdots \longrightarrow & Y_3 & \xrightarrow{\partial_3^2} & Y_2 & \xrightarrow{\partial_2^2} & Y_1 & \xrightarrow{\partial_1^2} & Y_0
\end{array}$$

For such a map of chain complexes, we define the **mapping cone** Cone$(g_\bullet)$ to be the sequence defined by $P_0 = Y_0$, $P_i = X_{i-1} \oplus Y_i$ for $i = 1, 2, 3...$ along with maps

$$\pi_1 : P_1 \to P_0 \qquad\qquad \pi_i : P_i \to P_{i-1}$$

$$(x, y) \mapsto g_0(x) + \partial_1^2(y) \qquad (x, y) \mapsto \big(-\partial_{i-1}^1(x), g_{i-1}(x) + \partial_i^2(y)\big)$$

This is illustrated in the following diagram:



Lemma 2.1.1: *Let R be a left noetherian ring and $X, Y \in \mathfrak{M}(R)$. Any R-homomorphism $g : X \to Y$ can be lifted to a map of chain complexes $g_\bullet$ between any projective resolution of X and any projective resolution of Y.*

Proof. Let $\cdots \xrightarrow{\partial_2^1} X_1 \xrightarrow{\partial_1^1} X_0 \xrightarrow{\partial^1} X \to 0$ and $\cdots \xrightarrow{\partial_2^2} Y_1 \xrightarrow{\partial_1^2} Y_0 \xrightarrow{\partial^2} Y \to 0$ be the resolutions in question.

Since X_0 is projective, we may use Definition 1.1.5.ii) to find g_0 such that $\partial^2 \circ g_0 = g \circ \partial^1$. By exactness, we get $\partial^2 \circ g_0 \circ \partial_1^1 = g \circ \partial^1 \circ \partial_1^1 = 0$ so that

$$\text{Im } g_0 \circ \partial_1^1 \subseteq \ker \partial^2 = \text{Im } \partial_1^2$$

This means $g_0 \circ \partial_1^1$ makes sense as a map $X_1 \to \text{Im } \partial_1^2$ which allows us to sidestep the fact that ∂_1^2 is not onto Y_0 (of course, it is onto its image) and still use the projectivity of X_1 to find g_1. This reasoning extends recursively, allowing us to find all the g_i. □

Lemma 2.1.2: *If $X \xrightarrow{g} Y \xrightarrow{h} M \to 0$ is exact, then the mapping cone of $g_\bullet : X_\bullet \to Y_\bullet$ (supplied by the lemma above) where $X_\bullet \to X \to 0$ and $Y_\bullet \to Y \to 0$ are finite projective resolutions of X and Y yields the following finite projective resolution of M*

$$0 \longrightarrow P_k \xrightarrow{\pi_k} P_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} P_0 \xrightarrow{h \circ \partial_1^2} M \longrightarrow 0 \tag{2.1}$$

(Where k is large enough so that both X_k and Y_{k+1} are zero – and hence $P_{k+1} = 0$.)

Proof. We use the same notation as in Definition 2.1.2 throughout. The maps π_i are obviously R-homomorphisms and $P_i = X_{i-1} \oplus Y_i \in \mathfrak{P}(R)$.

To see that the cone is exact at each P_i for $i > 0$, we use exactness of $X_\bullet$ and $Y_\bullet$ and the definition of $g_\bullet$ as a map of chain complexes:

$$\pi_i \circ \pi_{i+1}(x, y) = \Big(\partial_i^1 \circ \partial_{i+1}^1(x), -g_i \circ \partial_{i+1}^1(x) + \partial_{i+1}^2\big(g_{i+1}(x) + \partial_{i+2}^2(y)\big)\Big) =$$

$$= \Big(0, -g_i \circ \partial_{i+1}^1(x) + \partial_{i+1}^2 \circ g_{i+1}(x)\Big) = (0, 0)$$
$$\implies \text{Im } \pi_{i+1} \subseteq \ker \pi_i$$

Now suppose $(x, y) \in \ker \pi_i$. In the first component, this means $x \in \ker \partial_i^1 = \text{Im } \partial_{i+1}^1$. Let x' be such that $\partial_{i+1}^1(x') = x$. In the second component, using the definition of $g_\bullet$ again,

$$0 = g_i(x) + \partial_{i+1}^2(y) = g_i \circ \partial_{i+1}^1(x') + \partial_{i+1}^2(y) = \partial_{i+1}^2(g_{i+1}(x') + y),$$

hence $g_{i+1}(x') + y \in \ker \partial_{i+1}^2 = \text{Im } \partial_{i+2}^2$ and there exists y' such that $\partial_{i+2}^2(y') = g_{i+1}(x') + y$. Then $\pi_{i+1}(-x', y') = (x, g_{i+1}(-x') + g_{i+1}(x') + y) = (x, y)$, proving $\text{Im } \pi_{i+1} \supseteq \ker \pi_i$. So (2.1) is exact at P_i for every $i > 0$.

At P_0, we need $\ker h \circ \partial_1^2 = \text{Im } \pi_1$.

$$(h \circ \partial_1^2) \circ \pi_1(x, y) = h \circ \partial_1^2(g_1(x) + \partial_2^2(y)) = h(g \circ \partial_1^1(x) + 0) = 0$$

$$\implies \ker h \circ \partial_1^2 \supseteq \text{Im } \pi_1$$

Suppose $h \circ \partial_1^2(x) = 0$, then $\partial_1^2(x) \in \ker h = \text{Im } g$ and there is an $x' \in X$ such that $g(x') = \partial_1^2(x)$, since ∂_1^1 is surjective $\partial_1^1(x'') = x'$ for some $x'' \in X_1$. But then

$$\partial_1^2(x) = g \circ \partial_1^1(x'') = \partial_1^2 \circ g_1(x'') \implies x - g_1(x'') \in \ker \partial_1^2 = \text{Im } \partial_2^2$$

so there exists an $x''' \in Y^2$ such that $\partial_2^2(x''') = x - g_1(x'')$. We now have $\pi(x'', x''') = g_1(x'') + x - g_1(x'') = x$, proving $\ker h \circ \partial_1^2 \subseteq \text{Im } \pi_1$ and the exactness at P_0.

Since h and ∂_1^2 are surjective, so is $h \circ \partial_1^2$, giving exactness at M. □

Lemma 2.1.3 (Swan)**:** *Let R be a left noetherian ring. If N is an $R[t]$-submodule of an $M \in \mathfrak{M}^R(R[t])$ then there exist $X, Y \in \mathfrak{M}^R(R[t])$ and an exact sequence $X \to Y \to N \to 0$.*

Proof. Take $M_0 \in \mathfrak{M}(R)$ such that $M \simeq R[t] \otimes_R M_0$. As in the proof of Lemma 2.0.1, elements of M can be viewed as formal polynomials over M_0, of the form $\sum_i t^i \otimes m_i$.

If we define $M_k = \sum_{i=0}^k R \cdot t^i \otimes M_0$, then $\bigcup_k M_k = M$. Put $N_k = M_k \cap N$. Now $R[t]$ is noetherian by the Hilbert Basis theorem and since M is a finitely generated $R[t]$-module, it is also noetherian. So $N \subset M$ is finitely generated and there exists an n large enough so that N_{n+1} contains an $R[t]$-generating set of N.

We claim $X = R[t] \otimes_R N_n$ and $Y = R[t] \otimes_R N_{n+1}$ are the modules we want. Obviously, these are extended from $R[t]$ and finitely generated as submodules of the noetherian M.

It remains to define homomorphisms ϕ and ψ to form an exact sequence

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} N \longrightarrow 0$$

So let $\psi(t^i \otimes n) = t^i \cdot n$ for $n \in N^{n+1}$. This definition extends uniquely to all of Y (using the fact that the t^is generate $R[t]$ as an R-module and the universal property of the tensor product). This ψ is onto N by the choice of n.

To define ϕ, we note $tM_k \subseteq M_{k+1}$ and hence also $tN_k \subseteq N_{k+1}$. We may therefore define $\phi(t^i \otimes n) = t^{i+1} \otimes n - t^i \otimes tn$. Again, the t^is generate $R[t]$ and the map $(t^i, n) \mapsto t^{i+1} \otimes n - t^i \otimes tn$ (extended linearly in the first component) is easily seen to be R-balanced, so $\phi : X \longrightarrow Y$ is an R-homomorphism.

Finally, we check that $X \to Y \to N \to 0$ is exact:

1. We have already seen that ψ is an epimorphism.
2. $\psi(\phi(t^i \otimes n)) = \psi(t^{i+1} \otimes n - t^i \otimes tn) = t^{i+1} \cdot n - t^i \cdot t \cdot n = 0$, hence $\psi \circ \phi = 0$ by linearity, which then yields $\text{Im } \phi \subseteq \ker \psi$.
3. Recalling the reasoning with which this proof began, we may view any element of $Y = R[t] \otimes_R N_{n+1}$ as a polynomial over N_{n+1}, i.e. as $\sum_{i=0}^k t^i \otimes n_i$. Let us therefore suppose $\psi\left(\sum_{i=0}^k t^i \otimes n_i\right) = 0$. We shall prove that this element lies in $\text{Im } \phi$ by induction on k.

For the base case, we have $0 = \psi(1 \otimes n) = n$ and obviously $n = 0 \in \text{Im } \phi$. Suppose $\mathbf{n} \in Y$ satisfies $\mathbf{n} \in \ker \psi \Rightarrow \mathbf{n} \in \text{Im } \phi$ whenever $\mathbf{n}$ is "of degree" less than k. Take an element of Y of degree k, say

$$\sum_{i=0}^{k} t^i \otimes n_i, \text{ where } n_i = \sum_{j=0}^{n+1} t^j \otimes a_{ij} \in N_{n+1}, (a_{ij} \in M_0)$$

Suppose this lies in $\ker \psi$, then

$$0 = \psi \left(\sum_{i=0}^{k} t^i \otimes n_i \right) = \sum_{i=0}^{k} t^i \cdot n_i = \sum_{i=0}^{k} \sum_{j=0}^{n+1} t^{i+j} \otimes a_{ij}$$

this implies $a_{k,n+1} = 0$, since $t^{k+n+1} \otimes a_{k,n+1}$ is the only term containing this power of t. So we conclude $n_k \in N_n$!

$$\sum_{i=0}^{k} t^i \otimes n_i - \phi(t^{k-1} \otimes n_k) = \sum_{i=0}^{k-1} t^i \otimes n_i - t^{k-1} \otimes tn_k \in \ker \psi$$

and this is of degree less than k, so the induction hypothesis yields $\sum_{i=0}^{k-1} t^i \otimes n_i - t^{k-1} \otimes tn_k = \phi(\mathbf{x})$ for some $\mathbf{x} \in X$. This finally implies

$$\sum_{i=0}^{k} t^i \otimes n_i = \phi(\mathbf{x} + t^{k-1} \otimes n_k)$$

Since this was done for an arbitrary element of Y, we may write $\text{Im } \phi \supseteq \ker \psi$.

The sequence is exact at both N (from 1.) and Y (2. and 3.). $\square$

Theorem 2.1.1 (Swan)**:** *If R is left regular then $R[t]$ is left regular.*

In fact, if R is regular then, given any $M \in \mathfrak{M}(R[t])$, we can find a resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where $P_i \in \mathfrak{P}^R(R[t])$.

Proof. The noetherian condition carries from R to $R[t]$ by the Hilbert Basis Theorem.

So it is enough to prove the second statement. By Corrolary 1.1.1, there exists a short exact sequence

$$0 \longrightarrow M' \xrightarrow{\iota} R[t]^k \longrightarrow M \longrightarrow 0,$$

where ι is just the inclusion homomorphism. Suppose we find a resolution of the desired form for M':

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \xrightarrow{\pi} M' \to 0$$

By Corrolary 1.2.1, we have $R[t]^k \simeq R[t] \otimes_R R^k \in \mathfrak{P}^R(R[t])$. Since π is an epi- and ι a monomorphism, then $\text{Im } \iota \circ \pi = \text{Im } \iota$ and $\ker \iota \circ \pi = \ker \pi$, so we know

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \xrightarrow{\iota \circ \pi} R[t]^k \to M \to 0$$

is exact. So it will suffice to find the resolution for M'.

M' is given as a submodule of $R[t]^k \in \mathfrak{M}^R(R[t])$, by Lemma 2.1.3, there exists an exact sequence

$$X \xrightarrow{g} Y \xrightarrow{h} M' \longrightarrow 0,$$

where $X, Y \in \mathfrak{M}^R(R[t])$, that is $X \simeq R[t] \otimes_R X'$ and $Y \simeq R[t] \otimes_R Y'$.

R is regular by assumption, so the resolutions exist for X' and Y':

$$0 \to X'_k \to \cdots \to X'_0 \to X' \to 0$$

$$0 \to Y'_l \to \cdots \to Y'_0 \to Y' \to 0,$$

where X'_i, Y'_j are all $\mathfrak{P}(R)$.

Since $R[t]$ is flat (Proposition 1.2.2), we may apply $R[t] \otimes_R -$ to both of these sequences to get exact sequences:

$$0 \to X_k \to \cdots \to X_0 \to X \to 0$$

$$0 \to Y_l \to \cdots \to Y_0 \to Y \to 0,$$

writing $X_i = R[t] \otimes_R X'_i$ and $Y_j = R[t] \otimes_R Y'_j$. Then X_i, Y_i are extended from R by their definition, so $X_i, Y_j \in \mathfrak{P}^R(R[t])$ by Corrolary 1.2.1. Applying the mapping cone construction of Lemma 2.1.2 to these, we get the desired resolution for M. ($P_i \simeq X_{i-1} \oplus Y_i$ is extended from R by Theorem 1.2.1.) □

Remark 2.1.2*:* Note that if the lengths of the R-resolutions of X' and Y' are bounded by a fixed $n_0 \in \mathbb{N}$, then the length of the $R[t]$ resolution which we found for M is bounded by $n_0 + 1$.

2.2 Stably Free Modules

Definition 2.2.1: A module $P \in \text{Mod}-R$ is called **stably free of type** k, if $P \oplus R^k \simeq F$ is free.

Remark 2.2.1*:*

1. A stably free module is necessarily projective. To see when the converse holds will be the object of this section.

2. $P \oplus R^k = F$ is equivalent to the existence of a split short exact sequence $0 \to P \to F \to R^k \to 0$. Hence P is stably free if and only if $P \simeq \ker f$ for some epimorphism $f : F \twoheadrightarrow R^k$, where F is free.

3. If $P = \ker(f : F \twoheadrightarrow R^k)$ is stably free and not finitely generated, then it is free: Suppose $F = P \oplus R^k$ is free on the basis $\{b_i \mid i \in I\}$, then I is infinite since P is not finitely generated. R^k is finitely generated, its generators obtained as linear combinations of the images of only finitely many b_is, hence there exists a finite subset $I_0 \subset I$ such that, letting $R^{(I_0)} \simeq F_0 \subset F$, $f \restriction F_0$ is already onto R^k, giving us the (split) short exact sequence

$$0 \to (P \cap F_0) \subset F_0 \xrightarrow{f \restriction F_0} R^k \to 0$$

and hence $F_0 \simeq (P \cap F_0) \oplus R^k$. Furthermore, $F \simeq P + F_0$ and

$$R^{(I \setminus I_0)} \simeq F/F_0 \simeq (P + F_0)/F_0 \simeq P/(P \cap F_0) \Leftrightarrow P \simeq R^{(I \setminus I_0)} \oplus (P \cap F_0)$$

Since $I \setminus I_0$ is infinite, $R^{(I \setminus I_0)} \simeq E \oplus R^k$ for some free E, proving that $P \simeq E \oplus R^k \oplus (P \cap F_0) \simeq E \oplus F_0$ is free.

Henceforth, we will deal with *finitely generated* stably free modules.

Definition 2.2.2: Let R be a ring. For $P \in \mathfrak{P}(R)$ let (P) denote the isomorphism type of P. Take G to be the free abelian group (a free $\mathbb{Z}$-module) on the basis $\{(P) \mid P \in \mathfrak{P}(R)\}$ and H the subgroup of G generated by elements of the form

$$(P \oplus Q) - (P) - (Q).$$

The **Grothendieck group of** R is defined as $K_0R = G/H$. We write $[P] = (P) + H \in K_0R$.

Remark 2.2.2*:* It is worth noting that $\{(P) \mid P \in \mathfrak{P}(R)\}$ is actually a set: A finitely generated R-module is isomorphic to a quotient of R^n by a submodule of R^n (Corrolary 1.1.1). We may choose the representatives of the isomorphism types to be these quotients. The class of submodules of R^n (denoted by $L(R^n)$) is a set (a subset of $\mathcal{P}(R^n)$). The class of isomorphism types of finitely generated R modules corresponds one-to-one with the *set* $\bigcup_{i \in \mathbb{N}} L(R^n)$.

Lemma 2.2.1: *Let* $P_i \in \mathfrak{P}(R)$ *for* $i = 1, 2, ..., n$. *If* $0 \to P_3 \to P_2 \to P_1 \to 0$ *is exact, then* $[P_2] = [P_3] + [P_1]$. *More generally, if*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to 0$$

is exact, then $\sum_{i=0}^{n} (-1)^i [P_i] = 0$.

Proof. Since P_1 is projective, the short exact sequence in the first statement splits, so that $P_2 \simeq P_3 \oplus P_1$, hence $(P_2) - (P_3) - (P_1) \in H$ and $[P_2] = [P_3] + [P_1]$ by definition.

The second statement is proved by induction. We use $n = 3$ (above) as the base case. For $n > 3$, the sequence can be divided into shorter exact sequences, as follows

$$0 \longrightarrow P_n \cdots P_5 \longrightarrow P_4 \xrightarrow{\partial_4} \text{Im } \partial_4 \longrightarrow 0$$

$$0 \longrightarrow P_3/\ker \partial_3 \xrightarrow{\partial_3^*} P_2 \xrightarrow{\partial_2} P_1 \longrightarrow 0$$

Where ∂_3^* is the homomorphism induced by ∂_3 in the homomorphism theorem. The second sequence identifies $P_3/\ker \partial_3$ as a direct summand of a f.g. projective module, making it also f.g. projective. The (split) short exact sequence $0 \to \text{Im } \partial_4 = \ker \partial_3 \to P_3 \to P_3/\ker \partial_3 \to 0$ then proves that $\text{Im } \partial_4 \in \mathfrak{P}(R)$ and

$$[P_3] = [\text{Im } \partial_4] + [P_3/\ker \partial_3].$$

The sequences above are exact, the inductive hypothesis gives

$$\cdots + [P_5] - [P_4] + [\text{Im } \partial_4] = 0 \quad \text{and} \quad [P_3/\ker \partial_3] - [P_2] + [P_1] = 0.$$

The three equations together prove the statement.

The missing cases (e.g. $n = 0$, $n = 1$) are covered by the fact that an exact sequence can be extended with zero modules (which are f.g. projective) on either end. □

Proposition 2.2.1: *Let $P, P' \in \mathfrak{P}(R)$, then the following statements are equivalent:*

i) $[P] = [P']$ *in* K_0R.

ii) There exists $T \in \mathfrak{P}(R)$ *such that* $P \oplus T \simeq P' \oplus T$.

iii) P *and* P' *are* ***stably isomorphic****, i.e. there exists* $k \in \mathbb{N}$ *such that* $P \oplus R^k \simeq P' \oplus R^k$.

Proof. ii) $\Rightarrow$ i) is obvious and ii) $\Leftrightarrow$ iii) follows from the definition of T as a f.g. projective module: there exist $k \in \mathbb{N}, Q \in \text{Mod}-R$ such that $R^k \simeq T \oplus Q$.

i) $\Rightarrow$ ii) We have $[P] - [P'] = 0 \Rightarrow (P) - (P') \in H$ so that

$$(P) - (P') = \sum_i \Big[(P_i \oplus Q_i) - (P_i) - (Q_i)\Big] - \Big[(P_i' \oplus Q_i') - (P_i') - (Q_i')\Big]$$

for suitable $P_i, Q_i, P_i', Q_i' \in \mathfrak{P}(R)$. Rearranging:

$$(P) + \sum_i \Big[(P_i' \oplus Q_i') + (P_i) + (Q_i)\Big] = (P') + \sum_i \Big[(P_i \oplus Q_i) + (P_i') + (Q_i')\Big] \quad (2.2)$$

Now G is free on the symbols (P), so that $\sum(M_\alpha) = \sum(N_\beta)$ implies that the symbols on either side are identical, they can differ only by their order. So the terms on either side of (2.2) coincide, meaning the respective modules are isomorphic (they are the same isomorphism type). Since $\oplus$ is commutative, this means we may set

$$T \simeq \bigoplus_i \Big[(P_i' \oplus Q_i') \oplus (P_i) \oplus (Q_i) \simeq \bigoplus_i \Big[(P_i \oplus Q_i) \oplus (P_i') \oplus (Q_i')$$

to get $P \oplus T \simeq P' \oplus T$. ☐

Corrolary 2.2.1: $P \in \mathfrak{P}(R)$*, is stably free if and only if* $[P] = n \cdot [R]$ *for some* $n \in \mathbb{Z}$.

Every finitely generated projective R-module is stably free if and only if $K_0R = \mathbb{Z} \cdot [R]$.

Proof. Suppose P is stably free, then $P \oplus R^k \simeq R^n$ and so $[P] + k[R] = n[R] \Longrightarrow [P] = (n-k) \cdot [R]$.

Suppose $[P] = n \cdot [R]$, take an integer r such that $n + r \geq 0$, then

$$[P \oplus R^r] = [P] + r \cdot [R] = (n+r) \cdot [R] = [R^{n+r}]$$

and Proposition 2.2.1 gives $P \oplus R^{r+k} \simeq R^{n+r+k}$. P is stably free.

The second statement follows immediately. ☐

Theorem 2.2.1 (Grothendieck)**:** *Let R be left regular ring, then the map*

$$F : K_0R \to K_0R[t]$$
$$[P] \mapsto \Big[R[t] \otimes_R P\Big]$$

is a group isomorphism. If any $P \in \mathfrak{P}(R)$ *is stably free, then any* $Q \in \mathfrak{P}(R[t])$ *is stably-free.*

Proof. F is a group homomorphism, since by Theorem 1.2.1 we can write

$$F([P]+[P'])=[R[t]\otimes_R(P\oplus P')]=[R[t]\otimes_R P]+[R[t]\otimes_R P']=F([P])+F([P']).$$

Injectivity of F follows immediately from Lemma 2.0.1 and surjectivity is proved using Swan's Theorem 2.1.1 as follows: Take $Q\in\mathfrak{P}(R[t])$ and find a resolution

$$0\to Z_n\to\cdots\to Z_0\to Q\to 0$$

where $Z_i\in\mathfrak{P}^R(R[t])$ (which directly implies $[Z_i]\in \text{Im } F$), then by Lemma 2.2.1 we have $[Q]=\sum_i(-1)^i[Z_i]\in \text{Im } F$.

The second statement follows from Corrolary 2.2.1. Assuming $K_0R=\mathbb{Z}\cdot[R]$, we have $F(k\cdot[R])=k\cdot\Big[R[t]\Big]$ and hence $K_0R[t]=\mathbb{Z}\cdot\Big[R[t]\Big]$. □

Inductively applying Swan's Theorem 2.1.1 and Grothendieck's Theorem 2.2.1 proves:

Corrolary 2.2.2: *Suppose R is a left regular ring and any f.g. projective R-module is stably free. Then any f.g. projective $R[t_1,...,t_n]$-module is stably free.*

Suppose $R=\mathbf{k}$ is a field. Then $\mathbf{k}$ is noetherian and any $\mathbf{k}$-module is free, since every vector space has a basis. So any $\mathbf{k}$-module V is stably free and the trivial exact sequence $0\to V\to V\to 0$ gives regularity of $\mathbf{k}$. The corrolary then yields:

Corrolary 2.2.3: *If $\mathbf{k}$ is a field then any f.g. projective $\mathbf{k}[t_1,...,t_n]$-module is stably free.*

2.3 Integral Extensions

In this section we restrict ourselves to *commutative* rings, recall the definition and basic properties of integral extensions, and use them to prove the clever Lemma 2.3.2.

The result in Lemma 2.3.1 is closely related to the important Going-up Theorem of commutative algebra; the Lemma is a direct corrolary and can also be used to prove the Theorem. [8]

Definition 2.3.1: An $f\in R[t]$ is called **monic** if its leading coefficient is 1. A $g\in R[t]$ is called **unitary** if its leading coefficient is invertible, i.e. there exists an invertible $c\in R$ such that $f=c^{-1}\cdot g$ is monic.

Definition 2.3.2: If R is a subring of a commutative ring S, we say S is an **extension** of R. An $s\in S$ is **integral** over R if s is a root of some monic polynomial in $R[t]$. S is said to be an **integral** extension of R, if every element of S is integral.

We record the basic properties of integral extensions. The proofs can be found in Matsumura [8].

Proposition 2.3.1: *Let S be an extension of R.*

1. *The elements of S which are integral over R form a ring.*

2. $R[s_1, ..., s_n] \subset S$ *is an integral extension of* R *if and only if it is finitely generated as an* R*-module.*

Lemma 2.3.1: *Let* S *be an integral extension of a ring* R *and let* I *be an ideal in* R*. Then* $IS = S \Longrightarrow I = R$.

Proof. $IS = S$ is equivalent to the existence of $i_k \in I$ and $s_k \in S$ such that $\sum_k i_k s_k = 1$. Put $T = R[s_1, ..., s_n]$, then it still holds that $IT = T$. By Proposition 2.3.1, $T \in \mathfrak{M}(R)$ so that $T = R \cdot b_1 + \cdots + R \cdot b_m$ for some set of generators $\{b_i \in T \mid i = 1, ..., m\}$. Then $T = IT = I \cdot b_1 + \cdots + I \cdot b_n$, since $IR = I$ by the definition of an ideal. This implies that there exist $i_{kl} \in I$ such that $b_k = \sum_{l=1}^{n} i_{kl} \cdot b_l$ for each k. In matrix form:

$$\begin{pmatrix} i_{11} & \cdots & i_{n1} \\ \vdots & \ddots & \vdots \\ i_{1n} & \cdots & i_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Let $A = (i_{kl})$, $b = \begin{pmatrix} b_1 \cdots b_n \end{pmatrix}^T$, id_n be the identity matrix, and C the matrix of cofactors for $id_n - A$. Then $(id_n - A) \cdot b = 0 \Longrightarrow C(id_n - A) \cdot b = \det(id_n - A) \cdot id_n \cdot b = 0$ which means $\det(id_n - A) b_k = 0$ for each k. But the b_ks generate S, so $\det(id_n - A) = 0$. The determinant is calculated using just the ring operations on the entries, so we see that $\det(id_n - A) + I = \det(id_n) + I = 1 + I$ since $r \mapsto r + I$ is a ring homomorphism and the entries of A lie in I. Then $0 + I = 1 + I \Longrightarrow 1 \in I \Longrightarrow I = R$. □

Lemma 2.3.2: *Let* I *be an ideal in* $R[t]$ *containing a monic polynomial and* J *an ideal in* R*. If* $I + J[t] = R[t]$ *then* $(I \cap R) + J = R$.

Proof. Take $\alpha' : R \to R[t]/I,\ r \mapsto r + I$, where $r + I$ is the coset of the constant polynomial. Then $\ker \alpha' = R \cap I$ and the homomorphism theorem gives $\alpha : R/(I \cap R) \hookrightarrow R[t]/I$ so we may regard $R/(I \cap R)$ as a subring of $R[t]/I$ by identifying it with its image in α.

$R[t]/I$ is an integral extension of $R/(I \cap R)$: By Proposition 2.3.1.i), it is enough to check that $t + I \in R[t]/I$ is integral over $R/(I \cap R)$. Let $t^n + \sum a_i t^i \in I$ denote the monic polynomial given in the hypothesis. Then $a_i + I \in R/(I \cap R)$ and, since $f \mapsto f + I$ is a ring homomorphism, we have

$$(t + I)^n + \sum_{i=0}^{n-1} (a_i + I)(t + I)^i = t^n + \sum_{i=0}^{n-1} a_i t^i + I = 0 + I$$

so that $t + I$ is a root of $x^n + \sum (a_i + I) x^i \in \Big(R/(I \cap R)\Big)[x]$.

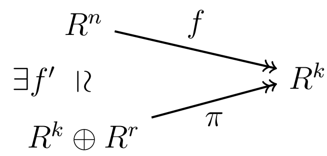
Furthermore, expanding out the equality in the hypothesis

$$I + J[t] = R[t] \Leftrightarrow \sum j_k t^k + I = 1 + I \text{ for some } j_k \in J \Leftrightarrow \sum (j_k + I)(t^k + I) = 1 + I$$

and since $t^k + I \in R[t]/I$, we have $\Big(J/(I \cap R)\Big) \cdot \Big(R[t]/I\Big) = R[t]/I$. Lemma 2.3.1 applied to the ring extension $R[t]/I \supseteq R/(I \cap R)$ and the ideal $J/(I \cap R)$ of $R/(I \cap R)$ yields $J/(I \cap R) = R/(I \cap R)$ which just means that there exists a $j \in J$ such that $j + (I \cap R) = 1 + (I \cap R)$, or equivalently $J + (I \cap R) = R$. □

2.4 Hermite Rings

Proposition 2.4.1: *Let* $P = \ker(f : R^n \twoheadrightarrow R^k) \in \mathfrak{M}(R)$ *be a finitely generated stably free* R*-module. Then* P *is free if and only if there exists an* r *and an isomorphism* $f' : R^n \to R^k \oplus R^r$ *such that* $f' \circ \pi = f$ *where* $\pi : R^k \oplus R^r \twoheadrightarrow R^k$ *is the canonical projection.*



Proof. ($\Leftarrow$) Suppose such an f' exists, then $P = \ker f = \ker f' \circ \pi = \ker \pi = R^r$.

($\Rightarrow$) Suppose P is free and $g : P \to R^r$ is an isomorphism for some r. P is a direct summand in R^n by assumption, write $R^n = R^n/P \oplus P$. By the homomorphism theorem, f factors into an isomorphism $R^n/P \to R^k, r + P \mapsto f(r)$. The map

$$f' : (R^n/P) \oplus P \to R^k \oplus R^r$$
$$(r + P, p) \mapsto \big(f(r), g(p)\big)$$

is the desired isomorphism. □

We say R has the **invariant basis number (IBN)** property, if $R^n \simeq R^m \Rightarrow n = m$ for any n, m. IBN does not hold in general. We know it holds for vector spaces and this allows us to prove the same for commutative rings.

Proposition 2.4.2: *If* R *is a non-trivial commutative ring then* R *has the IBN property.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R, and suppose $R^n \simeq R^m$, then taking $R/\mathfrak{m} \otimes_R -$ on both sides gives $R/\mathfrak{m} \otimes_R R^n \simeq (R/\mathfrak{m})^n \simeq (R/\mathfrak{m})^m$ by Theorem 1.2.1. $R/\mathfrak{m}$ is a field, so this isomorphism implies $n = m$. □

Since free bases have the same convenient property as vector space bases, that a map defined on the basis extends uniquely to a homomorphism, we see that any homomorphism $R^n \to R^k$ can be represented by an $n \times k$ matrix. The epimorphism $f : R^n \twoheadrightarrow R^k$, which defines a stably free module P, is necessarily split (there exists a homomorphism g such that $f \circ g = id_{R^k}$), hence the matrix A representing such a homomorphism is always right invertible.

Assuming R to be commutative in Proposition 2.4.1 just gives $n = k + r$. So a stably free $P \in \mathfrak{P}(R)$ of type k for R commutative can be equivalently defined as the kernel (or *solution space*) of a right invertible $n \times k$ matrix. Proposition 2.4.1 now has an equivalent statement in terms of matrices.

Proposition 2.4.3: *Let* R *be a commutative ring and* A *a right invertible* $n \times k$ *matrix with entries in* R*. The stably free module* $\ker A$ *is free if and only if* A *can be completed to a square invertible matrix.*

Definition 2.4.1: A $1 \times n$ matrix $\begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix}$ over a ring R is called a **unimodular row** if it is right invertible, i.e. if there exist $s_i \in R$ such that $r_1 s_1 + r_2 s_2 + \cdots r_n s_n = 1$. We denote $\mathrm{Un}_n(R)$ the set of unimodular rows of length n with entries in R.

Putting all this together (and using induction) we arrive at three equivalent properties:

Definition 2.4.2: A commutative ring R is called **Hermite** if it satisfies any of the following equivalent conditions:

i) Any stably free $P \in \mathfrak{M}(R)$ is free.

ii) Any stably free $P \in \mathfrak{M}(R)$ of type 1 is free.

iii) Any unimodular row over R can be completed to a square invertible matrix.

Let G be a subgroup of $\mathrm{GL}_n(R)$ and let G act on $\mathrm{Un}_n(R)$ by matrix multiplication on the right. We write $(\cdots) \sim_G (\cdots)$ to indicate that two rows are in the same orbit.

Lemma 2.4.1: *A row* $\begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix} \in \mathrm{Un}_n(R)$ *is completable to a square invertible matrix if and only if* $\begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix} \sim_{\mathrm{GL}_n(R)} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$.

Proof. $\begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix} \sim_{\mathrm{GL}_n(R)} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ if and only if

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \cdot A = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix}$$

for some $A \in \mathrm{GL}_n(R)$. Multiplying a matrix on the left by $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ amounts to picking out the first row of that matrix, so A *is* the completion of $\begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix}$ to a square invertible matrix. □

2.4.1 Elementary Matrices

We denote $E_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by matrices of the form $id_n + r \cdot e_{ij}$ $(i, j \in \{1, ..., n\}, i \neq j)$, where e_{ij} has 1 in the (i, j) entry and 0 in all other entries. These matrices are invertible (the inverse being $id_n - re_{ij}$), so $E_n(R)$ is well-defined.

If $a = (a_1, ..., a_n) \in \mathrm{Un}_n(R)$, then $a \cdot (id_n + r \cdot e_{ij}) = (a_1, ..., a_j + ra_i, ..., a_n)$. So we have the basic relation

$$(a_1, ..., a_n) \sim_{E_n(R)} (a_1, ..., a_j + ra_i, ..., a_n) \tag{2.3}$$

Lemma 2.4.2: *If* $a = (a_1, ..., a_n) \in \mathrm{Un}_n(R)$ *contains a unimodular row of shorter length, then* $a \sim_{E_n(R)} (1, 0, ..., 0)$.

Proof. The hypothesis just means that $\sum_{j=0}^{k} r_j a_{i_j} = 1$ for some $k < n$. We can find an a_k such that $k \neq i_j$ for any $j = 1, ..., k$. Successively applying $id_n + (1 - a_k) r_j e_{i_j,k}$ to a, we get

$$a \sim_{E_N(R)} \left(a_1, ..., a_k + (1 - a_k) \cdot \sum_{j=0}^{k} r_j a_{i_j}, ...a_n \right) = (a_1, ..., a_{k-1}, 1, a_{k+1}, ..., a_n)$$

Another series of elementary matrices brings the rest of the entries to 0. Finally, applying $id_n + e_{k,1}$ puts 1 in the first entry and $id_n - e_{1,k}$ then changes the 1 in the kth entry to 0 yielding $a \sim_{E_n(R)} (1, 0, ..., 0)$. □

Lemma 2.4.3: *Suppose R is a commutative, finite-dimensional algebra over a field* $\mathbf{k}$*, then* $a \sim_{E_n(R)} (1, ..., 0)$ *for any* $a = (a_1, ..., a_n) \in \mathrm{Un}_n(R)$.

Proof. Any ideal in R is also a $\mathbf{k}$-vector subspace of R. Hence R is artinian, because an infinite chain of strictly decreasing ideals would mean an infinite chain of subspaces, which cannot happen for a finite-dimensional R.

Since R is artinian, any set of ideals in R has a minimal element with respect to inclusion. Take the set of finite products of maximal ideals in R and denote the minimal element of this set by $\mathfrak{m}_1 \cdots \mathfrak{m}_k$ (for $\mathfrak{m}_i \subset R$ maximal, pairwise distinct ideals). Then $\mathfrak{m}_1, ..., \mathfrak{m}_k$ are the *only* maximal ideals in R. For suppose $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m} \cdot \mathfrak{m}_1 \cdots \mathfrak{m}_k \subseteq \mathfrak{m}_1 \cdots \mathfrak{m}_k$ implies $\mathfrak{m} \cdot \mathfrak{m}_1 \cdots \mathfrak{m}_k = \mathfrak{m}_1 \cdots \mathfrak{m}_k$ by minimality and it follows that $\mathfrak{m} = \mathfrak{m}_i$ for some i.

So R has finitely many maximal ideals[1] and hence

$$R/\mathrm{Rad}(R) = R \Bigg/ \left(\bigcap_{i=1}^{k} M_i \right) \simeq \bigoplus_{i=1}^{k} R/\mathfrak{m}_i = \bigoplus_{i=1}^{k} \mathbf{k}_i \tag{2.4}$$

where $\mathbf{k}_i$ are fields. This is trivial for $k = 1$ and for $k > 1$ it follows from the Chinese Remainder Theorem since $\mathfrak{m}_i$ are pairwise coprime by maximality.

Now, $(a_1, ..., a_n) \in \mathrm{Un}_n(R)$ amounts to $Ra_1 + \cdots + Ra_n = R$. Denote $I = a_2R + \cdots a_nR$ and $J = \mathrm{Rad}(R)$. The above obviously implies $(a_1 + J) \cdot R/J + I/J = R/J$ so there exist $r \in R, b \in I$ such that

$$(a_1 + J) \cdot (r + J) + (b + J) = 1 + J \tag{2.5}$$

Since R/J is a finite direct product of fields, it will help to look at (2.5) componentwise. Suppose $a_1 + J \mapsto (a_{1,1}, ..., a_{1,k})$, $r + J \mapsto (r_1, ..., r_k)$, $b + J \mapsto (b_1, ..., b_k)$ in the isomorphism of (2.4). Then (2.5) is equivalent to $\forall i = 1, .., k : a_{1,i} r_i + b_i = 1$ which forces

$$\forall i = 1, .., k : a_{1,i} = 0 \Rightarrow b_i = 1.$$

Define $r' \in R$ componentwise: There exists $r' \in R$ such that $r' + J \mapsto (r'_1, ..., r'_k)$ where $r'_i = 1$ if $a_{1,i} = 0$ and $r'_i = 0$ if $a_{1,i} \neq 0$ (reduction modulo J is onto). Then $r'b \in I$ and $a + r'b + J$ is non-zero in each component.

It follows that $a + r'b$ is a unit in R: If it were not a unit, there would exist a maximal ideal $\mathfrak{m}$ of R containing $a + r'b$ and $a + r'b + J$ would be zero in the component corresponding to $\mathfrak{m}$.

[1] R is said to be semilocal. [8]

We have proved that a_1+I contains a unit, i.e. there exists a unit $u \in R^\times$ and $r_2, ..., r_k \in R$ such that $u = a_1 + \sum_{i=2}^k r_i a_i$. Then $a \sim_{E_n(R)} (u, a_2, ..., a_n)$ and this contains the unimodular row (u) of length 1, so $(u, a_2, ..., a_n) \sim_{E_n(R)} (1, 0..., 0)$ by Lemma 2.4.2. □

Lemma 2.4.4: *Let R be a commutative ring, S a commutative R-algebra, and $f_1, f_2 \in R[t]$. Suppose $c \in R \cap (f_1 \cdot R[t] + f_2 \cdot R[t])$ is not a zero divisor. Then for any $b, b' \in S$ we have $b \equiv b' \mod cS \implies \exists A \in \mathrm{SL}_2(S) : (f_1(b), f_2(b)) \cdot A = (f_1(b'), f_2(b'))$*

Proof. Write $c = f_1 g_1 + f_2 g_2$ for $g_1, g_2 \in R[t]$. Since c is not a zero divisor, the localization $A[c^{-1}]$ is defined and we can write

$$A = \frac{1}{c} \cdot \begin{pmatrix} g_1(b) & -f_2(b) \\ g_2(b) & f_1(b) \end{pmatrix} \cdot \begin{pmatrix} f_1(b') & f_2(b') \\ -g_2(b') & g_1(b') \end{pmatrix}$$

Now in S/cS we have $b = b'$ so that

$$\begin{pmatrix} g_1(b) & -f_2(b) \\ g_2(b) & f_1(b) \end{pmatrix} \cdot \begin{pmatrix} f_1(b') & f_2(b') \\ -g_2(b') & g_1(b') \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = 0$$

So the entries of this product are all multiples of c and hence the the entries of A belong to S. Furthermore, $\det A = 1/c^2 \cdot c \cdot c = 1$. So $A \in \mathrm{SL}_2(A)$ as needed and

$$\begin{pmatrix} f_1(b) & f_2(b) \end{pmatrix} \cdot A = (1, 0) \cdot \begin{pmatrix} f_1(b') & f_2(b') \\ -g_2(b') & g_1(b') \end{pmatrix} = \begin{pmatrix} f_1(b') & f_2(b') \end{pmatrix}$$

(The calculation is done in $A[c^{-1}]$ but the result holds in A as a subring of $A[c^{-1}]$.) □

Lemma 2.4.5: *Let R be a commutative ring, S a commutative R-algebra, and $f \in \mathrm{Un}_n(R[t])$. Denote $f(b) = \begin{pmatrix} f_1(b) & \cdots & f_n(b) \end{pmatrix} \in \mathrm{Un}_n(S)$ for $b \in S$. Then the set $I_{f,S,G} = \{c \in R \mid b = b' \mod cS \implies f(b) \sim_G f(b')\}$ is an ideal in R for any subgroup G of $\mathrm{GL}_n(S)$.*

Proof. Suppose $c, c' \in I_{f,S,G}$, we need to prove $rc + r'c' \in I_{f,S,G}$. Suppose $b - b' = (rc + r'c')s$ for $b, b', s \in S$, then $b - rcs = b' + r'c's$ and we have by assumption

$$f(b) \sim_G f(b - rcs) = f(b' + r'c's) \sim_G f(b')$$

Hence, $f(b) \sim_G f(b')$. □

To make use of Lemma 2.4.4, we will view $\mathrm{SL}_2(S)$ as a subgroup of $\mathrm{SL}_n(S)$ for $n \geq 2$ by the embedding

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & id_{n-2} \end{pmatrix}$$

The following theorem is due to Suslin. [6] It provides a different proof of Serre's conjecture from the one that Suslin published in 1976, this one being less involved.

Theorem 2.4.1: *Let R be an integral domain, $f = (f_1, ..., f_n) \in \mathrm{Un}_n(R[t])$ (for $n \geq 2$) with f_1 unitary, and let S be a commutative R-algebra and $b, b' \in S$.*

Then $f(b) \sim_G f(b')$, where G is the subgroup of $\mathrm{GL}_n(S)$ generated by $\mathrm{SL}_2(S)$ and the subgroup E_n of elementary row transformations.

Proof. For $n = 2$, this follows directly from Lemma 2.4.4, so suppose $n \geq 3$. We will prove that $I = I_{f,S,G}$ from the Lemma 2.4.5 contains 1, the condition $b = b'$ mod $1 \cdot S$ is then vacuous, hence $f(b) \sim_G f(b')$ will hold for all $b, b' \in S$.

It will be enough to find $c \in I \setminus \mathfrak{m}$ for an arbitrary $\mathfrak{m} \in \max R$. By the isomorphism theorems:

$$R' = R[t] \Big/ (\mathfrak{m}[t] + R[t] \cdot f_1) \simeq \Big(R[t]/\mathfrak{m}[t]\Big) \Big/ (\mathfrak{m}[t] + R[t] \cdot f_1/\mathfrak{m}[t])$$

Consider the isomorphism

$$\alpha : R[t]/\mathfrak{m}[t] \to (R/\mathfrak{m})[t]$$
$$\sum a_i t^i + \mathfrak{m}[t] \mapsto \sum (a_i + \mathfrak{m}) t^i$$

(defined using the first isomorphism theorem). Since f_1 is unitary, $\overline{f_1} = \alpha(f_1 + \mathfrak{m}[t])$ is non-zero and we can write

$$\beta : R[t]/\mathfrak{m}[t] \to (R/\mathfrak{m})[t]/\overline{f_1}$$
$$g + \mathfrak{m}[t] \mapsto \overline{g} \mod \overline{f_1}$$

The kernel contains those $g(t) + \mathfrak{m}[t]$ for which $\overline{g}$ is a multiple of $\overline{f}$. Using the fact that α is an isomorphism, we have $\overline{g} = \overline{f_1} \cdot \overline{h} \Leftrightarrow g + \mathfrak{m}[t] = f_1 h + \mathfrak{m}[t]$ for some $h \in R[t] \Leftrightarrow g + \mathfrak{m}[t] \in (\mathfrak{m}[t] + R[t] \cdot f_1)/\mathfrak{m}[t]$. Which means $\ker \beta = (\mathfrak{m}[t] + R[t] \cdot f_1)/\mathfrak{m}[t]$ yielding an isomorphism

$$R' \simeq (R/\mathfrak{m})[t] \Big/ \overline{f_1}$$

This is a finite-dimensional commutative algebra over the field $R/\mathfrak{m}$ (the cosets modulo f_1 are represented by polynomials of degree $< \deg f_1$). We apply Lemma 2.4.3 as follows: Reducing f modulo $\mathfrak{m}[t] + R[t]f_1$, we get $(f_2 + \mathfrak{m}[t] + R[t]f_1, ..., f_n + \mathfrak{m}[t] + R[t]f_1) \in \mathrm{Un}_{n-1}(R')$ and the lemma allows us to find a matrix $E' \in E_n(R')$ such that

$$(f_2 + \mathfrak{m}[t] + R[t]f_1, ..., f_n + \mathfrak{m}[t] + R[t]f_1) \cdot E' = (1 + \mathfrak{m}[t] + R[t]f_1, 0, ..., 0)$$

We can find a matrix $E \in E_{n-1}(R[t])$ such that reducing each entry of E yields back E', this is obvious for an elementary matrix $I + re_i j$ and extends to $E_n(R[t])$ by homomorphism. Define

$$(g_2, ..., g_n) = (f_2, ..., f_n) \cdot E \tag{2.6}$$

Then $g_2 = 1 \mod \mathfrak{m}[t] + R[t]f_1 \Longrightarrow (R[t]g_2 + R[t]f_1) + \mathfrak{m}[t] = R[t]$ and applying Lemma 2.3.2, $(R[t]g_2 + R[t]f_1) \cap R + \mathfrak{m} = R$ which in turns gives $c \in (R[t]g_2 + R[t]f_1) \cap R$ such that $c \notin \mathfrak{m}$. We can now finish the proof by showing $c \in I$.

Take $b, b' \in S$ such that $b = b' \mod cS$, by the choice of $c \in R[t]g_2 + R[t]f_1$, this means

$$g_i(b) - g_i(b') \in cS \subseteq f_1(b)S + g_2(b)S \tag{2.7}$$

for each $i \geq 2$. We note c is not a zero-divisor (allowing Lemma 2.4.4) because R is a domain and we embed $E_{n-1}(S) \subseteq E_n(S)$ by $A \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)$.

$$\begin{aligned} f(b) &\sim_{E_n(S)} (f_1(b), g_2(b), ..., g_n(b)) && \text{by (2.6) using the matrix } E(b) \\ &\sim_{E_n(S)} (f_1(b), g_2(b), g_3(b')..., g_n(b')) && \text{by (2.7)} \\ &\sim_{\mathrm{SL}_n(S)} (f_1(b'), g_2(b'), g_3(b')..., g_n(b')) && \text{Lemma 2.4.4} \\ &\sim_{E_n(S)} f(b') \end{aligned}$$

The final step just reverses the first. We have $c \in I$ by definition. □

Applying this theorem for $S = R[t]$, $b' = t$, and $b = 0$ yields

Corrolary 2.4.1: *Let R be an integral domain and $f = (f_1, ..., f_n) \in \mathrm{Un}_n(R[t])$ for $n \geq 2$ and f_1 unitary, then $f \sim_G f(0)$, where G is generated by $E_n(R[t])$ and $\mathrm{SL}_2(R[t])$.*

Remark 2.4.1*:* Lemma 2.4.4 holds even if c is a zero divisor. The assumption that R is an integral domain is not actually needed in Theorem 2.4.1 and Corrolary 2.4.1.

2.4.2 Nagata's Lemma and the Finished Proof

The following Lemma comes from Nagata [9], proven here in more detail. We define $\deg f$ for a polynomial in many variables as the maximum sum of the exponents of variables appearing in any term of f.

Lemma 2.4.6 (Nagata)**:** *Let $\mathbf{k}$ be a field, $n \in \mathbb{N}$. Then for any non-zero $f \in \mathbf{k}[t_1, ..., t_n]$, there exist natural numbers $m_2, ..., m_n$ and $c \in \mathbf{k} \setminus \{0\}$ such that*

$$f(t_1, t_2 + t_1^{m_2}, t_3 + t_1^{m_3}, ..., t_n + t_1^{m_n}) = c \cdot h(t_1, ..., t_n)$$

where h is monic as a polynomial in $\Big(\mathbf{k}[t_2, ..., t_n]\Big)[t_1]$, i.e. $h(t_1, .., t_n) = t_1^{m_1} +$ terms with lower powers of t_1.

In this setting $f(t_1, t_2 + t_1^{m_2}, t_3 + t_1^{m_3}, ..., t_n + t_1^{m_n})$ is unitary in $\Big(\mathbf{k}[t_2, ..., t_n]\Big)[t_1]$ and the map $g(t_1, ..., t_n) \mapsto g(t_1, t_2 + t_1^{m_2}, t_3 + t_1^{m_3}, ..., t_n + t_1^{m_n})$ is an automorphism of $\mathbf{k}[t_1, ..., t_n]$.

Proof. Take $m > \deg f$ and set $m_j = m^{j-1}$, $j = 1, ..., n$. For any tuple of non-negative integers $i = (i_1, ..., i_n)$ we have the monomial $M_i = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$, and define $w(M_i) = \sum_{j=1}^n m_j \cdot i_j$, the *weight* of M_i.

By the choice of m, weight orders the monomials of degree at most $\deg f$ "reverse" lexicographically, i.e. $w(M_i) \leq w(M_{i'}) \Leftrightarrow i \leq_{Lex'} i'$. (by Lex' we mean the entries in the tuples are read back to front.) this is because m is so large that the i_js appearing in $w(M) = \sum_{j=1}^n m^{j-1} \cdot i_j$ cannot interfere with each other (i coincides with the "m-ary" expansion of $w(M_i)$ written backwards). The weights of two monomials are equal if and only if the corresponding tuples are equal.

We write f out as

$$f(t_1, ..., t_n) = \sum_{i=(i_j)_{j=1}^n} a_i \, t^{i_1} \cdots t^{t_n} = \sum_{M \text{monomial}} a_M \cdot M$$

The discussion above means that there is a monomial M_0 of *strictly* largest weight among those appearing in f, i.e. among those M for which $a_M \neq 0$. Hence

$$f(t_1, t_2 + t_1^{m_2}, t_3 + t_1^{m_3}, ..., t_n + t_1^{m_n}) = a_{M_0} t_1^{w(M_0)} + \text{ terms with lower powers of } t_1$$

and setting $c = a_{M_0} \neq 0$ completes the first statement.

Since $c \neq 0$ and $\mathbf{k}$ is a field, c is a unit and $c \cdot h$ is unitary. Finally, taking $g(t_1, ..., t_n)$ to $g(t_1, t_2 - t_1^{m_2}, t_3 - t_1^{m_3}, ..., t_n - t_1^{m_n})$ is inverse to the map in the second statement. Both maps are obviously ring homomorphisms, so they are mutually inverse isomorphisms. □

Theorem 2.4.2: *If* $\mathbf{k}$ *is a field, then* $\mathbf{k}[t_1, ..., t_n]$ *is Hermite.*

Proof. We prove $S = \mathbf{k}[t_1, ..., t_n]$ is Hermite by induction on n, the number of variables. For $n = 0$, we have $S = \mathbf{k}$ a field, any $\mathbf{k}$-module (vector space) is free, so $\mathbf{k}$ is Hermite.

For $n > 0$ we take a unimodular row $f = (f_1, ..., f_m) \in \mathrm{Un}_m(S)$. We wish to prove that $f \sim_{\mathrm{GL}_m(S)} (1, 0, \ldots, 0)$ and apply Lemma 2.4.1. If $f_1 = 0$, then $(f_2, ..., f_m)$ is a shorter unimodular row and we are done by Lemma 2.4.2.

Suppose $f_1 \neq 0$. We interpret f as a unimodular row over $(\mathbf{k}[t_2, .., t_n])[t_1]$ and apply Lemma 2.4.6 to bring f_1 to a unitary polynomial. Since the change of variables given by the lemma is an automorphism on S, and since the target $(1, 0, ..., 0)$ is unchanged by it, we can just as well assume f_1 is already unitary.

Since $\mathbf{k}[t_2, .., t_n]$ is an integral domain, Corrolary 2.4.1 applies, yielding

$$f(t_1, t_2, ..., t_n) \sim_{\mathrm{GL}_m(S)} f(0, t_2, ..., t_n)$$

which is a unimodular row of polynomials in $n - 1$ variables.

By induction, $f \sim_{\mathrm{GL}_m(S)} (1, 0, ..., 0)$ and so f can be completed to a square invertible matrix by Lemma 2.4.1. Since $f \in \mathrm{Un}_m(S)$ was arbitrary, this proves that S is Hermite. □

The proof of Serre's Conjecture is thus obtained:

Theorem 2.4.3 (Quillen-Suslin)**:** *Let* $\mathbf{k}$ *be a field and* $S = \mathbf{k}[t_1, ..., t_n]$*, then any* $P \in \mathfrak{P}(S)$ *is free.*

Proof. Any $P \in \mathfrak{P}(S)$ is stably free by Corrolary 2.2.3. Since S is Hermite, any stably free module is free. □

3. The Bass-Quillen Conjecture

This chapter recounts some of the progress made on the Bass-Quillen Conjecture since 1976—when it replaced the newly settled Serre's Conjecture.

The period 1976 – 2006 is covered in Lam [6]. This chapter contains a summary of that account, supplemented with some missing definitions and extended by the results of Dorin Popescu from 2020. It contains only some sketches of proofs, serving as a suggestion for further reading rather than a self-contained text.

The Bass-Quillen Conjecture is the following supposed generalisation of Theorem 2.4.3:

Bass-Quillen Conjecture (BQ)**:** Let R be a regular ring. Any finitely generated projective module over $R[t_1, ..., t_n]$ is extended from R.

Note that Lemma 2.0.1 extends easily to the case of multiple variables, yielding for any $R[t_1, ..., t_n]$-module M:

$$M \text{ is extended from } R \Longleftrightarrow M \simeq R[t_1, ..., t_n] \otimes_R M \Big/ \langle t_1, ..., t_n \rangle M$$

3.1 Preliminaries

This section will introduce the usual definition of *regular rings*, explain in what sense BQ is a generalisation of Serre's Conjecture, and list some fundamental theorems which have been used to chip away at BQ.

R is taken to be a commutative ring throughout.

Definition 3.1.1: Let R be a commutative ring. We say a chain of *strict* inclusions of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ has **length** n. The **Krull dimension** of R is defined as the supremum of the lengths of such chains in spec R.

A noetherian local ring $(R, \mathfrak{m})$ is called a **regular local ring** if its Krull dimension is equal to the minimal number of generators of $\mathfrak{m}$.

A noetherian ring R is called **regular** if $R_\mathfrak{p}$ is a regular local ring for every $\mathfrak{p} \in$ spec R.

Remark 3.1.1*:* A theorem of Serre (Theorem 19.2 in Matsumura [8]) states that a noetherian R is regular in this sense if and only if R has **finite global dimension**, i.e. iff there exists a fixed $n_0 \in \mathbb{N}$ such that every $M \in \mathfrak{M}(R)$ has a projective resolution by at most n_0 terms.

So the "usual" definition of regularity given here implies *left* regularity in the sense of Definition 2.1.1. In the case of a local ring, the opposite implication holds as well.

Assuming Serre's Characterization of regular rings, the proof given in this paper of Swan's Theorem 2.1.1 actually proves that "R regular $\Rightarrow$ $R[t]$ regular" holds for this stronger form of regularity as well.

The notion of an $R[t_1, ..., t_n]$-module being extended from R was instrumental in proving Theorem 2.4.3. It is no accident that it is used to formulate BQ:

Lemma 3.1.1: *Suppose R has the property that any f.g. projective module is free. Then any $P \in \mathfrak{P}(R[t_1, ..., t_n])$ is extended from R if and only if any $P \in \mathfrak{P}(R[t_1, ..., t_n])$ is free.*

Proof. Follows from the fact that M is R-projective if and only if $R[t_1, ..., t_n] \otimes_R M$ is $R[t_1, ..., t_n]$-projective. $\square$

By Theorem 1.2.2, a local ring has this extra property, so BQ can be stated locally as:

(BQ'): Let R be a regular local ring. Any $P \in \mathfrak{P}(R[t_1, ..., t_n])$ is free.

We actually have BQ $\iff$ BQ' by the following local-global theorem of Quillen: [6] (p.160)

Theorem 3.1.1 (Quillen's Patching Theorem)**:** *Let R be a commutative ring, A a (possibly non-commutative) R-algebra, and let M be a finitely presented $A[t_1, ..., t_n]$-module.*

If $M_\mathfrak{m} \in \mathfrak{M}^{A_\mathfrak{m}}\Big(A_\mathfrak{m}[t_1, ..., t_n]\Big)$ for every $\mathfrak{m} \in \max R$, then $M \in \mathfrak{M}^A\Big(A[t_1, ..., t_n]\Big)$.

Recall that any finitely generated projective module is finitely presented. The case $A = R$ and $M \in \mathfrak{P}(R[t_1, ..., t_n])$ is what we are after. This reductive argument is used repeatedly.

Definition 3.1.2: We say R satisfies the **extension property** (E_k) for some $k \geq 1$ if any $P \in \mathfrak{P}(R[t_1, ..., t_k])$ is extended from R. If R satisfies (E_k) for each $k \in \mathbb{N}$, then we say R has the property (E).

Quillen's Patching Theorem just says that (E) (as well as any (E_k)) is a property which can be checked *locally*.

Denote $R\langle t\rangle$ the localization of $R[t]$ at the multiplicative set of monic polynomials. This ring is the subject of a technical theorem of Horrocks: [6] (p.171)

Theorem 3.1.2 (Affine Horrocks' Theorem)**:** *Let R be a commutative ring and $P \in \mathfrak{P}(R[t])$. If $R\langle t\rangle \otimes_{R[t]} P$ is extended from a f.g. projective R-module, then $P \in \mathfrak{P}^R\Big(R[t]\Big)$.*

Finally, another important method developed by Quillen is so-called Quillen induction, which allows one to check the property (E) for some family $\mathfrak{F}$ of commutative rings.

Theorem 3.1.3: *Suppose a class $\mathfrak{F}$ of commutative rings satisfies the properties:*

Q_1 : *If $R \in \mathfrak{F}$, then $R\langle t\rangle \in \mathfrak{F}$,*

Q_2 : *If $R \in \mathfrak{F}$ and $\mathfrak{m} \in \max R$, then $R_\mathfrak{m} \in \mathfrak{F}$,*

Q_3 : *Any local ring $R \in \mathfrak{F}$ has the property (E_1).*

Then any $R \in \mathfrak{F}$ satisfies (E).

Proof. Sketch: Each (E_k) is proved by induction on k. (E_1) follows from Q_2 and Q_3 by Quillen Patching. For $n > 1$, write $A = R[t_2, ..., t_n]$ and take $P \in \mathfrak{P}\Big(A[t_1]\Big)$.

If P is shown to be extended from A, i.e. $P \simeq A[t_1] \otimes_A P_0$ where $P_0 \in \mathfrak{P}R[t_2, ..., t_n]$ then the inductive hypothesis says P_0 (and hence also P) is extended from R.

By Theorem 3.1.2, it is enough to prove:

$$A\langle t_1\rangle \otimes_{t_1} P \text{ is extended from an f.g. projective } R\text{-module.} \tag{3.1}$$

Consider the subring $S = R\langle t_1\rangle[t_1, ..., t_n]$ of $A\langle t_1\rangle$. Q_1 and the inductive hypothesis imply that any module in $\mathfrak{P}(S)$ is extended from $R\langle t_1\rangle$. A technical argument now yields (3.1). [6] (p. 178) □

We end this section with definitions needed to cite Popescu's later work, the first is taken from Popescu [10]:

Definition 3.1.3: Let A be a commutative associative R-algebra. A is **of finite type** if their exists a finite set of elements $a_1, ..., a_n \in A$ such that any $a \in A$ can be expressed as a polynomial in the a_is with coefficients from R. A is **essentially of finite type** if it is the localization of a finite-type algebra.

A has **geometrically regular fibres** if $K \otimes_{R/\mathfrak{p}} A\big/\mathfrak{p}A$ is regular for any $\mathfrak{p} \in \max R$ and any field extension K of $R/\mathfrak{p}$.

A is **(essentially) smooth** if it is R-flat, has geometrically regular fibres, and is (essentially) of finite type.

Another term used by Popescu is a "filtered inductive limit", perhaps more commonly referred to as a *filtered colimit.*

Definition 3.1.4: A non-empty category K is **filtered** if every finite diagram in K has a cocone.

A **filtered colimit** is the colimit of some diagram $F : K \to L$ where K is a filtered category.

3.2 Partial Results

Remark 3.2.1*:* Using Theorem 1.2.2.i), we know that Theorem 2.2.1 applies to the local case BQ', proving stably-freeness.

Of course, the simplest special case of BQ' is Serre's Conjecture itself, since a field is a local ring $(\mathbf{k}, 0)$ of Krull dimension zero.

The case of Krull dimension ≤ 2 came soon after.

Theorem 3.2.1 (Quillen-Suslin)**:** *If a regular ring R has Krull dimension ≤ 2, then it has the property (E).*

Proof. Sketch: By Quillen Induction on

$$\mathfrak{F} = \{\text{commutative rings of Krull dimension } \leq 2\}.$$

One shows that the Krull dimension of R is equal to that of $R\langle t\rangle$ and no smaller than that of $R[S^{-1}]$ (for any multiplicative set S), implying Q_1 and Q_2. The property Q_3 takes some doing: The case for R of Krull dimension $= 2$ is proved using Theorem 3.1.2, Krull dimension $= 1$ is due to Shesadri, and Krull dimension $= 0$, i.e. where R is a field, has been covered. □

Another special case, keeping the theme of small Krull dimension:

Theorem 3.2.2 (Big Rank Theorem)**:** *Let R be a regular ring of finite Krull dimension d and $P \in \mathfrak{P}(R[t_1, ..., t_n])$. If $P_\mathfrak{m}$ is of rank $> d$ for every $\mathfrak{m} \in \max R$, then P is extended from R.*

Proof. Sketch: Quillen patching is used to reduce to the case where R is local. In this case Theorem 2.2.1 applies, proving P is stably-free. Theorem 2.4.1 was used in Chapter 2 to prove that $\mathbf{k}[t_1, ..., t_n]$ is Hermite, and can similarly be used to prove that a noetherian ring of Krull dimension d is d-Hermite, i.e. that any stably free module of rank $\geq d$ is free. □

The statements and full proofs of the previous two results can be found in Lam [6]. The next big result, known as the Geometric Case of BQ, was obtained in 1981 by Hartmut Lindel. This was later improved upon by Richard Swan; the following statement of the theorem appears in Popescu [10].

Theorem 3.2.3 (Lindel, Swan)**:** *Let $(R, \mathfrak{m})$ be a regular local ring essentially of finite type over $\mathbb{Z}$ and write $p = \text{char } R\big/\mathfrak{m}$.*

i) If $p = 0$ (in R, i.e. $p \cdot 1_R = 0$), then R is essentially smooth over its prime field

ii) If $p \notin \mathfrak{m}^2$, then R is essentially smooth over $\mathbb{Z}$

iii) If either i) or ii) holds for R, then R has the property (E).

Dorin Popescu later [12] proved that any regular local ring is a filtered colimit of rings satisfying the assumption of Theorem 3.2.3, which implies the following by an argument due to Swan:

Theorem 3.2.4 (Popescu)**:** *Let $(R, \mathfrak{m})$ be a regular local ring. If* char $R/\mathfrak{m} = 0$ *or* char $R/\mathfrak{m} \notin \mathfrak{m}^2$*, then R has the property (E).*

Proof. Sketch: One shows that if R is a filtered colimit of R_α and each R_α satisfies (E), then so does R. □

This is the most general special case proven to date. An exposition of Popescu's work on *Néron Desingularization*, which lead him to this result, can be found e.g. in [13]. The following proposition justifies talking about the prime field of R in Theorem 3.2.4.i):

Proposition 3.2.1: *Let $(R, \mathfrak{m})$ be a local ring. Then* char $R/\mathfrak{m} = 0$ *(in R) if and only if R contains a field.*

Proof. We make use of the characterization of the units in a local ring $(R, \mathfrak{m})$ as the elements of $R \setminus \mathfrak{m}$ (see Remark 1.2.10).

($\Rightarrow$) Suppose char $R/\mathfrak{m}$ is actually 0, that is $n = 1 + \cdots + 1 \neq 0$ for any finite sum. Then any $n \in \mathbb{N}$ lies outside of $\mathfrak{m}$, hence is invertible in R. So any non-zero integer $(\pm n)$ is invertible. R contains $\mathbb{Q}$ as a subring.

If char $R/\mathfrak{m} = p$. Then $p = 0$ in R implies char $R \leq p$ and the opposite inequality holds trivially. So the characteristic of R is the prime number p and hence R contains the prime field $\mathbb{Z}_p$.

($\Rightarrow$) If a local ring R contains a field $\mathbf{k}$, then the non-zero elements of $\mathbf{k}$ lie in $R \setminus \mathfrak{m}$. So $R/\mathfrak{m}$ contains a field isomorphic to $\mathbf{k}$ and hence char $R/\mathfrak{m} =$ char $\mathbf{k} =$ char R. Then of course $1_R \cdot \text{char } R/\mathfrak{m} = 0$ in R. □

This proposition with Lemma 3.1.1 also allows us to state Theorem 3.2.4.i) in a way which more clearly generalises Serre's Conjecture:

Corrolary 3.2.1: *If a regular local ring R contains a field, then any $P \in \mathfrak{P}(R)$ is free.*

3.2.1 Popescu's Reduction

We turn our attention to a new (2020) result of Popescu [10], which builds on all of the previous work and reduces the Bass-Quillen Conjecture to the following question:

(P): Let $(R, \mathfrak{m})$ be a regular local ring essentially smooth over $\mathbb{Z}_{(p)}$, where $p = \text{char } R/\mathfrak{m}$, and take $b \in \mathfrak{m}^2$ arbitrary. Does the ring $R\big/(p-b)R$ have the property (E)?

Popescu's main result is that checking (E) for this special kind of ring already proves (E) for *any* regular ring. The key ingredient in Popescu's result is that he completes the analysis of Theorem 3.2.3.i),ii) for the other cases of p.

Theorem 3.2.5: *If $(R, \mathfrak{m})$ is a regular local ring and $0 \neq p \in \mathfrak{m}^2$, then R is a filtered colimit of rings R_α essentially smooth over the rings $A_\alpha\big/(p - b_\alpha)A_\alpha$, where each A_α satisfies the requirements of (P).*

Corrolary 3.2.2 (Popescu)**:** *If P is answered affirmatively, then BQ is true.*

Proof. Sketch: Quillen Patching reduces BQ to the local case BQ'. By Theorem 3.2.4, one can assume $0 \neq p \in \mathfrak{m}^2$ and apply Theorem 3.2.5. Each $A_\alpha\big/(p - b_\alpha)A_\alpha$ satisfies (E), since (P) is assumed true. Popescu then proves, that 1) (E) carries over from a ring to any essentially smooth algebra over that ring and 2) (as before) if R is a filtered colimit of R_α and each R_α satisfies (E), then so does R. □

Conclusion

While building up the proof of Serre's Conjecture in the first two chapters, we came upon many of the essential facts of Commutative Algebra and Algebraic Geometry, as well as the origins of Homological Algebra (Section 2.1) and Algebraic K-theory (Section 2.2). Moreover, even the calculus of unimodular rows, sampled in Section 2.4, has become a subject in its own right.

Regarding the Bass-Quillen Conjecture, Hartmut Lindel wrote "Since there does not seem to be much hope for a general solution..." and went on to introduce his breakthrough 1981 result. [11]

Despite this early pessimism, already present since the first attempts at the Bass-Quillen conjecture, a steady flow of new developments has continued to the present day, in the form of solutions in special cases, as discussed in Chapter 3, and analogues such as that of Asok et al. [14] from 2018.

All this to say that, though long since proven and resolved, Serre's Conjecture, with its various offshoots and its historical value, still remains a vital and fascinating subject.

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